Beyond series expansions: mathematical structures for the susceptibility of the square lattice Ising model

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Dedicated to A J Guttmann on the occasion of his 60-th birthday

Abstract. We first study the properties of the Fuchsian ordinary differential equations for the three and four-particle contributions $\chi^{(3)}$ and $\chi^{(4)}$ of the square lattice Ising model susceptibility. An analysis of some mathematical properties of these Fuchsian differential equations is sketched. For instance, we study the factorization properties of the corresponding linear differential operators, and consider the singularities of the three and four-particle contributions $\chi^{(3)}$ and $\chi^{(4)}$, versus the singularities of the associated Fuchsian ordinary differential equations, which actually exhibit new "Landau-like" singularities. We sketch the analysis of the corresponding differential Galois groups. In particular we provide a simple, but efficient, method to calculate the so-called "connection matrices" (between two neighboring singularities) and deduce the singular behaviors of $\chi^{(3)}$ and $\chi^{(4)}$. We provide a set of comments and speculations on the Fuchsian ordinary differential equations associated with the n-particle contributions $\chi^{(n)}$ and address the problem of the apparent discrepancy between such a holonomic approach and some scaling results deduced from a Painlevé oriented approach.

1. Introduction

It is a great pleasure to write this paper, in honour of the sixtieth birthday of Prof. A.J. Guttmann, on the mathematical structures deduced from series expansion of the susceptibility of the Ising model. One of us (JMM) has profited from many years of fruitful collaboration and correspondence with him and his group, and many times from his friendly hospitality.

1.1. Notation and a few results on the Ising susceptibility

Since the (monumental) work of T.T. Wu, B.M. McCoy, C.A. Tracy and E. Barouch [1], it has been known that the expansion in n-particle contributions to the zero field susceptibility of the square lattice Ising model at temperature T can be written as an infinite sum:

$$\chi(T) \,=\, \sum_{n=1}^{\infty} \chi^{(n)}(T) \tag{1.1}$$

of $(n-1)$-dimensional integrals [2, 3, 4, 5, 6, 7]. This sum is restricted to odd (respectively even) n for the high (respectively low) temperature case. While the first contribution in the sum, $\chi^{(1)}$, is obtained directly without integration, and the second one, $\chi^{(2)}$, is given in terms of elliptic integrals, *no closed forms* for the higher order contributions were known, despite the well-defined forms of the integrands in these $(n-1)$-dimensional integrals. It is only recently that the differential equations for $\chi^{(3)}$ and $\chi^{(4)}$ have been found [8, 9, 10, 11, 12]. In the following we will use, as in [8, 9, 10, 11], the following normalization:

$$\chi^{(n)} \, = \, \frac{(1-s^4)^{1/4}}{s} \cdot \tilde{\chi}^{(n)} \tag{1.2}$$

in order to focus on the $\tilde{\chi}^{(n)}$'s which are only functions of the variable $w = s/(1+s^2)/2$.

As far as singular points are concerned (physical or non-physical singularities in the complex plane), and besides the known $s=\pm 1$ and $s=\pm i$ singularities, B. Nickel showed [6] that $\chi^{(2n+1)}$ is singular[1] for the following finite values of $s = sh(2J/kT)$ lying on the $|s| = 1$ *unit circle* ($m = k = 0$ excluded):

$$\begin{aligned} 2 \cdot \Big(s + \frac{1}{s}\Big) \, &= \, u^k + \frac{1}{u^k} + u^m + \frac{1}{u^m} \\ u^{2n+1} \, &= \, 1, \qquad\qquad -n \, \le \, m, \; k \, \le \, n \end{aligned} \tag{1.3}$$

When n increases, the singularities of the higher-particle components of $\chi(s)$ *accumulate on the unit circle* $|s| = 1$. The existence of such a *natural boundary* for the total susceptibility $\chi(s)$, shows that $\chi(s)$ is *not D-finite* (not holonomic) as a function[2] of s.

A significant amount of work had already been performed to generate isotropic series coefficients for $\chi^{(n)}$ (by B. Nickel [6, 7] up to order 116, then to order 257 by A.J. Guttmann[3] *et al.*). More recently, W. Orrick *et al.* [15, 16], have generated coefficients[4] of the susceptibility of the Ising model [15, 16] $\chi(s)$ up to order 323 (resp. 646) for high (resp. low) temperature series in s, using a quadratic double recursion (*non-linear Painlevé difference equations*) for the correlation functions [15, 16, 17, 18, 19]. As a consequence of this remarkable non-linear Painlevé difference equation and its associated double recursion, the computer algorithm had a $O(N^6)$ *polynomial growth* for the calculation of the series expansion instead of the exponential growth that one might expect at first sight.

However, in such a *non-linear*, non-holonomic, *Painlevé-oriented* approach, one obtains results directly for the *total* susceptibility $\chi(s)$ which *do not* satisfy any finite order linear differential equation. It thus prevents the easy disentanglement of the contributions of the various holonomic $\chi^{(n)}$'s.

On the other hand, the individual $\chi^{(n)}$'s *do satisfy* finite order linear differential equations, leaving some hope to understand the susceptibility from a deeper knowledge of the mathematical structures of the successive differential equations of the n-particle sequence of the $\chi^{(n)}$'s.

2. The Fuchsian ODE's for $\chi^{(3)}$ and $\chi^{(4)}$

2.1. The Fuchsian ODE for $\chi^{(3)}$

With an original method[5] [9] that allows us to write $\tilde{\chi}^{(3)}$ as fully integrated sums, we generated a long series of 490 terms. Using a dedicated program we searched for the finite order linear differential

[1] The singularities being logarithmic branch points of order $\epsilon^{2n(n+1)-1} \cdot ln(\epsilon)$ with $\epsilon = 1 - s/s_i$ where s_i is one of the solutions of (1.3).

[2] See in particular, the paper of I. Enting and A.J. Guttmann [13]. One has probably the same situation for the *three-dimensional* Ising model [14].

[3] Private communication.

[4] The short-distance terms were shown to have the form $(T-T_c)^p \cdot (log|T-T_c|)^q$ with $p \ge q^2$.

[5] It is worth noting that in the details of this method we encountered many times the occurrence of generalizations of hypergeometric functions to *several variables*, extremely similar to the ones occurring in Feynman diagrams [20] (Appell, Lauricella, Kampé de Feriet, ...). We will discuss these quite interesting questions elsewhere.

equation with polynomial coefficients in the variable $w = s/2/(1+s^2)$, by steadily increasing the order. We finally succeeded [8] in finding the following linear differential equation of order *seven* for $\tilde{\chi}^{(3)}$ with only 359 terms of our long series:

$$\sum_{n=0}^{7} a_n \cdot \frac{d^n}{dw^n} F(w) \;=\; 0 \tag{2.1}$$

with:

$$\begin{aligned}
&a_7 \;= \\
&w^7 \cdot (1-w)\,(1+2\,w)\,(1-4\,w)^5\,(1+4\,w)^3\,(1+3\,w+4\,w^2)\cdot P_7(w), \\
&a_6 \;=\; w^6 \cdot (1-4\,w)^4\,(1+4\,w)^2 \cdot P_6(w), \\
&a_5 \;=\; w^5 \cdot (1-4\,w)^3\,(1+4\,w)\cdot P_5(w), \\
&a_4 \;=\; w^4 \cdot (1-4\,w)^2 \cdot P_4(w), \qquad\qquad a_3 \;=\; w^3 \cdot (1-4\,w)\cdot P_3(w) \\
&a_2 \;=\; w^2 \cdot P_2(w), \qquad\quad a_1 \;=\; w \cdot P_1(w), \qquad\quad a_0 \;=\; P_0(w)
\end{aligned} \tag{2.2}$$

where [8]:

$$\begin{aligned}
P_7(w) \;&=\; 1568 + 15638\,w - 565286\,w^2 - 276893\,w^3 \;+\; \cdots \\
&\quad +158329674399744\,w^{27} + 39582418599936\,w^{28}
\end{aligned} \tag{2.3}$$

The polynomials $P_7(w), P_6(w) \cdots, P_0(w)$ are polynomials of degree respectively 28, 34, 36, 38, 39, 40, 40 and 36 in w, and are given in [8]. The singular points of this differential equation correspond to the roots of the polynomial corresponding to the highest order derivative $F^{(7)}(w)$.

The linear differential equation (2.1) is an equation of the *Fuchsian type since there are no singular points, finite or infinite, other than regular singular points*. With this property, using the Frobenius method [21, 22], it is straightforward to obtain the critical exponents, from the indicial equations, in w, for each regular singular point. These are given[6] in Table 1.

