On the diagonals of rational functions: the minimal number of variables

S. Hassani[‡], **J.M. Maillard**[£], **N. Zenine**[§]

[‡] Retired: Centre de Recherche Nucléaire d'Alger, Algeria
[£] LPTMC, UMR 7600 CNRS, Université de Paris, Tour 23, 5ème étage, case 121, 4 Place Jussieu, 75252 Paris Cedex 05, France
§Centre de Recherche Nucléaire d'Alger, 2 Bd. Frantz Fanon, BP 399, 16000 Alger, Algeria

Abstract.

From some observations on the linear differential operators occurring in the Lattice Green function of the d-dimensional face centred and simple cubic lattices, and on the linear differential operators occurring in the n-particle contributions $\chi^{(n)}$ to the magnetic susceptibility of the square Ising model, we forward some conjectures on the diagonals of rational functions. These conjectures are also in agreement with exact results we obtain for many Calabi-Yau operators, and many other examples related, or not related to physics.

Consider a globally bounded power series which is the diagonal of rational functions of a certain number of variables, annihilated by an irreducible minimal order linear differential operator homomorphic to its adjoint. Among the logarithmic formal series solutions, at the origin, of this operator, call n the highest power of the logarithm. We conjecture that this diagonal series can be represented as a diagonal of a rational function with a minimal number of variables N_v related to this highest power n by the relation $N_v = n+2$.

Since the operator is homomorphic to its adjoint, its differential Galois group is symplectic or orthogonal. We also conjecture that the symplectic or orthogonal character of the differential Galois group is related to the parity of the highest power n, namely symplectic for n odd and orthogonal for n even.

We also sketch the case where the denominator of the rational function is not irreducible and is the product of, for instance, two polynomials. We recall that the linear differential operators occurring in the n-particle contributions $\chi^{(n)}$ to the magnetic susceptibility of the square Ising model factorize in a large number of direct sums and products of factors. The analysis of the linear differential operators annihilating the diagonal of rational function where the denominator is the product of two polynomials, sheds some light on the emergence of such mixture of direct sums and products of factors. The conjecture $N_v = n+2$ still holds for such reducible linear differential operators.

PACS: 05.50.+q, 05.10.-a, 02.10.De, 02.10.Ox

Key-words: diagonal of rational function, multi-Taylor expansion, multivariate series expansions, multinomial coefficients, lattice Green function, face-centred cubic lattice, simple cubic lattice, Calabi-Yau operators, Fuchsian linear differential equations, homomorphisms of linear differential operators, differential Galois groups, magnetic susceptibility of the square Ising model

7 February 2025

1. Introduction

Diagonals of rational functions have been seen to occur naturally [1] for n-fold integrals in physics, field theory, enumerative combinatorics, etc. On many examples and cases, striking properties emerged that are worthy to be understood.

When we seek for a characterization of the diagonal of rational functions representation of a D-finite *globally bounded* power series†, one may think of the *least number of variables* N_v occurring in the rational function‡.

Given a diagonal of a rational function R_1 of N variables (which is necessarily [1] a D-finite *globally bounded* power series) there is a rational function R with a *minimal number* N_v of variables. One aim of this paper amounts to showing that this number of variables $N_v \le N$ is simply related to the logarithmic singular behavior, at the origin, of the formal series of the (minimal order) linear differential operator§ annihilating the diagonal.

We illustrate our conjecture with the analysis of the lattice Green function (LGF) of the d-dimensional simple cubic (s.c) and face centred cubic (f.c.c) lattices [3, 4, 5, 6, 7, 8, 9], as well as results obtained for many Calabi-Yau operators [10], and an accumulation of other examples related, or not related, to physics, displayed (or not displayed) in this paper.

The differential operators for LGF of the simple cubic (s.c) and face centred cubic (f.c.c) lattices are *irreducible*. These differential operators are *homomorphic to their adjoint* and, consequently, their differential Galois groups are (included in) symplectic or orthogonal groups. All these lattice Green functions can, obviously, be cast into diagonal of rational functions.

This irreducibility is in sharp contrast with the differential operators of the n-particle contributions $\chi^{(n)}$¶ to the magnetic susceptibility of the square Ising model [11, 12, 13, 14, 15, 16, 17], which have a large set of factors. Here, the symplectic, or orthogonal, character of the differential Galois group concerns the *factors* occurring in the factorization of the differential operators annihiliting the $\chi^{(n)}$'s. Furthermore, we observe, for the n-particle contributions $\chi^{(n)}$ of the susceptibility of the square Ising model, that, for each block of factors in the differential operator which has a unique factorization, (e.g. for a block of three factors $L_n \cdot L_p \cdot L_q$), we have *alternately* orthogonal and symplectic groups.

We will show that these characteristics can be seen on the diagonals of rational functions, with simple enough expressions, that may lead to a better understanding of their occurrences.

With P and Q multivariate polynomials (with $Q(0, ..., 0) \neq 0$), the formal series of the (minimal order) differential operator annihilating the diagonal of the rational functions P/Q^r (with r an integer), correspond to a *finite dimensional* vectorial space related††, as shown by Christol [18, 19, 20], to the de Rham cohomology. There is a

† A diagonal of a rational function is necessarily [1] a D-finite *globally bounded* power series. Conversely, according to Christol's conjecture [2], a D-finite *globally bounded* power series should be the diagonal of a rational function.

‡ It is obvious, however, that the rational function, whose diagonal gives a given *globally bounded* series, is far from unique. This series can be the diagonal of many rational functions, even with different numbers of variables.

§ All the differential operators, in this paper, are linear. We will omit "linear" in the sequel.

¶ The $\chi^{(n)}$ have a very convoluted form of algebraic fractions as integrands. They are shown to be diagonal of rational functions [1].

†† We are thankful to P. Lairez for having clarified this point.

homomorphism between the (minimal order) differential operators for the diagonal of P/Q^k and for the diagonal of $1/Q$. Therefore, without too much loss of generality (see section 3 of [21]), we will basically restrict ourselves, in this paper, to rational functions in the form $R = 1/Q$, where Q is an irreducible multivariate polynomial, with $Q(0, ..., 0) \neq 0$. We will also consider, for pedagogical reason, the case where the denominator Q factors in only *two* polynomials, $Q = 1/Q_1/Q_2$, where at least one of the Q_j's depends on *all* the variables¶.

In the sequel, we notice that, generically, for irreducible Q over the rationals, the resulting (minimal order) differential operator, annihilating the diagonal of $1/Q$, seems to be systematically irreducible†. In contrast, a factorization of the differential operator occurs for factorizable Q.

1.1. *The formal solutions*

The differential operators annihilating diagonals of rational functions are very selected differential operators [21]. The formal solutions, at the origin, of such differential operators (call it L_q) can be organized as the union of different sets (1), (2), ... of formal series which makes the monodromy at $t = 0$ crystal clear:

$$\begin{aligned} &S_0, \qquad S_0 \cdot \ln(t) + S_{1,0}, \qquad S_0 \cdot \frac{\ln(t)^2}{2!} + S_{1,0} \cdot \ln(t) + S_{2,0}, \\ &\cdots, \\ &S_0 \cdot \frac{\ln(t)^n}{n!} + S_{1,0} \cdot \frac{\ln(t)^{n-1}}{(n-1)!} + S_{2,0} \cdot \frac{\ln(t)^{n-2}}{(n-2)!} + \cdots + S_{n,0}, \end{aligned} \tag{1}$$

$$\begin{aligned} &T_0, \qquad T_0 \cdot \ln(t) + T_{1,0}, \qquad T_0 \cdot \frac{\ln(t)^2}{2!} + T_{1,0} \cdot, \ln(t) + T_{2,0}, \\ &\cdots, \\ &T_0 \cdot \frac{\ln(t)^m}{m!} + T_{1,0} \cdot \frac{\ln(t)^{m-1}}{(m-1)!} + T_{2,0} \cdot \frac{\ln(t)^{m-2}}{(m-2)!} + \cdots + T_{m,0}, \end{aligned} \tag{2}$$

etc.

In each of these "sets" of formal solutions, the series (up to a t^α overall, $\alpha = 0, 1/2, ...$) are analytical at $t = 0$. One of the series S_0, T_0, ... is the diagonal of the rational function annihilated by the (minimal order) differential operator L_q. It is therefore a globally bounded series. The other analytic solution series have, at first sight, no reason to be globally bounded series.

1.2. *Conjecture on the number of variables*

Our main conjecture corresponds to the *exact value* of the *minimal number* N_v of variables of the *rational* functions required to represent a given diagonal of rational function series. Consider a rational function R_1 with N variables, and the (minimal order) differential operator annihilating $\mathrm{Diag}(R_1)$ (i.e. the diagonal of R_1), *assumed to be homomorphic to its adjoint*‡ (i.e. its differential Galois group is included either in

¶ The case where Q_1 and Q_2 depend on different sets of variables corresponds to a Hadamard product and will not be considered in this paper.

† For examples of irreducible Q and non irreducible operators see Appendix D.

‡ It has been noticed in several of our papers [21, 22], that diagonal of rational (or algebraic) functions almost systematically yield differential operators which are *homomorphic to their adjoint*. The rare cases breaking this "self-adjoint duality" were seen to correspond to candidates to rule out Christol's

a symplectic, or in an orthogonal group). Among the formal solutions, at the origin, of this differential operator, there is one formal solution which has the *highest* log-power, n, i.e. behaves as $\ln(t)^n$. We conjecture that the diagonal of R_1 identifies with the diagonal of a rational function R which depends on a minimal number of variables N_v, where N_v is simply related to the *highest* log exponent¶ n by the following relation♯ (with $N_v \le N$):

$$N_v \;=\; n+2. \qquad (3)$$

Note that the existence of such rational functions with a minimal number of variables N_v (*regardless of the number of monomials* and degrees), *does not prevent* the existence of other rational functions with more variables, giving the same diagonal. For instance, when the general term of the series writes as nested sums of binomials [1, 23], it is straightforward to obtain a first rational function [1]. This first rational function has often more variables than this minimal number N_v. Also note that a balanced ratio of factorials coefficients can be written in various binomial forms, thus yielding many rational functions.

1.3. Some remarks

Let us give some remarks on the definition of what we call the rational functions and on the number of variables occurring there.

We have seen in previous papers (see for instance section 4.3 and appendix G in [24]), that an exact result on the diagonal of a rational function of some variables x, y, ..., depending on *parameters*, can straightforwardly be generalised to the same rational function, but where the *parameters* become arbitrary (rational or algebraic) *functions* of the product $t = x\,y\,z\,\cdots$ of the variables of the rational function. Consequently we extend the definition of rational functions to rational functions where the variables are rescaled† by arbitrary (rational or algebraic) functions of the product $t = x\,y\,z\,\cdots$

In the sequel, some of the rational functions depend on N-th root of variables (e.g. $u^{1/4}$, $v^{1/6}$, ...). The calculation of the diagonal of such a function, is equivalent to the calculation of the diagonal of this function where all the variables are raised to some power. In the following we will say, by abuse of language, in such cases, that we have a rational function *even if it contains* N-th *root of some variables.*

Also, we have seen examples of rational functions where some monomials or variables in the rational function do not contribute to the diagonal, meaning that the rational function has in fact lesser number of variables. We used, for this situation, the term "effective" number of variables in section 2.4 of [21]. Consider the rational function $1/Q$ where $Q = 1 - y - z - x\,z - x\,u - x\,z\,u - x\,y\,u$, which depends on four variables. The diagonal of $1/Q$ is "blind"‡ on the monomial $x\,z$ and the product $x\,u$ stands for one variable, reducing Q to $\tilde{Q} = 1 - y - z - v - y\,v - z\,v$ which depends only on three variables. We should note that we have not seen this situation in our examples coming from physics or geometry.

conjecture [2, 20]. The question of this self-adjoint symmetry breaking is adressed in section 5.2 of [21].

¶ The crucial role played by the highest log exponent corresponds to the concept of monodromy filtration (see paragraph 4.2 page 40 of [20]).

♯ We do not have a conjecture for the minimum number of variables N_v, for diagonals of *algebraic* functions. An open question is to see if we could actually have $N_v = n + 1$ in the diagonal of *algebraic* functions case.

† See for instance equations (54) and (55) below.

‡ Changing the monomial $x\,z$ into $\mu\,x\,z$, the resulting diagonal will not depend on the parameter μ.

1.4. Complements and additional speculations on conjecture (3)

Among the different sets (1), (2), ... of formal series solutions of the (minimal order) differential operator, one set, for instance (1), corresponds to the highest power of the logarithms, namely n. We also conjecture that the series with no logarithm in this set, here S_0, *will necessarily be a globally bounded series*, and, thus, according to Christol's conjecture [2], will be a diagonal of a rational function, with, according to the conjecture (3), the minimal number of variables $N_v = n + 2$. The other sets, for instance (2), will, in general, have a power of logarithms m which is less than the highest power n: $m < n$. In this $m < n$ case we also conjecture that the power series (up to a t^α overall, $\alpha = 0, 1/2, \cdots$) with no logarithm in this set, here T_0 for (2), *cannot be a globally bounded series*. Consequently, *it cannot be a diagonal of a rational function*.

In the cases where another set (2), is such that its power of logarithms, m, is actually equal to the highest power of logarithm n, we conjecture that the power series (up to a t^α overall, $\alpha = 0, 1/2, \cdots$) with no logarithm in this set, here T_0, *will necessarily also be a globally bounded series*, and, thus, according to Christol's conjecture [2], will be a diagonal of a rational function, with, according to the conjecture (3), the minimal number of variables $N_v = n + 2$.

Let us recall that the definition of diagonal of rational functions is based on multi-Taylor expansions around the origin [1]. Therefore, the seeking of the maximum exponent of the logarithm in the formal solutions *is also made at the origin.* The maximum exponent of the logarithms in the formal solutions around the other singularities can also be considered, and this is checked for many of our examples below.

1.5. Conjecture on the parity of the number of variables and the differential Galois groups

Another conjecture is related to the symplectic, or orthogonal, character of the differential Galois group† of the irreducible differential operator annihilating the diagonal of the rational function with N_v variables.

Since we assume that the (minimal order) irreducible differential operator annihilating the diagonal of the rational function is *homomorphic to its adjoint*, its differential Galois group is necessarily symplectic or orthogonal [22, 25].

We forward a conjecture stating that the parity of the "minimal" number N_v of variables in the irreducible denominator Q dictates the character either symplectic Sp (N_v is an odd number) or orthogonal SO (N_v is an even number) of the differential Galois group of the (minimal order) differential operator annihilating the diagonal of the rational function. This conjecture is that the group is orthogonal for N_v even and symplectic for N_v odd:

$$N_v \text{ even (resp. odd)} \qquad \longrightarrow \qquad SO \,(\text{resp. } Sp) \tag{4}$$

The illustrative examples displayed in this paper in favour of these two conjectures (3), (4), are chosen for pedagogical reasons, but also for their interest *per se*. We have

† An irreducible differential operator L_q, of order q, has, generically, a symmetric square ($sym^2(L_q)$) of order $N_s = q\,(q+1)/2$ and an exterior square ($ext^2(L_q)$) of order $N_e = q\,(q-1)/2$. If $sym^2(L_q)$ (resp. $ext^2(L_q)$) annihilates a rational solution, or is of order $N_s - 1$ (resp. $N_e - 1$), the differential operator L_q is included in the *orthogonal group* $SO(q, \mathbb{C})$ (resp. *symplectic group* $Sp(q, \mathbb{C})$) that admits an invariant quadratic (resp. alternating) form.

two different kinds of examples: the ones corresponding to (irreducible) differential operators which have *Maximum Unipotent Monodromy*¶ (MUM), and the ones that do not have MUM. In the MUM case, we have a simple relation $n = q - 1$ between the power of logarithm n in (3), and the order q of the differential operator L_q. The conjecture (3) thus becomes, in the MUM case, a simple relation $N_v = q + 1$ between the number of variables and the order of the operator. For the non-MUM examples, the number of variables is *not* related to the order of the differential operators, but to the highest power of the logarithms n, in other words, *it is related to the monodromy matrix at the origin.*

The paper is organized as follows. We recall in section 2 the definition of the diagonal of a rational function, and in section 3 we recall the results of the Lattice Green Functions of the d-dimentional (face centred cubic, simple cubic and diamond) lattices, to illustrate the two conjectures (3), (4). Section 4 presents some illustrative examples for non factorizable multivariate polynomials Q. We give, in section 5, the polynomials Q for some Calabi-Yau equations [10] of geometric origin, obtaining, for each case, the polynomial Q with the minimum number of variables $N_v = 3 + 2 = 5$. Section 6 deals with the situation where the denominator polynomial Q factorizes as $Q = Q_1 Q_2$, giving either a direct sum, or a unique factorization, of the (minimal order) differential operator annihilating the diagonal of $1/Q$. In section 7, we give examples of diagonal of algebraic functions which are N-th root of rational functions and equivalent rational functions of the form $1/Q$, giving the same diagonals. Finally, in section 8, we discuss the homomorphisms-to-adjoint assumption on a counter-example candidate to Christol's conjecture [2].

