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Holonomic functions of several complex variables and singularities of anisotropic Ising n-fold integrals

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Abstract

Focusing on examples associated with holonomic functions, we try to bring new ideas on how to look at phase transitions, for which the critical manifolds are not points but curves depending on a spectral variable, or even fill higher dimensional submanifolds. Lattice statistical mechanics often provides a natural (holonomic) framework to perform singularity analysis with several complex variables that would, in the most general mathematical framework, be too complex, or simply could not be defined. In a learn-by-example approach, considering several Picard–Fuchs systems of two-variables 'above' Calabi–Yau ODEs, associated with double hypergeometric series, we show that D-finite (holonomic) functions are actually a good framework for finding properly the singular manifolds. The singular manifolds are found to be genus-zero curves. We then analyze the singular algebraic varieties of quite important holonomic functions of lattice statistical mechanics, the n-fold integrals $\chi^{(n)}$, corresponding to the n-particle decomposition of the magnetic susceptibility of the anisotropic square Ising model. In this anisotropic case, we revisit a set of so-called Nickelian singularities that turns out to be a two-parameter family of elliptic curves. We then find the first set of non-Nickelian singularities for $\chi^{(3)}$ and $\chi^{(4)}$, that also turns out to be rational or elliptic curves. We underline the fact that these singular curves depend on the anisotropy of the Ising model, or, equivalently, that they depend on the spectral parameter of the model. This has important consequences on the physical nature of the anisotropic $\chi^{(n)}$s which appear to be highly composite objects. We address, from a birational viewpoint, the emergence of families of elliptic curves, and that of Calabi–Yau manifolds on such problems. We also address the question of singularities of non-holonomic functions with a discussion on the accumulation of these singular curves for the non-holonomic anisotropic full susceptibility χ.

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1. Introduction

Singularities are known to play a crucial role in physics (particle physics [1], Landau singularities [2, 3], critical phenomenon theory, renormalization group, dynamical systems). They are the 'backbone' of many physical phenomena; in the same way cohomology can be introduced in mathematics as a 'skeleton' describing the most fundamental part of so many mathematical problems[4].

Seeking for the singular points and/or critical manifolds of models in lattice statistical mechanics is a necessary preliminary step toward any serious study of the lattice models. If the model is Yang–Baxter integrable, then there is a canonical parametrization of the model in algebraic varieties [5], and the critical manifolds will also be algebraic varieties. If one does not expect the model to be 'integrable' (or even that the integrability of the model requires too much work to be performed), finding the singular manifolds of the model is an attempt to obtain, at least, one exact result for the model. Recalling the standard-scalar Potts model [6, 7], it is worth keeping in mind that its singular manifolds (corresponding to second-order phase transitions or first-order phase transitions) are selected co-dimension-1 algebraic varieties where the model is actually Yang–Baxter integrable. The crucial role played by the (standard-scalar) Potts model in the theory of critical phenomena is probably at the origin of some 'conformal theory' mainstream prejudice identifying criticality with integrability for two-dimensional models.

A large number of papers [8–10] have tried (under the assumption of a unique phase transition) to obtain critical, and more generally singular[5], manifolds of lattice models as algebraic varieties preserved by some (Kramers–Wannier-like) duality, thus providing, at least, one exact (algebraic) result for the model, and, hopefully, algebraic subvarieties candidates for Yang–Baxter integrability of the models. The relation between singular manifolds of lattice statistical models and integrability is, in fact, *much more complex*. Along this line it is worth recalling two examples.

The first example is the 16-vertex model which is, generically, *not Yang–Baxter integrable*, but is such that the birational symmetries of the CP_{15} parameter space of the model are *actually integrable*[6], thus yielding a canonical parametrization[7] of the model in terms of *elliptic curves* [11]. This parametrization gives natural candidates for the singular manifolds of the model, namely the vanishing condition of the corresponding j-invariant (which is actually the

[4] And not surprisingly, cohomology is naturally introduced in the singularity theory [4].

[5] If the wording 'critical' still corresponds to singular in mathematics, it tends to be associated with second-order phase transitions exclusively. The singular condition for the standard-scalar q-state Potts model corresponds to second-order phase transitions for $q < 4$ and first-order transitions for $q > 4$.

[6] We have called such models 'quasi-integrable': they are *not* Yang–Baxter integrable but the birational symmetries of their parameter space correspond to *integrable mappings* [11].

[7] A foliation of CP_{15} in elliptic curves.

vanishing condition of a homogeneous polynomial of degree 24 in 16 homogeneous parameters of the model, with the polynomial being the sum of a very large[8] number of monomials [11]). This co-dimension-1 algebraic variety is, probably, not Yang–Baxter integrable.

The second example is the triangular q-state Potts model with 3-spin interactions on the up-pointing triangles [16, 17] for which the critical manifold has been obtained as a simple co-dimension-1 algebraic variety [18]. This co-dimension-1 algebraic variety is a remarkably selected one: it is preserved by a 'huge' set of birational transformations [19, 20]. Recalling the previous 'conformal theory' prejudice on standard-scalar q-state Potts models, it is worth mentioning that, *even restricted to this singular codimension-1* algebraic variety, the model *is not*[9] *Yang–Baxter integrable*.

