Algebraic Statistical Mechanics: Selected Non-holonomic functions in lattice statistical mechanics and enumerative combinatorics

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We are going to show that:

Diagonal of rational functions are like a gold mine.
They are selected non-holonomic functions that are another gold mine.

And you know what ?
It does not require excavating deeper and deeper ...

So many people have a defeatist attitude towards non-holonomic functions: they think nothing can be done on non-holonomic functions.

This is defeatist nonsense

As far as non-holonomic functions are concerned:

Defeatism, resignation and other misleading theorems on non-linear ODEs

This defeatist attitude on non-holonomic functions is in fact a **defeatist attitude on non-linear ODEs**.

Along this line, too often **Rubel's universal non-linear ODE** is recalled to discourage any "*non-linear differential Padé*" search. Rubel's universal non-linear polynomial differential equation reads:

$$\begin{aligned} 3\,y_1^4\,y_2\,y_4^2 \;&-4\,y_1^4\,y_3^2\,y_4 \;+6\,y_1^3\,y_2^2\,y_3\,y_4 \;+24\,y_1^2\,y_2^4\,y_4 \\ -12\,y_1^3\,y_2\,y_3^3 \;&-29\,y_1^2\,y_2^3\,y_3^2 \;+12\,y_2^7 \;=\; 0, \end{aligned}$$

where the y_n's are the n-th derivative $d^n y/dx^n$.

They are other universal non-linear homogeneous polynomial differential equations (with **piecewise polynomial solutions**), obtained by Duffin:

$$16\,y_4\,y_1^2 \;-32\,y_4\,y_2\,y_1 \;+17\,y_2^3 \;=\; 0.$$

Rubel's universal equation

Rubel's non-linear differential equation (and other piecewise polynomial approximation on the real axis) correspond to a **homogeneous polynomial differential equation** such that any continuous function can be approximated, **on the real axis**, by a solution of this "universal" equation.

This kind of **real analysis theorem do not mean** that any function of a complex variable is "almost" solution of a non-linear differential equation in the complex plane, which would mean that any "non-linear differential Padé" approach would be pointless.

Ising n-fold integrals : the $\chi^{(n)}$'s

The magnetic susceptibility of the two-dimensional Ising model can be written as an **infinite sum** of n-folds integrals **holonomic functions**:

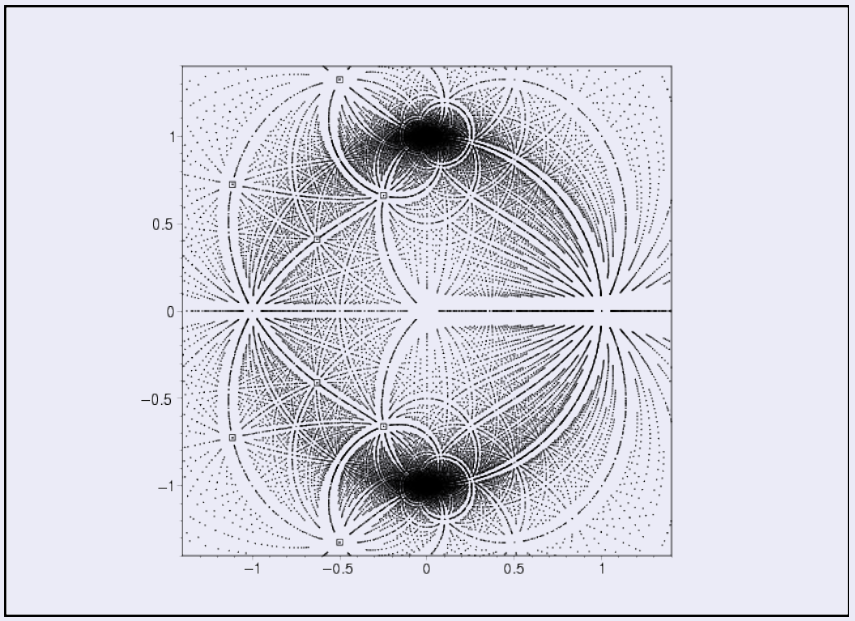
$$\chi(w) \;=\; \sum_{n=1}^{\infty} \chi^{(n)}(w).$$

The **magnetic susceptibility** χ is **not a holonomic function**, it is **not D-finite**: χ is **not solution of a linear differential equation**. It is much more involved.

The full susceptibility χ has a (unit circle) **natural boundary**, in the complex k-plane.

$|k| = 1$ is a natural boundary of $\chi(k)$.

Accumulation of the singularities of the linear ODEs for the $\chi^{(n)}$ in the k complex plane



Remark: for a holonomic function, there is a difference between the **singularities of that function**, and the **singularities of the linear differential operator** annihilating the function !!

Ising n-fold integrals : $\chi^{(5)}$

The five-particle contribution $\tilde{\chi}^{(5)}$ of the susceptibility of the Ising model is solution of an order-33 linear differential operator which has a **direct-sum** factorization (DFactorLCLM in Maple). The selected linear combination

$$\tilde{\chi}^{(5)} \; - \frac{1}{2}\,\tilde{\chi}^{(3)} \; + \frac{1}{120}\,\tilde{\chi}^{(1)},$$

is solution of an order-29 (globally nilpotent) linear differential operator

$$L_{29} \; = \quad L_5 \cdot L_{12} \cdot \tilde{L}_1 \cdot L_{11},$$

where:

$$L_{11} \; = \quad (Z_2 \cdot N_1) \oplus V_2 \oplus (F_3 \cdot F_2 \cdot L_1^s).$$

Z_2 in $\chi^{(2)}$: a modular form

The solution of the linear differential operator Z_2 can be expressed in terms of the ${}_2F_1$ hypergeometric function **up to a modular invariant pull-back**:

$$\mathcal{S} = \Big(\Omega \cdot \mathcal{M}_x\Big)^{1/12} \times {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}]; [1];\ \mathcal{M}_x\Big), \qquad \text{where:}$$

$$\Omega = \frac{1}{1728}\,\frac{(1-4\,x)^6\,(1-x)^6}{x\cdot(1+3\,x+4\,x^2)^2\,(1+2\,x)^6},$$

$$\mathcal{M}_x = 1728\,\frac{x\cdot(1+3\,x+4\,x^2)^2\,(1+2\,x)^6\,(1-4\,x)^6\,(1-x)^6}{(1+7\,x+4\,x^2)^3\cdot P^3},$$

$$P = 1+237\,x+1455\,x^2+4183\,x^3+5820\,x^4+3792\,x^5+64\,x^6.$$

It is a **modular form**.

