Necessary versus sufficient conditions for exact solubility of statistical models on lattices

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It is shown that under rather mild conditions the triangle relation represents a necessary condition for the existence of commuting transfer matrices of arbitrary size. The cases of spin models and vertex models are treated separately.

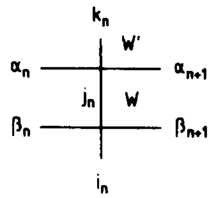
I. INTRODUCTION

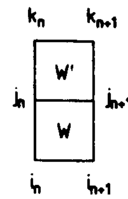
The problem of the parametrization of the models is one of the most important in exactly solvable models. All the solutions known in the literature are parametrized by elliptic, trigonometric, or even rational functions, but solutions involving curves of genus bigger than one or even surfaces are still unknown. There is not proof (nor even good arguments) that only genus one curves should occur in the solutions of the Yang–Baxter equations; it is possible to argue (see Appendix A) that one has to deal with algebraic varieties, but it seems very difficult to prove that it is necessary to deal with Abelian varieties. For that reason our approach is a very general one: there are no assumptions like the existence of a unique spectral parameter or the reduction of the Boltzmann weight to a simple transposition for a special value of the parameters. Therefore the proof is completely algebraic. The reader should be told that it is certainly possible to find simpler but less general proofs of the previous equivalence.

II. THE MAIN RESULT

A. Statement of the theorem

Following many authors (see, e.g., Refs. 1–11), it is quite simple to show that the star-triangle relation (for the Boltzman weights W, W', W'') implies the commutation of the transfer matrices with periodic boundary conditions $T_N(W)$ and $T_N(W')$, whatever their size N. The proof leads to a distinction between the case of the vertex models (see Fig. 1) and the case of the spin models (see Fig. 2). The configurations of the spin $i_1 \cdots i_N$, $k_1 \cdots k_N$ in Figs. 1 and 2 are fixed and we sum all the configurations of the remaining spins $(j_1 \cdots j_N, \alpha_i, \beta_i)$. These two figures represent the product of the two transfer matrices $T_N(W)$ and $T_N(W')$ for vertex and spin models, respectively. In the case of the Potts model (with spins belonging to Z_q), the transfer matrices are thus $q^N \times q^N$ matrices with coefficients $T_N(W)_{i_1,\ldots,i_N; j_1,\ldots,j_N}$ and $T_N(W')_{j_1,\ldots,j_N; k_1,\ldots,k_N}$. The commutation of the transfer matrices means that for any configuration of the spins $i_1 \cdots i_N$, $k_1 \cdots k_N$, the partition function of the two graphs on both sides of the equality are equal. Let us introduce the two matrices $M_{\alpha_n,\beta_n;\alpha_{n+1},\beta_{n+1}}(i_n, k_n)$ and $M_{j_n,j_{n+1}}(i_n,i_{n+1};k_n,k_{n+1})$ associated with the two following graphs





These two matrices (associated with the vertex and spin models, respectively) are $q^2 \times q^2$ (resp. $q \times q$) matrices and there are q^2 (resp. q^4) of them [as many as the number of configurations for (i_n,k_n) and $(i_n,i_{n+1};k_n,k_{n+1})$]. From now on these matrices will be denoted by M_{I_n} and $M_{I_n,I_{n+1}}$ $[I_n = (i_n,k_n)]$. We add a prime to denote the same matrices with the two Boltzmann weights W and W' permuted. With these notations the commutation of $T_N(W)$ and $T_N(W')$ is equivalent to

$$\mathrm{Tr}\,(M_{I_1}M_{I_2}\cdots M_{I_n}) = \mathrm{Tr}\,(M'_{I_1}M'_{I_2}\cdots M'_{I_N}) \tag{1}$$

and

$$\mathrm{Tr}\,(M_{I_1I_2}M_{I_2I_3}\cdots M_{I_NI_1}) = \mathrm{Tr}\,(M'_{I_1I_2}M'_{I_2I_3}\cdots M'_{I_NI_1}) \tag{2}$$

for any configuration of the I_n's, that is to say for any configurations of the i_n's and k_n's that index the coefficients of the matrices $T_N(W)T_N(W')$ and $T_N(W')T_N(W)$.