We have actually found two remarkable *rational and algebraic* solutions of (2.1), namely:

$$S_1 \;=\; \frac{w}{1-4\,w}, \qquad\qquad S_2 \;=\; \frac{w^2}{(1-4\,w)\,\sqrt{1-16\,w^2}} \tag{2.4}$$

We also found a solution behaving like w^3, that we denote S_3:

$$\begin{aligned}
S_3 \;&=\; w^3 + 3\,w^4 + 22\,w^5 + 74\,w^6 + 417\,w^7 + 1465\,w^8 \\
&\quad +7479\,w^9 + 26839\,w^{10} \;+\; \cdots
\end{aligned} \tag{2.5}$$

and S_9 associated with the physical solution $\tilde{\chi}^{(3)}$:

$$\begin{aligned}
S_9 \;&=\; w^9 + 36\,w^{11} + 4\,w^{12} + 884\,w^{13} + 196\,w^{14} \\
&\quad +18532\,w^{15} + 6084\,w^{16} \;+\; \cdots
\end{aligned}$$

and three solutions $S_1^{(2)}$, $S_2^{(2)}$ and $S_1^{(3)}$ with *logarithmic terms*, behaving, for small w, as follows:

$$\begin{aligned}
S_1^{(2)} \;&=\; \ln(w)\cdot(S_1 - 4\,S_2 + 16\,S_3 - 216\,S_9) - 32\,w^4 \cdot c_2, \\
S_2^{(2)} \;&=\; \ln(w)\cdot(S_2 - 2\,S_3 + 24\,S_9) \;+\; 8\,w^4 \cdot d_2, \\
S_1^{(3)} \;&=\; 3\,\ln(w)^2 \cdot (S_1 + 5\,S_2 - 2\,S_3) \\
&\quad -6\,\ln(w)\cdot(2\,S_3 - S_1^{(2)} - 9\,S_2^{(2)}) - 19\,w^4 \cdot e_3
\end{aligned}$$

[6] In the variable s the local exponents for $w = \pm 1/4$ are twice those given.

$w = 0$	$s = 0$	$\rho = 9, 3, 2, 2, 1, 1, 1$
$w = -1/4$	$s = -1$	$\rho = 3, 2, 1, 0, 0, 0, -1/2$
$w = 1/4$	$s = 1$	$\rho = 1, 0, 0, 0, -1, -1, -3/2$
$w = -1/2$	$1 + s + s^2 = 0$	$\rho = 5, 4, 3, 3, 2, 1, 0$
$w = 1$	$2 - s + 2s^2 = 0$	$\rho = 5, 4, 3, 3, 2, 1, 0$
$1 + 3\,w + 4\,w^2 = 0$	$(2s^2 + s + 1)(s^2 + s + 2) = 0$	$\rho = 5, 4, 3, 2, 1, 1, 0$
$1/w = 0$	$1 + s^2 = 0$	$\rho = 3, 2, 1, 1, 1, 0, 0$
$w = w_P$, 28 roots	$s = s_P$, 56 roots	$\rho = 7, 5, 4, 3, 2, 1, 0$

Table 1. Critical exponents ρ for each regular singular point. w_P is any of the 28 roots of $P_7(w)$. We have also shown the corresponding roots in the s variable.

where c_2, d_2, e_3 denote functions analytical at $w = 0$: $c_2 \;=\; 1 + 167/96\,w + 2273/96\,w^2 + \cdots$, $d_2 \;=\; 1 + 5/2\,w + 103/4\,w^2 + 315/4\,w^3 + \cdots$, and $e_3 \;=\; 1 + 7693/456\,w + 575593/11400\,w^2 + 2561473/5700\,w^3 + \cdots$

Remark: Besides the known regular "Nickel-type" singularities (1.3) mentioned above, we remark the occurrence of the roots of the polynomial P_7 of degree 28 in w, which just correspond to *apparent singularities* [8] and the occurrence of the two *quadratic numbers* $1 + 3\,w + 4\,w^2 \;=\; 0$ which *are not of the form* (1.3). The two quadratic numbers *are not* on the s-unit circle: $|s| = \sqrt{2}$ and $|s| = 1/\sqrt{2}$.

These new quadratic singularities that are not of the "Nickel type" (1.3), are *singularities of the Fuchsian differential equation associated with* $\chi^{(3)}$, but *are not singularities of the ("physical") solution* $\chi^{(3)}$ itself [11] ! More generally, for arbitrary $\chi^{(n)}$'s, the "non-Nickel" singularities [23], are singularities of the Fuchsian ODE's, but *does not seem to be singularities [12] of the* $\chi^{(n)}$'s. This is a very strong motivation to get more informations on the next $\chi^{(n)}$'s ($\chi^{(5)}$, $\chi^{(6)}$, ...), in particular their singularities and the singularities of the associated [12] Fuchsian ODE's.

The best series analysis of the $\chi^{(n)}$'s, which does not take into account the existence of the corresponding ODE, *will not be able to find such singularities*[7]. These singularities are *beyond any series analysis of the* $\chi^{(n)}$'s.

It is probably not necessary to underline the fundamental role of the understanding of singularities in lattice statistical mechanics, as far as physics is concerned. Along this line let us mention the recent comment by B. Nickel [23] on our two first papers [8, 9] where he gives strong arguments that these new sets of singularities should actually (necessarily) correspond to *pinch singularities* [23] totally similar to the well-known *Landau singularities* encountered in *Feynman diagrams* [23, 24, 25, 26, 27]. Such "Landau singularities" can be seen as some frontier between different perturbation regimes and their implications on the physics are too numerous to be listed here. As far as terminology is concerned we should not call these pinched "non-Nickel" singularities "Landau singularities" but rather "Landau-like" singularities: they are singularities of a mathematical structure associated with the $\chi^{(n)}$'s (namely the

[7] A simple Padé analysis of $\chi^{(n)}$'s series expansions will not see such "non-Nickel" singularities. A very careful diff-Padé analysis, taking into account the Fuchsian character of these ODE's, should be able to see such "non-Nickel" singularities, when a straight simple diff-Padé analysis will not.

x-singularity	s-singularity	Critical exponents in x	P
0	$0, \infty$	$8, 3, 3, 2, 2, 1, 1, 0, 0, -1/2$	1
1	± 1	$3, 2, 1, 1, 0, 0, 0, 0, -1, -3/2$	3
4	$\pm\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$	$8, 7, 13/2, 6, 5, 4, 3, 2, 1, 0$	0
∞	$\pm i$	$5/2, 3/2, 3/2, 1/2, 1/2, 1/2, 1/2, 0, -1/2, -1/2$	3
x_P, 17 roots	s_P, 68 roots	$10, 8, 7, 6, 5, 4, 3, 2, 1, 0$	0

Table 2. Critical exponents for each regular singular point. P is the maximum power of the logarithmic terms for each singularity. x_P is any of the 17 roots of $P_{10}(x)$.

associated Fuchsian ODE) and *not* automatically singularities of the $\chi^{(n)}$'s themselves.

2.2. The Fuchsian ODE for $\chi^{(4)}$

It is clear that the four particle contribution $\tilde{\chi}^{(4)}$ is even[8] in w. We thus introduce, in the following, the variable $x = 16\, w^2$. The same expansion method [9] is applied with some symmetries and tricks specific to $\tilde{\chi}^{(4)}$ to generate 216 terms in the variable x. We succeeded in obtaining the differential equation for $\tilde{\chi}^{(4)}$ that is given in $x = 16\, w^2$ by

$$\sum_{n=1}^{10} a_n(x) \cdot \frac{d^n}{dx^n} F(x) \;=\; 0 \qquad (2.6)$$

with

$$\begin{aligned}
&a_{10} = -512\, x^6\, (x-4)\, (1-x)^6\, P_{10}(x), \\
&a_9 = 256\, (1-x)^5\, x^5 P_9(x), \qquad a_8 = -384\, (1-x)^4\, x^4 P_8(x), \\
&a_7 = 192\, (1-x)^3\, x^3 P_7(x), \qquad a_6 = -96\, (1-x)^2\, x^2 P_6(x), \\
&a_5 = 144\, (1-x)\, x\, P_5(x), \qquad a_4 = -72\, P_4(x), \qquad a_3 = -108\, P_3(x), \\
&a_2 = -54\, P_2(x), \qquad a_1 = -27\, P_1(x)
\end{aligned} \qquad (2.7)$$

where $P_{10}(x), P_9(x) \cdots, P_1(x)$ are polynomials of degree respectively 17, 19, 20, 21, 22, 23, 24, 23, 22 and 21, and are given in [10]. The critical exponents of this ODE are given in Table 2.