2. Diagonals and multinomial expansion

The diagonal of a rational function R, dependent on (for example) three variables, is obtained by (multi-Taylor) expanding R around the origin

$$R(x, y, z) \;\; = \;\; \sum_m \sum_n \sum_l a_{m,n,l} \cdot x^m \, y^n \, z^l, \qquad (5)$$

keeping only the terms such that $m = n = l$. The diagonal reads, with $t = xyz$:

$$\mathrm{Diag}\Big(R(x, y, z)\Big) \;\; = \;\; \sum_m a_{m,m,m} \cdot t^m. \qquad (6)$$

In most of the examples of this paper the rational function will have the form $1/Q$ with

$$Q \;\; = \;\; 1 \;\; - (T_1 + T_2 + \cdots + T_n), \qquad (7)$$

where the T_j's are monomials. The expansion of the rational function $1/Q$ reads:

$$\frac{1}{Q} \;\; = \;\; \sum \frac{(k_1 + k_2 + \cdots + k_n)!}{k_1! \, k_2! \cdots k_n!} \cdot T_1^{k_1} \, T_2^{k_2} \cdots T_n^{k_n}. \qquad (8)$$

Calculating the diagonal of $1/Q$ amounts to distributing the powers k_j on the variables occurring in the monomials, and putting equal the exponents of each variable.

¶ Maximum unipotent monodromy: the critical exponents at the origin are all equal.

With the example¶ $Q = 1 - x - y - z - u^{1/2} - v^{1/5}$, depending on five variables, or n-th root of variables, the expansion of $1/Q$, around the origin, reads:

$$\frac{1}{Q} \;=\; \sum_{k_i=0,\, i=1,\cdots 5}^{\infty} \frac{(k_1 + k_2 + k_3 + k_4 + k_5)!}{k_1!\, k_2!\, k_3!\, k_4!\, k_5!} \cdot x^{k_1}\, y^{k_2}\, z^{2\,k_3}\, u^{k_4/2}\, v^{k_5/5}. \qquad (9)$$

The diagonal will extract the terms with the same power. Introducing the integer p, such that $k_1 = k_2 = k_3 = k_4/2 = k_5/5 = p$, the only terms contributing to the diagonal are

$$\sum_{p=0}^{\infty} \frac{(p + p + p + 2\,p + 5\,p)!}{p!\; p!\; p!\; (2\,p)!\, (5\,p)!} \cdot x^p\, y^p\, z^p\, u^p\, v^p \;=\; \sum_{p=0}^{\infty} \frac{(10\,p)!}{p!^3\, (2\,p)!\, (5\,p)!} \cdot t^p. \qquad (10)$$

Let us mention how we obtain the (minimal order) differential operator annihilating the diagonal. Once a long series is obtained, we use the guessing method to obtain†† the differential equation (ODE). We make use of the "ODE formula" forwarded in section 3.1 of [14] (and with details in section 1.2 of [17]) to ensure that we actually deal with *the minimal order* differential equation. For the differential operators of high order, which are not irreducible, the factorization is obtained using the method of factorization of differential operators modulo primes (see Section 4 in [15] and Remark 6 in [22]).

3. The lattice Green function of the simple cubic, diamond and face centred cubic lattices

For the lattice Green functions (LGF) of the *simple cubic* lattice of dimension d, the rational function have $N_v = d + 1$ variables. The corresponding differential operators L_n, ($n = d$, see Appendix A.1.1) annihilating the LGF have Maximum Unipotent Monodromy (MUM), and thus, the formal solution with the highest log-power at the origin corresponds to $\ln^{n-1}(t)$. The relation of conjecture (3) is satisfied.

Similarly, one may consider the LGF of the *diamond lattice* (Appendix A.1.2) in 3, 4 and 5 dimensions, where the rational functions depend on respectively 4, 5 and 6 variables, and check that the formal solutions with the highest log-power, at the origin, are respectively in $\ln(t)^2$, $\ln(t)^3$ and $\ln(t)^4$, in agreement with conjecture (3). All these differential operators also have MUM.

The lattice Green functions (LGF) of the *face-centred cubic* lattice of dimension d are diagonals of rational functions of the form $1/Q$ for the dimension $d < 7$, see [3, 4, 5, 6, 7], and we have produced the corresponding (minimal order) differential operators [8, 9] for $d = 7, 8, \cdots, 12$. These differential operators *no longer*‡ *have MUM.* According to the parity of the dimension d, which is related to the number of variables by $N_v = d + 1$, the differential Galois group of these differential operators are included in Sp (N_v odd) or SO (N_v even) in agreement with our conjecture (4). We have checked (see Appendix A.2) that, among the formal solutions at the origin of these differential operators, one solution is in front of $\ln(t)^n$, where n is the highest log-power. This exponent is in agreement with the relation $N_v = n + 2$ of the

¶ For "polynomials" with N-th root of variables, see, for instance, (31) in section 5.
†† Alternately, we can use the creative telescoping [21] to get this differential operator as a telescoper. This method often requires more computing time.
‡ Except for $d = 2, 3, 4$ see Table A.1 in Appendix A.

conjecture (3). For the exponent of the logarithmic formal solutions at all the other singularities see Appendix A.2.

Beyond this "LGF laboratory", let us give some examples with irreducible (non factorizable) denominator Q.

4. Examples with non factorizable denominator Q

4.1. A LGF-like non-MUM example

In the examples with the lattice Green functions, one may imagine that the occurrence of the differential Galois groups Sp or SO is related to the dimension of the lattice. Let us go beyond this relation by considering the structure function

$$\lambda \, = \, c_1 c_2 c_3 + c_1 c_2 c_4 + c_1 c_3 c_4 + c_2 c_3 c_4, \qquad\qquad c_j \, = \, \cos \phi_j, \tag{11}$$

considered by Guttmann in [5], *which does not correspond to any known obvious lattice.* The (minimal order) differential operator annihilating

$$G(t) \, = \, \frac{1}{(2\pi)^4} \int \frac{d\phi_1 \, d\phi_2 \, d\phi_3 \, d\phi_4}{1 \, - t \cdot \lambda}, \tag{12}$$

is of order 8 and is irreducible. Its exterior square is of order 27, instead of the generic order $28 \, = \, (8 \times 7)/2$. Its differential Galois group is thus included in the simplectic group $Sp(8, \mathbb{C})$. The lattice Green function $G(t)$ is the diagonal of the rational function $1/Q$, depending on 5 variables $(z_0, z_1, z_2, z_3, z_4)$

$$Q \, = \, 1 \, - t \cdot \lambda, \tag{13}$$

such that $c_j \, = \, z_j + 1/z_j \quad j \, = \, 1, \, \cdots \, 4$ and $t = \, z_0 z_1 z_2 z_3 z_4$.

The eight formal solutions, at the origin, of the order-eight differential operator read

$$\begin{aligned}
& S^{(0)}, \qquad S^{(0)} \cdot \ln(t) \, + S_{2,0}, \qquad S^{(0)} \cdot \frac{\ln(t)^2}{2!} \, + S_{3,1} \cdot \ln(t) \, + S_{3,0}, \\
& S^{(0)} \cdot \frac{\ln(t)^3}{3!} \, + S_{4,2} \cdot \frac{\ln(t)^2}{2!} \, + S_{4,1} \cdot \ln(t) \, + S_{4,0}, \\
& t^{1/2} \cdot S^{(1/2)} \qquad t^{1/2} \cdot S^{(1/2)} \cdot \ln(t) \, + t^{1/2} \cdot S_{6,0}, \\
& t^{1/3} \cdot S^{(1/3)}, \qquad t^{2/3} \cdot S^{(2/3)},
\end{aligned} \tag{14}$$

where the $S_{i,j}$'s are analytical series at $t \, = \, 0$, and where the other series begin as $1 \, + \cdots$ The series $S^{(0)}$ is *globally bounded*, and corresponds to the diagonal of the rational function. All the other analytical series $S^{(1/2)}$, $S^{(1/3)}$, $S^{(2/3)}$, $S_{i,j}$ in (14) are *not globally bounded*.

The diagonal $S^{(0)} \, = \, \mathrm{Diag}(1/Q)$ is the diagonal of a rational function of $N_v = \, 3 \, + 2 \, = \, 5$ variables, in agreement with conjecture (3).

Around all the other singularities, the highest log-power is in $\ln(t)^1$, and each formal series, in front of $\ln(t)^1$, is *not globally bounded*.

The generalization of (11) to 6 variables amounts to considering:

$$\lambda \, = \, c_1 c_2 c_3 c_4 + c_1 c_2 c_3 c_5 + c_1 c_2 c_4 c_5 + c_1 c_3 c_4 c_5 + c_2 c_3 c_4 c_5, \qquad c_j \, = \, \cos \phi_j. \tag{15}$$

The corresponding differential operator is expected to have its differential Galois group included in the orthogonal group. The differential operator is of order 9. Its differential Galois group is indeed (included) in $SO(9, \mathbb{C})$, (its symmetric square being of order 44, instead of the generic order $45 \, = \, (9 \times 10)/2$).

The formal solution of this order-nine differential operator with the highest log-power at the origin, corresponds to $\ln(t)^4$, giving, according to the conjecture (3), $N_v = 4+2 = 6$ as the minimal number of variables occurring in the rational function.

4.2. 3-D fcc example: reduction to three variables

The diagonal of $1/Q$ where

$$Q \, = \, 1 \, - xyzu \cdot \Bigl((x + \frac{1}{x}) \cdot (y + \frac{1}{y}) + (x + \frac{1}{x}) \cdot (z + \frac{1}{z}) + (y + \frac{1}{y}) \cdot (z + \frac{1}{z}) \Bigr), \quad (16)$$

is a polynomial depending on *four variables* x, y, z, u, reproduces the 3-dimensional face-centred cubic lattice Green function. The (minimal order) differential operator annihilating this diagonal is of order three, and its differential Galois group is included in $SO(3, \mathbb{C})$. The most singular formal solution is in $\ln(t)^2$, in agreement with $2+2$ variables for the rational function, in agreement with conjecture (3).

Let us reduce the number of variables of the polynomial Q, given in (16), to *three variables*, by *fixing* $u = 1$. The polynomial Q becomes:

$$Q_1 \, = \, 1 \, - xyz \cdot \Bigl((x + \frac{1}{x}) \cdot (y + \frac{1}{y}) + (x + \frac{1}{x}) \cdot (z + \frac{1}{z}) + (y + \frac{1}{y}) \cdot (z + \frac{1}{z}) \Bigr). \quad (17)$$

The (minimal order) differential operator, annihilating this diagonal, is of order six, and its differential Galois group is included in $Sp(6, \mathbb{C})$. The formal solution with the highest log-power at the origin is in $\ln(t)^1$, in agreement with $1+2 = 3$ variables for the rational function.

4.3. Another diagonal representation of the 3-D fcc LGF

The LGF of the three-dimensional fcc lattice *can also be seen* as the diagonal of $1/Q$, where the polynomial denominator Q depends on four variables:

$$Q \, = \, 1 \, - x^2yzu \cdot (1 + 4\,xyzu) \, - (1+u) \cdot (y+z). \quad (18)$$

The diagonal of $1/Q$, where Q depends on four variables, gives (with $t = xyzu$):

$$\mathrm{Diag}_{(xyzu)}\Bigl(\frac{1}{Q}\Bigr) \, = \, {}_3F_2\Bigl([\frac{1}{2}, \frac{1}{3}, \frac{2}{3}], [1,1], \, 108 \cdot t^2 \cdot (1+4t) \Bigr). \quad (19)$$

The formal solution of the corresponding differential operator, with the highest log-power at the origin, is in $\ln(t)^2$, in agreement with the minimal number of variables of $4 = 2+2$, according to conjecture (3).

Taking the diagonal of $1/Q$ with Q given in (18) on *only the 3 variables* (x, y, z), which means that the variable u *is seen as a parameter*, one obtains (with $s = xyz$):

$$\mathrm{Diag}_{(xyz)}\Bigl(\frac{1}{Q}\Bigr) \, = \, {}_2F_1\Bigl([\frac{1}{3}, \frac{2}{3}], [1], \, 27 \cdot u \cdot (1+u)^2 \cdot (1+4\,us) \cdot s^2 \Bigr). \quad (20)$$

The formal solution of the corresponding differential operator with the highest log-power at the $s = 0$ origin, is in $\ln(s)^1$, in agreement with $3 = 1+2$ variables, according to conjecture (3).

Here, for the *same* rational function $1/Q$, we clearly see that the fact that the diagonal is taken on 4 or 3 variables, produces a singular formal solution behaving in $\ln(t)^2$ or $\ln(s)$.

4.4. A reflexive polytope example

From the 210 dIfferential operators arising from reflexive 4-polytopes [23, 26] (Laurent polynomials of 4 variables) with symplectic differential Galois group Sp, let us consider the example‡, depending on 5 variables, which corresponds to the rational function $1/Q$ with polynomial denominator

$$Q \,\,=\,\, 1 \,\, - x\, y\, z\, u\, v \cdot S, \tag{21}$$

where:

$$S \,\,=\,\, x + y + z + u \,\, + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{u} \\ + yz + \frac{x}{z} + \frac{u}{z} + \frac{x}{yz} + \frac{z}{xu} + \frac{u}{yz} + \frac{xu}{yz} + \frac{yz}{x} + \frac{yz}{u} + \frac{xu}{z^2 y}. \tag{22}$$

The diagonal of $1/Q$ is annihilated by an irreducible order-six differential operator whose differential Galois group is symplectic (included in $Sp(6, \mathbb{C})$). The formal solution, at the origin, of this order-six differential operator, with the highest log-power, behaves as $\ln(t)^3$. This is in agreement with the number of variables $N_v = 3 + 2 = 5$ occurring in the polynomial Q, according to conjecture (3).

Around all the other singularities, the maximum power of the log in the formal solutions is 1, and the series are *non-globally bounded*.

4.5. An Apery generalization example

Let us recall the series with Apery numbers†:

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \cdot t^n \,\,=\,\, \sum_{n=0}^{\infty}\sum_{k=0}^{n} \left(\frac{(n+k)!}{k!^2\,(n-k)!}\right)^2 \cdot t^n. \tag{23}$$

This series is actually the diagonal of the rational function $1/Q$ with four variables [20]:

$$Q \,\,=\,\, (1\,\, - x_1\,\, - x_2) \cdot (1\,\, - x_3\,\, - x_4) \,\,\, - x_1\, x_2\, x_3\, x_4. \tag{24}$$

More generally, let us consider the series given by (with m a positive integer):

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{n}{k}^m \binom{n+k}{k}^m \cdot t^n \,\,=\,\, \sum_{n=0}^{\infty}\sum_{k=0}^{n} \left(\frac{(n+k)!}{k!^2\,(n-k)!}\right)^m \cdot t^n. \tag{25}$$

This series is the diagonal of the rational function $1/Q_m$, with¶

$$Q_m \,\,=\,\, 1 \,\,\, - \prod_{j=0}^{m-1} \left(x_{2j+1} \, + x_{2j+2} \, + x_{2j+1} \cdot \, x_{2j+2}\right), \tag{26}$$

depending on $2m$ variables. For any value of m, the number of variables being even, the differential operator, corresponding to the diagonal of $1/Q_m$, has a differential Galois group which should be included in the orthogonal group SO, according to conjecture (4).

For $m = 3$, we have 6 variables in (26),

$$Q_3 \,\,=\,\, 1 \,\,\, - (x_1 + x_2 + x_1 x_2) \cdot (x_3 + x_4 + x_3 x_4) \cdot (x_5 + x_6 + x_5 x_6), \tag{27}$$

‡ Polytope v18.10805, topology 28 in [23, 26].
† The sequence is related to Apery's proof on the irrationality of $\zeta(3)$.
¶ Use the multinomial expansion, then equate the exponents of the variables.

and the diagonal of the rational function $1/Q_3$ reads:

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{n}{k}^3 \binom{n+k}{k}^3 \cdot t^n \; = \; \sum_{n=0}^{\infty}\sum_{k=0}^{n} \Bigl({{(n+k)!} \over {k!^2\,(n-k)!}} \Bigr)^3 \cdot t^n$$
$$= \; 1 \; + 9\,t \; + 433\,t^2 \; + 36729\,t^3 \; + \, \cdots \qquad (28)$$

This series (28) is annihilated by an order-nine differential operator L_9. The symmetric square of L_9 is of order 44, instead of the generic order $45 \, = \, (9 \times 10)/2$. The differential Galois group of this order-nine differential operator L_9 is (included) in the orthogonal group $SO(9, \mathbb{C})$. The formal solution of L_9 with the highest log-power, at the origin, behaves as $\ln(t)^4$, in agreement with the conjecture (3): $N_v \, = \, 4+2 \, = \, 6$.

