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Algebraic invariants in soluble models

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Abstract. This paper exhibits, on the known solutions of the triangle relation for the exactly solved models, some simplifying methods of recovering their parametrisation in terms of algebraic varieties. The relation to the automorphy group, generated by the inverse and spatial symmetries of the model, is also analysed.

1. Introduction

The triangle relation has appeared to be a central element in the resolution of exactly solved two-dimensional models of statistical physics and field theory (Baxter 1982a, Zamolodchikov and Zamolodchikov 1979, Faddeev 1982). By iterated use of the triangle relation, it can be shown that this property between local transfer matrices leads to the commutation of global transfer matrices, which is an essential step in the resolution of the model. However, although the latter derivation can be extended to show the commutativity of transfer matrices of general spin models and of three-dimensional models (Zamolodchikov 1981, Bazhanov and Stroganov 1981, Jaekel and Maillard 1982a), there still remains the problem of solving the general triangle and tetrahedron relations. The great number of unknowns and equations makes it practically impossible to solve them in the general case. Only reduced solutions corresponding to particular symmetries or restrictions, like symmetric vertex models or the hard-hexagon model, are known (Baxter 1982a). However, the parametrisations of these solutions show the same particular structure; the parameter space is foliated into algebraic curves (which can be indexed by the values of some algebraic invariants), each curve being then described in the same fashion by a spectral parameter; within each curve, three local matrices satisfy the triangle relation when their three spectral parameters are in a particular configuration. It thus appears natural to look for solutions of the general triangle or tetrahedron equations in two steps; firstly to exhibit the algebraic varieties and then to parametrise these varieties. This is the approach we apply here to the known soluble models, with the aim of putting into evidence simplifying methods which can be used for more general and three-dimensional models. For the first step we exploit directly the essential consequence of the triangle relation, which is the passage from a local to a global commutativity.

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Studying the commutativity of transfer matrices of increasing size, one is led to derive a whole set of algebraic conditions which can be computed in an easy way. This set of necessary conditions for the transfer matrices to commute moreover expresses itself in the form of a foliation of the parameter space into algebraic varieties. For the second step, we introduce an auxiliary problem which states the stability of pure tensor products under the action of the local transfer matrix. It is thus possible firstly to recover the previous foliation into varieties and secondly to give a parametrisation of the varieties which leads to the simple expression of the triangle relation in terms of spectral parameters. This last approach can also be seen to provide an explicit realisation of the Zamolodchikov algebra (Zamolodchikov and Zamolodchikov 1979), in the form of coherent states.

Another simplifying way, which circumvents the triangle relation, has also been used in the resolution of soluble models. It consists in taking advantage of another local relation which also leads to global properties: the inverse relation (Baxter 1982b), which can be coupled to the spatial symmetries of the model. However, although these symmetries generate an infinite discrete group (the automorphy group), and thus an infinite number of constraints, these are not sufficient by themselves to provide the resolution of the model, which necessitates the introduction of supplementary analytic properties (Baxter 1980, Jaekel and Maillard 1983). It then appears interesting to compare this more general approach with the preceding one. As will be seen in this paper, the automorphy group is compatible with the triangle relation and in particular with the associated foliation into algebraic varieties. It also agrees with the stability of the pure tensor products under the action of the local transfer matrix and, moreover, gives a representation of this action in terms of a translation by an element of the group. This last property will also be used to give a simple representation of the pseudo-vacuum associated with the transfer matrices, a first step towards the Bethe ansatz (1931).

2. Commuting transfer matrices and triangle relation

2.1. Algebraic invariants

Usually, the triangle relation is used to derive the commutation of 'large' transfer matrices, i.e. in the thermodynamic limit when their size N goes to infinity. Nonetheless, the same commutation is quite deducible whatever the value of N is. Hence, it is quite feasible, for the smallest values of N $(1, 2, 3, 4, \ldots)$, to examine the conditions which are implied by the commutation and thus obtain necessary conditions for satisfying the triangle relation.