2.3. Seeking for the simplest Fuchsian ODE: paradox and subtleties

As far as finding the ODE's satisfied by the $\chi^{(n)}$'s is concerned one can, of course, find several ODE's for a given $\chi^{(n)}$. Actually, denoting $\mathbf{L}_q^{(n)}$ the differential operator of *smallest order* q (order seven for $\chi^{(3)}$, ten for $\chi^{(4)}$, ...) associated with a given $\chi^{(n)}$ ($\mathbf{L}_q^{(n)}(\tilde{\chi}^{(n)}) = 0$), the other ODE's can be seen to be associated with differential operators of order p of the form $\mathbf{L}_p^{(n)} = \mathbf{L}_p^q \cdot \mathbf{L}_q^{(n)}$, $p > q$. At this step one should try to fight the prejudice that $\mathbf{L}_p^{(n)}$, being the product of $\mathbf{L}_p^q$ and (the smallest order)

[8] This is also the case for any $\tilde{\chi}^{(2n)}$ (see for instance [7]).

operator $\mathbf{L}_q^{(n)}$, is necessarily more complicated than these two differential operators: in some cases it may actually be simpler ! Among these various ODE's and associated differential operators, some are singled-out: the differential operator $\mathbf{L}_m^{(n)}$ such that one *does not have any apparent singularity anymore* is clearly singled-out. Note that the requirement to have no apparent singularities, or just a polynomial corresponding to apparent singularities of small degree (quadratic ...), corresponds to ODE's such that *the number of coefficients* in a series necessary to guess the ODE, *is actually much smaller* than the number required to guess the differential operator $\mathbf{L}_q^{(n)}$ corresponding to the ODE of smallest order. However, the requirement to have no apparent singularity is *not the optimal condition to find (guess ...) the ODE with the minimal number of coefficients*. The optimal order p for guessing the (Fuchsian) ODE verified by a given $\chi^{(n)}$ and its associated differential operator $\mathbf{L}_p^{(n)}$, is, in fact, an integer such that $q \le p \le m$.

For instance, for $\chi^{(4)}$ one can introduce the following differential operators $\mathcal{L}_m = \mathbf{L}_m^{(4)}$:

$$\mathcal{L}_{14} = \mathbf{L}_{14}^{10} \cdot \mathcal{L}_{10}, \quad \mathcal{L}_{13} = \mathbf{L}_{13}^{10} \cdot \mathcal{L}_{10} \\ \mathcal{L}_{12} = \mathbf{L}_{12}^{10} \cdot \mathcal{L}_{10}, \quad \mathcal{L}_{11} = \mathbf{L}_{11}^{10} \cdot \mathcal{L}_{10} \tag{2.8}$$

where $\mathcal{L}_{10}$ is the differential operator of smallest order which requires 242 coefficients to be guessed; $\mathcal{L}_{14}$ is a differential operator *with one apparent singularity* that requires 137 coefficients to be guessed, $\mathcal{L}_{13}$ is the differential operator which requires the *smallest number of coefficients to be guessed* namely 134 coefficients (and it has a quadratic apparent polynomial). $\mathcal{L}_{12}$ requires 143 coefficients and has a quartic apparent polynomial. $\mathcal{L}_{11}$ requires 161 coefficients and it has an apparent polynomial of degree seven. Note that $\mathbf{L}_{12}^{10}$ *is not divisible* by $\mathbf{L}_{11}^{10}$ (and similarly $\mathbf{L}_{13}^{10}$ by $\mathbf{L}_{11}^{10}$, etc ...).

Therefore, we see that there are, at least, three singled-out ODE's: the one of smallest order q, the one with no apparent singularity of order m, and the "*optimal one to be guessed*" which is of order p with $q \le p \le m$.

2.4. Singled-out solutions of the Fuchsian ODE for $\chi^{(4)}$ and associated factorizations of differential operators

The order ten Fuchsian differential equation (2.6) has three remarkably simple *algebraic solutions*, namely the constant solution, $\mathcal{S}_0(x) = constant$, and:

$$\mathcal{S}_1(x) = \frac{8 - 12x + 3x^2}{8(1-x)^{3/2}}, \qquad \mathcal{S}_2(x) = \frac{2 - 6x + x^2}{2(1-x)\sqrt{x}}.$$

These solutions correspond to solutions of some differential operators of order one. Denote these three order one differential operators respectively L_0, L_1 and L_2.

A truly remarkable finding[9] *is the following solution of the Fuchsian differential equation* (2.6):

$$\mathcal{S}_3(x) = \frac{1}{64}\, x^2 \cdot {}_2F_1\Big(\frac{5}{2}, \frac{3}{2}; 3; x\Big) \tag{2.9}$$

which is nothing but the two-particle contribution to the magnetic susceptibility, i.e., $\tilde{\chi}^{(2)}$ associated with an operator of order two, N_0 such that $N_0(\mathcal{S}_3) = 0$.

We do not see this result as a coincidence, but rather as *the emergence of some "Russian-doll" structures*[10] for the infinite set of the $\chi^{(n)}$'s, and therefore as a first example of a structure bearing on the *whole* susceptibility χ.

[9] Do note that we had a similar, though less spectacular, property for the differential operator of $\tilde{\chi}^{(3)}$: solution S_1 in (2.4) is nothing but $\tilde{\chi}^{(1)}$ and is thus also a solution of that operator.

[10] For instance $\chi^{(3)}$ could be a solution of the differential operator associated with $\chi^{(15)}$, but *also* of the differential operator associated with $\chi^{(9)}$.

The second solution of the order-two operator N_0 is simply given[11] by:

$$\tilde{\mathcal{S}}_3(x) \;=\; \frac{\pi}{16}\, x^2 \cdot {}_2F_1\Big(\frac{3}{2}, \frac{5}{2}; 2; 1-x\Big) \qquad (2.10)$$

which can also be written around $x = 0$ as ($\mathcal{S}_3(x)$ analytic):

$$\tilde{\mathcal{S}}_3(x) = \; \mathcal{S}_3(x) \cdot \ln(x) + \frac{1}{12\,\pi} \cdot \sum_{k=0}^{\infty} x^k \frac{d}{dk} \Big(\frac{\Gamma(k-1/2)\,\Gamma(k+1/2)}{\Gamma(k-1)\,\Gamma(k+1)} \Big)$$

It is obvious on the simple form (2.10) that this second solution *is not singular* at the ferromagnetic critical point $x = 1$, but *is actually singular* at $x = 0$.

We found another solution that reads

$$\begin{aligned} \mathcal{S}_4(x) \;&=\; \frac{4\,(x-2)\sqrt{4-x}}{x-1} + 16 \ln \frac{x}{(2+\sqrt{4-x})^2} \\ &\quad + 16\, \mathcal{S}_1(x) \cdot \ln g(x) \;-\; 16\,\sqrt{x}\, \mathcal{S}_2(x) \;\cdot {}_2F_1\Big(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; \frac{x}{4}\Big) \end{aligned} \qquad (2.11)$$

with:

$$g(x) = \; \frac{1}{x}\Big((8-9x+2x^2) \; + 2\,(2-x) \cdot \sqrt{(1-x)(4-x)} \Big) \qquad (2.12)$$

The differential equation corresponding to $\mathcal{S}_4(x)$ has also $\mathcal{S}_0$, $\mathcal{S}_1(x)$ and $\mathcal{S}_2(x)$ as solutions.

Denoting by $\mathcal{L}_{10}$ the differential operator corresponding to (2.6), one may search for simple solution of its adjoint $\mathcal{L}_{10}^*$. We found a solution corresponding to an operator of order two (denoted N_8^*) which is a combination of elliptic integrals with polynomials [10] of degree 21

$$\mathcal{S}_1^*(x) = \; \frac{x\,(1-x)^6(4-x)}{3840000 \cdot P_{10}(x)} \cdot \Big(q_1(x)\, K(x) + q_2(x)\, E(x) \Big)$$

where:

$$K(x) = {}_2F_1\Big(\frac{1}{2}, \frac{1}{2}; 1; x\Big), \qquad E(x) = {}_2F_1\Big(-\frac{1}{2}, \frac{1}{2}; 1; x\Big).$$

All these solutions of (2.6) can be used to build the complete factorization of the differential operator $\mathcal{L}_{10}$ that we write as

$$\mathcal{L}_{10} \;=\; N_8 \cdot M_2 \cdot G(L) \;=\; M_1 \cdot N_9 \cdot G(L) \;=\; M_1 \cdot L_{24} \cdot G(N)$$

where $G(L)$ is a shorthand notation of a differential operator of order four, factorizable in four order one operators[12] and where $G(N)$ is an operator of order five, factorizable in one operator of order two and three operators of order one. $G(N)$ has 24 different factorizations involving eight differential operators of order two, and 24 order one operators with twelve appearing in $G(L)$. All these 36 factorizations (2.13) are given in [10].

From the 24 factorizations of the order five differential operator $G(N)$, six are divisible by the differential operator N_0. One such factorization reads $G(N) = L_{13} \cdot L_{17} \cdot L_{11} \cdot N_0$ implying the *direct sum* decomposition:

$$\mathcal{L}_{10} \;=\; \mathcal{L}_8 \oplus N_0, \qquad \mathcal{L}_8 \;=\; M_2 \cdot G(L) \qquad (2.13)$$

[11] Or in terms of the MeijerG function [28, 29] as $\pi/12 \cdot$ MeijerG $([[], [1/2, 3/2]], [[2, 0], []], x)$ (see [10]). We thank P. Abbott for pointing out a missprint in the corresponding formula given in a previous version of [10].