The maximum exponent of the log's in the formal solutions around all the other singularities $t \, = \, t_j$ (with the exception of $t \, = \, \infty$) is 2, and the series, in front of $\ln(t-t_j)^2$, are *not globally bounded*.

For the singularity $t \, = \, \infty \, = \, 1/s$, the highest log-power is *also* 4, i.e. the same value as the maximum log-power around $t \, = \, 0$, so we expect the corresponding series in front of $\ln(s)^4$ to be *globally bounded*. The first terms of the series read

$$s \, \cdot \, \Bigl(1 \; + 9\,s \; + 433\,s^2 \; + 36729\,s^3 \; + \, \cdots \Bigr). \qquad (29)$$

Similarly, for $m \, = \, 4$, we have 8 variables in (26). The differential operator, annihilating the diagonal of the rational function $1/Q_4$, is of order 15. Its differential Galois group is (included in) the orthogonal group $SO(15, \mathbb{C})$. The formal series solution, at the origin, of the order fifteen differential operator, with the highest log-power, behaves as $\ln(t)^6$ in agreement with the conjecture (3): $N_v \, = \, 6+2 \, = \, 8$.

5. Some Calabi-Yau examples

We consider, in this section, some examples of Calabi-Yau equations from [10], where the expressions of the general term of the series are known in closed form. These order-four differential operators are irreducible, their differential Galois groups are (included in) symplectic groups $Sp(4, \mathbb{C})$. They all have the MUM property [10], and their formal solutions with the highest log-power behave as $\ln(t)^3$. Therefore, according to conjecture (3), these differential operators should annihilate rational functions of $N_v \, = \, 3+2 \, = \, 5$ variables.

5.1. The first 19 Calabi-Yau operators in Almkvist et al. Table [10]

We have considered the first 19 Calabi-Yau differential operators of [10]. These Calabi-Yau equations (except #9) have a geometric origin [27, 28, 29, 30, 31]. The general term of the series are (or can be) written *as nested sums of products of binomials* [23], known to correspond to diagonals of rational functions [1, 23]. The aim, here, is to give the rational function *with only 5* variables.

Assuming some $1/Q$ form for the rational function, the polynomials Q are obtained from closed formulae (given in [10]) of the general term written as *ratio of factorials*, using a “guessing” procedure sketched in Appendix B. The procedure in Appendix B, amounts, from the general term given in [10], to going back up to the expression coming out from a *multinomial* expansion. Eventually one finds that all the polynomials Q can be written with five variables. We give the corresponding multivariate polynomials Q obtained from this guessing procedure, in Table 1.

Note that, for some Q's, we still say that Q is "polynomial" even if it contains N-th root of some variables ($u^{1/4}$, $v^{1/6}$, ...). Section 2 shows that the diagonal of the rational function with N-th root of variables is actually a power series (not a Puiseux series). In the following, instead of some "polynomials" containing N-th root of variables, we give equivalent polynomials still with 5 variables.

Table 1. Rational functions $1/Q$ for some Calabi-Yau series $\sum A_n t^n$, where A_n is the general term of the series given in [10]. We follow the numbering # of [10].

#	A_n	$1-Q$	N_v
1	$\frac{(5n)!}{n!^5}$	$x+y+z+u+v$	5
2	$\frac{(10n)!}{n!^3(2n)!(5n)!}$	$x+y+z+u^{1/2}+v^{1/5}$	5
3	$\frac{(2n)!^4}{n!^8}$	$(x+z)(1+y)(1+v)(1+u)$	5
4	$\left(\frac{(3n)!}{n!^3}\right)^2$	$(x+yu+zv)\,(y+xz+uv)$	5
5	$\frac{(2n)!^2(3n)!}{n!^7}$	$x+y+z+u+v(x+y)(z+u)$	5
6	$\frac{(2n)!(4n)!}{n!^6}$	$x+y+z+u+v(z+u)$	5
7	$\frac{(8n)!}{n!^4(4n)!}$	$x+y+z+u+v^{1/4}$	5
8	$\frac{(6n)!}{n!^4(2n)!}$	$x+y+z+u+v^{1/2}$	5
9	$\frac{(2n)!(12n)!}{n!^4(4n)!(6n)!}$	$x+y+z(x+y)+u^{1/4}+v^{1/6}$	5
10	$\left(\frac{(4n)!}{n!^2(2n)!}\right)^2$	$\left(x+yu+(zv)^{1/2}\right)\left(y+xz+(uv)^{1/2}\right)$	5
11	$\frac{(3n)!(4n)!}{n!^5(2n)!}$	$x+y+z+u+v(y+z+u)$	5
12	$\frac{(4n)!(6n)!}{n!^3(2n)!^2(3n)!}$	$x+y+(1+z)(u^{1/2}+v^{1/2})$	5
13	$\left(\frac{(6n)!}{n!(2n)!(3n)!}\right)^2$	$\left(x+(yu)^{1/2}+(zv)^{1/3}\right)\left(y+(xz)^{1/2}+(uv)^{1/3}\right)$	5
14	$\frac{(2n)!(6n)!}{n!^5(3n)!}$	$x+y+z+u(x+y)+v^{1/3}$	5
15	$\frac{(3n)!}{n!^3}\binom{n}{k}^3$	$x+y+z\,(x+v)+u\,(y+v)$	5
16	$\binom{2n}{n}\binom{n}{k}^2\binom{2k}{k}\binom{2n-2k}{n-k}$	$x+y+z+u+v(xyz+xyu+xzu+yzu)$	5
17	$\left(\frac{n!}{j!k!(n-j-k)!}\right)^3$	$(x+y+z)\,(v+xu+yz)\,(uv+zv+xyu)$	5
18	$\binom{2n}{n}\binom{n}{k}^4$	$u+(x+y)\,(x+v)\,(z+v)\,(y+z)$	5
19	$\binom{n}{k}^3\binom{n+k}{n}\binom{2n-k}{n}$	$(x+y)\,(u+x)\,(u+v)\,(y+z)\,(v+z+y+u)$	5

5.2. The Q's are multivariate polynomials of variables and N-th root of variables

For some Q's which are polynomials with N-roots of variables, a straighforward change of variables, such $(x,\,y,\,\cdots)\,\rightarrow\,(x^n,\,y^n,\,\cdots)$, may be introduced.

For instance, for Calabi-Yau number 2 (that we denote CY_2),

$$
\begin{aligned}
CY_2 \;&=\; \sum_{n=0}^{\infty} \frac{(10n)!}{n!^3\,(2n)!\,(5n)!}\cdot t^n \;=\; {}_4F_3\left([\frac{1}{10},\frac{3}{10},\frac{7}{10},\frac{9}{10}],[1,1,1],\,2^8\,5^5\cdot t\right) \\
&=\; 1\;+15120\,t\;+3491888400\,t^2\;+1304290155168000\,t^3\;+\;\cdots \qquad (30)
\end{aligned}
$$

which is actually the diagonal of $1/Q$, with

$$Q \; = \; 1 \; - (x+y+z \; + u^{1/2} \; + v^{1/5}), \tag{31}$$

the series is in $t = xyzuv$. If one wants to get rid of the N-th roots $u^{1/2}$ and $v^{1/5}$, one can rather consider the polynomial

$$Q \; = \; 1 \; - (x^{10}+y^{10}+z^{10} \; + u^5 \; + v^2), \tag{32}$$

yielding for the diagonal of $1/Q$ where Q is given by (32), the series (30) where t is changed into t^{10}. Other polynomials with five variables can be found. With the polynomial

$$Q \; = \; 1 \; - (x+y+z \; + x^4y^4u^4v^5 \; + xzvu^2), \tag{33}$$

or the other polynomial

$$Q \; = \; 1 \; - \left(xyzu+xzuv+yzuv \; + xyvu^2+xyzv^3\right), \tag{34}$$

the series (30) is reproduced with t changed respectively into t^6 and t^9.

5.3. The rational function versus the pullbacked solution

From the Calabi-Yau equations in [10], two (or more) may have the *same Yukawa coupling* †, which means [33] that one solution can be written in terms of the other one. Let us see *how this property appears* in the multivariate rational functions $1/Q$.

5.3.1. Calabi-Yau number 79

The series for Calabi-Yau number 79 reads

$$\begin{aligned} CY_{79} \; &= \; \sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{n}{k} \cdot \frac{(5k)!}{k!^5} \cdot t^n \\ &= \; 1 \; + 121\,t \; + 113641\,t^2 \; + 168508561\,t^3 \; + \; \cdots \end{aligned} \tag{35}$$

and has the *same* Yukawa coupling [10] as CY_1. It reads

$$\begin{aligned} CY_{79} \; &= \; \frac{1}{1-t} \cdot CY_1\left(\frac{t}{1-t}\right) \\ &= \; \frac{1}{1-t} \cdot {}_4F_3\left([\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5}],[1,1,1],\; 5^5 \cdot \frac{t}{1-t}\right), \end{aligned} \tag{36}$$

which is the diagonal of $1/Q_{79}$ where Q_{79} reads (with $t = xyzuv$):

$$Q_{79} \; = \; (1\,-t) \cdot \left(1 \; - \left(\frac{x}{1-t} \; + y+z+u+v)\right)\right). \tag{37}$$

The diagonal of $1/Q$, with Q given in (37), is the same series as in (35).

5.3.2. Calabi-Yau number 128

Another example is Calabi-Yau number 128 which series reads

$$CY_{128} \; = \; \sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{2n}{n} \cdot \binom{n}{k} \cdot \frac{(5k)!}{k!^3\,(2k)!} \cdot t^n \tag{38}$$

$$= \; 1 \; + 122\,t \; + 114126\,t^2 \; + 169305620\,t^3 \; + 307902541870\,t^4 \; + \; \cdots$$

† See, e.g. [32, 33, 34] for the definition of the Yukawa coupling, and Appendix G.

and has the *same* Yukawa coupling [10] as CY_1, in terms of which it writes:

$$CY_{128} = \frac{1}{\sqrt{1-4t}} \cdot CY_1\left(\frac{t}{1-4t}\right)$$
$$= \frac{1}{\sqrt{1-4t}} \cdot {}_4F_3\left([\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}], [1,1,1],\ 5^5 \cdot \frac{t}{1-4\,t}\right). \qquad (39)$$

The Calabi-Yau series (38) or (39), is the diagonal of $1/Q_{128}$, where the denominator Q_{128} has an "algebraic term" *depending only on the variable* $t = xyzuv$:

$$Q_{128} = \sqrt{1-4t} \cdot \Big(1 - (z+u+v)\Big) - x - y. \qquad (40)$$

6. Factorizable denominator Q: examples with two factors

6.1. The differential operators occurring in the $\chi^{(n)}$'s of the Ising model

The rational functions, considered in the previous sections, are in the form $1/Q$, where the denominator Q is a *non factorizable* multivariate polynomial, the (minimal order) differential operator annihilating these diagonals, being irreducible, contrary to the differential operators annihilating integrals corresponding to the n-fold integrals $\chi^{(n)}$'s of the susceptibility of the Ising model [12, 13, 14, 15, 16, 17]. These integrals are very convoluted forms of algebraic fractions, which have been shown to be diagonal of rational functions [1], but are far from being in the form $1/Q$.

All the factors, occurring in the differential operators for the $\chi^{(n)}$'s, have been shown to have differential Galois groups either symplectic or orthogonal (see [22] and references therein). Furthermore, focusing on the blocks of factors (occurring in the differential operators corresponding to the $\chi^{(n)}$'s) which have a *unique factorization*, i.e. that write as, (e.g. for three factors) $L_n\, L_p\, L_q$, it appears that if L_q is in the orthogonal group (resp. symplectic group), the left factor L_p is in the symplectic group (resp. orthogonal group) and so on, in an alternating way. Appendix C gives the situation for all the factors occurring in the differential operators corresponding to $\chi^{(3)}, \cdots, \chi^{(6)}$.

This occurence of products, as well as *direct sums* of products, for factors of the differential operators annihilating these diagonals, seems to be related to the fact that the denominator Q is *not* irreducible. In the sequel, we address this case, *restricting for pedagogical reason, to only two factors*. We will see, in these examples with *two* factors, that conjecture (3) is still valid. The fact that the denominator Q is not irreducible yields differential operators that are not irreducible: thus one cannot simply introduce a differential Galois group for these differential operators. One has to consider the differential Galois group of *each factor* of these reducible differential operators. We will see that an alternating symplectic/orthogonal structure seems to systematically occur (as Appendix C shows for the $\chi^{(n)}$'s).

6.2. Foreword: two factors

In all the examples of the form $1/Q = 1/Q_1/Q_2$ displayed below, and many more not given in this paper, we have systematically obtained the following results. If the number of variables, and the variables for Q_1 and Q_2 are the *same*, the differential operator, annihilating the diagonal of $1/Q = 1/Q_1/Q_2$, is of the form

$$\Big(\mathcal{L}^{(1)} \oplus \mathcal{L}^{(2)}\Big) \cdot N, \qquad (41)$$

where the differential operators $\mathcal{L}^{(1)}$ and $\mathcal{L}^{(2)}$ are *homomorphic to the differential operators annilating, respectively, the diagonal of* $1/Q_1$ and $1/Q_2$, and where the differential operator N, at the right, is some "dressing"¶. We have not found any simple interpretation of this "dressing".

In contrast, if the number of variables for Q_1 and Q_2 is different, the variables for (for instance) Q_2 being a subset of the set of variables for Q_1, the differential operator, annihilating the diagonal of $1/Q = 1/Q_1/Q_2$, is of the form

$$\mathcal{L}^{(1)} \cdot N, \tag{42}$$

where the differential operator $\mathcal{L}^{(1)}$ is *homomorphic to the differential operator annilating the diagonal of* $1/Q_1$, and where, again, we have not found any simple interpretation†† of this "dressing" differential operator N.

6.3. First example

As a first example, consider the diagonal of $1/Q$

$$Q \; = \; 1 \; - x - y - z \; + y^2 - z^2 + xy + yz \; + yz^2 + xz^2 + z^3, \tag{43}$$

which is annihilated by an *irreducible* order-ten differential operator L_{10} which has a differential Galois group (included) in $Sp(10, \mathbb{C})$. The formal solution with the highest log-power at the origin is in $\ln(t)^1$, in agreement with $3 = 1 + 2$ variables.

Changing the monomial $-y$ into $-2\,y$ in (43), one obtains a polynomial $\tilde{Q}$ that, now, factorizes:

$$\tilde{Q} \; = \; \Big(1 \, - (x+y+z)\Big) \cdot (1 \, - y - z^2). \tag{44}$$

The diagonal of $1/\tilde{Q}$ is, now, annihilated by an order-six differential operator with the unique‡ factorization $L_6 \, = \, M_2 \cdot M_4$, where the order-two differential operator M_2 has a differential Galois group (included) in $Sp(2, \mathbb{C})$, and where the order-four differential operator M_4 has a differential Galois group (included) in† $SO(4, \mathbb{C})$. While the (left) differential differential operator M_2 is *homomorphic* to the order-two differential operator annihilating the diagonal of $1/\,(1\, - (x+y+z))$, we have *no interpretation* for the "dressing" right factor M_4 *with respect to* the right factor of $\tilde{Q}$.

This example shows that changing the coefficients in front of the monomials to a value that makes the polynomial Q factorizable leads to a *reducible* differential operator. A more subtle situation is addressed in Appendix D.

Remark 6.1. The formal solution of L_6 (and of M_2) with the highest log-power at the origin is in $\ln(t)^1$, in agreement with the fact that $\tilde{Q}$ depends on $3 \, = \, 1 + 2$ variables. The "dressing" differential operator M_4 has *no logarithmic* formal solution around *all* the singularities, and all these formal solutions are globally

¶ For Q_1 and Q_2 polynomial of three variables, the "dressing" operator N seems to always have algebraic solutions.

††In particular, in the case where both operators have the same order and the same singularities, the "dressing" operator N in (42) is not homomorphic to the differential operator annihilating the diagonal of $1/Q_2$.

‡ No direct sum factorization.

† Its symmetric square has the rational solution $(175\,t+48)/t/(3125\,t^2+1644\,t+128)$. Its symmetric cube (of order 20) has a rational solution, $(3125\,t^2 + 1644\,t + 128)/t$. Its symmmetric fourth power (of order 35), has a rational solution, $(3125\,t^2 + 1644\,t + 128)/t^2$, and its symmmetric fifth power (of order 56), again, has a rational solution, $(3125\,t^2 + 1644\,t + 128)/t^2$, suggesting a differential Galois group smaller than $SO(4, \mathbb{C})$.

bounded. According to Christol's conjecture [2] these formal solutions are diagonal of rational functions and according to our conjecture (3), the rational functions depend on only *two* variables and therefore, should be *algebraic series*.