Let us give an explicit example of the approach by using an asymmetric eight-vertex model. This can be seen either as a vertex model, with the following local transfer matrix L_{ij}^{kl} $(i, j, k, l = \pm 1,\ ij = kl)$ (figure 1(a))

$$\begin{vmatrix} w_1 & 0 & 0 & w_8 \\ 0 & w_3 & w_6 & 0 \\ 0 & w_5 & w_4 & 0 \\ w_7 & 0 & 0 & w_2 \end{vmatrix},$$

or as a spin model, with a Boltzmann weight $w(a, b, c, d)$, $a, b, c, d = \pm 1$, associated to every elementary square, which also depends on eight homogeneous parameters

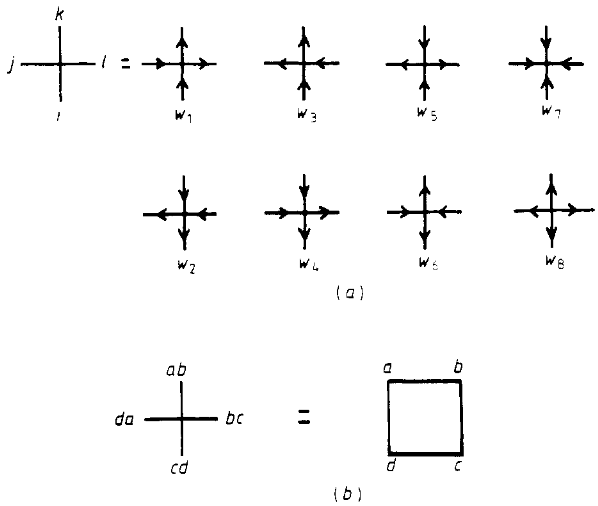


Figure 1. Definition of the weights of the asymmetric eight-vertex model (*a*) as a vertex model and (*b*) as a spin model.

and such that (figure 1(*b*)):

$$w(a, b, c, d) = w(-a, -b, -c, -d). \tag{R}$$

When one writes the commutation of two transfer matrices consisting of N identical local matrices L, in the case of a vertex model, or of N identical local weights, in the case of a spin model (see figure 2), one obtains the following results (the parameters of the second transfer matrix will be denoted by primes).

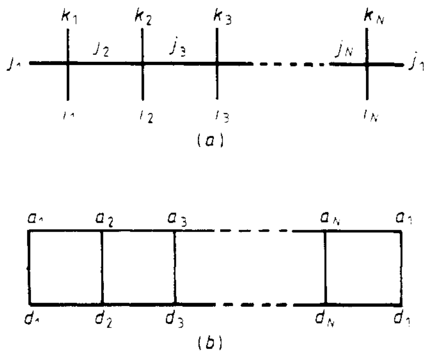


Figure 2. Transfer matrices of size N with periodic boundary conditions: (*a*) for a vertex model and (*b*) for a spin model.

(i) For $N = 1$ the matrices always commute.

(ii) For $N = 2$ they commute only if the following identities are satisfied

$$\begin{aligned} &(w_1^2 + w_3^2 - w_2^2 - w_4^2)/w_6 w_7 = (w_1'^2 + w_3'^2 - w_2'^2 - w_4'^2)/w_6' w_7' \\ &w_5 w_8 / w_6 w_7 = w_5' w_8' / w_6' w_7'. \end{aligned} \tag{1}$$

(iii) For $N=3$ one can order the basis vectors (i_1, i_2, i_3) or (a_1, a_2, a_3), in the following way:

$$(+++), (+--), (-+-), (--+), (---), (-++), (+-+), (++-).$$

The problem reduces to the commutation of two matrices of the form (for a vertex model)

$$\begin{vmatrix} A+B & 0 \\ 0 & C+D \end{vmatrix}$$

or (for a spin model)

$$\begin{vmatrix} A & B \\ B & A \end{vmatrix}$$

where A, B, C, D all have the same form

$$\begin{vmatrix} a & b & b & b \\ c & d & e & f \\ c & f & d & e \\ c & e & f & d \end{vmatrix}.$$