[12] The order four operator $G(L)$ has six different factorizations involving thirteen differential operators of order one.

2.5. More factorizations for the differential operator of $\chi^{(3)}$

Let us come back to $\chi^{(3)}$. One can change[13] S_3 into $S_3^{(new)} \, = \, S_3 \, - \, 16\, S_9$. This amounts to changing the w^9 coefficient in S_3 from 7479 to 7463 (and the following coefficients consequently). Actually B. Nickel found[14] that the new solution $S_3^{(new)}$ is actually *a solution of a fourth order differential equation*.

Focusing on this differential operator one can[15] find the general solution of this order four differential operator, which we call N_4, which reads:

$$Sol \, = \, \alpha_1 \cdot S_1 \, + \, \alpha_2 \cdot S_2 \, + \, S_2 \cdot \int \frac{Sol_2}{S_2} \cdot dw \qquad (2.14)$$

where Sol_2 denotes the general solution of an order two differential operator [11] Z_2:

$$\begin{aligned}
Z_2 \, = \, & \frac{d}{dw^2} \, - \, \frac{A}{P} \cdot \frac{d}{dw} \, + \, \frac{B}{P}, \\
P = \, & w \cdot (1-4\,w)^2\,(1+4\,w)\,\Big(1+3\,w+4\,w^2\Big)\,(1-w)\,(1+2\,w) \cdot P_1 \\
P_1 = \, & 96\,w^4+104\,w^3-18\,w^2-3\,w+1 \\
A = \, & (1-4\,w)\,\Big(49152\,w^{10}+72704\,w^9-48384\,w^8-94272\,w^7 \\
& -40368\,w^6-4488\,w^5+1080\,w^4-108\,w^3-111\,w^2-6\,w+1\Big) \\
B = \, & 98304\,w^{10}+98304\,w^9-239616\,w^8-307200\,w^7-71552\,w^6 \\
& -4224\,w^5-3192\,w^4-1520\,w^3-276\,w^2+48\,w+4.
\end{aligned}$$

Let us make a first remark: the degree four polynomial P_1 corresponds to *apparent singularities* of the order two differential operator Z_2. This differential operator appears when solving the order four differential operator N_4 which has seven apparent singularities roots of a polynomial of degree seven. One thus sees that the degree 28 polynomial (denoted P_7 in [8], see equation (9) in [8]), corresponding to 28 apparent singularities, has been replaced, as far as N_4 is concerned, by a degree seven polynomial or, as far as Z_2 is concerned, by the degree four polynomial P_1. The apparent singularities of these various differential operators are clearly highly volatile.

The linear combination (2.14) of four simple solutions is in complete agreement with the (Dfactor/DEtools) factorization of this order four differential operator N_4:

$$\begin{aligned}
N_4 \, = \, & \, N_3 \oplus L_1, \qquad L_1(S_1) \, = \, 0 \\
N_3 \, = \, & \, Z_2 \cdot N_1, \qquad N_1(S_2) \, = \, 0.
\end{aligned} \qquad (2.15)$$

From the existence of this order four differential operator one immediately deduces an order three differential operator L_3, such that the order seven differential operator L_7 factorizes as follows:

$$L_7 \, = \, L_6 \oplus \, L_1, \quad \text{with:} \quad L_6 \, = \, L_3 \cdot N_3 \, = \, L_3 \cdot Z_2 \cdot N_1 \qquad (2.16)$$

Let us note that the four solutions (2.14) are *totally decoupled* from the linear combination $S_9 \, + \mu \cdot S_1$ such that $L_6(S_9 \, + \mu \cdot S_1) \, = \, 0$.

Let us also mention that at the singular points $w \, = \, 1$, $w \, = \, -1/2$, and at the two quadratic roots w_1, w_2 of $1 + 3w + 4w^2 \, = \, 0$, the solution carrying a logarithmic term is in fact a solution of $Z_2 \cdot N_1$. Therefore, the three solutions of the differential operator $L_3 \cdot Z_2 \cdot N_1$, emerging from L_3, are analytical

[13] In some of our calculations the linear combination $S_3 \, - \, 12\, S_9$ which amounts to changing the w^9 coefficient in S_3 from 7479 to 7467 has also been used.

[14] Private communication.

[15] Using for instance the dsolve and DFactor command of DEtools [30]. See also [31, 32, 33].

at the non physical singular points $w = 1$, $w = -1/2$, and at the quadratic roots of $1 + 3w + 4w^2 = 0$. The solutions of the differential operator L_3, itself, are of "elliptic integral type" [11]. At the singular point $w = 1/4$, one also notes that the differential operator $Z_2 \cdot N_1$ is responsible of the $(1 - 4w)^{-1}$ behavior. One will then expect the "ferromagnetic constant" I_3^+ to be localized in the blocks of the connection matrix corresponding to the solutions of the order three differential operator $Z_2 \cdot N_1$ at the point $w = 1/4$.

Note that finding such complete factorization for the differential operator L_7 (or $\mathcal{L}_{10}$ for $\chi^{(4)}$) is of great help to get the *differential Galois group* (which identifies here with the *monodromy group*), and, more precisely, when computing the *connection matrices* necessary to write the monodromy matrices written in the *same* (Frobenius series solutions) basis.

3. Differential Galois group, singular behaviour and asymptotics

3.1. Differential Galois group

From this factorization (2.15) of L_7 one can deduce, immediately, the form[16] in some "well-suited global basis" of solutions, of the 7×7 matrices representing the differential Galois[17] group [11, 35, 36, 37] of L_7, namely:

$$\begin{bmatrix} a_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & h_1 & g_1 & g_2 & 0 & 0 & 0 \\ 0 & h_2 & g_3 & g_4 & 0 & 0 & 0 \\ 0 & H_1 & H_2 & H_3 & G_1 & G_2 & G_3 \\ 0 & H_4 & H_5 & H_6 & G_4 & G_5 & G_6 \\ 0 & H_7 & H_8 & H_9 & G_7 & G_8 & G_9 \end{bmatrix} \qquad (3.1)$$

where the 2×2 and 3×3 block matrices $\mathcal{G}al(Z_2)$ and $\mathcal{G}al(L_3)$

$$\mathcal{G}al(Z_2) \, = \, \begin{bmatrix} g_1 & g_2 \\ g_3 & g_4 \end{bmatrix}, \qquad \mathcal{G}al(L_3) \, = \, \begin{bmatrix} G_1 & G_2 & G_3 \\ G_4 & G_5 & G_6 \\ G_7 & G_8 & G_9 \end{bmatrix}$$

correspond respectively to the differential Galois groups of the differential operators Z_2 and L_3. The 3×3 matrix $\mathcal{G}al(N_3)$, corresponding to the differential Galois group of $N_3 \, = \, Z_2 \, N_1$, and the 3×3 block-matrix corresponding to the fact that we have a *semi-direct product* of the differential Galois group of L_3 and N_3 in $L_6 \, = \, L_3 \cdot N_3$, read:

$$\mathcal{G}al(N_3) \, = \, \begin{bmatrix} b_1 & 0 & 0 \\ h_1 & g_1 & g_2 \\ h_2 & g_3 & g_4 \end{bmatrix}, \qquad \mathcal{H}(N_3; L_3) \, = \, \begin{bmatrix} H_1 & H_2 & H_3 \\ H_4 & H_5 & H_6 \\ H_7 & H_8 & H_9 \end{bmatrix}$$

The semi-direct product structure is also clear on the 3×3 matrix $\mathcal{G}al(N_3)$. The 3×3 matrix block $\mathcal{H}(N_3; L_3)$ (resp. the entries h_1 and h_2 in (3.1)) can, in principle be calculated exactly from[18] the

[16] One has to be careful with the definition one takes for these matrices, which will be associated with the choice of a vector set of solutions, as a line vector or as a column vector. Different choices change these matrices into their transpose.

[17] Since our ODE is a Fuchsian ODE the differential Galois group identifies with the monodromy group generated by the monodromy matrices written in the same basis: there are no Stokes matrices [34] associated with irregular singular points.

