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NEW LOCAL RELATIONS FOR LATTICE MODELS*

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We describe new local relations leading to non-trivial (non-homogeneous) equations for the row-to-row transfer matrices of arbitrary size for two dimensional I.R.F. and vertex models. We sketch the connection between this relation and the Yang-Baxter equations, and we describe the example of the hard hexagon model.

1. Introduction

It is remarkable that most of the exact results of statistical mechanics on lattices are consequences of some simple local relation on the Boltzmann weights of the models. These local relations imply algebraic relations, for example on transfer matrices of arbitrary size and consequently, in the thermodynamic limit, on relevant quantities like the partition function of the model.

The paradigm of these local relations is the Yang-Baxter relation.[1] It bears indeed on local Boltzmann weights associated to each face of the square lattice for models with interactions around the faces (I.R.F. models),[1,2] respectively each site for vertex models, even in non-homogeneous cases (Z-invariance property[3]). What makes it a key to the exact solvability of lattice models is that it is a sufficient and to some extent necessary condition for the commutation of transfer matrices of *arbitrary size.*[4]

Another example is given by the "disorder solutions", which also arise from a local disorder criterion (see for instance Ref. 5): a simple decimation procedure enables one to obtain results on global quantities (partition function. . . .) from this local condition. For vertex models these disorder conditions are very similar to the local relations required to build the (global) Bethe ansatz of the exactly solvable models (see Eq. (C 34 a,b) of Ref. 6, the Frobenius relations on theta functions,[7,8] and Eq. (1), Eq. (17) of Ref. 9).

Beyond the case where the models are exactly solvable another local relation called the *inversion relation* has been used to produce exact functional equations.[10] It is noticeable that here again one goes, for appropriate boundary conditions, from a local to a (global) relation on transfer matrices of arbitrary

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size. The importance of these local conditions leading to global consequences is perfectly clear. They are a way to get exact functional relations which facilitate the calculation of the partition function of the model (inversion "trick").[1,11]

Among the relations on transfer matrices used in the analysis of statistical mechanical models, the one obtained by R. J. Baxter and P. A. Pearce for the hard hexagon model (Eq. (3.3) or (3.5) of Ref. 12) i.e.

$$T_N T_N' = 1 + T_N''$$

is especially appealing. Indeed it is *valid for all sizes* of the transfer matrices, it is not homogeneous and therefore gives non-trivial functional equations and new information on the spectrum (interfacial tension of the model, see Ref. 12, and it has a very promising algebraic aspect.

Our point is to show that it is a consequence of *a simple local relation* on the Boltzmann weights of the model. We give a general form of this local relation for the q-state I.R.F. (resp. vertex) models, and describe it with some detail in the specific case of the hard hexagon model.

It is an open question to decide what the status of this new relation is, with respect to the integrability of the model (equivalently the Yang-Baxter equations), and what are the possible connections with the aforementioned inversion relation.

2. The q-state I.R.F. Model

In this section we introduce the notations used to describe the q-state I.R.F. model. The model is defined on a square lattice with a spin sitting at each site and taking q values $(0, 1, \ldots, q-1)$. The statistical weight of a configuration is obtained from the elementary Boltzmann weight W associated to a face. For a spin configuration

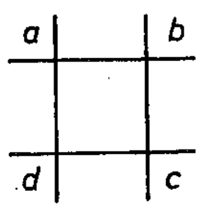

Fig. 1. Spin configuration around a face.

the corresponding weight is

$$W(\text{face}) = W(a, b, c, d)\ .$$

The partition function for the $N \times P$ lattice (P lines of length N) with periodic boundary conditions is then:

$$Z = \sum_{\text{configurations}} \prod_{\text{faces}} W(\text{face}) \ ,$$

where the sum is over all the spin configurations and the product is over all faces of the lattice.

It is standard to introduce the row-to-row transfer matrix T_N associated to a horizontal line

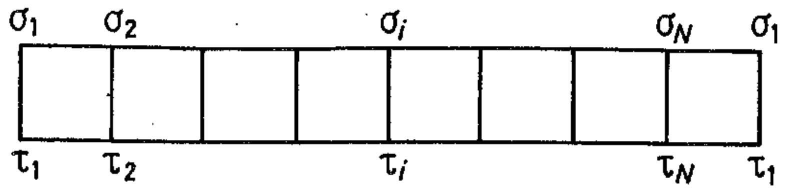

Fig. 2. Row-to-row transfer matrix.

The matrix elements of T_N are given by

$$(T_N)^{\sigma_1, \ldots, \sigma_N}_{\tau_1, \ldots, \tau_N} = \prod_{i=1}^{N} W(\sigma_i, \sigma_{i+1}, \tau_{i+1}, \tau_i) \ ,$$

with the periodic boundary conditions

$$\sigma_{N+1} = \sigma_1, \qquad \tau_{N+1} = \tau_1.$$

As usual the partition function is written as

$$Z = \mathrm{tr}(T_N^P) \ .$$

Suppose now that W and W' are two Boltzmann weights for the q-state I.R.F. model. Let T_N and T'_N be the corresponding transfer matrices, the product of these two matrices reads:

$$\begin{aligned}(T_N T'_N)^{\sigma_1, \ldots, \sigma_N}_{\sigma'_1, \ldots \sigma'_N} &= \sum_{\tau_1, \ldots, \tau_N} \prod_{i=1}^{N} W(\sigma_i, \sigma_{i+1}, \tau_{i+1}, \tau_i) \cdot W'(\tau_i, \tau_{i+1}, \sigma'_{i+1}, \sigma'_i) \\ &= \mathrm{tr}\left(\prod_{i=1}^{N} X(\Sigma_i, \Sigma_{i+1}) \right) \qquad (1)\end{aligned}$$

where Σ_i denotes the pair (σ_i, σ_i'), and for each Σ_i, Σ_{i+1}, $X(\Sigma_i, \Sigma_{i+1})$ denotes the $q \times q$ matrix with entries:

$$X_{\alpha,\beta}(\Sigma_i, \Sigma_{i+1}) = W(\sigma_i, \sigma_{i+1}, \beta, \alpha) \cdot W'(\alpha, \beta, \sigma_{i+1}', \sigma_i') .$$