This particular shape of the matrices is a consequence of the translation invariance of the lattice once the periodic boundary conditions have been taken into account. The two 4×4 matrices equal to zero which appear in the vertex model come from the null terms in the local matrix L_{ij}^{kl}, $ij=kl$. The equality of the diagonal matrices (A) and of the antidiagonal matrices (B) for the spin model is a consequence of the symmetry (R). In the latter case, an appropriate change of basis also leads to the form

$$\begin{vmatrix} A+B & 0 \\ 0 & A-B \end{vmatrix}.$$

In both cases a further change of basis by the matrix P

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & \omega & \omega^2 \\ 0 & 1 & \omega^2 & \omega \end{vmatrix},$$

with $\omega^3=1$, then reduces the matrices $A+B$, $A-B$, $C+D$ to the form

$$\begin{vmatrix} E & 0 \\ 0 & F \end{vmatrix}$$

where E and F are 2×2 matrices, with F diagonal. The commutation of such matrices implies the equation

$$\frac{w_1^3+w_3^3-w_1w_4^2-w_2^2w_3-(w_2+w_4)(w_5w_6+w_7w_8)}{(w_1+w_3)w_5w_8}=\text{idem}(w') \tag{2}$$

and the equation obtained by substituting

$$w_1\leftrightarrow w_4 \qquad w_2\leftrightarrow w_3 \qquad w_5\leftrightarrow w_7 \qquad w_6\leftrightarrow w_8 \tag{S}$$

(crossing or spatial symmetry). In the case of the spin model, one obtains the same equation (2), but also its image under

$$w_3 \to -w_3 \qquad w_4 \to -w_4 \qquad w_5 \to -w_5 \qquad w_6 \to -w_6. \tag{T}$$

One should notice that the transformation under crossing of equation (2), which has not been obtained in the case of the spin model, corresponds to antiperiodic boundary conditions ($a_{N+1} = -a_1$). Yet the symmetry (R) of this model allows one to derive, from the triangle relation, not only the commutation of transfer matrices with periodic boundary conditions, but also that with antiperiodic conditions (figure 3). At the order $N = 3$, the latter property provides the crossed image of equation (2), and its transform under (T), which gives a total of four invariants.

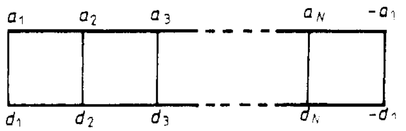


Figure 3. Transfer matrix for a spin model with antiperiodic boundary conditions.

Though described on symmetric models, the method extends without difficulty to general sixteen-vertex or IRF (interaction round a face) spin models (Baxter 1980) (in particular the reduction to 2×2 matrices through diagonalisation by P). In all cases the method provides a set of algebraic invariants which constitute necessary conditions for satisfying the triangle relation. Let us note in the case of the asymmetric eight-vertex model that, already at the $N = 3$ order, the number of invariants is equal to the number of parameters which the transfer matrix depends on (one can set $w_7 = w_8$ according to the weak graph duality) and that it increases at higher order. Hence, in order to satisfy the triangle relation, it is necessary to make these invariants dependant through degeneracies linked to symmetries or other constraints. The choice of the latter can thus be guided by a systematic determination of these invariants. One can see that, for the symmetric eight-vertex model (letting $w_1 = w_2 = a$, $w_3 = w_4 = b$, $w_5 = w_6 = c$, $w_7 = w_8 = d$), one is left with the two known invariants

$$(a^2 + b^2 - c^2 - d^2)/ab \qquad ab/cd;$$

for the asymmetric six-vertex model ($w_1 = a$, $w_2 = a'$, $w_3 = b$, $w_4 = b'$, $w_5 = w_6 = c$, $w_7 = w_8 = 0$), the invariants reduce to

$$(aa' - bb' - c^2)/ab \qquad a'b'/ab$$

(which are obtained at the $N = 4$ order).

Similarly for the hard-hexagon model (with the notations of Baxter and Pearce 1982), one obtains

$$z^{-1/2}(1 - z\,\mathrm{e}^{L+M}) \qquad \text{at the } N = 2 \text{ order}$$

$$z^{-1/2} - z^{1/2}(\mathrm{e}^{L} + \mathrm{e}^{M}) \qquad \text{at the } N = 3 \text{ order}$$

$$z^{-1/2}(\mathrm{e}^{-L} + \mathrm{e}^{-M} - \mathrm{e}^{-(L+M)} - z\,\mathrm{e}^{L+M}) \qquad \text{at the } N = 4 \text{ order.}$$

For the Potts model also, the study of the $N = 3$ and $N = 4$ orders exhibits the two varieties $(\mathrm{e}^{K} - 1)(\mathrm{e}^{K'} - 1) = q$ and $(\mathrm{e}^{K} + 1)(\mathrm{e}^{K'} + 1) = 4 - q$ which are known to correspond to the critical ferromagnetic and antiferromagnetic curves (with the notations of Baxter (1982c)).