We want to establish an equivalence between the existence of a star-triangle relation and the commutation of the transfer matrices $T_N(W)$ and $T_N(W')$ for arbitrary size N; in other words, we want to show that when relation (1) [resp. (2)] is satisfied—for all I_n's and N—there necessarily exists a star-triangle relation. With the above notations it is equivalent to saying that there exists a $q^2 \times q^2$ matrix R (resp. q^2 $q \times q$ matrices R_I) such that

$$RM_{I_n} = M'_{I_n}R \tag{3}$$

$$(\text{resp. } R_I M_{IJ} = M'_{IJ} R_J). \tag{4}$$

B. Proof in the case of the vertex models

The case of the vertex models is the simpler case to deal with. Switching to a slightly more convenient notation $(I_n \to n)$, the question is easily seen to be reduced to the following theorem.

Theorem 1: Let $\mathscr{M}$ and $\mathscr{M}'$ be two subalgebras of $M_n(\mathbb{C})$

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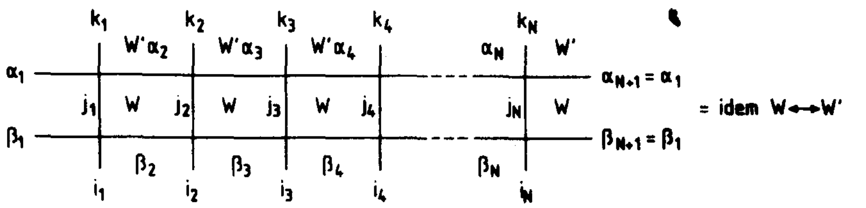

FIG. 1. Pictorial representation of the commutation of the two transfer matrices of size N, $T_N(W)$, and $T_N(W')$ in the case of vertex models.

(the $n\times n$ complex matrices) and φ: $\mathscr{M} \to \mathscr{M}'$, a surjective algebra homomorphism satisfying the following property:

$$\forall\, M_1,\ldots,M_k \in \mathscr{M},$$

$$\mathrm{Tr}\,(M_1\cdots M_k) = \mathrm{Tr}\,[\varphi\,(M_1\cdots M_k)].$$

Suppose further that there is no nontrivial invariant subspace of $E\approx\mathbb{C}^n$ under the action of $\mathscr{M}$, then, there exists $R\in \mathrm{GL}_n(\mathbb{C})$ so that

$$\forall M\in\mathscr{M},\quad M'\,(\equiv\varphi\,(M)) = RMR^{-1}.$$

In Appendix A we discuss the problem of the existence of nontrivial invariant subspaces.

Proof: For any $M\in\mathscr{M}$, the corresponding spectral projection operators are elements of $\mathscr{M}$ (being polynomials in M).

We shall need the following lemma.

Lemma: There exists in $\mathscr{M}$ a matrix with (n) distinct eigenvalues.

Proof of the lemma: For any M in $\mathscr{M}$, we set

$$\nu(M)\equiv\Sigma\ (\text{dim. spectral subspace} - 1)$$

and $\nu\equiv\inf\nu(M)$, $M\in\mathscr{M}$.

The lemma is then equivalent to $\nu=0$. Suppose $\nu\neq 0$ and take $M\in\mathscr{M}$ such that $\nu(M)=\nu$; select further V a spectral subspace of M of dimension $\geqslant 2$ and let π be the associated spectral projection operator. One has the following simple proposition.

Proposition: $\forall M_\lambda\in\mathscr{M}$, $\pi M_\lambda\pi$ has only one eigenvalue when considered as an operator on V.

For if not, consider operators of the form

$$\widetilde{M} = (1-\pi)\,M\,(1-\pi) + \pi(N-k)\,\pi,$$

where $k\in\mathbb{C}$ and $\pi N\pi$ has more than one eigenvalue on V. Then, for suitable k, $\nu(\widetilde{M}) < \nu(M) = \nu$, a contradiction.

Any M_λ in $\mathscr{M}$ can thus be written as

$$\pi M_\lambda\pi = k_\lambda 1_\nu + N_\lambda,$$

$k_\lambda\in\mathbb{C}$, N_λ nilpotent on V (1_ν is the identity operator on V). Adding $(1-k_\lambda)1_\nu$, we obtain a family of operators on V of the form $1_\nu + N_\lambda$, stable under multiplication. The Engel theorem[12] provides us with a vector $v\in V$ such that $N_\lambda v = 0$ $(\forall\lambda)$. The subspace $(Mv)_{M\in\mathscr{M}}$ is then a nontrivial invariant subspace for $\mathscr{M}$, contradicting the assumption. This finishes the proof of the lemma.