The matrix element $X_{\alpha,\beta}(\Sigma_i, \Sigma_{i+1})$ is the weight of the configuration

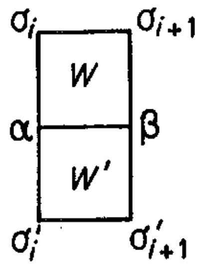

Fig. 3. Pictorial representation of $X_{\alpha,\beta}(\Sigma_i, \Sigma_{i+1})$

In (1) taking the trace over q-dimensional space ensures the periodicity of the boundary condition if one takes $\Sigma_{N+1} = \Sigma_1$.

The hard hexagon model[13] is a special case of this q-state I.R.F. model, for $q = 2$ (the spin takes values 0 or 1), with

$$W(a, b, c, d) = \begin{cases} mz^{(a+b+c+d)/4} e^{Lac+Mbd} t^{-a+b-c+d} & \text{if } ab = bc = cd = da = 0 . \\ 0 \quad \text{otherwise} & \end{cases}$$

In this expression, z is the activity, L and M are diagonal interactions, m is a trivial normalization factor, and t is a parameter that cancels out of the partition function. The parameter space consists in only five homogeneous parameters corresponding to the five allowed spin configurations around the face.

$$\begin{aligned} \omega_1 &= W(0000) \\ \omega_2 &= W(1000) = W(0010) \\ \omega_3 &= W(0100) = W(0001) \\ \omega_4 &= W(1010) \\ \omega_5 &= W(0101) . \end{aligned}$$

The model happens to be exactly solvable on a subvariety of this parameter space (see Eq. (23) of Refs. 13 and 14, and Sec. 4.1). It is then parametrized through elliptic functions by a constant λ (related to the so-called shift operator) and a spectral parameter u. The existence of this parametrization is the signature of the generalized star-triangle relation (Yang-Baxter for I.R.F. models) and is also particularly appropriate to write the matrix equation for the normalized partition function:

$$T_N(u,\lambda)\cdot T_N(u+\lambda,\lambda) = 1 + T_N(u+3\lambda,\lambda) \ . \tag{2}$$

Together with quasi-periodic properties and analyticity properties of the row-to-row transfer matrix, this relation enables one to calculate in principle every eigenvalue for finite N.

Equation (2) is valid for all N. It yields a functional equation for the whole spectrum of the transfer matrix and for the normalized partition function (largest eigenvalue):

$$Z_N(u,\lambda)\cdot Z_N(u+\lambda,\lambda) = 1 + Z_N(u+3\lambda,\lambda) \ . \tag{3}$$

The free energy, interfacial tension, and correlation length may then be calculated, with the help of these equations.

3. A New Local Relation on the Boltzmann Weights

Coming back to the derivation of the relation $TT' = 1 + T''$ for the dimensionless transfer matrix of the hard hexagon model, we see that although it looks model dependent (see p. 900 of Ref. 12), it stems from purely algebraic relations (more specifically matrix algebra, properties on eigenvectors, . . .) on the set of matrices $X(\Sigma_i, \Sigma_{i+1})$. The fact that algebraic considerations on the local objects $X(\Sigma_i, \Sigma_{i+1})$ yield matrix relations *valid for all sizes* N is an indication that here again a local property leads to a global one. We give here a simple algebraic relation on the local Boltzmann weights for the q-state I.R.F. model which is *sufficient* to ensure the relation between transfer matrices for arbitrary values of N. We shall show in the next section that it is fulfilled in the case of the hard hexagon model.

The local property of the model is the existence of a triplet of Boltzmann weights W, W', W'', of a collection of $q\times q$ matrices $P(\Sigma_i)$, and of functions $\Gamma_{m,n}(\Sigma,\Sigma')$ such that

$$P^{-1}(\Sigma_i)\cdot X(\Sigma_i,\Sigma_{i+1})\cdot P(\Sigma_{i+1}) = Y(\Sigma_i,\Sigma_{i+1}) \tag{4}$$

with

$$\begin{aligned} Y_{\alpha,\beta}(\Sigma_i,\Sigma_{i+1}) = {} & \delta_{\alpha,1}\delta_{\beta,1}\Delta(\Sigma_i,\Sigma_{i+1}) + \delta_{\alpha,0}\delta_{\beta,0}W''(\Sigma_i,\Sigma_{i+1}) \\ & + \sum_{m>n}\delta_{\alpha,m}\delta_{\beta,n}\Gamma_{m,n}(\Sigma_i,\Sigma_{i+1}) \end{aligned} \tag{5}$$

where

a) $W''(\Sigma_i,\Sigma_{i+1}) = W''(\sigma_i,\sigma_{i+1},\sigma'_{i+1},\sigma'_i)$

b) $\Delta(\Sigma_i,\Sigma_{i+1}) = \lambda\delta_{\sigma_i,\sigma'_i}\delta_{\sigma_{i+1},\sigma'_{i+1}}$ on the *allowed configurations* of a row of the transfer matrices.

