II. THE INVERSION RELATION : SOME SIMPLE EXAMPLES

Introduction.

This part headed II is the continuation of part I. It concentrates on the analytical discussion of the inversion relation.

1. Inverse functional relation for exactly solvable models.

1.1 ANALYTICAL APPROACH TO THE POTTS AND BAXTER MODEL. — Let us take a simple example, the Potts model, to show how the exact expression of its partition function can be obtained by means of the inversion relation. We consider the anisotropic two-dimensional Potts model on a square lattice : the exponential of the two coupling constants of this model will be called b and c. As in part I we can claim to have the following inverse functional equation for the partition function and its analytical continuation (Jaekel and Maillard [1]) :

$$Z(b, c) . Z(1/b, 2 - q - c) = (c - 1)(1 - q - c) .$$

We can of course add to this equation the obvious symmetry equation $Z(b, c) = Z(c, b)$.

In order to analyse these two simple functional equations it is convenient to introduce new appropriate variables x and y :

$$x = (b - q_+)/(b - q_-)$$
$$y = (c - q_+)/(c - q_-)$$

with

$$q_\pm = 1 - \frac{q}{2} \pm \frac{1}{2}\sqrt{q(q-4)} .$$

The partition function of the Potts model is known exactly at the critical temperature $T = T_c$

$$\text{i.e. } (b - 1)(c - 1) \doteq q$$
$$\text{i.e. } x . y = - q_+ .$$

On this critical variety there is a only one variable, say x, and the two preceding equations can be

Article published online by EDP Sciences and available at http://dx.doi.org/10.1051/jphys:01985004603033501

rewritten :

$$Z(x)\,Z\left(\frac{1}{x}\right) = -\,qq_+\,\frac{\left(1+\dfrac{x}{q_+}\right)\left(1+\dfrac{1}{q_+ x}\right)}{(1-x)\left(1-\dfrac{1}{x}\right)}$$

$$Z(x) = Z\left(\frac{-\,q_+}{x}\right).$$

An iteration between these two functional equations gives an exact expression of the partition function in terms of simple infinite products :

$$Z(x) = \sqrt{-\,qq_+}\,\frac{P(x)\,P\left(\dfrac{-\,q_+}{x}\right)}{P(-\,q_+\,x)\,P(q_+^2/x)}$$

with

$$P(x) = \prod_{n=0}^{\infty}\left(\frac{1+q_+^{2n-1}\,x}{1-q_+^{2n}\,x}\right)$$

which can be verified to be the exact expression of the critical Potts model (known by the Bethe ansatz method) [2].

Let us give a more systematic exposure of this exact calculation : the two involutions $I : x \to \dfrac{1}{x}$ and $S : x \to \dfrac{-\,q_+}{x}$ generate an infinite discrete group G which satisfies the exact sequence

$$0 \to Z \to G \to Z_2 \to 0\,.$$

Let us introduce $L(x) = \dfrac{1+\dfrac{x}{q_+}}{1-x}$ and for every function φ the notation $\varphi_g = \varphi(g(x))$, where g denotes an element of the group G ; with these notations the two functional equations can be written as :

$$Z.Z_{\mathrm{I}} = -\,qq_+\,L.L_{\mathrm{I}}$$

and

$$Z = Z_{\mathrm{S}}\,.$$

Obviously

$$Z = \sqrt{-\,qq_+}\,\frac{P.P_{\mathrm{S}}}{P_{\mathrm{SI}}.P_{\mathrm{SIS}}}$$

with

$$P = \prod_{n=0}^{\infty} L(\mathrm{SI})^{2n}$$

is a solution to both equations. This systematic procedure can be applied to more complicated models where we have an elliptic instead of a rational, uniformization.

