Article

Plea for diagonals and telescopers of rational functions

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Abstract: This paper is a plea for diagonals and telescopers of rational, or algebraic, functions using creative telescoping, in a computer algebra experimental mathematics learn-by-examples approach. We show that diagonals of rational functions (and this is also the case with diagonals of algebraic functions) are left invariant when one performs an infinite set of birational transformations on the rational functions. These invariance results generalize to telescopers. We cast light on the almost systematic property of homomorphism to their adjoint of the telescopers of rational, or algebraic, functions. We shed some light on the reason why the telescopers, annihilating the diagonals of rational functions of the form P/Q^k and $1/Q$, are homomorphic. For telescopers with solutions (periods) corresponding to integration over non-vanishing cycles, we have a slight generalization of this result. We introduce some challenging examples of generalization of diagonals of rational functions, like diagonals of transcendental functions, yielding simple ${}_2F_1$ hypergeometric functions associated with elliptic curves, or (differentially algebraic) lambda-extension of correlation of the Ising model.

Keywords: Diagonals of rational or algebraic functions, creative telescoping, globally bounded series, modular forms, multi-Taylor expansions, multivariate series expansions, magnetic susceptibility of the Ising model, lattice Green functions, Fuchsian linear differential equations, homomorphisms of differential operators, self-adjoint operators, Poincaré duality, differential Galois groups

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1. Introduction: plea for a computer algebra experimental mathematics learn by example approach

A paper in the honor of Professor Richard Kerner must be a paper on theoretical physics, mathematical physics, physical mathematics, applied mathematics, applicable mathematics or even experimental mathematics [1]. These different domains have large overlaps and, quite often, their differences, or shades, are slightly irrelevant, only corresponding to social membership to different "mathematical tribes". This computer algebra paper will actually be a plea for *diagonals and telescopers of rational (or algebraic) functions* and for *creative telescoping*, with a computer algebra experimental mathematics learn-by-examples approach.

1.1. Honor, pride and prejudice

The Journal of Mathematical Physics defines mathematical physics as "the application of mathematics to problems in physics and the development of mathematical methods suitable for such applications and for the formulation of physical theories". An alternative definition would also include those mathematics that are inspired by physics (also known as physical mathematics). Mathematical physics clearly raises the question of the watershed

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between mathematics and physics (especially in France ...). Does “Mirror Symmetry” [2–5] which is a relationship between geometric objects called Calabi–Yau manifolds, belong to algebraic geometry or theoretical physics? Does “Special Relativity” belong to physics or mathematics, Einstein or Poincaré? “Einstein was reluctant to acknowledge that the Michelson-Morley experiment had a significant influence on his road to special relativity” [6]. In fact, “once Maxwell’s equations are properly understood mathematically, special relativity is an inevitable consequence” [6]. Physical mathematics is sometimes viewed with suspicion by both physicists *and* mathematicians. On the one hand, mathematicians regard it as deficient, for lack of proper mathematical rigor. In the years since this “mathematical physics debate” erupted [7] there have been many spectacular successes scored by physical mathematics, thanks to the “unreasonable effectiveness” of physics in the mathematical sciences. Dyson famously proclaimed: “As a working physicist, I am actualy aware of the fact that the marriage between mathematics and physics, which was so enormously fruitful in past centuries, has recently ended in divorce”. This “divorce” is particularly serious in France, because of the overwhelmingly leading figure of Alexander Grothendieck and the huge influence of the Bourbaki group, which raises the question of rigor versus creativity (“We should not confuse rigor with rigor mortis”, Isadore Singer, see [6]). Recalling Pierre Cartier [8], the Bourbaki group has been criticized by several mathematicians, including its own former members, for a variety of reasons. “ Criticisms have included the choice of presentation of certain topics within the Éléments [9] at the expense of others, *dislike of the method of presentation* for given topics, dislike of the group’s working style, and a perceived elitist mentality around Bourbaki’s project and its books, especially during the collective’s most productive years in the 1950s and 1960s. There is essentially *no analysis* beyond the foundations: *nothing about partial differential equations*, nothing about probability. There is also *nothing about combinatorics*, nothing about algebraic topology [10], *nothing about concrete geometry*. Anything connected with *mathematical physics is totally absent* from Bourbaki’s text.” Dieudonné (founding member), later, regretted that Bourbaki’s success had contributed to a snobbery for pure mathematics in France, *at the expense of applied mathematics* [11,12]. In an interview (to Marian Schmidt in 1990), he said: "It is possible to say that there was no serious applied mathematics in France for forty years after Poincaré. There was even a snobbery for pure mathematics. When one noticed a talented student, one would tell him “You should do pure math.” On the other hand, one would advise a mediocre student to do applied mathematics while thinking, “It’s all that he can do! ...”. Apart from french mathematicians (when in doubt, blame the french), this snobbery for pure mathematics met with harsh criticism from Vladimir Arnold in his deliciously polemical paper [13] “Sur l’éducation mathématique”.

Quantum groups emerged from one (Yang-Baxter integrable) explicit example, namely Quantum Toda, and *not* from an ex-nihilo abstract, formal, construction of a noncommutative algebra formalism, and other $C^\star$-algebras, dressed with coassociative coproducts. In theoretical physics we get used to the emergence of *modular forms* [14] and sometimes automorphic forms [15] like *Shimura forms* [16]. If a physicist asks a mathematician for more information on these structures he will probably only get the academical Poincaré upper half-plane definition and formalism which is totally and utterly useless for him and he will not recognize the representation of *modular forms* and *Shimura forms* which naturally emerges in physics [16,17] in terms of pullbacked ${}_2F_1$ hypergeometric functions. In theoretical physics we are flooded by *elliptic curves*, K3 surfaces, Calabi-Yau manifolds [3,18–23]. If a physicist tries to discuss with a mathematician of the elliptic curve he just discovered (he has even calculated the j-invariant, or the Hauptmodul, of this elliptic curve ...), he might be severely rebuked that he has absolutely no right to talk of an elliptic curve because an elliptic curve must have a “specified point”, or will be seen with suspicion because his elliptic curve does not correspond to the complete intersection of quadrics [24,25] framework mathematicians like to consider in their theorems. Along this (slightly polemical ...) line, pure mathematicians will, often, refuse to provide *representation* of their formalism, in particular they will refuse to provide *examples*. If a physicist, eager to understand a mathematical concept, asks for an example of an algebraic variety, an example of holonomic function, or an example of functor, some mathematicians will, maliciously, reply: a point, the constant function and the oblivion functor. In such a frustrating "dialogue of the deaf" between physicists and mathematicians, mathematical physics is probably the perfect place to be criticized by physicists to be too abstract, or too mathematical, and also by mathematicians for a lack of rigor, a lack of mathematical proofs.

Jean-Louis Verdier performed his thesis under the direction of Alexandre Grothendieck. He was a member of the Bourbaki group. He passed away in August 1989. At this step, one of us (JMM) would like to seize the opportunity of this experimental mathematics paper in honor of Professor Richard Kerner, to express his deep regrets for the numerous fruitful conversations with Jean-Louis Verdier and his very generous pedagogical explanations. A discussion with him was not flooded with "Derived Categories" or "p-adic cohomology", but with simple examples and representations of the mathematical concepts. A really good mathematician can provide examples, he is not afraid, or ashamed, to provide examples and representations. For Jean-Louis Verdier mathematics was not an obfuscation contest.

This paper is an experimental mathematics [1] paper with a learn-by-example approach: we get puzzling *exact* results from computer algebra (Maple, Mathematica), and we hope mathematicians will be interested to provide proofs of these results, in a proper framework. Furthermore, these exact results, useful for physics, raise a lot of fascinating new questions at the crossroad of different domains of mathematics.

1.2. Diagonal of rational functions, creative telescoping, birational transformations and effective algebtraic geometry

Diagonals of rational functions (or diagonals of algebraic functions) have been shown to emerge naturally [26] for n-fold integrals in physics (corresponding to solutions of linear differential operators of quite high order [27,28]), field theory, enumerative combinatorics [29,30], seen as "Periods" [31] of algebraic varieties (corresponding to the denominators of these rational functions). The fact that diagonals of rational, or algebraic, functions occur frequently in physics, explains many unexpected mathematical properties encountered in physics, that are far more obvious from a physics viewpoint. Physicists are clearly very interested to see if the critical exponents of the *three-dimensional* Ising model are, or are not, *rational* numbers. In contrast, since many lattice Green functions in any dimension [32] are diagonals of rational functions, their critical exponents are necessarily rational numbers in *any* dimension. Along this line, the linear differential operators, annihilating these "Periods", are *globally nilpotent* [33,34], and, consequently, the *critical exponents* of *all* the (regular) singular points of these operators are necessarily *rational numbers* (Katz theorem states that *globally nilpotent* linear differential operators are *fuchsian* with *rational exponents*, see for instance [35]). These n-fold integrals are also *globally bounded* [26,36] series, which means that they can be recast into (finite radius of convergence) series with *integer coefficients*. Furthermore, these series, with integer coefficients, reduce *modulo every prime* to *algebraic functions*. The calculation of the linear differential operators annihilating these n-fold integrals of algebraic functions can be systematically performed using the *creative telescoping method* [37–39] which corresponds, essentially, to successive *differential algebra eliminations which are blind to the cycles* over which one performs the n-fold integrals. At first sight one expects the analysis of these n-fold integrals to require, as in the S-matrix theory [40], a lot of complex analysis of *several complex variables*, but one quickly discovers, with creative telescoping, that one needs *differential algebra*, possibly *algebraic geometry* [41], because of the crucial role of an algebraic variety and, surprisingly one finds out almost "arithmetical" properties (like in the Grothendieck–Katz p-curvature conjecture which is a local-global principle for linear ordinary differential equations, related to differential Galois theory). More experimentally, this time, one finds out that *almost all* the diagonals of rational, or algebraic, functions, corresponding, or not, to physics, are annihilated by linear differential operators which are *homomorphic to their adjoint*, and consequently, their differential Galois groups are (or are a subgroup of) selected $Sp(n, \mathbb{C})$ symplectic or $SO(n, \mathbb{C})$

orthogonal groups [34,42,43]. More generally, one finds out that the *telescopers of almost all* the rational, or algebraic, functions are *also* homomorphic to their adjoint [42]. A physicist, already surprised to see the emergence of all these mathematical concepts in his backyard, will have the prejudice that these *selected differential Galois groups* are probably a consequence of some "sampling bias", these diagonals and telescopers being, in fact, related to (Yang-Baxter) integrable models, like the $\chi^{(n)}$ components of the susceptibility of the Ising model [27,28], or beyond, Calabi-Yau manifolds, Mirror Symmetries, Picard-Fuchs systems, and other theory "integrable" in some way (Yang-Mills ...). In contrast, a mathematician will have the prejudice that this is nothing but the *Poincaré duality* [44] since we have a canonical algebraic variety for all these diagonals or telescopers [41]. Experimentally it is quite hard to find telescopers, or linear differential operators, that are not homomorphic to their adjoint, i.e. that do not have selected symplectic, or orthogonal, differential Galois groups [34,42,43]. Christol conjectured [45,46] that every D-finite globally bounded series is the diagonal of a rational function. If one considers Christol's conjecture [45–49], one can seek for $_nF_{n-1}$ hypergeometric series with integer coefficients that are candidates to be counter-examples to Christol's conjecture [45–48]. Among these candidates a sub-set has actually been seen [49] to be diagonals of rational, or algebraic, functions like for instance $_3F_2([2/9, 5/9, 8/9], [2/3, 1], x)$, or $_3F_2([1/9, 4/9, 7/9], [1/3, 1], x)$. The fact that the others, like the original example of G. Christol, $_3F_2([1/9, 4/9, 5/9], [1/3, 1],\, 3^6\, x)$, are, or are not, diagonals of rational or algebraic functions remains an open question. It turns out that the linear differential operators of these $_nF_{n-1}$ candidates precisely provide such rare examples of linear differential operators (annihilating diagonals of rational, or algebraic, functions), that are *not* homomorphic to their adjoint. The existence of such examples (curiously related to Christol's conjecture ...) shows that seeing the emergence of such selected differential Galois groups [42] for diagonals of rational, or algebraic, functions cannot simply be seen as some consequence of the *Poincaré duality*. The Poincaré duality works for *any* algebraic variety: the diagonal of *any* rational, or algebraic, function should always yield "self-dual" linear differential operators in the sense that they are homomorphic to their adjoint. This is *not* the case. Could it be that the physicist's prejudice is right and that, trying to be generic in our computer algebra experiments, we were, in fact, just exploring diagonals of selected subsets of rational, or algebraic, functions related to some kind of "integrable" physics?

Like Monsieur Jourdain (in "Le Bourgeois Gentilhomme", Molière) speaking "prose" without noticing himself, physicists often perform some fundamental mathematics when they work on their n-fold integrals without noticing these n-fold integrals are, in fact, diagonals of rational, or algebraic, functions. In fact diagonals of rational, or algebraic, functions, and more generally telescopers, are a perfect subject of analysis in mathematical physics: they are, essentially, not well-known by mathematicians and by physicists (even if physicists speak "diagonal" without noticing ...), and even when these concepts are superficially known, they are not taken seriously by mathematicians, probably because the definition is so simple, and the calculations are just "computer algebra" which is not highly regarded in the "mathematical food chain". This is in contrast with the fact that almost every calculation of a diagonal of rational, or algebraic, function, or calculation of a telescoper, yields interesting, or remarkable, sometimes even puzzling exact results, providing answers in physics and mathematics, but also raising new interesting questions, that could be called "speculative mathematics".

In a learn-by-example approach we are going to address the previous questions of "duality-breaking" of some telescopers of rational, or algebraic, functions, and we will also sketch some remarkable *birational symmetries* [24,25,50,51] of the diagonals and telescopers of rational, or algebraic, functions.

2. Definition of the diagonals of rational, or algebraic, functions. Definition of telescopers.

The purpose of this paper is not to provide an introduction to *creative telescoping* [37–39,52–55] but, rather, to provide many (non-trivial) pedagogical examples of telescopers using extensively the "HolonomicFunctions" package [56]. One can obtain these telescopers using Chyzak's algorithm [55] or Koutschan's semi-algorithm [56] (semi-algorithm because the termination is not proven). For the examples displayed in this paper, Koutschan's package [56] is more users-friendly or efficient.

Creative telescoping [37–39,52–55] is a methodology to deal with parametrized symbolic sums and integrals that yields differential/recurrence equations for such expressions. This methodology became popular in computer algebra in the past twenty five years. By "telescoper" of a rational function, say $R(x,y,z)$, we here refer to the output of the creative telescoping program [56]. The telescoper T represents a linear differential operator that is satisfied by the diagonal $Diag(R)$, and *also all the other "periods"*.

The paper is essentially dedicated to solutions of telescopers of rational functions which are *not necessarily* diagonals of rational functions. These solutions correspond to "periods" [31] of algebraic varieties over some cycles which are *not necessarily vanishing cycles* [57], like in the case of *diagonals* of rational functions.

The reader interested in the connection between the process of taking diagonals, calculating telescopers, and the notion of "Periods", de Rham cohomology (i.e. differential forms) and other Picard-Fuchs equations can read the thesis of Pierre Lairez [58] (see also [31]).

2.1. Definition

Let us recall that the diagonal of a rational function in (for example) three variables is obtained through its multi-Taylor expansion [19,20]

$$R(x,y,z) \quad = \quad \sum_m \sum_n \sum_l a_{m,n,l} \cdot x^m\, y^n\, z^l, \tag{1}$$

by extracting the "diagonal" terms, i.e. the powers of the product $p = xyz$:

$$Diag\Big(R(x,y,z)\Big) \quad = \quad \sum_m a_{m,m,m} \cdot p^m. \tag{2}$$

In order to avoid a proliferation of variables, the variable p, the diagonal (2) depends on, is, in the following, simply denoted x (see below (3)). Extracting these diagonal terms essentially amounts to finding constant terms [59] in several complex variables expansions, i.e. amounts to performing a residue calculation in several complex variables expansions

$$\begin{aligned} Diag\Big(R(x,y,z)\Big) \quad &= \quad \int_{\mathcal{C}} \frac{1}{y\,z} \cdot R\Big(\frac{x}{y\,z},\, y,\, z\Big) \cdot dy\; dz \\ &= \quad \frac{1}{2\,i\,\pi} \int \frac{1}{2\,i\,\pi} \int \; \sum_m \sum_n \sum_l a_{m,n,l} \cdot x^m\, y^{n-m}\, z^{l-m} \cdot \frac{dy}{y}\, \frac{dz}{z} \quad = \quad \sum_m a_{m,m,m} \cdot x^m, \end{aligned} \tag{3}$$

or equivalently

$$Diag\Big(R(x,y,z)\Big) \quad = \quad \int_{\mathcal{C}} \frac{1}{y\,z} \cdot R\Big(\frac{x}{y},\, \frac{y}{z},\, z\Big) \cdot dy\, dz, \tag{4}$$

where $\mathcal{C}$ denotes a vanishing cycle [57], where $\int_{\mathcal{C}}$ is a symbolic notation for the n-fold integral with the well-suited pre-factors, and where the diagonal (4) is seen as a function of the remaining variable x. This is the very reason why diagonals of rational, or algebraic, functions can be interpreted as n-fold integrals [26]. More generally, with n variables, one can write the diagonal of a rational function of n-variables as the residue in $n-1$ variables $x_2, \, \cdots, \, x_n$:

$$Diag\Big(R(x_1, \, x_2, \, \cdots, \, x_n)\Big) \tag{5}$$
$$= \, \frac{1}{2\, i\, \pi} \int \cdots \frac{1}{2\, i\, \pi} \int \, \frac{1}{x_2 \, \cdots \, x_n} \cdot R\Big(\frac{x_1}{x_2 \, \cdots \, x_n}, \, x_2, \, \cdots, \, x_n\Big) \cdot dx_2 \, \cdots \, dx_n.$$

If the definition of the diagonal of a rational or algebraic function is very simple, it *does not mean* that calculating such a diagonal is simple ! By "calculating" we mean finding that the series, corresponding to the diagonal, is the series expansion of some known special function [60–63] (an algebraic function [64], a pullbacked ${}_2F_1$ hypergeometric function which turns out to be a *modular form* [16,65,66], a ${}_nF_{n-1}$ hypergeometric function, a Heun function [67], etc). Most of the time, it means, since diagonals of rational, or algebraic, functions are selected (Fuchsian [27,28,68], G-nilpotent operators, globally bounded series [36]) D-finite functions, finding the linear differential operator annihilating the diagonal series, even if we are not able to "solve" this linear differential equation. Finding this linear differential operator can be performed by first getting large series expansion of the diagonal and then finding, by a "guessing" approach, the linear differential operator, or getting the linear differential operator from a more global differential algebra approach, called creative telescoping.