[18] In mathematical words the solutions of L_3 are in the Picard-Vessiot extension of L_6.

differential Galois group of N_3, and the differential Galois group of L_3 (resp. the differential Galois group of L_2 and the differential Galois group of N_1). The calculation of these off-diagonal blocks, associated with the fact that we have a semi-direct product instead of a product (a product of differential operators instead of a direct sum of differential operators), relies on the fact that *one knowns how to actually calculate the morphisms between differential operators* (see for instance P. Berman and M. Singer [39] or P. Berman [40]). It is worth noting that such calculations are *totally effective*. We will, not however, give more details of these later calculations [39, 40, 38]: they are only necessary if one wishes to perform exhaustively the complete analysis of the differential Galois group of L_7: we just perform, here, a "warm-up", only sketching the analysis of the differential Galois group of L_7, which reduces (up to a product by the field of complex numbers $\mathcal{C}$, because L_7 is the direct sum of L_6 and L_1) to the differential Galois group of L_6, which is, thus, just seen as a semi-direct product of the differential Galois group of L_3, of the differential Galois group of L_2 and of the differential Galois group of N_1 (namely the field of complex numbers $\mathcal{C}$). At first sight the differential Galois group of L_3 can be either the group $SL(3, \mathcal{C})$, or the group[19] $PSL(2, \mathcal{C})$, or the semi-direct product of $\mathcal{C}$ and of the permutation group Σ_3 of three elements. Some calculations that will not be detailed here[20], show that the differential Galois group of L_3 is actually $SL(3, \mathcal{C})$ because the Wronskian of L_3 is rational and that the differential Galois group of L_2 is $SL(2, \mathcal{C})$ also because its Wronskian is rational. The differential Galois group of L_7 thus reduces (up to a product by $\mathcal{C}$) to semi-direct products of $SL(3, \mathcal{C})$, $SL(2, \mathcal{C})$ and $\mathcal{C}$. This is a quite general description of the differential Galois group, that may satisfy mathematicians, but, as far as physics is concerned, we need to actually calculate specific elements of the differential Galois group, in particular the "non-local" *connection matrices*, in order to have some understanding of the analytical properties of the solutions [11] (dominant and sub-dominant singular behaviours, asymptotics, ..., see section (3.3) below). As far as explicit calculations are concerned, we have, however, learned, from this quite general analysis, that a well-suited "global" basis for writing explicitely any kind of " non-local" connection matrices, should be : $S_1, \; S_2, \; S_1^{(2;new)}, \; S_3^{(new)}$ (corresponding to the solutions of (2.14), on the N_4 side) together with the further solutions $S_9, \; S_2^{(2)}, \; S_1^{(3)}$ coming from the right division of L_7 by N_4.

In particular it is shown in [11], and this will be sketched in the next section that the *connection matrix* between the singularity points 0 and $1/4$ (matching the well-suited local series-basis near $w = 0$ and the well-suited local series-basis near $w = 1/4$) is a matrix of the form (3.1) where the entries are expressions in terms of $\ln(2)$, $\ln(6)$, $\sqrt{3}$, π^2, $1/\pi^2$, ... and a constant I_3^+ introduced in equation (7.12) of [1]:

$$\frac{1}{2\,\pi^2} \cdot \int_1^\infty \int_1^\infty \int_1^\infty dy_1\, dy_2\, dy_3 \Big(\frac{y_2^2 \, - 1}{(y_1^2 \, - 1)\,(y_3^2 \, - 1)} \Big)^{1/2} \cdot Y^2 \; =$$
$$= \; .000814462565662504439391217128562721997861158118508 \cdots$$
$$Y \; = \; \frac{y_1 \, - y_3}{(y_1 \, + y_2)\,(y_2 \, + y_3)\,(y_1 \, + y_2 \, + y_3)}.$$

This constant can actually be written in term of the Clausen function Cl_2 :

$$I_3^+ \; = \; \frac{1}{2\pi^2} \cdot \Big(\frac{\pi^2}{3} \, + 2 \, - 3\sqrt{3} \cdot Cl_2(\frac{\pi}{3}) \Big) \qquad (3.2)$$

where Cl_2 denotes the Clausen function :

$$Cl_2(\theta) \; = \; \sum_{n=1}^{\infty} \frac{\sin(n\,\theta)}{n^2}$$

[19] Such a $PSL(2, \mathcal{C})$ situation would correspond to building differential equations such that one of their solutions is actually the product of two solutions of a second order differentiable equation. One can easily build such examples using the $diffeq * diffeq$ command in Gfun [41].

[20] A publication with Jacques-Arthur Weil on these questions will follow [42].

Similarly one can consider the (Frobenius series) solutions of the differential operator associated with $\chi^{(4)}$ around $x = 0$ and around the ferromagnetic and antiferromagnetic critical point $x = 1$ respectively. Again the corresponding *connection matrix* (matching the solutions around the singularity points $x = 0$ and the ones around the singularity point $x = 1$) have entries which are expressions in terms of π^2, rational numbers but also of constants like constant I_4^- introduced in [1] which can actually be written in term of the Riemann zeta function, as follows :

$$I_4^- \, = \, \frac{1}{16\pi^3} \cdot \Big(\frac{4\,\pi^2}{9} - \frac{1}{6} - \frac{7}{2} \cdot \zeta(3) \Big) \tag{3.3}$$

The derivation of these two results (3.2), (3.3) has never been published[21] but these results appeared in a conference proceedings [43]. We have actually checked that I_3^+ and I_4^- we got in our calculations of connection matrices displayed in [12] (and in the next section) as floating numbers with respectively 421 digits and 431 digits accuracy, *are actually in agreement* with the previous two formula. These two results (3.2), (3.3) provide a clear answer to a question that will be addressed in the next section of how "complicated and transcendental" some of our constants occurring in the entries of the connection matrices can be. These two remarkable exact formulas (3.2), (3.3) are not totally surprising when one recalls the deep link between zeta functions, polylogarithms and hypergeometric series [44, 45, 46, 47]. Along this line, and keeping in mind that we see all our Ising susceptibility calculations as a "laboratory" for other more general problems (Feynman diagrams, ...), we should also recall the various papers of D. J. Broadhurst [48, 49, 50, 51] where $Cl_2(\frac{\pi}{3})$ and $\zeta(3)$ actually occur in a Feynman-diagram-hypergeometric-polylogarithm-zeta framework (see for instance equation (163) in [48]).

3.2. Connection matrices and monodromy matrices

Connecting various sets of Frobenius series-solutions well-suited to the various sets of regular singular points amounts to solving a linear system of 36 unknowns (the entries of the connection matrix). We have obtained these entries in floating point form with a very large number of digits (400, 800, 1000, ...). We have, then, been able to actually "recognize" these entries obtained in floating form with a large number of digits [11]. As an axample the connection matrix for the order six differential operator L_6 matching the Frobenius series-solutions around $w = 0$ and the ones around $w = 1/4$, namely $C(0, 1/4)$ reads:

$$C(0,\, 1/4) \, = \tag{3.4}$$

$$\left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -\frac{9\sqrt{3}}{64\pi} & 0 & 0 & 0 \\ 0 & -\frac{3\pi\sqrt{3}}{32} & 0 & 0 & 0 & 0 \\ 5 & \frac{1}{3} - 2 \cdot I_3^+ & \frac{3\sqrt{3}}{64\pi} & 0 & 0 & \frac{1}{16\pi^2} \\ -\frac{5}{4} & -\frac{3\pi\sqrt{3}}{32} & \frac{45\sqrt{3}}{256\pi} & 0 & \frac{1}{32} & 0 \\ \frac{29}{16} - \frac{2\pi^2}{3} & \frac{15\pi\sqrt{3}}{64} & -\frac{225\sqrt{3}}{1024\pi} - \frac{3\pi\sqrt{3}}{64} & \frac{\pi^2}{64} & 0 & 0 \end{array} \right]$$

Not surprisingly[22] a lot of π's pop out in the entries of these connection matrices. We will keep track of the π's occurring in the entries of connection matrices through the introduction of the variable $\alpha \, = \, 2\,i\,\pi$.

From the local monodromy matrix $Loc(\Omega)$, expressed in the $w \, = \, 1/4$ well-suited local series-basis, and from the connection matrix (3.4), the monodromy matrix around $w \, = \, 1/4$, expressed in terms of

[21] We thank C. A. Tracy for pointing out the existence of these two results (3.2), (3.3) and reference [43].