Of course the indices 0 and 1 in (5) are chosen arbitrarily and could be replaced by any other values among $0, 1, \ldots, q-1$.

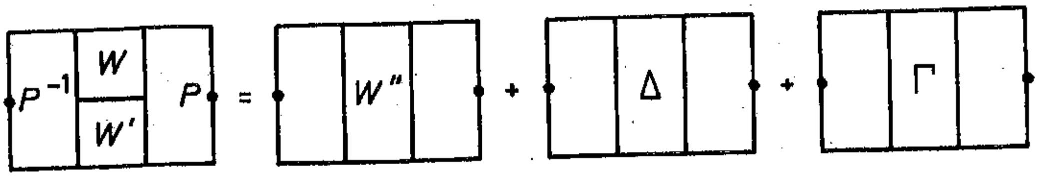

Fig. 4. Pictorial representation of the local relation for I.R.F. models.

Relation (4,5) uses a gauge type transformation which preserves the trace in relation (1). Thus

$$\begin{aligned}
(T_N T'_N)^{\sigma_1,\ldots,\sigma_N}_{\sigma'_1,\ldots,\sigma'_N} &= \mathrm{tr}\left(\prod_{i=1}^{N} X(\Sigma_i, \Sigma_{i+1})\right) \\
&= \mathrm{tr}\left(\prod_{i=1}^{N} Y(\Sigma_i, \Sigma_{i+1})\right) \\
&= \prod_i \Delta(\Sigma_i, \Sigma_{i+1}) + \prod_i W''(\Sigma_i, \Sigma_{i+1}) \\
&= \lambda^N (I)^{\sigma_1,\ldots,\sigma_N}_{\sigma'_1,\ldots,\sigma'_N} + (T''_N)^{\sigma_1,\ldots,\sigma_N}_{\sigma'_1,\ldots,\sigma'_N}
\end{aligned} \tag{6}$$

$(T''_N)^{\sigma_1,\ldots,\sigma_N}_{\sigma'_1,\ldots,\sigma'_N}$ are the matrix elements of the row-to-row transfer matrix with periodic boundary conditions for the Boltzmann weight W''. Due to the definition of $\Delta(\Sigma_i, \Sigma_{i+1})$, I is nothing but the identity matrix on the *allowed*[a] *configurations* of a row of the model.

4. The Example of the Hard Hexagon Model

In this section, we examine the hard hexagon model restricted to the subvariety of the space of parameters where the Boltzmann weights satisfy the generalized star-triangle relation. We show that for any such W, there exist W', W'' and a matrix P such that (4), (5) holds. In Baxter and Pearce[12] W, W', and W'' correspond respectively to the values u, $u+\lambda$, and $u+3\lambda$ of the spectral parameter.

4.1. *Description of the model*

The model is exactly solvable on the algebraic subvariety of the parameter space given by the incomplete intersection of a quadric, a cubic, and a quartic in

[a] A similar subtlety has already been encountered for the inversion relation of the hard hexagon model (see for instance p. 14 of Ref. 12 and p. 421 of Ref. 1). In general using the inversion relation amounts to taking the inverse of a matrix associated to the Boltzmann weight of the model. For the hard hexagon model this matrix is not invertible but one may define an inverse if one restricts the matrix to the allowed configurations of the model.

$\mathbb{C}P_4$. If we denote

$$F_1 = \frac{\omega_1^2 - \omega_4\omega_5}{\omega_2\omega_3}$$

$$F_2 = \frac{\omega_4\omega_3^2 + \omega_5\omega_2^2 - \omega_1\omega_4\omega_5}{\omega_1\omega_2\omega_3} \tag{7}$$

$$F_3 = \frac{\omega_1\omega_2^2\omega_5 + \omega_1\omega_3^2\omega_4 - \omega_4^2\omega_5^2 - \omega_2^2\omega_3^2}{\omega_2\omega_3\omega_4\omega_5}$$

then the equation of this subvariety is:

$$F_1 = C_1 \ ,$$

$$F_2 = C_2 \ ,$$

$$F_3 = C_3$$

where C_1, C_2, C_3 are three constants verifying

$$C_1C_2 = 1 \ , \qquad C_3 = C_1 + C_2 \ .$$

The constraints on the constants C_i come from the relation on the F_i's:

$$F_1F_2 - 1 = (F_3 - F_1 - F_2)\frac{\omega_4\omega_5}{\omega_2\omega_3} \ .$$

Without these constraints, the subvariety would have reduced to one point up to the transformation:

$$\begin{cases} \omega_1 \to \omega_1 \\ \omega_2 \to t\,\omega_2 \\ \omega_3 \to t^{-1}\omega_3 \\ \omega_4 \to t^2\omega_4 \\ \omega_5 \to t^{-2}\omega_5 \end{cases} . \tag{8}$$

The transformation (8) leaves the transfer matrix unchanged, and should be factored out.

