THE ADLER–VAN MOERBEKE MODEL. LAX REPRESENTATION AND POISSON STRUCTURE ☆

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We study the classical integrability of the Adler–van Moerbeke model, describing in particular cases the motion on an ellipsoid with a central force. Classical integrability is associated with a generalized structure for the Poisson brackets of the Lax operator. The already known set of conserved quantities for this model turns out to follow straightforwardly from this structure.

1. Introduction

We have described in a previous paper [1] an example, drawn from classical mechanics, of the general D-matrix structure for classically integrable systems a la Liouville introduced in refs. [2–5]. This structure describes the Poisson brackets between components of the L-matrix, occurring in the Lax form of the equations of motion:

$$\dot{L}(\lambda)=[L(\lambda),M(\lambda)],\quad \lambda\in\mathbb{C} \text{ is the spectral parameter}, \tag{1}$$

It was shown [3] that for any integrable system a la Liouville (that is, when the eigenvalues of L are in involution), the Poisson brackets of L can be described by a single "D-matrix" ("R–S couple" in the terminology of another approach described in ref. [2]), such that

$$\{L(\lambda)\otimes L(\mu)\} = [D(\lambda,\mu),L(\lambda)\otimes\mathbf{1}]-[D^{\mathrm{T}}(\mu,\lambda),\mathbf{1}\otimes L(\mu)], \tag{2a}$$

where as usual

$$\{L(\lambda)\otimes L(\mu)\}_{ij}{}^{kl}=\{L(\lambda)_i{}^k, L(\mu)_j{}^l\} \tag{2b}$$

and D acts in the tensor product of spaces. D has no special symmetry property; if it is antisymmetric one recovers the R-matrix formalism [6]. Besides, it may also depend on the dynamical variables [2,5].

The example described in ref. [1] allowed us to construct a D-matrix for the Poisson bracket of the L-operator associated to the Moser–Uhlenbeck model, a particular case of which is the Neumann model of a particle on a sphere submitted to harmonic forces. We study here another example in classical mechanics, introduced by Adler and van Moerbeke [7], which contains the case of a particle on an n-ellipsoid with a central force.

2. The Adler–van Moerbeke model

Let us recall the main features of this model as they are described in ref. [7]. The general equations of motion derive from the hamiltonian H:

$$H=\tfrac{1}{4}\sum_{i\neq j}(b_i-b_j)/(a_i-a_j)(J_{ij})^2-\mu\sum_i b_i x_i y_i, \tag{3}$$

where a, b are any set of positive numbers, $a_i\neq a_j$ for $i\neq j$, $J_{ij}=x_iy_j-x_jy_i$, x_i, y_i are n canonically conjugate variables, μ is any real number and can in fact be reabsorbed in a redefinition of the set (a). The equations of motion read

$$\dot{x}_k=\partial H/\partial y_k,\quad \dot{y}_k=-\partial H/\partial x_k. \tag{4}$$

These equations for (x_i, y_i) can be described by introducing the compact notation

$$X=(x_i),\quad Y=(y_i),\quad K=(K_{ij}),$$

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$$K_{ij} = (b_i - b_j)/(a_i - a_j)(J_{ij}),$$

$$A = (A_{ij}), \quad A_{ij} = a_i \delta_{ij}, \quad B = (B_{ij}), \quad B_{ij} = b_i \delta_{ij}. \quad (5a)$$

The equations of motion are then

$$\dot{X} = -KX - BX, \quad \dot{Y} = -KY + BY. \quad (5b)$$

They admit n integrals of motion in involution under the Poisson bracket

$$G_k = -2x_k y_k + \sum_{l \neq k} (J_{kl})^2/(a_k - a_l). \quad (6)$$

Since $\sum_k b_k G_k = 2H(x, y)$, the (x, y) system is integrable in the sense of Liouville. The motion of a particle on an n-ellipsoid with a central force corresponds to the choice $b_i = (a_i)^{-1}$ (see ref. [7]).

