Modular invariance in lattice statistical mechanics

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Dedicated to G. Lochak on the occasion of his 70th birthday

ABSTRACT. We show that non-trivial symmetries, and structures, originating from lattice statistical mechanics, provide a quite efficient way to get interesting results on elliptic curves, algebraic varieties, and even arithmetic problems in algebraic geometry. In particular, the lattice statistical mechanics approach underlines the role played by the modular j-invariant, and by a reduction of elliptic curves to a symmetric biquadratic curve (instead of the well-known reduction to a canonical Weierstrass form). This representation of elliptic curves in terms of a very simple symmetric biquadratic is such that the action of a (generically infinite) discrete set of birational transformations, which corresponds to important non-trivial symmetries of lattice models, can be seen clearly. This biquadratic representation also makes a particular symmetry group of elliptic curves (group of permutation of three elements related to the modular j-invariant) crystal clear. Using this biquadratic representation of elliptic curves, we exhibit a remarkable polynomial representation of the multiplication of the shift of elliptic curves (associated with the group of rational points of the curve). The two expressions g_2 and g_3, occurring in the Weierstrass canonical form $y^2 = 4\,x^3 - g_2\,x - g_3$, are seen to present remarkable covariance properties with respect to this infinite set of commuting polynomial transformations (homogeneous polynomial transformations of three homogeneous variables, or rational transformations of two variables).

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1 Introduction

Elliptic curves naturally occur in the study of integrable models of (1+1 dimensional) field theory or integrable models of lattice statistical mechanics. This very fact may look "suspect" to "down-to-earth" physicists, who would certainly think that any reasonably analytical parameterization would do the job. How could the mathematical beauties of elliptic curves be related to "true" physics ? After Einstein, even "down-to-earth" physicists are ready to believe that differential geometry has something to do with nature, but they are clearly reluctant to believe that this could also be the case with algebraic geometry. Let us sketch very briefly the reasons of this occurrence of algebraic geometry in integrable lattice models.

Let us consider models of lattice statistical mechanics with *local* Boltzmann weights. The Yang-Baxter equations are known to be a sufficient condition (and to some extent, necessary condition [1]) for the commutation of transfer matrices in lattice statistical models. Moreover, it has been shown that the commutation of transfer matrices *necessarily* yields a parameterization of the R-matrices in terms of *algebraic varieties* [2]. In general, even if these models are not Yang-Baxter integrable, the set of the so-called *inversion relations* [3, 4], combined together with the geometrical symmetries of the lattice, yield a (generically infinite) discrete set of *non-linear* symmetries of the models. For such lattice models with *local Boltzmann weights*, it is straightforward to see that these *non-linear* symmetries are represented in terms of *rational* transformations of the (homogeneous) parameters of the parameter space of the model (the various Boltzmann weights). Furthermore, since this (generically infinite) discrete set of rational transformations is generated by rational involutions, like the matricial inversion corresponding to the inversion relations, and by simple linear transformations of finite order (involutive permutations of the homogeneous parameters of the parameter space of the model), one thus gets a canonical (generically infinite) discrete set of *non-linear* symmetries represented in terms of *birational transformations* of the parameter space of the model. Let us recall that a birational transformation of several (complex) variables is a rational transformation such that its inverse is *also* a rational transformation. Birational transformations play a crucial role in the study of algebraic varieties as illustrated by the Italian geometers of the last century[1]. To sum up, the *locality* of the Boltzmann weights necessarily yields, for regular lattices (square lattices, triangular lattices, cubic lattices, ...), *birational* symmetries of the parameter space of the model which can be seen as a (complex) *projective space* (homogeneous parameters).

Coming back to the (Yang-Baxter) integrable framework, it has been shown that these *birational* symmetries are actually *discrete symmetries* of the Yang-Baxter equations [6, 7]. *Generically*, this set of birational symmetries is an *infinite set*. Combining these facts together, one gets the following result: the Yang-Baxter integrability is *necessarily* parameterized in terms of *algebraic varieties* having a generically infinite set of *discrete birational symmetries*[2]. Let us also recall that an algebraic variety with an infinite set of (birational) automorphisms *cannot be an algebraic variety of the so-called "general type"* [2]. When the algebraic varieties are algebraic *curves* the occurrence of this infinite set of (birational) automorphisms implies that the algebraic curves are *necessarily* of genus zero (rational curves) *or genus one* (elliptic curves). These are the reasons why an elliptic parameterization occurs, for instance, in the (Yang-Baxter) integrable symmetric eight-vertex model, also called the Baxter model [8].

Such iteration of birational transformations yields, quite naturally, to problems of *rational points on algebraic varieties*. Actually, when one performs, on a computer, an iteration of the previous birational transformations associated with lattice models, in order to visualize these orbits and get some "hint" on the integrability of the model [9], one always iterates rational numbers: in a typical computer "experiment" (even with a precision of 5000 digits ...), a "transcendental point" of the parameter space is in fact *represented by a rational point*. When one visualizes the previously mentioned elliptic curves [6, 10, 11], or some Abelian surfaces [9], as orbits of the iteration of $\widehat{K}$, one visualizes, in fact, *an infinite set of rational points of an elliptic curve or of an*

[1]More than anyone else, the creation of the Italian school of projective and algebraic geometry is due to Cremona [5].

[2]This set of discrete birational symmetries can be seen as generated by the iteration of a birational transformation, thus canonically associating a *discrete dynamical system* [9].

Abelian surface. Lattice statistical mechanics thus yields naturally to consider *arithmetic problems of algebraic geometry.* The description of the *rational points on algebraic varieties* has always been a fascinating problem[3] for mathematicians [12, 13, 14, 15, 16, 17, 18]. Beyond the general framework of the Hilbert's tenth problem, one can hope to be able to say something in some special cases, in particular in the case of *rational points on Abelian varieties*[4].

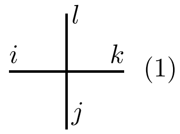
In this paper, we will show that the *modular invariance* of elliptic curves, and more precisely the so-called j-invariant [19] of elliptic curves (also called Klein's absolute invariant, or "hauptmodul" ...), does play an important role in the symmetries of the Baxter model, in the much more general (and, generically, non Yang-Baxter integrable) sixteen-vertex model [10, 20], and, beyond, in a large class of (non Yang-Baxter integrable) two-dimensional models of lattice statistical mechanics and field theory (in section (6), we will give the example of the four-state chiral Potts model). We will also underline the existence of *an infinite set of commuting homogeneous polynomial transformations preserving the modular invariant* j. We will also see that the birational symmetries we introduce, enable to build the *group of rational points* of some Abelian varieties, and in the case of elliptic curves, also preserve the j-invariant. These results emerge, very simply, from the analysis of an important *representation of elliptic curves in term of a simple symmetric biquadratic* that occurs, naturally, in lattice statistical mechanics (propagation property, see below (19)). This fundamental symmetric biquadratic can actually be generalized in the case of surfaces, or higher dimensional (Abelian) varieties [21].

[3]In 1970 Au. V. Matiyasevich showed that Hilbert's tenth problem is unsolvable, i.e. there is no general method for determining when algebraic equations have a solution in whole numbers.

[4]The Birch and Swinnerton-Dyer conjecture [12] asserts that the size of the group of rational points of an Abelian variety is related to the behavior of an associated zeta function $\zeta(s)$ near the point $s = 1$. The conjecture asserts that if $\zeta(1)$ is not equal to 0, then there is only a finite number of such rational points.

2 Birational symmetries associated with the Baxter model and the sixteen-vertex model

Let us consider a quite general vertex model namely the sixteen-vertex model [10, 20]. Pictorially this can be represented as follows:



(1)

where i and k (corresponding to direction (1)) take two values, and similarly for j and l. The sixteen homogeneous parameters corresponding to the sixteen Boltzmann weights of the model can be displayed in a 4×4 matrix R or equivalently in a block matrix of four 2×2 matrices A, B, C, D:

$$R \;=\; \begin{pmatrix} A & B \\ C & D \end{pmatrix} \;=\; \begin{pmatrix} a_1 & a_2 & b_1 & b_2 \\ a_3 & a_4 & b_3 & b_4 \\ c_1 & c_2 & d_1 & d_2 \\ c_3 & c_4 & d_3 & d_4 \end{pmatrix} \tag{2}$$

An important integrable subcase of this quite general vertex model is the symmetric eight vertex model, also called Baxter model [8], which corresponds to the following 4×4 R-matrix:

$$R = \begin{pmatrix} a & 0 & 0 & d \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ d & 0 & 0 & a \end{pmatrix} \tag{3}$$

Let us introduce the partial transposition t_1 associated with direction (1):

$$(t_1 R)^{ij}_{kl} = R^{kj}_{il} \qquad \text{that is :} \qquad t_1 : \; \begin{pmatrix} A & B \\ C & D \end{pmatrix} \longrightarrow \begin{pmatrix} A & C \\ B & D \end{pmatrix} \tag{4}$$

and the two following transformations on matrix R, the matrix inverse $\widehat{I}$ and the homogeneous matrix inverse I:

$$\widehat{I} : \; R \longrightarrow \; R^{-1}\,, \qquad\qquad I : \; R \longrightarrow \; \det(R) \cdot R^{-1} \tag{5}$$

Let us also introduce the (generically) infinite order homogeneous, and inhomogeneous, transformations, respectively:

$$K \, = \, t_1 \cdot I \, , \qquad \text{and:} \qquad \widehat{K} \, = \, t_1 \cdot \widehat{I} \tag{6}$$

For such vertex models of lattice statistical mechanics, transformations $\widehat{I}$ and t_1 are consequences of the *inversion relations* [3, 4] and the geometrical symmetries of the lattice, in the framework of integrability and *beyond* integrability. These involutions generate a *discrete* group of (*birational*) automorphisms of the Yang-Baxter equations [6, 7] and their higher dimensional generalizations [22]. They also generate a discrete group of (birational) automorphisms of the algebraic varieties canonically associated with the Yang-Baxter equations (or their higher dimensional generalizations) [2]. In the generic case where the birational transformation $\widehat{K} \, = \, t_1 \cdot \widehat{I}$ is an *infinite order* transformation, this set of birational automorphisms corresponds, essentially, to the iteration of $\widehat{K}$.

Suppose that one iterates a *rational point* M_0, all the points of the orbit of this initial point by the iteration of $\widehat{K}$, namely the points $\widehat{K}^N(M_0)$ *will also be rational points*. When this infinite set of rational points densifies an algebraic curve, one thus gets, necessarily, an *elliptic curves with an infinite set of rational points* [23].

When the infinite order transformation $\widehat{K}$ *densifies d-dimensional algebraic varieties* [9], one can deduce the equations of these algebraic varieties exactly [10]. One can even argue that these algebraic varieties should be *Abelian varieties*. Let us denote $\mathcal{V}_{\mathcal{K}}(M_0)$ the set of points of the form $\widehat{K}^N(M_0)$, where M_0 is a given initial point, and N denotes a relative integer. Let us consider two points, M and M', belonging to $\mathcal{V}_{\mathcal{K}}(M_0)$ (that is such that there exist two relative integers N and N' such that $M \, = \, \widehat{K}^N(M_0)$ and $M' \, = \, \widehat{K}^{N'}(M_0)$). One can straightforwardly define a notion of *addition* of two such points $M'' \, = \, M + M'$, associating to $M \, = \, \widehat{K}^N(M_0)$, and $M' \, = \, \widehat{K}^{N'}(M_0)$, the point M'' given by $M'' \, = \, \widehat{K}^{N+N'}(M_0)$. Therefore, one can define a notion of Abelian addition of two arbitrary points belonging to $\mathcal{V}_{\mathcal{K}}(M_0)$. The identity point is the origin M_0: $M \, + \, M_0 \, = \, M$, since $M \, + M_0 \, = \, \widehat{K}^{(N+0)}(M_0) \, = \, \widehat{K}^N(M_0) \, = \, M$. The inverse of point M, with respect to this Abelian addition, is the point $-M \, = \, \widehat{K}^{-N}(M_0)$. If $\mathcal{V}_{\mathcal{K}}(M_0)$ densifies an algebraic variety $\mathcal{V}$, one deduces (by continuity of the birational transformation K) that $\mathcal{V}$ is an Abelian variety (an algebraic variety which is also an Abelian group). Suppose that the iteration of an initial rational point *densifies* an algebraic variety of dimension d, one deduces that necessarily this d-dimensional algebraic variety is an Abelian variety with an *infinite (dense) set of rational points*. In these explicit examples, the relation between the birational transformation $\widehat{K}$ and the *group of rational points* of algebraic varieties is straightforward.

Factorization properties and complexity of the calculations. The *integrability* of the birational mapping $\widehat{K}$, or, equivalently, of the homogeneous polynomial transformation K, is closely related to the occurrence of *remarkable factorization schemes* [9, 21]. In order to see this, let us consider a 4×4 initial matrix $M_0 = R$, and the successive matrices obtained by iteration of transformation $K = t_1 \cdot I$, where t_1 is defined by (4). Similarly to the factorizations described in [9, 21], one has, for arbitrary n, the following factorizations for the iterations of K:

$$M_{n+2} = \frac{K(M_{n+1})}{f_n^2}, \qquad f_{n+2} = \frac{\det(M_{n+1})}{f_n^3}, \tag{7}$$
$$\widehat{K}_{t_1}(M_{n+2}) = \frac{K(M_{n+2})}{\det(M_{n+2})} = \frac{M_{n+3}}{f_{n+1} f_{n+3}}$$

where the f_n's are homogeneous polynomials in the entries of the initial matrix M_0, and the M_n's are "reduced matrices" with homogeneous polynomial entries [21, 9].