One can for instance consider the Baxter model where the four canonical parameters, namely a, b, c, d are uniformized as follows :

$$a = \rho.\theta(2\,\eta).\theta(v-\eta).H(v+\eta)$$
$$b = \rho.\theta(2\,\eta).H(v-\eta).\theta(v+\eta)$$
$$c = \rho.H(2\,\eta).\theta(v-\eta).\theta(v+\eta)$$
$$d = \rho.H(2\,\eta).H(v-\eta).H(v+\eta)$$

(θ and H are the usual elliptical functions : see Gradstein, Ridzyk [3]). If K and K' denote the two periods of the elliptical function, introducing

$$x = \mathrm{e}^{i\pi\eta/K}\,,\quad z = \mathrm{e}^{i\pi v/K}\,,\quad q = \mathrm{e}^{-\pi K'/K}$$

we have

$$\theta(v) = \prod_{n=1}^{\infty}(1-q^{2n})(1-q^{2n-1}\,z)(1-q^{2n-1}/z)$$

$$H(v) = 2\,q^{1/2}(z^{1/2}-z^{-1/2}) \times \times \prod_{n=1}^{\infty}(1-q^{2n})(1-q^{2n}\,z)(1-q^{2n}/z)\,.$$

With these new notations the inversion relation introduced in part I is $z \to \dfrac{x^2}{z}$ or $v \to 2\,\eta - v$ and the symmetry $K_1 \leftrightarrow K_2$ is $z \to 1/z$ or $v \to -\,v$.

Following Baxter we introduce $A(x) = \prod_{r=0}^{\infty}(1-q^n\,z)$ (Baxter [4])

$$F(z) = \prod_{m=0}^{\infty}\frac{A(x^{4m+1}\,z)}{A(x^{4m+3}\,z)}\,.$$

Baxter's solution is

$$Z(z) = A\left(\frac{qz}{x}\right)\cdot A\left(\frac{q}{xz}\right)\frac{F(x^2\,z)\,F\left(\dfrac{x^2}{z}\right)}{F(qz)\,F(q/z)}\cdot\left(\frac{-\,c}{\rho.H(2\,\eta)}\right) = Z(1/z)$$

which satisfies the inverse functional equation

$$Z(z)\,Z(x^2/z) = = A(xz)\,A\left(\frac{x^3}{z}\right)A\left(\frac{qz}{x^3}\right)A\left(\frac{q}{xz}\right)\cdot\frac{cc_{\mathrm{I}}}{(-\,\rho H(2\,\eta))^2}\,.$$

The term on the right-hand side of this equation can be shown, using identities on the θ and H function, to be equal to $cc_{\mathrm{I}}\left(1+\dfrac{b}{c}\right)\left(1-\dfrac{b}{c}\right)$: one recovers the $(2\,i\,\mathrm{sh}\,(2\,K_1+2\,M)\,\mathrm{sh}\,(2\,K_1-2\,M))^{1/2}$ factor of part I. Baxter's exact expression can be obtained easily by this systematic procedure by writing $L(z)$ as proportional to

$$A(xz).A\left(\frac{qz}{x^3}\right).$$

Another simple example is a subcase of the Baxter model : the symmetrical six-vertex model for which we have a rational uniformization ($d = 0$) :

$$a = \rho \sin(2\eta - \lambda\theta)$$
$$b = \rho \sin \lambda\theta$$
$$c = \rho \sin 2\eta$$

$$Z(\theta)\, Z(-\theta) = cc_{\mathrm{I}}\left(\frac{\sin(2\eta - \lambda\theta)\sin(2\eta + \lambda\theta)}{\sin^2 2\eta}\right)$$
$$= cc_{\mathrm{I}}\frac{(\sin^2 2\eta - \sin^2 \lambda\theta)}{\sin^2 2\eta}$$
$$= cc_{\mathrm{I}}\left(1 - \frac{b^2}{c^2}\right)$$
$$Z(\theta) = Z(2\eta - \lambda\theta).$$

On this last example one sees that there are at least two ways of cutting the right-hand side of the inverse functional relation into two parts L and L_{I} :

$$L(\theta) = \frac{\sin(2\eta - \lambda\theta)}{\sin 2\eta}$$

or

$$L(\theta) = \frac{\sin 2\eta - \sin \lambda\theta}{\sin 2\eta}.$$

In fact both choices lead to the same expression for $Z(\theta)$. This means in abstract terms that if L is multiplied by $\frac{f_{\mathrm{SI}} \cdot f_{\mathrm{ISI}}}{f_{\mathrm{S}} \cdot f_{\mathrm{SI}}}$ for any f, or more generally, by $\frac{g}{g_{\mathrm{I}}}$ with $g_{\mathrm{I(SI)}^2} = g$, we ought to get the same results for Z.