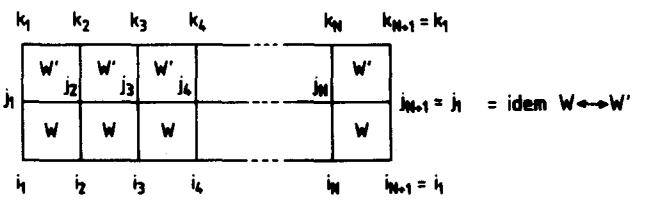

FIG. 2. Pictorial representation of the commutation of two transfer matrices of size N in the case of spin models.

Returning to the main proof, we let $M\in\mathscr{M}$ be a matrix such that $\nu(M)=0$ and let $(\pi_i)_{i=1}^{i=n}$ be the corresponding one-dimensional projection operators

$$\pi_i\cdot\pi_j = \pi_j\cdot\pi_i = \delta_{ij};\quad \sum_{i=1}^{i=n}\pi_i = 1_E.$$

Let $\pi_i' = \varphi\,(\pi_i)$ and choose $e_i\in\mathrm{Ran}\,\pi_I$, $e_i'\in\mathrm{Ran}\,\pi_i'$. Setting $e_i'\equiv\widetilde{R}e_i$ we shall prove that $\widetilde{R}$ is the intertwining operator up to scaling, that is,

$$R = D\widetilde{R},\quad D = \mathrm{diag}\,(\alpha_1,\ldots,\alpha_n),$$

for some nonzero α_i's.

To prove this, for M_λ in $\mathscr{M}$, we denote by $(m_{ij}^{(\lambda)})$ and $(m_{ij}'^{(\lambda)})$ the matrices of M_λ and $M'_\lambda\equiv\varphi\,(M_\lambda)$, with respect to the bases $(e_i)_{i=1}^{i=n}$ and $(e_i')_{i=1}^{i=n}$, respectively. The existence of D (which implies the theorem) is then equivalent to the existence of nonzero numbers $(\alpha_i)_{i=1}^{i=n}$ such that

$$m_{ij}'^{(\lambda)} = m_{ij}^{(\lambda)}\alpha_i/\alpha_j.$$

The existence of the α_i's is now proved in a sequence of simple assertions.

Assertion 1:

$$\beta_{ij}\equiv m_{ij}'^{\lambda}/m_{ij}^{(\lambda)}$$

is independent of λ.

For all $M_\lambda, M_\mu\in\mathscr{M}$, $\quad\forall\, i,j$,

$$m_{ij}^{(\lambda)}\cdot m_{ji}^{(\mu)} = \mathrm{Tr}\,(\pi_i M_\lambda\pi_j M_\mu\pi_i),$$

hence

$$m_{ij}^{(\lambda)}\cdot m_{ji}^{(\mu)} = m_{ij}'^{(\lambda)}\cdot m_{ji}'^{(\mu)}$$

or

$$m_{ij}'^{(\lambda)}/m_{ij}^{(\lambda)} = m_{ji}^{(\mu)}/m_{ji}'^{(\mu)},$$

which demonstrates the validity of the assertion.

Assertion 2: $\forall\, i,j$, $\beta_{ij}\neq 0,\infty$, i.e., $\forall i,j$, $\exists M_\lambda\in\mathscr{M}$, $m_{ij}^{(\lambda)}\neq 0$.

In fact, if there existed a pair (i,j) such that $m_{ij}^{(\lambda)}=0$ for any M_λ in $\mathscr{M}$, then the subspace $(Me_j)_{M\in\mathscr{M}}$ would be a nontrivial (it would not contain e_i) invariant subspace for $\mathscr{M}$.

Assertion 3: There exist n nonzero α_i's such that $\beta_{ij} = \alpha_i/\alpha_j$.

Setting $\alpha_i\equiv\beta_{i1}$, it only remains to show that

$$\forall\, i,j,k,\quad \beta_{ij}\cdot\beta_{jk} = \beta_{ik}\ .$$

This can be written (dropping the superscript λ) as

$$\frac{m_{ij}'}{m_{ij}}\cdot\frac{m_{jk}'}{m_{jk}} = \frac{m_{ik}'}{m_{ik}} = \frac{m_{ki}}{m_{ki}'}$$

(the last equality comes from the proof of Assertion 1) or

$$m'_{ij}\cdot m'_{jk}\cdot m'_{ki}=m_{ij}\cdot m_{jk}\cdot m_{ki}.$$

This finishes the proof of the theorem.

The result has thus been established in the case of vertex models.