Let us denote by α_n the homogeneous degree of the determinant of matrix M_n, and by β_n the homogeneous degree of polynomial f_n, and let us introduce $\alpha(x)$, $\beta(x)$ which are the generating functions of these α_n's, β_n's:

$$\alpha(x) = \sum_{n=0}^{\infty} \alpha_n \cdot x^n, \qquad \beta(x) = \sum_{n=0}^{\infty} \beta_n \cdot x^n, \tag{8}$$

From these factorization schemes, one sees that a *polynomial growth* of the iteration calculations occurs (quadratic growth of the degrees). Actually, one can easily get linear relations on the exponents α_n, β_n and exact expressions for their generating functions and for the α_n's and β_n's:

$$\alpha(x) = \frac{4(1+3x^2)}{(1-x)^3}, \qquad \beta(x) = \frac{4x}{(1-x)^3},$$
$$\alpha_n = 4\,(2\,n^2+1), \qquad \beta_n = 2n\,(n+1) \tag{9}$$

On the other hand, one has a whole hierarchy of recursions *integrable*, or compatible with integrability [21]. For instance, on the f_n's, one has:

$$\frac{f_n\, f_{n+3}^2 - f_{n+4}\, f_{n+1}^2}{f_{n-1}\, f_{n+3}\; f_{n+4} - f_n\, f_{n+1}\, f_{n+5}} = \frac{f_{n+1}\, f_{n+4}^2 - f_{n+5}\, f_{n+2}^2}{f_n\, f_{n+4}\; f_{n+5} - f_{n+1}\, f_{n+2}\, f_{n+6}} \qquad (10)$$

In an equivalent way, introducing the variable $x_n = \det(\widehat{K}^n(R)) \cdot \det(\widehat{K}^{n+1}(R))$, one gets another *hierarchy* of recursions on the x_n's, (see [21]), the simplest one reading:

$$\frac{x_{n+2} - 1}{x_{n+1}\, x_{n+2}\, x_{n+3} - 1} \quad = \quad \frac{x_{n+1} - 1}{x_n\, x_{n+1}\, x_{n+2} - 1} \cdot x_n\, x_{n+1}\, x_{n+2}^2 \qquad (11)$$

Equation (11) is equivalent to (10) since $x_n = (f_n^3 \cdot f_{n+2})/(f_{n+1}^3 \cdot f_{n-1})$. It can be seen that these recursions (10), and (11), are *integrable* ones [21]. In order to see this, one can introduce [11] a new (homogeneous) variable:

$$q_n \;=\; \frac{f_{n+1} \cdot f_{n-1}}{f_n^2} \qquad \text{then :} \qquad x_n \;=\; \frac{q_{n+1}}{q_n} \qquad (12)$$

and end up, after some simplifications, with the following *biquadratic* relation between q_n and q_{n+1}:

$$q_n^2 \cdot q_{n+1}^2 \,+\, \mu \cdot q_n \cdot q_{n+1} \,+\, \rho \cdot (q_n + q_{n+1}) \,-\, \nu \quad = \quad 0 \qquad (13)$$

which is clearly an integrable recursion (elliptic curve) [21]. In terms of the f_n's, the three parameters ρ, ν and μ read:

$$\rho \;=\; \frac{{f_1}^2\, f_4 \,-\, {f_2}^2\, f_3}{f_1^3 f_3 - f_2^3}\,, \qquad \nu \;=\; \frac{f_2\, f_4 -\, f_1\, {f_3}^2}{f_1^3\, f_3 - f_2^3}, \qquad (14)$$
$$\mu \;=\; \frac{{f_2}^5 \,-\, {f_3}^2 {f_1}^3 - {f_1}^5 f_4 + f_3\, {f_2}^3}{f_2\, f_1 \cdot (f_1^3\, f_3 - f_2^3)}$$

For the most general sixteen-vertex model, the expressions of ρ, ν and μ are quite large in terms of its sixteen homogeneous parameters and, thus, will not be given here. Let us just give an idea of these expressions in the simple Baxter limit:

$$\rho \;=\; (ab + cd)^2 \cdot (ab - cd)^2 \cdot (a^2 + b^2 - c^2 - d^2)^2\,,$$
$$\nu \;=\; -\nu_1 \cdot \nu_2 \cdot \nu_3 \cdot \nu_4 \qquad (15)$$

$$
\begin{aligned}
\nu_1 &= b^2dc - b^2a^2 + cda^2 - d^3c + c^2d^2 - c^3d\,, \\
\nu_2 &= b^2dc + b^2a^2 + cda^2 - d^3c - c^2d^2 - c^3d\,, \\
\nu_3 &= b^3a + b^2a^2 + ba^3 - d^2ba - c^2ba - c^2d^2\,, \\
\nu_4 &= b^3a - b^2a^2 + ba^3 - d^2ba - c^2ba + c^2d^2, \\
\mu &= -b^2a^6 + 2\,a^4b^2c^2 - a^4c^2d^2 - 4\,a^4b^4 + 2\,a^4d^2b^2 - a^2c^4b^2 + 2\,a^2c^4d^2 \\
&\quad +2\,a^2c^2b^4 + 2\,a^2c^2d^4 + 2\,d^4b^2c^2 - a^2b^6 + 2\,a^2b^4d^2 - a^2d^4b^2 - c^6d^2 \\
&\quad +2\,c^4d^2b^2 - 4\,c^4d^4 - d^6c^2 - c^2d^2b^4
\end{aligned}
$$

3 Sixteen-vertex model and Baxter model: revisiting the elliptic curves

Considering the (non-generically Yang-Baxter integrable) sixteen vertex model [20], one finds that a *canonical* parameterization in terms of *elliptic curves* occurs in the sixteen homogeneous parameter space of the model [10]. In the Yang-Baxter integrable subcase, the Baxter model, this elliptic parameterization, deduced from the previous iteration of K, or $\widehat{K}$, (Baxterisation procedure, see [24]), is *actually the elliptic parameterization introduced by R. J. Baxter to solve the Baxter model* [25].

In fact, several elliptic curves (associated with different "spaces") occur: one corresponding to the factorization analysis of the previous section (see (11), (13)), another one obtained from the iteration of the birational transformation $\widehat{K}^2$ in the sixteen homogeneous parameter space of the model [10], and, as will be seen below in the next section, another deduced from the so-called "propagation property" (see (16) below). An analysis of the relations between these various elliptic curves shows that they *actually identify*, as we will be seen below.

3.1 Propagation property for the sixteen-vertex model and the Baxter model

One of the "keys" to the Bethe Ansatz is the existence (see equations (B.10), (B.11a) in [8]) of vectors which are *pure tensor products* (of the form $v \otimes w$) and which R maps onto pure tensor products $v' \otimes w'$. This key property[5] was called *propagation property* by R. J. Baxter, and corresponds to the existence of a *Zamolodchikov algebra* [28] for

[5] Which is "almost" a sufficient condition for the Yang-Baxter equation. In the case of the Baxter model this non trivial relation corresponds to some intertwining relation of the product of two theta functions, which is nothing but the *quadratic Frobenius relations* on theta functions [26, 27].

the Baxter model[6]. Let us consider the sixteen-vertex model (2). The "propagation" equation reads here:

$$R\,(v_n \otimes w_n) \;=\; \mu \cdot v_{n+1} \otimes w_{n+1} \qquad \text{or:} \tag{16}$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} w_n \\ p_n \cdot w_n \end{pmatrix} \;=\; \mu \cdot \begin{pmatrix} w_{n+1} \\ p_{n+1} \cdot w_{n+1} \end{pmatrix} \qquad \text{with :}$$

$$v_n = \begin{pmatrix} 1 \\ p_n \end{pmatrix}, \qquad w_n = \begin{pmatrix} 1 \\ \tilde{p}_n \end{pmatrix}, \quad v_{n+1} = \begin{pmatrix} 1 \\ p_{n+1} \end{pmatrix}, \quad w_{n+1} = \begin{pmatrix} 1 \\ \tilde{p}_{n+1} \end{pmatrix}$$

Actually, one can associate an *algebraic curve* (or "vacuum curve", or "propagation curve") of equation:

$$\det(A \cdot p_{n+1} \,- C \,-\, D \cdot p_n +\, B \cdot p_n\, p_{n+1}) \;=\; 0 \tag{17}$$

which form is invariant by t_1, $\widehat{I}$ and, thus, also by $\widehat{K}$ or $\widehat{K}^2$. As a byproduct this provides a *canonical algebraic curve for such vertex models*, namely curve (17). More precisely, the eliminations of p_n, p_{n+1} (resp. $\tilde{p}_n$, $\tilde{p}_{n+1}$) yields the two *biquadratic* relations [10]:

$$\begin{aligned} &l_4 \,+\, l_{11} \cdot p_n \,-\, l_{12} \cdot p_{n+1} + \, l_2 \cdot p_n^2 \,+ l_1 \cdot p_{n+1}^2 \,-\, (l_9 + l_{18}) \cdot p_n \cdot p_{n+1} \\ &\qquad - l_{13} \cdot p_n^2 \cdot p_{n+1} \,+\, l_{10} \cdot p_n \cdot p_{n+1}^2 \,+\, l_3 \cdot p_n^2 \cdot p_{n+1}^2 \;=\; 0\,, \\ &l_7 \,+\, l_{16} \cdot \tilde{p}_n \,-\, l_{15} \cdot \tilde{p}_{n+1} + \, l_8 \cdot \tilde{p}_n^2 \,+\, l_5 \cdot \tilde{p}_{n+1}^2 \,-\, (l_9 - l_{18}) \cdot \tilde{p}_n \cdot \tilde{p}_{n+1} \\ &\qquad - l_{17} \cdot \tilde{p}_n^2 \cdot \tilde{p}_{n+1} \,+ l_{14} \cdot \tilde{p}_n \tilde{p}_{n+1}^2 \,+\, l_6\, \tilde{p}_n^2\, \tilde{p}_{n+1}^2 \;=\; 0 \end{aligned} \tag{18}$$

where the l_i's are quadratic expressions of the sixteen independent entries (2) of the R-matrix [10].

Some $\widehat{K}^2$-invariants can be deduced from $SL(3)$ "gauge-invariance", namely a quadratic expression in the l_i's $l_1 \cdot l_2 + l_3 \cdot l_4 - l_{10} \cdot l_{11} - l_{12} \cdot l_{13} + (l_9 + l_{18})^2$, a cubic expression (in the l_i's) which is nothing but a 3×3 *determinant*, and a quartic one. Eighteen (algebraically related) *quadratic* polynomials $(p_1, \ldots, p_{18})$ which are linear combinations of the l_i's, and transform very simply under t_1 and I, have been found [10]. Introducing the ratio of these covariants p_i's, one gets *invariants* of $\widehat{K}^2$, thus giving the equations of the elliptic curves: the elliptic curves are

[6] The existence of a Zamolodchikov algebra is, at first sight, a *sufficient* condition for the Yang-Baxter equations to be verified. Theta functions of g variables *do satisfy quadratic Frobenius relations* [27], consequently *yielding a Zamolodchikov algebra parameterized in terms of theta functions of several variables.* However this Zamolodchikov algebra is apparently *not sufficient* for Yang-Baxter equations to be satisfied [29].

given by the intersection of fourteen quadrics in the associated fifteen-dimensional projective parameter space of the model [10].

In the subcase of the Baxter model, the two previous "vacuum curves" actually identify, the two "propagation" biquadratics (18) of the Baxter model [8, 25] reading:

$$\begin{aligned}
&(J_x - J_y) \cdot (p_n^2 \, p_{n+1}^2 + 1) - (J_x + J_y) \cdot (p_n^2 + p_{n+1}^2) + 4 \cdot J_z \cdot p_n \cdot p_{n+1} \\
&= (p_n^2 - 1)(p_{n+1}^2 - 1) \, J_x - (p_n^2 + 1)(p_{n+1}^2 + 1) \, J_y + 4 \, J_z \, p_n \, p_{n+1} \\
&= \Gamma_1(p_n, \, p_{n+1}) = 0 \qquad (19)
\end{aligned}$$

where J_x , J_y, and J_z are the three well-known homogeneous quadratic expressions of the XYZ quantum Hamiltonian [8]:

$$J_x = a \cdot b + c \cdot d \, , \; J_y = a \cdot b - c \cdot d \, , \; J_z = \frac{a^2 + b^2 - c^2 - d^2}{2} \qquad (20)$$

It is known that simple "propagation" curves, like (19), have the following elliptic parameterization [8, 25]:

$$p_n = sn(u_n, k) \, , \quad p_{n+1} = sn(u_{n+1}, k) \quad \text{where :} \quad u_{n+1} = u_n \pm \eta \qquad (21)$$

where $sn(u, k)$ denotes the elliptic sinus of modulus k, and η denotes some "*shift*". With this elliptic parametrization one sees that the birational transformation $\widehat{K}$ has a very simple representation on the spectral parameter u, namely *a simple shift*: $u \longrightarrow u \pm \eta$.

3.2 Reduction of a sixteen-vertex model to a K^2-effective Baxter model

Let us note that the R-matrix of the sixteen-vertex model *can actually be decomposed* [7] as:

$$R_{sixteen} \quad = \quad g_{1L}^{-1} \otimes g_{2L}^{-1} \cdot R_{Baxter} \cdot g_{1R} \otimes g_{2R} \qquad (22)$$

where R_{Baxter} denotes the R-matrix of an "effective" Baxter model and g_{1R}, g_{2R}, g_{1L}, g_{2L} are 2×2 matrices. The sixteen homogeneous parameters of the sixteen-vertex model are thus decomposed into four homogeneous parameters of an "effective" Baxter model, and four times

[7] Finding, for a given R-matrix of the sixteen vertex model, the elements of this decomposition, namely R_{Baxter} and g_{1R}, g_{2R}, g_{1L}, g_{2L}, is an extremely difficult process that will not be detailed here. Conversely, one can show easily that the matrices of the form (22) *span the whole space* of 4×4 matrices.

three parameters (four homogeneous parameters) of the various 2×2 matrices: g_{1R}, g_{2R}, g_{1L}, g_{2L}. Let us note, on the sixteen-vertex model [10], that a remarkable $sl_2 \times sl_2 \times sl_2 \times sl_2$ symmetry exists which generalizes the well-known weak-graph "gauge" symmetry [30]:

$$\begin{aligned} \widehat{K}^2(g_{1L}^{-1} \otimes g_{2L}^{-1} \cdot R_{sixteen} \cdot g_{1R} \otimes g_{2R}) = \\ = \quad g_{1L}^{-1} \otimes g_{2L}^{-1} \cdot \widehat{K}^2(R_{sixteen}) \cdot g_{1R} \otimes g_{2R} \end{aligned} \tag{23}$$

These symmetries are symmetries of the propagation curve (17).

Using this very decomposition (22), and the previous symmetry relation (23), one actually gets:

$$\widehat{K}^2(R_{sixteen}) = \quad g_{1L}^{-1} \otimes g_{2L}^{-1} \cdot \widehat{K}^2(R_{Baxter}) \cdot g_{1R} \otimes g_{2R} \tag{24}$$

The matrices g_{1R}, g_{2R}, g_{1L}, g_{2L} of decomposition (22) can thus be seen as *constants of motion* of the iteration of $\widehat{K}^2$.