In fact, the calculation of the p-curvature of the differential operator M_4 gives zero†† for every prime p, in agreement with algebraic series (according to Grothendieck-Katz conjecture [35]).

6.4. Second example

Consider, now, the denominator Q

$$Q \,\, = \,\, \Bigl(1 \, - (x+y+z+u)\Bigr) \cdot (1 \, - xy \, - zu)\,, \tag{45}$$

and the diagonal of the rational function $1/Q$:

$$\mathrm{Diag}\Bigl(\frac{1}{Q}\Bigr) \,\, = \,\, 1 \,\, + 30\,t \,\, + 2958\,t^2 \,\, + 428652\,t^3 \,\, + 72819090\,t^4 \,\, + \,\, \cdots \tag{46}$$

This diagonal is annihilated by an order-five differential operator that factorizes as $L_5 \, = \, L_3 \cdot L_2$, where L_3 has a differential Galois group (included) in $SO(3, \mathbb{C})$, and L_2 has a differential Galois group (included) in $Sp(2, \mathbb{C})$. The differential operator L_3 is homomorphic with the order-three differential operator annihilating the diagonal of $1/\,(1 \, - (x+y+z+u))$. Note that the formal solution of L_5 with the highest log-power at the origin, is in $\ln(t)^2$, indicating that we should deal with a minimum number of $4 \, = \, 2 \, + 2$ variables, in agreement with (3).

Here again, while the left factor of the differential operator L_5, and the differential operator annihilating the diagonal of the reciprocal of the first factor of Q, are actually related by operator homomorphism, we have *no interpretation* on the right factor L_2 with respect to the right factor of Q. The differential operator L_2 has a differential Galois group (included) in $Sp(2, \mathbb{C})$, and carries $\ln(t)^1$ as the formal solution with the highest log-power, meaning that it is, *per se*, given by the diagonal of a rational function with $3 \, = \, 1 + 2$ variables:

$$\begin{aligned} \mathrm{sol}(L_2) \,\, &= \,\, \frac{1}{\sqrt{1 \, - 4t}} \cdot {}_2F_1\Bigl([\frac{1}{4}, \frac{3}{4}], [1], \frac{64}{9}\,t\Bigr) \\ &= \,\, \frac{1}{\sqrt{1 - 4t}} \cdot \mathrm{Diag}\Bigl(\frac{1}{1 \, - c_1\,x - c_2\,y \, - c_3/3 \cdot \, z^{1/2}}\Bigr) \quad \text{where:} \qquad c_1 c_2 c_3^2 \, = \, 1. \end{aligned} \tag{47}$$

Remark 6.2. One should note that the second factor in the polynomial (46), *in itself*, depends on *only two* variables $x\,y$ and $z\,u$. However, in the product of polynomials given in (46), all the four variables contribute to the diagonal. This is not the kind of situation of section 6.2 where the two polynomials should depend *separetly* on the same number of the same variables for which the diagonal is not blind (see section 1.3). The resulting differential operator $L_5 \, = \, L_3 \cdot L_2$, annihilating the diagonal of $1/Q$, with the differential operator L_2, shows that *there is* a product of polynomials $Q_1\,Q_2$ where Q_1 and Q_2 depend respectively on four and three variables and such that the diagonals of $1/Q$ and of $1/Q_1/Q_2$ identify.

In the previous examples, the denominator polynomial Q factorizes into two polynomials $Q = \, Q_1\,Q_2$, where the number of variables occurring in each polynomial

†† We thank A. Bostan for calculating the first p-curvatures, thus showing that these p-curvatures are zero for $7 \le p \le \, 73$.

is different. Appendix E addresses the case when the product $Q_1 Q_2$ switches from the situation where both polynomials Q_j carry the *same number* of variables, to the situation where one of them has a smaller number of variables. Appendix F considers a denominator $Q = Q_1 Q_2$, depending on one parameter b, and where the polynomials Q_1 and Q_2 depend, respectively, on 4 and 3 variables. We address in Appendix F the situation where, for one particular value of the parameter b, the polynomial Q reduces to a polynomial depending on *only* three variables instead of the four variables we started with.

7. Rational versus algebraic functions and powers of rational functions

In this section we consider some examples of diagonals of *algebraic* functions, square root of rational functions, and powers of rational functions.

7.1. A modular form example

Let us consider the order-two differential operator L_2 annihilating the hypergeometric function ${}_2F_1([1/12, 5/12], [1], 1728\, t)$:

$$L_2 \; = \; (1-1728\, t) \cdot t \cdot D_t^2 \; + (1-2592\, t) \cdot D_t \; - 60. \tag{48}$$

This order-two differential operator L_2 is homomorphic to its adjoint provided‡ one considers a simple square-root algebraic extension:

$$(1-1728\, t)^{1/2} \cdot adjoint(L_2) \; = \; L_2 \cdot (1-1728\, t)^{1/2}. \tag{49}$$

This hypergeometric function corresponds to the diagonal

$${}_2F_1\left([\frac{1}{12}, \frac{5}{12}], [1], \; 1728\, t\right) \; = \; \mathrm{Diag}\,(R)\,, \tag{50}$$

of the *algebraic function* R (depending on *two* variables with $t = xy$) given by:

$$R \; = \; \left(\sqrt{1\, - 1728\, xy} \; - \sqrt{432} \cdot (x-y)\right)^{-1/6}. \tag{51}$$

According to conjecture (3), we would like to be able to write (50), not as a diagonal of an *algebraic* function of *two* variables, but as a *rational* function of *three* variables.

The procedure in [36] by Denef and Lipschitz gives, on this example, a *rational* function $\tilde{R}$ depending on *four* variables (x, y, u, v) such that (50), the diagonal of $\tilde{R}$ on the four variables (x, y, u, v), identifies with the diagonal of the *algebraic* function R on the variables (x, y). In this procedure, the denominator of $\tilde{R}$ comes in a factorized form by *construction*, and each factor *does not contain* all the variables. For an algebraic function with n variables $(x_1, x_2, \cdots, x_n)$, the Denef and Lipschitz procedure [36] will give, as denominator, the product $Q(x_1, x_2, \, \cdots, \, x_n, u_1) \cdot$ $Q(x_1, x_2, \cdots, x_n, u_2) \cdots \; Q(x_1, x_2, \cdots, x_n, u_n)$.

For the hypergeometric function ${}_2F_1([1/12, 5/12], [1], 1728\, t)$, the procedure sketched in Appendix B, is not applicable. However, one may imagine that, introducing a pullback, the general term may be cast in the appropriate form of *ratio*

‡ Alternatively one can use the "Homomorphisms" command of DEtools in Maple on the *symmetric square* of $adjoint(L_2)$ and L_2.

of factorials. One actually has (with a double expansion, index summation change, and summation of the inner sum):

$$
\begin{aligned}
& {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], \; 1728\, t \cdot (1 \, - 432\, t)\Big) \\
&\quad = \quad \sum_{n=0}^{\infty} \sum_{k=0}^{n} \, (-4)^k \cdot \begin{pmatrix} k \\ n-k \end{pmatrix} \cdot \frac{(1/12)_k \, (5/12)_k}{(1)_k \; k!} \cdot (-432\, t)^n \\
&\quad = \quad \sum_{n} \frac{(6\, n)!}{n! \; (2\, n)! \; (3\, n)!} \cdot t^n. \qquad (52)
\end{aligned}
$$

The last multinomial form leads to (with $t \, = \, xyz$):

$$
{}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], \; 1728\, t \cdot (1 - 432\, t)\Big) \quad = \quad \mathrm{Diag}\Big(\frac{1}{1 \, - x \, - y^{1/2} \, - z^{1/3}}\Big). \qquad (53)
$$

The order-two differential operator, annihilating the diagonal (53), is not only *homomorphic* with its adjoint, it is self-adjoint. Its differential Galois group is (included) in $Sp(2, \mathbb{C})$, and the formal solution of the differential operator with the highest log-power carries a $\ln(t)^1$, in agreement with the $3 \, = \, 1 + 2$ variables of the diagonal in (53), and conjecture (3).

Remark 7.1. From (53) we easily get that

$$
{}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], \; 1728\, t\Big) \quad = \quad \mathrm{Diag}\Big(\frac{1}{1 \, - \alpha \cdot x \, - y^{1/2} \, - z^{1/3}}\Big), \qquad (54)
$$

$$
\text{where} \quad \alpha \; = \; \frac{1 - (1 \, - 1728\, t)^{1/2}}{864\, t}, \qquad \text{with} \quad t \; = \; x\, y\, z. \qquad (55)
$$

Up to an algebraic function of the product $t \; = \; x\, y\, z$, we thus have a representation of (50) as a diagonal of a rational function of *three* variables.

Remark 7.2. Note that the pullback in (53) is precisely the one that matches (53) with one of a *modular form* of Appendix B in [37]:

$$
{}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], \, 1728 \cdot \, t \cdot \, (1 \, - 432\, t)\Big) \quad = \; {}_2F_1\Big([\frac{1}{6}, \frac{5}{6}], [1], \, 432\, t\Big). \qquad (56)
$$

7.2. From the LGF of 3-D s.c to Calabi-Yau number 69

Recall the rational function $1/Q$, depending on four variables, and corresponding¶ to the lattice Green function of the 3-dimensional simple cubic (Appendix A.1.1),

$$
Q \;\; = \;\; 1 \;\; - \Big(x + y + z \;\; + u \cdot \, (xy + xz + yz)\Big), \qquad (57)
$$

$$
\mathrm{Diag}\Big(\frac{1}{Q}\Big) \;\; = \;\; \mathrm{HeunG}\Big(\frac{1}{9}, \frac{1}{12}, \frac{1}{4}, \frac{3}{4}, 1, \frac{1}{2}, \; 4 \cdot \, t\Big)^2. \qquad (58)
$$

Considering, now, the diagonal of the reciprocal of $Q^{1/2}$,

$$
\mathrm{Diag}\Big(\frac{1}{Q^{1/2}}\Big) \;\; = \;\; 1 \;\; + \frac{9}{4}\, t \;\; + \frac{1575}{64}\, t^2 \;\; + \frac{107415}{256}\, t^3 \;\;\; + \; \cdots \qquad (59)
$$

¶ This Heun function (58) can be written as pullbacked ${}_2F_1$ hypergeometric function with algebraic pullbacks (see equations (12), (13), (14) in [37]).

one obtains an annihilating irreducible order-four differential operator, with a differential Galois group included in the symplectic group $Sp(4, \mathbb{C})$, and with $\ln(t)^3$ highest log-power formal solution. According to conjecture (3), a rational function $1/Q_{eq}$, *depending on* $5 = 3 + 2$ *variables* should exist. It actually reads:

$$Q_{eq} \; = \; 1 \; - \Bigl(x+y+z \; + u \cdot \, (xy+xz+yz) \; + \frac{v^{1/2}}{4}\Bigr). \qquad (60)$$

The diagonal of $1/Q_{eq}$ identifies with the diagonal of $1/Q^{1/2}$. With the rescale $v^{1/2}/4 \; \rightarrow \; v^{1/2}$ (i.e. $t \; \rightarrow \; 16\,t$) it corresponds to the Calabi-Yau series (number 69 in [10]):

$$CY_{69} \; = \; \sum_{n=0}^{\infty} \frac{(4n)!}{n!^2\,(2n)!} \cdot \sum_{k}^{n} \binom{n}{k}^2 \binom{2k}{k} \cdot t^n. \qquad (61)$$

7.3. From the LGF of 3-D f.c.c to Calabi-Yau number 14

The diagonal of the rational function $1/Q$ with denominator

$$Q \; = \; 1 \; - (x+y+z \; + u \cdot \, (y+z))\,, \qquad (62)$$

reads:

$$\mathrm{Diag}\Bigl(\frac{1}{Q}\Bigr) \; = \; {}_3F_2\left([\frac{1}{2}, \frac{1}{3}, \frac{2}{3}], [1,1], \, 108\,t\right). \qquad (63)$$

This hypergeometric series (63) actually occurs‡ for the lattice Green function of the 3-dimensional *face-centred cubic* lattice. Let us, now, consider a *rational number* α, and let us introduce the *algebraic function* $1/Q(\alpha)$:

$$Q(\alpha) \; = \; \Bigl(1 \; - (x+y+z \; + u \cdot \, (y+z))\Bigr)^{\alpha}. \qquad (64)$$

One actually obtains for the diagonal of $1/Q(\alpha)$:

$$\mathrm{Diag}\Bigl(\frac{1}{Q(\alpha)}\Bigr) \; = \; {}_4F_3\left([\frac{1}{2}, \frac{\alpha}{3}, \frac{\alpha}{3}+\frac{1}{3}, \frac{\alpha}{3}+\frac{2}{3}], \; [1,1,1], \; 108\,t\right). \qquad (65)$$

For $\alpha = 1/2$, the diagonal given in (65)

$$\mathrm{Diag}\Bigl(\frac{1}{Q(1/2)}\Bigr) \; = \; 1 \; + \frac{15}{4}\,t \; + \frac{31185}{256}\,t^2 \; + \frac{6381375}{1024}\,t^3 \; + \; \cdots \qquad (66)$$

is annihilated by an irreducible order-four differential operator, homomorphic to its adjoint, with $\ln(t)^3$ as the maximum power in its formal solutions around the origin. This globally bounded power series (66) is actually the diagonal of a rational function $1/\tilde{Q}(1/2)$ with $5 \; = \; 3 \; + 2$ variables where the polynomial denominator reads:

$$\tilde{Q}(1/2) \; = \; 1 \; - \Bigl(x+y+z \; + u \cdot \, (y+z) \; + \frac{v^{1/3}}{4}\Bigr). \qquad (67)$$

The diagonal of $1/\tilde{Q}(1/2)$ identifies with the series given in (66). The polynomial (67) (with the change $v^{1/3}/4 \; \rightarrow \; v^{1/3}$) is the same polynomial as the one given in Table 1, which corresponds to Calabi-Yau equation number 14.

Remark 7.3. The relations between the square root of the rational function of the LGF of simple cubic (resp. face centred cubic) and the rational function corresponding to Calabi-Yau number 69 (resp. Calabi-Yau number 14) are similar to the known relation [5, 38, 39] between the LGF of the $d-$dim diamond and the LGF of the $(d+1)-$dim simple cubic lattices. Other examples are given in Appendix G.

‡ The solution of the LGF of 3-D f.c.c is given in (19), a pullbacked form of (63).

7.4. Power of rational functions and homomorphisms

Consider the polynomial Q dependent on the parameter μ:

$$Q \,\, = \,\, 1 \,\, - \big(x + y \,\, + z^2 \,\, + \mu \cdot xz\big)\,. \tag{68}$$

The diagonal of $1/Q$ is annihilated by an irreducible order-four differential operator L_4, and the diagonal of the square $1/Q^2$ is annihilated by an irreducible order-four differential operator N_4. These two differential operators are actually *homomorphic* [21], giving

$$\mathrm{Diag}\Big(\frac{1}{Q^2}\Big) \,\, = \,\, W_3(\mu) \cdot \mathrm{Diag}\Big(\frac{1}{Q}\Big), \tag{69}$$

where $W_3(\mu)$ is an intertwiner of order three. For $\mu = 0$, the order-three differential operator W_3 reduces to an order-one intertwiner:

$$W_3(\mu = 0) \,\, = \,\, \frac{5}{2} \cdot t \cdot D_t \,\, + 1. \tag{70}$$

Let us take $\mu = 0$, and generalize the polynomial Q to (with r a positive integer):

$$\tilde{Q} \,\, = \,\, 1 \,\, - (x + y \,\, + z^r)\,. \tag{71}$$

The diagonal of $1/\tilde{Q}$ is annihilated by an order $2\,r$ differential operator $L_{2r}^{(1)}$. The differential operator, annihilating the diagonal of $1/\tilde{Q}^n$, is also of order $2\,r$. Let us call it $L_{2r}^{(n)}$. The differential operator $L_{2r}^{(n)}$ is homomorph with the differential operator $L_{2r}^{(n+1)}$, giving:

$$\mathrm{Diag}\Big(\frac{1}{\tilde{Q}^{n+1}}\Big) \,\, = \,\, \Big(\frac{2r+1}{r\,n} \cdot t\,D_t \,\, + 1\Big) \cdot \mathrm{Diag}\Big(\frac{1}{\tilde{Q}^n}\Big), \tag{72}$$

As far as conjecture (3) is concerned, the rational functions $1/\tilde{Q}$ and $1/\tilde{Q}^n$, have the same minimum number of variables, therefore the formal solutions of the corresponding differential operators *have the same maximum log-power.*

7.5. From reciprocal of square root to rational functions

Let us recall§ the diagonals $\mathrm{Diag}(1/Q^{1/2})$ given in (59) and (66), where Q depends on four variables, and which identify with the diagonals of $\mathrm{Diag}(1/Q_{eq})$ given in (60) and (67)

$$\mathrm{Diag}\left(\frac{1}{Q^{1/2}}\right) \,\, = \,\, \mathrm{Diag}\left(\frac{1}{Q_{eq}}\right), \tag{73}$$

where
$$Q_{eq} \,\, = \,\, Q \,\, - \frac{1}{4}\,u^\alpha, \tag{74}$$

with an additional fifth variable u, and where α is a rational (not integer) number. Let us show how this occurs with the procedure of Appendix B, and show whether the form (74) is general.