C. Proof in the case of the spin models

The algebra is more intricate in this case; this is why, in order to be able to obtain a neat mathematical statement we shall restrict ourselves here to the case $q=2$. For general q, however, the result must still be valid except for very particular values of the matrices (Boltzmann weight). So let us consider the $q=2$ case. Here we can replace the cumbersome indexation $I_n=(i_n,k_n)$ $(i_n=\pm 1,\quad k_n=\pm 1)$ by an index i running through the values 1,2,3,4.

We are thus given a set of sixteen 2×2 matrices $(i,j=1,2,3,4)$ with positive coefficients, but we can only form "chain products" of the form $M_{i_1i_2}M_{i_2i_3}\cdots M_{i_{n-1}i_n}$, returning to the same index i_1. We shall apply Theorem 1 to the algebra generated by multiplying chains starting, and finishing, with the same *fixed* index, but we first need to find a condition that ensures that the hypothesis on the nonexistence of invariant subspaces is satisfied. Since the matrices are 2×2, this is equivalent to the nonexistence of a common eigenspace; we shall also see below that the condition is independent of the length of the chains we consider.

The only possibility we need to explore is the following: Whatever i_1 $(i_1=1,2,3,4)$, there exists a common eigenvector V_{i_1} for the matrices $M_{i_1i_2}\cdot M_{i_2i_3}\cdots M_{i_{n-1}i_n}\cdot M_{i_ni_1}$ (with variable $i_2,\ldots,i_n$).

The M_{ij}'s induce homographic transformations on $\mathbb{P}^1(\mathbb{C})$, which we still call M_{ij} when there is no risk of confusion. The existence of the four vectors V_i is then equivalent to the existence of four points F_i $(i=1,2,3,4)$ for $\mathbb{P}^1(\mathbb{C})$ such that

$$M_{i_1i_2}\cdots M_{i_ni_1}(F_{i_1})=F_{i_1}$$

multiplying by $M_{i_ni_1}$ we get

$$M_{i_ni_1}M_{i_1i_2}\cdots M_{i_{n-1}i_n}\cdot(M_{i_ni_1}(F_{i_1}))=M_{i_ni_1}(F_{i_1}).$$

This shows that we can assume that the F_i's are permuted under the action of M_{ij}'s:

$$\forall\, i,j\in\{1,2,3,4\},\quad M_{ij}(F_j)=F_i.$$

Also, recalling that the M_{ij}'s have real positive coefficients, we find that each M_{ij} has two real fixed points, one negative and one positive (possibly ∞), and that the real positive axis (including ∞) is stable under their action. This shows that the F_i's are all positive or all negative real numbers. In the latter case, we can replace all the M_{ij}'s by $S\,{}^tM_{ij}S$ $[S\equiv(\begin{smallmatrix}0&1\\1&0\end{smallmatrix})]$ and this allows us to assume that the F_i's are all real positive. In Appendix B we describe a pair of families (M_{ij}) and (M'_{ij}) arising in this fashion, which *do not* satisfy the intertwining property to be shown below; they are seen to be essentially the only possible ones.

Let us now state our result in the case of spin models.

Theorem 2: Let (M_{ij}) and (M'_{ij}) be two families of sixteen 2×2 matrices such that the following hold.

(i) All M_{ij}'s and M'_{ij}'s have positive elements and are invertible matrices.

(ii) $\forall n,\ \forall i_1,\ldots,i_n\in\{1,2,3,4\}$,

$$\mathrm{Tr}\,(M_{i_1i_2}\cdots M_{i_{n-1}i_n}M_{i_ni_1})=\mathrm{Tr}\,(M'_{i_1i_2}\cdots M'_{i_{n-1}i_n}M'_{i_ni_1}).$$

(iii) There do not exist four—all positive or all negative—real numbers F_i $(i=1,2,3,4)$, some of the F_i's possibly ∞ so that $M_{ij}(F_j)=M_i$, M_{ij} being viewed as a projective transformation on $\mathbb{P}^1(\mathbb{C})$. Then there exist four matrices R_i $(i=1,2,3,4)$, $R_i\in\mathrm{GL}_2(\mathbb{C})$ with the property

$$\forall\, i,j\in\{1,2,3,4\},\quad R_iM_{ij}=M'_{ij}R_j.$$

Remarks:

(1) Assumption (iii) can be made on any one of the two families; if it holds for one, it will also be satisfied by the other.

