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LETTER TO THE EDITOR

An $\alpha = \frac{1}{2}$ singularity in the vicinity of a disorder variety and its random walk interpretation

A Georges†, D Hansel‡, P Le Doussal† and J M Maillard§

† Laboratoire de Physique Théorique de l'Ecole Normale Supérieure, Laboratoire propre du CNRS associé à l'Université de Paris-Sud, 24 rue Lhomond, 75231 Paris Cedex 05, France
‡ Centre de Physique Théorique de l'Ecole Polytechnique, Groupe de Recherche du CNRS, No 48, Route de Saclay, 91128 Palaiseau, Cedex, France
§ Laboratoire de Physique Théorique et Hautes Energies, Laboratoire associé au CNRS, et à l'Université Pierre et Marie Curie, 4 Place Jussieu, 75230 Paris Cedex 05, France

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Abstract. We exhibit a new solution of the anisotropic triangular Ising model. It is based on diagrammatics well suited to the study of the vicinity of the so-called disorder varieties. A systematic expansion in this vicinity reveals an unexpected $\alpha = \frac{1}{2}$ singularity. This hidden singularity is, in turn, interpreted as the genuine critical behaviour of a related random walk process.

The aim of this letter is to describe a new solution of the triangular anisotropic Ising model, based on a new diagrammatic expansion (Georges *et al* 1986) suggested by the study of the vicinity of the so-called 'disorder varieties' (Stephenson 1970). Some simple classes of these diagrams have a good physical interpretation in terms of random walk type problems, which sheds some light on an 'exotic' $\alpha = \frac{1}{2}$ singularity hidden in the well known solution of the triangular Ising model, as a special limiting case (Blöte and Hilhorst 1982).

Let us write the Boltzmann weight associated with each elementary cell of the triangular lattice (figure 1) as

$$W = \exp(K_1\sigma_2\sigma_3 + K_2\sigma_1\sigma_3 + K_3\sigma_1\sigma_2) = (\lambda/2)(1 + T_1\sigma_2\sigma_3 + T_2\sigma_1\sigma_3 + T_3\sigma_1\sigma_2) \qquad (1)$$

where $\lambda = 2\cosh K_1 \cosh K_2 \cosh K_3(1 + t_1t_2t_3)$. The variables $t_i = \tanh K_i$ are well suited to a high temperature expansion, while $T_i = (t_i + t_jt_k)/(1 + t_it_jt_k)$ $(i, j, k = 1, 2, 3)$ are three new variables well suited to an expansion in the vicinity of the disorder varieties. Indeed, the equations of the three disorder varieties are given by $T_i = 0$ ($i = 1$, 2, 3 respectively). The partition function reads

$$Z = 2^{-N}\lambda^N \sum_{\{\sigma\}} \prod_{\Delta} (1 + T_1\sigma_2\sigma_3 + T_2\sigma_1\sigma_3 + T_3\sigma_1\sigma_2). \qquad (2)$$

The new diagrammatic expansion (Georges *et al* 1986) originates from the expansion of the product over all elementary cells. Z is thus given as the sum of all closed diagrams, connected or not, which pass at most once on each hatched triangle.

Because of this constraint, this new diagrammatics differs in nature from the usual high temperature one. In particular, no self-intersection is allowed. At first sight this forbids a complete resummation making use of the well known Vdovichenko counting

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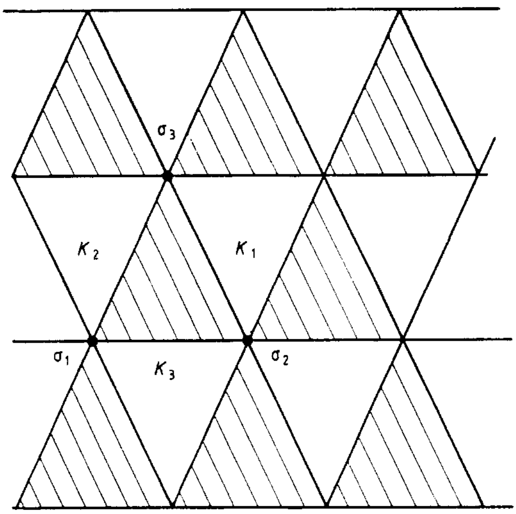