3. The coadjoint orbit formulation and integrability

The Adler–van Moerbeke model can be described as a particular application of the Adler–Kostant–Kirillov scheme [8] for the Lie group $\mathrm{Gl}(n)$. Any element of its Lie algebra G can be decomposed as a pair (a, s) of an antisymmetric matrix a and a symmetric matrix s, with Lie bracket

$$[(a, s); (a', s')]$$
$$= ([a, a'] + [s, s']; [a, s'] + [a', s]). \quad (7)$$

Considered simply as a vector space, G is then identified with its dual G* using the bilinear form $\langle (a, s); (a', s') \rangle = \mathrm{Trace}(aa') + \mathrm{Trace}(ss')$.

A Poisson bracket is then defined for the functions on the dual G* by Kirillov's formula [6]:

$$\{\mathrm{F}, \mathrm{G}\}_{\mathrm{KK}}(L) = \langle L, [\mathrm{dF}, \mathrm{dG}] \rangle \quad (8)$$

(L is a point of G*, $\mathrm{dF}(L)$ is identified with an element of G.) This defines a symplectic structure on each coadjoint orbit of the action of G on G* [8], and one considers then the hamiltonian actions on these orbits defined by this symplectic structure. Note that the adjoint and coadjoint action are identical up to a sign for this Lie algebra. One then has to construct adequate hamiltonians and coadjoint orbits in G*, that is, hamiltonians belonging to a set of quantities in involution under the Kirillov bracket, and orbits of sufficiently low dimensions to describe a tractable physical system.

The family of integrable hamiltonians is obtained as follows: one starts from coadjoint-invariant functions F on the dual algebra (such as the trace of any integer power of the argument); one then defines a suitable loop-algebra-valued function $L(\lambda; a, s)$ of the point (a, s) of the dual algebra which one plugs as the argument in the invariant function F. The coefficients of the development in powers of λ provide the demanded set of integrals of motion in involution thereby leading to an integrable system, and L is the Lax operator for this system (see (1)). This follows in ref. [7] from a "computational miracle" which enables to transform the Kirillov bracket of the functions of (a, s) of the form $F(f(a, s))$ into a combination of expressions like $[\nabla F(f), f]$ which vanish due to the ad*-invariance of F. This is in fact a consequence of the existence of a D-matrix structure as we shall comment on later.

A particularly interesting set of low-dimensional coadjoint orbits of this group is parametrized in the following form. We take a and s as

$$a = x \otimes y - y \otimes x = J,$$
$$s = \kappa x \otimes x + x \otimes y + y \otimes x = Q, \quad (9)$$

x, y being n-dimensional vectors and κ any real number. Calculating now the coadjoint action of an arbitrary couple (a', s') on (a, s) induced by the Lie bracket (7), one can show that the form (9) is preserved with the following action on x and y:

$$\delta x = ax + sx, \quad \delta y = ay - sy - \kappa sx. \quad (10)$$

Notice now that the redefinitions

$$y \to y + cx, \quad x \to x, \qquad y \to 1/dy, \quad x \to dx \quad (11)$$

leave a invariant and transform s into $s + [2c + \kappa(d^2 - 1)] x \otimes x$; moreover, the coadjoint action of G on (a, s) leaves the trace of s invariant; hence the couple (x, y) with arbitrary choice of vectors contains in fact two degrees of freedom more than the point (a, s) of a coadjoint orbit, and this coadjoint orbit is of dimension $(2n-1)$. The two supplementary degrees of freedom can be eliminated by first setting the scalar product of x and y to zero using the invariance (11) with $2c + \kappa(d^2 - 1) = 0$; the trace of s then becomes the norm of x and is an invariant of the orbit. When $\kappa = 0$ this is not possible, but instead one has invariance of a and s under the redefinitions $y \to cy$, $x \to x/c$; in this case one sets the norm of x to 1 by this redefinition, and the conserved quantity is now the scalar product of x and y. Since one can always set $\kappa=0$ by the redefinition (11) with $d \neq 1$ and $c=-\kappa(d^2-1)/2$, we shall from now on choose this parametrization of the coadjoint orbits denoted as (J, Q). Notice that the symplectic structure induced by the Kirillov bracket on this particular set of orbits is the same, up to a factor $\frac{1}{2}$, as the one induced by the canonical Poisson structure on the variables x, y by the parametrization (9) (see ref. [7]), hence the hamiltonian flows on these orbits can be described equivalently in the two frameworks.