In the first two models however — the Potts and Baxter models — it was possible to make different choices for L

$$\left(\text{respectively} \quad L = \frac{1 + \dfrac{x}{q_+}}{1 - \dfrac{1}{x}} \quad \text{and} \quad A(xz)\, A(q/xz)\right),$$

leading to a non-defined expression for Z : the right choice corresponds to a Wiener-Hopf factorization where L and L_{I} are analytical in two disconnected domains in the complex plane. So far the procedure described is well defined and uniquely determines the solutions with the smallest number of singularities (of course an infinite number of solutions can be found for these two equations but the number of singularities is larger : CDD ambiguities (Castillejo, Dalitz, Dyson [5])).

1.2 INVERSE FUNCTIONAL RELATION ON THE ANISOTROPIC TWO-DIMENSIONAL ISING MODEL ; DIAGRAMMATIC APPROACH. — Let us consider a subcase of the Baxter model and of the anisotropic Potts model, the well known two dimensional Ising model.

Introducing the resummed high temperature expansion shown in part I for the normalized partition function

$$\Lambda(K_1, K_2) = Z(K_1, K_2)\,(\mathrm{ch}\, K_1\, \mathrm{ch}\, K_2)^{-1}$$

we can write $\ln \Lambda$ as follows (Baxter [17])

$$\ln \Lambda(K_1, K_2) = \sum_{n=1}^{\infty} (\mathrm{th}^2\, K_1)^n . Q_n(\mathrm{th}^2\, K_2)$$

where Q_n is a rational function of $\mathrm{th}^2 K_2$. Using this expansion Baxter has shown that, assuming only $\mathrm{th}^2 K_2 = 1$ singularities in Q_n (this can be checked directly on the Onsager solution), the inversion relation and the symmetry relation determine *the rational function Q_n order by order : the partition function is therefore determined completely*. This approach is not exact as in the two preceding examples but perturbative; however in all of these examples the partition function is completely determined with the help of certain analytical assumptions : on the spectral variable x or θ in the case of two-dimensional critical Potts model or the Baxter model, or on one of the two coupling constants in the case of the anisotropic two dimensional Ising model.

2. Inverse functional relation : near the critical point.

2.1 DETERMINATION OF THE CRITICAL MANIFOLD. — We will only mention that the inverse and symmetry relations enable the critical manifold of many models to be determined exactly. In the case of anisotropic Potts model on a square lattice for instance, we get the equations $xy = -q_+$ and $xy = q_+$ for the ferromagnetic and antiferromagnetic critical varieties respectively (Maillard, Rammal [6]). For triangular and honeycomb lattices, the ferromagnetic critical varieties are $xyz = q_+^2$ and $xyz = -q_+$ respectively. We should point out that the Kramers-Wannier duality is not used to determine these critical manifolds. When this manifold is determined exactly it is tempting to try using inverse and symmetry relations to analyse the neighbourhood of this manifold.

2.2 INVERSION RELATION IN THE SCALING LIMIT. — We have seen that the inversion relation enables the partition function of the Potts model to be calculated exactly at criticality; outside the critical temperature it can no longer be calculated, but some exact information can be obtained in the neighbourhood of this critical temperature : latent heat (for $q \geqslant 4$) [2], critical exponents, and magnetization discontinuity (Baxter [7]). One could ask whether the inversion relation can supply information (on the partition function, for instance) in the critical temperature region (within the scaling limit); in other words the problem is that of the connection between the IR (or the infinite group G) and the renormalization group.

Instead of taking the non-critical Potts model, which is complicated, let us consider the problem on simpler models. Those for which the renormalization group is exact and an IR exists are not very instructive since the two are seen to be compatible; we shall therefore consider the Baxter model for which an IR exists and the renormalization group is not trivial. Within the scaling limit the preceding parameters v, η, K, K' are degenerate. Following Baxter we introduce the Landen transformation

$$K_\lambda = \frac{1}{2}(1+k).K'$$
$$K'_\lambda = (1+k).K$$

(k is the module of elliptic functions) and introduce U and μ such that

$$U = -i\pi v/K \cdot \frac{K'_\lambda}{K_\lambda}$$

and

$$\mu = -i\pi\eta/K \cdot \frac{K'_\lambda}{K_\lambda}$$

(μ and U have finite values within the scaling limit).