Figure 1. The three coupling constants of the anisotropic triangular Ising model.

rule (Vdovichenko 1965) which applies only to Markovian processes. Indeed this counting rule associates with each vertex an additional factor $e^{i\theta/2}$ where θ is the angle of rotation of the path as it passes through the vertex. This trick enables us to avoid double counting of self-intersecting diagrams and allows a simple application of the exponentiation theorem. One is then left, after Fourier transform, with an expression of $\ln Z$ in terms of the determinant of the 6×6 transition matrix describing the Markovian process at each vertex. The difficulty encountered for the new expansion is that the absence of any self-intersection is *a priori* a non-Markovian feature. This can be overcome by a careful inspection of the three possible self-intersecting situations which could violate our diagrammatic rules. They are depicted in figure 2. Situation (a) can be forbidden in a Markovian way in the recursion process defining the transition matrix. The non-Markovian situations (b) and (c) correspond to diagrams whose self-intersection numbers are of opposite parities, and thus cancel in the sum when one makes use of the phase factor trick. This leads to

$$\ln Z = N \ln \lambda + \tfrac{1}{2} \sum_{p,q=0}^{N} \ln \det(\mathbb{1} - \Lambda_{p,q}) \tag{3}$$

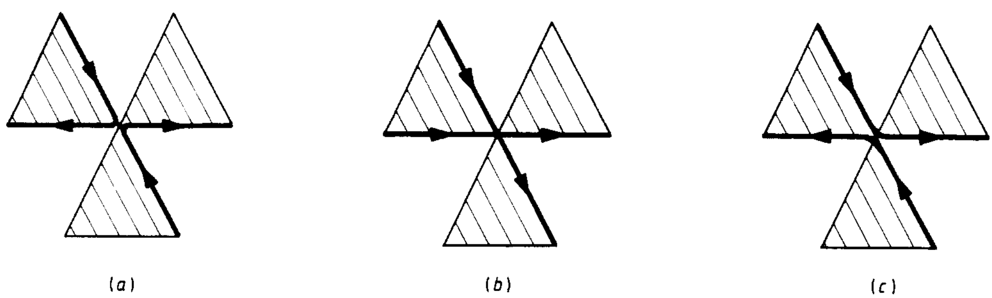

Figure 2. The three forbidden oriented paths at a vertex (up to rotations).

where $\Lambda_{p,q}$ is the 6×6 Fourier transformed transition matrix

$$\Lambda_{p,q}=\begin{bmatrix} T_1\varepsilon^{-p} & \omega T_2\varepsilon^{-q} & \omega^2 T_3\varepsilon^{p-q} & 0 & 0 & \omega^{-1}T_3\varepsilon^{q-p} \\ \omega^{-1}T_1\varepsilon^{-p} & T_2\varepsilon^{-q} & \omega T_3\varepsilon^{p-q} & 0 & 0 & \omega^{-2}T_3\varepsilon^{q-p} \\ 0 & \omega^{-1}T_2\varepsilon^{-q} & T_3\varepsilon^{p-q} & \omega T_1\varepsilon^{p} & \omega^2 T_2\varepsilon^{p} & 0 \\ 0 & \omega^{-2}T_2\varepsilon^{-q} & \omega^{-1}T_3\varepsilon^{p-q} & T_1\varepsilon^{p} & \omega T_2\varepsilon^{p} & 0 \\ \omega^2 T_1\varepsilon^{-p} & 0 & 0 & \omega^{-1}T_1\varepsilon^{p} & T_2\varepsilon^{p} & \omega T_3\varepsilon^{q-p} \\ \omega T_1\varepsilon^{-p} & 0 & 0 & \omega^{-2}T_1\varepsilon^{p} & \omega^{-1}T_2\varepsilon^{p} & T_3\varepsilon^{q-p} \end{bmatrix} \tag{4}$$

where $\varepsilon=e^{i(2\pi/N)}$ and $\omega=e^{i\pi/6}$. Note that the occurrence of two zeros for each line is a consequence of the exclusion rule of the new diagrammatics. In the thermodynamic limit, this gives the following expression for the partition function per site:

$$\ln\lambda+(1/8\pi^2)\int_0^{2\pi}\int_0^{2\pi} dq_1\,dq_2 \ln[1+T_1^2+T_2^2+T_3^2+2(T_2T_3-T_1)\cos q_1 + 2(T_1T_3-T_2)\cos q_2+2(T_1T_2-T_3)\cos(q_1+q_2)]. \tag{5}$$