Consider the multivariate polynomial $Q = 1 - (T_1 + T_2 + \cdots + T_n)$, where the T_j's are monomials. The expansion of $1/Q^{1/2}$ reads:

$$\begin{aligned} \frac{1}{Q^{1/2}} \,\, &= \,\, \sum \frac{1}{4^{k_0}} \frac{(2\,k_0)!}{k_0!^2} \cdot (T_1 \, + T_2 \,\, + \cdots + T_n)^{k_0} \\ &= \,\, \sum \frac{1}{4^{k_0}} \frac{(2\,k_0)!}{k_0!^2} \cdot \frac{(k_1 + k_2 + \,\cdots\, + k_n)!}{k_1!\,k_2! \,\cdots\, k_n!} \cdot T_1^{k_1}\,T_2^{k_2}\,\cdots\,T_n^{k_n}. \end{aligned} \tag{75}$$

§ See also examples in Appendix G.

With the constraint $k_0 \, = \, k_1 + k_2 \, + \cdots + k_n$, there will be one more factorial (i.e. $k_0!$) in the denominator with respect to the expansion of $1/Q$ given in (8). One has then one additional monomial in the polynomial Q_{eq} if we consider the right hand side of (75) as an expansion of $1/Q_{eq}$.

By performing the diagonal on (75), one factorial in the denominator $k_j!$ takes a value like $k_j! \, = \, (r\, p + \cdots)!$, where p is the running index of the diagonal and r an integer. The additional monomial in Q_{eq} will be $v^{1/r}$. Now depending on the monomials T_j, there is no reason why the additional variable will not occur elsewhere.

Hereafter, let us display some examples of polynomials Q and Q_{eq} (with μ a parameter) where the relation (73) holds but with an expression for Q_{eq}, different from the simple expression given in (74):

$$Q \, = \, 1 \, - \big(x + y + z \, + \mu\, x^5\, y\big), \tag{76}$$

$$Q_{eq} \, = \, 1 \, - \Big(x + y + z \, + \mu\, x^5\, y\, u^{5/3} \, + {{1} \over {4}} u^{1/3}\Big), \tag{77}$$

$$Q \, = \, 1 \, - \big(x + y + z \, + xy + yz \, + \mu\, x^3\, y\, z\big), \tag{78}$$

$$Q_{eq} \, = \, 1 \, - \Big(x + y + z \, + (xy + yz) \cdot \, u^{1/3} \, + \mu\, x^3\, y\, z\, u^{4/3} \, + {{1} \over {4}} u^{1/3}\Big). \tag{79}$$

Such relations and relation (74) single out *square roots*, and *cannot* be simply generalised to any N-th roots of polynomials. This special role played by reciprocal of *square roots* comes, in fact, from the emergence of ratio of factorials of multiple argument in the relation:

$$ {{(1/2)_n} \over {n!}} \, = \, {{1} \over {4^n}} \, {{(2\, n)} \over {n!^2}}. \tag{80}$$

In view of the expressions of the polynomials Q_{eq} a natural question arises for the case where the polynomial Q is factorizable (in two factors, for instance) $Q \, = \, Q_1\, Q_2$. Does the additional variable u occurs in one factor, or in both, or as an additive monomial ? In the last situation, the equivalent polynomial Q_{eq} may come out *non factorizable*, and the equivalent differential operator will be *irreducible*.

Let us consider the polynomial $\tilde{Q} \, = \, (1 \, - x - y - z) \cdot (1 \, - y - z^2)$ given in (44). Here the variables in the second factor of $\tilde{Q}$ are a subset of the variables in the first factor. The diagonal of $1/\tilde{Q}$ is annihilated by an order-six differential operator with the unique factorization $L_6 \, = \, M_2 \cdot M_4$.

The first terms of the diagonal of $1/\tilde{Q}^{1/2}$ are ($t \, = \, xyz$):

$$\mathrm{Diag}\Big({{1} \over {\tilde{Q}^{1/2}}}\Big) \, = \, 1 \, + {{9} \over {4}}\, t \, + {{1695} \over {64}}\, t^2 \, + {{26215} \over {64}}\, t^3 \, + {{120986775} \over {16384}}\, t^4 \, + \cdots \tag{81}$$

This globally bounded power series is annihilated by an order-eight differential operator L_8. The differential operator L_8 is *irreducible* with a differential Galois group included in $SO(8, \mathbb{C})$. Among the formal solutions, at the origin, of the differential operator L_8, two series behave as $\ln(t)^2$, (i.e. the maximum log power). One of these solutions is the series given in (81) which, according to our conjectures, should be diagonal of a rational function with four variables ($N_v \, = \, 2 \, + 2$).

In order to find such a rational function $1/Q_{eq}$, we follow the procedure in Appendix B with the general term of the diagonal of $1/\tilde{Q}^{1/2}$, to obtain:

$$Q_{eq} \, = \, 1 \, \, - x - y - z \, \, + {{1} \over {4}} \cdot \, \big(2\, xy + y^2 \, + 2\, yz \, - 4\, z^2\big) \cdot \, u^{1/3}$$

$$+ \frac{1}{2} z^2 \cdot (2x + y + 2z) \cdot u^{2/3} \; - \frac{1}{4} u^{1/3}. \tag{82}$$

Consider, now, the polynomial $Q = (1-x-y-z) \cdot \big(1-x-y-z^2\big)$ where both factors in Q carry the same number of variables. The diagonal of $1/Q$ is annihilated by an order-seven differential operator with the direct sum factorization $L_7 = (L_2 \oplus L_4) \cdot N_1$. The diagonal of $1/Q^{1/2}$

$$\mathrm{Diag}\Big(\frac{1}{Q^{1/2}}\Big) \; = \; 1 \; + 3t \; + \frac{195}{4} t^2 \; + 665 t^3 \; + \frac{820575}{64} t^4 \; + \; \cdots \tag{83}$$

is annihilated by an *irreducible* order-eight differential operator with differential Galois group included in $SO(8, \mathbb{C})$. One of the formal solutions, at the origin, of the differential operator, behaves as $\ln(t)^2$, (i.e. the maximum log power). The power series in front of $\ln(t)^2$ identifies with the series given in (83), and should be the diagonal of a rational function Q_{eq} with four variables. The polynomial Q_{eq} reads:

$$Q_{eq} \; = \; 1 \; - (x+y) \cdot (z-1) \cdot (z+2) \; + (x+y)^2 \cdot u^{1/2}$$
$$+ z \cdot \big(z^2 - z - 1\big) \cdot u^{-1/2} \; - \frac{1}{4} \cdot u^{1/2}. \tag{84}$$

8. On the homomorphism to the adjoint assumption

All the examples, displayed in this paper (and many others not given here), confirm the conjecture (3). One assumption for this conjecture to hold, is that the (minimal order) differential operator is *homomorphic to its adjoint*, thus yielding symplectic, or orthogonal, differential Galois groups [22, 25, 40]. This assumption may look as an innocent caveat since, as we underlined in several papers [40], the (minimal order) differential operators annihilating diagonals of rational functions are, almost systematically, homomorphic to their adjoint†.

However, some examples of diagonals, whose corresponding (minimal order) differential operators are *not* homomorphic to their adjoint, have been seen to correspond to ${}_3F_2$ candidates to be counterexamples to Christol's conjecture [2, 20]. Such a candidate, for instance, reads:

$${}_3F_2\Big([\frac{2}{9}, \frac{5}{9}, \frac{8}{9}], \, [1, \, \frac{2}{3}], \; 27t\Big) \; = \; \mathrm{Diag}\Big(\frac{(1-x-y)^{1/3}}{1-x-y-z}\Big). \tag{85}$$

This hypergeometric function is the diagonal of a quite simple *algebraic function* of three variables. The order-three differential operator annihilating (85) is *not* homomorphic to its adjoint. Its differential Galois group is $SL(3, \mathbb{C})$. One of the formal solutions (at the origin) of the order-three differential operator has the highest logarithmic power $\ln(t)^1$.

Note that a representation of this hypergeometric function (85) as diagonal of *rational* function of more than *three variables*, is possible [2] using Denef-Lipschitz formulation [36], but a representation as a diagonal of a *rational* function of $1+2 = 3$ variables does not seem possible.

Thus this example does not seem to satisfy relation (3). It is outside the framework of this paper, for which the assumption to be homomorphic to its adjoint is not superfluous, *but necessary.*

† For reducible differential operator, each factor is homomorphic to its adjoint.

9. Conclusion

In this paper, we addressed some properties of the diagonal of rational functions. We restricted the analysis to the *rational* functions of the form $1/Q$.

It seems that the minimal number of variables N_v required to represent a globally bounded D-finite series as a diagonal of rational function, is simply related to the highest power n of the logarithmic formal solutions of the (minimal order) differential operator annihilating the diagonals, by $N_v = n + 2$, provided one *assumes* that this differential operator is *homomorphic to its adjoint*.

Furthermore it is observed that the symplectic, or orthogonal, character of the differential Galois group seems to be related to the *parity* of this highest power n of the logarithmic formal solution.

In the situation where the polynomial Q factorizes in two polynomial factors as $Q = Q_1 \cdot Q_2$, the differential operator annihilating the diagonal of $1/Q$ has either a direct sum or a unique factorization, depending on whether both polynomials Q_j carry all (the same) variables, or not. Furthermore, in the case of a unique factorization, the successive factors in the differential operator are included in the symplectic and orthogonal differential Galois groups, in alternance. Even in these factorized cases conjecture (3) remains valid.

All the results, and educated guess statements, of this paper are just conjectures. One would like to have a demonstration of these various conjectures. At first sight, one can imagine that the number of variables N_v is related to some "complexity measurement" of the (minimal order) differential operator annihilating the diagonal. A very naive measure of the complexity is the order of the operator, and the MUM examples, for which we have a simple relation between the highest power n and the order, seem to confirm such a naive view-point. We show, in this paper, that *this is not the case*. The relation the number of variables N_v is *not related with the order* but with highest log-power n. Along this line, and as far as a demonstration of the main conjecture (3) is concerned, let us underline that the crucial role played by the highest log exponent corresponds to the concept of monodromy filtration†, which can be introduced even if one is not totally sure that the differential operator is minimal order.

Acknowledgments We thank A. Bostan and J-A Weil, for discussions on ODE's with algebraic solutions. We thank A. Bostan for some p-curvature calculations. We thank G. Christol, for monodromy filtration discussions. We thank C. Koutschan for LGF discussions.

Appendix A. Linear differential operators corresponding to Lattice Green functions

Appendix A.1. MUM cases

Appendix A.1.1. Simple cubic lattice Green functions in d dimensions

The lattice Green function of the d-dimensional simple cubic lattice is given by the multiple integral of a rational function with numerator 1, and non factorizable denominator depending on d variables, and on the parameter t.

† See paragraph 4.2 page 40 of [20].

The diagonal $1/Q$, with polynomial Q given by

$$Q \;=\; 1 \;\; - \sum_{j=1}^{d} x_j \;\; - x_{d+1} \cdot \sum_{j=1}^{d} \prod_{i \neq j}^{d} x_i, \qquad \text{(A.1)}$$

depending on $N_v = d+1$ variables, reproduces the *simple cubic lattice* of dimension d. The diagonal of $1/Q$ is annihilated by an irreducible differential operator of order d, *having MUM*. The maximum power of the logarithmic formal solutions of the differential operators up to $d = 8$, and the corresponding rational functions follow the two conjectures (3), (4).

Appendix A.1.2. The diamond lattice Green function

The LGF of the 3-dimensional diamond lattice is given by the diagonal of $1/Q$, where

$$Q \;=\; 1 \;\; - xyzu \cdot S(x,y,z) \cdot S\Bigl(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\Bigr), \qquad \text{(A.2)}$$

with : $$S(x,y,z) \;=\; x + \frac{1}{x} \;\; + z \cdot \left(y + \frac{1}{y}\right), \qquad \text{(A.3)}$$

With the fourth variable u, the number of variables is $N_v = 4$. The corresponding order-three differential operator is irreducible, its differential Galois group is included in the orthogonal group $SO(3, \mathbb{C})$, and its formal solutions with the highest log-power behave as $\ln(t)^2$, in agreement with $N_v = 2 + 2 = 4$ variables.

The LGF of the 4-dimensional diamond lattice is given by the diagonal of $1/Q$, depending on $N_v = 5$ variables. The corresponding order-four differential operator is irreducible. Its differential Galois group is (included) in the symplectic group $Sp(4, \mathbb{C})$. Its formal solution with the highest log-power behaves as $\ln(t)^3$, in agreement with $N_v = 3 + 2 = 5$ variables.

Likewise, the LGF of the 5-dimensional diamond lattice is given by the diagonal of $1/Q$, which depends on $N_v = 6$ variables. The corresponding order-five differential operator is irreducible. Its differential Galois group is (included) in the orthogonal group $SO(5, \mathbb{C})$, and with the formal solution with the highest log-power at the origin behaving as $\ln(t)^4$, in agreement with $N_v = 4 + 2 = 6$ variables.

All these examples are in agreement with the two conjectures (3) and (4).

Appendix A.2. Non-MUM examples: Face centred cubic lattice Green functions in d dimensions

The lattice Green function of the d-dimensional face-centred cubic lattice is given by the multiple integral of a rational function with numerator 1, and non factorizable denominator depending on d variables. For these lattice Green we have *not* MUM (except for $d = 2, 3, 4$).

The diagonal $1/Q$ where Q is the polynomial

$$Q(x_1, x_2, \cdots, x_d, x_{d+1})$$
$$= \;\; 1 \;\; - \left(\prod_{j=1}^{d+1} x_j\right) \cdot \sum_{j>i}^{d} \left(x_i + \frac{1}{x_i}\right) \cdot \left(x_j + \frac{1}{x_j}\right), \qquad \text{(A.4)}$$

depending on $N_v = d+1$ variables, reproduces the face centred cubic lattice Green function of dimension d.

From $d = 2$ to $d = 12$, we denote the corresponding differential operators (the subscript being the order), G_2^{2Dfcc}, G_3^{3Dfcc} [3], G_4^{4Dfcc} [5], G_6^{5Dfcc} [6], G_8^{6Dfcc} [7], G_{11}^{7Dfcc} [8], and (G_{14}^{8Dfcc}, G_{18}^{9Dfcc}, G_{22}^{10Dfcc}, G_{27}^{11Dfcc}, G_{32}^{12Dfcc}) [9].

The differential operators, up to $d = 9$, are known to be irreducible [8, 9]. The differential operators, up to $d = 12$, have formal solutions with the highest log-power at the origin behaving as $\ln(t)^n$. All the differential operators (and their corresponding rational functions) (see Table A1) are in agreement with the two conjectures (3) and (4).

All the differential operators up to† $d = 11$ have *non globally bounded* formal solutions around all the singularities $t \neq 0$. There is an exception for G_3^{3Dfcc} which has, at the singularity $t = -1/4$, a *globally bounded* formal solution in front of $\ln(t + 1/4)^2$, i.e. the *same* maximum as around the singularity $t = 0$.

Table A1. Minimal number of variables, order, maximum exponent of $\ln(t)^n$ of the formal solutions and differential Galois group for the LGF of the fcc lattice of dimension $d = 2, 3, \cdots, 12$.

d	2	3	4	5	6	7	8	9	10	11	12
N_v	3	4	5	6	7	8	9	10	11	12	13
Order	2	3	4	6	8	11	14	18	22	27	32
n	1	2	3	4	5	6	7	8	9	10	11
Sp or SO	Sp	SO	Sp	SO	Sp	SO	Sp	SO	Sp	SO	Sp

Appendix B. The minimum number of variables in the rational function

For general coefficient of the series written as nested sums of binomials, one may use the integral representation of the binomial

$$\binom{n}{k} \; = \; \frac{1}{2\pi i}\int_C \frac{(1+z)^n}{z^k}\,\frac{dz}{z}, \qquad \text{(B.1)}$$

to write down the rational function [1]. The calculations are straithforward [1], and one obtains, this way, the rational function with as many variables as binomials plus one more variable. Furthermore, the denominator polynomial Q will be in a non factorized form, for more than one summation in the general term of the series.