If R_{Baxter} belongs to a "special" manifold, or algebraic variety, then $R_{sixteen}$, given by (22), will *also belong* to a "special" manifold, or algebraic variety: for instance, if R_{Baxter} belongs to a *finite order* algebraic variety for the iteration of $\widehat{K}^2$, namely $\widehat{K}^{2\,N}(R_{Baxter}) = \eta \cdot R_{Baxter}$, then $R_{sixteen}$ will also belong to a finite order algebraic variety for the iteration of $\widehat{K}^2$: $\widehat{K}^{2\,N}(R_{sixteen}) = \eta \cdot R_{sixteen}$. If R_{Baxter} belongs to a *critical variety* then $R_{sixteen}$ given by (22) should also belong to a critical variety. This last result does not come from the fact that g_{1L}, g_{2L}, g_{1R}, g_{2R} are symmetries of the partition function (they are not, except in the "gauge" case [30]: $g_{1L} = g_{1R}$ with $g_{2L} = g_{2R}$): they are symmetries of $\widehat{K}^2$ which is a symmetry of the critical manifolds. Therefore they are symmetries of the phase diagram (critical manifolds, ...) even if they are not symmetries of the partition function.

A decomposition, like (22), is closely associated to the *parametrization of the sixteen-vertex model* in terms of *elliptic curves* [10]: given R_{Baxter}, g_{1L}, g_{2L}, g_{1R} and g_{2R}, one can easily deduce $R_{sixteen}$. Conversely, given $R_{sixteen}$, it is extremely difficult to get R_{Baxter}, g_{1L}, g_{2L}, g_{1R} and g_{2R}, however, and remarkably, it is quite simple to get R_{Baxter}. Since g_{1L}, g_{1R}, g_{2L}, g_{2R} are $\widehat{K}^2$-invariants, one can try to relate, directly, the "$\widehat{K}^2$-effective" covariants J_x, J_y and J_z with the $\widehat{K}^2$-invariants related to the recursion on the x_n's or the q_n's, namely ρ, μ, ν, or $\kappa = 4 \cdot (\nu + \mu^2)/\rho$. In terms of these well-suited algebraic covariants, the previous parameters read:

$$\rho = 4\,J_z^2\,J_x^2\,J_y^2\,, \qquad \mu = -2\cdot(J_z^2J_x^2+J_z^2J_y^2+J_x^2J_y^2)\,, \tag{25}$$
$$\kappa = \frac{4\cdot\nu + \mu^2}{\rho} = 4\cdot(J_z^2 + J_x^2 + J_y^2)$$
$$\nu = -(J_z^2\,J_x^2 + J_z^2J_y^2 + J_x^2J_y^2)^2 + 4\cdot(J_z^2 + J_x^2 + J_y^2)\cdot J_z^2J_x^2J_y^2$$

Since these expressions are symmetric polynomials of $J_x^2,\ J_y^2,\ J_z^2$, it is easy to see that the $\widehat{K}^2$-effective covariants $J_x,\ J_y,\ J_z$ can be straightforwardly obtained[8] from a cubic polynomial $P(u)$:

$$P(u) = 4\cdot u^3 - \kappa\cdot u^2 - 2\cdot\mu\cdot u - \rho = \tag{26}$$
$$= 4\cdot(u-J_x^2)\cdot(u-J_y^2)\cdot(u-J_z^2)$$

This is remarkable, because trying to get the "effective" $J_x,\ J_y,\ J_z$, by brute-force eliminations from (22), yields huge calculations.

4 The modular invariant j

For every elliptic curve $\mathcal{E}$ with a singled-out point P_0 on this elliptic curve, there is a closed immersion $\mathcal{E} \longrightarrow \mathbf{CP_2}$ such that the image of this curve has the following simple form:

$$y^2 = x\cdot(x-1)\cdot(x-\lambda) \tag{27}$$

The parameter λ is called the λ *elliptic modulus* [31]. One can also recall the well-known canonical Weierstrass form [31]:

$$y^2 = 4\,x^3 - g_2\,x - g_3 \tag{28}$$

The j-invariant [19, 31] reads alternatively:

$$j = 256\cdot\frac{(1-\lambda+\lambda^2)^3}{\lambda^2\,(1-\lambda)^2} = 1728\cdot\frac{g_2^3}{g_2^3 - 27\,g_3^2} \tag{29}$$

The j-invariant *classifies elliptic curves up to isomorphisms.* It is clear, by direct computation, that $j(\lambda) = j(1/\lambda)$ and $j(\lambda) = j(1-\lambda)$. A

[8] In fact, one only gets, from (26), the *squares* of the $J_x,\ J_y,\ J_z$, but the critical manifold, as well as the finite order conditions (see below), only depend on $J_x^2,\ J_y^2,\ J_z^2$.

permutation group Σ_3 is generated by $\lambda \longrightarrow 1/\lambda$ and $\lambda \longrightarrow 1-\lambda$. The orbits of λ under the action of Σ_3 read:

$$\lambda\,, \qquad \frac{1}{\lambda}\,, \qquad 1-\lambda\,, \qquad \frac{1}{1-\lambda}\,, \qquad \frac{\lambda}{1-\lambda}\,, \qquad \frac{1-\lambda}{\lambda} \qquad (30)$$

We note that j is a rational function of degree 6 of the λ elliptic modulus (i.e. $\lambda \longrightarrow j$ defines a finite morphism $\mathbf{CP_1} \longrightarrow \mathbf{CP_1}$ of degree 6). Furthermore this is a *Galois covering*, with Galois group Σ_3 under the action described above: $j(\lambda) = j(\lambda')$ if, and only if, λ and λ' differ by an element of Σ_3.

The group $Aut(\mathcal{E}, P_0)$, of automorphisms of $\mathcal{E}$ leaving P_0 fixed, is not generic when λ is equal to one of the other expressions in (30). This happens for the following singled-out values of λ:

$$\begin{aligned}
&\lambda \,=\, \frac{1}{2}\,, \quad \text{or:} \quad \lambda \,=\, 2\,, \quad \text{or:} \quad \lambda \,=\, -1 \\
&\quad <=> \quad j \,=\, 1728 \quad <=> \quad g_3 \,=\, 0 \qquad (31)\\
&\lambda \,=\, -\omega\,, \quad \text{or:} \quad \lambda \,=\, -\omega^2\,, \quad \text{with:} \quad 1+\omega+\omega^2 \,=\, 0 \\
&\quad <=> \quad j \,=\, 0 \quad <=> \quad g_2 \,=\, 0
\end{aligned}$$

The Fermat curve, $x^3+y^3 \,=\, z^3$, actually corresponds to the last situation: $\lambda \,=\, -\omega$, $j \,=\, 0$ (writing $x \,=\, -1/3+z$ maps this Fermat curve onto a Weierstrass form with $g_2 \,=\, 0$). The other singled-out values of λ are $\lambda \,=\, 0$ and $\lambda \,=\, 1$, for which $j \,=\, \infty$, and for which the elliptic curve degenerates into a *rational curve* ($g_2^3 \,-\, 27\, g_3^2 \,=\, 0$).

Remark: A Weierstrass form (27), or (28), is known to have an elliptic parametrization in terms of Weierstrass elliptic functions $\wp(z)$. Let us recall the simple biquadratic (19), and its simple elliptic sinus parameterization (21), for which the action of the birational transformations $\widehat{K}$ is very simply represented as a simple shift $u \longrightarrow u \pm \eta$. In contrast with the biquadratic (19), one sees that the Weierstrass representation of elliptic curves, is certainly well-suited to describe the "moduli space" of the curve (the j-invariant), but it is unfortunately *totally blind to a crucial discrete symmetry of our integrable models*, namely the birational transformations $\widehat{K}$, or equivalently, the shifts on the spectral parameter: the fact that a point P_0 is singled-out simplifies the analysis of the moduli space, "quicking out" the automorphisms corresponding to the translations of the spectral parameter, but it is "too universal" to enable a description of the infinite set of birational transformations $\widehat{K}$ closely related to the group of rational points of the elliptic curve (see below). *One purpose of this paper is to promote a representation of the elliptic curves in term of the biquadratic* (19), rather than a Weierstrass form (27), since (19) also encapsulates, very simply, the fundamental j-invariant describing the moduli space, gives a crystal clear representation of the Σ_3 Galois covering, and, especially, does provide a simple description of the infinite set of birational transformations $\widehat{K}$, as a shift (see section (5.2)).

The modular invariant j is of basic importance because it can be shown that every modular function is expressible as a *rational* function of j. The expansion of j, in terms of the nome of the elliptic functions $q = exp(i\pi\tau)$, reads:

$$j(q) \,=\, 1728 \cdot J(q) \,=\, \sum_{n=-2}^{\infty} j_n \cdot q^{2\,n} \,=$$
$$= \eta(\tau)^{-24} \cdot \Bigl(1 + 240 \sum_{n=1}^{\infty} \frac{n^3\, exp(2\pi i n\tau)}{1 - exp(2\pi i n\tau)}\Bigr)^3$$
$$= \,\frac{1}{q^2} \,+ 744 \,+ 196884\, q^2 \,+\, 21493760\, q^4 \,+\, +864299970\, q^6$$
$$+ 20245856256\, q^8 \,+\, \cdots$$

where $\eta(\tau)$ is the Dedekind eta function. Peterson and Rademacher [32, 33] have derived the following asymptotic formula for the Fourier coefficients j_n in (32):

$$j_n \,=\, 2^{-1/2}\, n^{-3/4}\, exp(4\,\pi\, n^{1/2})\,, \qquad \text{as:} \qquad n \longrightarrow \infty \tag{32}$$

Remark: The j-invariant is sometimes defined in the litterature [19] by $j \,=\, 4/27 \cdot (1-\lambda+\lambda^2)^3/\lambda^2/(1-\lambda)^2$, the factor $4/27$ replacing the factor 256 in (29). This factor is in fact quite arbitrary: the $4/27$ is such that the singled-out values for the λ elliptic modulus, $\lambda \,=\, 2\,,\ 1/2$, actually correspond to $j \,=\, 1$, while the 256 choice corresponds to a more mathematical choice [31] which makes things work in characteristic 2, despite appearances to the contrary !

It turns out that the j-function is also important in the classification theorems for finite simple groups[9] and that the factors of the orders of

[9] It has been shown by Thompson [34] that there exists a remarkable connection

the sporadic groups are also related. Seen as a function of the nome q, the j-function is a meromorphic function of the upper-half of the q-complex plane. It is invariant with respect to the special linear group $SL(2,\, Z)$. Two cases of *complex multiplications* are well-known: $\tau\, =\, i$ corresponding to $g_3\, =\, 0$, and $\tau\, =\, \omega$ (where $\omega^3 = 1$) corresponding to $g_2\, =\, 0$. Even though there is a good criterion for complex multiplication in terms of τ, the connection between τ and j is not easy to compute. If we are given a curve by its equation in $\mathbf{CP_2}$ or its j-invariant, it is not easy to tell whether it has complex multiplication or not [35, 36]. Along this line [37], for some special values of the nome q, the j-invariant $j(q)$ can be an *algebraic number*, sometimes a *rational number* and *even an integer*[10]. The determination of j as an algebraic integer in the quadratic field $\mathbf{Q}(j)$ is discussed in Greenhill [38], Gross and Zaiger [39] and Dorman [40]. If one considers the *Heegner's numbers*[11], one gets [41, 42] the following nine remarkable integer values for j:

$$\begin{aligned}
&j(-e^{-\pi\,\sqrt{3}})\,=\,0^3\,,\quad j(-e^{-\pi})\,=\,12^3\,,\quad j(-e^{-\pi\,\sqrt{7}})\,=\,-15^3\,,\\
&j(e^{-\pi\,\sqrt{8}})\,=\,20^3\,,\quad j(-e^{-\pi\,\sqrt{11}})\,=\,-32^3\,,\quad j(-e^{-\pi\,\sqrt{19}})\,=\,-96^3\,,\\
&j(-e^{-\pi\,\sqrt{43}})\,=\,-960^3\,,\quad j(-e^{-\pi\,\sqrt{67}})\,=\,-5280^3\\
&j(-e^{-\pi\,\sqrt{163}})\,=\,-640320^3
\end{aligned}$$

For elliptic curves over the rationals, Mordell proved that there are finite number of integral solutions. The Mordell-Weil Theorem says that the *group of rational points* [23] of an elliptic curve over $\mathbf{Q}$, is finitely generated. Actually, a remarkable connection between rational elliptic curves and modular forms is given by the Taniyama-Shimura conjecture, which states that any rational elliptic curve is a modular form in disguise[12]. In the early 1960's B. Birch and H.P.F. Swinnerton-Dyer conjectured that if a given elliptic curve has an *infinite number of rational points* then the associated L-function has value 0 at a certain fixed point. In 1976

between the coefficients j_n and the degrees of the irreducible characters of the Fisher-Griess "monster group".

[10]The latter result is the end result of a massive theory of complex multiplication and the first step of Kronecker's so-called "Jugendtraum".

[11]The Heegner's numbers have a number of fascinating connections with amazing results in prime number theory. Beyond connection (33) between the j-function, e, π and algebraic integers, they also explain why Euler's prime generating polynomial $n^2 - n + 41$ is so surprisingly good at producing prime numbers.

[12]This result was the one proved by Andrew Wiles in his celebrated proof of Fermat's Last Theorem.

Coates and Wiles showed[13] that elliptic curves, with complex multiplication, having an infinite number of rational points, have L-function which are zero at the relevant fixed points (Coates-Wiles Theorem).

Let us show the remarkable situations for which the j-invariant *becomes a rational number*, do play a relevant role in lattice statistical mechanics.