Let us now take as hamiltonian acting on these coadjoint orbits the following object:

$$H(B; J, Q)=\tfrac{1}{4}[-\mathrm{Tr}(JK+2BQ)]\,, \tag{12}$$

where J and Q are defined in (9) and K and B are defined in (5). H can indeed be expressed as a combination of the previously described quantities in involution as we shall see soon. The equations of motion for J and P induced by Kirillov's bracket are then

$$\mathrm{d}Q/\mathrm{d}t=[Q, K]+[J, B]\,,$$
$$\mathrm{d}J/\mathrm{d}t=[Q, B]+[J, K]\,. \tag{13}$$

This is precisely the evolution induced on J, Q by the equations of motion of x and y in the general Adler–van Moerbeke model (compare for instance the coadjoint action on x, y in (10) with the AvM equations (5)).

Finally eq. (13) can be combined in the Lax form (1) with

$$L=A(\lambda^2-1)+J\lambda+Q\,, \quad M=J+\lambda B\,. \tag{14}$$

4. The Poisson structure of the Lax operator L

Let us now describe the Poisson brackets of the Lax operator $L(\lambda)$ with itself, given the canonical Poisson structure of the dynamical variables (x, y), or the equivalent Kirillov bracket structure of (K, Q). We obtain that

$$\{L(\lambda)\otimes L(\mu)\}$$
$$=[R_1(\lambda,\mu), L(\lambda)\otimes\mathbf{1}]+[R_2(\mu,\lambda), \mathbf{1}\otimes L(\mu)]\,, \tag{15}$$

where $\{L(\lambda)\otimes L(\mu)\}$ lives in the space of endomorphisms of the tensor product E⊗E and E carries the fundamental representation of the Lie algebra G. The matrices $R_{1,2}$ are expressed in terms of two natural endomorphisms of E⊗E: the permutation operator $T: u\otimes v \to v\otimes u$ and the unitary covariant "Casimir" operator $C: u\otimes v \to \langle u, v\rangle \sum e_i \otimes e_i$. They read

$$R_{1,2}(\lambda,\mu)=F_{1,2}(\lambda,\mu)T+G_{1,2}(\lambda,\mu)C\,, \tag{16a}$$

where

$$F_1(\lambda,\mu)=(1-\lambda^2)/(\lambda-\mu)\,,$$
$$G_1(\lambda,\mu)=-(1-\lambda^2)/(\lambda+\mu)\,,$$
$$F_2(\lambda,\mu)=(1-\mu^2)/(\lambda-\mu)\,,$$
$$G_2(\lambda,\mu)=(1-\mu^2)/(\lambda+\mu)\,. \tag{16b}$$

As it was the case for the Neumann–Moser model [1], $R_{1,2}$ do not depend only on the difference of spectral parameters, but also on their sum. Again there is no way to rewrite this structure as a genuine R-matrix structure by means of a redefinition of L. Finally, one encounters here the same type of combination of a permutation plus a Casimir operator; this particular type of D structure shall be examined in a forthcoming paper [9].

Note that the existence of particular functions solving some characteristic functional equations which allowed the manipulations in ref. [7] is equivalent to the existence of the coefficient functions F, G as solutions of related functional equations enabling to define the D matrix as above: in other words and as it should be expected, the fundamental justification of the construction in ref. [7] is the existence of the D structure for the L matrices.