Since the tetrahedron relation is known to imply the commutation of transfer matrices of arbitrary size, in the same way as the triangle relation does, the same approach applies without any major modification in three dimensions. It then leads to a set of necessary conditions which are expressed in the form of algebraic invariants for vertex or spin models. (For instance, the symmetric 20-vertex three-dimensional model possesses invariants similar to those of the symmetric six-vertex model, which it generalises.)

2.2 Pure tensor products stability

The foliation of the parameter space into algebraic varieties constitutes only a part of the complete parametrisation of the transfer matrices, which one will need in order to find the common eigenvectors and the corresponding eigenvalues. In order to complete this description, one may remember that, in the Bethe ansatz, the triangle relation between local transfer matrices can be interpreted as the compatibility conditions between the generators of the group of permutations (Yang 1967). This property invites us to look for a 'good' representation of each of the generators, i.e. of each local transfer matrix (which corresponds to a transposition between neighbours). For that purpose, let us introduce the following auxiliary problem, denoted by (A), which we shall for simplicity illustrate on the symmetric six-vertex model. Its local transfer matrix L can be written as a 4×4 matrix (figure 1(*a*)). We require the matrix L and four vectors e, f, e', f' to satisfy the following relation:

$$Le\otimes f=\lambda e'\otimes f' \tag{A}$$

i.e. a pure tensor product is transformed by the action of L into another pure tensor product. This condition can also be written

$$\lambda\begin{vmatrix}1\\ r'\\ p'\\ p'r'\end{vmatrix}=\begin{vmatrix}a&0&0&0\\0&b&c&0\\0&c&b&0\\0&0&0&a\end{vmatrix}\begin{vmatrix}1\\ r\\ p\\ pr\end{vmatrix}\qquad\text{or }\lambda\begin{vmatrix}e'\\ p'e'\end{vmatrix}=\begin{vmatrix}A&B\\C&D\end{vmatrix}\begin{vmatrix}e\\ pe\end{vmatrix} \tag{3}$$

where

$$A=\begin{vmatrix}a&0\\0&b\end{vmatrix}\qquad B=\begin{vmatrix}0&0\\c&0\end{vmatrix}\qquad C=\begin{vmatrix}0&c\\0&0\end{vmatrix}\qquad D=\begin{vmatrix}b&0\\0&a\end{vmatrix}.$$

e' is easily eliminated and hence a non-trivial solution e exists if, and only if, the following equation is satisfied:

$$\det[p'A+pp'B-C-pD]=0$$

$$\Leftrightarrow (p^2+p'^2)ab-pp'(a^2+b^2-c^2)=0. \tag{4}$$

One remarks that the correspondence between p and p' is fixed by the ratio

$$(a^2+b^2-c^2)/ab.$$

This expression coincides with the algebraic invariant associated to the symmetric six-vertex model. This expresses the compatibility of problem (A) with the commutation of transfer matrices (as in § 2.1). Requiring this expression to be equal to the constant $t+1/t$, one sees that two different p' are in correspondence with each p,

$p'^{+}=pt$ and $p'^{-}=p/t$. The equation of the variety can also be written in the form $(a-bt)(a-b/t)=c^2$, giving

$$\begin{aligned} &a-bt=cx && a-b/t=c/x \\ \Leftrightarrow\ &\frac{a}{c}=\frac{t/x-x/t}{t-1/t} && \frac{b}{c}=\frac{1/x-x}{t-1/t}. \end{aligned} \tag{5}$$

Referring to equation (3), one obtains the following expressions for e, f, e', f':

$$\begin{aligned} &r^{+}=(t/x)p && p'^{+}=tp && r'^{+}=(1/x)p \\ &r^{-}=(x/t)p && p'^{-}=(1/t)p && r'^{-}=xp. \end{aligned} \tag{6}$$

Conversely, solving equation (3) for a, b, c in terms of the coordinates (6) of vectors e, f, e', f' just provides the required rational parametrisation (5), which, as one can see is precisely the one deduced from the resolution of the Yang–Baxter equations.