2.2. *Telescopers*

For pedagogical reason let us sketch creative telescoping [37–39,52–55] in the case of a rational function of three variables. By "telescoper" of a rational function, say $R(x, y, z)$, we here refer to the output of the creative telescoping program [56], applied to the transformed rational function $\hat{R} \, = \, R(x/y, y/z, z)/(yz)$. Such a telescoper is a linear differential operator T in x and ∂x, such that

$$T \cdot \Big(\frac{1}{y\, z} \cdot R\Big(\frac{x}{y}, \, \frac{y}{z}, \, z\Big)\Big) \, + \frac{\partial U}{\partial y} \, + \frac{\partial V}{\partial z} \, = \, 0, \tag{6}$$

where the so-called "certificates" $U, \, V$ are rational functions in $x, \, y, \, z$. These rational functions are often quite large rational functions. This equation is called the *telescoping equation*. Extracting the diagonal of a rational function amounts to calculating residues in several complex variables, namely

$$Diag\Big(R(x, \, y, \, z)\Big) \, = \, \int_{\mathcal{C}} \frac{1}{y\, z} \cdot R\Big(\frac{x}{y}, \, \frac{y}{z}, \, z\Big), \tag{7}$$

where the cycle $\mathcal{C}$ is a vanishing cycle [57]. Performing the previous integration over a cycle $\mathcal{C}$ on the LHS of the telescoping equation (6) one will get (with the reasonable assumption that the linear differential operator T commutes with the integration):

$$T \cdot Diag\Big(R(x, \, y, \, z)\Big) \, + \int_{\mathcal{C}} \Big(\frac{\partial U}{\partial y} \, + \frac{\partial V}{\partial z}\Big) \, = \, 0. \tag{8}$$

Again (with reasonable assumptions) one can expect the second term in (8) to be equal to zero, thus yielding the equation:

$$T \cdot Diag\Big(R(x, \, y, \, z)\Big) \, = \, T \cdot \int_{\mathcal{C}} \frac{1}{y\, z} \cdot R\Big(\frac{x}{y}, \, \frac{y}{z}, \, z\Big) \, = \, 0. \tag{9}$$

In other words, the telescoper T represents a linear differential operator annihilating the diagonal $Diag(R)$. For the calculation of a diagonal, the cycle $\mathcal{C}$ has to be a *vanishing cycle* (residue calculation). Note that the creative telescoping calculations giving as an output the telescoper T and the two "certificates" U and V, essentially amounts to performing differential algebra calculations (similar to integration by part for several complex variables). Since these creative telescoping calculations are differential algebra eliminations, *they are totally and utterly blind to the cycle* $\mathcal{C}$. Consequently, even if one performs an integration over a *non-vanishing cycle*, the telescoper T will also be such that

$$T \cdot \mathcal{P} \, = \, 0 \qquad \hbox{where:} \qquad \mathcal{P} \, = \, \int_{\mathcal{C}} \, {{1} \over {y\, z}} \cdot \, R\Bigl({{x} \over {y}}, \, {{y} \over {z}}, \, z\Bigr), \tag{10}$$

this integral being *not necessarily equal* to the diagonal $Diag(R(x, \, y, \, z))$ (which could be, for instance, equal to zero). Equation (10) means that the telescoper annihilates *all the periods* $\mathcal{P}$.

The paper is essentially dedicated to solutions of telescopers of rational functions which are not necessarily diagonals of rational functions. These solutions correspond to *periods* [31] of algebraic varieties over some cycles *which are not necessarily vanishing cycles* like in the case of diagonals of rational functions.

To sum-up: In order to calculate the diagonal of a rational function one can try, in a very down-to-earth way, to get large enough series expansions of this diagonal from multi-series expansion, and then try some guessing approach to obtain the linear differential operator annihilating the diagonal of a rational function, or one can perform the creative telescoping approach that will provide this telescoper even if the diagonal is zero, or cannot be nicely defined because the rational function does not have a multi-Taylor expansion: in that case the telescoper annihilates periods corresponding to *all* the cycles, in particular non-vanishing cycles.

2.3. Diagonals versus telescopers: vanishing cycles versus non-vanishing cycles

2.3.1. Diagonals versus telescopers: a first example

Let us first consider the following rational function of three variables

$$R(x,y,z) \quad = \quad {{1} \over { -x \, -y \, \, -z^2 }}. \tag{11}$$

This rational function does not have a multi-Taylor expansion, and thus we cannot define the diagonal of the rational function. This rational function has, however, a telescoper which is a linear differential operator of order one, namely $5\,\theta \, +2$, where $\theta \, = x\,D_x \, = x\,d/dx$ is the homogeneous derivative. Let us now consider a slightly more general rational function:

$$R(x,y,z) \quad = \quad {{1} \over {\alpha \, \, \, -x \, -y \, \, -z^2 }}. \tag{12}$$

This rational function (12) has a multi-Taylor expansion, and one can, thus, get the first terms of the diagonal of this rational function (12):

$$Diag\Bigl(R(x,y,z)\Bigr) \quad = \quad {{1} \over {\alpha}} \, \, + {{30} \over {\alpha^6}} \cdot x^2 \, \, + {{3150} \over {\alpha^{11}}} \cdot x^4 \, \, + {{420420} \over {\alpha^{16}}} \cdot x^6 \, \quad + \, \, \cdots \tag{13}$$

The α-dependent rational function (12) has an order-four α-dependent telescoper $L_4(\alpha)$

$$\begin{aligned} x^2 \cdot \, L_4(\alpha) \quad &= \quad -5 \cdot \, x^2 \cdot \, (5\,\theta \, +2) \cdot \, (5\,\theta \, +4) \cdot \, (5\,\theta \, +6) \cdot \, (5\,\theta \, +8) \\ &\qquad +16 \cdot \, \alpha^5 \cdot \, \theta^2 \cdot \, (\theta \, -1)^2, \end{aligned} \tag{14}$$

which has the following ${}_4F_3$ hypergeometric function solution:

$${{1} \over {\alpha}} \cdot \, {}_4F_3\Bigl([{{1} \over {5}}, \, {{2} \over {5}}, \, {{3} \over {5}}, \, {{4} \over {5}}], \, \, [{{1} \over {2}}, \, {{1} \over {2}}, \, 1], \, \, {{3125} \over {16\, \alpha^5}} \cdot \, x^2\Bigr). \tag{15}$$

The series expansion of this ${}_4F_3$ hypergeometric function (15) is in agreement with the series expansion (13). In the $\alpha \rightarrow 0$ limit the order-four α-dependent telescoper $L_4(\alpha)$ becomes the direct-sum:

$$-5 \cdot x^4 \cdot \Big((5\,\theta\, + 2) \oplus (5\,\theta\, + 4) \oplus (5\,\theta\, + 6) \oplus (5\,\theta\, + 8) \Big). \tag{16}$$

We thus see, in this $\alpha \rightarrow 0$ limit, that one recovers, among the different factors in (16), the order-one telescoper of the rational function (11), namely $5\,\theta\, + 2$. This first example being a bit too simple, or degenerate, let us consider another example.

2.3.2. Diagonals versus telescopers: a second example

Let us now consider the rational function of three variables:

$$R(x, y, z) \quad = \quad \frac{1}{-x\, - y - z\, - x^5\, y}. \tag{17}$$

This rational function has a telescoper L_4, which is a linear differential operator of order four, which reads:

$$\begin{aligned} L_4 \quad = \quad & -(800000\, x^5\, - 27) \cdot x^4\, D_x^4 \, - (11200000\, x^5\, + 27) \cdot x^3\, D_x^3 \\ & -15 \cdot (2800000\, x^5\, - 1) \cdot x^2\, D_x^2 \, - 60 \cdot (700000\, x^5\, - 1) \cdot x\, D_x \\ & -12 \cdot (437500\, x^5\, + 9), \end{aligned} \tag{18}$$

or, introducing the homogeneous derivative $\theta = x\, D_x$,

$$\begin{aligned} L_4 \quad = \quad & -50000 \cdot x^5 \cdot (2\,\theta\, + 7)\,(2\,\theta\, + 5)\,(2\,\theta\, + 3)\,(2\,\theta\, + 1) \\ & +3 \cdot (3\,\theta\, + 1)\,(3\,\theta\, - 4)\,(\theta\, - 3)^2. \end{aligned} \tag{19}$$

The rational function (17) does *not* have a multi-Taylor expansion. We have a problem to define the diagonal of the rational function (17). The analytic solutions of (18), or (19), are thus just "Periods" of the rational function (17), i.e. integrals over a non-vanishing cycle of the rational function (17). A solution of (18), or (19), is, for instance, the hypergeometric function:

$$x^3 \cdot {}_4F_3\Big([\frac{7}{10}, \frac{9}{10}, \frac{11}{10}, \frac{13}{10}], \, [1, \frac{4}{3}, \frac{5}{3}], \, \frac{800000}{27} \cdot x^5 \Big). \tag{20}$$

If one finds that the concept of diagonal is easier to understand, compared to"Periods" over non-vanishing cycles that are not really defined (we just know they exist), such a result may look a bit too abstract, and thus slightly frustrating. In fact one can recover some contact with the easier concept of diagonals, performing some kind of "desingularization". Let us consider the more general α-dependent rational function of three variables:

$$R(x, y, z) \quad = \quad \frac{1}{\alpha\, - x\, - y - z\, - x^5\, y}. \tag{21}$$

It has a telescoper which is a linear differential operator of order four $M_4(\alpha)$. The first terms of the diagonal of that rational function (21) can easily be calculated. We have calculated this order four linear differential operator $M_4(\alpha)$. It is a bit too large to be given here. However one remarks that this α-dependent order four linear differential operator $M_4(\alpha)$, is actually related to the previous order-four linear differential operator L_4, in the $\alpha \rightarrow 0$ limit:

$$M_4(0) \quad = \quad -675000000\, x^{11} \cdot L_4. \tag{22}$$

To sum-up: The telescoper corresponding to "Periods" over a non-vanishing cycles can be obtained from a one-parameter telescoper having clear-cut diagonal solutions ("Periods" over a vanishing cycle).

2.4. The Devil is in the detail: the number of variables

Let us consider the diagonal of the following rational function of four variables:

$$\frac{1}{1 \, - \alpha \, x - y - z \, \, - \beta \cdot x \, u}. \tag{23}$$

Its telescoper is, *for any value of* α, and for $\beta \neq 0$, the order-two linear differential operator

$$L_2 \, = \, (1 - 27 \, \beta \cdot x) \cdot x \, D_x^2 \, + (1 - 54 \, \beta \cdot x) \cdot D_x \, - 6 \, \beta, \tag{24}$$

which has the following hypergeometric ${}_2F_1$ solution:

$${}_2F_1\Big([\frac{1}{3}, \frac{2}{3}], [1], \, 27 \, \beta \cdot x\Big). \tag{25}$$

Recalling the definition of the diagonal of a rational function based on multi-Taylor expansion, it is easy to see, on this almost trivial example, that the various powers of the product $t \, = \, x \, y \, z \, u$ that the diagonal extracts, require the occurrence of the variable u which only occurs, in the denominator of (23), through the product $x \, u$ yielding automatically the occurrence of the variable x. Consequently, any further occurrence of the variable x, from the $-\alpha \, x$ monomial in the denominator of (23), is excluded. This explains why the diagonal of (23) is actually blind to the $-\alpha \, x$ term. In other words, the diagonal of the four variables rational function (23) is, *in fact the diagonal of a rational function of three variables* y, z, and the product $x \, u$.

Remark 2.1: To take into account this problem, we will introduce the concept of "effective number" of variables. In the previous example the number of variables is four but the "effective number" of variables is three.

2.5. Understanding the complexity of the diagonal of a rational function

2.5.1. Order of the linear differential operator and number of variables

The simplest example of diagonal of rational function of n variables, corresponds to the diagonal of the rational function

$$\frac{1}{1 \, - x_1 - x_2 - x_3 \, \cdots \, - x_n}. \tag{26}$$

The diagonal of (26) is annihilated by an order-$(n-1)$ linear differential operator with a ${}_{n-1}F_{n-2}$ hypergeometric solution:

$${}_{n-1}F_{n-2}\Big([\frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \cdots, \frac{n-1}{n}], [1, 1, \cdots, 1], \, n^n \cdot x\Big). \tag{27}$$

This simple example may provide the prejudice that, for a given globally bounded series (36), the number of variables of the rational function is related to the (minimal) order of the linear differential operator annihilating the series. One should note, however, for the class of the above example, that the corresponding linear differential operator has the Maximally Unipotent Monodromy property (MUM) which means that all its indicial exponents (at the origin) of the operator are equal (see for instance [22,32]) .

This result is reminiscent of the well-known ${}_4F_3([1/5, 2/5, 3/5, 4/5], [1, 1, 1], x)$ Candelas et al. hypergeometric series emerging in [3] for a particular Calabi-Yau manifold. The simplest Calabi-Yau series (see for instance [18]) are ${}_4F_3$ hypergeometric series like

${}_4F_2([1/2, 1/2, 1/2, 1/2], [1, 1, 1], x)$, or ${}_4F_2([1/5, 2/5, 3/5, 4/5], [1, 1, 1], x)$ (see equation 3.11 in [3]).

Let us recall that Calabi-operators [22], annihilating Calabi-Yau series [18], are (self-adjoint) order-four linear differential operators which have the Maximally Unipotent Monodromy property (MUM) at $x = 0$: if one considers their formal series expansions at $x = 0$, among the four formal series expansions, one is analytic (it actually corresponds to our diagonals of rational functions), another one is a formal series with a $\ln(x)^1$, another one is a formal series with a $\ln(x)^2$, and the last one is a formal series with a $\ln(x)^3$. Along this line ((26) yielding (27)), one would expect that the diagonal of rational function representation of a Calabi-Yau series (solution of an order-four linear differential operator) should require, at least *five* variables for the rational function.

2.5.2. Order of the linear differential operator and degree in the variables

Let us now consider the diagonal of the following rational function of three variables

$$\frac{1}{1 \; - x \, - \alpha \, y \; \, - z^2}, \tag{28}$$

whose diagonal writes as a simple ${}_4F_3$ hypergeometric solution:

$${}_4F_3\Big([\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}], [1, \frac{1}{2}, \frac{1}{2}], \frac{5^5}{2^4} \cdot \alpha^2 \cdot x^2 \Big). \tag{29}$$

In contrast with the example (26), here, we just need, for the rational function, *three* variables, instead of the expected *five* variables. Note however, that the order-four linear differential operator L_4, annihilating this hypergeometric solution (29), does *not* have MUM. As usual, this order-four linear differential operator is homomorphic to it adjoint with a very simple order-two intertwiner:

$$L_4 \cdot \Big(x \, D_x^2 \, + D_x \Big) \;\; = \;\; \Big(x \, D_x^2 \, + D_x \Big) \cdot adjoint(L_4). \tag{30}$$

One thus expects [43] this order-four linear differential operator L_4 to have a symplectic differential Galois group included in $Sp(4, \mathbb{C})$. Actually the exterior square of this o.rder-four operator L_4 has a simple rational function solution [43], namely $1/x/(5^5 \cdot \, x^2 \, - 2^4)$.

Let us now consider the diagonal of the following rational function of three variables:

$$\frac{1}{1 \; - x \, - \alpha \, y \; \, - z^3}. \tag{31}$$

The linear differential operator annihilating this diagonal is an order-six linear differential operator with a quite simple ${}_6F_5$ hypergeometric solution:

$${}_6F_5\Big([\frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}], [1, \frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}], \frac{7^7}{3^6} \cdot \alpha^3 \cdot x^3 \Big). \tag{32}$$

Let us restrict to $\alpha = 1$. The order-six linear differential operator, annihilating the diagonal of (31), does *not* have MUM. One has *three* series analytic at $x = 0$, one of the form $x \cdot (1 + 2377375/6561 \, x^3 \, + \cdots)$, one of the form $x^2 \cdot (1 + 16509584/32805 \, x^3 \, + \cdots)$, and the third one being the diagonal of the rational function which is the expansion of (32):

$$\begin{aligned} 1 \, &+ 140 \, x^3 \, + 84084 \, x^6 \, + 64664600 \, x^9 \, + 55367594100 \, x^{12} \, + 50356110752640 \, x^{15} \\ &+ 47606217704845800 \, x^{18} \, + 46236665756994672960 \, x^{21} \quad + \cdots \end{aligned} \tag{33}$$

One also has three other formal series solutions with a $\ln(x)^1$, but no $\ln(x)^2$ or $\ln(x)^3$.

As usual, this order-six linear differential operator is homomorphic to its adjoint with a very simple order-four intertwiner:

$$L_6 \cdot \Bigl(x^2\, D_x^4 + 4\, x\, D_x^3 + 2\, D_x^2\Bigr) \;=\; \Bigl(x^2\, D_x^4 + 4\, x\, D_x^3 + 2\, D_x^2\Bigr) \cdot \, adjoint(L_6). \qquad (34)$$

One expects [43] this order-six linear differential operator L_6 to have a symplectic differential Galois group included in $Sp(6, \mathbb{C})$. Actually the exterior square of this order-six linear differential operator L_6 has a simple rational function solution [43], namely $1/x/(7^7 \cdot x^3 - 3^6)$.