Let us consider, for instance, the general term of the Calabi-Yau series (number 16 in [10])

$$\begin{aligned} CY_{16} \; &= \; \sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{2n}{n}\cdot\binom{n}{k}^2\cdot\binom{2k}{k}\cdot\binom{2n-2k}{n-k}\cdot t^n \\ &= \; 1 \; + 8\,t \; + 168\,t^2 \; + 5120\,t^3 \; + \cdots \qquad \text{(B.2)} \end{aligned}$$

Using the integral representation (B.1), each binomial, and the variable t, brings one variable. One obtains

$$CY_{16} \; = \; \mathrm{Diag}\left(\frac{1}{Q_1\cdot Q_2}\right), \qquad \text{(B.3)}$$

† For $d = 12$, the differential operator G_{32}^{12Dfcc} is known only modulo some primes.

where:

$$Q_1 \,=\, 1 \;\; - z_0 z_2 z_3 z_4 \cdot \, (1+z_1)^2 \cdot \, (1+z_2) \cdot \, (1+z_3) \cdot \, (1+z_5)^2,$$
$$Q_2 \,=\, 1 \;\; - z_0 z_5 \cdot \, (1+z_1)^2 \cdot \, (1+z_2) \cdot \, (1+z_3) \cdot \, (1+z_4)^2. \qquad (B.4)$$

This diagonal of rational function representation of the Calabi-Yau CY_{16} depends on six variables with a *factorizable* denominator. This is a diagonal of rational function representation of CY_{16}, but this simple nested sum of binomials *does not provide the minimal number of variables representation* for the diagonal.

Let us now show how the rational function $1/Q$, in Table 1, is obtained for Calabi-Yau CY_{16}.

The procedure amounts to casting the general term into the original form of the *multinomial* theorem, instead of the previous nested sum of binomials form. This means making the numerator with one factorial by introducing more summations. Converting to factorials, and using the formula

$$\frac{(2n)!}{n!^4} \;=\; \sum_{k=0}^{n} \frac{1}{k!^2\,(n-k)!^2}, \qquad (B.5)$$

in the general term of CY_{16}, one obtains¶:

$$CY_{16} \;=\; \sum_{n,k,k_1,k_2} \frac{(2n)!}{k_1!^2\; k_2!^2\; (k-k_1)!^2\; (n-k-k_2)!^2} \cdot t^n. \qquad (B.6)$$

The general term carries one factorial in the numerator, and eight factorials in the denominator. There are, then, 8 monomials in the multinomial expansion of the unknown polynomial denominator $Q \;=\; 1 \; - (T_1 + T_2 + \cdots + T_8)$.

These monomials correspond to each factorial as

$$T_1 \to\; k_1, \qquad T_2 \to\; k_1, \qquad T_3 \to\; k_2, \qquad T_4 \to\; k_2,$$
$$T_5 \to\; k-k_1, \qquad T_6 \to\; k-k_1, \qquad (B.7)$$
$$T_7 \to\; n-k-k_2, \qquad T_8 \to\; n-k-k_2$$

and satisfy

$$\frac{T_1T_2}{T_5T_6} \,=\, 1, \qquad \frac{T_3T_4}{T_7T_8} \,=\, 1, \qquad \frac{T_5T_6}{T_7T_8} \,=\, 1, \qquad T_7T_8 \,=\, t, \qquad (B.8)$$

which give:

$$T_2 \,=\, \frac{t}{T_1}, \qquad T_4 \,=\, \frac{t}{T_3}, \qquad T_6 \,=\, \frac{t}{T_5}, \qquad T_8 \,=\, \frac{t}{T_7}. \qquad (B.9)$$

The variable t is the product of all the variables. There are four unfixed T_j, the number of variables should be greater, or equal, to 4. Guided by our conjectures, let us, first, assume there are five variables, i.e. $t = xyzuv$. Choosing $T_1 = x$, $T_3 = y$, $T_5 = z$ and $T_7 = u$, we obtain the polynomial Q given in Table 1. On may also choose $T_1 = x$, $T_3 = y$, $T_5 = xu$ and $T_7 = yv$, to obtain

$$Q \,=\, 1 \;\; - x \cdot (1 + u + uz + uzv) \;\; - y \cdot (1 + v + vz + vzu)\,, \qquad (B.10)$$

and the diagonal of $1/Q$, then identifies with CY_{16}.

Assume, now, that we start with 6 variables, i.e. $t = xyzuvw$, and fix $T_1 = x$, $T_3 = y$, $T_5 = z$ and $T_7 = u$. One obtains the polynomial

$$Q \,=\, 1 \;\; - \Big(x + y + z + u \;\; + vw \cdot\, (xyz + yzu + zux + uxy)\Big), \qquad (B.11)$$

¶ This is the original form of the general term given in Section 8.1 of [31].

where it is clear that the product vw stands for *just one* variable.

Remark B.1. One should note that the conditions (B.8) are satisfied with 4 variables giving $Q \, = \, 1 \, - (x+y+z+u \, + xyz+yzu+zux+uxy)$, the diagonal of which *does not* identity with CY_{16}. The corresponding differential operator is of order four, is irreducible, has his differential Galois group included in $Sp(4, \mathbb{C})$, and the most logarithmic singular formal solution is in $\ln(t)^2$.

Remark B.2. Assume that, in the numerator of (B.6), one has $a^{k_1} \, (2n)! \, t^n$, the first condition in (B.8) changes to

$$\frac{T_1 T_2}{T_5 T_6} \, = \, 1 \qquad \longrightarrow \qquad \frac{T_1 T_2}{T_5 T_6} \, = \, a \tag{B.12}$$

and the denominator polynomial Q becomes:

$$1 \, - \Big(x+y+z+u \, \, + v \cdot \, (xzu+xyu+xyz+a\,yzu) \Big), \tag{B.13}$$

Appendix C. The factors occurring in the $\chi^{(n)}$'s of the Ising model

The differential operators corresponding to the n-particles contributions to the magnetic susceptibility of the Ising model, namely the $\chi^{(n)}$'s, and especially their decomposition in *products and direct sums of many factors* are recalled below (the subscript denote the order of the differential operator):

$\chi^{(1)}$, $\qquad L_1$,

$\chi^{(2)}$, $\qquad L_2$,

$\chi^{(3)}$, $\qquad L_7 \, = \, L_1 \oplus (Y_3 \cdot \, Z_2 \cdot \, N_1)$,

$\chi^{(4)}$, $\qquad L_{10} \, = \, L_2 \oplus (\tilde{L}_4 \cdot \, L_4^{(4)})$, $\qquad L_4^{(4)} \, = \, L_{1,3} \, (L_{1,2} \oplus L_{1,1} \oplus \, D_t)$,

$\chi^{(5)}$, $\qquad L_{33} \, = \, L_7 \oplus \, (L_5 \, L_{12} \cdot \, \tilde{L}_1) \cdot \, \Big(V_2 \oplus \, (Z_2 \cdot \, N_1) \oplus \, (F_3 \cdot \, F_2 \cdot \, L_1^s) \Big)$,

$\chi^{(6)}$, $\qquad L_{52} \, = \, L_{10} \oplus \, (L_6 \cdot \, \tilde{L}_2 \cdot \, L_{21}) \cdot \, \Big(\Big(D_t \, - \frac{1}{t} \Big) \oplus L_3 \oplus \, L_4^{(4)} \oplus \, (L_4 \, \tilde{L}_3 \cdot \, L_2^e) \Big)$,

All the factors, occurring in the differential operators, have been shown to be such that their differential Galois group is either symplectic or orthogonal (see [22] and references therein).

The blocks of factors (of order greater than 1) with unique factorization (with their corresponding differential Galois group), are:

$Y_3 \, Z_2$,	$Y_3 \rightarrow SO(3, \mathbb{C})$,	$Z_2 \rightarrow Sp(2, \mathbb{C})$	
$L_5 \, L_{12}$,	$L_5 \rightarrow SO(5, \mathbb{C})$,	$L_{12} \rightarrow Sp(12, \mathbb{C})$,	
$F_3 \, F_2$,	$F_3 \rightarrow SO(3, \mathbb{C})$,	$F_2 \rightarrow Sp(2, \mathbb{C})$,	
$L_6 \, \tilde{L}_2 \, L_{21}$,	$L_6 \rightarrow SO(6, \mathbb{C})$,	$\tilde{L}_2 \rightarrow Sp(2, \mathbb{C})$,	$L_{21} \rightarrow SO(21, \mathbb{C})$,
$L_4 \, \tilde{L}_3 \, L_2^e$,	$L_4 \rightarrow Sp(4, \mathbb{C})$,	$\tilde{L}_3 \rightarrow SO(3, \mathbb{C})$,	$L_2^e \rightarrow Sp(2, \mathbb{C})$.

Each time we deal with a block of factors with unique factorization, the differential Galois groups of the differential operators inside the block, are, *in alternance*, (included in) symplectic, or orthogonal, differential Galois groups.

Let us note that we have shown [1] that the $\chi^{(n)}$'s are actually diagonals of rational functions. However, for each differential operator factor in the blocks, it is quite hard to find the corresponding rational (or algebraic) function whose diagonal is annihilated by this differential operator factor. Conjecture (3), can just be used to determine the minimum number of variables in the rational function. This should, also, be consistent with the fact that *only* the series in front of $\ln(t)^n$, with n the maximum exponent, are globally bounded.

Consider, for instance, the order-twelve differential operator L_{12} occurring in the factorization of L_{33}, the differential operator annihilating $\chi^{(5)}$. Among the formal solutions of L_{12} around the origin, there are two series in front of $\ln(t)^3$, (the exponent $n = 3$ being the maximum), *which are, both, globally bounded.* According to conjecture (3), these two globally bounded series solutions of the differential operator L_{12}, *should be diagonal of rational functions* of $N_v = 3 + 2 = 5$ variables. Around the singularity $t = \infty = 1/s$, there are also two series in front of $\ln(s)^3$, (the same value of the exponent), and these series are *also globally bounded.* At the singularity $t = 1/4$, there is one series in front of $\ln(t - 1/4)^3$ among the formal solutions, and this series is *actually globally bounded.* Around each of the other singularities (with rational values), i.e. $t = -1/4, -1/2, -1, 1$, the maximum of power log in the formal solutions is, respectively, 2, 1, 1, 1, and all these series are *not globally bounded.* For the singularities $t = t_s$, roots of polynomials of degree 2, 3 and 4, we only checked that, for each, the formal solutions behave at the most as $\ln(t - t_s)^1$.

Appendix D. Other examples

Here, we give some examples that contradict, at first sight, the affirmation that the diagonal of $1/Q$ with non factorizable (resp. factorizable) Q over the rationals, is annihilated by an irreducible (resp. factorizable) differential operator.

The three rational functions $1/Q_j$ with Q_j given by

$$Q_1 \; = \; 1 \; + (x + y + z \; + xy + yz \; - x^3yz), \tag{D.1}$$

$$Q_2 \; = \; 1 \; - (x + y + z \; + x^4y), \tag{D.2}$$

$$Q_3 \; = \; 1 \; - \left(x + y + z \; + x^2 \cdot (y + z)\right), \tag{D.3}$$

rule out the irreducibility statement. The diagonal of the rational function $1/Q_j$ is annihilated by an order-four differential operator $L_4^{(j)}$ which factorizes as a *direct sum* of two order-two differential operators ($L_2^{(j)}$ and $M_2^{(j)}$):

$$L_4^{(j)} \; = \; L_2^{(j)} \oplus M_2^{(j)}. \tag{D.4}$$

The three examples follow exactly the same features. For the three examples, one has

$$\mathrm{sol}(L_2^{(j)}) \; = \; \mathrm{Diag}\left(\frac{1 + x}{Q_j}\right), \qquad \mathrm{sol}(M_2^{(j)}) \; = \; \mathrm{Diag}\left(\frac{1 - x}{Q_j}\right), \tag{D.5}$$

and for the three examples, their solutions $\mathrm{sol}(L_2^{(j)})$ and $\mathrm{sol}(M_2^{(j)})$ can be written as pullbacked hypergeometric function ${}_2F_1([1/12, 5/12], [1], \bullet)$ with an algebraic prefactor (see [41], for Q_1).

Note however, that when the polynomials Q_j are considered with *generic values* of the coefficients in front of the monomials, the resulting differential operators, annihilating the diagonal of $1/Q_j$, are of order four, and are *irreducible.*

Introducing a parameter μ in Q_1

$$Q_1(\mu) \; = \; 1 \; + (x + y + z \; + xy + yz \; - \mu \cdot x^3yz), \tag{D.6}$$

and using the method of factorization of differential operators modulo primes (see Section 4 in [15] and Remark 6 in [22]), one finds that the differential operator annihilating the diagonal $\mathrm{Diag}(1/Q_1(\mu))$ factorizes for only two values of μ. The trivial $\mu = 0$ where $Q_1(0) = (1 + x + z)(1 + y)$, and the particular value $\mu = 1$. The example with $Q_1(\mu = 1)$ is presented in [41] and analyzed via the notion of split Jacobian [42].

Note that either irreducible or factorizable in direct sum, the differential operators $L_2^{(j)}$, $M_2^{(j)}$ have their differential Galois groups included in symplectic groups, and their formal solution with the highest log-power behaves as $\ln(t)^1$, indicating a minimal number $N_v = 3 = 1 + 2$ of variables occurring in the rational function $1/Q_j$. This is also the case for the order-four irreducible differential operator corresponding to the diagonal of $1/Q_1(\mu)$, for generic values of the parameter μ.

Appendix E. Direct sum versus unique factorization for the diagonal of $1/Q$, with $Q = Q_1 Q_2$

The examples of section 6 dealt with a polynomial Q that factorizes as $Q_1^{(c)} Q_2$, where the polynomial $Q_1^{(c)}$ contains *all* the variables, and the polynomials Q_2 has a smaller number of variables. Call L_q, the differential operator annihilating the diagonal of $1/Q_1^{(c)}$. The differential operator, annihilating the diagonal of $1/Q_1^{(c)}/Q_2$, appears with a unique factorization as $N_q \cdot M_n$, where N_q is homomorphic with L_q. There is no known relation between the differential operator, corresponding to $1/Q_2$, and the "dressing" differential operator M_n.

Appendix E.1. Direct sum

Let us consider the situation where Q factorizes as $Q_1^{(c)} Q_2^{(c)}$, and where both polynomials $Q_1^{(c)}$ and $Q_2^{(c)}$ carry all the variables. Call L_q, L_n and $\mathcal{L}_m$ the differential operators annihilating respectively, $\mathrm{Diag}(1/Q_1^{(c)})$, $\mathrm{Diag}(1/Q_2^{(c)})$ and $\mathrm{Diag}(1/Q_1^{(c)}/Q_2^{(c)})$. Since the polynomials $Q_1^{(c)}$ and $Q_2^{(c)}$ are on an equal footing in terms of number of variables, one may expect the differential operator $\mathcal{L}_m$ to have a left factor homomorphic to L_q as well as another left factor homomorphic to L_n. This means that $\mathcal{L}_m$ has a factorization in *direct sum* between the homomorphic differential operators of L_q and L_n.