It is natural to arrange the weights $X_{\alpha,\beta}(\Sigma_i, \Sigma_{i+1})$ in a matrix where the rows (resp. columns) are labelled by $(\sigma_i, \alpha, \sigma_i')$ (resp. $(\sigma_{i+1}, \beta, \sigma'_{i+1})$) (see Fig. 3). The exclusion rules of the model lead us to restrict to a 5×5 submatrix corresponding to the following allowed configurations:

$$A = (0,0,0), \quad B = (0,1,0), \quad C = (1,0,1), \quad D = (0,0,1), \quad E = (1,0,0) \ .$$

This 5×5 matrix has the form:

$$\chi = \begin{pmatrix} x_{AA} & x_{AB} & x_{AC} & x_{AD} & x_{AE} \\ x_{BA} & 0 & x_{BC} & x_{BD} & x_{BE} \\ x_{CA} & x_{CB} & 0 & 0 & 0 \\ x_{DA} & x_{DB} & 0 & 0 & x_{DE} \\ x_{EA} & x_{EB} & 0 & x_{ED} & 0 \end{pmatrix}$$

with

$$\begin{array}{lll} x_{AA} = \omega_1\omega_1' , & x_{BA} = x_{AC} = \omega_3\omega_2' , & x_{CA} = x_{AB} = \omega_2\omega_3' \\ x_{DA} = \omega_1\omega_3' & x_{EA} = \omega_2\omega_1' & x_{CB} = \omega_4\omega_5' \\ x_{DB} = \omega_2\omega_5' & x_{EB} = \omega_4\omega_3' & x_{BC} = \omega_5\omega_4' \\ x_{AD} = \omega_1\omega_2' & x_{BD} = \omega_3\omega_4' & x_{ED} = \omega_2\omega_2' \\ x_{AE} = \omega_3\omega_1' & x_{BE} = \omega_5\omega_2 & x_{DE} = \omega_3\omega_3' \end{array}$$

The symmetry (8) induces the following transformations on the x_{pq}'s:

$$\begin{array}{lll} x_{AA} \rightarrow x_{AA} , & x_{DA} \rightarrow t^{-1}x_{DA} , & x_{BA} = x_{AC} \rightarrow x_{BA} = x_{AC} , \\ x_{EA} \rightarrow t\, x_{EA} , & x_{CB} \rightarrow x_{CB} , & x_{CA} = x_{AB} \rightarrow x_{CA} = x_{AB} , \\ x_{DB} \rightarrow t^{-1}x_{DB} , & x_{EB} \rightarrow t^{-3}x_{EB} , & x_{BC} \rightarrow x_{BC} , \\ x_{AD} \rightarrow t\, x_{AD} , & x_{BD} \rightarrow t\, x_{BD} , & x_{ED} \rightarrow t^2 x_{ED} , \\ x_{AE} \rightarrow t^{-1}x_{AE} , & x_{BE} \rightarrow t^{-1}x_{BE} , & x_{DE} \rightarrow t^{-2}x_{DE} . \end{array}$$

4.2. *Construction of the matrices P*

In the following, we give a step-by-step construction of the matrices $P(\Sigma)$ which ensure relation (4), (5).

Define the family of matrices $P_1(\Sigma_i)$ by

$$P_1(0,0) = \begin{pmatrix} 1 & -s \\ r & 1 \end{pmatrix}$$

and

$$P_1(\Sigma) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{otherwise} .$$

In $P_1(0,0)$, s and r are chosen in such a way that $\binom{-s}{1}$ and $\binom{1}{r}$ are eigenvectors of

$$Q = \begin{pmatrix} x_{AA} & x_{AB} \\ x_{BA} & 0 \end{pmatrix} .$$

In particular, s verifies

$$x_{BA} \cdot s^2 + x_{AA} \cdot s - x_{AB} = 0 \ . \tag{9}$$

We shall denote by μ the eigenvalue corresponding to $\binom{-s}{1}$ (resp. μ' for $\binom{1}{r}$). If the x_{pq} verify

$$x_{DA} \cdot x_{AC} + x_{DB} \cdot x_{BC} = 0 \tag{10}$$

$$x_{EA} \cdot x_{AC} + x_{EB} \cdot x_{BC} = 0 \ , \tag{11}$$

and if

$$s = -\frac{x_{AC}}{x_{BC}} \tag{12}$$

is a solution of Eq. (9), then transformation (4) acting on χ yields:

$$\chi = \begin{pmatrix} x'_{AA} & 0 & 0 & x'_{AD} & x'_{AE} \\ 0 & x'_{BB} & x'_{BC} & x'_{BD} & x_{BE} \\ x'_{CA} & x'_{CB} & 0 & 0 & 0 \\ x'_{DA} & 0 & 0 & 0 & x'_{DE} \\ x'_{EA} & 0 & 0 & x'_{ED} & 0 \end{pmatrix}$$

where

$$x'_{BB} = \mu$$

$$x'_{DE} = x_{DE}$$

$$x'_{ED} = x_{ED} \ .$$

Equation (10) (resp. (11)) implies $x'_{DB} = 0$ (resp. $x'_{EB} = 0$). They are trivially fulfilled if one chooses $W = W(u)$ and $W' = W(u + \lambda)$ in the elliptic parametrization of the model (Eq. 2.12 of Ref. 12).

Equation (12) yields

$$x'_{AC} = 0 \ .$$

Equation (9) together with (12) demand an algebraic relation between the x_{pq}'s:

$$x_{AA} x_{AC} x_{BC} + x_{AB} x_{BC}^2 + x_{AC}^2 x_{BA} = 0 \ . \tag{13}$$

This is actually verified if one uses the elliptic parametrization of the model[12] and the following identity on theta functions with period 5λ:

$$\theta(\lambda)^2\cdot\theta(2\lambda+u)\cdot\theta(2\lambda-u)-\theta(u)^2\cdot\theta(\lambda)\cdot\theta(2\lambda)$$
$$-\theta(\lambda-u)\cdot\theta(\lambda+u)\cdot\theta(2\lambda)^2=0\;. \qquad (14)$$

Notice also that s is independent of u, while r does depend on u.