People working on lattice statistical mechanics (or condensed matter theory) have some (*lex parsimoniae*[10]) prejudice that there exists a concept of 'singularities of a model', with the singularities of the partition function being, 'of course', the same as the singularities of the full susceptibility. Furthermore, they also have another prejudice, namely that singularity manifolds are simple sets, such as points (self-dual), straight lines, smooth co-dimension-1 manifolds, the maximum complexity being encountered with the phase diagram of the Ashkin–Teller model [21], with the emergence of tricritical points [22, 23], forgetting less common (and more sophisticated or involved) critical behavior like the Kosterlitz–Thouless transition [24], the massless phase in the classical XY model or in Z_N models (see for instance [25, 26]), or the massless phase in the three-state superintegrable chiral Potts model [27] or in the XXZ quantum chain [27–29], the Griffiths–McCoy singularities [30, 31] in random systems and the much more complex phase diagrams of commensurate–incommensurate models [32–35]. This Ockham's razor simplicity prejudice is clearly not shared by people working on singularity theory in algebraic geometry and discrete dynamical systems [4, 36, 37] (see also Arnold's viewpoint on singularity theory and catastrophe theory [38]).

In fact, singular manifolds in lattice statistical mechanics (or condensed matter theory) have no reason to be simple co-dimension-1 sets (or even stratified spaces). For lattice models of statistical mechanics, where the parameter space corresponds to *several* (complex) variables, there is a gap between a physicist's viewpoint that roughly amounts to seeing singular manifolds as simple *mutatis mutandis* generalizations of singularities of one complex variable, conjecturing singular manifolds as algebraic varieties [8–10], and the mathematician's viewpoint that is reluctant to introduce the concept of singular manifolds for functions of several complex variables. (It is not clear that the functions one studies are even defined in a Zariski space.)

Singular manifolds can be well defined in a framework that is, in fact, quite natural and *emerges quite often in theoretical physics*, namely the *holonomic functions* [39] corresponding to n-fold integrals of a holonomic integrand. (Most of the time, in theoretical physics, the integrand is simply rational or algebraic.) In Sato's $\mathcal{D}$-module theory [40], a holonomic system is a highly *overdetermined* system, such that the solutions locally form a vector space of *finite dimension* (instead of the expected dependence on some arbitrary functions). Furthermore,

[8] In [11], this polynomial of degree 24 in 16 unknowns is seen as the double discriminant of a biquadratic. It is nothing but a hyperdeterminant [12–14] (Schäfli's hyperdeterminant [15] of format $2 \times 2 \times 2 \times 2$). It has 2894 276 terms.

[9] It is not Yang–Baxter integrable in the natural embedding of the model (namely a parameter space made up of the three (anisotropic) nearest-neighbor edge interactions and the 3-spin interaction on the up-pointing triangle). Of course, it is always conceivable that upon increasing the parameter space the selected critical algebraic subvariety becomes embedded in a Yang–Baxter family. However, the hyperbolic character [16, 17, 19, 20] of the set of birational automorphisms of this algebraic subvariety seems to exclude an *Abelian variety* for the larger (integrable) variety. Furthermore, random matrix analysis also seemed to exclude an integrability of this subvariety.

[10] Ockham's razor.

holonomic functions naturally correspond to systems with *fixed regular singularities*. It is crucial to avoid movable singularities. For *non-holonomic functions*, only the ones that *can be decomposed as an infinite sum of holonomic functions* (like χ, the full susceptibility of the square[11] Ising model [46]) give some hope for interesting and/or rigorous studies of their singularities.

For one complex variable, the holonomic (or D-finite [42, 43]) functions are solutions of linear ODEs with polynomial coefficients in the complex variable. The (regular) singularities can be seen immediately as solutions of the head polynomial coefficient of the linear ODE, up to apparent singularities [44]. If one takes a representation of the linear ODE as a linear differential system, one gets rid of the apparent singularities, and one also sees, quite immediately, the singularities in such systems. More generally, for holonomic functions of several complex variables, one can define, and see, quite clearly, the singular manifolds of the corresponding systems of PDEs. In a learn-by-example approach, we will show how one can find, and see, these singular algebraic varieties.

The paper is organized as follows. After briefly recalling the framework of the isotropic $\chi^{(n)}$s, we will first study various examples of Picard–Fuchs systems of two variables associated with hypergeometric series, and generalizing some known Calabi–Yau ODEs [45]. We will show how the singular manifolds can be obtained from the holonomic systems and from simpler asymptotic calculations. We will then obtain singular manifolds for quite important holonomic functions of lattice statistical mechanics, the *n*-fold integrals, $\chi^{(n)}$s (corresponding to the decomposition of the magnetic susceptibility of the anisotropic square Ising model [46]), describing a set of (so-called) Nickelian singularities, and then getting, from a 'Landau singularity [1, 2] approach', the first set of other (non-Nickelian) singularities. We will underline the dependence of the singularity manifolds in the *anisotropy* of the Ising model. This has important consequences for understanding the mathematical, as well as the physical, nature of the anisotropic $\chi^{(n)}$s. The question of the accumulation of these singular manifolds for the anisotropic full susceptibility, χ, will be discussed. We will finally comment on the emergence of families of elliptic curves for the singularity manifolds and the (birational) reason for the *occurrence of Calabi–Yau manifolds* on such problems.