Remark 2.2: This result can be generalised. Let us consider the rational function:

$$\frac{1}{1 \, - x \, - \, y \, - z^n}. \qquad (35)$$

The linear differential operator $L_{2n}^{(1)}$, annihilating this diagonal, is an order-$(2\,n)$ linear differential operator with a quite simple ${}_{2n}F_{2n-1}$ hypergeometric solution:

$$\begin{aligned} &{}_{2n}F_{2n-1}\Bigl([\frac{1}{2\,n\,+1}, \, \frac{2}{2\,n\,+1}, \, \frac{3}{2\,n\,+1}, \, \cdots, \, \frac{2\,n}{2\,n\,+1}], \\ &\qquad [1, \, \frac{1}{n}, \frac{1}{n}, \frac{2}{n}, \frac{2}{n}, \cdots, \frac{n-1}{n}, \, \frac{n-1}{n}], \; \frac{(2\,n\,+1)^{(2\,n+1)}}{n^{2\,n}} \cdot x^n\Bigr). \end{aligned} \qquad (36)$$

Let us also consider the linear differential operators $L_{2n}^{(m)}$ annihilating the diagonal of the rational function:

$$\Bigl(\frac{1}{1 \, - x \, - \, y \, - z^n}\Bigr)^m. \qquad (37)$$

One finds (using the Homomorphisms command in Maple) the following homomorphisms between successive linear differential operators $L_{2n}^{(m)}$:

$$Homomorphisms\Bigl(L_{2n}^{(m)}, \, L_{2n}^{(m+1)}\Bigr) \;=\; (2\,n\,+1) \cdot x \cdot D_x \,+ m \cdot n. \qquad (38)$$

In other words one has the relations:

$$L_{2n}^{(m+1)} \cdot \Bigl((2\,n\,+1) \cdot \theta \,+ m \cdot n\Bigr) \;=\; Z_1(m) \cdot L_{2n}^{(m)}, \qquad (39)$$

where $Z_1(m)$ is an order-one linear differential operator. The linear differential operator $L_{2n}^{(1)}$ is simply homomorphic to its adjoint:

$$\begin{aligned} &Homomorphisms\Bigl(adjoint(L_{2n}^{(1)}), \, L_{2n}^{(1)}\Bigr) \;= \\ &\qquad \frac{1}{x^{n-1}} \cdot \theta^2 \cdot (\theta\, -1)^2 \cdot (\theta\, -2)^2 \cdot (\theta\, -3)^2 \, \cdots \, \Bigl(\theta \,-\, (n\, -2)\Bigr)^2. \end{aligned} \qquad (40)$$

Remark 2.3: With the previous, rather simple, examples we see that the order of the linear differential operator annihilating the diagonal of a rational function, is *not* related to the number of variables of the rational function (or even to the number of "effective" variables see section 2.4). Furthermore, a given *globally bounded* series can be seen to be the diagonals of an infinite number of rational functions of a certain number of variables, but also, in the same time, of other infinite number of rational functions with a different number of variables. For a given globally bounded series we can find the (minimal order) linear differential operator annihilating this series. Having this (minimal order) linear differential operator, the question is: can we find the *minimal number of variables* necessary to see this globally bounded series as the diagonal of a rational function of that number of variables? We will address these questions in a forthcoming paper [69].

3. Diagonals of rational functions: should we restrict to rational functions of the form $1/Q$?

With P and Q multivariate polynomials (with $Q(0) \neq 0$), the diagonals of the rational functions P/Q^k are, for fixed polynomial Q, and for arbitrary integer k, a *finite dimensinal* vectorial space related, as shown by Christol [45,46], to the de Rham cohomology (we are thankful to P. Lairez for having clarified this point). There are so many cohomologies in mathematics. For non-mathematicians let us just say that the introduction of a cohomology often amounts to seeing that "something" you expect, at first sight, to be infinite, for instance the number of solutions of a system of PDE's (partial differential equations), is in fact a *finite* set (for instance for D-finite systems of PDE's). For physicists, not familiar with de Rham cohomology, let us just say that this can be seen as a consequence of the fact that these P/Q^k rational functions are solutions of *D-finite systems*, which means that these systems of PDE's have a *finite set of solutions* of the form P/Q^k. Being in such a "finite box" will force the telescopers of the diagonals of P/Q^k and $1/Q$, to be related (by homomorphisms). This requires to find a "cyclic vector" in mathematicians wording.

Experimentally, if one considers the (minimal order) linear differential operators for the diagonal of P/Q^k and for the diagonal of $1/Q$, these two linear differential operators are *actually homomorphic*. Note that this experimental result, valid for diagonals (i.e. integrals over vanishing cycles), is *no longer* valid for telescopers of rational functions with analytic solutions corresponding to "periods", n-fold integrals, over non-vanishing cycles. In this case we have a slight generalization of that homomorphism between telescopers P/Q^k and telescopers $1/Q$, that will be described in the sequel (see section 5.2 below).

It is true that the analysis of lattice Green functions (LGF) [70–74] in physics naturally yields to diagonals of rational functions in the form $R = 1/Q$, where Q is a polynomial. However, the other n-fold integrals, emerging in physics, are much more complex (for instance the $\chi^{(n)}$ terms of the susceptibility of the two-dimensional Ising model [28]). The lattice Green functions [32,32,70–75] and some Occam's razor simplicity argument are not sufficient to justify a bias of studying, quite systematically, rational functions of the form $R = 1/Q$ (as we often do). In fact these de Rham cohomology arguments are the reason why, for diagonals (and diagonals only), one can restrict to rational functions in the form $R = 1/Q$, but since these arguments may look too esoteric for physicists, let us, in a learn-by-example, pedagogical approach, provide examples showing that telescopers of rational functions in the form $R = 1/Q^k$ are homomorphic to telescopers of rational functions in the form $R = 1/Q$, and then that telescopers of rational functions in the form $R = P/Q$ are homomorphic to telescopers of rational functions in the form $R = 1/Q$.

3.1. *Diagonals of rational functions:* $R = 1/Q^k$ *reducing to* $1/Q$

Let us denote Q the polynomial:

$$Q \, = \, 1 \, + x\,y + y\,z + z\,x \, + 3 \cdot (x^2 + y^2 + z^2). \tag{41}$$

Let us denote $L_4^{(n)}$ the telescopers of $Diag(1/Q^n)$:

$$L_4^{(n)} \cdot Diag\Big(\frac{1}{Q^n}\Big) \, = \, 0. \tag{42}$$

One remarks that these telescopers are all of order four. One actually finds the following homomorphisms between successive telescopers (42):

$$Homomorphisms\Big(L_4^{(n)}, \, L_4^{(n+1)}\Big) \, = \, 3\,x \cdot D_x \, + 2\,n. \tag{43}$$

In other words one has the relations:

$$L_4^{(n+1)} \cdot (3\theta + 2n) \; = \; Z_1(n) \cdot L_4^{(n)}, \qquad (44)$$

where $Z_1(n)$ is an order-one linear differential operator, the intertwining relation (44) yielding:

$$L_4^{(n+1)} \cdot (3\theta + 2n) \; \cdots \; (3\theta + 6) \cdot (3\theta + 4) \cdot (3\theta + 2) \\ = \; Z_1(n) \; \cdots \; Z_1(3) \cdot Z_1(2) \cdot Z_1(1) \cdot L_4^{(1)}. \qquad (45)$$

One deduces:

$$2^n \cdot n! \cdot Diag\Big(\frac{1}{Q^{n+1}}\Big) \\ = \; (3\theta + 2n) \; \cdots \; (3\theta + 6) \cdot (3\theta + 4) \cdot (3\theta + 2) \cdot Diag\Big(\frac{1}{Q}\Big). \qquad (46)$$

In other words the diagonal of $1/Q^{n+1}$ can be simply deduced from the diagonal of $1/Q$.

Remark 3.1: The product $(3\theta + 2n) \; \cdots \; (3\theta + 6) \cdot (3\theta + 4) \cdot (3\theta + 2)$, in the intertwining relation (45), *is in fact a direct sum*:

$$(3\theta + 6) \cdot (3\theta + 4) \cdot (3\theta + 2) \\ = \; 27\, x^3 \cdot LCLM\Big((3\theta + 6), \; (3\theta + 4), \; (3\theta + 2)\Big). \qquad (47)$$

One has, for instance, the relations:

$$\begin{aligned} 2 \cdot Diag\Big(\frac{1}{Q^2}\Big) \; &= \; (3\theta + 2) \cdot Diag\Big(\frac{1}{Q}\Big) \\ 8 \cdot Diag\Big(\frac{1}{Q^3}\Big) \; &= \; (3\theta + 4) \cdot (3\theta + 2) \cdot Diag\Big(\frac{1}{Q}\Big) \\ 48 \cdot Diag\Big(\frac{1}{Q^4}\Big) \; &= \; (3\theta + 6) \cdot (3\theta + 4) \cdot (3\theta + 2) \cdot Diag\Big(\frac{1}{Q}\Big) \\ 384 \cdot Diag\Big(\frac{1}{Q^5}\Big) \; &= \; (3\theta + 8) \cdot (3\theta + 6) \cdot (3\theta + 4) \cdot (3\theta + 2) \cdot Diag\Big(\frac{1}{Q}\Big). \end{aligned} \qquad (48)$$

Of course, since the telescoper of $Diag\big(\frac{1}{Q}\big)$ is an order four linear differential operator, the order-$(k-1)$ product in front of $Diag\big(\frac{1}{Q}\big)$ in (48) can be, for $Diag\big(\frac{1}{Q^k}\big)$, reduced to an order-three linear differential operator (the simple products $(3\theta + 2 \cdot (k-1)) \cdots (3\theta + 4) \cdot (3\theta + 2)$ in (48) being taken "modulo" L_4, for $k \geq 5$).

3.2. *Diagonals of rational functions: $R = P/Q$ reducing to $1/Q$*

Experimentally one finds, quite often, that the telescoper of a rational function of the form $R = P/Q$ and the telescoper of the simple rational function $1/Q$ with its numerator normalized to 1, are homomorphic. The intertwiner M occurring in the homomorphisms of these two telescopers yields a relation of the form

$$Diag\Big(\frac{P}{Q}\Big) \; = \; M \cdot Diag\Big(\frac{1}{Q}\Big), \qquad (49)$$

yielding the prejudice that the diagonals of the rational functions of the form P/Q should reduce to the "simplest" diagonal, namely $Diag(1/Q)$. In fact things are slightly more subtle, as will be seen below. In fact one is looking for a *cyclic vector*, and the cyclic vector is not necessarily $Diag(1/Q)$ (see relation (58) and (59) below).

Sticking with the polynomial (41), one has

$$L_4^{(1)} \cdot Diag\Big(\frac{1}{Q}\Big) \,\, = \,\, 0, \tag{50}$$

and considering the diagonal of $x\,y/Q$, one obtains an order-five differential operator with unique factorization:

$$L_4^{(xy)} \cdot D_x \cdot Diag\Big(\frac{x\,y}{Q}\Big) \,\, = \,\, 0. \tag{51}$$

The homomorphisms between $L_4^{(1)}$ and $L_4^{(xy)}$ amounts to seeking for linear differential operators that map the solutions of one differential operator into the other. These relations are

$$L_4^{(xy)} \cdot Q_3 \,\, = \,\, K_3 \cdot L_4^{(1)}, \tag{52}$$

and

$$L_4^{(1)} \cdot J_3 \,\, = \,\, P_3 \cdot L_4^{(xy)}, \tag{53}$$

where the intertwiners Q_3, K_3, J_3 and P_3 are linear differential operators of order three.

Note that the above two relations show [23] that the linear differential operator $J_3 \cdot Q_3$ (resp. $Q_3 \cdot J_3$) leaves the solutions of $L_4^{(1)}$ (resp. $L_4^{(xy)}$) unchanged,

$$J_3 \cdot Q_3 \cdot Diag\Big(\frac{1}{Q}\Big) \,\, = \,\, Diag\Big(\frac{1}{Q}\Big) \tag{54}$$

$$= \,\, 1 \,\, - 195\,x^2 \, + 135225\,x^4 \, - 143647728\,x^6 \, + 182699446545\,x^8 \, -252437965534755\,x^{10} \, + 364803972334074000\,x^{12} \,\, + \,\, \cdots$$

and:

$$Q_3 \, \cdot \, J_3 \cdot D_x \cdot Diag\Big(\frac{x\,y}{Q}\Big) \,\, = \,\, D_x \cdot Diag\Big(\frac{x\,y}{Q}\Big) \tag{55}$$

$$= \,\, 16\,x \,\, - 38400\,x^3 \, + 71593536\,x^5 \, - 126120445440\,x^7 \, +218901889206000\,x^9 \, - 378463218115207680\,x^{11} \,\, + \,\, \cdots \tag{56}$$

Equivalently, the adjoint of $P_3 \cdot K_3$ (resp. the adjoint of $K_3 \cdot P_3$) leaves the solutions of the adjoint of L_4 (resp. the adjoint of $L_4^{(xy)}$) unchanged.

Introducing the differential operator D_x on both sides of (53), and using (51), one obtains:

$$L_4^{(1)} \cdot J_3 \cdot D_x \cdot Diag\Big(\frac{x\,y}{Q}\Big) \,\, = \,\, P_3 \cdot (L_4^{(xy)} \cdot D_x) \cdot Diag\Big(\frac{x\,y}{Q}\Big). \tag{57}$$

The RHS of (57) cancels and therefore, the LHS of (57), according to (50), leads to

$$Diag\Big(\frac{1}{Q}\Big) \,\, = \,\, J_3 \cdot D_x \cdot Diag\Big(\frac{x\,y}{Q}\Big). \tag{58}$$

Also, acting by both sides of (52) on $Diag(1/Q)$, using (50), and (51) in mind leads to:

$$D_x \cdot Diag\Big(\frac{x\,y}{Q}\Big) \,\, = \,\, Q_3 \cdot Diag\Big(\frac{1}{Q}\Big). \tag{59}$$

With these relations we see that the derivative of the diagonal of xy/Q simply reduces to the diagonal of $1/Q$, but the diagonal of xy/Q does not simply reduce to the diagonal of $1/Q$. Here $1/Q$ is not the "cyclic vector".

4. Diagonals of algebraic functions

4.1. Diagonals of algebraic functions: a first example

Let us consider the algebraic functions:

$$A(x,y) \; = \; \frac{1}{\Big(1 - \alpha \cdot (x+y)\Big)^{1/n}} \qquad n \, = \, 2, 3, \, \cdots \tag{60}$$

The telescopers of these algebraic functions are order-two linear differential operators with the simple ${}_2F_1$ hypergeometric solution:

$$\begin{aligned} & {}_2F_1\Big([\frac{1}{2n}, \frac{n+1}{2n}], [1], 4 \cdot \alpha^2 \cdot x\Big) \\ & \quad = \; 1 \; + \frac{n+1}{n^2}\,\alpha^2 x \; + \frac{(1+n)\cdot(1+2n)\cdot(1+3n)}{4 \cdot n^4}\,\alpha^4 x^2 \quad + \; \cdots \end{aligned} \tag{61}$$

Note that, among these ${}_2F_1$ hypergeometric functions, the $n = 2$, $n = 3$, $n = 4$, $n = 6$ cases correspond to *modular forms* (see Appendix B in [16]).

These hypergeometric series can be seen to be, as it should, the diagonals of the algebraic functions (60). In particular, for $n = 2$, one gets:

$$\begin{aligned} & {}_2F_1\Big([\frac{1}{4}, \frac{3}{4}], [1], \; 4 \cdot \alpha^2 \cdot x\Big) \\ & \quad = \; \Big(\frac{1}{1 - 3\,\alpha^2\,x}\Big)^{1/4} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], \, \frac{27}{4} \cdot \frac{\alpha^4 \cdot x^2 \cdot (1 - 4\,\alpha^2\,x)}{(1 - 3\,\alpha^2\,x)^3}\Big) \\ & \quad = \; 1 \; + \frac{3}{4}\,\alpha^2 x \; + \frac{105}{64}\,\alpha^4 x^2 + \frac{1155}{256}\,\alpha^6\,x^3 + \frac{225225}{16384}\,\alpha^8\,x^4 \quad + \; \cdots \end{aligned} \tag{62}$$

For $n = 2$ it is natural to associate the denominator of (60), with the algebraic surface

$$z^2 \; = \; 1 \; - \alpha \cdot (x+y), \tag{63}$$

and, following ideas developped in [41], since calculating the diagonal of the function (60) for $n = 2$, amounts, in the multi-Taylor expansion, to extracting the terms depending only on the product $p = x\,y$, take the intersection of the algebraic surface (63) with the surface $p = x\,y$. This amounts, for instance, to eliminating $y = p/x$ in (63), thus getting the algebraic curve

$$-\alpha \cdot x^2 \; - x\,z^2 \; - \alpha \cdot p \; + x \; = \; 0, \tag{64}$$

which turns out to be an *elliptic curve* (genus-one). Calculating the j-invariant of the elliptic curve (64), one deduces the following Hauptmodul

$$\mathcal{H} \; = \; \frac{1728}{j} \; = \; \frac{27}{4} \cdot \frac{\alpha^4 \cdot p^2 \cdot (1 - 4\,\alpha^2\,p)}{(1 - 3\,\alpha^2\,p)^3}, \tag{65}$$

which is actually the Hauptmodul pullback in (62). This example gives some hope that the effective algebraic geometry approach of diagonals of rational functions, detailed in [41], could also work with diagonals of algebraic functions.

For $n \neq 2$ it is tempting to associate the denominator of (60), with the algebraic surface

$$z^n \; = \; 1 - \alpha \cdot (x+y), \tag{66}$$

and after the elimination $y \, = \, p/x$ in (63), the algebraic curve

$$-\alpha \cdot x^2 \, - x\, z^n \, - \alpha \cdot p \, + x \, = \, 0, \tag{67}$$

but such algebraic curves turn out to be of genus $g = n - 1$. Understanding the emergence of *modular forms* for the $n = 3$, $n = 4$, $n = 6$ subcases of (61) from (respectively) genus 2, 3, and 5 algebraic curves, is an open (and challenging) problem.

Remark 4.1: From the definition of the diagonals of a rational, or algebraic, functions it is straightforward to see that the diagonals of the algebraic functions (60) are series of the variable $\alpha^2 \, x$. Consequently, the previous calculations for a particular value of α, are sufficient to recover the previous results valid for arbitrary α. For that reason we will, in the next example, take specific values of the parameters.

4.2. Diagonals of algebraic functions: a second example

Let us consider the algebraic functions:

$$A(x,y) \,\, = \,\, \frac{1}{\Big(1 - 3 \cdot (x+y) + 5 \cdot (x^2+y^2)\Big)^{1/n}}, \qquad n \, = \, 2, 3, \, \cdots \tag{68}$$

For $n = 2$ the telescoper of the algebraic function (68) is an order-two linear differential operator with the pullbacked ${}_2F_1$ hypergeometric solution:

$$\begin{aligned} & \frac{1}{(1-30\,x)^{1/2}} \cdot {}_2F_1\Big([\frac{1}{4}, \frac{3}{4}], [1], \, -\frac{4 \cdot (11 - 200\,x) \cdot x}{(1-30\,x)^2}\Big) \\ & = \, \frac{1}{(1-27\,x+300\,x^2)^{1/4}} \\ & \qquad \times {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], \, \frac{27}{4} \cdot \frac{x^2 \cdot (11-200\,x)^2 \cdot (1-16\,x+100\,x^2)}{(1-27\,x+300\,x^2)^3}\Big) \\ & = \,\, 1 \, + \frac{27}{4}\,x \, + \frac{4305}{64}\,x^2 + \frac{199395}{256}\,x^3 + \frac{167040825}{16384}\,x^4 \, + \, \cdots \end{aligned} \tag{69}$$

From multi-Taylor series expansion, it is straightforward to see that the hypergeometric series is actually the diagonal of the algebraic function (68) for $n = 2$.