Define next the family (related to the transformations (8)) of matrices $P_2(\Sigma_i)$ by

$$P_2(\Sigma)=p_2(\Sigma)\cdot\begin{pmatrix}1&0\\0&1\end{pmatrix}\;.$$

The action of P_2 on χ' leads to χ''. The coefficients $p_2(\Sigma)$ may be adjusted so as to ensure

$$x''_{EA}=x''_{AD}\quad\text{and}\quad x''_{AE}=x''_{DA}\;,$$

that is to say $W''(1000)=W''(0010)$ and $W''(0100)=W''(0001)$ meaning that W'' fulfills the symmetries of the hard hexagon model, and

$$x''_{BC}=x''_{CB}=x''_{BB}\;.$$

This adjustment is possible because

$$x'_{EA}\cdot x'_{AE}=x'_{DA}\cdot x'_{AD} \qquad (15)$$

and

$$x'^2_{BB}=x'_{BC}x'_{CB}\;. \qquad (16)$$

Relation (16) is a direct consequence of (14) and Eq. (15) comes from another relation on the theta functions with period 5λ:

$$\theta(u)\cdot\theta(3\lambda+u)\cdot\theta(2\lambda)-\theta(\lambda+u)\cdot\theta(2\lambda+u)\cdot\theta(\lambda)+\theta(\lambda-u)^2\cdot\theta(2\lambda)=0\;. \qquad (17)$$

At this point, one will obtain the desired form (4), (5) with the action of $P=P_1\cdot P_2\cdot P_3$, where

$$P_3(1,1)=\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

and

$$P_3(\Sigma)=\begin{pmatrix}1&0\\0&1\end{pmatrix}\quad\text{otherwise.}$$

Remark 0. There are simple relations worth noticing:

$$\mu = \frac{x_{BC} \cdot x_{CB}}{x_{AA}}, \quad \mu\mu' = -x_{BA} \cdot x_{AB}, \quad \frac{-s}{r} = -\frac{x_{AB}}{x_{BA}}, \quad s^2 = \frac{\theta(2\lambda)}{\theta(\lambda)}.$$

Remark 1. The previous proof breaks the parity invariance since the form of χ' is not stable by transposition. The transposition on χ, i.e. the exchange $x_{pq} \leftrightarrow x_{qp}$ is the mirror symmetry exchanging left and right (parity transformation). It corresponds to $\omega_2 \leftrightarrow \omega_3$, $\omega_4 \leftrightarrow \omega_5$. In terms of the parametrization, it reads $u \leftrightarrow \lambda - u$ (changing the sign of ω_1). If one takes into account the specific properties of the transfer matrix (commutation property and periodicity), relation (2) is compatible with the parity symmetry: actually, the matrix relation

$$T(u) \cdot T(u') = \mathbf{1} + T(u'') \tag{18}$$

(with $u' = u + \lambda$, $u'' = u + 3\lambda$) becomes

$$T(v') \cdot T(v) = \mathbf{1} + T(v'')$$

with $v = -u$, $v' = v + \lambda$, $v'' = v - 2\lambda$. These two relations are identical since

$$T(v) = T(v + 5\lambda) \quad \text{and} \quad [T(v), T(v')] = 0 .$$

Remark 2. The interest of the above detailed calculations is to shed some light on the links between the Yang-Baxter relation and the new local relation (5). We actually see that the specific construction of the matrices P we used will work if the x_{pq}'s verify a number of algebraic equations (Eqs. (10), (11), (13), (15), (16)). These equations are invariant by the symmetry (8). One could imagine that this set of equations (defining an algebraic subvariety of the parameter space of W and W') has solutions beyond the elliptic parametrization of the integrable case. In fact the parameter space for χ is, due to the symmetry (8), of dimension six, and we have five conditions on the x_{pq}, leading to an algebraic curve (known to contain an elliptic curve).

Remark 3. The previous calculation is just a *block triangulation* with a 2×2 block corresponding to Δ and a 3×3 block

$$\begin{pmatrix} x''_{AA} & x''_{AD} & x''_{AE} \\ x''_{DA} & 0 & x''_{DE} \\ x''_{EA} & x''_{ED} & 0 \end{pmatrix}$$

where $x''_{EA} = x''_{AD}$ and $x''_{AE} = x''_{DA}$ forming the weight W''. The zeroes in this 3×3 matrix reflect the exclusions for the hard hexagon model.