2. Holonomic functions of one complex variable: $\chi^{(n)}$s for the isotropic Ising model

Let us start with the simplest holonomic, or D-finite [42], functions, namely the holonomic functions of *one* complex variable, by recalling important holonomic functions of lattice statistical mechanics, the *n*-fold integrals, $\chi^{(n)}$, of the isotropic square lattice Ising model [44, 47, 48]. These *n*-fold integrals correspond to the decomposition of the full susceptibility of the model as an *infinite sum* [46] of the *n*-particle contributions $\chi^{(n)}$. The singularities of these $\chi^{(n)}$s have been completely described and can be seen to be a very rich and complex set of points [3, 49]. In particular, one finds, in some well-suited variable *k*, which is the modulus of the elliptic function parametrizing the two-dimensional Ising model, that the unit circle $|k| = 1$ is a *natural boundary* for the full susceptibility χ of the Ising model [49]. The singularities of $\chi^{(n)}$s accumulate on the unit circle. This is the reason why we have this unit circle *natural boundary* [48–52] for the full magnetic susceptibility χ. Singularities *also* accumulate *inside* the unit circle (see figures 1, 2, 3, 4 of [49]), probably becoming an *infinite set of points dense in the open disc* $|k| < 1$. They also accumulate outside the unit *k*-circle $|k| > 1$, probably becoming another infinite set of points *also dense outside the unit circle* $|k| > 1$. This accumulation of singular points of the linear ODEs of $\chi^{(n)}$s is thus

[11] We have similar decompositions as an infinite sum of *n*-fold integrals for the full susceptibility of the triangular or honeycomb Ising models for which dramatic extensions of their series expansion have been obtained recently [41].

(probably) dense in the *whole k-complex plane*. In other words, we do have an infinite set of singularities *dense in the whole k-complex plane*. This seems to confirm the mathematician's reluctance to consider singular manifolds of functions of several complex variables that are not holonomic: even in the very simple case of *one* complex variable, we already seem to encounter serious trouble. The full susceptibility χ, which is an infinite sum [46] of these $\chi^{(n)}$s, does not even seem to be defined in a Zariski space. Recalling these results [49], the common wisdom identifying the singularities of the partition function and the singularities of the full susceptibility is no longer obvious.

There is, however, an important subtlety here: these singularities are *singularities of the linear ODEs* of $\chi^{(n)}$s, but not of the (series expansions of) $\chi^{(n)}$s given by holonomic n-fold integrals. When one considers the k-series expansions of $\chi^{(n)}$s, one finds that singularities inside the unit circle in the open disc, $|k| < 1$, are *not singularities of these series* [49]. This is quite a non-trivial result. This is also the case for the k-series expansion for the full susceptibility χ which is the infinite sum of $\chi^{(n)}$s. For the full susceptibility χ, the accumulation of $\chi^{(n)}$s singularities on the unit circle makes this unit circle a *natural boundary* [49]. Switching from high-temperature series expansions to low-temperature series, we have a similar result for $|k| > 1$. We thus have quite a drastic difference between the singularities of the n-fold integrals $\chi^{(n)}$, which are solutions of linear ODEs (they are D-finite or holonomic, see below), and the full susceptibility χ, which is *not* the solution of a linear ODE (it is *not* holonomic).

Before generalizing to several complex variables with the case of $\chi^{(n)}$s for the *anisotropic* square Ising model with two complex variables, let us consider, in a learn-by-example approach, simple Picard–Fuchs systems associated with hypergeometric series of *two* complex variables.

3. The first simple Picard–Fuchs system with two variables

Let us consider the double hypergeometric series, *symmetric in x* and *y*

$$H_0(x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(3m+3n)!}{n!^3\, m!^3} \cdot x^n \cdot y^m \tag{1}$$

$$= \sum_{n=0}^{\infty} \frac{(3n)!}{n!^3} \cdot {}_3F_2\left(\left[n+1,\, n+\frac{1}{3},\, n+\frac{2}{3}\right],\, [1,\, 1];\, 27y\right) \cdot x^n$$

$$= 1 + 6 \cdot (x+y) + (90 \cdot (x^2+y^2) + 720 \cdot xy) + (1680 \cdot (x^3+y^3) + 45\,360 \cdot xy \cdot (x+y)) + (34\,650 \cdot (x^4+y^4) + 2217\,600 \cdot xy \cdot (x^2+y^2) + 7484\,400 \cdot x^2y^2) + \cdots . \tag{2}$$

This series reduces, when $y = x$, to

$$\sum_{n=0}^{\infty} \left[\frac{(3n)!}{(n!)^3} \sum_{k=0}^{n} \binom{n}{k}^3\right] \cdot x^n = 1 + 12 \cdot x + 900 \cdot x^2 + 94\,080 \cdot x^3 + 11\,988\,900 \cdot x^4 + 1704\,214\,512 \cdot x^5 + 260\,453\,217\,024 \cdot x^6 + \cdots, \tag{3}$$

which is the solution analytic at $x = 0$ of the *Calabi–Yau operator* Ω of order 4 introduced by Batyrev and van Straten (section 7.1 of [45], see also the ODE number 15 in [53]):

$$\Omega = \theta^4 - 3x \cdot (7\theta^2 + 7\theta + 2) \cdot (3\theta + 1) \cdot (3\theta + 2) - 72x^2 \cdot (3\theta + 5) \cdot (3\theta + 4) \cdot (3\theta + 2) \cdot (3\theta + 1),$$

$$\text{where} \quad \theta = x \cdot \frac{d}{dx}. \tag{4}$$

The double hypergeometric series (1) is the *unique* analytical (in x and y) solution of the Picard–Fuchs system corresponding to the two partial linear differential operators:

$$\Omega_x = \theta_x^3 - x \cdot (3\,\theta_x + 3\,\theta_y + 1) \cdot (3\,\theta_x + 3\,\theta_y + 2) \cdot (3\,\theta_x + 3\,\theta_y + 3),$$
$$\Omega_y = \theta_y^3 - y \cdot (3\,\theta_x + 3\,\theta_y + 1) \cdot (3\,\theta_x + 3\,\theta_y + 2) \cdot (3\,\theta_x + 3\,\theta_y + 3), \qquad (5)$$
$$\text{where} \qquad \theta_x = x \cdot \frac{\partial}{\partial x}, \qquad \theta_y = y \cdot \frac{\partial}{\partial y}.$$