Obviously the inversion and symmetry relations can be restricted to the singular part of the partition function (which characterizes the critical behaviour) :

$$(-\beta f)_{\text{sing}}(U) = (-\beta f)_{\text{sing}}(-U)$$
$$(-\beta f)_{\text{sing}}(U) + (-\beta f)_{\text{sing}}(2\mu - U) = 0 .$$

The singular part of the partition function has been calculated exactly (Baxter [4], Appendice E page 226) and gives to the lowest order :

$$(-\beta f)_{\text{sing}}(U) = 4\cos\frac{\pi U}{2\mu}\cdot\cot\frac{\pi^2}{2\mu}\cdot q^{\pi/\mu} + \cdots$$

which satisfies the two preceding equations. Assuming some Kadanoff-type form such as

$$-\beta f_{\text{sing}} = a(U,\mu).q^{2-\alpha(\mu)} + \cdots$$

(q is the « temperature » variable, which vanishes as T tends to T_c and does not depend on the spectral parameter U as does the amplitude a) we get the two equations

$$a(U,\mu) = a(-U,\mu)$$
$$a(U,\mu) + a(2\mu - U,\mu) = 0 .$$

Paradoxically the inverse and symmetry relations constrain the sophisticated information in $(-\beta f)_{\text{sing}}$, the amplitude $a(U, \mu)$, but give no information on the critical exponent $2-\alpha$. Conversely we have seen that the IR, associated with the symmetry relation and an analyticity assumption, determines exactly the partition function and thus the critical exponent α. The question thus remains open as to whether a more sophisticated approach to the IR, not roughly confined to the neighbourhood of criticality, can give informations on the critical exponents.

For instance, the critical exponents of the 2-d Potts model are known exactly (conjecture of den Nijs [8], Nienhuis *et al.* [9]) and their expressions are quite simple when the variable q_+, appropriate for the infinite group G, is used. One should mention that these exact expressions for the critical exponents can be obtained assuming the conformal covariance of a 2-d field theory. These expressions are nothing more than the Kac determinant formula (see for instance Dotsenko [10]). It is amusing to notice that the theory can be built only for special values of q, which already attracted attention in graph theory and statistical mechanics $\left(\text{the so-called Tutte-Beraha numbers : } q = 2 + 2\cos\frac{2\pi}{n},\ n \text{ integer [11]}\right)$. These special values of q are exactly such that the infinite group G degenerates into a finite group ! Moreover, for these values, the critical exponents are rational numbers. It seems important to understand the relation between these different mathematical structures (conformal covariance, exact integrability, or say in Baxter's terminology Z-invariance, the infinite group G).

3. Characterization of the constraints associated with the IR.

Since the inversion relation is constraining, a question arises : how strong are these constraints and what is their nature ?

From this last point of view it can be interesting to compare the consequences on the expansions asso-

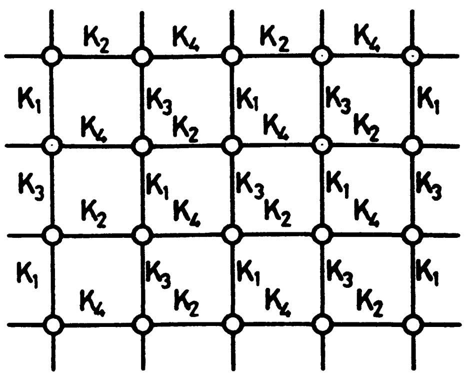


Fig. 1.

ciated with the inverse relation and those with some other linear functional relation associated with the integrability structure of the model (generalization of the Fisher relation). Let us consider the Ising model associated with figure 1 (Utiyama model).

We know that the condition necessary for the two diagonal transfer matrices $T(K_1, K_2)$ and $T(K_3, K_4)$ to commute is

$$\mathrm{sh}\, 2\, K_1\, \mathrm{sh}\, 2\, K_2 = \mathrm{sh}\, 2\, K_3\, \mathrm{sh}\, 2\, K_4 = k \qquad (1)$$

(Stephen Mittag [12]).