4.1 The modular invariant j for the two-dimensional Ising model

Despite the celebrated Onsager's solution for the partition function of the two-dimensional Ising model, and the remarkably simple expression of its spontaneous magnetization, in term of the modulus k of the elliptic functions parametrising this simple free-fermion model, or, in term of the previous λ elliptic modulus:

$$M \, = \, (1 \, - \, k^2)^{1/8} \, = \, (1 \, - \, \lambda)^{1/8} \, , \qquad \text{where:} \qquad \lambda \, = \, k^2 \qquad (33)$$

there exists no closed formula for the susceptibility [43]. For the anisotropic square lattice, the previous modulus k reads $k \, = \, sh(2\,K_1) \cdot sh(2\,K_2)$, where K_1 and K_2 correspond to the two nearest neighbor coupling constants of a square lattice [20]. For two-dimensional Ising models on other lattices (triangular, honeycomb, checkerboard, ...) the spontaneous magnetization has the same expression (33) where k, or λ, are more involved expressions (of the nearest neighbor coupling constants), but are still the modulus of the elliptic functions parametrising the models [20]. An exact, but formal, expression of the susceptibility as an infinite sum of integrals over n variables was derived some years ago by Wu et al [44]. However the relative "intractability of the integrals", appearing there, has impeded progress in clarifying the analyticity properties of the susceptibility as a function of the parameters of the model. Let us consider an anisotropic square lattice, namely $\lambda \, = \, k^2 \, = \, (sh(2\,K_1) \cdot sh(2\,K_2))^2$. Wu et al have reduced the susceptibility χ to an infinite sum of $2\,n+1$ "particle contributions", which reads, in the isotropic limit $sh(2\,K_1) \, = \, sh(2\,K_2) \, = \, s$:

$$\beta^{-1} \cdot \chi \, = \, \sum_{n\,=0}^{\infty} \chi^{(2n+1)} \, = \, \frac{(1-s^4)^{1/4}}{s} \cdot \sum_{n\,=0}^{\infty} \widehat{\chi}^{(2n+1)} \qquad \text{for:} \quad s < 1 \; (34)$$

where $\widehat{\chi}^{(2n+1)}$ are integrals over $2n+1$ "angles". Some extensive calculations have, however, been performed by Nickel [45, 46] on these successive terms providing expansions of these $\widehat{\chi}^{(2n+1)}$ in s (up to order 112

[13] V. Kolyavagin has extended this result to modular curves.

for $\widehat{\chi}^{(3)}$), and these expansions have been extended by Orrick et al [43]. From the examination of $\widehat{\chi}^{(3)}$ as a double integral, one can deduce that it is related to a hyperelliptic function, and Nickel [45, 46] remarked that, beyond the known ferromagnetic and antiferromagnetic critical points, $s = 1$ and $s = -1$, $\widehat{\chi}^3$ has (at least ...) the following "unphysical" singularities: $s = \pm i$, $s = exp(2i\pi/3)$ (that is $s^2 = -1$, $1+s+s^2 = 0$), *but also* $s = (1 \pm i\sqrt{15})/4$. As far as elliptic parametrization of a model is concerned, one knows that $\lambda = \pm 1$, $\lambda = -\omega$ ($\omega^3 = 1$), and also $\lambda = 2$ and $\lambda = 1/2$, are singled out (see (31)), and are, thus, natural "candidates" for singular points of $\widehat{\chi}^{(3)}$. However, if one writes the λ elliptic modulus as a function of s, one sees that these "unphysical" singularities do not correspond to such singled out values. Let us write the j-invariant as a function of s. It reads:

$$j \, = \, 256 \cdot \frac{\left(s^8 + 1 - s^4\right)^3}{s^8 \left(s^4 \, - 1\right)^2} \qquad (35)$$

One easily finds that $s = exp(2i\pi/3)$ and $s = (1 \pm i\sqrt{15})/4$ *actually correspond to the following singled-out rational values* for j: $s = exp(2i\pi/3) \longrightarrow j = 2048/3$, $s = (1 \pm i\sqrt{15})/4 \longrightarrow j = -1/15$.

Therefore one actually sees that the singular loci for the modulus of elliptic curve, and the remarkable situations for which the j-invariant *becomes a rational number*, do play an important role for a model which is the paradigm of exactly solvable models, the two-dimensional Ising model. This is a general situation: the j-invariant *must* play an important role to describe the analytical properties of lattice models, since it "encapsulates" *all the symmetries* of the parameter space of the model. This will be seen, in the next section, with the example of a quite general model depending on sixteen homogeneous parameters, and having the Ising model and the Baxter model as a subcase, namely the sixteen vertex model [20].

4.2 The modular invariant j for the sixteen-vertex model

The elliptic curves corresponding to the orbits of $\widehat{K}^2$ in the parameter space of the sixteen vertex model, as well as the two biquadratics (18), together with the elliptic curves associated to the factorization analysis of the previous section (like (10), (11) or (13)), share the *same* modular invariant j, (also called Klein's absolute invariant, or "hauptmodul", or j-invariant [47], ...) which can be simply written, for the sixteen-vertex model, in terms of the "effective" $J_x^2,\ J_y^2,\ J_z^2$ deduced from (26) (using of course (14)):

$$j \;=\; 256 \cdot \frac{(J_x^4 + J_y^4 + J_z^4 - J_z^2 J_y^2 - J_z^2 J_x^2 - J_y^2 J_x^2)^3}{\left(J_y^2 - J_x^2\right)^2 (J_z^2 - J_x^2)^2 \left(J_z^2 - J_y^2\right)^2} \tag{36}$$

This modular invariant j can also be written (see (29)) $j \;=\; 256 \cdot (1 - \lambda + \lambda^2)^3/\lambda^2/(1-\lambda)^2$, or $j \;=\; 1728 \cdot g_2^3/(g_2^3 \,- 27\, g_3^2)$, where the λ elliptic modulus is given by:

$$\lambda = \frac{J_z^2 - J_y^2}{J_x^2 \,-\, J_z^2} \tag{37}$$

and where g_2, and g_3, correspond to a reduction of the elliptic curve into a Weierstrass canonical form (28): $y^2 \;=\; 4\,x^3 \,- g_2\, x \,- g_3$. Recalling the Σ_3 Galois group invariance of j, that is the invariance under the change of the elliptic modulus λ into $1 -\, \lambda$, or $1/\lambda$, and their combinaisons (see (30)), one sees easily that these homographic transformations (30) on the elliptic modulus λ, are just *homographic representations of the* Σ_3 *group of permutation* of $J_x^2,\ J_y^2,\ J_z^2$. The expressions of $g_2,\ g_3$, and the discriminant $g_2^3 \,- 27\, g_3^2$ (see (28)), read respectively:

$$\begin{aligned}
g_2 \;&=\; 12 \cdot (J_x^4 + J_y^4 + J_z^4 - (J_x^2\, J_y^2 + J_x^2\, J_z^2 + J_y^2\, J_z^2))\,, \\
g_3 \;&=\; 4 \cdot (J_x^2 - 2\, J_y^2 + J_z^2) \cdot (J_x^2 + J_y^2 - 2\, J_z^2) \cdot (-2\, J_x^2 + J_y^2 + J_z^2) \\
g_2^3 \,- 27\, g_3^2 \;&=\; 11664 \cdot (J_x^2 - J_y^2)^2 \cdot (J_z^2 - J_y^2)^2 \cdot (J_x^2 - J_z^2)^2
\end{aligned} \tag{38}$$

Based on the classical theory of algebraic invariants (Hilbert's "syzygys", see [48]), an irreducible basis of *algebraic invariants* has been built for the sixteen-vertex model [30, 49, 50, 51, 52, 53, 54, 55, 56, 57]: these algebraic invariants take into account the weak-graph "gauge" (similarity ...) $sl_2 \times sl_2$ symmetries of the sixteen vertex model [58]. The modular invariant j is, of course, invariant under the previous $sl_2 \times sl_2$ similarity symmetries, but it is actually *also invariant* [10] under the much larger set of $sl_2 \times sl_2 \times sl_2 \times sl_2$ symmetries. These last symmetries are not invariances of the partition function, like the previous similarity symmetries [30, 58], but only symmetries of the parameter space (or symmetries of the birational symmetries of the model, "symmetries of second order" ...). Furthermore, this modular invariant is also invariant under the infinite discrete set of birational transformations $\widehat{K}^N$, corresponding to the Baxterisation procedure [24]. It is thus invariant under a *continuous* group of *linear* transformations $sl_2 \times sl_2 \times sl_2 \times sl_2$ and, *in the same time*, under an *infinite discrete* group of *non-linear* transformations. In a forthcoming section (5.2), it will also be seen to be invariant under another remarkable *infinite set of (non-invertible) polynomial transformations* (see (44)).

The modular invariant j thus "encapsulates" *all* the symmetries of the sixteen-vertex model. It is invariant under three *very different sets of symmetries, which seem, at first sight, hardly compatible*: the $sl_2 \times sl_2 \times sl_2 \times sl_2$ continuous "gauge-like-symmetries", the infinite discrete set of birational transformations $\widehat{K}^N$ (closely related to the group of rational points), and finally, an infinite discrete set of homogeneous polynomial (or rational, but not birational) transformations (see (5.2)).

The modular invariant j can also be calculated directly in terms of the μ, ρ and ν (see (14)):

$$j \, = \, -\frac{\left(\mu^4 + 8\,\mu^2\nu + 16\,\nu^2 + 24\,\rho^2\mu\right)^3}{\rho^4\,(\mu^4\nu + 8\,\mu^2\nu^2 + 16\,\nu^3 + \rho^2\mu^3 + 36\,\rho^2\mu\,\nu + 27\,\rho^4)} \qquad (39)$$

For the sixteen vertex model, using (14), the modular invariant j can be calculated exactly and becomes, in terms of the sixteen homogeneous parameters of the model, the ratio of two "huge" homogeneous polynomials that will not be written here.

5 Biquadratic representation of elliptic curves and polynomial representation of natural integers.

5.1 Biquadratic (19) versus biquadratic (35), for the Baxter model

The "Baxterisation process"[24] is associated with the iteration of $\widehat{K}$, or, rather, $\widehat{K}^2$. Since, as far as $\widehat{K}^2$ is concerned, one can reduce a sixteen-vertex model to an "effective" Baxter model, one can try to revisit, directly, the relation between the biquadratic (13) and the "propagation curve" (18) for the Baxter model. For the Baxter model, relation (13) becomes the biquadratic:

$$\begin{aligned} q_n^2\, q_{n+1}^2 \, &- \, 2\,(J_z^2 J_x^2 + J_z^2 J_y^2 + J_x^2 J_y^2)\, q_n\, q_{n+1} \, + \, 4 J_z^2\, J_x^2\, J_y^2 \, \cdot (q_n + q_{n+1}) \\ &+ (J_z^2\, J_x^2 \, + J_z^2 J_y^2 \, + J_x^2 J_y^2)^2 \, - \, 4 \cdot (J_z^2 \, + J_x^2 \, + \, J_y^2) \cdot J_z^2 J_x^2 J_y^2 = 0 \quad (40) \end{aligned}$$

which should be compared with the "propagation" biquadratic (18) of the Baxter model [8, 25]:

$$(J_x \, - \, J_y)\,(p_n^2\, p_{n+1}^2 \, + 1) \, - \, (J_x \, + \, J_y)\,(p_n^2 + p_{n+1}^2) \, + \, 4\, J_z\, p_n\, p_{n+1}$$

$$= (p_n^2 - 1)(p_{n+1}^2 - 1) J_x - (p_n^2 + 1)(p_{n+1}^2 + 1) J_y + 4 J_z p_n p_{n+1}$$
$$= \Gamma_1(p_n, p_{n+1}) = 0 \qquad (41)$$

Let us recall that the "propagation" curve (19), has the simple elliptic parameterization (21), namely $p_n = sn(u_n, k)$, $p_{n+1} = sn(u_{n+1}, k)$ whith $u_{n+1} = u_n \pm \eta$, where $sn(u,k)$ denotes the elliptic sinus of modulus k and η denotes some "shift". The modulus [59] k is equivalent to the modulus $\lambda = k^2$ which has the simple expression (37) in terms of J_x^2, J_y^2 and J_z^2.

At first sight, it seems that one has two different elliptic curves (biquadratics associated with the Baxter model), namely (40) which is symmetric under permutations of J_x^2, J_y^2 and J_z^2, and (41) which *breaks this* Σ_3 *permutation symmetry.* It can, however, be seen that they are actually equivalent (up to involved birational transformations). Actually, let us also consider the same "propagation curve" (41), but now between p_{n+1} and p_{n+2}, and let us eliminate p_{n+1} between these two algebraic curves. One gets, after the factorization of $(p_n - p_{n+1})^2$ a biquadratic relation between p_n and p_{n+2} of the *same form* as (41):

$$2 J_z^2 \cdot (J_y^2 - J_x^2) \cdot (p_n^2 p_{n+2}^2 + 1) + 2 J_x^2 J_y^2 \cdot (p_n^2 + p_{n+2}^2)$$
$$+ 4 \cdot (J_x^2 J_y^2 - J_z^2 J_x^2 - J_z^2 J_y^2) \cdot p_n \cdot p_{n+2} =$$
$$(p_n^2 - 1)(p_{n+2}^2 - 1) J_x^{(2)} - (p_n^2 + 1)(p_{n+2}^2 + 1) J_y^{(2)} + 4 p_n p_{n+2} J_z^{(2)}$$
$$= \Gamma_2(p_n, p_{n+2}) = 0 \qquad (42)$$

where $J_x^{(2)}$, $J_y^{(2)}$ and $J_z^{(2)}$ are given below (see (44)). The two biquadratic curves (40) and (42) are, in fact, (birationally) equivalent, their shift η and modular invariant [59] being equal. Actually, one can find directly an homographic transformation

$$q_n = \frac{\alpha \cdot p_n + \beta}{\gamma \cdot p_n + \delta}, \qquad q_{n+1} = \frac{\alpha \cdot p_{n+2} + \beta}{\gamma \cdot p_{n+2} + \delta} \qquad (43)$$

which maps (40) onto (42), the parameters α, ... δ of the homographic transformation (43) being quite involved. Of course, similarly, the biquadratic (42) is (birationally) equivalent to *five other* equivalent biquadratics deduced from (42) by permutations of J_x, J_y and J_z.