As in the previous subsection we introduce the algebraic surface

$$z^2 \, = \, 1 \, - 3 \cdot (x+y) \, + 5 \cdot (x^2+y^2), \tag{70}$$

and, again, eliminate $y \, = \, p/x$ in (70), thus getting the algebraic curve

$$5\,x^4 \, - x^2\,z^2 \, - 3\,x^3 \, + 5\,p^2 \, - 3\,p\,x \, + x^2 \,\, = \,\, 0, \tag{71}$$

which turns out to be an *elliptic curve* (genus-one). Calculating the j-invariant of the elliptic curve (71), one deduces the following Hauptmodul

$$\mathcal{H} \,\, = \,\, \frac{1728}{j} \,\, = \,\, \frac{27}{4} \cdot \frac{p^2 \cdot (11-200\,p)^2 \cdot (1-16\,p+100\,p^2)}{(1-27\,p+300\,p^2)^3}, \tag{72}$$

which is actually the Hauptmodul pullback in (69). Again, this last example gives some hope that the effective algebraic geometry approach of diagonals of rational functions, detailed in [41], could also work with diagonals of algebraic functions. For $n \neq 2$, it is tempting to introduce the algebraic surface

$$z^n \,\, = \,\, 1 \, - 3 \cdot (x+y) \, + 5 \cdot (x^2+y^2), \tag{73}$$

and, again, eliminate $y = p/x$ in (70), thus getting the algebraic curve

$$5\,x^4 \,-x^2\,z^n \,-3\,x^3 \,+5\,p^2 \,-3\,p\,x \,+x^2 \,=\, 0, \tag{74}$$

which is an algebraic curve of genus $g = 2\,n - 3$ for n even, and $g = 2\,n - 2$ for n odd. For $n = 3$ (genus 4) the telescoper of the algebraic function (68) is an (irreducible) *order-three* linear differential operator *which is not homomorphic to its adjoint*. The interpretation of such non-self-dual order-three linear differential operators from these higher genus algebraic curves is a totally open problem.

5. Understanding the emergence of selected differential Galois groups for diagonals of rational functions

Experimentally one finds that almost all the linear differential operators annihilating the diagonal of a rational, or algebraic, function are homomorphic to their adjoint [42]. For instance, recalling an example in [42]

$$\begin{aligned} &{}_4F_3\Big([\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}], [\frac{1}{2}, 1, 1], \frac{729}{4}\cdot x\Big) \,=\, Diag\Big(\frac{1}{1-(1+u)\cdot(x+y+z)}\Big) \\ &\quad = \, 1 \,+18\,x \,+1350\,x^2 \,+\cdots \end{aligned} \tag{75}$$

we find the corresponding order-four linear differential operator

$$x\cdot L_4 \,=\, 2\cdot x\cdot (3\,\theta+2)^2\cdot(3\,\theta+1)^2 \,-81\cdot\theta^3\cdot(2\,\theta-1), \tag{76}$$

which can be seen to be non-trivially homomorphic to its adjoint:

$$L_4\cdot\Big(\theta+\frac{1}{2}\Big) \,=\, \Big(\theta+\frac{1}{2}\Big)\cdot adjoint(L_4). \tag{77}$$

Beyond diagonals of a rational, or algebraic, functions, one also finds experimentally, that almost all the telescopers of rational or algebraic functions are homomorphic to their adjoint. This homomorphism to the adjoint property is so systematic that, following a mathematician's prejudice one can imagine that this is nothing but the *Poincaré duality*. The Poincaré duality [44] works for *any* algebraic variety: the diagonal of *any* rational, or algebraic, function should yield self-dual linear differential operators in the sense that they are homomorophic to their adjoint. This is *not* the case. It turns out that the linear differential operators of some ${}_nF_{n-1}$, candidates to rule-out Christol's conjecture [45,46,49], precisely provide such rare examples of linear differential operators annihilating diagonal of rational or algebraic functions that are *not* homomorphic to their adjoint. Among these candidates a large set has been seen to actually be diagonals of rational, or algebraic, functions [49,76].

5.1. *A recall on Christol's conjecture*

Let us recall one of the ${}_3F_2$ hypergeometric candidates introduced to rule out Christol's conjecture:

$$\begin{aligned} &{}_3F_2\Big([\frac{2}{9}, \frac{5}{9}, \frac{8}{9}], [\frac{2}{3}, 1], 27\cdot x\Big) \\ &\quad = \, 1 \,+\frac{40}{9}\cdot x \,+\frac{5236}{81}\cdot x^2 \,+\frac{7827820}{6561}\cdot x^3 \,+\frac{1444588600}{59049}\cdot x^4 \,+\cdots \end{aligned} \tag{78}$$

It is a globally bounded series (change $x \rightarrow 3^3\cdot x$ to get a series with integer coefficients). In fact it actually corresponds [49] to the diagonal of the algebraic function:

$$\frac{(1\,-y\,-z)^{1/3}}{1-x-y-z}. \tag{79}$$

The telescoper of the algebraic function (79) is the order-three linear differential operator which has (78) as a solution. This order-three linear differential operator is *not* homomorphic to its adjoint. We have a $SL(3, \mathbb{C})$ differential Galois group.

Other similar examples are, for instance:

$$ {}_3F_2\Big([\frac{1}{9}, \frac{4}{9}, \frac{7}{9}], [\frac{2}{3}, 1], 27 \cdot x\Big) = Diag\Big(\frac{(1 - y - 2z)^{2/3}}{1 - x - y - z}\Big), \tag{80} $$

or

$$ {}_3F_2\Big([\frac{2}{9}, \frac{5}{9}, \frac{8}{9}], [\frac{5}{6}, 1], 27 \cdot x\Big) = Diag\Big(\frac{(1 - y - 2z)^{1/3}}{1 - x - y - z}\Big), \tag{81} $$

or even the ${}_4F_3$ hypergeometric function:

$$ {}_4F_3\Big([\frac{2}{9}, \frac{5}{9}, \frac{8}{9}, \frac{1}{2}], [\frac{1}{3}, \frac{5}{6}, 1], 27 \cdot x\Big) = Diag\Big(\frac{(1 - x)^{1/3}}{1 - x - y - z}\Big). \tag{82} $$

Again these three diagonals (80), (81) and (82) are solutions of telescopers that *are not* homomorphic to their adjoint.

These examples are taken in a list of 116 potential counter-examples constructed in 2011 by Bostan et al. [26]. Note that, more recently, 38 cases in that list of 116, have actually been found to be diagonals of algebraic functions [76]. The two relations (80) and (81) can be generalized [76,77] as follows:

$$ \begin{aligned} {}_4F_3\Big([\frac{1-(R+S)}{3}, \frac{2-(R+S)}{3}, \frac{3-(R+S)}{3}, \frac{1-S}{2}], \\ [\frac{1-(R+S)}{2}, \frac{2-(R+S)}{2}, 1], 27 \cdot x\Big) \\ = Diag\Big(\frac{(1-x)^R \cdot (1 - x - 2y)^S}{1 - x - y - z}\Big), \end{aligned} \tag{83} $$

where R and S are rational numbers. These diagonals are annihilated by the order-four linear differential operator:

$$ \begin{aligned} 2 \cdot x \cdot (S - 1 - 2\theta) \cdot (S + R - 3\theta) \cdot (S + R - 1 - 3\theta) \cdot (S + R - 2 - 3\theta) \\ -\theta^2 \cdot (S + R + 1 - 2\theta) \cdot (S + R - 2\theta). \end{aligned} \tag{84} $$

This order-four linear differential operator is *not* homomorphic to its adjoint. Other more involved similar relations can be found in section 2.1 of chapter 2 of [76].

Experimentally we found, after quite systematic calculations of thousands of telescopers of rational, or algebraic, functions, that the telescopers are (almost always) homomorphic to their adjoint, or if they are not irreducible, that each of the factors of these telescopers are homomorphic to their adjoint. Such previous examples like (78), (79), or (80) and (81), curiously related to Christol's conjecture, provide the *rare* examples of diagonals of algebraic functions such that their corresponding telescopers are *not* homomorphic to their adjoint. We have similar results with the algebraic function:

$$ \frac{x^{1/3}}{1 - x - y - z}. \tag{85} $$

In order to understand this "duality-breaking" (the telescoper is not self-adjoint up to homomorphisms), it is tempting to introduce the (algebraic) function:

$$ \frac{1}{1 - x - y - z - \alpha \cdot x^{1/3}}. \tag{86} $$

However, in order to avoid the introduction of rational functions of n-th roots of variables, we will (changing x, y, z into x^3, y^3, z^3) rather introduce the diagonal of the following rational function:

$$\frac{1}{1 \quad - x^3 - y^3 - z^3 \quad - \alpha \cdot x}. \tag{87}$$

5.2. Understanding the emergence of selected differential Galois groups for almost all the diagonal of rational functions

The linear differential operator annihilating the diagonal of the rational function (87) is a (quite large) order-eight linear differential operator $L_8(\alpha)$, depending on the parameter α, which is homomorphic to its adjoint with an order-six intertwiner. This order-eight linear differential operator $L_8(\alpha)$ is irreducible except at $\alpha = 0$. For $\alpha = 1$, $\alpha = 2$, $\alpha = 3$ the order-eight linear differential operator $L_8(\alpha)$ is homomorphic to its adjoint with an order-six intertwiner. The differential Galois group should, thus, be included in $Sp(8, \mathbb{C})$. This is confirmed when calculating [43] the exterior square of $L_8(\alpha)$. This exterior square has a rational function solution $P_a/x/Q_a$, where the polynomials P_a and Q_a read:

$$\begin{aligned} P_a \;&=\; (4\,\alpha^3 - 27) \cdot (20\,\alpha^3 - 81) \,+ 18 \cdot (-6561 \,- 891\,\alpha^3 \,+ 500\,\alpha^6) \cdot x^3 \,+ 1594323\,x^6, \\ Q_a \;&=\; 387420489\,x^9 \,- 531441 \cdot (81 + 100\,\alpha^3) \cdot x^6 \\ &\quad +(1594323 \,- 2972133\,\alpha^3 \,+ 729000\,\alpha^6 \,- 50000\,\alpha^9) \cdot x^3 \,\,- 27 \cdot (4\,\alpha^3 - 27)^2. \end{aligned} \tag{88}$$

Let us now take the $\alpha \to 0$ limit of the order-eight linear differential operator $L_8(\alpha)$. In this limit the order-eight linear differential operator just becomes the direct-sum

$$L_2 \,\oplus\, L_3 \,\oplus\, M_3, \tag{89}$$

where the order-two linear differential operator L_2 has the ${}_2F_1$ hypergeometric solution

$${}_2F_1\Big([\frac{1}{3}, \frac{2}{3}], [1], 27\,x^3\Big), \tag{90}$$

where the order-three linear differential operator L_3 has the ${}_3F_2$ hypergeometric function solution

$${}_3F_2\Big([\frac{5}{9}, \frac{8}{9}, \frac{11}{9}], \,[\frac{2}{3}, 1], 27\,x^3\Big), \tag{91}$$

and where the order-three linear differential operator M_3 has the ${}_3F_2$ hypergeometric function solution:

$${}_3F_2\Big([\frac{7}{9}, \frac{10}{9}, \frac{13}{9}], \,[\frac{1}{3}, 1], 27\,x^3\Big). \tag{92}$$

These two order-three linear differential operators, similarly to the previous example (78), are *not* homomorphic to their adjoint: they *break the self-adjoint duality* (up to homomorphisms of operators), and thus have a $SL(3, \mathbb{C})$ differential Galois group.

These two hypergeometric series are exactly on the same footing as (78): they are *globally bounded series* (just change $x^3 \to 3^3\,x^3$ in order to get a series with integer coefficients), and their respective order-three linear differential operators are *not* homomorphic to their adjoint, their differential Galois group being $SL(3, \mathbb{C})$. Let us note, however, that the order-three linear differential operator L_3 *is actually homomorphic to the adjoint of* M_3, and of course the order-three linear differential operators M_3 is homomorphic to the adjoint of L_3.

If, in an algebraic geometry perspective [41], one sees the fact that all our linear differential operators, annihilating diagonals of rational functions, are homomorphic to their adjoint as the differential algebra expression of the Poincaré duality on the algebraic varieties corresponding to the denominators of our rational functions [41], the fact that this Poincaré duality is broken for L_3 or M_3 *is, in fact, restored in the bigger picture* (87) with the linear differential order-eight operator. In the $\alpha \, \rightarrow \, 0$ limit we see that these two linear differential operators breaking the duality, actually emerge in a dual pair, thus *restoring the duality*. For instance, if one focuses on $L_6 \, = \, L_3 \oplus M_3$ in (90), one finds easily that this order-six linear differential operator is homomorphic to its adjoint. Its exterior square has the following rational function solution:

$$\frac{4 \, + 621 \, x^3}{(1 \, - 27 \, x^3)^3 \cdot \, x}. \tag{93}$$

Since these calculations are in the $\alpha \, \rightarrow \, 0$ limit, let us expand, in α, the rational function (87):

$$\begin{aligned} \frac{1}{1 \, - x^3 - y^3 - z^3 \, - \alpha \cdot \, x} \, &= \, \frac{1}{1 \, - x^3 - y^3 - z^3} \, + \frac{x}{(1 \, - x^3 - y^3 - z^3)^2} \cdot \, \alpha \\ &+ \frac{x^2}{(1 \, - x^3 - y^3 - z^3)^3} \cdot \, \alpha^2 \, + \frac{x^3}{(1 \, - x^3 - y^3 - z^3)^4} \cdot \, \alpha^3 \\ &+ \frac{x^4}{(1 \, - x^3 - y^3 - z^3)^5} \cdot \, \alpha^4 \, + \, \cdots \end{aligned} \tag{94}$$

The diagonal of a sum is clearly the sum of the diagonals. Thus the diagonal of the LHS of (94) will be the sum of the various rational function terms in α^n in the RHS of (94). The diagonal of the α^1 term in the α-expansion (94)

$$\frac{x}{(1 \, - x^3 - y^3 - z^3)^2}, \tag{95}$$

is clearly equal to zero, since the diagonal extracts, in the multi-Taylor series, the terms in the product $p \, = \, x \, y \, z$, or, in this case, the terms in the product $x^3 \, y^3 \, z^3$. Similarly the diagonal of the α^2 term in the α-expansion (94)

$$\frac{x^2}{(1 \, - x^3 - y^3 - z^3)^3}, \tag{96}$$

is also zero, but the diagonal of the α^3 term

$$\frac{x^3}{(1 \, - x^3 - y^3 - z^3)^4}, \tag{97}$$

is not zero. Actually this last diagonal reads:

$$\begin{aligned} &-\frac{1}{9} \cdot \, \frac{1 \, + 216 \, x^3}{(1 - 27 \, x^3)^3} \cdot \, x \cdot \, \frac{d}{dx} \, {}_2F_1\Big([\frac{1}{3}, \, \frac{2}{3}], \, [1], \, 27 \, x^3\Big) \\ &\qquad -18 \cdot \, \frac{x^3}{(1 - 27 \, x^3)^2} \cdot \, {}_2F_1\Big([\frac{1}{3}, \, \frac{2}{3}], \, [1], \, 27 \, x^3\Big). \\ &= \, -20 \, x^3 \, - 1680 \, x^6 \, - 92400 \, x^9 \, - 4204200 \, x^{12} \, - 171531360 \, x^{15} \, + \, \cdots \end{aligned} \tag{98}$$

It is annihilated by an order-two operator M_2.

We have a different story with telescopers. Since the telescoper of a sum of rational functions is the direct sum (LCLM) of the telescopers of these rational functions (or at least is a rightdivisor of the LCLM of the telescopers) let us consider the telescopers of the first five terms in the RHS of (94). The telescoper of the first term is, of course, the order-two linear differential operator L_2 annihilating the diagonal of this rational function. The telescoper of the second term (in α^1), is the previous order-*three* linear differential operator

L_3. The telescoper of the third term (in α^2) is exactly the previous M_3. The telescoper of the fourth term (in α^3), is the order-two linear differential operator M_2. The telescoper of the sum of the first orders in α in the expansion (94)

$$\frac{1}{1-x^3-y^3-z^3} + \frac{x}{(1-x^3-y^3-z^3)^2}\cdot\alpha + \frac{x^2}{(1-x^3-y^3-z^3)^3}\cdot\alpha^2, \qquad (99)$$

is actually the LCLM of the three telescopers L_2, L_3 and M_3 *which is precisely the* $\alpha \to 0$ *limit of the order-eight linear differential operator !*

5.3. *Revisiting* $1/Q \to P/Q^k$ *for telescopers*

The next terms in the α-expansion (94), namely the terms in α^{4+3n} with $n = 0, 1, \cdots$

$$\frac{x^{4+3n}}{(1-x^3-y^3-z^3)^{5+3n}}, \qquad (100)$$

have telescopers *actually homomorphic* to the telescoper L_3 for (95). Similarly, considering in the α-expansion (94), namely the terms in α^{5+3n} with $n = 0, 1, \cdots$

$$\frac{x^{5+3n}}{(1-x^3-y^3-z^3)^{6+3n}}, \qquad (101)$$

have telescopers *actually homomorphic* to the telescoper M_3 for (96). Finally, the terms in α^{3+3n} with $n = 0, 1, \cdots$

$$\frac{x^{3+3n}}{(1-x^3-y^3-z^3)^{4+3n}}, \qquad (102)$$

have telescopers *homomorphic* to the telescoper L_2, generalizing the result (98) for $n = 0$. This last sequence of telescopers can be understood from the ideas sketched in sections (3.1) and (3.2) for diagonals (changing for instance (x,y,z) into (x^3,y^3,z^3)). However, we see that these ideas *do not work anymore* when we compare the telescopers for (100) (resp. the telescopers for (101)) with the telescopers for (102). These different telescopers are *not homomorphic*. They correspond to *three different sequences* of telescopers of different nature, corresponding to three hypergeometric function of *quite different nature*:

$${}_2F_1\Big([\frac{1}{3},\frac{2}{3}],[1],27x^3\Big),\quad {}_3F_2\Big([\frac{7}{9},\frac{10}{9},\frac{13}{9}],[\frac{1}{3},1],27x^3\Big),\quad {}_3F_2\Big([\frac{5}{9},\frac{8}{9},\frac{11}{9}],[\frac{2}{3},1],27x^3\Big).$$

Along this line similar α-dependent examples are sketched in Appendix A.