It is possible to show, when relation (1) is satisfied, that the following linear combination of two-point correlation functions is a function only of k (Maillard note CEA [13]) :

$$\mathrm{ch}\, 2\, K_1\, \mathrm{ch}\, 2\, K_2 - \mathrm{sh}\, 2\, K_1\, \mathrm{ch}\, 2\, K_2 \langle \sigma_i\, \sigma_m \rangle - \mathrm{sh}\, 2\, K_2\, \mathrm{ch}\, 2\, K_1 \langle \sigma_j\, \sigma_m \rangle = f(k)\,. \qquad (2)$$

For $K_1 = K_3$, $K_2 = K_4$ we recover equation (13b) of Baxter and Enting [14]; this equation, which plays an important part in their exact solution of the two-dimensional Ising model, is related to the star triangle relation. It is easily verified for the anisotropic two-dimensional Ising model :

$$\langle \sigma_i\, \sigma_m \rangle = \frac{2}{\pi} \coth 2\, K_1\, \mathrm{ch}\, 2\, K_1\, \mathrm{ch}\, 2\, K_2 \times \left[\Pi_1(\mathrm{sh}^2\, 2\, K_1, k) - \frac{1}{\mathrm{ch}^2\, 2\, K_1} \cdot K(k) \right]$$

$$\langle \sigma_j\, \sigma_m \rangle = \frac{2}{\pi} \coth 2\, K_2\, \mathrm{ch}\, 2\, K_2\, \mathrm{ch}\, 2\, K_1 \times \left[\Pi_1(\mathrm{sh}^2\, 2\, K_2, k) - \frac{1}{\mathrm{ch}^2\, 2\, K_2} \cdot K(k) \right]$$

where Π_1 is an elliptic integral of the third kind and K of the first kind.

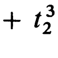
With these expressions for the two-point correlation function and the relation for Π_1 [1] :

$$\Pi_1(\nu, k) + \Pi_1\left(\frac{k^2}{\nu}, k\right) = K(k) + \frac{\pi}{2}\left[(1 + \nu)\left(1 + \frac{k^2}{\nu}\right)\right]^{-1/2}, \qquad (3)$$

the right-hand of relation (3) is equal to $-\frac{2}{\pi}(k^2 - 1) \times K(k)$, (i.e. an expression dependent only on k).

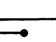
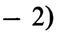
Clearly equation (2) is a quite non-trivial relation (in the theory of elliptic functions).

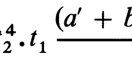
Let us compare on a resummed expansion the constraints from the inversion relation and relation (2). To the lowest order the correlation functions give :

$\langle \sigma_i\, \sigma_m \rangle =$ $t_2 +$  $t_2 \cdot \frac{2\, t_1^2}{1 - t_1^2} +$

$$+\, t_2^3 \cdot \frac{a + bt_1^2 + ct_1^4 + dt_1^6}{(1 - t_1^2)^3} + \cdots$$

and

$\langle \sigma_j\, \sigma_m \rangle =$  $t_1 +$  $\frac{2\, t_1}{(1 - t_1^2)^2}\, t_2^2 +$

$+$  $\left((-2) \sum_{p \geqslant 1} pt_1^{2p+1}\vdots\right) \frac{-2\, t_1^3}{(1 - t_1^2)^2}\, t_2^2$

$$+\, t_2^4 . t_1 \frac{(a' + b'\, t_1^2 + c'\, t_1^4 + d'\, t_1^6)}{(1 - t_1^2)^4} + \cdots$$

Let us denote the terms of t_2^3 and t_2^4 of these two correlation functions

$$\alpha' + \beta'\, \mathrm{ch}\, 2\, K_1 + \gamma\, \mathrm{ch}^2\, 2\, K_1 + \delta'\, \mathrm{ch}^3\, 2\, K_1$$

and

$$\mathrm{sh}\, 2\, K_1 (\alpha + \beta\, \mathrm{ch}\, 2\, K_1 + \gamma\, \mathrm{ch}^2\, 2\, K_1 + \delta\, \mathrm{ch}^3\, 2\, K_1)$$

respectively; equation (2) leads to the relations

$$\beta' + \delta' = 0 \qquad \beta = 2\, \alpha' \qquad \beta + 2\, \gamma' = 0 \qquad \delta = 0\,.$$