Ising n-fold integrals : $\chi^{(6)}$

Similarly $\tilde{\chi}^{(6)}$ is solution of an order-52 linear differential operator which has a **direct-sum** factorization: the selected linear combination

$$\tilde{\chi}^{(6)} - \frac{2}{3}\tilde{\chi}^{(4)} + \frac{2}{45}\tilde{\chi}^{(2)},$$

is solution of an order-46 (globally nilpotent) linear differential operator

$$L_{46} = L_6 \cdot L_{23} \cdot L_{17},$$

where: $$L_{17} = \tilde{L}_5 \oplus L_3 \oplus (L_4 \cdot \tilde{L}_3 \cdot L_2),$$

$$\tilde{L}_5 = \left(\frac{d}{dx} - \frac{1}{x}\right) \oplus \Big(L_{1,3} \cdot (L_{1,2} \oplus L_{1,1} \oplus D_x)\Big).$$

The "Quarks" in $\chi^{(5)}$ and $\chi^{(6)}$

Quasi-trivial order-one (globally nilpotent) linear differential operators: $\tilde{L}_1,\ N_1,\ L_1^s,\ L_{1,n} \quad \longrightarrow \quad D_x - \frac{1}{N} \cdot \frac{d\ln(R(x))}{dx}$

$V_2,\ L_2,\ L_3,\ L_5$ and L_6 are respectively equivalent (homomorphic) to L_K, to the symmetric square of L_K and to the *symmetric fourth and fifth power* of L_K, where L_K is the second order linear differential operator annihilating the **complete elliptic integral** $K = {}_2F_1([1/2,\ 1/2],\ [1],\ k^2)$.

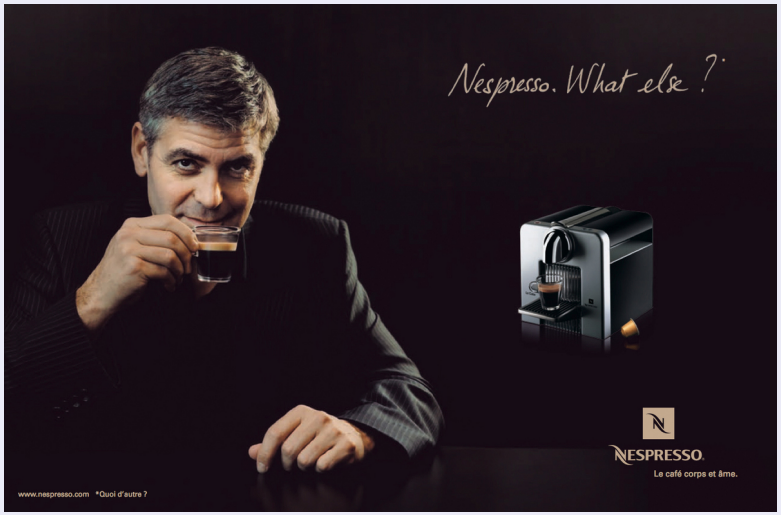
$F_2,\ F_3,\ \tilde{L}_3$ do correspond to **modular forms**: F_3 and $\tilde{L}_3$ are homomorphic to the symmetric square of order-two operators associated with the (fundamental) **modular curve** $X_0(2)$, and F_2 is related to Z_2 (and thus h_6, Apéry, ...).

Remains to understand the "very nature" of:

$$L_4 \qquad \text{and:} \qquad L_{12},\ L_{23}$$

Trying to understand the "very nature" of L_4 and: L_{12}, L_{23}

L_{23} is (at first sight ..) beyond current computational resources ... The order-12 operator L_{12} is almost beyond computational resources: it was already a "tour-de-force" to show that it is an **irreducible** operator. Let us focus on L_4, and try to reduce it to operators associated to elliptic curves. Any integrable expert would have an (educated ...) prejudice that the Ising model must be nothing but the theory of **elliptic curves** and other modular forms, **what else ?**



The puzzling L_4: preliminary results on L_4

Preliminary calculations show that L_4 **cannot be reduced to elliptic functions**, modular forms, and it is not ${}_4F_3$-solvable *if one restricts to rational pull-backs*.

Is this operator going to be a **counter-example** to our favourite "mantra" that the Ising model is nothing but the theory of elliptic curves and other modular forms ?

Computing the exterior square of the linear differential operator L_4, one finds an order-six linear differential operator with the *direct sum decomposition*

$$\mathrm{ext}^{(2)}(L_4) \;=\; M_5 \cdot \tilde{N}_1 \;=\; M_1 \cdot N_5 \;=\; N_5 \oplus \tilde{N}_1,$$

where $\tilde{N}_1$ has a **rational function solution**. Therefore L_4 has a **symplectic differential Galois group** $Sp(4, \mathbb{C})$.

L_4 is a Hadamard product of two elliptic curves:

it is a Calabi-Yau operator !

Seeking for ${}_4F_3$ hypergeometric functions up to homomorphisms, and assuming an **algebraic pull-back** with the *square root extension*, $(1-16\cdot w^2)^{1/2}$, we actually found that the solution of L_4 *can be expressed in terms of a selected* ${}_4F_3$

$$
\begin{aligned}
&{}_4F_3\Big([1/2,\,1/2,\,1/2,\,1/2],[1,1,1];\;z\Big)\\
&\quad = \quad {}_2F_1\Big([1/2,\,1/2],[1];\;z\Big)\star {}_2F_1\Big([1/2,\,1/2],[1];\;z\Big),
\end{aligned}
$$

where: $$z \;=\; \Big(\frac{1+(1-16\cdot w^2)^{1/2}}{1-(1-16\cdot w^2)^{1/2}}\Big)^4 \;=\; k^4$$

where the pull-back z *is nothing but* the fourth power of the **modulus** k of the elliptic functions !

Our linear differential operators must be "special", but what does it mean to be "special" ?

This raises the question of the "**modularity**" in these problems: beyond the occurrence of many **modular forms**, we also see, for **order-four** operators, the emergence of **Calabi-Yau ODEs**.

• The differential Galois groups of these linear differential operators are not the generic $SL(N, C)$ groups, but **selected orthogonal or symplectic** groups.