5.2 Polynomial representations of the multiplication of the shift by an integer.

The previously described elimination of p_{n+1}, changing Γ_1 into Γ_2, amounts to eliminating u_{n+1} between $u_n \longrightarrow u_{n+1} = u_n \pm \eta$ and

$u_{n+1} \longrightarrow u_{n+2} = u_{n+1} \pm \eta$ thus getting $u_n \longrightarrow u_{n+2} = u_n \pm 2 \cdot \eta$, together with two times $u_n \longrightarrow u_{n+2} = u_n$. Considering the coefficients of the biquadratic (42) one thus gets, very simply, a *polynomial representation* of the shift doubling $\eta \longrightarrow 2 \cdot \eta$:

$$\begin{array}{lcl} J_x & \longrightarrow & J_x^{(2)} = -J_z^2\, J_x^2 + J_z^2\, J_y^2 - J_x^2\, J_y^2 \\ J_y & \longrightarrow & J_y^{(2)} = J_z^2\, J_x^2 - J_z^2\, J_y^2 - J_x^2\, J_y^2 \\ J_z & \longrightarrow & J_z^{(2)} = -J_z^2\, J_x^2 - J_z^2\, J_y^2 + J_x^2\, J_y^2 \end{array} \tag{44}$$

The modulus (37) is (as it should), *invariant* by (44), which represents the shift doubling transformation[14]. Of course there is nothing specific with the shift doubling: similar calculations can be performed to get *polynomial representations* of $\eta \longrightarrow M \cdot \eta$, for *any integer* M. Actually, it can be seen that the multiplication of the shift by a *prime* number, $N \neq 2$, has the following *polynomial representation* $(J_x,\, J_y,\, J_z) \rightarrow (J_x^{(N)},\, J_y^{(N)},\, J_z^{(N)})$:

$$\begin{array}{lcl} J_x^{(N)} & = & J_x \cdot P_x^{(N)}(J_x,\, J_y,\, J_z) \\ J_y^{(N)} & = & J_y \cdot P_y^{(N)}(J_x,\, J_y,\, J_z) \;=\; J_y \cdot P_x^{(N)}(J_y,\, J_z,\, J_x) \\ J_z^{(N)} & = & J_z \cdot P_z^{(N)}(J_x,\, J_y,\, J_z) \;=\; J_z \cdot P_x^{(N)}(J_z,\, J_x,\, J_y) \end{array} \tag{45}$$

where the $P_x^{(N)}(J_x,\, J_y,\, J_z)$'s (and thus $P_y^{(N)}(J_x, J_y, J_z)$, $P_z^{(N)}(J_x, J_y, J_z)$) are polynomials of J_x^2, J_y^2 and J_z^2. For instance one has the following *polynomial representation* $(J_x,\, J_y,\, J_z) \rightarrow (J_x^{(3)},\, J_y^{(3)},\, J_z^{(3)})$ of the multiplication of the shift by three:

$$\begin{array}{l} J_x^{(3)} = J_x \cdot (-2\, J_z^2 J_y^2 J_x^4 - 3\, J_y^4 J_z^4 + 2\, J_y^2 J_z^4 J_x^2 + J_y^4 J_x^4 + 2\, J_y^4 J_z^2 J_x^2 + J_z^4 J_x^4) \\ J_y^{(3)} = J_y \cdot (2\, J_z^2 J_y^2 J_x^4 - 3\, J_z^4 J_x^4 + J_y^4 J_x^4 - 2\, J_y^4 J_z^2 J_x^2 + J_y^4 J_z^4 + 2\, J_y^2 J_z^4 J_x^2) \\ J_z^{(3)} = J_z \cdot (J_y^4 J_z^4 + 2\, J_y^4 J_z^2 J_x^2 - 3\, J_y^4 J_x^4 - 2\, J_y^2 J_z^4 J_x^2 + 2\, J_z^2 J_y^2 J_x^4 + J_z^4 J_x^4) \end{array} \tag{46}$$

The multiplication of the shift by three (46), can be obtained using the previous elimination procedure, namely eliminating y between $\Gamma_2(x,\, y)$ and $\Gamma_1(y,\, z)$ (or equivalently eliminating y between $\Gamma_1(x,\, y)$ and $\Gamma_2(y,\, z)$), thus yielding a resultant which factorizes into two biquadratics of the same form as the two previous ones, namely $\Gamma_1(x,\, z)$

[14] A general calculation, corresponding to eliminations between two biquadratics Γ_1 of same modulus (37), but different shifts η and η' can also be performed. It will be given elsewhere.

and $\Gamma_3(x,\, z)$:

$$(x^2\, -1)\cdot(z^2\, -1)\cdot J_x^{(3)} - (x^2\, +1)\cdot(z^2\, +1)\cdot J_y^{(3)} +\; 4\cdot J_z^{(3)}\cdot x\cdot z$$
$$=\; \Gamma_3(x,\, z) =\; 0 \qquad (47)$$

where $J_x^{(3)}$, $J_y^{(3)}$ and $J_z^{(3)}$ are polynomials in J_x, J_y and J_z . This provides a *polynomial representation* $(J_x,\, J_y,\, J_z) \,\rightarrow\, (J_x^{(3)},\, J_y^{(3)},\, J_z^{(3)})$ of the multiplication of the shift by three. This polynomial representation is of the form (45), where

$$P_x^{(3)} = -2\, J_z^2\, J_y^2\, J_x^4 - 3\, J_y^4\, J_z^4 + 2\, J_y^2\, J_z^4\, J_x^2 + J_y^4\, J_x^4 + 2\, J_y^4\, J_z^2\, J_x^2 + J_z^4\, J_x^4 \qquad (48)$$

The modulus (37) is (as it should) invariant by the polynomial representation (48) of the multiplication of the shift by three (48).

The multiplication of the shift by four has the following polynomial representation $(J_x,\, J_y,\, J_z) \,\rightarrow\, (J_x^{(4)},\, J_y^{(4)},\, J_z^{(4)})$:

$$J_x^{(4)}(J_x, J_y, J_z) = -4\, J_x^6\, J_y^6\, J_z^4 - 6\, J_z^8\, J_y^4\, J_x^4 + 4\, J_z^8\, J_y^6\, J_x^2 + 4\, J_z^8\, J_y^2\, J_x^6 \qquad (49)$$
$$+4\, J_z^6\, J_y^6\, J_x^4 - 4\, J_z^2\, J_x^8\, J_y^6 + 10\, J_z^4\, J_x^8\, J_y^4 - 4\, J_z^6\, J_x^8\, J_y^2 - J_x^8\, J_y^8 - J_z^8\, J_y^8$$
$$-J_z^8\, J_x^8 + 4\, J_z^2\, J_y^8\, J_x^6 - 4\, J_z^6\, J_x^6\, J_y^4 + 4\, J_z^6\, J_y^8\, J_x^2 - 6\, J_z^4\, J_y^8\, J_x^4$$
$$J_y^{(4)}(J_x, J_y, J_z) = J_x^{(4)}(J_y, J_z, J_x)\,, \quad J_z^{(4)}(J_x, J_y, J_z) = J_x^{(4)}(J_z, J_x, J_y)$$

which can be obtained, either by the elimination of y between $\Gamma_2(x,\, y)$ and $\Gamma_2(y,\, z)$ (and extracting a $(x\,-\,z)^2$ factor in the resultant), or, equivalently, by the elimination of y between $\Gamma_1(x,\, y)$ and $\Gamma_3(y,\, z)$, or the elimination of y between $\Gamma_3(x,\, y)$ and $\Gamma_1(y,\, z)$ (and extracting a Γ_2 factor in the resultant). Again, one gets $\Gamma_4(x,\, z)$:

$$(x^2\, -1)\cdot(z^2\, -1)\cdot J_x^{(4)} - (x^2\, +1)\cdot(z^2\, +1)\cdot J_y^{(4)} +\; 4\cdot J_z^{(4)}\cdot x\cdot z$$
$$= \Gamma_4(x,\, z) =\; 0 \qquad (50)$$

where $J_x^{(4)}$, $J_y^{(4)}$ and $J_z^{(4)}$ are given above. It can easily be verified that (49) can be obtained directly combining (44) with itself.

The multiplication of the shift by five has a polynomial representation $(J_x,\, J_y,\, J_z) \,\rightarrow\, (J_x^{(5)},\, J_y^{(5)},\, J_z^{(5)})$ of the form (45), where:

$$P_5(J_x,\, J_y,\, J_z) \;=\; 5\, {J_z}^{12}{J_y}^{12} + \left({J_z}^2 - {J_y}^2\right)^6 {J_x}^{12}$$
$$-10\, {J_y}^{10}{J_z}^{10}\left({J_z}^2 + {J_y}^2\right){J_x}^2 + 36\, {J_y}^6{J_z}^6\left({J_z}^2 + {J_y}^2\right)\,({J_z}^2 - {J_y}^2)^2{J_x}^6$$

$$-{J_z}^4{J_y}^4\left(29\,{J_z}^4+54\,{J_z}^2{J_y}^2+29\,{J_y}^4\right)\,({J_z}^2-{J_y}^2)^2{J_x}^8$$
$$+{J_y}^8{J_z}^8\,(4\,J_z\,J_y+3\,{J_z}^2-3\,{J_y}^2)\,(3\,{J_y}^2+4\,J_z\,J_y-3\,{J_z}^2)\,{J_x}^4$$
$$+2\,{J_z}^2{J_y}^2\left({J_y}^2+3\,{J_z}^2\right)\left(3\,{J_y}^2+{J_z}^2\right)\left({J_z}^2+{J_y}^2\right)\left({J_z}^2-{J_y}^2\right)^2{J_x}^{10}$$

The modulus (37) is, again, invariant by this last polynomial representation of the multiplication of the shift by five. One remarks that $P_x^{(5)}(J_x,\, J_y,\, J_z)$ singles out J_x and is invariant under the permutation $J_y \leftrightarrow J_z$ and, similarly, $P_y^{(5)}(J_x,\, J_y,\, J_z)$ singles out J_y and is invariant under the permutation $J_x \leftrightarrow J_z$ and $P_z^{(5)}(J_x,\, J_y,\, J_z)$ singles out J_z and is invariant under the permutation $J_x \leftrightarrow J_y$. One has similar results for the polynomial representation of the multiplication of the shift by $M = 6,\, 7,\, 9,\, 11,\, \cdots$

Let us denote by Γ_N a biquadratic corresponding to $u \rightarrow u \pm N\cdot\eta$:

$$(x^2-1)\cdot(z^2-1)\cdot J_x^{(N)} - (x^2+1)\cdot(z^2+1)\cdot J_y^{(N)} + 4\cdot J_z^{(N)}\cdot x\cdot z$$
$$= \Gamma_N(x,\, z) = 0 \tag{51}$$

In general, it should be noticed that the elimination of y between $\Gamma_M(x,\, y)$ and $\Gamma_{M'}(y,\, z)$, yields a resultant which is factorized into $\Gamma_{(M+M')}(x,\, z)$ and $\Gamma_{(M-M')}(x,\, z)$ (for $M \geq M'$). When seeking for a new $\Gamma_N(x,\, z)$ there may be many $(M,\, M')$ enabling to get $\Gamma_N(x,\, z)$ (that is such that $N = M + M'$). One can verify that all these calculations give, as it should, the *same* result (in agreement with a polynomial representation of $u \rightarrow u \pm M\cdot\eta \pm M'\cdot\eta$ giving $u \pm (M+M')\cdot\eta$ or : $u \pm (M-M')\cdot\eta$. Let us denote T_N these homogeneous polynomial representations of the multiplication of the shift by the natural integer N. In the same spirit, one can verify, for $N = M\cdot M'$ ($N,\, M,\, M'$ natural integers), that:

$$\begin{aligned} T_N(J_x,\, J_y,\, J_z) &= (T_M)^{M'}(J_x,\, J_y,\, J_z) = \\ &= T_M(T_M(T_M(\cdots T_M(J_x,\, J_y,\, J_z)\cdots))) = \\ &= (T_{M'})^M(J_x,\, J_y,\, J_z) = T_{M'}(T_{M'}(T_{M'}(\cdots T_{M'}(J_x,\, J_y,\, J_z)\cdots))) \end{aligned} \tag{52}$$

One can, for instance, easily verify that T_2 and T_3 *commute*, as well as T_2 and T_5. Similarly one can verify, in a brute-force way, that T_3 and T_5 commute. These commutation relations are true for T_N and T_M, for any N and M. One thus has a *polynomial representation of the natural integers together with their multiplication.* One verifies easily that the homogeneous polynomial transformations T_M are all of degree M^2, in J_x, J_y, J_z, for $M = 2, 3, \cdots 11$.

Remark: One can easily verify that the λ elliptic modulus (37) is *actually invariant* by the (polynomial representation of the) *shift doubling* (44), and by (46) the (polynomial representation of the) multiplication of the shift by three, *as well as all the other polynomial representations of the multiplication of the shift by any integer.* The canonical Weierstrass form (28) "encodes" the modular invariant j, or simply the modulus λ. However it is too "universal": a crucial (physical) symmetry, and a crucial information, namely the shift η has been been lost under this "universal" canonical Weierstrass form. We want here to underline a representation of elliptic curves in terms of the *biquadratic curve* (19), which is also very simple, but is *actually such that the action of the group of birational transformations (group of rational points) is crystal clear*, and such that the Σ_3 symmetry of elliptic curves (Galois covering of the j-invariant), namely the permutation group of three elements, is also clear. Furthermore, very simple calculations on this very biquadratic curve (eliminations by resultants) enable to find these polynomial representations ((44), (46), ...), which are fundamental to represent physically (and mathematically) relevant *finite order conditions* (see below).

5.2.1 Finite order conditions and associated algebraic varieties

Let us show that one can deduce the (projective) *finite order* conditions $K^M(R) = \zeta \cdot R$, from the previous polynomial representations. Our motivation is that the corresponding algebraic varieties are "good candidates" for new free-(para?)-fermions, or new equivalents of the integrable chiral Potts model [60]. Completely similar calculations can thus be performed, yielding an infinite set of "good candidates" for (higher genus) star-triangle integrability, enabling, in particular, to recover the higher genus integrable solution of Baxter-Perk-Au-Yang [64]. Of course, for the sixteen-vertex model, we do not expect that one of this infinite set of finite order conditions could yield new Yang-Baxter integrable subcases of the sixteen-vertex model. We just consider the sixteen-vertex model for heuristic reasons.

Since one knows that the (projective) *finite order conditions* of $\widehat{K}^2$ often play a singled-out role for integrability, and, in particular, since one knows [65] that the free-fermion conditions of the asymmetric eight vertex model correspond to $K^4(R) = \zeta \cdot R$, one can, as an exercise, try to systematically write, for the sixteen vertex model, the (projective) finite order conditions $K^{2\,N}(R) = \zeta \cdot R$, with N natural integer.