To sum-up: The ideas sketched in subsections (3.1) and (3.2) for diagonals, can be generalized to telescopers (which may correspond to vanishing cycles i.e. diagonals), with the caveat that the unique "root" rational function $1/Q$, has to be replaced by a finite set of rational functions ($1/Q_1$, $1/Q_2$, $1/Q_3$ in our previous example).

6. An infinite number of birational symmetries of the diagonals and telescopers

Let us consider the simplest example of non-trivial diagonal of rational function, namely the diagonal of the rational function of three variables:

$$R(x,y,z) \;=\; \frac{1}{1-x-y-z}. \qquad (103)$$

Let us consider the *birational transformation* B:

$$B: \quad (x,\,y,\,z) \;\longrightarrow\; \Big(x,\; y\cdot(1+3x+7x^2),\; \frac{z}{1+3x+7x^2}\Big). \qquad (104)$$

It is birational because its compositional inverse is also a rational function:

$$(x,\,y,\,z) \quad \longrightarrow \quad \Big(x,\ \frac{y}{1\,+3\,x\,+7\,x^2},\ z\cdot\,(1\,+3\,x\,+7\,x^2)\Big). \tag{105}$$

Note that this birational transformation preserves the product $p \,=\, x\,y\,z$, as well as the neighbourhood of the point $(x,\,y,\,z) \,=\, (0,\,0,\,0)$. This birational transformation is an *infinite order* transformation. The composition of this transformation n times gives:

$$(x,\,y,\,z) \quad \longrightarrow \quad \Big(x,\ \ y\cdot\,(1\,+3\,x\,+7\,x^2)^n,\ \frac{z}{(1\,+3\,x\,+7\,x^2)^n}\Big). \tag{106}$$

The rational function (103), transformed by the (infinite order) birational transformation (104), reads:

$$\begin{aligned} R_B(x,y,z) \quad &= \quad R\Big(x,\ \ y\cdot\,(1\,+3\,x\,+7\,x^2),\ \ \frac{z}{1\,+3\,x\,+7\,x^2}\Big) \\ &= \quad \frac{1}{1\ \ -x\ \ -y\cdot\,(1\,+3\,x\,+7\,x^2)\ \ -z/(1\,+3\,x\,+7\,x^2)}. \end{aligned} \tag{107}$$

On the multi-Taylor expansion of (107) one finds easily that the diagonal of (103) and (107) are *actually identical*.

More generally, let us consider

$$B_x: \quad (x,\,y,\,z) \quad \longrightarrow \quad \Big(x,\ \ y\cdot\,Q_1(x),\ \frac{z}{Q_1(x)}\Big), \tag{108}$$

where $Q_1(x)$ is a rational function (see however section (6.4)) with a Taylor expansion such that $Q_1(0) \,\neq\, 0$. One also finds for any such rational function $Q_1(x)$, that the diagonal of (103) and (107) are actually identical. This can be seen from the multi-Taylor expansion of (107):

$$\begin{aligned} R_B(x,y,z) \quad &= \quad \sum_m\sum_n\sum_l a_{m,n,l}\cdot\,x^m\cdot\,y^n\cdot\,Q_1(x)^n\cdot\,z^l\cdot\,Q_1(x)^{-l} \\ &= \ \sum_m a_{m,m,m}\cdot\,(x\,y\,z)^m \ \ + \sum_{(m,n,l)\neq(m,m,m)} a_{m,n,l}\cdot\,x^m\cdot\,y^n\cdot\,z^l\cdot Q_1(x)^{n-l}. \end{aligned} \tag{109}$$

The second triple sum can be decomposed into the terms such that $n \neq l$, which cannot contribute to the diagonal (which extracts terms in $p \,=\, x\,y\,z$ and thus terms in the product $y\,z$), and the $n \,=\, l$ terms (such that the $Q_1(x)^{n-l}$ factor in (109) disappear):

$$\sum_{m\neq n} a_{m,n,n}\cdot\,x^m\cdot\,y^n\cdot\,z^n. \tag{110}$$

This last sum (110), which excludes the power of x to be equal to the power of the product $y\,z$, cannot contribute to the diagonal. We have thus proved that the diagonal of (103) and (107) are equal.

Of course there is nothing particular with the variable x. We can also introduce other birational transformations which single out respectively y and z:

$$B_y: \quad (x,\,y,\,z) \quad \longrightarrow \quad \Big(x\cdot\,Q_2(y),\ \ y,\ \ \frac{z}{Q_2(y)}\Big), \tag{111}$$

and

$$B_z: \quad (x,\,y,\,z) \quad \longrightarrow \quad \Big(x\cdot\,Q_3(z),\ \ \frac{y}{Q_3(z)},\ \ z\Big), \tag{112}$$

for any rational functions $Q_2(x)$ and $Q_3(x)$ with a Taylor expansion and such that $Q_2(0) \neq 0$ and $Q_3(0) \neq 0$. We can compose these birational transformations (108), (111) and (112), in any order and changing the various $Q_1(x)$, $Q_2(x)$ and $Q_3(x)$ at each step. We get that way a quite large *infinite* set of birational transformations preserving the product $p = x\,y\,z$ and the neighbourhood of the point $(x, y, z) = (0, 0, 0)$. Since the product $p = x\,y\,z$ is preserved, let us eliminate (for instance) the variable $z = p/x/y$. The three previous birational transformations (108), (111) and (112), on the three variables x, y, z, become birational transformations depending on a parameter p, of only two variables x, y:

$$\tilde{B}_x : \quad (x, y) \quad \longrightarrow \quad \Big(x, \; y \cdot Q_1(x)\Big), \tag{113}$$

$$\tilde{B}_y : \quad (x, y) \quad \longrightarrow \quad \Big(x \cdot Q_2(y), \; y\Big), \tag{114}$$

and

$$\tilde{B}_z : \quad (x, y) \quad \longrightarrow \quad \Big(x \cdot Q_3\Big(\frac{p}{x\,y}\Big), \quad y/Q_3\Big(\frac{p}{x\,y}\Big)\Big). \tag{115}$$

Composing these birational transformations of two variables (113), (114) and (115), in any order and changing the various $Q_1(x)$, $Q_2(x)$ and $Q_3(x)$ at each step, one gets that way a quite large subset of the (huge set of) *Cremona transformations* [50,78].

Remark 6.1: Of course there is nothing specific with the particularly simple example (103) of rational function. The previous birational transformations (113), (114) and (115), are symmetries of the diagonals of any rational function of three variables. Furthermore, there is nothing specific with rational function of three variables. We can generalize such birational transformations for diagonal of rational function of n variables, for any number of variables n.

6.1. *Non birational symmetries for diagonals*

6.1.1. Monomial transformation

Let us consider the (non-birational) *monomial* transformation:

$$M : \quad (x, y, z) \quad \longrightarrow \quad \Big(x, \, x^2\,y^2, \, y\,z^3\Big). \tag{116}$$

Let us perform this monomial transformation (116) on the rational function (103), one gets the new rational function:

$$R_M(x, y, z) \; = \; R\Big(x, \, x^2\,y^2, \, y\,z^3\Big) \; = \; \frac{1}{1 \; - x \; - x^2\,y^2 \; - y\,z^3}. \tag{117}$$

The calculation of the telescoper of (117) gives an order-two linear differentizal operator which has the ${}_2F_1$ hypergeometric series solution:

$$\begin{aligned} {}_2F_1\Big([\frac{1}{3}, \frac{2}{3}], [1], \, 27\,x^3\Big) \; &= \; 1 \; + 6\,x^3 \; + 90\,x^6 \; + 1680\,x^9 \; + 34650\,x^{12} \\ &+756756\,x^{15} \; + 17153136\,x^{18} \; + \; \cdots \end{aligned} \tag{118}$$

One verifies easily, on the multi-Taylor expansion of (117), that its diagonal is actually the ${}_2F_1$ hypergeometric series (118). The fact that the diagonal is the diagonal of (103), where x is changed into x^3, is a consequence of the fact that the product $p = x\,y\,z$ is changed into $p = x^3\,y^3\,z^3$ by the monomial transformation (116).

6.1.2. Non-birational transformation

Let us now consider the non-birational "monomial-like" transformation

$$B: \quad (x, y, z) \quad \longrightarrow \quad \Big(x, \ x^2 y^2 \cdot (1+3x), \ \frac{y z^3}{1+3x}\Big). \tag{119}$$

Let us perform this non-birational monomial transformation (119) on the rational function (103), one gets the new rational function

$$\begin{aligned} R_B(x, y, z) \, &= \, R\Big(x, \ x^2 y^2 \cdot (1+3x), \ \frac{y z^3}{1+3x}\Big) \\ &= \, \frac{1}{1 \, - x \, - x^2 y^2 \cdot (1+3x) \, - y z^3/(1+3x)}. \end{aligned} \tag{120}$$

The calculation of the telescoper of (120) gives an order-two linear differential operator which has, again, the ${}_2F_1$ hypergeometric series solution:

$$\begin{aligned} {}_2F_1\Big([\frac{1}{3}, \frac{2}{3}], [1] \, 27 x^3\Big) \, &= \, 1 \, + 6 x^3 \, + 90 x^6 + 1680 x^9 \\ &+34650 x^{12} + 756756 x^{15} + 17153136 x^{18} \, + \, \cdots \end{aligned} \tag{121}$$

One verifies easily on the multi-Taylor expansion of (120) that its diagonal is the ${}_2F_1$ hypergeometric series (121). This result can be understood from the results on (117) and the diagonal-preservation results on the birational transformations (108), (111) and (112).

Consequently we have another infinite set of (non-birational) transformations such that the diagonal of a rational function is changed into the diagonal of that rational function where x is changed into x^N.

6.2. *Birational symmetries for telescopers*

Recalling the creative telescoping equation (6) and (9), we have verified experimentally, on thousands of examples, that the previous birational transformations generated by (108), (111) and (112), are actually compatible with the creative telescoping equations (6) and (9). Note however, in the birationally transformed creative telescoping equations, that if the telescoper does remain invariant (*even if we are not in a context where the rational function has a multi-Taylor expansion*), the two "certificates" U and V are transformed in a very involved way (they become quite large rational functions).

6.2.1. Birational symmetries not preserving $(x, y, z) \, = \, (0, 0, 0)$

Let us consider the involutive birational transformation:

$$I: \quad (x, y, z) \quad \longrightarrow \quad \Big(\frac{1}{x}, \ \frac{1}{y}, \ x^2 y^2 z\Big). \tag{122}$$

This involutive birational transformation transforms the rational function (103) into:

$$R_I(x, y, z) \, = \, - \frac{x y}{x^2 y^3 z \, - x y \, + x + y}. \tag{123}$$

The calculation of the telescoper of (123) gives the same telescoper as the telescoper of (103), whose diagonal is the hypergeometric series:

$$\begin{aligned} &{}_2F_1\Big([\frac{1}{3}, \frac{2}{3}], [1], 27 x\Big) \\ &\quad = \, (1 - 24 x)^{-1/4} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], \ \frac{1728 x^3 \cdot (1 - 327 x)}{(1 - 24 x)^3}\Big) \\ &\quad = \, 1 + 6 x + 90 x^2 + 1680 x^3 + 34650 x^4 + 756756 x^5 + 17153136 x^6 \, + \, \cdots \end{aligned} \tag{124}$$

The hypergeometric series (124) (which is equal to the diagonal of (103)), is, here, just an analytical solution of the telescoper of (123), that is, a "Period" of (123) but corresponding to a non-vanishing cycle, since (123) does not have a multi-Taylor expansion.

6.2.2. Birational symmetries from collineations

Let us recall Noether's theorem [50,79,80] on the decomposition [81] of Cremona transformations. Noether's theorem shows that any Cremona transformation can be seen as the composition [50,81] of *collineation transformations* and of the Hadamard inverse transformation:

$$(x, y) \quad \longrightarrow \quad \Big(\frac{1}{x}, \frac{1}{y}\Big). \tag{125}$$

Let us consider Cremona transformations preserving $(x, y) = (0, 0)$:

$$(x, y) \quad \longrightarrow \quad \Big(\frac{x}{1 - x + 2y}, \frac{y}{1 - x + 2y}\Big). \tag{126}$$

With this theorem in mind, since we have already considered the involutive transformation (122) corresponding to the Hadamard inverse (125), let us just introduce the following birational transformation associated with the *collineation* (126):

$$(x, y, z) \quad \longrightarrow \quad \Big(\frac{x}{1 - x + 2y}, \frac{y}{1 - x + 2y}, z \cdot (1 - x + 2y)^2\Big). \tag{127}$$

Such a birational transformation (associated with collineations) is an (infinite order) transformation. It preserves $(x, y, z) = (0, 0, 0)$ and the product $p = xyz$. Let us perform this birational transformation (127) on the rational function (103). One gets a new rational function whose telescoper is an order-four linear differential operator L_4 which is the product of two order-two linear differential operator M_2 and N_2: $L_4 = M_2 \cdot N_2$. The order-two linear differential operator M_2 is (non-trivially) homomorphic to the order-two telescoper of the rational function (103). The second order-two linear differential operator N_2 corresponds to algebraic functions. For such transformations, associated with collineations, we see that *the telescoper is not preserved: we just have a (non-trivial) homomorphism property*. The example (127) is revisited in detail in Appendix B.4. More examples of birational symmetries for telescopers, associated with collineations, are given in Appendix B. These examples illustrate the complexity of the homomorphism.

6.3. Algebraic geometry comments on these birational symmetries

The diagonal of the rational function (103) is the hypergeometric series:

$$\begin{aligned} &{}_2F_1\Big([\frac{1}{3}, \frac{2}{3}], [1], 27x\Big) \\ &\quad = \quad (1 - 24x)^{-1/4} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], \frac{1728x^3 \cdot (1 - 327x)}{(1 - 24x)^3}\Big) \\ &\quad = \quad 1 + 6x + 90x^2 + 1680x^3 + 34650x^4 + 756756x^5 + 17153136x^6 + \cdots \end{aligned} \tag{128}$$

The algebraic curve, associated with the denominator of the rational function (103), is the *genus-one* algebraic curve (elliptic curve):

$$1 - x - y - \frac{p}{xy} = 0 \qquad \text{or:} \qquad -x^2y - xy^2 + xy - p = 0. \tag{129}$$

The calculation of its j-invariant gives the following Hauptmodul

$$\mathcal{H} \, = \, \frac{1728}{j} \, = \, \frac{1728\, p^3 \cdot \, (1\, -27\, p)}{(1\, -24\, p)^3}, \tag{130}$$

which is exactly the Hauptmodul pullback in (128).

Let us consider the rational function (107), the algebraic curve corresponding to eliminate $z \, = \, p/x/y$ in the denominator of (107) reads:

$$-49\, x^5\, y^2 \, -42\, x^4\, y^2 \, -7\, x^4\, y \, -23\, x^3\, y^2 \, +4\, x^3\, y \, -6\, x^2\, y^2 \\ +2\, x^2\, y \, -x\, y^2 \, +x\, y \, -p \, = \, 0. \tag{131}$$

This algebraic curve is a *genus-one* algebraic curve (elliptic curve) and the calculation of its j-invariant gives the same Hauptmodul pullback in (128) as the Hauptmodul (130) for (129). This is in agreement with the fact that the diagonal of (103) and (107) are equal. At first sight, the fact that (131) is an elliptic curve is not totally obvious, however it is a consequence of the fact that (129) and (131) are *birationally equivalent elliptic curves* (since one gets one from the other one from a birational transformation). *Consequently they should have the same j-invariant*.

This kind of remark will be seen as obvious, or slightly tautological, for an algebraic geometer, however, as far as down-to-earth computer algebra calculations of diagonals of rational functions or telescopers of rational functions are concerned, it becomes more and more spectacular for more complicated birational transformations generated by the composition of birational transformations like (108), (111) and (112).

More generally, the previous birational transformations preserving the product $p \, = \, x\,y\,z, \; p \, = \, x\,y\,z\,u, \, \ldots$ occurring in the diagonals, will preserve the algebraic geometry description of the diagonal of rational functions [41]. For instance the genus-two curves associated with split Jacobians situation we have encountered in [41] (which corresponds to products of elliptic curves), will be preserved by such birational transformations.

6.4. *Diagonal of transcendental functions*

Generalizing the rationals functions

$$R_B(x,y,z) \, = \, R\Big(x, \; y \cdot \, Q_1(x), \; \frac{z}{Q_1(x)}\Big) \, = \, \frac{1}{1 \, -x \, -y \cdot \, Q_1(x) \, -z/Q_1(x)}, \tag{132}$$

deduced from (107), using birational transformations like (108), one can consider, beyond, *transcendental* functions like

$$R_T(x,y,z) \, = \, R\Big(x, \; y \cdot \, \cos(x), \; \frac{z}{\cos(x)}\Big) \, = \, \frac{1}{1 \, -x \, -y \cdot \, \cos(x) \, -z/\cos(x)}. \tag{133}$$

One can easily verify, from the multi-Taylor expansion of the (simple) transcendental function (133), that its diagonal *is actually the same as the one* of (103), namely (128). This is not a surprise since the demonstration of the invariance of the diagonal by birational transformation sketched in section 6 (see (109)), just requires that $Q_1(0) \neq 0$ with $Q_1(x)$ behaving at the origin as a polynomial.

7. Conclusion

Diagonals of rational functions have been shown to emerge naturally for n-fold integrals in physics, field theory, enumerative combinatorics, seen as "Periods" of algebraic varieties (corresponding to the denominators of these rational functions). On the thousands of examples we have analyzed, corresponding to n-fold integrals of theoretical physics (in particular the $\chi^{(n)}$'s of the susceptibility of the Ising model, ...), or corresponding to rather academical diagonal of rational functions, we have seen the emergence of many striking properties, and we want to understand if these remarkable properties are inherited from the "physics", and, more precisely, the rather "integrable" framework of these examples (Yang-Baxter integrability, 2D Ising models, Calabi-Yau and other mirror symmetries, ...) or, on the contrary, are a consequence of the remarkable nature of diagonals of rational functions in the most general framework.