With the help of this parametrisation, one can now come back to the problem of solving the triangle relation. Because of the symmetries of the local transfer matrices (conservation of the total spin and spin reversal symmetry) and of the symmetries of the triangle relation itself (under transposition), the original equations which correspond to the equality of two 8×8 matrices reduce in fact to the equality of two identical antisymmetric 3×3 matrices (the three transfer matrices are respectively denoted by w, w', w''):

$$\begin{aligned} &ab'c''+cc'b''=ba'c'' && ac'b''+cb'c''=bc'a'' \\ &ac'c''+cb'b''=ca'a''. \end{aligned} \tag{7}$$

On the other hand, one can see that, once the condition $xx'=x''$ is satisfied, the property (A) implies the triangle relation for a set of two configurations, which are represented by figure 4 (the notations correspond to those of (3, 6)). These equalities stand for every value of p and hence for an infinite number of configurations. Still,

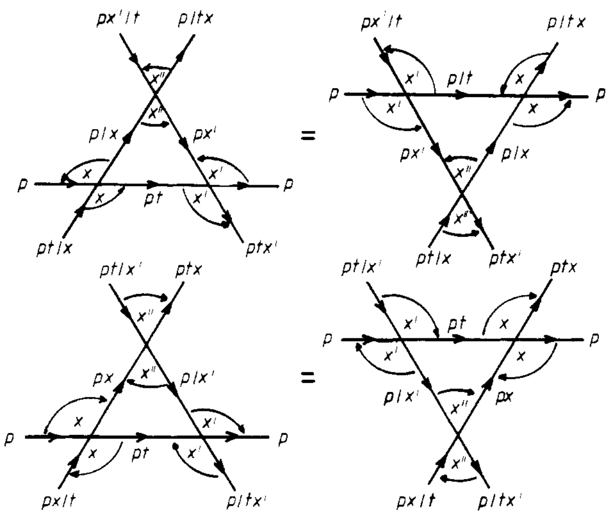


Figure 4. The two configurations with pure tensor products which satisfy the triangle relation.

one has to determine the corresponding independent configurations in the whole vector space. It is easily seen that a decomposition onto the homogeneous parts with respect to p $(1, p, p^2, p^3)$ identifies with the decomposition with respect to the total spin. Thus the two configurations of figure 4 force the triangle relation on a two-dimensional subspace of the three-dimensional vector space where the 3×3 antisymmetric matrix of (7) acts. Although these two configurations do not seem to be sufficient to imply the triangle relation generally, in fact one can check directly on the three equations (7) that they hold if and only if the condition $xx' = x''$ is satisfied.

The same method can be used to recover the two algebraic invariants of the symmetric eight-vertex model and to introduce a parametrisation, but this time, in terms of elliptic functions (of modulus k):

$$(p^2p'^2+1)cd-(p^2+p'^2)ab+pp'(a^2+b^2-c^2-d^2)=0. \tag{8}$$

Redefining the invariants by

$$\frac{a^2+b^2-c^2-d^2}{ab}=2\mathrm{cn}2\eta\,\mathrm{dn}2\eta \qquad \frac{ab}{cd}=\frac{1}{k\,\mathrm{sn}^2 2\eta} \tag{9}$$

one is led to the following solutions:

$$\begin{aligned} p&=k^{1/2}\mathrm{sn}2u & p'^{\pm}&=k^{1/2}\mathrm{sn}2(u\pm\eta)\\ r^{\pm}&=k^{1/2}\mathrm{sn}2(u\pm\theta) & r'^{\pm}&=k^{1/2}\mathrm{sn}2(u\pm\theta\mp\eta). \end{aligned} \tag{10}$$

Let us remark that the action on the vectors depends only on the invariant family (the parameter η). On the other hand, the relation between the two vectors e and f (e' and f') depends explicitly on the particular matrix inside the family and thus allows a parametrisation of the latter (the spectral parameter θ). In the case of the asymmetric six-vertex model two invariants are put into evidence

$$p^2a'b'+p'^2ab-pp'(aa'+bb'-c^2)=0 \tag{11}$$

which can be redefined as follows:

$$\frac{a'b'}{ab}=s^2 \qquad \frac{aa'+bb'-c^2}{ab}=s(t+1/t) \tag{12}$$

which implies

$$\begin{aligned} r^+&=\frac{t}{x}p & p'^+&=stp & r'^+&=\frac{s'^2}{s}\frac{1}{x}p\\ r^-&=\frac{x}{t}p & p'^-&=\frac{s}{t}p & r'^-&=\frac{s'^2}{s}xp \end{aligned} \tag{13}$$

leading to the following parametrisation:

$$\begin{aligned} \frac{a}{c}&=\frac{1}{s'}\frac{t/x-x/t}{t-1/t} & \frac{b}{c}&=\frac{s'}{s}\frac{1/x-x}{t-1/t}\\ \frac{a'}{c}&=s'\frac{t/x-x/t}{t-1/t} & \frac{b'}{c}&=\frac{s}{s'}\frac{1/x-x}{t-1/t}. \end{aligned} \tag{14}$$

It is verified in this case that the triangle relation reduces to the three equations (7) of the symmetric model and is solved by the condition $xx'=x''$.