Let us first give these finite order conditions for the Baxter model[15]. At first sight, writing down the (projective) finite order condition $K^{2\,N}(R) = \zeta \cdot R$, corresponds to write four homogeneous equations on the four homogeneous parameters a, b, c and d, yielding to points in the (projective) parameter space (codimension-three). In fact, the Baxter R-matrices of order two ($K^2(R) = \zeta \cdot R$) correspond to codimension two algebraic varieties. One easily checks that $c = d = 0$ are such matrices. Furthermore, recalling [65] that the free-fermion condition for the XYZ Hamiltonian, $J_z = 0$, corresponds to the finite order (projective) condition $K^4(R) = \zeta \cdot R$, one actually sees that the R-matrices of order four can actually correspond to a *codimension-one* algebraic variety (only *one* algebraic condition on the homogeneous parameters of the model). Recalling the polynomial representation (44) of the shift doubling, one can easily get convinced that $J_z^{(2)} = 0$ should correspond to $K^8(R) = \zeta \cdot R$, also yielding *a only one* algebraic condition on the homogeneous parameters of the model (codimension-one algebraic variety). This can be verified by a straight calculation. This is a general result: *all the finite order conditions* $K^{2\,N}(R) = \zeta \cdot R$ *correspond to codimension-one algebraic varieties*, *expect* $N = 1$. The idea here is the following: K, or $\widehat{K}$, corresponding, with some well-suited spectral parameter, to $\theta \longrightarrow \theta + \eta$, K^2, or $\widehat{K}^2$, must correspond to $\theta \longrightarrow \theta + 2 \cdot \eta$. A finite order condition of order M corresponds to a commensuration of η with a period of the elliptic curves: $\eta = \mathcal{P}/\mathcal{M}$, that is *just one* condition on the parameters of the model. Changing K into K^2 amounts to changing η into $2 \cdot \eta$, or equivalently, changing the order M into $2\,M$. The fact that the finite order conditions $K^{2\,N}(R) = \zeta \cdot R$ correspond to *codimension one* algebraic varieties is, thus, a *consequence of the foliation* of the parameter space in elliptic curves.

More generally, a polynomial condition $C_{2N}(J_x, J_y, J_z) = 0$, corresponding to $K^{2\,N}(R) = \zeta \cdot R$, has to be compatible with the polynomial representations of $\eta \longrightarrow M \cdot \eta$, for any integer M. This compatibility

[15] For this heuristic model it is not necessary to explain, beyond the free-fermion subcase, the usefulness of the finite order conditions any further: these algebraic varieties correspond exactly to the set of RSOS models [66].

is often, in fact, an efficient way to get these finite order conditions. For a prime integer $N \neq 2$ the algebraic varieties $P_x^{(N)}(J_x, J_y, J_z) = 0$, $P_y^{(N)}(J_x, J_y, J_z) = 0$, and $P_z^{(N)}(J_x, J_y, J_z) = 0$ give order $4N$ conditions:

$$K^{4N}(R) = \zeta \cdot R \tag{53}$$

Since the $P^{(N)}$'s (and the $J^{(N)}$'s for N even) are functions of J_x^2, J_y^2 and J_z^2, the order $4N$ conditions, $K^{4N}(R) = \eta \cdot R$, are also functions of the square J_x^2, J_y^2 and J_z^2. One can easily get infinite families of finite order conditions. For instance, iterating the shift doubling (44) (resp. (48)), and using this transformation on $J_z = 0$, one easily gets an *infinite number* of algebraic varieties corresponding to the finite order conditions of order 2^N (resp. 3^N). Combining (44) and (48), one gets straightforwardly the finite order conditions of order $2^N \times 3^M$.

A few miscellaneous examples of finite order conditions $K^N(R) = \zeta \cdot R$ as polynomial relations on J_x, J_y and J_z are given in Appendix A.

5.2.2 Covariance properties of g_2 and g_3

Let us now recall the homogeneous polynomial expressions (38) for g_2 and g_3, in terms of J_x, J_y and J_z, and show that they have remarkable covariance properties with respect to the previous homogeneous polynomial representations. These two (homogeneous) polynomial expressions are changed by (44) into:

$$(g_2\,,\; g_3) \quad \rightarrow \quad (t^4 \cdot g_2\,,\; t^6 \cdot g_3)\,, \quad \text{where:} \quad t \,=\, 2\,J_x\,J_y\,J_z \tag{54}$$

The condition $t = 0$ actually corresponds to the conditions of order four $K^4(R) = \mu \cdot R$. It is remarkable that the cofactor t of g_2 and g_3, corresponding to the shift doubling (44), is *closely related to a finite order condition* $K^4(R) = \mu \cdot R$. Similarly, the polynomial expressions g_2 and g_3 are changed by multiplication of the shift by three (46) into:

$$\begin{aligned}
&(g_2\,,\; g_3) \quad \rightarrow \quad (t^4 \cdot g_2\,,\; t^6 \cdot g_3)\,, \quad \text{where:} \quad t \,=\, t_0 \cdot t_x \cdot t_y \cdot t_z \qquad (55)\\
&t_0 \,=\, (J_x\,J_z + J_y\,J_z + J_x\,J_y)\,, \qquad t_x \,=\, (J_x\,J_z + J_x\,J_y - J_y\,J_z)\\
&t_y \,=\, (J_y\,J_z + J_x\,J_y - J_x\,J_z)\,, \qquad t_z \,=\, (J_x\,J_z + J_y\,J_z - J_x\,J_y)
\end{aligned}$$

Condition $t = 0$ actually corresponds to the conditions of order six $K^6(R) = \mu \cdot R$. Similarly, g_2 and g_3 are changed by the multiplication of the shift by five (see (51) in the following) into:

$$
\begin{aligned}
&(g_2\,,\, g_3) \quad \rightarrow \quad (t^4 \cdot g_2\,,\, t^6 \cdot g_3)\,, \qquad \text{where:} \qquad t \,=\, t_1 \cdot t_2 \cdot t_3 \cdot t_4 \\
&t_1 \,=\, (J_z + J_y)\,(-J_y + J_z)^2\, {J_x}^3 - J_y\, J_z\, (J_z + J_y)^2\, {J_x}^2 \\
&\qquad -{J_y}^2 {J_z}^2\, (J_z + J_y)\, J_x + {J_z}^3 {J_y}^3 \\
&t_2 \,=\, (-J_y + J_z)\,(J_z + J_y)^2\, {J_x}^3 - J_y\, J_z\, (-J_y + J_z)^2\, {J_x}^2 \\
&\qquad -{J_z}^2 {J_y}^2\, (-J_y + J_z)\, J_x + {J_z}^3 {J_y}^3 \\
&t_3 \,=\, -\,(J_z + J_y)\,(-J_y + J_z)^2\, {J_x}^3 - J_y\, J_z\, (J_z + J_y)^2\, {J_x}^2 \\
&\qquad +{J_y}^2 {J_z}^2\, (J_z + J_y)\, J_x + {J_z}^3 {J_y}^3 \\
&t_4 \,=\, -\,(-J_y + J_z)\,(J_z + J_y)^2\, {J_x}^3 - J_y\, J_z\, (-J_y + J_z)^2\, {J_x}^2 \\
&\qquad +{J_z}^2 {J_y}^2\, (-J_y + J_z)\, J_x + {J_z}^3 {J_y}^3
\end{aligned}
\tag{56}
$$

Condition $t = 0$ actually corresponds to the conditions of order ten $K^{10}(R) = \mu \cdot R$.

6 Elliptic parametrization of the four-state chiral Potts model

In view of the fundamental role played by the propagation property on a vertex model, one could have a prejudice that the previous remarkable symmetries, and structures, only exist in the framework of d-dimensional *vertex* models, but are much harder to exhibit on IRF models (interaction-(a)round-a-face models [1, 25]) or even on the simple *spin edge models*. Actually, in the case of q-state spin-edge models, the birational transformation $\widehat{K}$ is *still a discrete symmetry of the star-triangle relations* [7], but it is now the product of the matrix inversion of the $q \times q$ Boltzmann matrix, and of the Hadamard inverse which amounts to inverting all the entries of this matrix [7]. The corresponding homogeneous polynomial transformation K is the product of the homogeneous matrix inversion I (see (5)) of homogeneous degree $q-1$, and of the homogeneous Hadamard inverse H of homogeneous degree q^2-1. Denoting $m_{i,\,j}$ the entries of a $q \times q$ matrix M, the homogeneous Hadamard inverse reads:

$$
H: \qquad m_{i,\,j} \qquad \longrightarrow \qquad \frac{\prod_{k,\,l} m_{k,\,l}}{m_{i,\,j}} \tag{57}
$$

where the product is the product of the q^2 entries of the matrix. Therefore the associated homogeneous transformation $K = H \cdot I$ is of homogeneous degree $(q-1) \times (q^2-1)$. For a four-state spin edge model, the iteration of the homogeneous transformation $K = H \cdot I$ corresponds, for the most general model, to the iteration of a transformation of degree 45, instead of degree 3 for $K = t_1 \cdot I$ for the sixteen vertex model. Consequently, even the numerical iteration is more complicated. Actually this iteration is quite unstable numerically. As a consequence of this higher degree for the birational transformations one considers, the birational transformations $\widehat{K}$ corresponds most of the time to chaotic dynamical systems (*exponential growth of the complexity* of the iteration calculations, instead of a polynomial growth [9, 21, 24, 61, 62, 63]). However, the star-triangle relations are still compatible [6] with these associated homogeneous transformations $K = H \cdot I$, even if the spin-edge model is “much less favorable” to star-triangle integrability.

In this respect, let us recall the *chiral Potts model* [64, 67, 68], which is a generalization of the Ising model where more spin states are allowed. With this spin-edge model one encounters an algebraically much more complex situation: the difficulty with the *integrable chiral Potts model* [64, 67, 68] is that, instead of an elliptic parametrization, one discovers oneself faced with a curve of *higher genus* [60]. An infinite order set of birational symmetries of the curve is incompatible with the higher genus character of these integrable chiral Potts model star-triangle solutions [2, 60]. In fact the chiral Potts model is star-triangle *only when restricted* to a particular algebraic variety (namely $V_4 = 0$ in the following, see (66)), which happens to be a *finite order condition* for $\widehat{K}$, which is therefore compatible with the existence of a finite set of automorphisms on a higher genus algebraic curve [60]. This generic situation is not very favorable for the “quest” of integrable models. In this respect, the *four-state* chiral edge spin Potts model is remarkable since it will be seen to, surprisingly, yield an *elliptic parametrization* in the *whole parameter space of the (anisotropic) model*. Therefore, exact calculations, completely similar to the ones sketched in the previous sections (see (5.2)), can thus be performed, yielding an infinite set of “good candidates” for (higher genus) star-triangle integrability, enabling, in particular, to recover the $V_4 = 0$ (finite order) necessary condition for the higher genus integrable solution of Baxter-Perk-Au-Yang [64].

Let us consider a 4×4 cyclic matrix representing the Boltzmann weights for the four-state chiral edge spin Potts model [67, 68]:

$$M_0 \;=\; \begin{pmatrix} c_0 & c_1 & c_2 & c_3 \\ c_3 & c_0 & c_1 & c_2 \\ c_2 & c_3 & c_0 & c_1 \\ c_1 & c_2 & c_3 & c_0 \end{pmatrix} \qquad (58)$$

For such an edge spin model, the birational transformation K is now obtained as the product of the matrix inversion and the *Hadamard inverse* which amounts to changing all the entries c_i into their inverse $1/c_i$.

One can find two algebraically independent invariants under the matrix inversion and the Hadamard inverse, namely:

$$a \;=\; \frac{c_0\, c_1 \;-\; c_2\, c_3}{c_0\, c_3 \;-\; c_1\, c_2}, \qquad b \;=\; \frac{(c_0\, c_2 \;-\; c_1^2) \cdot (c_0\, c_2 \;-\; c_3^2)}{(c_0\, c_3 \;-\; c_1\, c_2)^2}, \qquad \hbox{or:} \qquad (59)$$

$$a \;=\; \frac{c_0\, c_1 \;-\; c_2\, c_3}{c_0\, c_3 \;-\; c_1\, c_2}, \qquad \hbox{and:} \qquad c \;=\; \frac{(c_0\, c_2 \;-\; c_1^2) \cdot (c_0\, c_2 \;-\; c_3^2)}{(c_0\, c_3 \;-\; c_1\, c_2) \cdot (c_0\, c_1 \;-\; c_2\, c_3)} \;=\; \frac{b}{a}$$

These two algebraic K-invariants *yield a foliation of the parameter space in terms of elliptic curves*. After some (tedious) calculations one finds that the corresponding modular invariant reads (see (29)) $j \;=\; 256\,(1-\lambda+\lambda^2)^3/\lambda^2/(1-\lambda)^2$, where λ is given by one of the six simple expressions:

$$\lambda \;=\; 16\, \frac{(b-a)\, b}{(a+1)^2\,(a-1)^2}, \qquad \qquad \lambda \;=\; 1/16\, \frac{(a^2-1)^2}{(b-a)\, b}$$

$$\lambda \;=\; \frac{(a^2-1)^2}{(-4\,b+1+2\,a+a^2)\,(4\,b+1-2\,a+a^2)}, \qquad (60)$$

$$\lambda \;=\; 16\, \frac{(a-b)\, b}{(-4\,b+1+2\,a+a^2)\,(4\,b+1-2\,a+a^2)}$$

$$\lambda \;=\; 1/16\, \frac{(-4\,b+1+2\,a+a^2)\,(4\,b+1-2\,a+a^2)}{(-b+a)\, b},$$

$$\lambda \;=\; \frac{(-4\,b+1+2\,a+a^2)\,(4\,b+1-2\,a+a^2)}{(a^2-1)^2}$$

In terms of the a and b invariants the modular invariant reads:

$$j \;=\; \frac{(j_{num})^3}{(4\,b-2\,a+1+a^2)^2\,(4\,b-2\,a-1-a^2)^2\, b^2\,(b-a)^2\,(a^2-1)^4} \qquad (61)$$

where: $\quad j_{num} \;=\; 1-4\,a^2+a^8-4\,a^6+6\,a^4-16\,b^2a^4+288\,b^2a^2$
$\qquad -16\,b^2+16\,ba^5-32\,ba^3+16\,ba+256\,b^4-512\,b^3a$

In order to write a (polynomial) representation of the shift doubling of this edge spin model one needs to find some equivalent of J_x, J_y and J_z. One can actually find the equivalent of J_x, J_y and J_z:

$$J_x \, = \, 2\,a\,, \qquad J_y \, = \, 2\cdot(2\,b\, -\, a)\,, \qquad J_z \, = \, a^2\, +\, 1 \qquad (62)$$

From these simple expressions one gets rational representations of shift doubling and the multiplication of the shift by three, four, five ... . The shift doubling reads $(a,\, b) \, \rightarrow \, (a_2,\, b_2)$:

$$a_2 \, = \, {{ba^2+a-b} \over {a^3-ba^2+b}}, \qquad b_2 \, = \, {{a^2\,(a-2\,b)^2} \over {(a^3-ba^2+b)^2}} \qquad \hbox{or:} \qquad (63)$$

$$(a_2,\, b_2) \, = \, ({{a^3-ba^2+b} \over {ba^2+a-b}},\, {{b\,\left(a^2+1\right)^2\,(a-b)} \over {(ba^2+a-b)^2}}) \qquad \hbox{or:}$$

$$(a_2, b_2) \, = \, ({{a^3-ba^2+b} \over {ba^2+a-b}},\, {{a^2\,(2\,b-a)^2} \over {(ba^2+a-b)^2}})$$

The multiplication of the shift by three reads $(a,\, b) \, \rightarrow \, (a_3,\, b_3)$ where:

$$a_3 \, = \, {{(a^4b^2+2\,a^2b^2-3\,b^2-a^5b-2\,a^3b+3\,ba-a^2)\cdot a} \over {b^2-ba+2\,a^2b^2-2\,a^3b-3\,a^4b^2+3\,a^5b-a^6}}, \qquad (64)$$

$$b_3 \, = \, {{(b-a^3+a^2b-a^2-a+2\,ba)^2\,(b-a^3+a^2b+a^2-a-2\,ba)^2\cdot b} \over {(ab-b^2-2\,a^2b^2+2\,a^3b+3\,a^4b^2-3\,a^5b+a^6)^2}}$$

The multiplication of the shift by five $(a,\, b) \, \rightarrow \, (a_5,\, b_5)$ is given in Appendix B. Note that one has a symmetry $a \leftrightarrow 1/a$. Changing $a \rightarrow 1/a$, the ratio b/a being fixed, amounts to changing $(a,\, b) \, \leftrightarrow \, (1/a,\, b/a^2)$. One easily verifies that changing $(a,\, b) \leftrightarrow (1/a,\, b/a^2)$ the a_N's and b_N's are also changed accordingly: $(a_N,\, b_N) \leftrightarrow (1/a_N,\, b_N/a_N^2)$.