The other formal series solutions of (5), around $(x, y) = (0, 0)$, have the form

$$H_0(x, y) \cdot \ln(x)^n \cdot \ln(y)^m + \cdots, \qquad (6)$$

where the maximum value reached by n and m is 2. They read for instance

$$H_0(x, y) \cdot \ln(x) + H_1(x, y), \qquad H_0(x, y) \cdot \ln(y) + H_1(y, x),$$
$$H_0(x, y) \cdot \ln(x) \cdot \ln(y) + H_1(y, x) \cdot \ln(x) + H_1(x, y) \cdot \ln(y) + H_3(x, y) + \cdots.$$

It is crucial to note that the dimension of the space spanned by these formal series is *finite*. In the case of the Picard–Fuchs system (5), the number of solutions (i.e. dimension) is 9. These nine formal solutions are given in appendix A. The double series analytic in x and y, $H_j(x, y)$ are either symmetric like $H_0(x, y)$, $H_3(x, y)$, or are not symmetric like $H_1(x, y)$.

Such holonomic systems are also called *D-finite* [42, 43] for the following reason: remarkably, they have a *finite* number of independent solutions, in contrast with generic systems of PDEs that have, generically, an infinite number of solutions. Systems of PDEs can also have no solution at all. Generically, the compatibility of the two operators Ω_x and Ω_y, requires some (slightly tedious) differential algebra calculations.

One can also see the system (5) as a (two-dimensional) recursion:

$$(n + 1)^3 \cdot c_{n+1,\,m} = b(n, m) \cdot c_{n,\,m},$$
$$(m + 1)^3 \cdot c_{n,\,m+1} = b(n, m) \cdot c_{n,\,m}, \qquad \text{where}$$
$$b(n, m) = (3\,(n+m) + 1) \cdot (3\,(n+m) + 2) \cdot (3\,(n+m) + 3). \qquad (7)$$

Here, the compatibility between the two partial differential operators Ω_x and Ω_y is easier to see at this (double) recursion level. Introducing

$$\alpha_1(n, m) = \frac{b(n, m)}{(n + 1)^3} = \frac{c_{n+1,\,m}}{c_{n,\,m}}, \qquad \alpha_2(n, m) = \frac{b(n, m)}{(m + 1)^3} = \frac{c_{n,\,m+1}}{c_{n,\,m}},$$

we have the identity

$$\alpha_2(n, m) \cdot \alpha_1(n, m + 1) = \alpha_1(n, m) \cdot \alpha_2(n + 1, m), \qquad (8)$$

which, from a recursion viewpoint, actually corresponds *to the compatibility between the two partial linear differential operators* Ω_x and Ω_y.

The discriminant of the two-parameter family of Calabi–Yau 3-folds reads[12] (see proposition 7.2.1 of [45])

$$(x + y)^3 - 3 \cdot (x^2 - 7\,x\,y + y^2) + 3 \cdot (x + y) - 1, \qquad (9)$$

or (without performing the $(x, y) \rightarrow (x/27, y/27)$ rescaling mentioned in [45])

$$\Delta = 19\,683\,(x+y)^3 - 2187 \cdot (y^2 + x^2 - 7\,x\,y) + 81 \cdot (x+y) - 1. \qquad (10)$$

This expression can easily be obtained as the resultant [12] in A (or equivalently in B) of the two (very simple) homogeneous binary cubics [45]:

$$27\,x \cdot (A+B)^3 - A^3 = 0, \qquad 27\,y \cdot (A+B)^3 - B^3 = 0. \qquad (11)$$

[12] Note a misprint in proposition 7.2.1 of [45]: $(x + y)$ must be changed to $3 \cdot (x + y)$.

3.1. Singular manifolds

What are the singularities of the double hypergeometric series like (1), and how do they compare with the singularities of the Picard–Fuchs system (5), assuming that the notion of singularities of such PDEs systems is well defined?

From a mathematical viewpoint, when introducing some 'canonical' system, equivalent to the Picard–Fuchs system, one should 'in principle' be able to see the singularities as simple poles of this equivalent system. Unfortunately, to our knowledge, the implementation of such a procedure, available as the formal calculation tools, is still underdeveloped[13] (see also [55, 56]).

A physicist's down-to-earth approach amounts to reducing the double hypergeometric series, like (1), to series in one (complex) variable imposing some relation between x and y, compatible with the $(x, y) = (0, 0)$ origin of the double series. Imposing, for example, $y = cx$ $(c = 2, 3, \ldots)$, or $y = cx^2$, one gets a series in one (complex) variable x and, then, in the second step, finds the corresponding linear ODE annihilating this series. The head polynomial of the corresponding linear differential operator gives (after getting rid of the apparent singularities) the singularities of these linear differential operators. An 'accumulation' of such results enables us to see that the singularities are always on the (genus-zero) algebraic curve $\mathcal{S}(x, y) = 0$, where

$$\mathcal{S}(x, y) = 3^9 \cdot (x+y)^3 - 3^7 \cdot (y^2 + x^2 - 7xy) + 3^4 \cdot (x+y) - 1, \qquad (12)$$

which is *nothing but the discriminant* (10) *of the two-parameter family of Calabi–Yau 3-folds previously mentioned* [45]. Remarkably, but not surprisingly, the *singular variety has an interpretation as a fundamental projective invariant* [12].