This paper is a plea for diagonals of rational, or algebraic, functions and more generally telescopers of rational or algebraic functions.

• We show that "periods" corresponding to non-vanishing cycles, obtained as solutions of telescopers of rational functions can sometimes be recovered from diagonals of rational functions corresponding to vanishing cycles, introducing an extra parameter. These two concepts are not that compartmentalized.

• When considering diagonals of rational functions we have shown that the number of variables of a rational function must, from time to time, be replaced by a notion of "effective number" of variables.

• We have shown that the "complexity" of the diagonals of a rational function, and for instance the order of the (minimal order) linear differential operator annihilating this diagonal, is not related to the number of variables, or "effective number" of variables of the rational function. In a forthcoming publication [69] we will try to understand what is the minimal number of variables necessary to represent a given D-finite globally bounded series as a diagonal of a rational function.

• We have shown that the algebraic geometry approach of the diagonals of rational functions, or of the telescopers of these rational functions, described in [41], can, probably, be generalized to diagonals of algebraic functions, or telescoper of algebraic functions. These are just preliminary studies and almost everything remains to be done.

• When studying diagonals of rational functions, our explicit examples enable to understand why one can actually restrict to rational functions of the form $1/Q$ provided the polynomial at the denominator is irreducible. The situation where the denominator Q factorizes clearly needs further analysis that will be displayed in a forthcoming paper [69]. The case of the calculations of telescopers is slightly different: one can (probably), again, reduce to rational functions of the form $1/Q$ but with a *finite set* of polynomials Q.

• We have shown that diagonals of rational functions (and this is also the case with diagonals of algebraic functions) are left invariant when one performs an *infinite set of birational transformations* on the rational functions. This remarkable result can, in fact, be generalized to *infinite set of rational transformations*, the diagonals of the transformed rational functions becoming the diagonal of the original rational function where the variable x is changed into x^n. These invariance results generalize to telescopers. More general (infinite) set of birational transformations are shown to correspond to more convoluted "covariance" property of the telescopers (see Appendix B).

• We provide some examples of diagonals of transcendental functions which can also yield simple ${}_2F_1$ hypergeometric functions associated with elliptic curves. The analysis of diagonal of transcendental functions is clearly an interesting new domain to study.

• Finally, when trying to understand the puzzling fact that telescopers of rational functions are almost always homomorphic to their adjoint, and thus have selected symplectic or orthogonal differential Galois groups, we understand a bit better the emergence of curious examples of telescopers that are *not* homomorphic to their adjoint, this (up to homomorphisms) *self-duality-breaking* ruling out a Poincaré duality interpretation of this quite systematic emergence of operators homomorphic to their adjoint. A "desingularization" of such puzzling cases, corresponding to the introduction of an extra parameter, shows that such operators now occur in dual (adjoint) pairs, thus restoring the duality (homomorphism to the adjoint). The limit when the extra parameter goes to zero, is the *direct sum* of different telescopers corresponding to the first rational function terms of the expansion of the extended rational function in term of this extra parameter. With section 5.2 we see that the puzzling (non self-adjoint up to homomorphism) order-three linear differential operator L_3 with $SL(3, \mathbb{C})$ differential Galois group, is better understood as a member of a triplet of three "quarks" (90), (91), and (92), which restores the duality. This may suggest that the quite strange ${}_3F_2$ hypergeometric functions (91) or (92), could be related to (90) which has a clear elliptic curve origin. After all, these functions are three periods of the same algebraic variety. The existence of such a relation between hypergeometric functions of totally and utterly different nature, is a challenging open question.

• In Appendix B the calculations of telescopers of rational functions, associated with very simple collineations, yield quite massive linear differential operators which factor into an order-two operator associated with an elliptic curve, and a "dressing" of products of factors which turn out to be direct sums of operators with algebraic function solutions. This occurrence of this "mix" between products and direct sums of a large number of operators (occurring, for instance, for the linear differential operators annihilating the $\chi^{(n)}$ components of the susceptibility of the Ising model [1,27,28]) will be revisited in a forthcoming paper [69].

Instead of pursuing one specific mathematical problem this paper can be seen as a journey into the amazing world of integer sequences, and differential equations. With all the examples displayed in this paper we provide some answers, sometimes some plausible scenarii, to many important questions naturally emerging when working on diagonals of rational or algebraic functions, or on telescopers of rational or algebraic, functions, related, or not related, to problems of physics or enumerative combinatorics. Like any fruitful concept, every answered questions does not "close" the subject but, on the contrary, often raises more new questions than the number of answered questions.

Diagonals of rational, or algebraic, functions, correspond to (globally bounded) series that can be recast into series with integer coefficients which are solutions of *linear* differential operators. When studying the two dimensional Ising model and its related *Painlevé equations*, one finds that the λ-extensions of the correlation functions [82,83] can also produce series with integer coefficients which are differentially algebraic [84] solutions of *non-linear* differential equations of the Painlevé type, these series being also such that their reduction modulo primes give algebraic functions, just like diagonals of rational or algebraic functions (for other examples of differentially algebraic series with integer coefficients see for instance [85]).

This paper tries to show that the concept of diagonals of rational, or algebraic, functions is a remarkably rich and fruitful concept providing tools for physics but also bridging, in a quite fascinating way, different domains of mathematics. The case of diagonal of transcendental functions, or of these λ-extensions seems to show that the "unreasonable richness" of diagonals and telescopers, may just be the top of an even more fascinating mathematical "iceberg" of mathematical physics.

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Appendix A. Other α-dependent example

Appendix A.1. A first very simple example

Another example, similar to the rational function (87) studied in section 5.2, is

$$\frac{1}{1 \ - x^2 - y^2 - z^2 \ - \alpha \cdot x\, y^2}. \tag{A.1}$$

Its telescoper is an order-four linear differential operator which becomes in the $\alpha \rightarrow 0$ limit the LCLM of two order two linear differential operators, one, L_2, corresponding to the hypergeometric solution (which is actually the $\alpha = 0$ diagonal)

$${}_2F_1\Big([\frac{1}{3}, \frac{2}{3}],\, [1],\, 27\, x^2\Big), \tag{A.2}$$

and an order-two linear differential operator M_2 having the solution

$$\frac{d}{dx}\, {}_2F_1\Big([\frac{1}{6}, \frac{5}{6}],\, [1],\, 27\, x^2\Big), \tag{A.3}$$

This order-two operator M_2 is not homomorphic to the order-two operator L_2. Let us consider the α expansion of (A.1)

$$\begin{aligned} \frac{1}{1 \ - x^2 - y^2 - z^2 \ - \alpha \cdot x\, y^2} \ &= \ \frac{1}{1 \ - x^2 - y^2 - z^2} \ + \frac{x\, y^2}{(1 \ - x^2 - y^2 - z^2)^2} \cdot \alpha \\ &+ \frac{x^2\, y^4}{(1 \ - x^2 - y^2 - z^2)^3} \cdot \alpha^2 \ \ + \ \cdots \end{aligned} \tag{A.4}$$

The diagonal of the term in α^1 in (A.4) is trivial: it is equal to zero. In contrast, the telescoper of the term in α^1 in (A.4) is actually nothing but the order-two linear differential operator M_2. The telescoper of the term in α^2 in (A.4) is an order-two linear differential operator homomorphic to the previous order-two linear differential operator L_2. Similarly to the calculations displayed in (87), the telescopers for the terms in $\alpha^{2\,n}$ in the expansion (A.4) yield order-two linear differential operators, homomorphic to L_2, when the telescopers for the terms in $\alpha^{2\,n\,+1}$ yield order-two operators, homomorphic to M_2.

Appendix A.2. Christol: breaking the duality symmetry

These results can be compared with ones for the diagonal of the rational function

$$\frac{1}{1 \ - x^4 - y^4 - z^4 \ - \alpha \cdot x}. \tag{A.5}$$

The linear differential operator annihilating the diagonal of the rational function (A.5) is an order-ten linear differential operator $L_{10}(\alpha)$ depending on the parameter α, which is homomorphic to its adjoint with an order-eight intertwiner. Consequently its differential Galois group is included in $Sp(10, \mathbb{C})$. This order-ten linear differential operator $L_{10}(\alpha)$ is irreducible except at $\alpha = 0$.

At $\alpha = 0$ it is the direct sum $LCLM(L_2, M_2, L_3, M_3)$, of two order-three linear differential operators and two order-two linear differential operators, namely L_2 corresponding to the solution

$$ {}_2F_1\Big([\frac{1}{3}, \frac{2}{3}], [1], 27\,x^4\Big) \\ = \quad 1 \ + 6\,x^4 \ + 90\,x^8 \ + 1680\,x^{12} \ + 34650\,x^{16} \ + 756756\,x^{20} \ + \ \cdots \tag{A.6} $$

as it should (this is the diagonal of (A.5) at $\alpha = 0$), and the other one, M_2, corresponding to the globally bounded series solution expressed in terms of HeunG functions (use Table page 24 of [67]):

$$ \frac{(1-24\,x^4)^2}{(1-27\,x^4)^2} \cdot HeunG\Big(\frac{9}{8}, \frac{97}{32}, \frac{7}{6}, \frac{5}{6}, 1, -1;\ 27 \cdot x^4\Big). \tag{A.7} $$

This linear differential operator M_2 is homomorphic to the order-two linear differential operator corresponding to the modular form (see Appendix B in [16]):

$$ {}_2F_1\Big([\frac{1}{6}, \frac{5}{6}], [1], 27\,x^4\Big). \tag{A.8} $$

Using the identity

$$ HeunG\Big(\frac{9}{8}, \frac{97}{32}, \frac{7}{6}, \frac{5}{6}, 1, -1;\ 27 \cdot x\Big) \ = \\ 4 \cdot (1-27\,x) \cdot \frac{(27\,x+2)}{(1-24\,x)^2} \cdot x \cdot \frac{d}{dx}\,{}_2F_1\Big([\frac{1}{6}, \frac{5}{6}], [1], 27\,x\Big) \\ + \frac{19\,x - 486\,x^2}{(1-24\,x)^2} \cdot {}_2F_1\Big([\frac{1}{6}, \frac{5}{6}], [1], 27\,x\Big), \tag{A.9} $$

we can rewrite (A.7) in terms of the modular form (A.8). One can thus write the solution of M_2 as:

$$ \frac{2+27\,x^4}{1-27\,x^4} \cdot x \cdot \frac{d}{dx}\,{}_2F_1\Big([\frac{1}{6}, \frac{5}{6}], [1], 27\,x^4\Big) \ + \frac{1+18\,x^4}{1-27\,x^4} \cdot x^4 \cdot {}_2F_1\Big([\frac{1}{6}, \frac{5}{6}], [1], 27\,x^4\Big) \\ = \quad 1 \ + \frac{315}{4}\,x^4 \ + \frac{225225}{64}\,x^8 \ + \frac{33948915}{256}\,x^{12} \ + \frac{75293843625}{16384}\,x^{16} \\ + \frac{9927744261435}{65536}\,x^{20} \ + \ \cdots \tag{A.10} $$

The order-three linear differential operator L_3 has the hypergeometric solution

$$ {}_3F_2\Big([\frac{7}{12}, \frac{11}{12}, \frac{15}{12}], [\frac{3}{4}, 1], 27\,x^4\Big), \tag{A.11} $$

while the order-three linear differential operator M_3 has the hypergeometric solution:

$$ {}_3F_2\Big([\frac{13}{12}, \frac{17}{12}, \frac{21}{12}], [\frac{1}{4}, 1], 27\,x^4\Big). \tag{A.12} $$

These two linear differential operators are such that L_3 is actually homomorphic to the adjoint of M_3, and, of course, M_3 is homomorphic to the adjoint of L_3, but L_3 is *not* homomorphic to the adjoint of L_3 (and M_3 is not homomorphic to the adjoint of M_3). We have again, a pair of dual linear differential operators.

Since these calculations are in the $\alpha \rightarrow 0$ limit, let us expand in α the rational function (A.5):

$$\begin{aligned} \frac{1}{1 - x^4 - y^4 - z^4 - \alpha \cdot x} \, &= \, \frac{1}{1 - x^4 - y^4 - z^4} \, + \frac{x}{(1 - x^4 - y^4 - z^4)^2} \cdot \alpha \\ &+ \frac{x^2}{(1 - x^4 - y^4 - z^4)^3} \cdot \alpha^2 \, + \frac{x^3}{(1 - x^4 - y^4 - z^4)^4} \cdot \alpha^3 \\ &+ \frac{x^4}{(1 - x^4 - y^4 - z^4)^5} \cdot \alpha^4 \, + \, \cdots \end{aligned} \tag{A.13}$$

Since the telescoper of a sum of rational functions is the direct sum (LCLM) of the telescopers of these rational functions, let us consider the telescopers of the first five terms in the RHS of (A.13). The telescoper of the first term is of course the order-two linear differential operator L_2 annihilating the diagonal of this rational function. The telescoper of the second term (in α^1), is the order-three linear differential operator L_3.The telescoper of the third term (in α^2), is the order-two linear differential operator M_2. The telescoper of the fourth term (in α^3), is exactly M_3. The telescoper of the sum of the first orders in α in the expansion (A.13)

$$\begin{aligned} &\frac{1}{1 - x^4 - y^4 - z^4} \, + \frac{x}{(1 - x^4 - y^4 - z^4)^2} \cdot \alpha \\ &\quad + \frac{x^2}{(1 - x^4 - y^4 - z^4)^3} \cdot \alpha^2 \, + \frac{x^3}{(1 - x^4 - y^4 - z^4)^4} \cdot \alpha^3, \end{aligned} \tag{A.14}$$

is actually the LCLM of the four telescopers L_2, M_2, L_3 and M_3 *which is precisely the $\alpha \rightarrow 0$ limit of the order-ten linear differential operator !*

Let us now consider the telescopers of the next α orders in the expansion (A.13). The telescoper of the last rational function in (A.13), namely $x^4/(1 - x^4 - y^4 - z^4)^5$, is an order-two linear differential operator N_2. One can thus write the solution of N_2 as:

$$\begin{aligned} \mathcal{D}_1 \, &= \, \frac{3}{48} \cdot \frac{1 + 540\,x^4 + 4374\,x^8}{(1 - 27\,x^4)^3} \cdot x \cdot \frac{d}{dx}\, {}_2F_1\Big([\frac{1}{3}, \frac{2}{3}], [1], \, 27\,x^4\Big) \\ &+ \frac{3}{2} \cdot \frac{(19 + 216\,x^4)}{(1 - 27\,x^4)^3} \cdot x^4 \cdot {}_2F_1\Big([\frac{1}{3}, \frac{2}{3}], [1], \, 27\,x^4\Big) \\ &= \, 30\,x^4 \, + 3780\,x^8 \, + 277200\,x^{12} \, + 15765750\,x^{16} \, + 771891120\,x^{20} \, + \, \cdots \end{aligned} \tag{A.15}$$

The telescoper of

$$\frac{x^8}{(1 - x^4 - y^4 - z^4)^9}, \tag{A.16}$$

is an order-two linear differential operator whose analytic solution reads:

$$\begin{aligned} \mathcal{D}_2 \, &= \, -\frac{3}{672} \cdot \frac{p_1}{(1 - 27\,x^4)^7} \cdot x \cdot \frac{d}{dx}\, {}_2F_1\Big([\frac{1}{3}, \frac{2}{3}], [1], \, 27\,x^4\Big) \\ &+ \frac{3}{28} \cdot \frac{p_2}{(1 - 27\,x^4)^7} \cdot x^4 \cdot {}_2F_1\Big([\frac{1}{3}, \frac{2}{3}], [1], \, 27\,x^4\Big) \\ &= \, 2970\,x^8 \, + 900900\,x^{12} \, + 137837700\,x^{16} \, + 14665931280\,x^{20} \\ &\qquad + 1236826871280\,x^{24} \, + 88597190167200\,x^{28} \, + \, \cdots \end{aligned} \tag{A.17}$$

where:

$$\begin{aligned} p_1 \, &= \, 1 \, - 714\,x^4 \, - 924372\,x^8 \, - 54587520\,x^{12} \, - 530141922\,x^{16} \, - 554824404\,x^{20}, \\ p_2 \, &= \, 1 \, + 27030\,x^4 \, + 2062098\,x^8 \, + 23960772\,x^{12} \, + 29170206\,x^{16}. \end{aligned} \tag{A.18}$$

If we consider, instead of the telescoper, the diagonal of the rational function (A.13), only the terms in α^{4n} $n = 0, 1, 2, \cdots$ will contribute, the other ones, corresponding to *non-vanishing cycles* [57], give zero contributions. Consequently we get for the diagonal of the rational function (A.13):

$$\mathrm{Diag}\Big(\frac{1}{1 - x^4 - y^4 - z^4 - \alpha \cdot x}\Big) = {}_2F_1\Big([\frac{1}{3}, \frac{2}{3}], [1], 27\,x^4\Big) + \mathcal{D}_1 \cdot \alpha^4 + \mathcal{D}_2 \cdot \alpha^8 + \cdots \tag{A.19}$$

$$\begin{aligned} = \; & 1 + (30\,\alpha^4 + 6) \cdot x^4 + (2970\,\alpha^8 + 3780\,\alpha^4 + 90) \cdot x^8 \\ & +(371280\,\alpha^{12} + 900900\,\alpha^8 + 277200\,\alpha^4 + 1680) \cdot x^{12} \\ & +(51482970\,\alpha^{16} + 185175900\,\alpha^{12} + 137837700\,\alpha^8 + 15765750\,\alpha^4 + 34650) \cdot x^{16} \\ & +(7571343780\,\alpha^{20} + 36141044940\,\alpha^{16} + 44975522592\,\alpha^{12} \\ & \quad +14665931280\,\alpha^8 + 771891120\,\alpha^4 + 756756) \cdot x^{20} + \cdots \\ = \; & 1 + 6\,x^4 + 90\,x^8 + 1680\,x^{12} + 34650\,x^{16} + 756756\,x^{20} + \cdots \\ & + \Big(30\,x^4 + 3780\,x^8 + 277200\,x^{12} + 15765750\,x^{16} + 771891120\,x^{20} + \cdots\Big) \cdot \alpha^4 \\ & + \Big(2970\,x^8 + 900900\,x^{12} + 137837700\,x^{16} + 14665931280\,x^{20} + \cdots\Big) \cdot \alpha^8 + \cdots \end{aligned}$$

Appendix B. Birational symmetries from collineations

Appendix B.1. Birational symmetries from collineations: a first example

Let us consider a collineation transformation not preserving $(x, y) = (0, 0)$:

$$(x, y) \quad \longrightarrow \quad \Big(\frac{2 + x + 3\,y}{1 - x + 2\,y}, \, \frac{1 + 5\,x + 7\,y}{1 - x + 2\,y}\Big), \tag{B.1}$$

and let us now introduce the following *birational transformation* associated with the collineation (B.1):

$$(x, y, z) \quad \longrightarrow \quad \Big(\frac{2 + x + 3\,y}{1 - x + 2\,y}, \, \frac{1 + 5\,x + 7\,y}{1 - x + 2\,y}, \, \frac{x\,y\,z \cdot (1 - x + 2\,y)^2}{(2 + x + 3\,y) \cdot (1 + 5\,x + 7\,y)}\Big), \tag{B.2}$$

which preserves the product $p = x\,y\,z$.