Remark: It might be tempting to represent the shift doubling in terms of the two invariants a and λ, since the last invariant λ remains unchanged by the shift doubling. In fact the shift doubling corresponds to $(a,\, \lambda) \, \rightarrow \, (a_2,\, \lambda)$ where a_2, λ and a are related by the algebraic relation:

$$(a_2+1)^2\,(a-1)^4\,(a+1)^4\cdot\lambda \, +16\,a^2\,(a_2\,a^2-1)\,(a^2-\, a_2) \, = \, 0 \qquad (65)$$

$$\hbox{or:} \qquad \lambda \, = \, 16\,{{a^2\,(a_2\,a^2-1)\,(a_2-a^2)} \over {(a_2+1)^2\,(a^2-1)^4}}$$

The λ elliptic modulus being invariant by the shift doubling, one can imagine to represent the shift doubling as a transformation bearing only on variable a, λ being a parameter. One sees that this transformation is not polynomial or even rational: it is algebraic.

The finite order conditions are easy to write from (63) or (64). For instance, the order N conditions ($N = 3, 4, 6, 8, 12$), read respectively:

$$V_3 \, = \, a^2b - a^2 + 2\,ba + b\,, \qquad\qquad V_4 \, = \, a - 2\,b\,, \qquad (66)$$
$$V_6 \, = \, V_6^{(1)} \cdot V_6^{(2)} \cdot V_6^{(3)} \quad \hbox{where:} \quad V_6^{(1)} \, = \, -a^3 + ba^2 - a^2 - a + 2\,ba + b$$
$$V_6^{(2)} \, = \, ba^2 - 2\,ba + b + a^2 \qquad V_6^{(3)} \, = \, -a^3 + ba^2 + a^2 - a - 2\,ba + b$$
$$V_8 \, = \, ba^5 - b^2a^4 - 6\,a^2b^2 - a^4 + 6\,a^3b + ba - b^2$$
$$V_{12} \, = \, a^8b^4 + 6\,a^4b^4 - 4\,a^6b^4 + b^4 - 4\,a^2b^4 + 8\,b^3a^3 - 12\,b^3a^5 - 2\,a^9b^3$$
$$-a^{11}b - 2\,a^9b + a^8 + 8\,a^7b^3 - 2\,b^3a + 2\,a^2b^2 - 2\,b^2a^4 + 16\,b^2a^6 - 2\,a^8b^2$$
$$+2\,a^{10}b^2 - a^3b - 2\,ba^5 - 10\,ba^7$$

Duality symmetry. The (Kramers-Wannier) duality [20] on this model is a transformation of order four [60]:

$$D: \quad (c_0, \, c_1, \, c_2, \, c_3) \quad \longrightarrow \quad \frac{1}{2} \cdot \big(c_0^{(D)}\,, \; c_1^{(D)}\,, \; c_2^{(D)}\,, \; c_3^{(D)}\big) \qquad (67)$$
$$\hbox{where:} \quad c_0^{(D)} \, = \, c_0 + c_1 + c_2 + c_3, \qquad c_1^{(D)} \, = \, c_0 + i\,c_1 - c_2 - i\,c_3,$$
$$c_2^{(D)} \, = \, c_0 - c_1 + c_2 - c_3, \qquad c_3^{(D)} \, = \, c_0 - i\,c_1 - c_2 + i\,c_3$$

which reads on the invariants a and b:

$$(a, \, b) \quad \longrightarrow \quad \Big(\frac{1 + i\,a}{a + i}\,, \quad i \cdot \frac{-4\,b + 1 + 2\,a + a^2}{2\,(a + i)^2}\Big) \qquad (68)$$

which is actually a transformation of order four. The duality commutes with the shift doubling, the multiplication of the shift by three, and, more generally, by the multiplication of the shift by any natural integer. The finite order conditions (66) are left invariant by the duality transformation. Transformation D^2 is simply $c_1 \longleftrightarrow c_3$. The duality (67) preserve *globally* a self-dual line: $c_0 \, = \, c_1 + c_2 + c_3$. Each point of this self-dual line is a fixed point of the duality (67) if one restricts oneself to $c_1 \, = \, c_3$.

In the $c_1 \, = \, c_3$ subcase of the model, the invariant a and d read respectively $a \, = \, 1, \; d \, = \, 0$, and the elliptic parametrization degenerates into a *rational parametrization*: the discriminant $g_2^3 - 27\, g_3^2$ is equal to zero. In this rational limit, $a = 1$, the previous rational representations of the multiplication of the shift by two, three, ... become simple *polynomial* representations:

$$b_2 = (1 - 2\, b)^2\,, \quad b_3 = b \cdot (-3 + 4\, b)^2\,, \quad b_5 = b \cdot (5 + 16\, b^2 - 20\, b)^2\,, \cdots \qquad (69)$$

Introducing the following change of variable:

$$b = \frac{(v + 1/v)^2}{4}$$

one finds that the previous *polynomial* representations (69) of the multiplication of the shift by N, simply read $v \rightarrow v^N$ (or $v \rightarrow v^{(-N)}$).

Terminology problem: From the point of view of discrete dynamical systems a mapping like (44) (or (45), and the mappings corresponding to other shift multiplications), could, at first sight, be called "integrable": the iteration of this (two-dimensional) mapping "densifies" algebraic curves (namely the conic $\lambda = $ *constant*) foliating the whole two-dimensional space, exactly as an integrable mapping does [11, 69]. One can even write *explicit analytical expressions* for the N-th iterate, *for any* integer N. However, this mapping is *not reversible*, the growth of the calculations [61, 62, 63] is exponential (2^N *exponential growth*, $\ln(2)$ topological entropy, ...). In fact this very example of "calculable" chaos is the exact equivalent of the situation encountered with the logistic map, $x \rightarrow \alpha \cdot x \cdot (1 - x)$, for $\alpha = 4$: one does not have a representation of a translation $\theta \rightarrow \theta + N \cdot \eta$, but a representation (see also the Bachet's duplication formula in Appendix C) of the iteration of a multiplication by 2: $\eta \rightarrow 2^N \cdot \eta$.

Another rational representation of the multiplication of the shift. These representations of the multiplication of the shift are just the previous polynomial representations of the multiplication of the shift, given in section (5.2), written in terms of the a and b's. Since J_x, J_y, and J_z are homogeneous parameters one can also, instead of (62), define them by:

$$J_x = 2\,, \qquad J_y = 2 \cdot (2\, \frac{b}{a} - 1) = 2 \cdot (2\, c - 1)\,, \qquad J_z = a + \frac{1}{a} \qquad (70)$$

From (70) it is clear that one has the $a \leftrightarrow 1/a$ symmetry. Let us introduce, instead of the invariants a and b, the invariants $c = b/a$ (see

(59)) and d, where d is defined by:

$$d \,=\, (a - 1/a)^2 \,=\, \frac{(c_1^2 - c_3^2)^2 \cdot (c_2^2 - c_0^2)^2}{(c_0\, c_3 \,-\, c_1\, c_2)^2 \cdot (c_0\, c_1 \,-\, c_2\, c_3)^2} \qquad (71)$$

Using (70), the expressions of g_2 and g_3 (see (38)) become:

$$\begin{aligned}
g_2 \,&=\, 12 \cdot \Bigl(d^2 \,-\, 16\, cd\,(c-1) \,+\, 256\, c^2\,(c-1)^2\Bigr)\,, \\
g_3 \,&=\, -8 \cdot (d - 16\,c + 16\,c^2) \cdot (d + 8\,c - 8\,c^2) \cdot (d + 32\,c - 32\,c^2)\,, \\
g_2^3 - 27\, g_3^2 \,&=\, 2985984 \cdot c^2\, d^2\,(c-1)^2 \cdot (d + 16\,c - 16\,c^2)^2\,, \qquad (72)\\
j \,&=\, \frac{(256\,c^4 - 512\,c^3 + 256\,c^2 \,-\, 16\,c^2\,d + 16\,cd + d^2)^3}{c^2\,d^2\,(c-1)^2\,(d - 16\,c^2 + 16\,c)^2}
\end{aligned}$$

The multiplication of the shift by two reads $(c,\, d) \,\rightarrow\, (c^{(2)},\, d^{(2)})$ where:

$$c^{(2)} \,=\, \frac{(2\,c-1)^2}{1 + dc - dc^2}\,, \qquad\qquad d^{(2)} \,=\, \frac{d \cdot (-1 + 2\,c)^2\,(4 + d)}{(dc^2 - dc - 1)^2} \qquad (73)$$

The multiplication of the shift by three reads $(c,\, d) \,\rightarrow\, (c^{(3)},\, d^{(3)})$ where:

$$\begin{aligned}
c^{(3)} &= \frac{c \cdot \bigl(dc^2 - 2\,dc - 4\,c + 3 + d\bigr)^2}{1 + d^2c - 4\,d^2c^2 + 6\,d^2c^3 - 3\,d^2c^4 + 6\,dc - 22\,dc^2 + 32\,dc^3 - 16\,dc^4}\,, \\
d^{(3)} &= \frac{d \cdot \bigl(dc^2 + 4\,c - 1\bigr)^2 \bigl(dc^2 - 2\,dc - 4\,c + 3 + d\bigr)^2}{(1 + d^2c - 4\,d^2c^2 + 6\,d^2c^3 - 3\,d^2c^4 + 6\,dc - 22\,dc^2 + 32\,dc^3 - 16\,dc^4)^2}
\end{aligned}$$

The multiplication of the shift by five is given in Appendix B.

7 Conclusion

Modular functions, and modular equations, have been employed in the theory of critical phenomena [70, 71] to study analytic properties of, for instance, the Ising model with pure triplet interaction on the triangular lattice. Application of modular functions to the hard hexagon model has also been noticed [66]. In particular, Tracy et al [72] have proved that the physical quantities computed by Baxter [25], are *modular functions* with respect to certain congruence groups. The group theoretic methods extensively used by Tracy et al [72] and Richey and Tracy [73] are more sophisticated, and modern, than those based on the "hauptmodul" functions of Klein and Fricke [47], and that we have presented here. However, the main advantage of the "hauptmodul" method is that it leads, more naturally, to *explicit algebraic and hypergeometric closed form expressions* for the various physical quantities of the model.

One purpose of this paper has been to promote the relevance of the j-invariant to describe the analytical properties of lattice models, and in particular the relevance of a (j-invariant well suited) representation of elliptic curves in terms of the *symmetric biquadratic curve* (19). This biquadratic representation of elliptic curves is as simple as the well-known canonical Weierstrass form, but *it is actually such that the action of the group of birational transformations (closely related to the group of rational points) is crystal clear*, and such that a symmetry of elliptic curves (related to the modular invariance), namely the permutation group of three elements is also clear. Furthermore, this *biquadratic representation enables to find, very simply, a remarkable infinite discrete set of homogeneous polynomials, representing the multiplication of the shift by an integer, and leaving j* invariant (see section (5.2)). Finally, the two expressions g_2 and g_3, occurring in the Weierstrass canonical form $y^2 = 4\,x^3 - g_2\,x - g_3$, have been shown to present remarkable covariance properties (see (54), (55), (56)) with respect to this infinite set of commuting homogeneous polynomial transformations.

The j-invariant "encapsulates" highly non-trivial symmetries of the lattice models (an $sl_2 \times sl_2 \times sl_2 \times sl_2$ symmetry [6], the birational symmetries $\widehat{K}^N$, infinite discrete set of non-invertible homogeneous polynomials, representing the natural integers together with their multiplication ...). One does not have an opposition between a "point of view of the physicist" and a " point of view of the mathematician": a remarkable symmetry of a lattice model *must* correspond to remarkable mathematical structures, and vice-versa ... From a mathematical point of view these two (infinite discrete) sets of transformations (birational and only rational) play a quite interesting role as far as *arithmetic properties of elliptic curves* are concerned. These mathematical structures can be easily deduced from the analysis of a biquadratic naturally occurring in lattice statistical mechanics. Fortunately such "canonical biquadratic" can be generalized when curves are replaced by *surfaces*, or higher dimensional varieties. It would be interesting to see if this approach could provide a way to get some results on the moduli space of surfaces.

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8 Appendix A: Finite order conditions as polynomial relations on J_x, J_y and J_z.