The (genus-zero) singular curve (12) can be parametrized by

$$x = \left(\tfrac{1}{6} + u\right)^3, \qquad y = \left(\tfrac{1}{6} - u\right)^3. \qquad (13)$$

or

$$x(u) = \left(\frac{5u+7}{6 \cdot (1-u)}\right)^3, \qquad y(u) = \left(\frac{7u+5}{6 \cdot (u-1)}\right)^3 = x\left(\frac{1}{u}\right), \qquad (14)$$

where the Atkin–Lehner-like involution $u \leftrightarrow 1/u$ could suggest a modular curve interpretation of (12).

The accumulation of calculations is quite tedious compared to the simplicity of the final result (12). It is far from obvious that (12) is the singularity manifold of the double series (1) or the singularity manifold of the Picard–Fuchs system (5). Let us find a Picard–Fuchs system for which it will become crystal clear that (12) is actually the singularity manifold of the system.

3.2. Other representations as PDE systems

In fact, the Picard–Fuchs partial differential system (5) can be recast into a system of two differential equations, each one being a linear ODE on *only one* variable. We consider[14] a linear combination of Ω_x and Ω_y and their derivatives and cancel the coefficients in front of

[13] See the Maple package [54] (in development) for computing closed form solutions of integrable connections for handling a D-finite partial differential system which is not written as a connection. Note that one must download the OreModules package and use its procedure called 'Connection'.

[14] For our purpose, we did not use the Groebner basis approach (use the pdsolve command on the system of equations obtained from the Rosenfeld–Groebner command in Maple).

the undesired derivatives. We obtain the following form:

$$\tilde{\Omega}_x = \sum_{n=0}^{9} P_n(x, y) \cdot D_x^n, \qquad \tilde{\Omega}_y = \sum_{n=0}^{9} Q_n(x, y) \cdot D_y^n,$$
$$\text{where} \quad D_x = \frac{\partial}{\partial x}, \qquad D_y = \frac{\partial}{\partial y}, \tag{15}$$

where $P_n(x, y)$ and $Q_n(x, y)$ are the polynomials of the two variables x and y. The partial differential operator $\tilde{\Omega}_x$ can be seen as a linear differential operator in x depending on a parameter y (and similarly $\tilde{\Omega}_y$ as a linear differential operator in y depending on a parameter x). The polynomials $P_n(x, y)$ appearing in $\tilde{\Omega}_x$ will not be given here. For $P_9(x, y)$, the monomial of highest degree in x and y is $x^{15} y^9$ (see (16) and (B.2) in appendix B), and for $P_8(x, y), \ldots, P_0(x, y)$, it reads, respectively, $x^{14} y^9$, $x^{13} y^9$, $x^{12} y^9$, $x^{11} y^9$, $x^{10} y^9$, $x^9 y^9$, $x^8 y^8$, $x^7 y^7$, $x^6 y^6$.

There is a 'price to pay' to recast the Picard–Fuchs partial linear differential system (5) into a system like (15). The partial linear differential operators $\tilde{\Omega}_x$ and $\tilde{\Omega}_y$ are *much more involved* than the operators Ω_x and Ω_y in (5), and of higher order in D_x or D_y. The operator $\tilde{\Omega}_x$ (resp. $\tilde{\Omega}_y$) is of *order* 9 with respect to D_x (resp. D_y), in agreement with the previously mentioned finite set (A.2) of *nine formal series solutions* of the Picard–Fuchs D-finite system (5). We have checked that these nine formal solutions (A.2) are indeed the solutions of $\tilde{\Omega}_x$ (resp. $\tilde{\Omega}_y$).

As a consequence of the exact symmetry interchange $x \leftrightarrow y$ of (1), the partial differential operator $\tilde{\Omega}_y$ is nothing but operator $\tilde{\Omega}_x$, where x and y are permuted. Not surprisingly, the head polynomials in (15) have the form

$$P_9(x, y) = x^6 \cdot \mathcal{P}_9(x, y) \cdot \mathcal{S}(x, y), \qquad Q_9(x, y) = y^6 \cdot \mathcal{P}_9(y, x) \cdot \mathcal{S}(x, y), \tag{16}$$

where $\mathcal{P}_9(x, y)$ is a polynomial of x and y, corresponding to the *apparent* singularities of the (y-dependent) linear differential operator $\tilde{\Omega}_x$. The expression of $\mathcal{P}_9(x, y)$ is given in appendix B.