Let us transform the simple rational function (103) with the birational transformation (B.2). It becomes the rational function:

$$\mathcal{R} = \frac{(1 - x + 2\,y) \cdot (2 + x + 3\,y) \cdot (1 + 5\,x + 7\,y)}{\mathcal{D}}, \tag{B.3}$$

where the denominator $\mathcal{D}$ reads:

$$\begin{aligned} \mathcal{D} = \; & x^4\,y\,z - 6\,x^3\,y^2\,z + 12\,x^2\,y^3\,z - 8\,x\,y^4\,z - 3\,x^3\,y\,z + 12\,x^2\,y^2\,z - 12\,x\,y^3\,z \\ & +3\,x^2\,y\,z - 6\,x\,y^2\,z - 35\,x^3 - 194\,x^2\,y - 323\,x\,y^2 - x\,y\,z - 168\,y^3 - 87\,x^2 \\ & -251\,x\,y - 178\,y^2 - 36\,x - 50\,y - 4. \end{aligned} \tag{B.4}$$

The intersection of the algebraic surface $\mathcal{D} = 0$ with the algebraic surface $p = x\,y\,z$, is an elliptic curve. One gets, almost instantaneously (using the *j_invariant* command in Maple with(algcurves)), the Hauptmodul of this elliptic curve:

$$\mathcal{H} \; = \; \frac{1728\,p^3 \cdot \, (1\, - 27\,p)}{(1\, - 24\,p)^3}. \qquad \text{(B.5)}$$

If one expects an *algebraic geometry interpretation* of the calculation of the diagonal of rational functions or telescopers [41], this Hauptmodul must be the same as the Hauptmodul (130) of the elliptic curve (129), since the two algebraic curves are birationaly equivalent, being related by a birational transformation namely (B.1). The calculation of the telescoper of (B.3) is really massive: it gives, after one month of computation, an order-eleven linear differential operator (we thank C. Koutschan for performing these slightly "extreme" computations). The result being too massive, let us consider other examples of birational transformations associated with collineations simpler than (B.2).

Remark B 1.1: The diagonal of the rational function (B.3) is a very simple series:

$$\begin{aligned} Diag\Big(\mathcal{R}\Big) \; &= \; -\frac{1}{2} \cdot \, \frac{1}{1\, + x/4} \\ &= \; -\frac{1}{2}\, + \frac{1}{8} \cdot x\, - \frac{1}{32} \cdot x^2\, + \frac{1}{128} \cdot x^3\, - \frac{1}{512} \cdot x^4\, + \, \cdots \end{aligned} \qquad \text{(B.6)}$$

Remark B 1.2: If one considers, instead of (B.3) the rational function with the same denominator(B.4) but where the numerator is normalised to 1,

$$\mathcal{R} \; = \; \frac{1}{\mathcal{D}}. \qquad \text{(B.7)}$$

The diagonal of (B.7) is the same as (B.6) up to a factor two:

$$Diag\Big(\mathcal{R}\Big) \; = \; -\frac{1}{4} \cdot \, \frac{1}{1\, + x/4}. \qquad \text{(B.8)}$$

The telescoper of (B.7) is an order-seven linear differential operator which factorises as follows:

$$L_7 \; = \; F_2 \cdot G_2 \cdot H_2 \cdot H_1 \qquad \text{with:} \qquad H_1 \; = \; D_x \, + \frac{1}{4\, + x}, \qquad \text{(B.9)}$$

where the order-two linear differential operator F_2 is quite large and is (non-trivially) homomorphic to the order-two linear differential operator L_2 which is the telescoper of the rational function (103), and where the order-two linear differential operators G_2 and H_2 have algebraic solutions. The diagonal (B.8) is solution of the order-one operator H_1. The homomorphism between F_2 and L_2 gives

$$F_2 \cdot X_1 \; = \; Y_1 \cdot L_2 \qquad \text{where:} \qquad X_1 \; = \; A(x) \cdot D_x \, + B(x), \qquad \text{(B.10)}$$

where $A(x)$ and $B(x)$ are rational functions. Consequently a solution $\mathcal{S}$ of the telescoper L_7 (but not of the product $G_2\,H_2\,H_1$ in (B.9)) will be related to the hypergeometric solution ${}_2F_1([1/3, 2/3], [1], 27\,x)$ of the order-two linear differential operator L_2, as follows:

$$X_1\Big({}_2F_1\Big([\frac{1}{3}, \frac{2}{3}], [1], 27\,x\Big)\Big) \; = \; G_2 \cdot H_2 \cdot H_1 \cdot \mathcal{S}. \qquad \text{(B.11)}$$

Remark B 1.3: Note that the diagonal of the rational function (B.3) is a very simple series (B.6). Therefore the solution $\mathcal{S}$ of the telescoper, associated with an elliptic curve of Hauptmodul (B.5) (see equation (B.11)) corresponds to a "period", an integral over a non-vanishing cycle, and is different from the integral over a vanishing cycle, namely the diagonal (B.6).

Remark B 1.4: The factorisation (B.9) is far from being unique. The product of the last three factors can be seen to be a direct sum:

$$G_2 \cdot H_2 \cdot H_1 \;=\; \tilde{G}_2 \oplus \tilde{H}_2 \oplus H_1, \tag{B.12}$$

where the two new order-two operators $\tilde{G}_2$ and $\tilde{H}_2$ are simpler, with, again, algebraic function solutions.

Appendix B.2. Birational symmetries from collineations. A simpler example

Let us consider the following birational transformation associated with a collineation:

$$(x,\, y,\, z) \quad \longrightarrow \quad \Big({{x\, +3\,y} \over {1\, -\, x\, +2\,y}},\ {{1\, +5\,x\, +y} \over {1\, -\, x\, +2\,y}},\ {{x\,y\,z \cdot \, (1\, -\, x\, +2\,y)^2} \over {(x\, +3\,y) \cdot \, (1\, +5\,x\, +7\,y)}}\Big), \tag{B.13}$$

which preserves the product $p \,=\, x\,y\,z$. Again, if one transforms the simple rational function (103) with the birational transformation (B.13), one gets the rational function of the form

$$\mathcal{R} \;=\; {{(1\, -\, x\, +2\,y) \cdot \, (x\, +3\,y) \cdot \, (1\, +5\,x\, +y)} \over {\mathcal{D}}}, \tag{B.14}$$

and, again, the intersection of the algebraic surface $\mathcal{D} \,=\, 0$ with the algebraic surface $p \,=\, x\,y\,z$, is an elliptic curve, corresponding to eliminate $z \,=\, p/x/y$ in $\mathcal{D} \,=\, 0$. One gets immediatly the same Hauptmodul (B.5) for this new elliptic curve.

The telescoper of the rational function (B.14) is an order-ten linear differential operator (we thank C. Koutschan for providing this order-ten linear differential operator). This telescoper is obtained using about nine days of computation time. It uses 286 evaluation points (in contrast with the 462 evaluation points required for (B.4)), and one uses in total 38 primes (of size $9 \cdot 22 \cdot 10^{18}$) to reconstruct the solution with Chinese remaindering. The telescoper of the rational function (B.14) factors as follows:

$$L_{10} \;=\; F_2 \cdot G_2 \cdot H_1, \cdot I_1 \cdot J_2 \cdot K_2, \tag{B.15}$$

The order-two linear differential operator F_2 in (B.15) is homomorphic to the order-two linear differential operator L_2 which is the telescoper of the rational function (103), and the order-two linear differential operators G_2, J_2 and K_2 have algebraic solutions.

Remark B 2.1: The factorisation of (B.15) is far from being unique. As usual we have a mix between product and direct-sum of factors. The order-ten operator being quite large it is difficult to get the direct-sum factorisation of L_{10} in (B.15). One finds, however, quite easily that L_{10} has two simple rational function solutions

$$ {{1} \over {(x-35) \cdot \, (4\,x\, +3)}}, \qquad \qquad {{x} \over {(x-35) \cdot \, (4\,x\, +3)}}, \tag{B.16}$$

corresponding to two order-one operators $L_1 \,=\, D_x \, + (8\,x - 137)/(4\,x + 3)/(x - 35)$ and $M_1 \,=\, D_x \, + (4\,x + 3)/(x + 21)/(x - 35) \, - 1/x$ and, thus, can be rightdivided by the LCLM of L_1 and M_1. In fact the product of the last factors at the right of the factorization of L_{10} can be seen to be a direct sum:

$$G_2 \cdot H_1 \, \cdot I_1 \cdot J_2 \cdot K_2 \;=\; L_1 \oplus M_1 \oplus \tilde{G}_2 \oplus \tilde{J}_2 \oplus K_2. \tag{B.17}$$

In contrast the product $F_2 \cdot G_2$ is *not* a direct sum. The order-two operators $\tilde{G}_2$ and $\tilde{J}_2$ are (much) simpler than G_2 and J_2, again with algebraic function solutions.

The result remaining still too large, let us consider another example of birational transformation associated with collineations, simpler than (B.2) or (B.13).

Remark B 2.2: If one considers, instead of (B.14) the rational function with the same denominator $\mathcal{D}$ but where the numerator is normalised to 1,

$$\mathcal{R} \, = \, \frac{1}{\mathcal{D}}. \tag{B.18}$$

The telescoper of the rational function (B.18) is an order-seven linear differential operator

$$L_7 \, = \, F_2 \cdot G_1 \cdot G_2 \cdot H_2, \tag{B.19}$$

where the order-two linear differential operator F_2 is (non-trivially) homomorphic to the order-two linear differential operator L_2 which is the telescoper of the rational function (103), and where the order-two linear differential operators G_2 and H_2 have simple algebraic solutions. This factorisation (B.19) is not unique. Introducing the order-one operator $\tilde{G}_1 \, = \, D_x \, + 1/x$, one can see that $\tilde{G}_1$ rightdivides L_7 and that the product of the three factors, at the right of the decomposition (B.19), can be written as a direct sum

$$G_1 \cdot G_2 \cdot H_2 \, = \, \tilde{G}_1 \oplus \tilde{G}_2 \oplus H_2, \tag{B.20}$$

where the solutions of $\tilde{G}_2$ are algebraic.

Remark B 2.3: In Appendix B we encounter many order-two linear differential operators with algebraic solutions/ Even for large order-two linear differential operators one can see quite easily (using hypergeometricsols in DEtools of Maple) that the *log-derivative* of these solutions are algebraic functions, but finding that the algebraic expression (minimal polynomial) of the solutions is much harder. Just showing that the solutions are algebraic without having their exact expressions, can be achieved by showing that their p-curvatures are zero, recalling the André-Christol conjecture that one must have a basis of globally bounded solutions, or looking for rational solutions of symmetric powers of the operators. In principle these algebraic functions solutions of order-two linear differential operators can be written as pullbacked ${}_2F_1$ hypergeometric functions, but again it is a difficult task [86].

Appendix B.3. Birational symmetries from collineations. An even simpler example

Let us consider the following birational transformation associated with a collineation:

$$\begin{aligned} (x, \, y, \, z) \quad &\longrightarrow \\ &\Big(\frac{x \, +3\, y}{1 \, - \, x \, +2\, y}, \, \frac{5\, x \, +7\, y}{1 \, - \, x \, +2\, y}, \, \frac{x\, y\, z \cdot (1 \, - x \, +2\, y)^2}{(x \, +3\, y) \cdot (5\, x \, +7\, y)}\Big), \end{aligned} \tag{B.21}$$

which preserves the product $p \, = \, x\, y\, z$, and also preserves the origin $(x, \, y, \, z) \, = \, (0, \, 0, \, 0)$. Again, if one transform the simple rational function (103) with the birational transformation (B.21), one gets the rational function of the form:

$$\mathcal{R} \, = \, \frac{(1 \, - \, x \, +2\, y) \cdot (x \, +3\, y) \cdot (5\, x \, +7\, y)}{\mathcal{D}}, \tag{B.22}$$

and, again, the intersection of the algebraic surface $\mathcal{D} \, = \, 0$ with the algebraic surface $p \, = \, x\, y\, z$, is an elliptic curve, corresponding to eliminate $z \, = \, p/x/y$ in $\mathcal{D} \, = \, 0$. One gets immediatly the same Hauptmodul (B.5) for this new elliptic curve. The telescoper of the rational function (B.22) is an order-ten linear differential operator

$$L_{10} \; = \; F_2 \cdot G_2 \cdot H_1 \cdot I_1 \cdot J_2 \cdot K_2, \tag{B.23}$$

where the order-two linear differential operator F_2 is a quite "massive" operator (30391 characters) which is (non-trivially) homomorphic to the order-two linear differential operator L_2 which is the telescoper of the rational function (103) and where the solutions of G_2, J_2 and K_2 are two *algebraic functions*. The order-two linear differential operator F_2 is of the form

$$F_2 \; = \; D_x^2 \; + \frac{A_1(x)}{D_1(x)} \cdot D_x \; + \frac{A_0(x)}{D_0(x)}, \tag{B.24}$$

where $A_1(x)$ and $A_0(x)$ are polynomials of degree 41 and 55 respectively, where $D_1(x)$ and $D_0(x)$ read

$$D_1(x) \; = \; \lambda(x) \cdot P_{14}(x) \cdot P_{20}(x), \qquad D_0(x) \; = \; x \cdot \lambda(x) \cdot P_{14}(x) \cdot P_{20}(x)^2, \tag{B.25}$$

with:

$$\begin{aligned} \lambda(x) \; = \; & (219024 - 6916931\, x - 23604075\, x^2) \cdot (7 - 225\, x) \cdot (5 - 243\, x) \\ & \times (1 - 27\, x) \cdot (35 - x) \cdot (21 + x) \cdot x, \end{aligned} \tag{B.26}$$

where $P_{14}(x)$ and $P_{20}(x)$ are polynomials of degree 14 and 20 respectively. The order-two operator linear differential G_2 yielding algebraic solutions is also a quite "large" linear differential operator.

Remark B 3.1: The factorisation of (B.23) is far from being unique. As usual we have a mix between product and direct-sum of factors. The order-ten linear differential operator being quite large it is difficult to get the direct-sum factorisation of L_{10} in (B.23). One finds, however, quite easily that L_{10} has two simple rational function solutions

$$\frac{1}{(x-35) \cdot (x+21)}, \qquad \qquad \frac{x}{(x-35) \cdot (x+21)}, \tag{B.27}$$

corresponding to two order-one operators $L_1 \; = \; D_x \, + 2\,(x-7)/(x+21)/(x-35)$ and $M_1 \; = \; D_x \, + 2\,(x-7)/(x+21)/(x-35) \; - 1/x$ and, thus, can be rightdivided by the LCLM of L_1 and M_1. More interestingly, the product $H_1 \cdot I_1 \cdot J_2 \cdot K_2$ in the decomposition (B.23) of L_{10}, can be seen as the direct sum of L_1, M_1, K_2 and two new (and simpler !) order-two linear differential operators $\tilde{G}_2$ and $\tilde{J}_2$:

$$G_2 \cdot H_1 \cdot I_1 \cdot J_2 \cdot K_2 \;\; = \;\; L_1 \oplus M_1 \oplus \tilde{G}_2 \oplus \tilde{J}_2 \oplus K_2. \tag{B.28}$$

In contrast note that the product $F_2 \cdot G_2$ in the decomposition (B.23) is *not* a direct-sum. It was easy to see that the log-derivative of the solutions of the order-two operator J_2 were algebraic functions, but harder to see that these solutions were actually algebraic. One now finds immediately that the solutions of $\tilde{J}_2$ are algebraic functions.

Remark B 3.2: If one considers, instead of (B.22) the rational function with the same denominator $\mathcal{D}$ but where the numerator is normalised to 1,

$$\mathcal{R} \; = \; \frac{1}{\mathcal{D}}. \tag{B.29}$$

Its telescoper is an order-seven linear differential operator

$$L_7 \; = \; F_2 \cdot G_1 \cdot G_2 \cdot H_2, \tag{B.30}$$

where the order-two linear differential operator F_2 is (non-trivially) homomorphic to the order-two linear differential operator L_2 which is the telescoper of the rational function (103), and where the order-two linear differential operators G_2 and H_2 have simple algebraic solutions.