• We also have properties of more *arithmetic* nature: the series expansions of these holonomic functions can be recast into series expansions with **integer** coefficients. For instance the $\tilde{\chi}^{(n)}(w) = 2^n \cdot w^{n^2} \cdot \kappa_n(w)$ expand as:

$$\begin{aligned}
\kappa_n(w) &= 1 + 4\,n^2 \cdot w^2 + 2 \cdot (4\,n^4 + 13\,n^2 + 1) \cdot w^4 \\
&\quad + \frac{p_6(n)}{3} \cdot w^6 + \frac{p_8(n)}{3} \cdot w^8 + \cdots \\
p_6(n) &= 8 \cdot (n^2 + 4)\,(4\,n^4 + 23\,n^2 + 3), \\
p_8(n) &= \cdot (32\,n^8 + 624\,n^6 + 4006\,n^4 + 8643\,n^2 + 1404),
\end{aligned}$$

The well-suited framework: diagonal of rational functions

We also found in enumerative combinatorics, many other selected linear differential operators, associated with **lattice Green functions**, which have **special differential Galois groups**, generalizing **Calabi-Yau operators** (Calabi-Yau 3-folds). All these linear differential operators are **globally nilpotent**: they are not only **Fuchsian**, they are such that their p-curvatures are nilpotent, and all their **critical exponents are rational numbers**, ... They are "**Derived from Geometry**": they annihilate n-fold integrals of **algebraic integrands** (in mathematician's wording "**Periods**"). These n-fold integrals are (or can be recast into) series with **integer coefficients** (globally bounded series). These two set of properties are, in fact, the consequence of the fact that these holonomic functions are actually **diagonal of rational functions**. *As Monsieur Jourdain talked prose, without knowing it*, n-fold integrals of physics are, without knowing it, **diagonal of rational functions**, which corresponds to a **quite remarkable set**.

Definition of the diagonal of series of several complex variables

Definition:

$$\mathcal{F}\Big(z_1,\, z_2,\, \ldots,\, z_n\Big) \;=\; \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\cdots\sum_{m_n=0}^{\infty} F_{m_1,m_2,\ldots,m_n}\cdot z_1^{m_1}\, z_2^{m_2}\,\cdots\, z_n^{m_n},$$

$$Diag\Big(\mathcal{F}\Big(z_1,\, z_2,\, \ldots,\, z_n\Big)\Big) \;=\; \sum_{m=0}^{\infty} F_{m,m,\ldots,m}\cdot z^m.$$

The result: if the **algebraic or rational integrand** of a n-fold integral has a **multi-Taylor expansion**, then this n-fold integral **is the diagonal of a rational function**.

Two by-products: Diagonal of rational functions are (or can be recast into) series with **integer coefficients**, which **reduce modulo any prime to algebraic functions**.

A pedagogical example of diagonal of rational functions.

Let us consider the **rational function of three complex variables** $\mathcal{F} = 1/(1 - z_2 - z_3 - z_1 z_2 - z_1 z_3)$. Its diagonal reads:

$$1 \quad + 4z \; + 36z^2 \; + 400z^3 \; + 4900z^4 + 63504z^5 \; + \; \cdots$$

which is nothing but the **complete elliptic integral** (first kind):

$$\sum_{m \geq 0} \binom{2m}{m}^2 \cdot z^m \quad = \quad {}_2F_1\Big([\frac{1}{2}, \frac{1}{2}], [1], 16\, z\Big)$$

This diagonal **modulo any prime** reduces to an **algebraic function**, for instance:

$$\begin{aligned} & Diag(\mathcal{F}) \mod 7 \; = \\ & = \quad 1 \; + 4\, z \; + z^2 + z^3 + 4z^7 + 2z^8 + 4z^9 \; + \cdots \\ & \qquad = \quad \frac{1}{\sqrt[6]{1 + 4z + z^2 + z^3}} \qquad \mod 7. \end{aligned}$$

Another example of diagonal of rational functions.

A less obvious example corresponds to the **modular form**:

$$\left(\frac{1}{1 - z_1 - z_2 - z_3 - z_1 z_2 - z_2 z_3 - z_3 z_1 - z_1 z_2 z_3}\right) = \frac{1}{1-z} \cdot {}_2F_1\left([\frac{1}{3}, \frac{2}{3}], [1]; \frac{54\, z}{(1-z)^3}\right).$$

Such **diagonals of rational functions** are **highly selected functions**: modulo **any prime** they reduce to **algebraic functions**.

They can be seen as the **simplest (transcendental) generalisations of algebraic functions**.

The integrands of the $\chi^{(n)}$ n-fold integral of the Ising model have a **multi-Taylor expansion** and are, thus, **diagonals of a rational function**.

The $\chi^{(n)}$'s are diagonal of rational functions.

Let us consider the series of $\tilde{\chi}^{(3)}/8/w^9$

$$1 + 36\,w^2 + 4\,w^3 + 884\,w^{13} + 196\,w^5 + 18532\,w^6 + \cdots$$

Let us now consider this very series **modulo the prime** $p = 2$. It reads the **lacunary** series

$$1 + w^8 + w^{24} + w^{56} + w^{120} + w^{248} + w^{504} + w^{1016} + \cdots,$$

In fact,*modulo the prime* $p = 2$, $H(w) = \tilde{\chi}^{(3)}/8$ is, actually, an **algebraic function**, solution of the quadratic equation:

$$H(w)^2 + w \cdot H(w) + w^{10} = 0 \mod 2.$$

Modulo $p = 3$. Indeed, $H(w)$ satisfies a polynomial equation of degree nine (the p_n are polynomials of degree less that 63):

$$p_9 \cdot H(w)^9 + w^6 \cdot p_3 \cdot H(w)^3 + w^{10} \cdot p_1 \cdot H(w) + p_0.$$

Non-holonomic functions modulo integers. The full susceptibility of the Ising model

Remarkably long series expansion (2041 **coefficients !!!**) were obtained for the **low-temp. full susceptibility** of the Ising model

$$\begin{aligned}\tilde{\chi}_L(w) &= 4\,w^4 + 80\,w^6 + 1400\,w^8 + 23520\,w^{10} + 388080\,w^{12} \\ &+6342336\,w^{14} + 103062976\,w^{16} + 1668639424\,w^{18} \\ &+26948549680\,w^{20} + \cdots\end{aligned}$$

to be compared with the series for $\tilde{\chi}^{(2)}(w)$ namely :

$$\begin{aligned}\tilde{\chi}_L^{(2)} &= 4\,w^4 \cdot {}_2F_1\Big([\frac{3}{2}, \frac{5}{2}], [3], 16\,w^2\Big) \\ &= 4\,w^4 + 80\,w^6 + 1400\,w^8 + 23520\,w^{10} + 388080\,w^{12} \\ &+6342336\,w^{14} + 103062960\,w^{16} + 1668638400\,w^{18} \\ &+26948510160\,w^{20} + \cdots\end{aligned}$$

Non-holonomic functions modulo integers. The full susceptibility of the Ising model

Since diagonal of rational functions also reduce to algebraic modulo powers of primes, let us consider the **low-temp.** expansion of $\tilde{\chi}^{(2)}$ **modulo** $2^5 = 32$:

$$\begin{aligned}\tilde{\chi}_L^{(2)}(w) &= 4\,w^4 + 16\,w^6 + 24\,w^8 + 16\,w^{12} + 16\,w^{16} + 16\,w^{20} \\ &+16\,w^{36} + 16\,w^{68} + 16\,w^{132} + 16\,w^{260} + 16\,w^{516} + 16\,w^{1028} + \cdots \\ &= 8\,x^8 + 16\,x^{16} - 12\,x^4 - 16\,x^5 + 16\,w^4 \cdot L(w),\end{aligned}$$

where $L(w)$ is the $\sum w^{2^n}$ **lacunary series**
$1 + w + w^2 + w^4 + w^8 + \cdots + w^{128} + w^{256} + w^{512} + w^{1024} + \cdots$
which satisfies the **functional equation**:

$$L(w) = \sum_{n=0}^{n=\infty} w^{2^n}, \qquad L(w) = w + L(w^2).$$

Modulo 2, this functional equation is an **algebraic relation** $L(w) = w + L(w)^2$.