3.3. *Operator factorizations*

One can actually go further in the analysis of these order-9 operators. The order-9 partial linear differential operator $\tilde{\Omega}_x$, in fact, factorizes into three order-1 operators and an order-6 operator:

$$\tilde{\Omega}_x = \left(D_x - \frac{\partial \ln(\tilde{r}_1(x, y))}{\partial x}\right) \cdot \left(D_x - \frac{\partial \ln(\tilde{r}_2(x, y))}{\partial x}\right)\left(D_x - \frac{\partial \ln(\tilde{r}_3(x, y))}{\partial x}\right) \cdot L_6(x, y), \tag{17}$$

where the order-6 operator $L_6(x, y)$ reads

$$L_6(x, y) = \frac{1}{p_6(x, y)} \cdot \sum_{n=0}^{6} p_n(x, y) \cdot D_x^n, \tag{18}$$

and where $\tilde{r}_1(x, y)$, $\tilde{r}_2(x, y)$ and $\tilde{r}_3(x, y)$ are the rational functions of x and y, while $p_6(x, y)$ has simple factorizations:

$$\tilde{r}_1(x, y) = \frac{\mathcal{P}_9(x, y)}{x^6 \cdot \mathcal{S}(x, y) \cdot q_1}, \qquad \tilde{r}_2(x, y) = \frac{q_1}{x^5 \cdot \mathcal{S}(x, y) \cdot q_2},$$
$$\tilde{r}_3(x, y) = \frac{q_2}{x^4 \cdot \mathcal{S}(x, y) \cdot \mathcal{P}_6(x, y)}, \qquad p_6(x, y) = x^4 \cdot \mathcal{S}(x, y) \cdot \mathcal{P}_6(x, y), \tag{19}$$

where $\mathcal{P}_9(x, y)$, $\mathcal{P}_6(x, y)$, q_1, q_2, are the polynomials of x and y given in appendix B. Not surprisingly the (x, y)-asymmetric polynomials $\mathcal{P}_6(x, y)$ and $\mathcal{P}_9(x, y)$ correspond respectively to apparent singularities of the order-6 and order-9 operators $L_6(x, y)$ and $\tilde{\Omega}_x$. The polynomials $p_n(x, y)$ appearing in $L_6(x, y)$ will not be given here. For $p_6(x, y)$, the monomial of highest degree in x and y is $x^{13} y^9$ (see (19) and (B.3) in appendix B) and for $p_5(x, y), \ldots, p_0(x, y)$, it reads, respectively, $x^{13} y^9$, $x^{12} y^9$, $x^{11} y^9$, $x^{10} y^9$, $x^9 y^9$, $x^8 y^8$, $x^7 y^7$.

Do note that the critical exponents of this order-6 operator $L_6(x, y)$ are *independent of* y. For instance at $x = 0$ the indicial polynomial reads $P(r) = r^3 \cdot (r-1)^3$. More remarkably, *on the singular variety* $\mathcal{S}(x, y) = 0$, the critical exponents of $L_6(x, y)$ are *also independent of* y. The indicial polynomial, at $\mathcal{S}(x, y) = 0$, reads $P(r) = r \cdot (r-1)^2 \cdot (r-2) \cdot (r-3) \cdot (r-4)$. The singular behavior at $\mathcal{S}(x, y) = 0$ is thus logarithmic. The Wronskians of this order-6 linear differential operator $L_6(x, y)$ and of the order-9 operator $\tilde{\Omega}_x$ are the rational functions of x and y, which read, respectively,

$$W(L_6(x, y)) = \frac{\mathcal{P}_6(x, y)}{x^{12} \cdot \mathcal{S}(x, y)^4}, \qquad W(\tilde{\Omega}_x) = \frac{\mathcal{P}_9(x, y)}{x^{27} \cdot \mathcal{S}(x, y)^7}. \tag{20}$$

In fact, the operator $L_6(x, y)$ is not only *Fuchsian* with *rational exponents* and *rational Wronskian*, it is actually *globally nilpotent* for any rational values of y. The p-curvature of this globally nilpotent order-6 operator is a nilpotent 6×6 matrix, which can be put into the following Jordan form[15], *not only for any rational value of* y, but, actually, *for any* y *being an algebraic number*:

$$\mathcal{C} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \text{where} \qquad \mathcal{C}^4 = 0. \tag{21}$$

Furthermore, the *exterior square* of $L_6(x, y)$ *is of order 14*, instead of the order-15 one should expect generically for an order-six irreducible operator. This remarkable property is related to the fact that $L_6(x, y)$ is *homomorphic to its (formal) adjoint*, with an order-2 intertwinner differential operator $I_2(x, y)$:

$$L_6(x, y) \cdot I_2(x, y) = \text{adjoint}(I_2(x, y)) \cdot \text{adjoint}(L_6(x, y)),$$
$$\text{where} \quad I_2(x, y) = 3^6 \cdot \frac{27x + 27y + 2}{\mathcal{S}(x, y)} \cdot D_x^2 + R_1(x, y) \cdot D_x + R_0(x, y), \tag{22}$$

where $R_1(x, y)$ and $R_2(x, y)$ are the rational functions of x and y.