[22] One can expect the entries of the connection matrices to be *evaluations* of (generalizations of) hypergeometric functions, or solutions of Fuchsian differential equations.

the $(w = 0)$-well suited basis reads:

$$24\,\alpha^4 \cdot M_{w=0}(1/4)(\alpha,\,\Omega) \;=\; \left[\begin{array}{cc} \mathbf{A} & \mathbf{0} \\ \mathbf{B} & \mathbf{C} \end{array}\right] \qquad (3.5)$$
$$= \; C(0,\,1/4)\cdot Loc(\Omega)\cdot C(0,\,1/4)^{-1}$$

where $\left[\begin{array}{c} \mathbf{A} \\ \mathbf{B} \end{array}\right]$ and $\left[\;\mathbf{C}\;\right]$ read respectively:

$$\left[\begin{array}{ccc} -24\,\alpha^4 & 0 & 0 \\ -48\,\alpha^4 & 24\,\alpha^4 & -144\,\alpha^2\Omega \\ 0 & 0 & 24\,\alpha^4 \\ -48\,\rho_1 & 32\,\Omega\,\rho_2 & 48\,\Omega\,\left(9\,\alpha^2+80\,\Omega\right) \\ 12\,\alpha^2\rho_3 & 4\,(75-4\,\alpha^2)\,\alpha^2\Omega & -300\,\alpha^2\Omega \\ -\,(87+8\,\alpha^2)\alpha^4 & 0 & 3\,(4\,\alpha^2-75)\,\alpha^2\Omega \end{array}\right],$$

with $\rho_1 \;=\; 5\,\alpha^4+8\,\Omega^2+8\,\Omega^2\alpha^2,\;\rho_2 \;=\; 4\,\Omega\,\alpha^2-75\,\Omega-15\,\alpha^2$ and $\rho_3 \;=\; 5\,\alpha^2+4\,\Omega+4\,\Omega\,\alpha^2$, and:

$$C \;=\; \left[\begin{array}{ccc} 24\,\alpha^4 & -384\,\alpha^2\Omega & 1536\,\Omega^2 \\ 0 & 24\,\alpha^4 & -192\,\alpha^2\Omega \\ 0 & 0 & 24\,\alpha^4 \end{array}\right]$$

Note that the transcendental constant I_3^+ has disappeared in the final exact expression of (3.5) *which actually depends only on* α and Ω. This $(\alpha,\,\Omega)$ way of writing the monodromy matrix (3.5) enables to get straightforwardly the $N-$th power of (3.5):

$$M_{w=0}(1/4)(\alpha,\,\Omega)^N \;=\; M_{w=0}(1/4)(\alpha,\,N\cdot\Omega)$$

Our "global" (800 digits, 1500 terms in the series, ...) calculations yield quite involved exact connection matrices. With such large, and involved, computer calculations there is always a risk of a subtle mistake or misprint. At this step, and in order to be "even more confident" in our results, note that the monodromy matrices must satisfy one matrix relation which will be an *extremely severe non-trivial check* on the validity of these eight matrices M_i, or more precisely their $(\alpha,\,\Omega)$ extensions. Actually it is known (see for instance Proposition 2.1.5 in [37]), that the *monodromy group*[23] of a linear differential equation (with r regular singular points) is generated by a set of matrices $\gamma_1,\,\gamma_2,\,\cdots,\,\gamma_r$ that satisfy $\gamma_1\cdot\gamma_2\cdots\gamma_r \;=\; \mathbf{Id}$, where $\mathbf{Id}$ denotes the identity matrix. The constraint that "some" product of all these matrices should be equal to the identity matrix, looks quite simple, but is, in fact, "undermined" by subtleties of complex analysis on how connection matrices between non neighboring singular points should be computed. The most general element of the monodromy group reads:

$$M_{P(1)}^{n_1}\cdot M_{P(2)}^{n_2}\cdot M_{P(3)}^{n_3}\cdot M_{P(4)}^{n_4}\cdot M_{P(5)}^{n_5}\cdot M_{P(6)}^{n_6}\cdot M_{P(7)}^{n_7}\cdot M_{P(8)}^{n_8} \qquad (3.6)$$

where P denotes are arbitrary permutation of the eight first integers. In other words, one of the products (3.6) must be equal to the identity matrix for some set of n_i's and for some permutation P.

[23] Which identifies in our Fuchsian case to the differential Galois group; we have regular singular points, and no irregular points with their associated Stokes matrices [34].

Let us introduce the following choice of ordering of the eight singularities, namely ∞, 1, $1/4$, w_1, $-1/2$, $-1/4$, 0, w_2 ($w_1 = (3 + i \cdot \sqrt{7})/8$ and $w_2 = w_1^*$ are the two quadratic number roots of $1 + 3\,w + 4\,w^2 = 0$), the first monodromy matrix M_1 is, thus, the monodromy matrix at infinity $\mathcal{M}(\infty)$, the second monodromy M_2 matrix being the monodromy matrix at $w = 1$, $\mathcal{M}(1)$, ... This is actually the particular choice of ordering of the eight singularities, such that a product like (3.6) is equal to the identity matrix[24]:

$$\begin{aligned} M_1 \cdot M_2 \cdot M_3 \cdot M_4 \cdot M_4 \cdot M_6 \cdot M_7 \cdot M_8 \,&=\, \mathbf{Id} \\ =\, \mathcal{M}(\infty) \cdot \mathcal{M}(1) &\cdot \mathcal{M}(1/4) \cdot \mathcal{M}(w_1) \\ \times \mathcal{M}(-1/2) \cdot \mathcal{M}(-1/4) &\cdot \mathcal{M}(0) \cdot \mathcal{M}(w_2) \end{aligned} \tag{3.7}$$

It is important to note that these relations (3.7) *are not verified* by the $(\alpha,\, \Omega)$ extension (like (3.5)) of the monodromy matrices M_i. If one consider relation (3.7) for the $(\alpha,\, \Omega)$ extensions of the M_i's, one will find that (3.7) is satisfied *only when* α *is equal to* Ω, but (of course[25]) this $\alpha = \Omega$ matrix identity *is verified for any value of* Ω, not necessarily equal to $2\,i\,\pi$.

A last remark is the following: right now, we have considered all the matrices (connection and therefore monodromy matrices expressed in a *unique* non-local basis) with respect to the ($w = 0$) well-suited basis of solutions. This is motivated by the physical solution $\tilde{\chi}^{(3)}$) which is known as a series around $w = 0$. In fact, we can switch to another $w = \tilde{w}$ well-suited basis of solutions. This amounts to considering the connection $C(\tilde{w}, w_s) = C^{-1}(0, \tilde{w}) \cdot C(0, w_s)$. For instance, we have actually performed the same calculations for the ($w = 1/4$) basis of series solutions. We have calculated all the connection matrices from the ($w = 1/4$) basis to the other singular point basis series solutions, and deduced the exact expressions of the monodromy matrices now expressed *in the same* ($w = 1/4$) *basis of series solutions*. It is worth noting that we get, this time, for the monodromy $M_{w=1/4}(w_s)$ around the singular point w_s and expressed in the ($w = 1/4$) basis, a matrix *whose entries depend rationally on* $\alpha = 2\,i\,\pi$, $\Omega = 2\,i\,\pi$, *but, this time, also (except for the monodromy matrix for* $w = 1$*) on the "ferromagnetic constant"* I_3^+. One verifies that the product of these $(\alpha,\, \Omega,\, I_3^+)$-dependant monodromy matrices in the *same order* as (3.7), is actually equal to the identity matrix when $\alpha = \Omega$, the matrix identity being valid for any value of $\alpha = \Omega$ (equal or not to $2\,i\,\pi$), and *for any value of* I_3^+ (equal or not to its actual value).

We have similar results for the monodromy matrices around singular point w_s, expressed in the ($w = \infty$) basis, but, now, the monodromy matrices $M_{w=\infty}(w_s)$ depend on $\alpha = 2\,i\,\pi$, $\Omega = 2\,i\,\pi$, and, this time, on some (as yet unrecognized) constants y_{41} and x_{42} (which can be found in the Appendix C in [11]). Again, the product of these monodromy matrices in the same order as (3.7), is actually equal to the identity matrix when $\alpha = \Omega$, the matricial identity being valid for any value of $\alpha = \Omega$ (equal or not to $2\,i\,\pi$) and for any values of y_{41} and x_{42}.

3.3. From the differential Galois group to the singularity behaviour of solutions and other asymptotics χ studies

The knowledge of these connection matrices gives, in particular, the decomposition of the $\chi^{(n)}$'s on some series well-suited to the analysis of the neighborhood of the critical points and which, consequently, have a well-defined critical behaviour near the critical points. One can, then, immediately deduce the singularity behaviour of the $\chi^{(n)}$'s.

Let us focus, for instance, on the critical behaviour of $\chi^{(3)}$ near the ferromagnetic critical point $w = 1/4$. Denoting $z = 1 - 4\,w$, the singular part of the "physical" solution S_9, associated with

[24] Of course, from this relation, one also has seven other relations deduced by cyclic permutations.

[25] A matrix identity like (3.7) yields a set of polynomial (with integer coefficients) relations on $\Omega = 2\,i\,\pi$. The number π being transcendental it is not the solution of a polynomial with integer coefficients. These polynomial relations have, thus, to be *polynomial identities valid for any* Ω.