Appendix E.2. From direct sum to unique factorization

Let us see how the direct sum *actually reduces to a simple product*, when one of the factors Q_j has less variables. Take the rational functions $1/Q_1$ and $1/Q_2$ with the polynomials:

$$Q_1 \; = \; 1 \; - (x+y+z), \qquad Q_2(\alpha) \; = \; 1 \; - (x+y \; + \alpha \cdot z \; + xy). \qquad \text{(E.1)}$$

The diagonal $\mathrm{Diag}(1/Q_1)$ (resp. $\mathrm{Diag}(1/Q_2(\alpha))$) is annihilated by an order-two differential operator $L_2^{(1)}$ (resp. $L_2^{(2)}$). The diagonal of $1/Q_1/Q_2(\alpha)$

$$\mathrm{Diag}\left(\frac{1}{Q_1 \, Q_2}(\alpha)\right) \qquad \text{(E.2)}$$
$$= \; 1 \; + (13 + 14\,\alpha) \cdot t \; + (241 + 273\,\alpha + 306\,\alpha^2) \cdot t^2 \; + \, \cdots$$

is annihilated by a (quite large†) order-six differential operator L_6, depending on α, and it factorizes *in a direct sum* as:

$$L_6 \,\, = \,\, \Big(N_2^{(1)} \oplus N_2^{(2)}\Big) \cdot H_2 \,\, = \,\, (N_2^{(1)} \cdot H_2) \oplus (N_2^{(2)} \cdot H_2). \qquad \text{(E.3)}$$

The order-two differential "dressing" operator H_2 has two *algebraic* solutions:

$$\mathrm{sol}(H_2) \,\, = \,\, \frac{1}{\sqrt{t} \cdot \sqrt{p_2}} \cdot \Big(const.\, A^{1/4} \,\, + const.\, (t \,\, - \alpha + 1) \cdot \, A^{-1/4}\Big), \qquad \text{where:}$$

$$A \,\, = \,\, p_1 \,\, + 4 \cdot (2\alpha - 3) \cdot \sqrt{\alpha - 1} \cdot \sqrt{t} \cdot \sqrt{p_2}, \qquad \text{where:}$$

$$p_1 \,\, = \,\, t^2 \,\, + 2 \cdot (\alpha - 1) \cdot (4\alpha - 5) \cdot (4\alpha - 7) \cdot t \,\, + (\alpha - 1)^2,$$

$$p_2 \,\, = \,\, t^2 \,\, + 2 \cdot (\alpha - 1) \cdot (8\alpha^2 - 24\alpha + 17) \cdot t \,\, + (\alpha - 1)^2. \qquad \text{(E.4)}$$

The order-two differential operator $N_2^{(1)}$ (resp. $N_2^{(2)}$) *is homomorphic with the differential operator* $L_2^{(1)}$ (resp. $L_2^{(2)}$). The differential operator $L_2^{(1)}$ does not depend on α, *but the intertwiner* $W_1^{(1)}$ *does*:

$$N_2^{(1)}(\alpha) \cdot W_1^{(1)}(\alpha) \,\, = \,\, \tilde{W}_1^{(1)}(\alpha) \cdot L_2^{(1)}. \qquad \text{(E.5)}$$

From the direct-sum decomposition (E.3), the diagonal of $1/Q_1/Q_2(\alpha)$ which is annihilated by the differential operator L_6, reads:

$$\mathrm{Diag}\Big(\frac{1}{Q_1\, Q_2(\alpha)}(\alpha)\Big) \,\, = \,\, \mathrm{sol}\Big(N_2^{(1)}(\alpha) \cdot H_2(\alpha)\Big) \,\, + \mathrm{sol}\Big(N_2^{(2)}(\alpha) \cdot H_2(\alpha)\Big). \qquad \text{(E.6)}$$

For $\alpha = 0$ the analytical solution (at $t = 0$) of $N_2^{(2)}(\alpha) \cdot H_2(\alpha)$ is the analytical solution (at $t = 0$) of $H_2(\alpha)$ for $\alpha = 0$, namely the algebraic series

$$\mathcal{A} \,\, = \,\, 1 \,\, + 21\, t \,\, + 561\, t^2 \,\, + 16213\, t^3 \,\, + 487521\, t^4 \,\, + 15015573\, t^5 \,\, + \,\, \cdots \qquad \text{(E.7)}$$

together with the constant function. The other solution of $H_2(\alpha)$, for $\alpha = 0$, is not analytic at $t = 0$:

$$S_2 \,\, = \,\, t^{-1/2} \cdot \big(1 \,\, + 13\, t \,\, + 321\, t^2 \,\, + 8989\, t^3 \,\, + \,\, \cdots\big). \qquad \text{(E.8)}$$

The analytical solutions (at $t = 0$) of $N_2^{(1)}(\alpha) \cdot H_2(\alpha)$ for $\alpha = 0$ are, the analytical solution (at $t = 0$) of $H_2(\alpha)$ for $\alpha = 0$, namely (E.7), together with the series:

$$\mathcal{S} \,\, = \,\, t \,\, + 40\, t^2 \,\, + 1400\, x^3 \,\, + 47110\, t^4 \,\, + 1560328\, t^5 \,\, + \,\, \cdots \qquad \text{(E.9)}$$

The diagonal series (E.2) for $\alpha = 0$ reads:

$$1 \,\, + 13\, t \,\, + 241\, t^2 \,\, + 5013\, t^3 \,\, + 110641\, t^4 \,\, + 2532949\, t^5 \,\, + \,\, \cdots \qquad \text{(E.10)}$$

which is nothing but the following combination of (E.7) and (E.9):

$$\mathrm{Diag}\Big(\frac{1}{Q_1 \cdot \, Q_2(\alpha) = 0}\Big) \,\, = \,\, \mathcal{A} \,\, - 8 \cdot \mathcal{S}. \qquad \text{(E.11)}$$

This series is, thus, annihilated by *only* the differential operator $N_2^{(1)} \cdot H_2$, for $\alpha = 0$, *with no need of the differential operator* $N_2^{(2)} \cdot H_2$ for $\alpha = 0$.

Remark E.1. Note that the "spurious" differential operator $N_2^{(2)} \cdot H_2$, actually factorizes as a direct sum for $\alpha = 0$

$$N_2^{(2)} \cdot H_2(\alpha = 0) \,\, = \,\, D_t \oplus \Big(L_1 \cdot H_2(\alpha = 0)\Big), \qquad \text{(E.12)}$$

where the order-one differential operator L_1 annihilates a rational solution.

† The polynomial coefficients of L_6 are of degree 21 in t and degree 22 in α.

Appendix F. Rational function from four variables to three variables as a parameter varies

The example in section 4.3 shows that, depending on the number (four or three) of variables on which the diagonal is performed, we obtain, in the formal solutions of the correspondig differential operators, the power 2 (i.e. $\ln(t)^2$) or the power 1 (i.e. $\ln(s)^1$) in agreement with conjecture (3).

Here, we consider an example (depending on one parameter) where the diagonal is performed on all the four variables, and the result, for some value of the parameter ($b = 2$ in Appendix F.2 below), will indicate that we deal, in fact, with only three variables in agreement with the power $\ln(t)^1$ obtained in the formal solutions.

Let us consider the diagonal of the following rational function of *four* variables x, y, z, u, and a parameter b

$$\mathrm{Diag}\Big(\frac{1}{Q_1\,Q_2}\Big) \;=\; 1 \;+20\cdot(b+2)\cdot t \;+756\cdot\Big(b^2+3\,b+6\Big)\cdot t^2 \\ +34320\cdot\Big(b^3+4\,b^2+10\,b+20\Big)\cdot t^3 \;+\;\cdots \tag{F.1}$$

where:

$$Q_1 \;=\; 1 \;-x-y-z-u, \qquad Q_2 \;=\; 1 \;-x-y-b\cdot z. \tag{F.2}$$

The power series (F.1) is annihilated (for generic values of b) by an order-six differential operator with the unique factorization

$$L_6(b) \;=\; L_3(b)\cdot L_2(b)\cdot L_1 \tag{F.3}$$

where the order-one differential operator, L_1, reads $D_t+1/2/t$.

The order-three differential operator $L_3(b)$ has the following hypergeometric solution:

$$\frac{1}{t^2\cdot(1-b\;+64\,b^2\cdot t)}\cdot {}_3F_2\Big([\frac{3}{2},\,\frac{7}{4},\,\frac{9}{4}],\,[1,\,2],\;256\cdot t\Big). \tag{F.4}$$

The formal series solutions of $L_3(b)$ have, at most, a $\ln(t)^2$.

The order-two differential operator $L_2(b)$ has the following ${}_2F_1$ hypergeometric solution:

$$\frac{1}{t}\cdot {}_2F_1\Big([\frac{3}{4},\,\frac{5}{4}],\,[1],\;\frac{64\cdot b^2}{b-1}\cdot t\Big). \tag{F.5}$$

The formal series solutions of $L_2(b)$ have, at most, a $\ln(t)^1$ logarithmic power.

The formal series solutions of $L_6(b)$ have, at most, a $\ln(t)^2$ logarithmic power. This is in agreement with conjecture (3), and the fact that (F.1) corresponds to a rational function of $2+2=4$ variables.

Let us introduce the order-three differential operator U_3, annihilating

$$\mathrm{Diag}\Big(\frac{1}{Q_1}\Big) \;=\; {}_3F_2\Big([\frac{1}{4},\,\frac{2}{4},\,\frac{3}{4}],\,[1,\,1],\,256\,t\Big). \tag{F.6}$$

This order-three differential operator reads:

$$U_3 \;=\; (1-256\,t)\cdot t^2\cdot D_t^3 \;+3\cdot(1-384\,t)\cdot t\cdot D_t^2 \\ +(1-816\,t)\cdot D_t \;-24. \tag{F.7}$$

The formal series solutions of U_3 have, at most, a $\ln(t)^2$ logarithmic power, in agreement with conjecture (3) and the fact that $1/Q_1$ is a rational function of four variables. The order-three differential operator $L_3(b)$, in the factorization given in (F.3), is *actually homomorphic* to the order-three differential operator U_3.

All the results above are for the generic values of the parameter b. In the sequel we consider the situation with the values of b for which the series (F.1) will be annihilated by a differential operator of order less than six. These values of b are† $b = 0$, $b = 1$ and $b = 2$.

Appendix F.1. The $b = 0$ and $b = 1$ cases

For the value of the parameter $b = 0$, the factorization of the order-six differential operator $L_6(b)$ becomes

$$L_6(b = 0) \;\; = \;\; \Bigl(\Bigl(D_t + \frac{1}{t}\Bigr) \cdot D_t\Bigr) \oplus \Bigl(N_3 \cdot L_1\Bigr). \qquad \text{(F.8)}$$

For $b = 0$ the series (F.1) is annihilated by the order-four differential operator $N_3 \cdot L_1$, whose formal solutions, at the origin carry the maximum exponent $\ln(t)^2$, indicating, according to conjecture (3), that we deal with a rational function with $4 = 2 + 2$ variables.

For the value of the parameter $b = 1$, the factorization of the order-six differential operator $L_6(b)$ reads:

$$L_6(b = 1) \;\; = \;\; \Bigl(D_t + \frac{1}{2t}\Bigr) \oplus \Bigl(D_t + \frac{3}{4t}\Bigr) \oplus \Bigl(D_t + \frac{5}{4t}\Bigr) \oplus M_3. \qquad \text{(F.9)}$$

For $b = 1$ the series (F.1) is annihilated by the order-three differential operator M_3, whose formal solutions at the origin carry the maximum exponent $\ln(t)^2$ indicating, according to conjecture (3), that we deal with a rational function with $2 + 2 = 4$ variables.

Appendix F.2. The $b = 2$ case

For $b = 2$ the factorization of the order-six differential operator $L_6(b)$ becomes:

$$L_6(b = 2) \;\; = \;\; V_3 \oplus \Bigl(L_2(b = 2) \cdot L_1\Bigr). \qquad \text{(F.10)}$$

The series (F.1), for the parameter $b = 2$, is annihilated by the order-three differential operator $L_2(b = 2) \cdot L_1$ with the hypergeometric solution:

$$\mathrm{sol}(L_2(b = 2) \cdot L_1) \;\; = \;\; {}_3F_2\Bigl([\frac{1}{2}, \frac{3}{4}, \frac{5}{4}], [1, \frac{3}{2}], 256\, t\Bigr) \qquad \text{(F.11)}$$

The formal solutions at the origin of $L_2(b = 2) \cdot L_1$ carry the maximum exponent $\ln(t)^1$ indicating, according to conjecture (3), that we deal with a rational function with $3 = 1 + 2$ variables, while the rational function we started with, was dependent on four variables. The power series (F.11) should, then, be the diagonal of a rational function depending on *only three* variables.

Introducing the polynomial $Q_3 = 1 - x - y - b \cdot u$, the partial fraction form of $1/Q_1/Q_2$ in the variable z, reads for $b = 2$:

$$\frac{1}{Q_1\, Q_2} \;\; = \;\; -\frac{1}{Q_1\, Q_3} \; + \frac{2}{Q_2\, Q_3}. \qquad \text{(F.12)}$$

† These values of b can be obtained using the method of factorization of differential operators modulo primes (see Section 4 in [15] and Remark 6 in [22]). They can also be obtained by reducing the four singularities $t = \infty, 0, 1/256, (b-1)/64/b^2$ of $L_6(b)$ to the three singularities $t = \infty, 0, 1/256$.

The diagonal of $1/Q_1/Q_2$, and the diagonal of $1/Q_1/Q_3$, are identical, by the symmetry $(z, u) \rightarrow (u, z)$. We end up with

$$\mathrm{Diag}\Big(\frac{1}{Q_1\,Q_2}\Big) \; = \; \mathrm{Diag}\Big(\frac{1}{Q_2\,Q_3}\Big) \tag{F.13}$$
$$= \; \mathrm{Diag}\Big(\frac{1}{(1\,-x\,-y\,-2\,z)\cdot(1\,-x\,-y\,-2\,u)}\Big).$$

It remains to show that this diagonal depends, in fact, *on only three variables* instead of four variables:

$$\mathrm{Diag}\Big(\frac{1}{Q_2\,Q_3}\Big) \; = \; \mathrm{Diag}\Big(\frac{1}{(1\,-x-y)^2}\cdot\sum_{i,j}\frac{(2z)^i\,(2u)^j}{(1\,-x-y)^{i+j}}\Big)$$
$$= \; \mathrm{Diag}\Big(\frac{1}{(1\,-x-y)^2}\cdot\sum_{k=0}^{\infty}\frac{(4\,z\,u)^k}{(1\,-x-y)^{2k}}\Big). \tag{F.14}$$

The sum on the index k gives:

$$\mathrm{Diag}\Big(\frac{1}{Q_2\,Q_3}\Big) \; = \; \mathrm{Diag}\Big(\frac{1}{(1\,-x-y)^2\;\;-4\,z\,u}\Big). \tag{F.15}$$

As far as the diagonal is concerned, the product $z\,u$ stands for *only one* variable. The diagonal given in (F.1) for $b = 2$, which is the series given in (F.11) is the diagonal of a rational function depending on *three* variables, as the power of the logarithm in the formal solutions of $L_2(b=2)\cdot L_1$ indicates.

Remark F.1. In the factorization of $L_6(b)$ for $b = 2$, given in (F.10), the differential operator $L_2(b=2)\cdot L_1$ annihilates the series (F.1) for the value $b = 2$. Therefore, the differential operator V_3 becomes "spurious". The analytical solution (at the origin) of the differential operator V_3 reads

$$\mathrm{sol}(V_3) \; = \; {}_3F_2\Big([\frac{2}{4},\,\frac{3}{4},\,\frac{5}{4}],\,[1,\,1],\,256\,t\Big), \tag{F.16}$$

to be compared with the solution (F.6) of the differential operator U_3. The differential operators U_3 and V_3 are *actually homomorphic*

$$(1-256\,t)\cdot t^2\cdot V_3\cdot\Big(t\cdot\,D_t\,+\frac{1}{4}\Big) \; = \; \Big(t\cdot D_t\,+\frac{5}{4}\Big)\cdot(1-256\,t)\cdot t^2\cdot U_3, \tag{F.17}$$

which shows (see section 7.4) that (F.16) is in fact:

$$\mathrm{sol}(V_3) \; = \; \mathrm{Diag}\Big(\frac{1}{Q_1^2}\Big). \tag{F.18}$$

At the value of $b = 2$, even if the differential operator V_3 is spurious with respect to the series (F.1) for $b = 2$, the differential operator $L_6(b)$ has kept, for $b = 2$, some "memory" of the rational function $1/Q_1$, through the spurious operator V_3.

Appendix G. Square root of rational functions versus Calabi-Yau equations

As in sections 7.2 and 7.3, we give, here, two more examples where the diagonal of square root of a rational function produces a Calabi-Yau equation.

Appendix G.1. The square root of the rational function of the LGF 4-D simple cubic

Recall the rational function $1/Q$, depending on 5 variables, and corresponding to the lattice Green function of the 4-dimensional simple cubic lattice:

$$Q \; = \; 1 \; - \Big(x+y+z+u \; + v \cdot (xyz+xyu+xzu+yzu)\Big), \qquad \text{(G.1)}$$

$$\mathrm{Diag}\Big(\frac{1}{Q}\Big) \; = \; 1 \; + 8\,t \; + 168\,t^2 \; + 5120\,t^3 \; + 190120\,t^4 \; + \; \cdots$$

Considering, now, the diagonal of the *square root* of this rational function (G.1)

$$\mathrm{Diag}\Big(\frac{1}{Q^{1/2}}\Big) \; = \; 1 \; + 3\,t \; + \frac{735}{16}\,t^2 \; + 1155\,t^3 \; + \frac{152927775}{4096}\,t^4 \; + \; \cdots$$

one obtains an annihilating irreducible order-five differential operator L_5, with a differential Galois group included in the orthogonal group $SO(5, \mathbb{C})$, and with $\ln(t)^4$ highest log-power formal solution at the origin. According to conjecture (3), a rational function $1/Q_{eq}$, *depending on* $6 = 4 + 2$ *variables* should exist. With the introduction of an extra variable w, it actually reads:

$$Q_{eq} \; = \; Q \; - \frac{w^{1/2}}{4}. \qquad \text{(G.2)}$$

The diagonal of the *algebraic* function $1/Q^{1/2}$ identifies with the diagonal of the *rational* function $1/Q_{eq}$.