Let us give a few miscellaneous examples of finite order conditions. The points of the Baxter model on the algebraic varieties:

$$V^{(3)}(J_x,\, J_y,\, J_z) \; = \; J_x\, J_y \, + \, J_z\, J_x \, + J_z\, J_y \, ,$$
$$\widehat{V}_z^{(3)}(J_x,\, J_y,\, J_z) \; = \; -J_x\, J_y \, + \, J_z\, J_x \, + J_z\, J_y$$

are actually such that $K^6(R) \, = \, \zeta \cdot R$. One has the following factorization property:

$$V^{(3)}(J_x^{(2)},\, J_y^{(2)},\, J_z^{(2)}) \; = \tag{74}$$
$$V^{(3)}(J_x,\, J_y,\, J_z) \cdot \widehat{V}_x^{(3)}(J_x,\, J_y,\, J_z) \cdot \widehat{V}_y^{(3)}(J_x,\, J_y,\, J_z) \cdot \widehat{V}_z^{(3)}(J_x,\, J_y,\, J_z) \; = \; 0$$

where $\widehat{V}_x^{(3)}(J_x,\, J_y,\, J_z) \, = \, \widehat{V}_z^{(3)}(J_y,\, J_z,\, J_x)$ and where $\widehat{V}_y^{(3)}(J_x,\, J_y,\, J_z)$ $= \, \widehat{V}_z^{(3)}(J_z,\, J_x,\, J_y)$. One has the relation:

$$\widehat{V}_z^{(3)}(J_x^{(2)},\, J_y^{(2)},\, J_z^{(2)}) \, - \, P_z^{(3)}(J_x,\, J_y,\, J_z) \; = \; 0$$

Note that the points of the Baxter model on the algebraic varieties $P_x^{(3)}(J_x,\, J_y,\, J_z) \, = \, 0$, or $P_y^{(3)}(J_x,\, J_y,\, J_z) \, = \, 0$, or $P_z^{(3)}(J_x,\, J_y,\, J_z) \, = \, 0$, are such that $K^{12}(R) \, = \, \zeta \cdot R$. The points of order six (namely $K^6(R) \, = \, \zeta \cdot R$) correspond to (74), their image by the shift doubling $(J_x,\, J_y,\, J_z) \, \rightarrow \, (J_x^{(2)},\, J_y^{(2)},\, J_z^{(2)})$ giving $P_x^{(3)}(J_x,\, J_y,\, J_z) \cdot P_y^{(3)}(J_x,\, J_y,\, J_z) \cdot P_z^{(3)}(J_x,\, J_y,\, J_z) \, = \, 0$, together, of course, with (74). The points of the Baxter model on the variety:

$$V_y^{(5)}(J_x,\, J_y,\, J_z) \; = \; J_z^2\, J_x\, J_y^3 \, - 2\, J_z^2\, J_x^2\, J_y^2 \, + J_z^2\, J_x^3\, J_y + J_z\, J_x^2\, J_y^3$$
$$-J_z\, J_x^3\, J_y^2 - J_z^3\, J_y^3 - J_z^3\, J_x\, J_y^2 + J_z^3\, J_x^2\, J_y + J_z^3\, J_x^3 - J_x^3\, J_y^3 \; = \; 0$$

are of order five $K^5(R) \, = \, \zeta \cdot R$. The points of the Baxter model on the algebraic variety:

$$\widehat{V}_y^{(5)}(J_x,\, J_y,\, J_z) \;\; = \;\; V_y^{(5)}(J_y,\, J_x,\, J_z) \;\; = \; 0 \;\; =$$

$$= \; J_z^2\, J_x^3\, J_y - 2\, J_z^2\, J_x^2\, J_y^2 \,+ J_z^2\, J_x\, J_y^3 \,+ J_z\, J_x^3\, J_y^2 - J_z\, J_x^2\, J_y^3 \,- J_z^3\, J_x^3$$
$$-J_z^3\, J_x^2\, J_y \,+ J_z^3\, J_x\, J_y^2 \,+ J_z^3\, J_y^3 \,- J_x^3\, J_y^3$$

are of order ten: $K^{10}(R) \;=\; \zeta\cdot R$. The point of the Baxter model on the variety $P_x^{(5)}(J_x,\, J_y,\, J_z) \,=\, 0$, $P_y^{(5)}(J_x,\, J_y,\, J_z) \,=\, 0$, or $P_z^{(5)}(J_x,\, J_y,\, J_z) \,=\, 0$, are of order *twenty*: $K^{20}(R) \,=\, \zeta\cdot R$. Let us note that:

$$V_y^{(5)}(J_x^{(2)},\, J_y^{(2)},\, J_z^{(2)}) \,+\, P_y^{(5)}(J_x,\, J_y,\, J_z) \,=\, 0\,,$$
$$\widehat{V}_y^{(5)}(J_x^{(2)},\, J_y^{(2)},\, J_z^{(2)}) \,+\, P_x^{(5)}(J_x,\, J_y,\, J_z) \,=\, 0$$

which is in agreement with the fact that $(J_x,\, J_y,\, J_z) \,\rightarrow\, (J_x^{(2)},\, J_y^{(2)},\, J_z^{(2)})$ represents the shift doubling.

9 Appendix B: rational representation of the multiplication of the shift by five.

The multiplication of the shift by five reads $(a,\, b) \,\rightarrow\, (a_5,\, b_5)$

$$a_5 \;=\; \frac{N_5^{(a)}}{a\cdot D_5^{(a)}}\,, \qquad\qquad b_5 \;=\; \frac{b\cdot (N_5^{(b)})^2\cdot (M_5^{(b)})^2}{a^2\cdot (D_5^{(b)})^2}\,, \qquad\qquad \text{where:}$$

$$N_5^{(a)} \;=\; -a^{10} + b^6 - b^3a^3 - ba^{17} + 36\, a^6b^6 - 108\, a^7b^5 + 128\, a^8b^4 - 76\, a^9b^3$$
$$+24\, a^{10}b^2 - 4\, a^{11}b - 6\, b^3a^5 + 5\, b^6a^{12} - 10\, b^6a^{10} - 9\, b^6a^8 - 15\, b^5a^{13}$$
$$+30\, b^5a^{11} + 27\, b^5a^9 + 20\, b^4a^{14} - 26\, b^4a^{12} - 21\, b^4a^{10} - 122\, a^6b^4$$
$$+2\, b^3a^{13} + 6\, b^2a^{16} + 6\, b^2a^{14} - 2\, ba^{15} - 29\, b^6a^4 + 18\, a^4b^4 - 15\, b^3a^{15}$$
$$+6\, b^6a^2 + 10\, a^9b - 3\, ab^5 + 3\, a^2b^4 + 87\, a^5b^5 + 9\, b^2a^{12} - 3\, ba^{13} - 3\, b^3a^{11}$$
$$-18\, a^3b^5 - 45\, a^8b^2 + 99\, a^7b^3$$

$$D_5^{(a)} \;=\; 5\, b^6 - 15\, b^3a^3 - ba^5 + 36\, a^6b^6 - 108\, a^7b^5 + 128\, a^8b^4 - 76\, a^9b^3$$
$$+24\, a^{10}b^2 - 4\, a^{11}b + 2\, b^3a^5 + b^6a^{12} + 6\, b^6a^{10} - 29\, b^6a^8 - 3\, b^5a^{13}$$
$$-18\, b^5a^{11} + 87\, b^5a^9 + 3\, b^4a^{14} + 18\, b^4a^{12} - 122\, b^4a^{10} - 3\, a^9b - 10\, b^6a^2$$
$$-15\, ab^5 + 20\, a^2b^4 + 27\, a^5b^5 - 21\, a^6b^4 - 45\, b^2a^{12} + 10\, ba^{13} + 99\, b^3a^{11}$$
$$-26\, a^4b^4 + 30\, a^3b^5 + 9\, a^8b^2 - 3\, a^7b^3 + 6\, b^2a^6 - 2\, ba^7 + 6\, b^2a^4 - a^{14}$$
$$-b^3a^{15} - 6\, b^3a^{13} - 9\, b^6a^4$$

$$N_5^{(b)} \;=\; 2\, b^3a - 4\, b^3a^3 - b^3a^2 + b^3 - a^4 + a^5 - a^6 + a^7 - a^8 - b^3a^4 + 5\, b^2a^5$$
$$-5\, ba^6 + 2\, b^3a^5 - 5\, b^2a^6 + 4\, ba^7 + b^3a^6 - b^2a^7 - 5\, a^4b + 2\, b^2a^4 + 2\, ba^5$$

$$+5\,a^3b^2+4\,a^3b-5\,a^2b^2-ab^2$$

$$M_5^{(b)} \;=\; -2\,b^3a+4\,b^3a^3-b^3a^2+b^3+a^4+a^5+a^6+a^7+a^8-b^3a^4$$
$$-5\,ba^6-2\,b^3a^5+5\,b^2a^6-4\,ba^7+b^3a^6-b^2a^7-2\,b^2a^4-2\,ba^5+5\,b^2a^5$$
$$-5\,a^4b+5\,a^3b^2-4\,a^3b+5\,a^2b^2-ab^2$$

$$D_5^{(b)} \;=\; 5\,b^6-15\,b^3a^3-ba^5+36\,a^6b^6-108\,a^7b^5+128\,a^8b^4-76\,a^9b^3$$
$$+24\,a^{10}b^2-4\,a^{11}b+2\,b^3a^5-a^{14}+b^6a^{12}+6\,b^6a^{10}-29\,b^6a^8-3\,b^5a^{13}$$
$$+87\,b^5a^9+3\,b^4a^{14}+18\,b^4a^{12}-122\,b^4a^{10}-b^3a^{15}-6\,b^3a^{13}-18\,b^5a^{11}$$
$$+27\,a^5b^5-21\,a^6b^4-45\,b^2a^{12}+10\,ba^{13}-2\,ba^7+6\,b^2a^4-15\,ab^5$$
$$+99\,b^3a^{11}-26\,a^4b^4+30\,a^3b^5+9\,a^8b^2-3\,a^7b^3+6\,b^2a^6+20\,a^2b^4$$
$$-9\,b^6a^4-10\,b^6a^2-3\,a^9b$$

Another rational representation of the multiplication of the shift. In term of the invariants c and d (see (59) and (71)), the multiplication of the shift by five reads $(c,\, d) \,\rightarrow\, (c^{(5)},\, d^{(5)})$ where:

$$c^{(5)} \;=\; \frac{c\cdot N_1^2}{D_1}\,, \qquad\qquad d^{(5)} \;=\; \frac{d\cdot N_1^2\cdot N_2^2}{D_1^2}\,, \qquad\qquad \text{where:} \quad (75)$$

$$N_1 \;=\; -5+20\,c-16\,c^2-5\,d-101\,dc^2+34\,dc-28\,d^2c^2+8\,d^2c-d^2$$
$$+184\,dc^3-176\,dc^4-2\,d^3c^5+d^3c^6+d^3c^4+20\,d^2c^5+64\,dc^5+56\,d^2c^3$$
$$-55\,d^2c^4$$

$$N_2 \;=\; -1+12\,c-16\,c^2+35\,dc^2+10\,d^2c^2-64\,dc^5-36\,d^2c^3+45\,d^2c^4$$
$$-4\,d^3c^5+d^3c^6+6\,d^3c^4-20\,d^2c^5+d^3c^2-120\,dc^3+144\,dc^4-4\,d^3c^3$$

$$D_1 \;=\; 1-610\,dc^2+50\,dc+35\,d^2c+3130\,d^2c^3-11005\,d^2c^4-480\,d^2c^2$$
$$+3680\,dc^3-12336\,dc^4+1060\,d^3c^3-150\,d^3c^2+24064\,dc^5-3920\,d^3c^4$$
$$+d^6c^4+d^4c+22080\,d^2c^5+15360\,d^2c^7+4160\,d^4c^{10}+8100\,d^3c^5$$
$$+256\,d^4c^{12}-27136\,dc^6-4096\,dc^8+6935\,d^4c^8+80\,d^5c^{12}+5760\,d^3c^7$$
$$+2160\,d^5c^8+5\,d^6c^{12}-4316\,d^4c^7-480\,d^5c^{11}-1536\,d^4c^{11}-16\,d^4c^2$$
$$-2070\,d^5c^9-3840\,d^2c^8+16384\,dc^7-390\,d^4c^4-9420\,d^3c^6-9\,d^6c^5$$
$$+1294\,d^5c^{10}+370\,d^4c^5+1136\,d^4c^6-25280\,d^2c^6+10\,d^3c-84\,d^6c^7$$
$$+670\,d^5c^6+20\,d^5c^4+126\,d^6c^8-1500\,d^5c^7-6720\,d^4c^9-1440\,d^3c^8$$
$$-30\,d^6c^{11}-125\,d^6c^9+80\,d^6c^{10}-174\,d^5c^5+36\,d^6c^6+120\,d^4c^3$$

10 Appendix C: Weierstrass versus biquadratic, Bachet's duplication formula

Let us consider the canonical Weierstrass form (28): $y^2 \,=\, 4\,x^3 - g_2\,x - g_3$. The Bachet's duplication formula discovered in 1621, amounts to saying that if $(x,\, y)$ is a solution of (28) then $(x_2,\, y_2)$ is also a solution of (28), where $(x_2,\, y_2)$ are given by:

$$x_2 \,=\, {{1} \over {16\, y^2}} \cdot (16\, x^4 + 8\, g_2\, x^2 + 32\, g_3\, x + {g_2}^2)\,, \quad y_2 \,=\, {{-1} \over {32 y^3}} \cdot y_{num} \text{ where:}$$
$$y_{num} \,=\, 64\, x^6 - 320\, g_3\, x^3 - 32\, {g_3}^2 - 20\, {g_2}^2 x^2 - 80\, x^4 g_2 + {g_2}^3 - 16\, g_3\, g_2\, x$$

We have here a rational representation of $\theta \,\rightarrow\, 2 \cdot \theta$. These results can be generalized to:

$$v^2 \,=\, u^3 \,+ a\, u^2 \,+ b\, u \,+ c$$

the duplication formula reading:

$$\begin{aligned} u_2 \,&=\, {{1} \over {4 v^2}} \cdot (u^4 - 2\, bu^2 - 8\, cu - 4\, ac + b^2)\,, \\ v_2 \,&=\, {{1} \over {8\, v^3}} \cdot V_{num} \qquad \text{where:} \\ V_{num} \,&=\, -u^6 - 20\, cu^3 + 8\, c^2 + 5\, b^2 u^2 - 2\, u^5 a - 5\, u^4 b \\ &\qquad +2\, b^2 ua \,- 4\, cab + b^3 - 8\, cua^2 - 20\, cau^2 + 4\, cbu \end{aligned}$$

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