(2) The F_i's can be replaced by vectors $V_i=(1,F_i)$ (or $V_i=0,1)$ if $F_i=\infty$ so that $M_{ij}V_j=\lambda_{ij}V_i$.

(3) The validity of (iii), intricate as it looks, is nonetheless very easy to check. In fact, each F_i is simply one of the two fixed points of M_{ii}; compute these, and check (iii) for the two disjoint sets of the positive fixed points and negative ones.

Proof of the theorem: Since (iii) is satisfied, we can apply Theorem 1 to the algebra generated by the $M_{i_1-i_2}\cdots M_{i_{n-1}i_n}M_{i_ni_1}$ (i_1 fixed), and we choose $n=2$ (any fixed n is allowed); we also set $i_1=1$, without loss of generality. Assumption (iii) means that the $(M_{ij}M_{jk}M_{k1})_{j,k=1,2,3,4}$ have no common eigenspace. Theorem 1 then asserts the existence of R_1 such that

$$\forall\, j,k,\quad R_1M_{1j}M_{jk}M_{k1}=M'_{ij}M'_{jk}M'_{k1}R_1.$$

Now, *define* R_j by $R_1M_{1j}=M'_{1j}R_j$, i.e., $R_j\equiv M'^{-1}_{ij}\cdot R_1\cdot M_{1j}$. We need to check that $R_iM_{ij}=M'_{ij}R_j$, $\forall\, i,j\in\{1,2,3,4\}$. But we can write

$$R_iM_{ij}=M'^{-1}_{1i}R_1M_{1i}M_{ij}=M'^{-1}_{1i}\cdot R_1M_{1i}M_{ij}M_{j1}\cdot M^{-1}_{j1}$$

$$=M'^{-1}_{1i}\cdot M'_{1i}M'_{ij}M'_{j1}R_1M^{-1}_{j1}$$

$$=M'_{ij}M'_{j1}R_1M^{-1}_{j1},$$

and thus we only need to prove that

$$R_jM_{j1}=M'_{j1}R_1,\quad \forall\, j\in\{1,2,3,4\}.$$

By the very definition of R_j, the left-hand side is equal to $M'^{-1}_{ij}R_1M_{1j}M_{j1}$ and the equality to be shown is therefore equivalent to

$$M'_{1j}R_1M_{1j}M_{j1}=M'_{j1}R_1\quad\text{or}\quad R_1M_{1j}M_{j1}=M'_{1j}M'_{j1}R_1,$$

which in turn can be reduced to

$$R_1M_{1j}M_{j1}M_{11}=M'_{ij}M'_{j1}R_1M_{11}.$$

Using the definition of R_1, the left-hand side is equal to $M'_{1j}M'_{j1}M'_{11}R_1$ and we only have to prove that $M'_{11}R_1=R_1M_{11}$. But, we already know that $M'^{3}_{11}R_1=R_1M^3_{11}$. Since both matrices M_{11} and M'_{11} have real positive coefficients, it is easy to show that the desired equality follows, finishing the proof of the theorem.

We thus arrive at (4), which is equivalent to the existence of a star-triangle relation for spin models.

We should note that in both cases (spin and vertex models) the star-triangle relation is implied by the commutation of the transfer matrices for only a *finite* number of sizes N.

This is similar to the result of Parke[13] according to which the existence of only three conserved quantities in involution implies the existence of an infinity of conserved quantities.

III. PROSPECTS

In the previous sections we have tried to specify the equivalence between the commutation of transfer matrices and the star-triangle relations. This amounts to reducing the complete integrability property to a simple local relation. On the other hand, the commutation of transfer matrices of specific sizes leads to the determination of algebraic invariants [cf. Appendix A(a)] that constitute constraining conditions. This explains the results of the search for models satisfying a star-triangle relation, namely, that there exist very few such models. For instance in the case of vertex models with two valued spins the general case is essentially given by the Baxter model and the free fermions models of Fan and Wu.[14] Such an analysis underlines the exceptional occurence of solvable models.

This study also calls for a generalization in dimension 3. In this respect we would like to establish a similar equivalence between the commutation of transfer matrices of finite sizes and the so-called tetrahedron relation[6,15]; this looks like a nontrivial extension. However these commutations of the transfer matrices of finite sizes ($[T_{N,M}(W), T_{N,M}(W')] = 0$ are still necessary conditions for the validity of the tetrahedron relation; in particular this includes the conditions that pertain to the two-dimensional models ($[T_N(W), T_N(W')] = 0; M = 1$), and these have been shown to imply the star-triangle relation. This imposes severe restrictions on the possible solutions of the tetrahedron relation that, in a way, appear as extensions of the—sparse—solutions of the star-triangle relation.