It is a straightforward but tedious matter to check that expression (5) identifies with the expression given by Houtappel (1950). One can also verify that, on the disorder variety (e.g. $T_3=0$), this expression simply reduces to $\ln\lambda$.

The critical variety is given by the vanishing condition of the argument of the logarithm for the zero mode $q_1=q_2=0$. This leads to the following very simple equation in terms of the T_i:

$$T_1+T_2+T_3=1. \tag{6}$$

It is a welcome remark that, in the isotropic case, the new variable T lies in the interval $[-\frac{1}{3},1]$, the critical value $T_c=\frac{1}{3}$ being the centre of the interval. This new diagrammatics is the natural one for the study of the vicinity of the disorder varieties (e.g. $T_3=0$). Indeed, remarkably, the expansion of (5) in powers of T_3 leads to algebraic expressions at *all* orders in T_3. This is in contrast with the expansion of $\ln Z$ in the high temperature variable t_3 which leads, even at zero and first order, to complicated elliptic functions. This is to be traced to the extreme simplification of the model on the disorder varieties (partition function, correlation functions, susceptibility, etc). From a diagrammatic point of view such an expansion amounts to considering the class of diagrams corresponding to a fixed order in T_3. Remarkably enough it is possible to exhibit closed-recursion relations in each of these classes. At first order, this has been described in Georges *et al* (1986).

This leads one to suspect that these classes of diagrams have a physical interpretation by themselves, which we now illustrate in the example of the first order in T_3. This corresponds to the class of closed diagrams (denoted $\mathscr{C}_1$ in the following) with one horizontal bond only (and necessarily connected at this order). Consider now two random walkers on a one-dimensional lattice, initially separated by two lattice spacings. The discretised time evolution of the walk can be represented as a second spatial dimension. At each step, each of the two walkers must either jump to a neighbouring site (with probability T_1 for a left jump, T_2 for a right jump) or die with probability $1-(T_1+T_2)$. The time evolution of the system naturally generates the triangular lattice, and a particular 'history' for the two walkers corresponds to a two-dimensional diagram of class $\mathscr{C}_1$, which closes if the two walkers meet. Figure 3 summarises this construction. Let us introduce the probability $P(n_1, n_2)$ that this meeting takes place after exactly

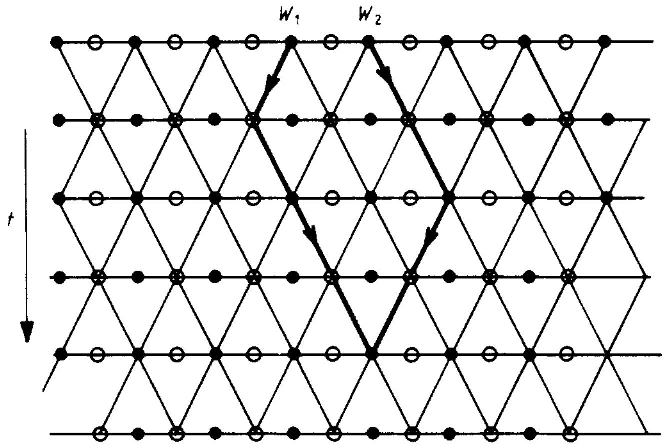

Figure 3. A particular discrete time history for two walkers on the one-dimensional lattice, represented as a diagram of the triangular lattice.