The differential operator L_5 has MUM, and is actually the exterior square of an order-four differential operator, $L_5 \, = \, \mathrm{ext}^2\,(L_4)$. The differential operator L_4 also has MUM, at $t \, = \, 0$, the indicial exponents being four times $1/2$. Let us introduce the differential operator N_4:

$$N_4 \; = \; L_4 \cdot \sqrt{t} \cdot (1\,-16\,t)^{1/4} \cdot \, (1\,-64\,t)^{1/4}. \qquad \text{(G.3)}$$

With the scaling $t \rightarrow 16\,t$, the differential operator N_4 reads (with $\theta = \, t\,D_t$):

$$\begin{aligned} N_4 \; = \; & \theta^4 \; - 4 \cdot t \cdot \big(960\,\theta^4 + 640\,\theta^3 + 574\,\theta^2 + 254\,\theta + 41\big) \\ & + 16 \cdot t^2 \cdot \big(356352\,\theta^4 + 451328\,\theta^2 + 475136\,\theta^3 + 199424\,\theta + 34257\big) \\ & - 2^{12} \cdot t^3 \cdot \big(1003520\,\theta^4 + 2007040\,\theta^3 + 2098048\,\theta^2 + 1043328\,\theta + 198453\big) \\ & + 2^{22} \cdot t^4 \cdot \big(356352\,\theta^4 + 950272\,\theta^3 + 1126912\,\theta^2 + 618752\,\theta + 124913\big) \\ & - 2^{37} \cdot t^5 \cdot (2\,\theta+1) \cdot \big(960\,\theta^3 + 2720\,\theta^2 + 2854\,\theta + 987\big) \\ & + 2^{48} \cdot t^6 \cdot (2\,\theta+1) \cdot (2\,\theta+3) \cdot (4\,\theta+3) \cdot (4\,\theta+5). \end{aligned} \qquad \text{(G.4)}$$

To be of Calabi-Yau type, the differential operator N_4 must satisfy some conditions [33, 43]. The differential Galois group of N_4 is included in $Sp(4, \mathbb{C})$, the differential operator N_4 has MUM, and at the infinity, the indicial exponents 1/2, 3/4, 5/4, 3/2 are such that $1/2 \, + 3/2 = \, 3/4 \, + 5/4 \, = \, 2$ is a rational. Morever, the coefficients of the power series given below in (G.5), and the instanton numbers given below in (G.11), should be integers.

The formal solutions, at the origin, of the differential operator N_4 are

$$S_0 \; = \; 1 \; + 164\,t \; + 66972\,t^2 \; + 38050160\,t^3 \; + \; \cdots \qquad \text{(G.5)}$$

and three formal solutions S_j, $(\, j = 1, 2, 3)$ in the form of the set (1) and behaving as $S_j = \; S_0 \, \ln(t)^j \; + \cdots$

Let $z = S_1/S_0$, the nome q, defined as $q = \exp(z)$, has the expansion

$$q \, = \, t \, + 360\, t^2 \, + 188244\, t^3 \, + 119619168\, t^4 \, + \, \cdots \tag{G.6}$$

and the mirror map reads:

$$t \, = \, q \, - 360\, q^2 \, + 70956\, q^3 \, - 14059968\, q^4 \, + \, \cdots \tag{G.7}$$

The Yukawa coupling defined by

$$K(q) \, = \, \frac{d^2}{dz^2} \left(\frac{S_2}{S_0} \right), \tag{G.8}$$

reads:

$$K(q) \, = \, 1 \, - 128\, q \, - 41984\, q^2 \, - 13919744\, q^3 \, - 4141162496\, q^4 \, + \, \cdots \tag{G.9}$$

The Yukawa coupling, expanded in a Lambert series, reads

$$K(q) \, = \, 1 \, + \sum_{j=1}^{\infty} n_j \, \frac{j^3\, q^j}{1 - q^j}, \tag{G.10}$$

and gives the "instantons numbers" n_j, $j = 1, 2, \cdots$

$$- 128, \, -5232, \, -\frac{1546624}{3}, \, -64705008, \, -7960717440, \, -1089730087792, \, \cdots \tag{G.11}$$

where, with $n_0 = 3$, the numbers $n_0\, n_j$ *are actually integers.*

The differential operator N_4 satifies the Calabi-Yau type conditions, and especially the series given in (G.5), (G.6), (G.7) and (G.9) have *integer* coefficients.

Appendix G.2. The square root of the rational function of the LGF 4-D body centred cubic

Let us consider the rational function $1/Q$, where the polynomial Q reads:

$$Q \, = \, 1 \, - (x+z) \cdot (1+y) \cdot (1+u) \cdot (1+v) \, . \tag{G.12}$$

This is the multivariate polynomial corresponding to the Calabi-Yau number 3 (see Table 1). The diagonal of $1/Q$ also identifies with the LGF of the 4-D body centred cubic lattice [4].

The diagonal of $1/\sqrt{Q}$ with Q given in (G.12), reads:

$$\mathrm{Diag}\left(\frac{1}{\sqrt{Q}} \right) \, = \, {}_5F_4 \left([\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}], \, [1,1,1,1], \, 256\, t \right). \tag{G.13}$$

This diagonal is annihilated by an irreducible order-five differential operator L_5, which has a differential Galois group included in the orthogonal group $SO(5, \mathbb{C})$. The operator L_5 has a $\ln(t)^4$ highest log-power formal solution at the origin. According to conjecture (3), a rational function $1/Q_{eq}$, *depending on* $6 \, = \, 4 + 2$ *variables*, should exist. It actually (with an extra variable w) reads $Q_{eq} = Q - w^{1/2}/4$. and the diagonal of $1/\sqrt{Q}$ actually identifies with the diagonal of $1/Q_{eq}$.

The order-five differential operator L_5 has MUM, and is actually the exterior square of an order-four differential operator, $L_5 \, = \, \mathrm{ext}^2\,(L_4)$. The differential operator L_4 has MUM, at $t = 0$, the indicial exponents being four times $1/2$.

Let us introduce the differential operator N_4:

$$N_4 \, = \, L_4 \cdot \sqrt{t} \cdot (1 \, - 256\, t)^{1/4} \, . \tag{G.14}$$

With the scaling $t \rightarrow 16\,t$, the differential operator N_4 reads (with $\theta = t\,D_t$):

$$N_4 \;=\; \theta^4 \; - 16 \cdot t \cdot \, \bigl(758\,\theta^4 + 512\,\theta^3 + 440\,\theta^2 + 184\,\theta \, + 25\bigr) \\ + 2^{12} \cdot t^2 \cdot \, \bigl(12288\,\theta^4 + 16384\,\theta^2 \, + 13056\,\theta^3 \, + 3840\,\theta \, + 401\bigr) \\ - 2^{24} \cdot t^3 \cdot \, (8\,\theta + 3)^2 \cdot \, (8\,\theta + 5)^2 \, . \qquad \text{(G.15)}$$

Similarly to the previous example, the differential operator N_4 given in (G.15) satisfies the Calabi-Yau type conditions.

References

[1] A. Bostan, S. Boukraa, G. Christol, S. Hassani, J.-M. Maillard, *Ising n-fold integrals as diagonal of rational functions and integrality of series expansions*, J. Phys. A: Math. Theor. **46** (2013)185202 (44pp); arXiv:1211.6645

[2] Y. Abdelaziz, C. Koutschan and J-M. Maillard, *On Christol's conjecture*, 2020 J. Phys. A: Math. Theor. 53 205201

[3] G.S. Joyce, *On the cubic modular transformation and the cubic lattice Green functions*, J. Phys. A: Math. Theor. **31** (1998) 5105-5115

[4] A. J. Guttmann, *Lattice Green functions and Calabi-Yau differential equations*, J. Phys. **A 42**: Mathematical and Theoretical, (2009) 232001 (6pp)

[5] A. J. Guttmann, *Lattice Green functions in all dimensions*, J. Phys. A: Math. Theor. **43**, (2010) 305205 (26pp) and arXiv:1004.1435

[6] D. Broadhurst 2009 *Bessel moments, random walks and Calabi-Yau equations*, unpublished, available at carma.newcastle.edu.au/jon/Preprints/Papers/SubmittedPapers/4step-walks/walk-broadhurst.pdf

[7] C. Koutschan, *Lattice Green's Functions of the higher-dimensional face-centred cubic lattices*, J. Phys. A: Math. Theor. **46** (2013) 125005 and http://www.koutschan.de/data/fcc/fcc.nb.pdf and arXiv:1108.2164

[8] N. Zenine, S. Hassani, J.-M. Maillard, *Lattice Green functions: the seven-dimensional face-centred cubic lattice*, J. Phys. A: Math. Theor. **48** (2015)035205 (19pp)

[9] S. Hassani, Ch. Koutschan, J.-M. Maillard, N. Zenine, *Lattice Green functions: the $d-$dimensional face-centered cubic lattice, $d = 8, 9, 10, 11, 12$*, J. Phys. A: Math. Theor. **49** (2016)164003 (30pp)

[10] G. Almkvist, C. van Enckevort, D. van Straten and W. Zudilin (2010), *Tables of Calabi-Yau equations*, arXiv:math0507430v2

[11] T. T. Wu, B. M. McCoy, C. A. Tracy and E. Barouch, *Spin-spin correlation functions for the two-dimensional Ising model: Exact theory in the scaling region*, Phys. Rev. **B 13** (1976) 316–374

[12] N. Zenine, S. Boukraa, S. Hassani, J.M. Maillard, *The Fuchsian differential equation of the square lattice Ising model $\tilde{\chi}^{(3)}$ susceptibility*, J. Phys. A: Math. Gen. **37**, (2004)9651-9668; arXiv:0407060

[13] N. Zenine, S. Boukraa, S. Hassani, J.M. Maillard, *Ising model susceptibility: Fuchsian differential equation for $\tilde{\chi}^{(4)}$ and its factorization properties*, J. Phys. A: Math. Gen. **38**, (2005)4149-4173; arXiv:0502155

[14] S. Boukraa, A.J. Guttmann, S. Hassani, I. Jensen, J.M. Maillard, B. Nickel, N. Zenine, *Experimental mathematics on the magnetic susceptibility of the square lattice Ising model*, J. Phys. A: Math. Theor. **41** (2008) 455202 (51pp); arXiv:0808.0763

[15] A. Bostan, S. Boukraa, A.J. Guttmann, S. Hassani, I. Jensen, J.M. Maillard, N. Zenine, *High order Fuchsian equations for the square lattice Ising model: $\tilde{\chi}^{(5)}$*, J. Phys. A: Math. Theor. **42** (2009) 275209 (32pp); arXiv:0904.1601

[16] B. Nickel, I. Jensen, S. Boukraa, A.J. Guttmann, S. Hassani, J.M. Maillard, N. Zenine, *Square lattice Ising model $\tilde{\chi}^{(5)}$ ODE in exact arithmetic*, J. Phys. A: Math. Theor. **43** (2010) 195205 (24pp); arXiv:1002.0161

[17] S. Boukraa, S. Hassani, I. Jensen, J.-M. Maillard and N. Zenine, *High order Fuchsian equations for the square lattice Ising model: $\chi^{(6)}$*, J. Phys. A: Math. Theor. **43** (2010) 115201 (22pp); arXiv: 0912.4968v1

[18] G. Christol, *Diagonales de fractions rationnelles et équations différentielles* , Groupe de travail d'analyse ultramétrique, tome 10, no 2 (1982-1983), exp. n0 18, p.1-10

[19] G. Christol, *Diagonales de fractions rationnelles et équations de Picard-Fuchs*, Groupe de travail d'analyse ultramétrique, tome 12, no 1 (1984-1985), exp. n0 13, p.1-12

[20] G. Christol, *Diagonals of rational fractions*, Newsletter of European Mathematical Society, (Sept. 2015) pp.37-43

[21] S. Hassani, J.M Maillard and N. Zenine *Plea for diagonals and telescopers of rational functions*, in *The Languages of Physics—A Themed Issue in Honor of Professor Richard Kerner on the Occasion of His 80th Birthday*, Universe 2024, 10(2), 71; https://doi.org/10.3390/universe10020071 - 02 Feb 2024 and arXiv:2310.06963 [math-ph]

[22] S. Boukraa, S. Hassani, J.-M. Maillard, *The Ising model and special geometries*, J. Phys. A: Math. Theor. **47** (2014) 225204 (31pp); arXiv:1402.6291

[23] P. Lairez, *Computing periods of rational integrals*, arXiv:1404.5069v2; Supplementary material http//pierre.lairez.fr/supp/periods/

[24] Y Abdelaziz, S Boukraa, C Koutschan and J-M Maillard, *Diagonals of rational functions, pullbacked $_2F_1$ hypergeometric functions and modular forms*, J. Phys. **A**: Math. Theor. 51 (2018) 455201 (30pp)

[25] S. Boukraa, S. Hassani, J.-M. Maillard, J.-A. Weil, *Differential algebra on lattice Green functions and Calabi-Yau operators*, J. Phys. A: Math. Theor. **47** (2014)095203 (37pp)

[26] V. Batyrev, M. Kreuzer, *Constructing new Calabi-Yau* $3-$*folds and their mirrors via conifold transitions*, Adv. Theor. Math. Phys. **14** (2010) 879-898

[27] D.R. Morrison, *Picard-Fuchs equations and mirror maps for hypersurfaces*, in: Essays on mirror manifolds, (Hong Kong, International Press, 1992) pp.241-264; arXiv:9111025

[28] A. Klemm, S. Theisen, *Considerations of one-modulus Calabi-Yau compactifications: Picard-Fuchs equations, Kahler potentials and mirror maps*, Nuclear Phys. **B 389** (1993) 153-180; arXiv:9205041

[29] A. Klemm, S. Theisen, *Mirrror maps and instanton sums for intersections in weighted projective space*, Mod. Phys. Lett. **A 9** (1994) 1807-1818; arXiv:9304034

[30] A. Libgober, J. Teitelbaum, *Lines on Calabi-Yau complete intersections, mirror symmetry and Picard-Fuchs equations*, Duke Math. J. Int. Math. Res. Notices **1** (1993) 29; arXiv:9301001

[31] V.V. Batyrev, D. van Straten, *Generalized hypergeometric functions and rational curves on Calabi-Yau complete intersections in toric varieties*, Comm. Math. Phys. **168** (1995) 493-533; arXiv:9307010

[32] B. H. Lian, S-T Yau; *Arithmetic properties of mirror map and quantum coupling*, Commun. Math. Phys. **176** (1996) 163–91; arXiv:hep-th/9411234v3

[33] G. Almkvist, W. Zudilin, 2006, Differential equations, mirror maps and zeta values, *Mirror Symmetry V (AMS/IP Studies in Advanced Mathematics vol. 38)* , (Providence RI: American Mathematics Society) pp. 481-515

[34] A Bostan, S Boukraa, S Hassani, M van Hoeij, J-M Maillard, J-A Weil and N Zenine, *The Ising model: from elliptic curves to modular forms and Calabi–Yau equations*, J. Phys. **A**: Math. Theor. **44** (2011) 045204 (44pp)

[35] N.M. Katz, (1972), *Algebraic solutions of differential equations (p-curvature and the Hodge filtration)*, Invent. Math. **18** pp. 1-118.

[36] J. Denef and L. Lipschitz, *Algebraic power series and diagonals* J. Number Theo. **26** (1987) 46-67

[37] Y. Abdelaziz, S. Boukraa, C. Koutschan, J-M. Maillard, *Heun functions and diagonals of rational functions*, J. Phys. A: Math. Theor. **53** (2020) 075206 (24pp); arXiv:1910.10761

[38] A. J. Guttmann, T. Prellberg, *Staircase polygons, elliptic integrals, Heun functions and lattice Green functions*, Phys. Rev. E **47**, (1993) R2233

[39] M.L. Glasser, E. Montaldi, *Staircase polygons and recurrent lattice walks*, Phys. Rev. E **48**, (1993) R2339-R2342

[40] A. Bostan, S. Boukraa, J-M. Maillard and J-A. Weil, *Diagonals of rational functions and selected differential Galois groups*, J. Phys. **A**: Math. Theor. **48** (2015) 504001 (29pp)

[41] Y. Abdelaziz, S. Boukraa, C. Koutschan, J.-M. Maillard *Diagonals of rational functions: from differential algebra to effective algebraic geometry*, Symmetry **14** (2022) 1297; arXiv:2002.00789

[42] A. Kumar, *Hilbert Modular Surfaces for square discriminants and elliptic subfields of genus 2 function fields*, Mathematical Sciences (2015) 2:24, Research in the mathematical Sciences, a Springer Open Journal; arXiv: 1412.2849

[43] C. van Enckevort, D. van Straten, 2006, Monodromy calculations of fourth order equations of Calabi-Yau type, *Mirror Symmetry V (AMS/IP Studies in Advanced Mathematics vol. 38)* , (Providence RI: American Mathematics Society) pp. 539-559