The above discussion may give the impression that the domain of validity of the star-triangle and tetrahedron relation is indeed very restricted.

However, if the commutation of transfer matrices allows their simultaneous diagonalization (Bethe ansatz), thereby leading to the calculation of the partition function, we can imagine weaker condition that still make this calculation possible. In fact there already exist simple examples that illustrate this idea; these are the so-called disorder (or crystal-growth) solutions.[16–18] These solutions lead unfortunately to simple analytical expressions for the partition function; however, we should notice that one condition for the existence of such disorder solutions is very similar to a constraintful relation occurring in the framework of exactly solvable models [compare Eq. (2.10) of Ref. 19 and the so-called Frobenius relation[20,21]].

More precisely, if we look carefully at the construction of the Bethe ansatz for the Baxter model,[2] we can see that only relations similar to the so-called Frobenius relations are used [Eqs. (C.34a) and (C.34b) of Ref. 2] and not the full Yang–Baxter structure. We could therefore imagine that a model involving a higher-dimensional theta function would not satisfy the Yang–Baxter equations,[22,23] but that it would actually be possible to build a Bethe ansatz for that model (because of the Frobenius relations) leading to a commutation of transfer matrices only in a *subspace* of the space on which the matrices act; the case of disorder solutions corresponds to a one-dimensional subspace.

IV. CONCLUSION

We have thus shown the equivalence between the existence of a star-triangle relation and that of a family of commuting transfer matrices of arbitrary size; this has been established under conditions mild enough to be almost always satisfied in physical cases. Moreover we have proved that it suffices to check the commutation of the transfer matrices for a finite number of sizes. It may be interesting to look for the three-dimensional generalization of the above results.

In two dimensions, the above equivalence fully legitimizes the tentatively exhaustive studies that are currently done on the star-triangle relation.[24,25] In this framework we have also touched upon the problem of finding simple, algebraic, necessary conditions for the existence of the star-triangle relation (see Appendix A). Such relations, which appear very stringent, are directly related to one of the major problems concerning exactly solvable models: that of the parametrization of these models (rational or elliptic uniformization, Abelian varieties).

Finally these studies on the star-triangle relation seem to show that this is really a rarity; it is thus desirable to extend the notion of integrability beyond it, and to introduce new local criteria.

ACKNOWLEDGMENT

It is a pleasure to thank A. Douady for very helpful conversations, in particular for providing a complete answer to question (a) of Appendix A.

APPENDIX A: ALGEBRAIC VARIETIES AND COMMUTATION OF MATRICES

In this appendix, we briefly describe a solution to three elementary but important questions. The approach is both theoretical and practical, in that it readily provides effective algorithms. However, being as the size of the different matrices involved is a very important feature of the problem for practical purposes, there may be more powerful methods of solution in a given situation.

Let A and B in $M_n(\mathbb{C})$ be two complex $n \times n$ matrices, which we also view as linear operators on $E \approx \mathbb{C}^n$ with basis $(e_i)_{i=1}^{i=n}$. The three questions are the following.

(a) Can we find a list of invariants that ensure commutation of A and B? By this we mean expressions $(\varphi_k)_{k=1}^{k=m}$, algebraic in the coefficients of A and B, such that

$$\{\varphi_k(A) = \varphi_k(B);\ k = 1,...,m\} \Leftrightarrow \{AB = BA\}.$$

(b) Can we find an easy way to detect a nontrivial invariant subspace under the action of A and B?

(c) This is the same question as (b) in the one-dimensional case, namely, when do A and B have a common (one-dimensional) eigenspace?

(a) We restrict ourselves to the case when A and B are both diagonalizable with distinct eigenvalues [we denote this subset of $M_n(\mathbb{C})$ by $\widetilde{M}_n\, \mathbb{C}$], that is, we discard the codimension one algebraic variety in $M_n(\mathbb{C})$ given by the vanishing of the discriminant of the characteristic polynomial; the invariants will have poles on this surface.

Now, let $M\in \widetilde{M}_n(\mathbb{C})$ have eigenvectors $(w_i)_{i=1}^{i=n}$. There is a natural map φ, given by the composition

$$\widetilde{M}_n(\mathbb{C}) \to \mathbb{C}^{n\times n} \to (\mathbb{P}^{n-1}(\mathbb{C}))^n/\sigma_n,$$

$$M \to (W_1,...,W_n) \to (\overline{W}_1,...,\overline{W}_n) \bmod \sigma_n,$$

where the bar denotes the natural fibration map $\mathbb{C}^n \to \mathbb{P}^{n-1}(\mathbb{C})$.