These are similar to those implied by the inversion relation (Jaekel and Maillard [1]) :

$$\langle \sigma_i\, \sigma_m \rangle (K_1, K_2) = \langle \sigma_i\, \sigma_m \rangle (K_1 + i\pi/2, -K_2)$$

and

$$\langle \sigma_j\, \sigma_m \rangle (K_1, K_2) + \langle \sigma_j\, \sigma_m \rangle (K_1 + i\pi/2, -K_2) = 2 \coth 2\, K_1$$

which yield :

$$\alpha' = \gamma' = 0 \quad \text{and} \quad \beta = \delta = 0$$

(i.e. $a' + d' = b' + c' = 0$ and $a = d$ $b = c$). To these relations can be added the obvious relation

$$a = 0 \Leftrightarrow \alpha' + \beta' + \gamma' + \delta' = 0\,.$$

These relations can be compared to those associated with the symmetry between K_1 and K_2 :

$$\frac{\partial}{\partial K_1} \langle \sigma_i\, \sigma_m \rangle = \frac{\partial}{\partial K_2} \langle \sigma_j\, \sigma_m \rangle$$

which yield

$$\beta' = 2\, \alpha - 1$$
$$\beta = \gamma'$$
$$2\, \gamma = 3\, \delta'$$
$$\delta = 0\,.$$

---

[1] See for example A. Cayley, *An Elementary Treatise on Elliptic Functions* (Dover Publication, New York) p. 120.

On this particular completely integrable model we can compare the relations obtained from (2) (which is connected to the star-triangle relation, i.e. to the complete integrability of the model) and those from the inversion relation and the ($K_1 \leftrightarrow K_2$) symmetry (the inverse and symmetry relations completely determine the partition function for this model). Although quite similar these constraints are not identical because relation (2) while constraining, is not enough so to determine the partition function completely, since it is only a consequence of the star-triangle relation.

For more complicated models, *a priori* non integrable for instance, the comparison between consequences on the model of the inversion relation and symmetry relation and other properties is not so clear and simple. This is partially studied (Jaekel and Maillard [15]) in the case of constraints associated with the Lee-Yang theorem on a model. In general all these conditions constitute an overlapping set of constraints and it is quite difficult to combine them all together to get some new non trivial property for the model. Of course the main problem is to characterize the « missing » information which should be added to the inverse and symmetry relations in order to determine the partition function completely. For example in the case of the three-dimensional anisotropic Ising model there is some evidence that, as in the two-dimensional case, only $\text{th}^2 K_1 = 1$ singularities might occur in the resummed high-temperature expansion associated with the model. This analytical property, if confirmed, would certainly be a very important factor in the three-dimensional Ising model and it would then be possible using the inversion and symmetry relations to determine its partition function completely with much less added information. For instance to get the anisotropic high-temperature expansion of this model up to the eighth order in $\text{th}\, K_i$ we have to specify only the coefficient of $\text{th}^2 K_1\, \text{th}^2 K_2\, \text{th}^2 K_3$ (see Jaekel and Maillard [1]). If this analyticity assumption is true we can then reach the tenth order with only the $\text{th}^2 K_1\, \text{th}^4 K_2\, \text{th}^4 K_3$ coefficient, the twelfth order with the $\text{th}^4 K_1\, \text{th}^4 K_2\, \text{th}^4 K_3$ coefficient, the fourteenth order with the $\text{th}^2 K_1\, \text{th}^6 K_2\, \text{th}^6 K_3$ and $\text{th}^4 K_1\, \text{th}^4 K_2\, \text{th}^6 K_3$ coefficients, the sixteenth order with the $\text{th}^4 K_1\, \text{th}^6 K_2\, \text{th}^6 K_3$ coefficient, the eighteenth order with the $\text{th}^2 K_1\, \text{th}^8 K_2\, \text{th}^8 K_3$, $\text{th}^4 K_1\, \text{th}^6 K_2\, \text{th}^8 K_3$, and $\text{th}^6 K_1\, \text{th}^6 K_2\, \text{th}^6 K_3$ coefficients, and so on : if this analyticity assumption fails the missing information would be much more important.