Non-holonomic functions modulo integers. The full susceptibility of the Ising model

One deduces the **functional equation on the full susceptibility**:

$$\begin{aligned}\tilde{\chi}_L^{(2)}(w^2) &= w^4 \cdot \tilde{\chi}_L^{(2)}(w) \\ &\quad + 8\,w^{10} \cdot (2\,w^{22} - 2\,w^{10} + w^6 - w^2 - 2) \qquad mod.\ 32.\end{aligned}$$

Let us compare $\tilde{\chi}_L^{(2)}$ and the **full susceptibility** modulo 32:

$$\tilde{\chi}_L^{(2)}(w) = \tilde{\chi}_L(w) + 16\,w^{16}. \tag{1}$$

In other words, modulo 32, one cannot see the difference between the **full susceptibility** and a **diagonal of rational function**, which actually reduces to an **algebraic function !!**
One deduces the **functional equation on the full susceptibility**:

$$\tilde{\chi}_L(w^2) = w^4 \cdot \tilde{\chi}_L(w) + 8\,w^{10} \cdot (w^6 - w^2 - 2) \qquad mod.\ 32.$$

Non-holonomic functions modulo integers. The full susceptibility of the Ising model

One finds that the **difference** of $\tilde{\chi}_L$ and $\tilde{\chi}_L^{(2)}$, is zero modulo $2, 4, 8, 16$, and equal to $16\,w^{16}$ modulo 32. Modulo 64 this difference is given by a **lacunary series**

$$\begin{aligned}\tilde{\chi}_L - \tilde{\chi}_L^{(2)} &= 32\,w^4 \cdot L(w) + 16\,w^{16} \\ &\quad + 32\,w^4 \cdot (w^{28} - w^8 - w^4 - w^2 - w - 1).\end{aligned}$$

If one includes $\tilde{\chi}_L^{(4)}$, the difference between $\tilde{\chi}_L$ and $\tilde{\chi}_L^{(2)} + \tilde{\chi}_L^{(4)}$, is seen to be zero modulo $2, 4, 8, 16, 32, 64$, and is given by a **lacunary series** modulo 128:

$$\begin{aligned}\tilde{\chi}_L - \tilde{\chi}_L^{(2)} - \tilde{\chi}_L^{(4)} &= 64\,w^4 \cdot L(w) \\ &\quad - 64\,w^4 \cdot (w^{16} + w^8 + w^4 + w^2 + w + 1).\end{aligned}$$

If one includes $\tilde{\chi}_L^{(6)}$, the difference between $\tilde{\chi}_L$ and $\tilde{\chi}_L^{(2)} + \tilde{\chi}_L^{(4)} + \tilde{\chi}_L^{(6)}$ is seen to be zero modulo $2, 4, \cdots 128, 256$.

Non-holonomic functions reducing to algebraic functions modulo integers

One cannot distinguish between the full susceptibility $\tilde{\chi}_L$ and the finite sum $\tilde{\chi}_L^{(2)} + \tilde{\chi}_L^{(4)} + \cdots + \tilde{\chi}_L^{(2n)}$ modulo 2^r (where r grows linearly with n), which reduces, modulo 2^r, to algebraic functions since it is the diagonal of a rational function.

Rational $\rightarrow$ Algebraic $\rightarrow$ Holonomic $\rightarrow$ Non-Holonomic.

There is a class of **non-holonomic** functions (highly relevant for physics !) which, **modulo integers**, cannot be distinguished from (holonomic) diagonal of rational functions, and, thus, **reduce to algebraic functions** modulo integers. **What are they ?**

In order to get some perspective (and more examples ...), let us switch to **enumerative combinatorics**.

More non-holonomic functions: enumerative combinatorics

In enumerative combinatorics we must recall **Tutte's study of colouring problems.** His work culminates in 1982, when he proved that the series counting q-coloured rooted triangulations by vertices, $H(w) = q \cdot (q-1) \cdot w^2 + q \cdot (q-1) \cdot (q-2) \cdot w^3 + \cdots$, satisfies a **non-linear (polynomial) differential equation**:

$$2\,q^2 \cdot (1-q) \cdot w + \left(q\,w + 10\,H(w) - 6\,w\,\frac{dH(w)}{dw}\right) \cdot \frac{d^2H(w)}{dw^2}$$
$$+\,q \cdot (4-q) \cdot \left(20\,H(w) - 18\,w\,\frac{dH(w)}{dw} + 9\,w^2\,\frac{d^2H(w)}{dw^2}\right) = 0.$$

$H(w)$ reduces to **algebraic functions** for all the well-known **Tutte-Beraha numbers**, and, in fact, for $q = 2 + 2\,cos(j\,\pi/m)$.

However, **the status of the $q = 4$ series is not clear**:
$H(w) = 12\,w^2 + 24\,w^3 + 168\,w^4 + 1656\,w^5 + 19296\,w^6 + \cdots$

Tutte's non-linear differential equation

The coefficients of the series $H(w) = \sum h_n w^n$ are the number h_n of *rooted triangulations with n vertices*. They satisfy a **remarkably simple (and entirely mysterious ...) quadratic recurrence relation**:

$$\begin{aligned} q \cdot (n+1)(n+2) \cdot h_{n+2} \\ = \quad q \cdot (q-4) \cdot (3n-1)(3n-2) \cdot h_{n+1} \\ +2 \cdot \sum_{i=1}^{n} i \cdot (i+1) \cdot (3n-3i+1) \cdot h_{i+1} \cdot h_{n-i+2}, \end{aligned}$$

with the initial conditions $h_0 = 0, \quad h_1 = 0, \quad h_2 = q \cdot (q-1)$.

Tutte's non-linear differential equation: the $q = 4$ case.