One can check that the double (x, y)-symmetric series (1), solution of the order-9 operator $\tilde{\Omega}_x$, is, in fact, annihilated by the order-6 linear differential operator $L_6(x, y)$ and, thus (by $x \leftrightarrow y$ symmetry) by the other order-6 operator

$$L_6(y, x) = \frac{1}{p_6(y, x)} \cdot \sum_{n=0}^{6} p_n(y, x) \cdot D_y^n. \tag{23}$$

At this step, we should recall that our purpose is to obtain the singularities of the system (5) and not to obtain an equivalent system for (5). Generically, systems of linear PDEs *cannot be strictly recast*[16] into a form like (15), even for D-finite systems[17]. The two order-6 operators

[15] Of characteristic polynomial $P(\lambda) = \lambda^6$ and of minimal polynomial $P_m(\lambda) = \lambda^4$.
[16] Non-holonomic systems cannot be recast into a form like (15). This is the case, for instance, of the system of linear operators $(\Omega_x, \Omega_y) = (D_x^2, D_x D_y)$, which has an *infinite number* of solutions, namely $c \cdot x + f(y)$, where $f(y)$ is an *arbitrary function of* y.
[17] For instance, the solutions of the D-finite system $(\Omega_x, \Omega_y) = (D_x^2 - y D_y^2, D_x D_y)$ are the solutions of the D-finite system $(\tilde{\Omega}_x, \tilde{\Omega}_y) = (D_x^3, y D_y^3 + D_y^2)$, but this last D-finite system has more solutions. One needs additional operators to have system equivalence.

$L_6(x,\ y)$ and $L_6(y,\ x)$ form a PDE system that is *not equivalent* (in the sense of equivalence of systems) to the Picard–Fuchs system (5). However, as far as the double series $H_0(x, y)$ is concerned, the three systems $(\Omega_x,\ \Omega_y)$, $(\tilde{\Omega}_x,\ \tilde{\Omega}_y)$ and $(L_6(x,\ y),\ L_6(y,\ x))$ can alternatively be considered.

Remark. Recovering the Calabi–Yau order-4 ODE (4) from the $y = x$ limit of the Picard–Fuchs system (5), or (15), is *not straightforward* (as one could naively imagine). Within the (down-to-earth) approach, which amounts, for instance, to restricting to the straight lines $y = c \cdot x$, where c is a constant, and finding the linear differential operator in x, one obtains an order-6 linear differential operator with coefficients that are polynomials in x, as well as in the constant c. One can then take the $c \to 1$ limit and actually recover the Calabi–Yau order-4 ODE (4). These calculations are displayed in appendix C. The (genus-zero) singular curve (12)

$$(1 - 108 \cdot (x+y)) \cdot (2 + 27 \cdot (x+y))^2 + 3^9 \cdot (x-y)^2 = 0, \qquad (24)$$

reduces, in the $y = x$ limit, to $(1 - 216x) \cdot (1 - 27x)^2 = 0$, namely the singularities corresponding to the order-4 Calabi–Yau ODE (4).

4. More Picard–Fuchs systems with two variables

Similar calculations can be performed with double hypergeometric series generalizing the analytic solution of another Calabi–Yau order-4 ODE (see appendix D below). One can perform exactly the same calculations *mutatis mutandis*.

4.1. More Picard–Fuchs system with two variables

Let us first consider a two-variable Picard–Fuchs system 'above' another Calabi–Yau ODE [45] (see the ODE number 16 in appendix A of [53]), corresponding to the following $(x,\ y)$-symmetric series with *binomial* coefficients:

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} & \binom{2n+2m}{n+m} \binom{n+m}{n}^2 \binom{2n}{n} \binom{2m}{m} \cdot x^n y^m \\ &= \sum_{m=0}^{\infty} \binom{2m}{m}^2 \cdot {}_3F_2\left(\left[\frac{1}{2}, \frac{1}{2}+m, \frac{1}{2}+m\right], [1,\ 1];\ 16y\right) \cdot x^m \\ &= 1 + 4\ (x+y) + (36\,(x^2+y^2) + 96\,x y) \\ &\quad + [2160\ (x^2 y + x y^2) + 400\ (x^3+y^3)] + [4900\ (x^4+y^4) \\ &\quad + 44\,800\ (x y^3 + x^3 y) + 90\,720\ x^2 y^2] + \cdots. \end{aligned} \qquad (25)$$

This hypergeometric double series is the solution of the Picard–Fuchs system of PDEs:

$$\begin{aligned} \Omega_x &= \theta_x^3 - 4x \cdot (2\theta_x + 1)\,(\theta_x + \theta_y + 1)\,(2\theta_x + 2\theta_y + 1), \\ \Omega_y &= \theta_y^3 - 4y \cdot (2\theta_y + 1)\,(\theta_x + \theta_y + 1)\,(2\theta_x + 2\theta_y + 1). \end{aligned} \qquad (26)$$

In the $y = x$ limit, this series reduces to

$$\begin{aligned} \sum_{n=0}^{\infty} & \left[\binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} \binom{2n-2k}{n-k}\right] \cdot x^n \\ &= \sum_{n=0}^{\infty} \binom{2n}{n}^2 \cdot {}_4F_3\left(\left[\frac{1}{2}, -n, -n, -n\right], \left[1,\ 1,\ -\frac{2n-1}{2}\right];\ 1\right) \cdot x^n \end{aligned}$$

$$= 1 + 8x + 168x^2 + 5120x^3 + 190\,120x^4 + 7939\,008\,x^5 + 357\,713\,664x^6 + 16\,993\,726\,464\,x^7 + 839\,358\,285\,480\,x^8 + \cdots \quad (27)$$

annihilated by the order-4 Calabi–Yau operator:

$$\theta^4 - 4x \cdot (5\theta^2 + 5\theta + 2) \cdot (2\theta + 1)^2 + 64x^2 \cdot (2\theta + 3) \cdot (2\theta + 1) \cdot (2\theta + 2)^2. \quad (28)$$

The recast of the PDE system for the double series (25) into the form (15) gives two (x, y)-symmetric linear differential operators of order 9. The singularities of the two order-9 linear differential operators are respectively $x \cdot (1 - 16x) = 0$ and $y \cdot (1 - 16y) = 0$ together with the quadratic condition:

$$\mathcal{S}_2(x, y) = 2^8 \cdot (x - y)^2 - 2^5 \cdot (y + x) + 1 = 0, \quad (29)$$

which has the simple rational parametrization

$$(x, y) = \left(\left(\tfrac{1}{8} - u\right)^2, \left(\tfrac{1}{8} + u\right)^2\right).$$

The singularities $\mathcal{S}_2(x, y) = 0$ are, here also, logarithmic, the local exponents being $0, 1, 1, 2, 3, \ldots, 7$.