As a final remark let us relate the problem introduced here to the Zamolodchikov algebra. In order to explain the S-matrix factorisation, Zamolodchikov and Zamolodchikov (1979) introduced particular symbols $A_i(\theta)$ representing particles and satisfying the following algebra

$$A_i(\theta_1)A_j(\theta_2)=\sum_{kl} S_{ij}^{kl}(\theta_1-\theta_2)A_l(\theta_2)A_k(\theta_1) \tag{15}$$

where S_{ij}^{kl} describes the two-body S matrix. One can give an explicit functional representation for the A symbols with the following action on functions defined on an algebraic variety (Takhtadjan and Faddeev 1979):

$$\begin{aligned} &A_i(\theta)f(u)=a_i(u-\theta)f(u-\eta) \\ &a_1(u)=\Theta_1(2u) \qquad a_2(u)=\Theta_4(2u) \end{aligned} \tag{16}$$

where Θ_1 and Θ_4 are the usual Jacobi Θ functions. With this representation, where the A symbols act on some coherent states (and using the relation $\mathrm{sn}u = k^{-1/2}\Theta_1(u)/\Theta_4(u)$), the equations defining the Zamolodchikov algebra (15) can easily be identified with those expressing the stability of the pure tensor product (A):

$$a_i(u-\theta_1)a_j(u-\theta_2-\eta)=\sum_{kl} S_{ij}^{kl}(\theta_1-\theta_2)a_k(u-\theta_1-\eta)a_l(u-\theta_2). \tag{17}$$

Note that the same restriction as the one previously encountered also exists concerning the independence of the symbols A and is also known to prevent the general derivation of the Yang–Baxter equations for the S-matrix, from the Zamolodchikov algebra.

3. Automorphy group

Another way of solving the soluble models has also been developed, which does not make use of the triangle relation and the commuting transfer matrices (Zamolodchikov 1979, Stroganov 1979, Baxter 1980). Instead, it exploits another relation which occurs simultaneously with the triangle relation in all the known soluble models, the inverse relation (which can also be interpreted as the unitarity relation in the case of S-matrix models). For instance, on the symmetric eight-vertex model, the inverse relation corresponds to the following transformation on the transfer matrix:

$$L_I=\begin{vmatrix} a/\Delta & 0 & 0 & -d/\Delta \\ 0 & b/\Delta' & -c/\Delta' & 0 \\ 0 & -c/\Delta' & b/\Delta' & 0 \\ -d/\Delta & 0 & 0 & a/\Delta \end{vmatrix} \qquad \begin{aligned} \Delta &= a^2-d^2 \\ \Delta' &= b^2-c^2. \end{aligned} \tag{I}$$

The partition function of the model (or the S matrix) can be seen to transform multiplicatively (by an automorphic factor) under the action of the inverse transformation. Similarly, the obvious spatial symmetry of the model (crossing symmetry of the S matrix) leaves the partition function (S matrix) invariant. For the symmetric eight-vertex model this corresponds to the following transformation on the transfer matrix:

$$a \leftrightarrow b. \tag{S}$$

Iterating these two transformations generates an infinite discrete group, under which the partition function transforms multiplicatively (the automorphy group). With some knowledge on the analytic behaviour of the partition function, this can lead to its determination.

3.1. *Algebraic invariants*

In view of the simplifying features introduced by this group in the resolution of the model, it appears interesting to confront it with the parametrisation of the triangle relation. Indeed, one can easily show that the triangle relation is preserved by the transformations corresponding to the inverse and spatial symmetries, and hence that the commutation of two transfer matrices associated with the triangle relation still holds if one transforms both matrices by inverse or spatial symmetry. Therefore, the corresponding images of the algebraic invariants which were deduced from the commutation of transfer matrices (for instance, those of § 2.1)) must also be algebraic invariants. The automorphy group thus infinitely multiplies the number of algebraic invariants. Of course, in the soluble cases these invariants are not independent, but can be expressed in terms of a finite number of basic invariants. It even occurs, in the simplest cases, that these basic invariants are themselves stable under the action of the group. This property then allows one to recover, in a simple way, the foliation of the parameter space, using the automorphy group.