5. Vertex Models

For vertex models one can introduce similar notations. To the vertex

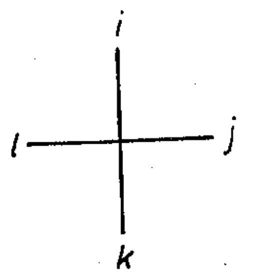

Fig. 5. Vertex configuration.

one associates the local Boltzmann weight $W(i, j, k, l)$. The row-to-row transfer matrix T_N reads

$$(T_N)^{i_1, \dots, i_N}_{i'_1, \dots, i'_N} = \sum_{k_1, \dots, k_N} \prod_p W(k_p, i_p, k_{p+1}, i'_p)$$

for the line configuration:

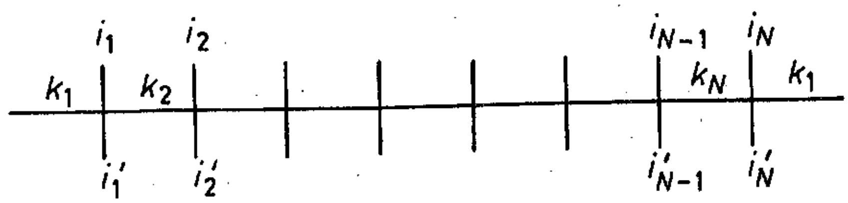

Fig. 6. Row-to-row transfer matrix of a vertex model.

If W and W' are two Boltzmann weights for the vertex model, and T_N and T'_N the corresponding transfer matrices with periodic boundary conditions, then:

$$\begin{aligned}
&(T_N T'_N)^{i_1, \dots, i_N}_{i'_1, \dots, i'_N} \\
&= \sum_{j_1, \dots, j_N} \sum_{k_1, \dots, k_N} \prod_p \left(\sum_{l_p} W(j_p, i_p, j_{p+1}, l_p) \cdot W'(k_p, l_p, k_{p+1}, i'_p) \right) \\
&= \mathrm{tr} \left(\prod_p X^{v}(i_p, i'_p) \right) .
\end{aligned} \tag{19}$$

The trace in formula (19) is taken over a q^2-dimensional space, and

$$j_{N+1} = j_1 \quad \text{and} \quad k_{N+1} = k_1 .$$

For each (i_p, i'_p), $X^v(i_p, i'_p)$ is a $q^2 \times q^2$ matrix with matrix elements

$$X^v_{j_p k_p, j_{p+1} k_{p+1}}(i_p, i'_p) = \sum_l W(j_p, i_p, j_{p+1}, l) \cdot W'(k_p, l, k_{p+1}, i'_p) \ .$$

The matrix element $X^v_{j_p k_p, j_{p+1} k_{p+1}}(i_p, i'_p)$ is the weight of the configuration

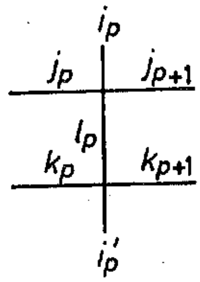

Fig. 7. Pictorial representation of $X^v(i_p, i'_p)$.

where one sums over the internal configurations l_p.

A local relation, similar to (5) can be introduced for the q-state vertex model: if there exists a triplet W, W', W'' for the Boltzmann weight of the vertex model and a $q^2 \times q^2$ matrix P such that

$$P^{-1} \cdot X^v(i_p, i'_p) \cdot P = Y^v(i_p, i'_p) \tag{20}$$

where $Y^v(i_p, i'_p)$ has matrix elements:

$$Y^v_{j_p k_p, j_{p+1} k_{p+1}}(i_p, i'_p) = \sum_{l,m} F^l_{j_p k_p} W''(l, i_p, m, i'_p) G^m_{j_{p+1} k_{p+1}} + H_{j_p k_p} \cdot \delta_{i_p, i'_p} \cdot I_{j_{p+1} k_{p+1}} \tag{21}$$

where F, G, H, I do not depend on i_p nor i'_p and satisfy the following relations

$$\sum_{j,k} I_{jk} F^l_{jk} = 0 \ , \quad \sum_{j,k} G^m_{jk} H_{jk} = 0 \tag{22}$$

$$\sum_{j,k} G^m_{jk} F^l_{jk} = \delta_{m,l} \ , \quad \sum_{j,k} H_{jk} I_{jk} = 1 \ . \tag{23}$$

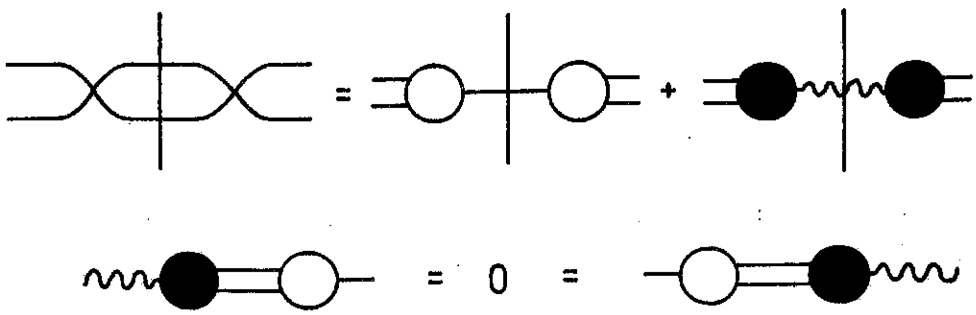

Fig. 8. Pictorial representation of relations (21) and (22) for vertex models.

Relation (21) yields

$$(T_N T'_N)^{i_1,\dots,i_N}_{i'_1,\dots,i'_N} = \lambda^N (I^v)^{i_1,\dots,i_N}_{i'_1,\dots,i'_N} + (T''_N)^{i_1,\dots,i_N}_{i'_1,\dots,i'_N}$$

where I^v coincides with the identity matrix on the configurations $(i_1,\dots,i_N)$.

Remark. The local relation (21) amounts to performing a block-diagonalization, but of course in the same way as we did for the hard hexagon model one might have block-triangulation only.