Appendix B.4. Birational symmetries from collineations. Another example

Let us consider the following birational transformation associated with a collineation:

$$(x, y, z) \quad \longrightarrow \quad \Big(\frac{x}{1 - x + 2y}, \ \frac{y}{1 - x + 2y}, \ z \cdot (1 - x + 2y)^2 \Big), \tag{B.31}$$

which preserves the product $p = x y z$, and also preserves the origin $(x, y, z) = (0, 0, 0)$. Again, if one transforms the simple rational function (103) with the birational transformation (B.31), one gets the rational function of the form:

$$\mathcal{R} = \frac{1 - x + 2y}{\mathcal{D}}, \tag{B.32}$$

and again the intersection of the algebraic surface $\mathcal{D} = 0$ with the algebraic surface $p = x y z$, is an elliptic curve, corresponding to eliminate $z = p/x/y$ in $\mathcal{D} = 0$. One gets immediatly the same Hauptmodul (B.5) for this new elliptic curve. The telescoper of the rational function (B.32) is an order-four linear differential operator

$$L_4 = F_2 \cdot G_2, \tag{B.33}$$

where the order-two linear differential operator F_2 is (non-trivially) homomorphic to the order-two linear differential operator L_2 which is the telescoper of the rational function (103) and where the solutions of G_2 are two *algebraic functions* of series expansion:

$$s_0 = 1 + \frac{105}{4} \cdot x + \frac{12753}{16} \cdot x^2 + \frac{876225}{32} \cdot x^3 + \frac{251403765}{256} \cdot x^4 + \cdots$$
$$s_1 = x + \frac{105}{4} \cdot x^2 + \frac{7385}{8} \cdot x^3 + \frac{2111725}{64} \cdot x^4 + \frac{155849463}{128} \cdot x^5 + \cdots \tag{B.34}$$

The series $s = s_1$ is, for instance, solution of the polynomial equation $P(s,x) = 0$, where $P(s,x)$ reads:

$$P(s,x) = 2847312 \cdot p(x)^3 \cdot s^6 + 158184 \cdot p(x)^2 \cdot s^4 + 5040 \cdot p(x)^2 \cdot s^3 + 2197 \cdot p(x) \cdot s^2 + 140 \cdot p(x) \cdot s + 4x \cdot (243x + 35), \tag{B.35}$$

with $p(x) = 243x^2 + 35x - 1$. The series expansions of the algebraic solutions of $P(s,x) = 0$ read:

$$\mathcal{S}(u) = u + \frac{448451640\,u^4 - 38438712\,u^3 - 20761650\,u^2 + 1377667\,u + 221830}{17710} \cdot x + 3 \cdot \frac{448451640\,u^4 - 38438712\,u^3 - 20761650\,u^2 + 1450531\,u + 221830}{2024} \cdot x^2 + \cdots$$

where $u = 0, -1/6, 1/6, 5/26, -4/39, -7/78$. One finds that

$$15 \cdot \mathcal{S}\Big(\frac{1}{6}\Big) + 8 \cdot \mathcal{S}\Big(-\frac{1}{6}\Big) + 13 \cdot \mathcal{S}\Big(-\frac{7}{78}\Big) = 0,$$
$$13 \cdot \mathcal{S}\Big(\frac{1}{6}\Big) + 8 \cdot \mathcal{S}\Big(-\frac{4}{39}\Big) + 15 \cdot \mathcal{S}\Big(-\frac{7}{78}\Big) = 0,$$
$$15825411 \cdot \mathcal{S}\Big(\frac{1}{6}\Big) - 1771 \cdot \mathcal{S}\Big(\frac{5}{6}\Big) + 29373604 \cdot \mathcal{S}\Big(-\frac{7}{78}\Big) = 0, \tag{B.36}$$

and that the two solutions (B.34) of G_2 read:

$$s_0 \;=\; \mathcal{S}(0), \qquad\quad s_1 \;=\; \frac{521}{32}\cdot \mathcal{S}\Big(\frac{1}{6}\Big) \;+\frac{611}{32}\cdot \mathcal{S}\Big(-\frac{7}{78}\Big). \tag{B.37}$$

The homomorphism between F_2 and L_2 gives

$$\begin{aligned} F_2\cdot X_1 \;&=\; Y_1\cdot L_2, \qquad\qquad \text{where:} \\ X_1 \;&=\; \alpha(x)\cdot \Big((3240\,x^2\,+6\,x\,+1)\cdot D_x\;+1080\,x\,-6\Big), \qquad\qquad \text{with:} \\ \alpha(x) \;&=\; \frac{81}{10\cdot\,(1\,-35\,x\,-243\,x^2)\cdot\,(1\,-27\,x)}. \end{aligned} \tag{B.38}$$

Consequently a solution $\mathcal{S}$ of the telescoper L_4 (but not of G_2 in (B.33)) will be related to the hypergeometric solution ${}_2F_1([1/3,2/3],[1],27\,x)$ of the order-two linear differential operator L_2, as follows:

$$X_1\Big({}_2F_1\Big([\frac{1}{3},\,\frac{2}{3}],[1],\,27\,x\Big)\Big) \;\;=\;\; G_2\cdot\mathcal{S}. \tag{B.39}$$

The formal series solutions of the order-four linear differential operator (B.33) are (of course ...) the two (algebraic) solutions (B.34) of G_2, together with a solution with a $\ln(x)^1$, and a series s_2, analytic at $x\,=\,0$:

$$s_2 \;=\; x^2\;+\frac{93}{2}\cdot x^3\;+\frac{31185}{16}\cdot x^4\;+\frac{2488035}{32}\cdot x^5\;+\frac{1953542437}{640}\cdot x^6\;+\;\cdots \tag{B.40}$$

Relation (B.39) is actually satisfied with $\mathcal{S}\,=\,5103\cdot s_2$. Note that the series for (B.39) is a series with *integer* coefficients:

$$\begin{aligned} \frac{1}{2}\cdot\frac{1}{5103}\cdot X_1\Big({}_2F_1\Big([\frac{1}{3},\,\frac{2}{3}],[1],\,27\,x\Big)\Big) \;\;&=\;\; 1\;+87\,x\;+5358\,x^2\;+282459\,x^3 \\ &+13662531\,x^4\;+626640714\,x^5\;+27758265651\,x^6\;+1200939383487\,x^7\;+\;\cdots \end{aligned}$$

Remark B 4.1: Note that the diagonal δ of the rational function (B.32) reads:

$$\begin{aligned} \delta \;=\; &1\;+4\,x\;+108\,x^2\;+1960\,x^3\;+43240\,x^4\;+965664\,x^5\;+22377600\,x^6 \\ &+528712272\,x^7\;+12698698320\,x^8\;+308814134200\,x^9\;+\;\cdots \end{aligned} \tag{B.41}$$

We expect this diagonal to be a solution of the order-four telescoper (B.33). This series is actually a linear combination of the three series s_0, s_1 and s_2, analytic at $x\,=\,0$:

$$\delta \;=\; s_0\,-\frac{89}{4}\cdot s_1\;-105\cdot s_2. \tag{B.42}$$

It is interesting to see how the three globally bounded series s_0, s_1 and s_2, conspire to give a series with integer coefficients, the diagonal (B.42).

Remark B 4.2: These results must be compared with the calculations for the rational function

$$\mathcal{R} \;=\; \frac{1}{\mathcal{D}}, \tag{B.43}$$

where the denominator $\mathcal{D}$ is the same as the one in (B.32). In this case where the numerator has been normalised to 1, the diagonal is the same as the diagonal of $1/(1-x-y-z)$, namely ${}_2F_1([1/3,2/3],[1],27\,x)$, and the telescoper is the same telescoper as the one for $1/(1-x-y-z)$.

Appendix B.5. Birational symmetries from collineations. Another example

Let us consider the following birational transformation associated with a collineation:

$$(x, \, y, \, z) \quad \longrightarrow \quad \Big(\frac{x \, +3\, y}{1 \, - \, x \, +2\, y}, \, \, \frac{y}{1 \, - \, x \, +2\, y}, \, \, \frac{x\, z \cdot \, (1 \, - x \, +2\, y)^2}{x \, +3\, y} \Big), \tag{B.44}$$

which preserves the product $p \, = \, x\, y\, z$, and also preserves the origin $(x, \, y, \, z) \, = \, (0, \, 0, \, 0)$. Again, if one transform the simple rational function (103) with the birational transformation (B.44), one gets the rational function of the form:

$$\mathcal{R} \, = \, \frac{(1 \, - \, x \, +2\, y) \cdot \, (x \, +3\, y)}{\mathcal{D}}, \tag{B.45}$$

and again the intersection of the algebraic surface $\mathcal{D} \, = \, 0$ with the algebraic surface $p \, = \, x\, y\, z$, is an elliptic curve, corresponding to eliminate $z \, = \, p/x/y$ in $\mathcal{D} \, = \, 0$. One gets immediatly the same Hauptmodul (B.5) for this new elliptic curve. The telescoper of the rational function (B.45) is an order-seven linear differential operator

$$L_7 \, = \, F_2 \cdot \, G_2 \cdot \, H_1 \cdot \, H_2, \tag{B.46}$$

where the order-two linear differential operator F_2 is (non-trivially) homomorphic to the order-two linear differential operator L_2 which is the telescoper of the rational function (103), and where the order-two linear differential operators G_2 and H_2 have algebraic solutions (one finds easily that the log-derivative of these solutions are algebraic functions) and where H_1 is an order-one linear differential operator. This homomorphism between F_2 and L_2 gives

$$F_2 \cdot \, X_1 \, = \, Y_1 \cdot \, L_2 \qquad \text{where:} \qquad X_1 \, = \, A(x) \cdot \, D_x \, + B(x), \tag{B.47}$$

where $A(x)$ and $B(x)$ are rational functions. Consequently a solution $\mathcal{S}$ of the telescoper L_7 (but not of the product $G_2 \cdot \, H_1 \cdot \, H_2$ in (B.46)) will be related to the hypergeometric solution ${}_2F_1([1/3, 2/3], [1], 27\, x)$ of the order-two linear differential operator L_2, as follows:

$$X_1 \Big({}_2F_1 \Big([\frac{1}{3}, \, \frac{2}{3}], [1], \, 27\, x \Big) \Big) \, = \, G_2 \cdot \, H_1 \cdot \, H_2 \cdot \, \mathcal{S}. \tag{B.48}$$

In that case the solution of $\mathcal{S}$ of the telescoper L_7 reads

$$\mathcal{S} \, = \, x^4 \, + \frac{13316825310791}{231428221515} \cdot x^5 \, + \frac{303601408305956 51}{11108554632720} \cdot x^6 \, + \, \cdots \tag{B.49}$$

and the expansion of (B.48) reads:

$$X_1 \Big({}_2F_1 \Big([\frac{1}{3}, \, \frac{2}{3}], [1], \, 27\, x \Big) \Big) \, = \, \frac{1}{x} \, + \frac{85390121841387522079}{629841285410317908} + \frac{90649281143332377215505300 2605}{77136236451492696817854192} \cdot \, x \, + \, \cdots \tag{B.50}$$

Remark B 5.1: The factorisation (B.46) is far from being unique. Introducing the order-one linear differential operator $L_1 \, = \, D_x \, + 4/(3 + 4\, x)$, one has the following direct-sum decomposition:

$$L_7 \, = \, L_1 \oplus L_6, \tag{B.51}$$

$$G_2 \cdot \, H_1 \cdot \, H_2 \, = \, L_1 \oplus \tilde{G}_2 \oplus H_2, \tag{B.52}$$

where L_6 is an order-six linear differential operator, and where the order-two linear differential operator operator $\tilde{G}_2$ is slightly simpler than G_2.

Remark B 5.2: If one considers, instead of (B.45), the rational function with the same denominator $\mathcal{D}$ but where the numerator is normalised to 1,

$$\mathcal{R} \,\, = \,\, \frac{1}{\mathcal{D}}. \tag{B.53}$$

its telescoper is an order-four linear differential operator

$$L_4 \,\, = \,\, F_2 \cdot G_2. \tag{B.54}$$

The order-two linear differential operator F_2 is (non-trivially) homomorphic to the order-two linear differential operator L_2 which is the telescoper of the rational function (103), and the order-two linear differential operator G_2 has simple algebraic solutions.

Appendix B.6. Birational symmetries from collineations. Another simpler example

Let us consider the following birational transformation associated with a collineation:

$$\begin{aligned} (x,\, y,\, z) \quad & \longrightarrow \\ & \Big(\frac{x\, + 3\, y}{1\, -\, x\, + 2\, y},\; \frac{1\, + y}{1\, -\, x\, + 2\, y},\; \frac{x\, y\, z \cdot \, (1\, -\, x\, + 2\, y)^2}{(x\, + 3\, y) \cdot \, (1\, + y)}\Big), \end{aligned} \tag{B.55}$$

which preserves the product $p \, = \, x\, y\, z$. Again, if one transform the simple rational function (103) with the birational transformation (B.55), one gets the rational function of the form:

$$\mathcal{R} \,\, = \,\, \frac{(1\, -\, x\, + 2\, y) \cdot \, (x\, + 3\, y) \cdot \, (1\, + y)}{\mathcal{D}}, \tag{B.56}$$

and again the intersection of the algebraic surface $\mathcal{D} \, = \, 0$ with the algebraic surface $p \, = \, x\, y\, z$, is an elliptic curve, corresponding to eliminate $z \, = \, p/x/y$ in $\mathcal{D} \, = \, 0$. One gets immediatly the same Hauptmodul (B.5) for this new elliptic curve.

The telescoper of the rational function (B.56) can now be calculated in only a few hours, and one gets an order-nine linear differential operator of the form

$$L_9 \,\, = \,\, F_2 \cdot G_2 \cdot H_1 \cdot H_2 \, \cdot I_2, \tag{B.57}$$

where the order-two linear differential operator F_2 is (non-trivially) homomorphic to the order-two linear differential operator L_2 which is the telescoper of the rational function (103), and where the order-two linear differential operators G_2, H_2 and I_2 have algebraic solutions and where H_1 is an order-one linear differential operator. This homomorphism between F_2 and L_2 gives

$$F_2 \cdot X_1 \,\, = \,\, Y_1 \cdot L_2 \qquad \text{where:} \qquad X_1 \,\, = \,\, A(x) \cdot D_x \, + B(x), \tag{B.58}$$

where $A(x)$ and $B(x)$ are quite large rational functions. Consequently a solution $\mathcal{S}$ of the telescoper L_9 (but not of the product $G_2 \cdot H_1 \cdot H_2 \, \cdot I_2$ in (B.57)) will be related to the hypergeometric solution ${}_2F_1([1/3, 2/3], [1], 27\, x)$ of the order-two linear differential operator L_2, as follows:

$$X_1 \Big({}_2F_1\Big([\frac{1}{3}, \frac{2}{3}], [1], 27\, x\Big)\Big) \,\, = \,\, G_2 \cdot H_1 \cdot H_2 \, \cdot I_2 \cdot \mathcal{S}. \tag{B.59}$$

If finding the emergence of the hypergeometric function ${}_2F_1([1/3, 2/3], [1], 27\, x)$ is easy to obtain from the (algebraic geometry) calculation of the Hauptmodul (B.5), (see (129)), the telescoper of (B.56), or equivalently, the solution $\mathcal{S}$ of that telescoper, requires to find many linear differential operators, namely the intertwinner X_1 and also the right factors G_2, H_1, H_2 and I_2. In contrast with the birational transformations described in section 6 (see (108), (111), (112)), which simply preserve the diagonals of the rational functions, we have here, with the birational transformation (B.55), again *two birationally equivalent underlying elliptic curves*, but a *much more convoluted "covariance"* requiring to find many linear differential operators. The "elliptic curve skeleton" (the j-invariant or the Hauptmodul) is preserved, but the right factors dressing G_2, H_1, H_2 and I_2 and the intertwiner X_1 are quite involved.

Remark B 6.1: In fact the order-nine operator (B.57) is a direct sum. It can be written in the form

$$L_9 \; = \; L_8 \, \oplus \, L_1, \tag{B.60}$$

$$G_2 \cdot H_1 \cdot H_2 \cdot I_2 \; = \; L_1 \, \oplus \, \tilde{G}_2 \, \oplus \, \tilde{H}_2 \, \oplus \, I_2, \tag{B.61}$$

where the order-one operator reads:

$$L_1 \; = \; D_x \, + \frac{4}{3 + 4\,x}, \tag{B.62}$$

where L_8 is an order-eight operator, and where the operators with a tilde are much simpler than the operators without a tilde.

Remark B 6.2: Again if one considers, instead of (B.56), the rational function with the same denominator $\mathcal{D}$, but where the numerator has been normalised to 1,

$$\mathcal{R} \; = \; \frac{1}{\mathcal{D}}, \tag{B.63}$$

one finds an order-seven telescoper which factorises as follows:

$$L_7 \; = \; F_2 \cdot G_1 \cdot H_2 \, \cdot \, I_2, \tag{B.64}$$

where the order-two linear differential operator F_2 is (non-trivially) homomorphic to the order-two linear differential operator L_2 which is the telescoper of the rational function (103), and where the order-two linear differential operators H_2 and I_2 have algebraic solutions.

Remark B 6.3: Again the factorisation (B.64) is far from being unique. Introducing the order-one linear differential operator $L_1 \; = \; D_x \, + 1/x$, one has the two following direct-sum decompositions

$$L_7 \; = \; L_6 \, \oplus \, L_1, \tag{B.65}$$

$$G_1 \cdot H_2 \, \cdot \, I_2 \; = \; L_1 \, \oplus \, \tilde{H}_2 \, \oplus \, I_2, \tag{B.66}$$

where the order-two linear differential operator $\tilde{H}_2$ is slightly simpler than H_2.

Remark B 6.4: As far as an *algebraic geometry approach* of diagonals and telescopers is concerned (see [41]), we see that the concept of telescopers, which describes *all* the periods, can be more interesting than the concept of diagonals which often yields to diagonals that can be almost trivial functions (being simple rational functions, or being simply equal to zero). The examples of Appendix B show that the differential algebra approach of creative telescoping cannot be totally replaced by an algebraic geometry approach [41]. The algebraic geometry approach provides very quickly some precious information on the telescoper (the Hauptmodul), but not the telescoper itself. In fact one might consider the opposite point of view: creative telescoping could be seen as a tool to get effective algebraic geometry results.

Remark B 6.5: The examples displayed in this appendix can be seen as an illustration of the “dialogue of the deaf” between mathematicians and physicists. Some mathematicians will point out the fact that the calculation of the Hauptmodul (B.5) underlines the essence of the problem, namely the existence of an underlying elliptic curve, and will see the explicit calculation of the telescoper, and all its periods, as a laborious and slightly useless piece of work. In particular they will consider the “dressing” right-factors occurring in the decompositions (B.15), (B.23), ... as a totally and utterly spurious information, and they will also probably see the explicit expression of the large order-two operators F_2 as superfluous, retaining only the order-two linear differential operator L_2, prefering to ignore, or forget, the intertwiner X_1 in (B.47) or (B.58). Along this line they may consider the other solutions of the telescoper, namely the “periods” (associated with non-vanishing cycles) that are not diagonals, as irrelevant. In contrast for a physicist, getting all the periods, and the explicit expression of the telescoper will be seen as essential Recalling the $\chi^{(n)}$ components of the susceptibility of the Ising model, it is essential to get the explicit expression of the linear differential operators (telescopers) annihilating these $\chi^{(n)}$'s even if these (large) linear differential operators [27,28] are products (and direct sums) of a large set of factors. In the framework of integrable models, beyond diagonals, a physicist will always seek for a linear differential operator corresponding to an elliptic curve (resp. K3 surface, Calabi-Yau manifold, ...) even if it is “buried” as a left factor of a large telescoper, like the F_2's in (B.15) or (B.23).

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