Clearly we have the following proposition.

Proposition: $\forall\, A, B\in \widetilde{M}_n(\mathbb{C}), AB = BA \Leftrightarrow \varphi(A) = \varphi(B)$.

It remains therefore to give an explicit description of the map φ. To this end we use the embedding

$$(\mathbb{P}^{n-1}(\mathbb{C}))^n/\sigma_n \to \mathbb{P}(S^nE)$$

$$(\overline{W}_1,...,\overline{W}_n) \bmod \sigma_n \to \overline{W_1\otimes\cdots\otimes W_n},$$

and consider the map $i\circ\varphi$: $M_n(\mathbb{C}) \to \mathbb{P}(S^nE)$. This is easily seen to be described by the following proposition.

Proposition: $(i\circ\varphi)(M)$, $\widetilde{M}\in M_n(\mathbb{C})$, represents the one-dimensional eigenspace of $M^{\otimes n}$ (the nth symmetric power of M) for the eigenvalue $\det(M)$.

Proof: If $MW_i = \mu_i W_i$, we have

$$\begin{aligned} M^{\otimes n} W_1\otimes\cdots\otimes W_n &= MW_1\otimes\cdots\otimes MW_n \\ &= \mu_1\cdots\mu_n \cdot W_1\otimes\cdots\otimes W_n \\ &= \det(M)\, W_1\otimes\cdots\otimes W_n. \end{aligned}$$

The recipe is thus the following: Compute $M^{\otimes n}$ acting on S^nE [of dimension $\binom{2n-1}{n}$] and find the eigenvector of this matrix for the eigenvalue $\det(M)$, which appears as a polynomial in the variables $(e_i)_{i=1}^{i=n}$, homogeneous of degree n. The quotients of the coefficients of this polynomial by any one of them represent the sought after invariants.

Example: $n = 2$, $M = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$, $\det(M) = ad - bc$,

$$M^{\otimes 2} = \begin{bmatrix} a^2 & 2ab & b^2 \\ ac & ad+bc & bd \\ c^2 & 2cd & d^2\end{bmatrix};$$

the eigenvector of $M^{\otimes 2}$ with eigenvalue $\det(M)$ is given by

$$\Phi = \lambda\,(be_1\otimes e_1 + (d-a)e_1\otimes e_2 - ce_2\otimes e_2).$$

This gives the (projective) invariants $(b, d-a, -c)$ and we may take

$$\varphi_1(M) = b/c;\quad \varphi_2(M) = (d-a)/c.$$

The validity of this result of course can be readily checked by direct computation.

Important remark: The φ's we have found are enormously redundant for $n>2$. In fact, there should be $n(n-1)$ [$=\dim(\mathbb{P}^{n-1}(\mathbb{C}))^n/\sigma_n$] of them, whereas our result gives $\binom{2n-1}{n} - 1$. It would be interesting to know what is the minimum possible number, a question equivalent to finding "better" embeddings of $(\mathbb{P}^{n-1}(\mathbb{C}))^n/\sigma_n$ in projective varieties. Can the optimum $(n(n-1))$ be achieved?

(b) This question is reduced to the next by the following obvious proposition.

Proposition: There is an equivalence between the following statements: (i) M has an invariant subspace of dimension j generated by $(U_1,...,U_j)$; and (ii) $\Lambda^j M$ [the jth exterior power of M, dimension $\binom{n}{j}$] has $U_1\wedge U_2\wedge\cdots\wedge U_j$ as an eigenvector.

(c) We make again the hypothesis that A and B are in $\widetilde{M}_n(\mathbb{C})$ and let $U_1,...,U_j$ (resp. $V_1,...,V_j$) be the eigenvectors—unique up to scalar multiplication—of A (resp. B). Then $U_1\otimes\cdots\otimes U_n$ and $V_1\otimes\cdots\otimes V_n$ are viewed as two polynomials in $\mathbb{C}[e_1,...,e_n]$. We have the following equivalence: (i) A and B possess a common eigenvector; and (ii) $U_1\otimes\cdots\otimes U_n$ and $V_1\otimes\cdots\otimes V_n$ have a linear factor in common.