The Poisson bracket relations between the matrix elements of the Lax operator L then imply the existence of conserved quantities in involution. In fact one has the following set of identities:

$$\{\mathrm{Tr}_{\mathrm{E}}(L^n(\lambda)), \mathrm{Tr}_{\mathrm{E}}(L^m(\mu))\}=\mathrm{Tr}_{\mathrm{E}\otimes\mathrm{E}}\{L^n(\lambda), L^m(\mu)\}$$
$$=nm\,\mathrm{Tr}_{\mathrm{E}\otimes\mathrm{E}}\{L^{n-1}(\lambda)\otimes L^{m-1}(\mu)$$
$$\times\{[R_1(\lambda,\mu); L(\lambda)\otimes\mathbf{1}]+[R_2(\mu,\lambda); \mathbf{1}\otimes L(\mu)]\}\}$$
$$=0 \quad \text{by cyclicity of the trace}\,. \tag{17}$$

Hence all the coefficients of $\mathrm{Tr}(L^n(\lambda))$ developed as $\sum c^n_k(J, P)\,.\,\lambda^k$ are in involution: $\{c^n_k, c^m_l\}=0$. The quantities G_k defined in (11) are combinations of these objects via a Vandermonde matrix:

$$c^m{}_{2m-2} = -n \sum_k a_k{}^{n-1}(G_k + a_k) . \tag{18}$$

Since all a_k are different the set $\{G_k,\ k=1,\ ...,\ n\}$ is equivalent to the set $\{c^m{}_{2m-2},\ m=1,\ ...,\ n\}$. The hamiltonians H $(B;\ J,\ Q)$ in (12) are obtained as $\frac{1}{2}(\sum_k b_k G_k)$, hence they are contained in the set of $\mathrm{Tr}(L^n)$. One recovers in this way the set of action variables of the model.

5. Relations between analytical and group-theoretic descriptions

Let us now consider the "spectral curve" [10] defined by

$$\mathrm{Det}(L(\lambda) - z) = 0 . \tag{19}$$

This curve is time independent since $\dot{L} = [L, M]$. Above any point (λ, z) of the curve sits the one-dimensional eigenspace Ker $(L(\lambda) - z)$. This defines a complex line bundle over the spectral curve. As time evolves, its Chern class remains constant, and it is shown in ref. [10] that its evolution is given by multiplication by a degree 0 line bundle, moving linearly on the jacobian of the curve. We shall show that this curve is birationally equivalent to a hyperelliptic curve. Dividing by λ^{2n} and setting $t = z/\lambda^2$ and $\lambda' = 1/\lambda$, eq. (28) reads

$$\mathrm{Det}(D + \lambda' J + (\lambda')^2 Q) = 0 ,$$

$$D = (1 - \lambda'^2) A - t\mathbf{1} . \tag{20}$$

The symmetry properties of J and Q imply that only the powers 0, 2 and 4 of λ' will appear in (20). Then the Grassmann algebra allows one to rewrite (20) as

$$\prod_k (a_k - w)$$

$$= (\lambda')^2/(1-\lambda'^2) \prod_k (a_k - w) \sum_k G_k/(a_k - w) ,$$

$$w = t/(1 - \lambda'^2) . \tag{21}$$

Redefining $s = \lambda' \prod_l (a_l - w)[1 + \sum_k G_k/(a_k - w)]$ finally gives the equation of the spectral curve as follows:

$$s^2 = P_{2g+2}(w)$$

$$= \prod_l (a_l - w)^2 \left(1 + \sum_k G_k/(a_k - w)\right) . \tag{22}$$

P_{2g+2} is a polynomial of degree $2g+2$ and eq. (22) describes therefore a smooth hyperelliptic curve of genus g since $a_k \neq a_l$ when $k \neq l$. This curve is such that the equations of motion actually linearize on its jacobian torus, as it was the case in the Neumann–Moser model [1].

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