n_1 left jumps and n_2 right jumps of the two walkers. One has

$$P(n_1, n_2) = N_1(n_1, n_2) T_1^{n_1} T_2^{n_2} \tag{7}$$

where $N_1(n_1, n_2)$ is precisely the number of diagrams in the class $\mathscr{C}_1$ with n_1 bonds of type 1 and n_2 of type 2. The total probability of meeting $\mathscr{P}(T_1, T_2)$ given by

$$\mathscr{P}(T_1, T_2) = \sum_{n_1, n_2} P(n_1, n_2) \tag{8}$$

is nothing but the generating function of the diagrams of class $\mathscr{C}_1$. It satisfies a simple algebraic equation

$$T_1 T_2 \mathscr{P}^2 + (T_1^2 + T_2^2 - 1)\mathscr{P} + T_1 T_2 = 0. \tag{9}$$

(The only acceptable solution is the one with the minus sign in front of the square root.) From this exact result one easily obtains the asymptotic behaviour for $N_1(n, n)$

$$N_1(n, n) \sim \mu^n n^{-\theta} \tag{10}$$

with $\mu = 2$ and $\theta = \frac{1}{2}$.

This kind of behaviour is frequently encountered for enumeration problems on lattices (for a review see Viennot (1984) or Fisher (1985)). One of many examples is directed site animals for which one also has $\theta = \frac{1}{2}$ (Dhar *et al* 1982, Hakim and Nadal 1983). From equation (9) we see that $\mathscr{P}$ has a non-analytic behaviour when $T_1 + T_2 \to 1$. In this limit the probability of death goes to zero, and the probability of meeting goes to one as $1 - \mathscr{P} \sim [1 - (T_1 + T_2)]^{1-\alpha}$ with $\alpha = \frac{1}{2}$. The averaged time of meeting, $\bar{\tau}$, when this meeting occurs, can be calculated exactly:

$$\bar{\tau} = \frac{\mathscr{P}}{[1 - (T_1^2 + T_2^2) - 2T_1 T_2 \mathscr{P}]} \tag{11}$$

and diverges in this limit like $[1 + (T_1 + T_2)]^{-1}$ which is the inverse of the death probability. For a non-zero death probability, $\mathscr{P} < 1$ and $\bar{\tau}$ remains finite.

In fact, the occurrence and the location of this transition are nothing but a consequence of the original transition of the Ising model itself; when $T_3 \to 0$ the critical variety (6) reduces to $T_1 + T_2 = 1$. These results seem in contradiction with common knowledge on the Ising model: first it is known that no intersection occurs, for finite values of the Ising coupling constants, between the critical and disorder varieties; secondly an algebraic singularity arises, instead of the usual logarithmic one.

The first point is a consequence of the use of the variables T_i, which map the intersection of the critical and disorder varieties (lying at infinite coupling constants) on to finite T_i values. These finite values correspond to the true physical transition for our random walk problem (where T_1, T_2 have a natural interpretation as probabilities).

Secondly, the $\alpha = \frac{1}{2}$ algebraic singularity was already known to occur for the triangular Ising model (Blöte and Hilhorst 1982) but only in the very special limit where all the K_i become infinite. This is nothing other than a special region in the neighbourhood of the disorder varieties. In this region the $\alpha = \frac{1}{2}$ singularity can be interpreted as a roughening transition. In fact, the vicinity of the disorder varieties is the only region of the parameter space where the usual logarithmic singularity is replaced by another singular behaviour (namely $\alpha = \frac{1}{2}$). This can be seen by performing a first integration in equation (5), which leads to a square root in the argument of the logarithm. The argument of the square root becomes a perfect square only on the disorder varieties $T_i = 0$. This explains why the successive coefficients of the expansion in T_3 are algebraic.

To conclude, we have shown that the vicinity of the disorder varieties naturally supports a specific diagrammatic expansion which leads to a new solution of the triangular Ising model. This also sheds some light on the kind of singularities encountered in the vicinity of these disorder varieties; while it is true that no singularity occurs when one is strictly restricted to the disorder varieties, our expansion does reveal a critical behaviour which corresponds to the region where the disorder variety is asymptotic to the critical one. This new critical behaviour ($\alpha = \frac{1}{2}$) hidden in the Ising model can indeed be seen as a genuine one for a new physical problem.

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