Let us consider again the symmetric eight-vertex model. It is easily seen that the only invariants under the action of the spatial symmetry, S, which can be built out of the statistical weights are

$$a+b \qquad ab \qquad c \qquad d$$

and functions thereof. Looking for algebraic invariants only, one observes that the following polynomials transform multiplicatively under the action of the inverse symmetry:

$$c \to -c/\Delta' \qquad d \to -d/\Delta$$
$$ab \to ab/\Delta\Delta' \qquad \Delta+\Delta' \to (\Delta+\Delta')\Delta\Delta'.$$

It is then easy to construct algebraic invariants under the actions of both symmetries:

$$ab/cd \qquad (\Delta+\Delta')/ab.$$

These two invariants precisely realise the particular foliation into elliptic curves that can be deduced by solving the triangle equations. This result can be better understood if one remembers that the two transformations, inverse I and spatial symmetry S, do not commute but generate an infinite discrete group. The latter contains a normal subgroup, which is isomorphic to $\mathbb{Z}$ and which corresponds to translations by the remarkable element SI. The algebraic invariants must then determine varieties, which, besides each point, must also contain all the images of this point under the action of the group. On the other hand, the analytic properties which are linked with the actions of the inverse and spatial symmetries, that is the equivalence of such transformations with the change of a certain analytic parameter θ, lead to the inclusion of all the images of a same point inside a same curve which is described by the spectral parameter θ. To sum up, the orbits under the action of the automorphy group can be considered as the backbones around which the algebraic curves, which will correspond to commuting transfer matrices, are built by a kind of algebraic continuation.

The same approach can also be used to recover the varieties which correspond to families of commuting transfer matrices, in the case of the exactly solved asymmetric six-vertex model.

3.2. *Pure tensor product stability*

Besides preserving the triangle relation and the associated algebraic invariants, the automorphy group can also be seen to act in a simple way on the pure tensor products, which we have introduced to represent the local transfer matrices. Let us recall their property in the case of the symmetric eight-vertex model. One looks for local transfer matrices which satisfy the property (A)

$$Le \otimes f = \lambda e' \otimes f'. \tag{A}$$

The existence of vectors e and e' satisfying this property is equivalent to the equation (8). First, a certain family can be chosen by fixing the values of the invariants which appear as the coefficients of the polynomial in (8). The equation then associates with this family two transformations which make each vector f correspond with two others $f'^{\pm}$. These transformations can also be represented by a unique chain linked to each vector:

$$\to f_{i-1} \to f_i \to f_{i+1} \to$$

f_{i-1} and f_{i+1} being the two images of f_i by (8). Consider now the family and its associated chain as being fixed. The property then uniquely associates a couple of vectors $e_i^{\pm}$ to each local matrix of the family

$$Le_i^{\pm} \otimes f_i = \lambda_i e_i'^{\pm} \otimes f_{i\pm 1}$$

where

$$e_i^{\pm} = \begin{vmatrix} 1 \\ r_i^{\pm} \end{vmatrix} \qquad r_i^{\pm} = \frac{ap_{i\pm1} - bp_i}{c - dp_i p_{i\pm1}} = \frac{cp_i p_{i\pm1} - d}{ap_i - bp_{i\pm1}}. \tag{18}$$

A simple computation shows that the vectors $e_i^{\pm}$, $e_i'^{\pm}$ build another chain:

$$r_i'^{\pm} = (ap_i - bp_{i\mp1})/(c - dp_i p_{i\mp1}) = (cp_i p_{i\mp1} - d)/(ap_{i\mp1} - bp_i) \tag{19}$$

or else

$$e_i'^{\pm} = e_{i\mp1}^{\pm}. \tag{20}$$

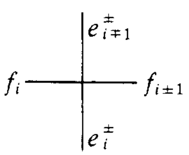
This last correspondence then allows one to follow the action of the inverse and spatial symmetries on the vectors $e_i^{\pm}$, obtaining the following relations

$$[e_i^{\pm}]_{\mathrm{S}} = -e_{i\pm1}^{\mp} \qquad [e_{i\pm1}^{\pm}]_{\mathrm{I}} = e_{i\mp1}^{\mp} \tag{21}$$