In the $q = 4$ case the previous series is a series with **integer** coefficients of finite radius of convergence:

$$H(w) = 12\,w^2 + 24\,w^3 + 168\,w^4 + 1656\,w^5 + 19296\,w^6 + 248832\,w^7 + 3437424\,w^8 + 49923288\,w^9 + \cdots$$

In the $q = 4$ case, **Tutte's non-linear differential equation** has **many other solutions**. With the initial conditions $h_0 = 0$ but $h_1 \neq 0$, one finds a **one-parameter family** of solutions of Tutte's non-linear equation :

$$H_A(w) = -w + A^3 \cdot \left(\frac{w}{A^2} + H\left(\frac{w}{A^2}\right)\right).$$

One can use this **one-parameter group of symmetry** of this non-linear differential equation to rewrite the equation in a simpler form, introducing the following change of variable:

: $$H(w) = -w + w^{3/2} \cdot G(w).$$

Tutte's non-linear differential equation: the $q = 4$ case.

In the $q = 4$ case, **Tutte's non-linear differential equation**can be rewritten as a simple **autonomous non-linear differential equation**:

$$(G(w) - 6\,G_1(w)) \cdot (3\,G(w) + 8\,G_1(w) + 4\,G_2(w)) = 3 \cdot 2^7.$$

where $G(w) = w^{-3/2} \cdot (H(w) + w)$ and:

$$G_1(w) = w \cdot \frac{dG(w)}{dw}, \qquad G_2(w) = w \cdot \frac{dG_1(w)}{dw}.$$

As far as the singular points are concerned, this change of function $G(w) = w^{-3/2} \cdot (H(w) + w)$ suggests that the exponent $3/2$ should play a selected role. A diff-Padé analysis gives a first set of singular points: one gets one real singularity $w_s = 0.04965\ldots$, and a bunch of complex singularities $0.202837\ldots \pm i \cdot 0.0964358\ldots$, $0.470420\ldots \pm i \cdot 0.37727\ldots$, etc ... all of them with the exponent $3/2$, the exponents at ∞ being $-1/3$, $-2/3$, $-4/3$, $-5/3$, ...

Tutte's non-linear differential equation: the $q = 4$ case.

In order to get **very long series**, it is more efficient to consider **Tutte's recurrence** for $q = 4$. Using this recurrence we have been able to get $N = 24000$ coefficients of the series. This is a 376 Megaoctets file. This series has a **finite radius of convergence** $r \simeq 0.04965 \ldots$, the coefficients growing like λ^N where $\lambda \simeq 20.1378 \ldots$ We seek for linear differential operators, annihilating the series of order Q in the homogeneous derivative $\theta = w \cdot d/dw$, and of degree D for the polynomial coefficients such that $(Q+1) \cdot (D+1) = N - 1500$. We failed to find a linear differential operator.

This seems to **exclude** the possibility that the $q = 4$ series **could be a holonomic function**.

Seeking for algebraic functions for the reduction modulo primes.

We have shown that the **full susceptibility of the Ising model**, which is a **non-holonomic** function, actually reduces to **algebraic functions modulo any powers of the prime** 2. It is tempting to see if **this enumerative combinatorics series, for** $q = 4$, **also reduces to algebraic functions modulo the first eight primes** $2,\ 3,\ \ldots\ 19$ (and powers of these primes).
Since we have developed many tools to find (Fuchsian) **linear differential operators annihilating a given series modulo primes**, let us first try (before seeking directly for algebraic relations on this series modulo primes), to see if this $q = 4$ series, **modulo the first eight primes, is solution of a linear differential operator**.

Seeking for algebraic functions for the reduction modulo primes.

To take into account the fact that all the integer coefficients of the series are divisible by $q \cdot (q-1) = 12$, we will rather consider the series divided by $12\,w^2$, which is also a **series with integer coefficients**:

$$S(w) = \frac{H(w)}{12\,w^2} = 1 + 2\,w + 14\,w^2 + 138\,w^3 + 1608\,w^4$$
$$+20736\,w^5 + 286452\,w^6 + 4160274\,w^7 + 62772488\,w^8 + \cdots$$

We **actually found linear differential operators** for this last series, modulo the first primes 3, ... 17. Introducing the **homogeneous derivative** $\theta = w \cdot d/dw$, the linear differential operators L_p read respectively:

$$L_3 = 2\,w + \theta + (w+1) \cdot \theta^2 \qquad mod.\ 3,$$
$$L_5 = 2\,w + (2+3\,w) \cdot \theta + (w+2) \cdot \theta^2 \qquad mod.\ 5,$$

Seeking for linear ODEs for the reduction modulo primes.

$$
\begin{aligned}
L_7 &= 3\,w^3 + (4+w^3)\cdot\theta + (3\,w^3+3)\cdot\theta^3 + (5+w^3)\cdot\theta^4,\\
L_{11} &= 9\,w^{15} + 5\,w^{10} + 5\,w^5 + (2\,w^{15} + 6\,w^{10} + 9\,w^5 + 6)\cdot\theta\\
&\quad + (2\,w^{15} + 8\,w^{10} + 7\,w^5 + 1)\cdot\theta^2 + (5\,w^{15} + 7\,w^{10} + w^5)\cdot\theta^3\\
&\quad + (6 + 4\,w^5 + w^{10} + 2\,w^{15})\cdot\theta^4 + (10\,w^{15} + 9\,w^{10} + 8\,w^5 + 10)\cdot\theta^5\\
&\quad + (8\,w^{15} + 8\,w^{10} + 5\,w^5 + 7)\cdot\theta^6 + (5\,w^{15} + 4\,w^5 + 6)\cdot\theta^7\\
&\quad + (w^{15} + w^5 + 8)\cdot\theta^8,
\end{aligned}
$$

and:

$$
L_{13} = \sum_{n=0}^{8} p_n(w)\cdot\theta^n, \qquad L_{17} = \sum_{n=0}^{13} q_n(w)\cdot\theta^n,
$$

It is quite a surprise to find **linear differential operators** on such a **typically non-linear, probably non-holonomic**, function.

Seeking for algebraic functions for the reduction modulo primes.

However, keeping in mind the exact results modulo powers of the prime 2 on the **full susceptibility of the Ising model**, it is natural to ask if such results modulo various primes, could correspond to reductions of this (probably non-holonomic) series to **algebraic functions modulo primes**. This is actually the case: we calculated the p-curvature of all these linear differential operators L_p, modulo the primes p, and found that they all have **zero** p**-curvature**.

Let us show that these series, modulo various primes, **are actually algebraic functions modulo primes**, by **finding directly the polynomial equations** $P(w, S(w)) = 0$ they satisfy $mod.\ p$.

Seeking directly for polynomial equations for the reduction modulo primes (and powers of primes).