$\chi^{(3)}$, reads:

$$\tilde{\chi}^{(3)}(\text{singular}) \; = \; 8 \cdot S_9(\text{singular}) \; \simeq \; \frac{I_3^+}{2\,z} \; + \ln(z) \cdot \text{Sing}_1$$
$$- \frac{1}{64\,\pi^2} \cdot \Bigl(\ln(z)^2 \; - 6\, \ln(2)\, \ln(z)\Bigr) \cdot \text{Sing}_2$$

where

$$\text{Sing}_1 \; = \; 23/3 \; + 227\, z/240 \; - 2047\, z^2/1680 \; - 88949\, z^3/92160 \; + \cdots$$
$$\text{Sing}_2 \; = \; 1 \; - z/8 \; + 3\, z^2/16 \; + \; 29\, z^3/512 \; + \cdots$$

and where I_3^+ is *actually the transcendental constant* (3.2), namely $I_3^+ \; = \; 0.000814462 \cdots$ These results agree with previous results of B. Nickel (see for instance the footnote of page 3904 in [6]), but the correction terms are new[26], in particular the terms in $\ln(2)/(64\,\pi^2)$. In the $\tau \; = (1/s - s)/2$ variable introduced in [15, 16] this reads

$$s^{1/2} \cdot \chi^{(3)} \;\; \simeq \;\; 2^{1/2} \cdot \tau^{1/4} \cdot \Bigl(\frac{I_3^+}{\tau^2} \; - \frac{\ln(\tau)^2}{16\,\pi^2} + \cdots \Bigr)$$

where the dots denote less divergent terms. Near the antiferromagnetic critical point $w \; = \; -1/4$ we get:

$$\tilde{\chi}^{(3)}(\text{singular}) \;\; \simeq \;\; \frac{-1}{32\pi^2} \cdot \ln^2(1 + 4\, w) \; + \; \cdots \tag{3.8}$$

From these results, and using some identities like[27]

$$\ln^2(1 - x) \; = \; \sum_{n=2}^{\infty} \frac{2}{n} \cdot \Bigl(\psi(n) \; + \; \gamma\Bigr) \cdot x^n$$

where γ denotes the Euler Gamma function, and $\psi(n)$ is the logarithmic derivative of the Γ function, and recalling, that, when $n \; \rightarrow \; \infty$, one has the asymptotic behaviour $\psi(n) \rightarrow \ln(n)$, one can find the following asymptotic behavior for the expansion coefficients $c(n)$ of $\tilde{\chi}^{(3)}/8/w^9$ near $w = 0$:

$$\frac{2}{4^8 \cdot 4^n} \cdot c(n) \;\; \simeq \;\; \frac{I_3^+}{2} \; - \frac{1}{32\pi^2} \cdot \frac{\ln(n)}{n} + \; \frac{b_1}{n}$$
$$+ \, (-1)^n \cdot \Bigl(\frac{b_2}{n} \; - \frac{1}{16\pi^2} \cdot \frac{\ln(n)}{n} \Bigr) \; + \; \cdots$$

with $b_1 \; \simeq \; 0.0037256$ and $b_2 \; \simeq \; 0.0211719$. This parity behavior explains why we had so much difficulty to actually find the “true” asymptotic behavior of these coefficients after obtaining around 500 coefficients (see equations (3.6) in [9]: one has 13.5×4^N for N even and 11×4^N for N odd, instead of the “true” $(4^7 \cdot I_3^+) \times 4^N \; \simeq \; 13.4 \times 4^N$ asymptotic behavior).

[26] All these results have also been found by B. Nickel (private communication).

[27] The asymptotic expansion of the coefficients of Taylor series can be obtained using various packages availables at `http://algol.inria.fr/libraries/software.html`. For instance one can use the command “equivalent” in gfun [41] (see [52] for more details. For more details on calculations of asymptotics see [53, 54, 55].

3.4. Generalization to $\chi^{(4)}$

All the previous calculations can be performed, *mutatis mutandis*, for $\chi^{(4)}$. For instance from the singular behaviour near $x = 1$:

$$\tilde{\chi}^{(4)}(\text{singular}) \quad \simeq \quad \frac{I_4^-}{1-x} + \frac{1}{384\,\pi^3} \cdot \ln^3(1-x) \, + \, \cdots \tag{3.9}$$

where I_4^- is given by (3.3) ($I_4^- \, = \, 0.000025448511065 86 \cdots$), one obtains the following asymptotic form for the series coefficients[28] $c(n)$ of $\tilde{\chi}^{(4)}$ near $x = 0$:

$$c(n) \quad \simeq \quad I_4^- - \frac{1}{128\,\pi^3} \cdot \frac{\ln^2(n)}{n} + a_2 \cdot \frac{\ln(n)}{n} + \frac{a_3}{n} + \cdots \tag{3.10}$$

where $a_2 \, \simeq \, 0.0012515$ and $a_3 \, \simeq \, -0.0021928$.

3.5. Non generic character of $\chi^{(4)}$

As we remarked in [9] the critical behavior of $\chi^{(3)}$ near the ferromagnetic point $w \, = \, 1/4$, *does not corresponds to the "dominant" exponent* one obtains in the set of indicial exponents of the indicial equation at $w \, = \, 1/4$. The same situation occurs exactly for $\chi^{(4)}$ as will be seen below. The singular behavior of the solutions of the differential equation can be read easily from Table 2. Near $x = 4$, they are $t^{13/2}$, where $t \, = \, 4 - x$. Near $x \, = \, \infty$, they are $t^{-1/2} \, \log^k(t)$, (with $k \, = \, 0, 1$), $t^{1/2} \, \ln^k(t)$ (with $k \, = \, 0, 1, 2, 3$). $t^{3/2} \, \ln^k(t)$ (with $k \, = \, 0, 1$) and $t^{5/2}$, where $t \, = \, 1/x$. Near $x \, = \, 1$, they are $t^{-3/2}$, t^{-1}, $\ln^k(t)$ (with $k \, = \, 1, 2, 3$) and $t \, \ln(t)$, where $t \, = \, 1 - x$. Note that these behaviors take place for the *general* solution of the differential equation (2.6).

As far as the *physical* solution is concerned, the dominant singular behavior at $x = 4$ (namely $t^{13/2}$), and $x \, = \, \infty$ (namely $t^{-1/2} \, \ln(t)$), are present in the physical solution $\tilde{\chi}^{(4)}$ confirming [7, 15]. At the ferromagnetic point ($x \, = \, 1$ which also corresponds to the anti-ferromagnetic point for $\chi^{(4)}$), with the dominant behavior $t^{-3/2}$, the growth of the coefficients would be $(3/2)_N \, /N! \, \sim \, \sqrt{N}$. Since this is not the case, the coefficients of the series for large values of N behaving like $C_N \, \sim \, 0.2544 \, \cdot \, 10^{-4}$, this $t^{-3/2}$ singular behavior *does not contribute* to the physical solution. Only the *subdominant singular behavior is actually present.*

As far as singularity analysis in lattice statistical mechanics and, more specifically, as far as the analysis of singular behavior near a physical singularity, like the ferromagnetic singularity $x \, = \, 1$, is concerned, we probably have a prejudice which consists in believing that "physical solutions" will correspond to generic solutions and, thus, will take into account the strongest possible singularity if a set of exponents is available (from the indicial equations). In fact the "physical solution" $\chi^{(4)}$ chooses to "avoid" this dominant singular behavior, and will, near the ferromagnetic singularity $x \, = \, 1$, correspond to "sub-dominant" singular behavior.

As far as asymptotic calculations are concerned, there are many tools and methods [53, 54, 55] which give (after some work) asymptotic results for the "generic" solutions of an ODE (or of a recursion). In the "sneaky" situation where physics does not correspond to the "generic" solutions associated with the strongest possible singularities, all the asymptotic calculations become much more difficult. In fact such calculations really correspond to this difficult class of "global" problems of finding a "non-local connection matrix".

3.6. $\tilde{\chi}^{(1)} + \tilde{\chi}^{(3)}$ versus $\tilde{\chi}$ at scaling

Thus far we have sketched some mathematical aspects of the solutions to the Fuchsian differential equations for $\tilde{\chi}^{(3)}$ and $\tilde{\chi}^{(4)}$. However, the physics implications of the solutions we have obtained call for

[28] More generally terms like $\ln^i(1-x)$ give coefficients growing asymptotically like $(-1)^i (\ln(n))^{(i-1)}/n$.

some remarks near the physical critical points. Taking, as an example, the ferromagnetic singularity for $\tilde{\chi}^{(3)}$, the sum of the first two n-particle terms behave near $\tau = (1/s - s)/2 \simeq 0$ as:

$$\begin{aligned} \tilde{\chi}^{(1)} + \tilde{\chi}^{(3)} \quad \simeq \quad & \frac{1 + I_3^+}{\tau^2} - \frac{\ln^2(\tau)}{16\,\pi^2} + \left(\ln(2) - \frac{23}{24}\right) \cdot \frac{\ln(\tau)}{4\,\pi^2} \\ & + \frac{11}{48} + \frac{3}{8} I_3^+ - \frac{1}{4\pi^2}\left(\ln^2(2) - \frac{23}{12}\ln(2) + \frac{14}{144}\right) + \cdots \end{aligned} \tag{3.11}$$