(which one can alternatively deduce from the invariances of the problem), and thus

$$[e_i^{\pm}]_{\mathrm{SI}} = -e_{i\mp1}^{\pm} \qquad [e_i^{\pm}]_{\mathrm{IS}} = -e_{i\pm1}^{\pm}. \tag{22}$$

Translation along the chain can thus be interpreted as the action of the translation (SI) of the automorphy group:



$$Le_i^{\pm} \otimes f_i = \lambda_i e_{i\mp1}^{\pm} \otimes f_{i\pm1} \qquad i \overset{\mathrm{SI}}{\to} i-1. \tag{A}$$

Such a representation allows one to give a purely algebraic description of the action of the local transfer matrix on pure tensor products, without recalling the elliptic functions (in such a parametrisation, the action of SI would be represented by the translation of the spectral parameter by the constant η).

An interesting feature of this formulation of the stability of pure tensor products is that it leads easily to a similar property for the global transfer matrix (see figure 5).

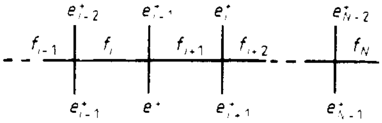


Figure 5. Action of the transfer matrix on the pseudo-vacuum.

Indeed, the set of vectors of the form $\otimes_i e_i^{\pm}$ is stable under the action of the whole transfer matrix:

$$T_{f_0 f_N} \bigotimes_{n=0}^{N} e_n^{\pm} = \left(\prod_{n=0}^{N} \lambda_n\right) \bigotimes_{n=0}^{N} e_{n\mp 1}^{\pm}. \tag{23}$$

As translation along the lattice is equivalent to translation by the element SI of the automorphy group, the action of the transfer matrix reduces to a translation by one lattice spacing. Similarly, the set of vectors constructed with the same ansatz, but where this time the translation along the lattice can correspond either to a forward translation (SI) or to a backward translation (IS) in the group, is also globally stable under the action of the transfer matrix. The complete Bethe ansatz then reduces to finding the linear combinations of the preceding vectors which diagonalise the transfer matrix (Kasteleyn 1975). Let us remark that the present use of pure tensor products is equivalent to the triangulation of the local transfer matrix and, in consequence, of the global transfer matrix. In the formalism of quantum field theory (Faddeev 1982), this approach corresponds to the construction of the pseudo-vacuum, by triangulating the monodromy matrix, and then to the construction of the real vacuum and of the excited states by the Bethe ansatz.

Finally, by relating the action of the automorphy group to the diagonalisation of the transfer matrix with the Bethe ansatz, this approach should also shed some light on another remarkable feature of the partition fuctions of soluble models; their final expression under the form of infinite products, each factor representing the action of an element of the automorphy group on some initial factor (Jaekel and Maillard 1982b).

4. Conclusion

This paper has tried to recover, in a pedestrian way, some properties on the local transfer matrix and the triangle relation, which belong to the field of algebraic geometry. A more precise representation of the solutions making use of pure tensor products can also be given by the methods of algebraic geometry, as Krichever (1981) has shown with the introduction of the so-called 'vacuous vectors'. In view of these results the triangle or tetrahedron relations seem to be more than mere systems of equations and appear to be linked to a richer structure defined on algebraic varieties. Corresponding methods should lead to an exhaustive classification of the solutions.

Though studied only on models with spins or arrows of the Ising type, the approach developed here also applies to N-state models like the one studied by Stroganov (1979). The algebraic invariants can also be quickly recovered for N-state spin models (like Potts models) but one must remark that the equivalent of the property of pure tensor product stability is not yet available for general spin models.

Since it only deals with small transfer matrices, the method is easily generalised to three dimensions, to derive algebraic invariants which represent as many necessary conditions for the commutation of the transfer matrices and for the tetrahedron relation. Looking for the compatibility of these equations thus provides a guide for determining under which symmetries the model might become a good candidate for solving the tetrahedron relation. In the same way, the stability of pure tensor products and its representation by a translation of the automorphy group could give hints for the construction of a Bethe ansatz in three dimensions.

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