5.1. *Generalizations*

In the same spirit we could write a more general local relation which will lead to other non-homogeneous relations of higher degree on transfer matrices for all sizes N.

If W_1, W_2, W_3 are three Boltzmann weights and T_1, T_2, T_3 are the corresponding transfer matrices, then

$$(T_1 \cdot T_2 \cdot T_3)^{i_1,\dots,i_N}_{i'_1,\dots,i'_N} = \operatorname{tr}\left(\prod_p X^v_{(13)}(i_p, i'_p)\right)$$

where the trace is taken over a q^3-dimensional space, and where $X^v_{(3)}$ is a $q^3 \times q^3$ matrix with matrix elements

$$(X^v)_{j_p k_p l_p,\, j_{p+1} k_{p+1} l_{p+1}}(i_p, i'_p) = \sum_{m_p}\sum_{n_p} W_1(j_p, i_p, j_{p+1}, m_p) \times W_2(k_p, m_p, k_{p+1}, n_p)\cdot W_3(l_p, n_p, l_{p+1}, i'_p)$$

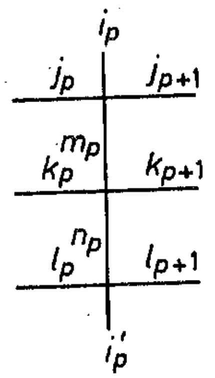

Fig. 9. Pictorial representation of X^v for three Boltzmann weights.

A natural generalization of (20), (21) could be the existence of weights W_1, W_2, W_3, and W_4, W_5, W_6 and of a $q^3 \times q^3$ matrix P such that

$$P^{-1} \cdot X^v_{(3)}(i_p, i'_p) \cdot P = Y^v_{(3)}$$

where $Y_{(3)}^{v}$ has matrix elements:

$$
\begin{aligned}
Y^{v}_{j_pk_pl_p,\,j_{p+1}k_{p+1}l_{p+1}}(i_p, i'_p) = & \sum_{q,r} F^{v}_{j_pk_pl_p} W_4(q, i_p, r, i'_p) G^{r}_{j_{p+1}k_{p+1}l_{p+1}} \\
& + \sum_{s,t} H^{s}_{j_pk_pl_p} W_5(s, i_p, t, i'_p) I^{t}_{j_{p+1}k_{p+1}l_{p+1}} \\
& + \sum_{u,v} J^{u}_{j_pk_pl_p} W_6(u, i_p, v, i'_p) K^{v}_{j_{p+1}k_{p+1}l_{p+1}}
\end{aligned}
\tag{24}
$$

where F, G, H, I, J, K do not depend on i_p nor i'_p and satisfy the relations

$$
\sum_{j,k,l} F^{q}_{jkl} G^{r}_{jkl} = \delta_{q,r}\,, \quad \sum_{j,k,l} H^{s}_{jkl} I^{t}_{jkl} = \delta_{s,t}\,, \quad \sum_{j,k,l} J^{u}_{jkl} K^{v}_{jkl} = \delta_{u,v} \tag{25}
$$

$$
\sum_{j,k,l} G^{r}_{jkl} H^{s}_{jkl} = 0, \quad \sum_{j,k,l} G^{r}_{jkl} J^{u}_{jkl} = 0, \quad \sum_{j,k,l} I^{t}_{jkl} F^{q}_{jkl} = 0 \tag{26}
$$

$$
\sum_{j,k,l} I^{t}_{jkl} J^{u}_{jkl} = 0, \quad \sum_{j,k,l} K^{v}_{jkl} F^{q}_{jkl} = 0, \quad \sum_{j,k,l} K^{v}_{jkl} H^{s}_{jkl} = 0\ .
$$

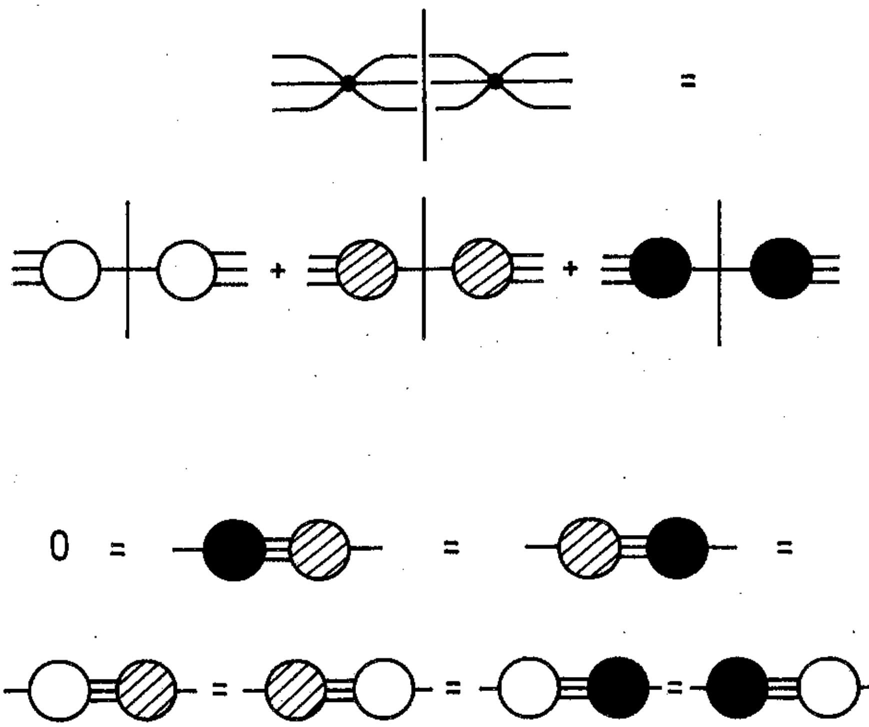

Fig. 10. Representation of relations (24) and (26).