The exact susceptibility, as reported in [16], yields for the normalized susceptibility $\tilde{\chi}$ near $\tau = (1/s - s)/2 \simeq 0$:

$$\begin{aligned} \tilde{\chi} \, = \, & \frac{s}{(1-s^4)^{1/4}} \cdot \chi \, = \, \frac{\left(\tau + \sqrt{1+\tau^2}\right)^{-1/2}}{(1+\tau^2)^{1/8}} \times \\ & \left(c_1\,\tau^{-2}\,F_+(\tau) \, + \, \frac{\tau^{-1/4}}{\sqrt{2}} \sum_{p=0}^{\infty} \sum_{q=p^2}^{\infty} b_+^{(p,q)} \cdot \tau^q \, \ln^p(\tau)\right) \end{aligned} \tag{3.12}$$

where $c_1 = 1.000815260\cdots$ is given with some 50 digits in [15]. $F_+(\tau)$ and $b_+^{(p,q)}$ are given in [15]. The constants $1 + I_3^+$ and c_1 verify $1 + I_3^+ + I_5^+ = c_1$ with nine digits, I_5^+, corresponding to $\chi^{(5)}$, is a constant given in [1] (and with some 30 digits in [6]). Thus, and as suggested in [1], the partial sums of the $\chi^{(n)}$ should converge rapidly to the full χ. Furthermore, adding $\chi^{(3)}$ term has resulted in a series expansion that reproduces the first 24 terms of χ to be compared with only eight first terms for $\chi^{(1)}$ series.

However, equation (3.12) shows a $\tau^{-1/4}$ divergence as an overall factor to the logarithmic singularities which is absent in (3.11). The same situation occurs for the low temperature regime when we compare the first two n-particle terms ($\tilde{\chi}^{(2)}$ and $\tilde{\chi}^{(4)}$) with the full $\tilde{\chi}$ at scaling [29]. This observation raises several profound issues: how does the logarithmic terms in the entire sum add up to make the $\tau^{-1/4}$ divergence be factored out ? If one assumes that the other $\tilde{\chi}^{(2n+1)}$ terms share the same singularity structure as $\tilde{\chi}^3$, in particular the occurrence (in variable τ or s) of only *integer* critical exponents at the ferromagnetic critical point, the $\tau^{-1/4}$ divergence, as an overall factor, implies the following correspondence:

$$\sum_{n=1}^{\infty} \sum_{m=0}^{N(n)} \alpha_{n,m} \cdot S_{n,m}(\tau) \, \ln^m(\tau) \quad \rightarrow \quad \tau^{-1/4} \cdot \sum_{p=0}^{\infty} \sum_{q=p^2}^{\infty} b_+^{(p,q)} \, \tau^q \, \ln^p(\tau)$$

where $S_{n,m}(\tau)$ is analytical at $\tau = 0$, the $\alpha_{n,m}$'s are numerical coefficients and $N(n)$ is the maximum power of logarithmic terms occurring in the solution around the ferromagnetic point of the differential equation of $\tilde{\chi}^{2n+1}$. This correspondence requires probably a *very particular structure* in the successive differential equations. Note that the discrepancy phenomenon we discuss here may be more widespread than that observed here. The challenging problem one faces here is to link *linear and non linear* descriptions of a physical problem, namely the description in terms of an infinite number of holonomic (linear) expressions for a physical quantity of a non linear nature. The difficulty to link holonomic versus non-linear descriptions of physical problems is typically the kind of problems one faces with the Feynman diagram approach of particle physics.

4. Conclusion

Most of the results we have displayed in this paper are, in large part, mathematical "subtleties" associated with the structure of the differential operators associated with the $\tilde{\chi}^{(n)}$'s and also surprising results like

[29] For the leading amplitude, $\tilde{\chi}^{(2)}$ and $\tilde{\chi}^{(4)}$ give $1/12\pi + I_4^- \simeq 1.0009593\cdots/12\pi$ which is very close to $1.0009603\cdots/12\pi$ for the full $\tilde{\chi}$ [6].

the fact that the Fuchsian ODE's of $\tilde{\chi}^{(3)}$ (resp. $\tilde{\chi}^{(4)}$) have simple *rational and algebraic solutions*, together with previously encountered solutions $\tilde{\chi}^{(1)}$ (resp. $\tilde{\chi}^{(2)}$).

In a miscellaneous list of these mathematical "subtleties" let us recall the occurrence of a large set of apparent singularities for the ODE of smallest order (or the fact that the simplest, and easiest to be guessed, ODE is *not* the one of smallest order), the existence of "Landau-like" pinched singularities, (that are not of the "Nickel form" (1.3)), that are singularities of the ODE *but not singularities of the "physical" solutions* $\chi^{(n)}$, the fact that the "physical" solutions $\tilde{\chi}^{(n)}$ are not "generic" solutions but "mathematically singled-out" ones (subdominant critical behavior[30], no singular behavior for the previous "Landau-like" pinched singularities), ...

The fact that $\tilde{\chi}^{(2)}$ is a solution of the differential operator associated with $\tilde{\chi}^{(4)}$ is probably the *most important result*, since it is a strong indication of some "Russian-doll-like" mathematical structure valid for the *infinite set* of differential operators associated with all the infinite sequence of $\chi^{(n)}$'s.

It is worth noting that almost all these mathematical structures, or singled-out properties, are far from being specific of the two-dimensional Ising model: they also occur on many problems of lattice statistical mechanics or, even[31], as A. J. Guttmann et al saw it recently, on enumerative combinatorics problems like, for instance, the generating function of the three-choice polygon [56].

With this specific study of the susceptibility of the Ising model, we think that, in the future, we should contemplate a "collision" between various mathematical domains and structures, namely the theory of *isomonodromic deformations* (Painlevé equations and their generalizations with unit circle natural boundaries like Chazy III equations, see [57, 58], or Garnier systems[32] [59]), the theory of holonomic systems (Fuchsian equations and in particular the so-called *rigid local systems* [60] and their geometrical interpretations [61]), the theory of modular forms [62, 63] (Eisenstein series [64, 65], Heegner numbers [66], complex multiplication for elliptic curves [66], modular equations [68, 69, 70] ...), and various ideas related to the analysis of Feynman diagrams (Landau singularities, generalizations of hypergeometric functions to several complex variables, Appell functions, Kampé de Fériet, Lauricella-like functions, polylogarithms [48, 49, 50, 51], Riemann zeta functions, multiple zeta values, ...). We also think that such "collisions" of concepts and structures are *not* a consequence of the free-fermion character of the Ising model, and that similar convergence should also be encountered on more complicated Yang-Baxter integrable models[33], the two-dimensional Ising model first popping out as a consequence of its simplicity. An open question is what could remain of these related structures for less Yang-Baxter integrable problems such as the calculations of Feynman diagrams ? Apparently a lot seems to remain [48, 49, 50, 51]. Another open problem which requires to be addressed is the (apparent ?) scaling discrepancy, we sketched in section (3.6), between the holonomic approach and more Painlevé oriented approach.

Acknowledgments
One of us (JMM) thanks Prof. A.J. Guttmann for his interest in our work, his extremely valuable advice and for his constant friendly support during the last decade, and more recently for an invitation in the ARC Centre of Excellence for Mathematics and Statistics of Complex Systems in Melbourne where this paper has been completed. We thank Jacques-Arthur Weil for valuable comments on differential Galois

[30] At the ferromagnetic critical point $w = 1/4$, the behavior $(1-4w)^{-3/2}$ *corresponding to the largest critical exponent for the ODE is actually absent in the physical solution* $\chi^{(3)}$.

[31] The wronskian of the corresponding differential equation is also rational, the associated differential operator factorizes in a way totally similar to the one described in sections (2.4), (2.5), large polynomial corresponding to apparent singularities also occur, ...

[32] The occurence of hypergeometric functions of several complex variables in the Garnier systems is worth noting (see page 10 and page 111 paragraph 31 in [59]).

[33] The comparison of the Riemann zeta functions equations obtained for the XXX quantum spin chain [67] with the evaluations of central binomial in [49] provides a strong indication in favor of similar structures on non-free fermion Yang-Baxter integrable models.

groups, and connection matrices. We would like to thank B. Nickel for his inspired comments on solution S_3. We thank C. A. Tracy for pointing out the results of reference [43]. We thank J. Dethridge for a highly optimized C++ program that gave an additional and very strong confirmation of our $\chi^{(4)}$ results. We would like to thank A. J. Guttmann, I. Jensen, and W. Orrick for a large set of useful comments on the singularity behaviour of physical solutions. (JMM) thanks R.J. Baxter for great hospitality and discussions when visiting the ANU (Canberra) and B. M. McCoy for his interest in our work and so many helpfull comments. (S. B) and (S. H) acknowledge partial support from PNR3-Algeria.

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