To check (ii), simply use $\mathbb{C}[e_1,...,e_n] \approx \mathbb{C}[e_1,..., \hat{e}_j,...,e_n][e_j]$ for some j (any j will do) and perform the Euclidean algorithm. (This, of course, relies heavily on the fact that we know *a priori* that the polynomials we are working on can be decomposed into a product of linear factors.)

APPENDIX B: DEGENERATE CASE FOR ISING SPIN MODEL

Here we describe two families of 2×2 matrices

$$M_{ij} = \begin{pmatrix} a_{ij} & b_{ij} \\ c_{ij} & d_{ij}\end{pmatrix}$$

(resp. M'_{ij}) with the following properties:

(i) The M_{ij}'s and M'_{ij}'s have real positive elements.

(ii) The M_{ij}'s (resp. M'_{ij}'s) generate $M_2(\mathbb{C})$ as a vector space.

(iii) $\forall n, \forall i_1,...,i_n$ $(1\leqslant i_k\leqslant 4)$,

$$\mathrm{Tr}(M_{i_1i_2}\cdots M_{i_{n-1}i_n}M_{i_ni_1}) = \mathrm{Tr}(M'_{i_1i_2}\cdots M'_{i_{n-1}i_n}M'_{i_ni_1}).$$

(iv) There *do not* exist matrices $(R_i)_{i=1}^{i=1}$ such that $R_iM_{ij} = M'_{ij}R_j$. The constructed families will be seen to be essentially the only ones possessing these properties.

We first choose four points $0<f_1<F_2<F_3<F_4\in\mathbb{P}^1(\mathbb{C})$ on the positive real axis (possibly with $F_4=\infty$), corresponding to four vectors V_1,V_2,V_3,V_4 [for example, take $V_i = (1,F_i)$ and $V_4 = (0,1)$ if $F_4=\infty$] and we also select 16 strictly positive numbers λ_{ij}. The M_{ij}'s and M'_{ij}'s will be constructed in order to satisfy

(a) $M_{ij}\cdot V_j = \lambda_{ij}V_i$; $M'_{ij}\cdot V_j = \lambda_{ij}\cdot V_i$,

(b) $\det M_{ij} = \det M'_{ij}$.

Proposition: (a) and (b) imply condition (iii).

In fact, $M_{i_1i_2}\cdots M_{i_{n-1}i_n}\cdot M_{i_ni_1}$ and $M'_{i_1i_2}\cdots M'_{i_{n-1}i_n}\cdot M_{i_ni_1}$ will have the same determinant and one eigenvalue in common, namely $\lambda_{i_1i_2}\cdots\lambda_{i_{n-1}i_n}\lambda_{i_ni_j}$.

Next, we prove the simple following lemma.

Lemma: $\forall\lambda,\mu;\lambda>0,\mu>0$; there exists a one parameter family of 2×2 matrices with real positive elements such that $\det M=\Delta$, Δ some fixed strictly positive number; and $M\binom{\lambda}{1} = \gamma\binom{\mu}{1}$, γ fixed, positive, with $\gamma^2>\Delta\lambda/\mu$.

Proof: The corresponding homographic transformation looks like

$$M(Z) = \mu + \alpha(Z-\lambda)/(cZ+d),\quad \alpha>0,\quad c>0,\quad d>0.$$

Now

$$M = \begin{pmatrix}\mu c+\alpha & d\mu-\alpha\lambda \\ c & d\end{pmatrix} \det M = \alpha(\lambda c+d),$$

$$\gamma = \lambda c+d.$$

We have therefore $\alpha\equiv\Delta\alpha$ and d *arbitrary* inside $(\Delta\lambda/\mu\gamma,\gamma)$ so that $d\mu-\alpha\lambda>0$ and $C\equiv(1/\lambda)(\gamma-d)>0$.

Repeat the above construction for all pairs F_i, F_j, keeping the d_{ij}'s as a set of variables. For the M_{ij}'s and M'_{ij}'s, we shall take matrices of this form, with different values of the d_{ij}'s. Conditions (i) and (iii) are automatically satisfied. Condition (ii) also is, except for very special values of the d_{ij}'s [and elementary calculations show that these can be chosen so that (iv) also holds]. In fact R_i (resp. R_j) intertwines M_{ii} and M'_{ii} (resp. M_{jj} and M'_{jj}) and we can choose d_{ij} such that $R_i M_{ij}$ and $M'_{ij} R_j$ are different for any R_i and R_j satisfying the intertwining property.

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