It is straightforward to get from the local relation (24) and the above equations (26), the relation:

$$
(T_1T_2T_3)^{i_1,\ldots,i_N}_{i'_1,\ldots,i'_N} = (T_4)^{i_1,\ldots,i_N}_{i'_1,\ldots,i'_N} + (T_5)^{i_1,\ldots,i_N}_{i'_1,\ldots,i'_N} + (T_6)^{i_1,\ldots,i_N}_{i'_1,\ldots,i'_N}\ .
$$

This equation is valid for arbitrary size N.

Remark 1. Of course one could imagine an unlimited number of extensions of all these local relations to a larger number of Boltzmann weights, leading in particular to higher degree relations on the transfer matrices ($TTTT = TT + T + \ldots$).

Remark 2. These various generalizations will also exist for the I.R.F. models.

6. Comments and Speculations

It is already very clear, from the existing literature on the Rogers-Ramanujan identities for the hard hexagon model and the RSOS models (see Refs. 15, 16), that relation (2) and the functional equations it yields on the partition function (3), are highly non-trivial.

It is thus natural to ask what the status of relation (4), (5) is, outside of any consideration of integrability. Indeed we have isolated, in the specific case of the hard hexagon model (sec.4), a number of algebraic constraints on the weights W, W'. These constraints do not identify at first sight with exact solvability.

One could first try to parametrize in general the algebraic variety defined by these constraints. This parametrization could go beyond an elliptic one.

If this parametrization is by curves one will get functional equations like (18). Even in the case where u' and u'' are obtained from u by translations (like $u' = u + \lambda$, $u'' = u + 3\lambda$) the set of non-periodic solutions, although not fully explored, is known to be quite remarkable and involved (theory of resurgent functions[17]). One may recall that the functional equation (3) on the partition function Z_N is actually identical to the one verified by Stokes multipliers for the irregular differential equation $y'' - (x^3 + \alpha)y = 0$.[17–19]

The more restrictive case of the periodic solutions furnishes a very promising setting for the study of the relations we have introduced. Indeed elliptic functions may verify Eq. (3), which amounts then to sums of products identities listed by Ramanujan and subsequently proved by Rogers. These equations appear in the RSOS models.[16]

We recall that the RSOS models have been introduced by Andrews *et al.* by revisiting the Bethe ansatz on the symmetric eight-vertex model[16] (the very construction of the Bethe Ansatz associates to the face centered at a vertex four relative integers l_i. The weights W of the RSOS model are expressed in terms of elliptic functions at points like $u + l_i\eta$ (see Eq. (1.2.5) and Eq. (1.2.6) of Ref. 16). If η is commensurate with the period K of the elliptic functions ($\eta = K^v/p$), this amounts to take $l_i \in \mathbb{Z}_p$ and not $\mathbb{Z}$. For different values of p, the partition function verifies different Rogers-Ramanujan identities (the hard hexagon model corresponds to $p = 5$)). What we have in mind is that the various local relations we introduced come from a unique underlying structure.

For the eight-vertex model, one can deduce non-linear finite difference equations such as (3) from a matrix equation

$$T(u)\,Q(u) = \varphi(u-\lambda)Q(u+\lambda) + \varphi(u)Q(u-\lambda) \tag{27}$$

(see Eq. (23) of Ref. 20). For $\lambda = \pi/3$ or $2\pi/3$, Eq. (27) leads to a linear system on the eigenvalues of Q (see Eq. (38) of Ref. 20), which has non-trivial solutions only if

$$t(u)t(u+\lambda)t(u+2\lambda) - t(u) - t(u+\lambda) - t(u+2\lambda) - 2 = 0 \tag{28}$$

where $t(u)$ is a normalized eigenvalue of $T(u)$.

For $5\lambda/\pi =$ integer, the matrix of the linear system is 5×5, and its rank reduces to three if $T(u) = T(u+\lambda)$ and

$$T(u)T(u+5\lambda) = \varphi(u-\lambda)\varphi(u+\lambda) - \varphi(u)T(u+3\lambda)\ .$$

Similar results exist also for the chiral Potts model.[21] One has a matrix equation analogous to Eq. (27) (see for instance Eq. (5.20) of Ref. 22), and an equation similar to (28) holds true (see Eq. (26) of Ref. 23 and Eq. (18) of Ref. 24). This shows in particular that such equations may be encountered for exactly solvable models parametrized by *curves of genus greater than one.*

Another interesting feature of the relations we have described is the similarity of the objects introduced in Eqs. (21), (24) with the ones appearing in two-dimensional conformal field theory (fusion algebra, see for example Ref. 25): this is visible in Figs. 8 and 10. In formula (21), G realizes the fusion of two spaces into a single one.

Finally, it should be very rewarding to explore, especially in the spirit of Ref. 26, the relations between our local relations and other exact symmetries (automorphy group[14,26,27]). The motivation is clear from the way the automorphy group acts in the elliptic case (it is generated by the translation: $u \to (u+\lambda)$ and the inversion $u \to -u$), and the role played by the translation in Eq. (3) or the shift operator in Eq. (5.20) of Ref. 22 or Eq. (26) of Ref. 23.

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