The series $S(w)$ satisfies, modulo $p = 3$, the polynomial relation:

$$w^2 \cdot S(w)^3 + 2\, S(w) + (1 + 2\,w + w^2 + w^5) = 0 \qquad mod.\ 3.$$

The series $S(w)$, modulo $p = 3^2$, satisfies, the **much more involved polynomial relation**:

$$\begin{aligned}
& w^3 \cdot (8\,w^{17} + 6\,w^{14} + 3\,w^{13} + 6\,w^{12} + 6\,w^{11} + 6\,w^{10} + 5\,w^8 \\
& \qquad +3\,w^6 + w^5 + 3\,w^4 + 3\,w^3 + 2\,w^2 + 6\,w + 3) \\
& + (5\,w^{15} + w^{12} + 5\,w^{11} + w^{10} + 5\,w^9 + 5\,w^8 \\
& \qquad +5\,w^6 + 5\,w^5 + 6\,w^3) \cdot S(w) \\
& +4\,w^5 \cdot (2\,w^5 + 2\,w^2 + w + 2) \cdot S(w)^2 \\
& +w^7 \cdot (w^{10} + 2\,w^7 + w^6 + 2\,w^5 + w^4 + w^3 + w + 1) \cdot S(w)^3 \\
& +w^7 \cdot (2\,w^5 + 2\,w^2 + w + 2) \cdot S(w)^4 \\
& +3\,w^7 \cdot S(w)^5 + w^9 \cdot (2\,w^5 + 2\,w^2 + w + 2) \cdot S(w)^6 = 0
\end{aligned}$$

Seeking directly for polynomial relations for the reduction modulo primes.

Modulo $p = 7$, we obtained the polynomial relation

$$\begin{aligned} w^4 \cdot S(w)^4 &+ w^2 \cdot (5\,w+1) \cdot S(w)^3 \\ &+ w \cdot (6\,w^2+5\,w+2) \cdot S(w)^2 + (w^2+2\,w+6) \cdot S(w) \\ &+2\,w^2+5\,w+1 = 0 \qquad mod.\ 7. \end{aligned}$$

Modulo $p = 11,\ 13,\ 17$ and 19 we also obtained four polynomial relations

$$\sum_{n=0}^{10} p_n(w) \cdot S(w)^n = 0, \qquad \sum_{n=0}^{14} q_n(w) \cdot S(w)^n = 0,$$

$$\sum_{n=0}^{24} r_n(w) \cdot S(w)^n = 0, \qquad \sum_{n=0}^{30} s_n(w) \cdot S(w)^n = 0,$$

We conjecture that this **non-holonomic function reduces to algebraic functions modulo every primes**, (or power of primes).

Another pedagogical non-holonomic example.

If the **product** of two holonomic functions is holonomic, the **ratio** of two holonomic functions is **not holonomic**.

Ratio of diagonals of rational functions, are (or can be recast into) series with *integer coefficients*, and **are actually such that**, **modulo any prime, they reduce to algebraic functions**.

Let us consider **non-holonomic functions** that are, **not only ratio of holonomic functions, but, in fact**, **ratio of diagonals of rational functions**:

$$R(x) \;=\; \frac{{}_2F_2\Big([\frac{1}{3},\,\frac{1}{3}],\,[1],\;27\,x\Big)}{{}_2F_2\Big([\frac{1}{2},\,\frac{1}{2}],\,[1],\;16\,x\Big)}.$$

Its series expansion is a series with **integer coefficients**:

$$1 \;-x\;+4\,x^2+208\,x^3+5549\,x^4+133699\,x^5+3142224\,x^6$$
$$+73623828\,x^7+1733029548\,x^8+41095725700\,x^9\;+\;\cdots$$

Another pedagogical non-holonomic example.

These two hypergeometric functions are diagonals of a rational function: their reductions modulo primes **must be algebraic functions**. For instance, modulo $p = 7$, they read:

$$
\begin{aligned}
{}_2F_2\Big([\frac{1}{2}, \frac{1}{2}], [1], \ 16\,x\Big) &= (1 + 4\,x + x^2 + x^3)^{-1/6} \qquad mod.\ 7, \\
{}_2F_2\Big([\frac{1}{3}, \frac{1}{3}], [1], \ 27\,x\Big) &= (1 + 3\,x + x^2)^{-1/6} \qquad mod.\ 7.
\end{aligned}
$$

Modulo the prime 7, the previous ratio $R(x)$ reduces, as it should, to the **ratio of the two previous (algebraic) reductions**:

$$
R(x) = \Big(\frac{1 + 4\,x + x^2 + x^3}{1 + 3\,x + x^2}\Big)^{1/6} \qquad mod.\ 7.
$$

In characteristic zero this ratio $R(x)$ **verifies a non-linear differential equation** that can be obtained from the two order-two linear ODEs verified by the two ${}_2F_1$.

The non-linear differential equation.

In characteristic zero this ratio $R(x)$ verifies the **non-linear (homogeneous polynomial) differential equation**:

$$
\begin{aligned}
&-2\,x^2\cdot\,(27\,x-1)\,(16\,x-1)\,\cdot\,\Big((27\,x-1)\,\cdot\,(16\,x-1)\cdot\,R_1\\
&\qquad\qquad-(72\,x+1)\cdot\,R\Big)\cdot\,R_3\\
&-2\,x\cdot\,\Big(3\,x\cdot\,(16\,x-1)\,(72\,x+1)\,(27\,x-1)\cdot\,R_1\\
&\qquad\qquad-(93312\,x^3-168\,x^2-297\,x+4)\cdot\,R\Big)\cdot\,R_2\\
&+2\,\cdot\,(29376\,x^3+5580\,x^2-221\,x+1)\cdot\,R\cdot\,R_1\\
&+3\,x^2\cdot\,(27\,x-1)^2\,(16\,x-1)^2\cdot\,{R_2}^2\\
&+(16\,x-1)\,(1944\,x^3-1569\,x^2+58\,x-1)\cdot\,{R_1}^2\\
&+(144\,x^2-432\,x+1)\cdot\,R^2\quad=\quad 0.
\end{aligned}
$$

where R denotes $R(x)$, and R_n denote d^nR/dx^n.