From a more general point of view it seems that the main problem arising from the use of the inverse relation with expansions is that the graphs to be introduced seem to *be the most difficult to count*, regardless of the method.

4. Conclusion.

The Kramers-Wannier duality was demonstrated in 1941 [16] on the two-dimensional Ising model, and work is still being devoted to searching for exact or approximate self-dual models in two, three or four dimensions.

In these two papers we have introduced another exact relation for a large number of models : the so-called inversion relation. The models for which an inverse functional relation can be written for the partition function far outnumber those for which a self-dual relation exists. Just one example is the 3-d anisotropic Ising model. Moreover the inversion relation is more powerful than the self-dual property : in some cases (exactly solvable models) it enables the partition function to be calculated exactly and quickly, and in every case introduces some constraints on the anisotropic high or low temperature expansions. Furthermore with the inversion relation concept, some very simple functional relations can be written for various quantities such as correlation function, magnetization, and susceptibility. For all of these reasons the inversion relation appears as a very fruitful concept destined for a career at least as brilliant as the K-W duality. It is quite simple to find many models for which an inversion relation exists, but the problem then is to use such a relation to best advantage; in this second paper we have tried to study this question. Three broad cases are distinguished : that of exactly solvable models, where we show how to get the exact expression of the partition function; that of the critical temperature neighbourhood where we examine the information we can get from the inversion relation on, for instance, the critical exponents and the amplitude of the singularity; lastly that of *a priori* non-solvable models where we try to characterize the nature and extent of the constraints imposed by the IR on the high-temperature expansion.

Clearly the IR can be useful in calculating and in checking control of expansions for non-solvable models. An important question still open is to define the missing information (analyticity hypothesis, asymptotic behaviour, and so on), we must add to the IR in order to describe the partition function more precisely. This seems to be a difficult and rather subtle problem.

Acknowledgments.

The author wishes to acknowledge illuminating discussions with Prof. R. J. Baxter.

References

[1] JAEKEL, M. T., MAILLARD, J. M., *J. Phys.* **A 15** (1982) 2241-58.

[2] BAXTER, R. J., TEMPERLEY, H. N. V., ASHLEY, S. E., *Proc. R. Soc.* **A 358** (1978) 535.

[3] GRADSTEIN, I. S., RIDZIK, I. M., *Table of Integrals, series and products* (Academic Press) 1980.

[4] BAXTER, R. J., *Ann. Phys.* **70** (1972) 193.

[5] CASTILLEJO, L., DALITZ, R. H., DYSON, F. J., *Phys. Rev.* **101** (1956) 453.

[6] MAILLARD, J. M., RAMMAL, R., *J. Phys.* **A 16** (1983) 353.

[7] BAXTER, R. J., *J. Phys.* **A 15** (1975) 3329-3340.

[8] DEN NIJS, M. P. M., *J. Phys.* **A 12** (1979) 1857.

[9] NIENHUIS, B., RIEDEL, E. K., SCHICK, M., *J. Phys.* **A 13** (1980) L189-92.

[10] DOTSENKO, V. S., FATEEV, V. A., Nordita preprint 84/8, March (1984).

[11] TUTTE, W. T., *Studies in Graph Theory Part II*, ed. D. R. Fulkerson (The Mathematical Association of America) Vol. 1, p. 323 (1972).

[12] STEPHEN, M. I., MITTAG, L., *J. Math. Phys.* **13** (1972) 1944.

[13] MAILLARD, J. M., Note CEA N-2332 (1983).

[14] BAXTER, R. J., ENTING, I. G., *J. Phys.* **A 11** (1978) 2463.

[15] JAEKEL, M. T., MAILLARD, J. M., *J. Phys.* **A 16** (1983) 1975.

[16] KRAMERS, H. A., WANNIER, G. H., *Phys. Rev.* **60** (1941) 252.

[17] BAXTER, R. J., *Fundamental problems in statistical mechanics V;* Proceedings of the 1980 Enschede Summer School; E.G.D. Cohen ed. (North-Holland, Amsterdam) 1980.