Reduction of ${}_nF_{n-1}$ hypergeometric functions modulo primes

In order to get some perspective, one can look at reduction of **holonomic functions** that are **diagonal of rational functions**. In general one gets **quickly and easily the algebraic functions**. Let us consider a very simple example of diagonal of rational functions. Let us consider the series expansions (with integer coefficients) of ${}_4F_3([1/2,\,1/2,\,1/2,\,1/2],\,[1,\,1,\,1],\,2^8\cdot x)$, which corresponds to a **Calabi-Yau linear differential operator**, and is the **diagonal of a rational function** since it is the *Hadamard product* of four time the algebraic function $(1-4\,x)^{-1/2}$. Modulo 23, this hypergeometric function becomes the **algebraic function** $1/P(x)^{1/22}$, where the polynomial $P(x)$ is a truncation of the series expansion of this hypergeometric function modulo 23:

$$\begin{aligned} P(x) \;=\; & 1 + 16\,x + 8\,x^2 + 12\,x^3 + x^4 + x^5 \\ & +3\,x^6 + 4\,x^7 + 18\,x^8 + 16\,x^9 + 12\,x^{10} + x^{11}. \end{aligned}$$

Reduction of ${}_3F_2([1/9,\,4/9,\,5/9],[1/3,\,1],3^6\,x)$ modulo primes

Along this **hypergeometric line** it is worth recalling the hypergeometric function ${}_3F_2([1/9,\,4/9,\,5/9],[1/3,\,1],3^6\,x)$ introduced by G. Christol, a few decades ago, to provide an example of holonomic G-series with **integer coefficients that may not be the diagonal of rational function**.
If one performs the same reductions modulo primes, one finds, **in contrast with the previous studies** of reductions modulo primes of diagonals of rational functions, that it becomes **almost impossible to see whether such series modulo primes are algebraic functions**. Even modulo 2, finding the polynomial relation is **quite hard**, because its degree is large: $(1+x^2)\cdot S^{64} - S = 0$. Of course checking the result is much simpler:

$$ {}_3F_2\Big([\frac{1}{9},\,\frac{4}{9},\,\frac{5}{9}],\,[\frac{1}{3},\,1],3^6\,x\Big) \;=\; (1\,+\,x^2)^{-1/63} \qquad mod.\ 2. $$

Reduction of holonomic functions modulo primes.

As far as the reduction of **holonomic functions modulo primes** is concerned, we seem to have the following situation: either the holonomic function is actually the *diagonal of a rational function*, the reduction to algebraic function modulo primes is thus garanted, and **one finds, very simply and quickly, these algebraic functions**, or the holonomic function is **not** "obviously" the diagonal of the rational function, and **getting these algebraic functions can be extremely difficult**.

This difficulty to find polynomial relations, **even modulo rather small primes**, for such a **holonomic function** (which is not obviously the diagonal of a rational function), **has to be compared with the rather easy way we obtained polynomial relations for a (probably non-holonomic) series solution of Tutte's $q = 4$ non-linear differential equation**.

To be or not to be holonomic ...

Towards a large class of selected non-holonomic functions.

We believe that this result on the $q = 4$ **solution of Tutte's non-linear differential equation** for the generating function of the q-coloured rooted triangulations by vertices, **is not an isolated curiosity**, but corresponds to a first pedagogical example of a **large class of remarkable non-holonomic functions in theoretical physics** (lattice statistical physics, enumerative combinatorics ...) that **reduce to algebraic functions modulo primes** (and power of primes). It is important to understand these remarkable non-holonomic functions: are they ratio of holonomic functions (having in mind **ratio of diagonals of rational functions**), or, more generally, rational (resp. algebraic) functions of diagonals of rational functions, do the non-linear differential equations they satisfy have the **Painlevé property**, etc ... ?

We need new tools, new algorithms !

Along this line, it is essential to build new tools, **new algorithms** to see whether a given (large) series is solution of a **non-linear differential equation**, and, in particular, of a **polynomial differential equation**. It must be clear that this kind of **"non-linear differential Padé" analysis**, should not be performed in the most general non-linear framework: **it must be performed with some assumptions**, ansatz, corresponding to the problem of theoretical physics one considers (Painlevé property assumption, regular singularities assumptions, autonomous assumptions), non-linear differential equations associated with *Schwarzian derivatives* or *modular forms*, ...).

We need new algorithms, longer series !

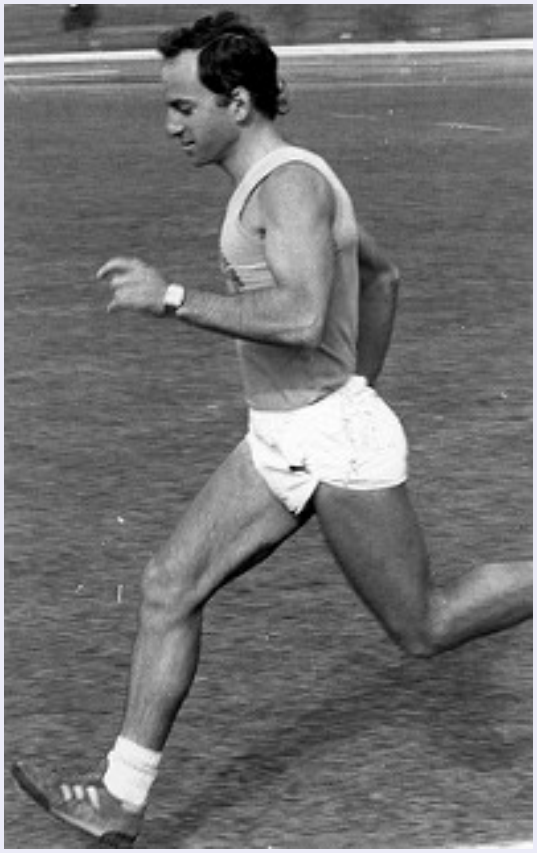
It is crucial to build new tools, **new algorithms to see whether a given (large) series is a ratio of holonomic functions** (having in mind **ratio of diagonals of rational functions**), or more generally could be rational (resp. algebraic) functions of diagonals of rational functions.

In lattice statistical mechanics such kind of results is clearly a strong incentive to obtain **longer series** (modulo some small primes $p = 3, \ldots$) for the **full susceptibility of the Ising model** to see if the susceptibility series **reduces, for instance modulo 3, to an algebraic function**.

This is not the end ...

The road to **non-holonomic functions** will certainely be a **hard one**, but it will be a **beautiful one**.

As far as **non-holonomic functions** are concerned:



The road is hard, but I am strong (Jean-Paul Sartre's Roads to Freedom trilogy, sung by Georgia Brown).
La route est dure mais qu'elle est belle. Le but est difficile mais qu'il est grand ! Allons ! Le départ est donné. Allocution radiodiffusée du Général de Gaulle (13 mai 1958).
THE END (of this talk)

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Differential Galois group for lattice Green functions ODEs

The 11-dimensional fcc operator is of order 27 (2464 coeff. are necessary to obtain the ODE), the 12-dimensional fcc operator is of order 32 (3618 coeff. are necessary). More generally, the operator of the d-dimensional fcc lattice is of order q given by

$$q = \frac{d^2}{4} - \frac{d}{2} + \frac{17}{8} - \frac{(-)^d}{8},$$

its differential Galois group being $SO(q, C)$ for d odd and $Sp(q, C)$ for d even, the order of U_1, the rightmost self-adjoint operator, being d, the order of the other self-adjoint operators U_n being 1 for d odd and 2 for d even.
Note that these higher order operators are **not MUM** (Maximal Unipotent Monodromy). The Lattice Green Function is a series with **integer coefficients**.

Differential Galois group for lattice Green functions ODEs

We have been able to find the linear differential operator of the **seven-dimensional fcc lattice Green function**. It is an order-11 operator.

$$G_{11}^{7Dfcc} = (U_5 \cdot U_4 \cdot U_3 \cdot U_2 \cdot U_1 + U_5 \cdot U_4 \cdot U_1 + U_5 \cdot U_2 \cdot U_1 + U_5 \cdot U_4 \cdot U_3 + U_3 \cdot U_2 \cdot U_1 + U_1 + U_3 + U_5) \cdot r(x),$$

where $r(x)$ is a rational function, where U_2, U_3, U_4 and U_5 are *order-one self-adjoint* operators, and where U_1 is an **order-seven self-adjoint** operator. G_{11}^{7Dfcc} is **non-trivially homomorphic to its adjoint**

$$adjoint(L_{10}) \cdot G_{11}^{7Dfcc} = adjoint\left(G_{11}^{7Dfcc}\right) \cdot L_{10}.$$

Integrable models are like a needle in the haystack

To be or not to be holonomic ...

Integrability: Holonomic functions.

Non-integrability: Non-holonomic functions, **However**

Non-holonomic functions like **Chazy III**, and also the susceptibility of the square Ising model are **non-holonomic**, but they **do belong to the "Integrability world"**. The $\chi^{(n)}$ decomposition of the χ susceptibility yields Calabi-Yau ODE (and manifolds) and highly selected linear differential operators (special differential Galois groups, etc ...). The $\chi^{(n)}$'s are **diagonal of rational functions**: they are the **class of transcendental functions which is the "closest" to algebraic functions** (modulo a prime they do reduce to algebraic functions). As far as the algorithmic complexity of the calculations of the χ series, these calculations are polynomial (in N^4, consequence of J.H.H. Perk's finite difference equations which can be viewed as a **finite difference generalization of Painlevé equations**). **Natural boundary** is not even characteristic of non-integrability (on the contrary !): think of Chazy III.

Non-holonomic functions ratio of holonomic functions

Along this line it is fundamental to recall that the **ratio** (not the product !) of **two holonomic** functions is **non-holonomic**

$$\frac{d^2y}{dx} + R(x) \cdot y = 0, \qquad \tau(x) = \frac{y_1}{y_2}, \qquad \{\tau(x), x\} = 2\,R(x).$$

The **Chazy III equation** is a third-order **non-linear** differential equation (it can also be rewritten using a **Schwarzian derivative**) that has a **natural boundary** for its solutions:

$$\frac{d^3y}{dx^3} = 2\,y\frac{d^2y}{dx^2} - 3\left(\frac{dy}{dx}\right)^2.$$

It has the **quasi-modular form** Eisenstein series E_2 has a solution

$$y = \frac{1}{2} \cdot \frac{\Delta'}{\Delta} = \frac{1}{2} \cdot E_2$$

where Δ is a selected holonomic function: a **modular form**.

Isogenies, Landen transformations, Modular curve

The Landen transformation corresponds to the *genus zero fundamental modular curve*

$$\begin{aligned} j^2 \cdot j'^2 \; &- (j + j') \cdot (j^2 + 1487 \cdot j\, j' + j'^2) \\ &+ 3 \cdot 15^3 \cdot (16\, j^2 - 4027\, j\, j' + 16\, j'^2) \\ &- 12 \cdot 30^6 \cdot (j + j') + 8 \cdot 30^9 = 0, \end{aligned}$$

which relates the two j-functions

$$j(k) = 256 \cdot \frac{(1 - k^2 + k^4)^3}{(1 - k^2)^2 \cdot k^4}, \quad j(k_L) = 16 \cdot \frac{(1 + 14\, k^2 + k^4)^3}{(1 - k^2)^4 \cdot k^2}.$$

Modular Forms

Let us consider the second order linear differential operator

$$\frac{d^2}{dz^2} + \frac{\left(z^2 + 56\,z + 1024\right)}{z \cdot (z+16)\,(z+64)} \cdot \frac{d}{dz} - \frac{240}{z \cdot (z+16)^2\,(z+64)}.$$

which has the (modular form) solution:

$$\begin{aligned} & {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1]; 1728\,\frac{z}{(z+16)^3}\Big) \\ = \; & 2 \cdot \Big(\frac{z+256}{z+16}\Big)^{-1/4} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1]; 1728\,\frac{z^2}{(z+256)^3}\Big). \end{aligned}$$

Fundamental modular curve

The **two pull-backs** in the previous modular form

$$u \;=\; u(z) \;=\; \frac{1728\; z}{(z+16)^3}, \qquad v \;=\; \frac{1728\; z^2}{(z+256)^3} \;=\; u\Big(\frac{2^{12}}{z}\Big).$$

are related by a *Atkin-Lehner involution* $z \quad \leftrightarrow \quad 2^{12}/z$, and correspond to a rational parametrization of the **fundamental modular curve** $X_0(2)$:

$$\begin{aligned} &5^9\, v^3\, u^3 \;-12\cdot 5^6\, u^2\, v^2\cdot(u+v) \\ &+375\; u\, v\cdot(16\, u^2 \;+16\, v^2 \;-4027\, v\, u) \\ &-64\,(v+u)\cdot(v^2+1487\, v\, u \;+u^2) \;+2^{12}\cdot 3^3\cdot u\, v \;=\; 0. \end{aligned}$$

relating two **Hauptmoduls** u and v.

Schwarzian derivative and natural boundary

It can be rewritten in terms of a Schwarzian derivative:

$$f^{(4)} \;=\; 2\,f'^2 \cdot \{f,\,x\} \;=\; 2\,f'\,f''' - 3\,f''^2 \qquad \text{with:} \quad y \;=\; \frac{df}{dx}.$$

It was introduced by Jean Chazy (1909, 1911) as an example of a third-order differential equation with a movable singularity that has a **natural boundary** for its solutions. It is also worth recalling the **Halphen-Ramanujan differential system**:

$$P' \;=\; \frac{P^2 - Q}{12}, \qquad Q' \;=\; \frac{P\,Q - R}{3}, \qquad R' \;=\; \frac{P\,R - Q^2}{2},$$

where $P = E_2$, $Q = E_4$, $R = E_6$ and X' denotes here the homogeneous derivative $q \cdot \frac{dX}{dq}$, and E_n the Eisenstein series.