These two (x, y)-symmetric order-9 operators also factorize in exactly the same way as (17), in three order-1 operators and an order-6 operator like (18). The exterior square of this order-6 operator is also of order 14 (instead of order 15 as one expects for a generic irreducible order-6 operator), and, again, this order-6 operator is homomorphic to its adjoint with a relation similar to (22), with the head coefficient in the order-2 intertwinner being replaced by $2^8\,(16x - 16y + 3)/\mathcal{S}_2(x, y)/(16x - 1)/x^2$. We also have relations similar to (20) for the various Wronskians.

4.2. Another Picard–Fuchs system above the Calabi–Yau operator (28)

Note that the Picard–Fuchs system of two variables 'above' the Calabi–Yau operator (28) is *not unique*. Other (x, y)-symmetric series reduce to the series (27) annihilated by (28), for instance, the double series expansion:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} 64^{n+m} \cdot \frac{(1/2)_n^3 \cdot (1/2)_m^3 \cdot (1/2)_{m+n}}{(1)_{n+m}^3 \cdot n!\ m!} \cdot x^n y^m \quad (30)$$

$$= \sum_{m=0}^{\infty} \left(\frac{(\frac{1}{2})_m}{m!}\right)^4 \times {}_4F_3\left(\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} + m\right], [m + 1, m + 1, m + 1];\ 64x\right) \cdot (64y)^m$$
$$= 1 + 4 \cdot (y + x) + 3 \cdot [27 \cdot (x^2 + y^2) + 2 \cdot x \cdot y] + 20 \cdot (y + x) \cdot [125 \cdot (x^2 + y^2) - 122 \cdot x \cdot y] + 35/16 \cdot [42\,875 \cdot (x^4 + y^4) + 162 \cdot x^2 \cdot y^2 + 500 \cdot xy \cdot (x^2 + y^2)] + 63/4 \cdot (y + x) \cdot [250\,047 \cdot (x^4 + y^4) - 248\,332 \cdot xy \cdot (x^2 + y^2) + 248\,602 \cdot x^2 y^2] + \cdots, \quad (31)$$

where $(a)_n$ is the usual Pochhammer symbol. This series can be found in Guttmann and Glasser [57] as a lattice Green function. It can also be seen as the expansion of a *Kampé de Fériet function* [58–61] (see appendix D):

$$F^{(1,3,3)}_{(3,0,0)}\left(\left[\frac{1}{2}\right], \left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right], \left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right]; [1, 1, 1], -, -;\ 64x,\ 64y\right). \quad (32)$$

The double series (30) is not a series with integer coefficients but it can be recast[18] into a series with *integer* coefficients if one performs the simple rescaling $(x,\ y)\ \rightarrow\ (4x,\ 4y)$. One obtains

$$1 + 16 \cdot\ (x\ + y) + [1296 \cdot (x^2 + y^2) + 96 \cdot x \cdot y] + 1280 \cdot (y + x) \cdot [125 \cdot (x^2 + y^2) - 122 \cdot x \cdot y] + [24\,010\,000 \cdot (x^4 + y^4)\ + 280\,000 \cdot x\,y \cdot (x^2 + y^2) + 90\,720 \cdot x^2 \cdot y^2] + 16\,128 \cdot (y + x) \cdot [250\,047 \cdot (x^4 + y^4) - 248\,332 \cdot x\,y \cdot (x^2 + y^2) + 248\,602 \cdot x^2 \cdot y^2] + \cdots. \qquad (33)$$

The recast of the PDE system for the double series (30) into the form (15) gives two (x, y)-symmetric linear differential operators, now of order 13.

The singular varieties of the two order-13 operators $\tilde{\Omega}_x$ and $\tilde{\Omega}_y$ are respectively[19] $x \cdot (x{-}y) {\cdot} (1 - 64\,x) = 0$ and $y \cdot (x{-}y) {\cdot} (1 - 64\,y) = 0$, together with an $(x,\ y)$-symmetric *genus-zero* biquadratic which reads

$$\tilde{S}_2(x,\ y) =\ 2^{12} \cdot\ x^2\,y^2\ \ - 2^7 \cdot\ x\,y \cdot\ (y + x)\ \ + (x - y)^2 =\ \ 0. \qquad (34)$$

The local exponents at the singularities of the order-13 partial linear differential operators are *independent* of y (respectively x).

This genus-zero curve (34) has the rational parametrization (well suited for series expansions near $(x,\ y) =\ (0,\ 0)$),

$$x(t) = u^2, \qquad\qquad y(t) = \Big(\frac{u}{1\ +\ 8\,u}\Big)^2, \qquad (35)$$

or the rational parametrization

$$x(u) = \Big(\frac{u\ +\ 1}{8}\Big)^2, \qquad\qquad y(u) = \Big(\frac{u\ +\ 1}{8\,u}\Big)^2 = x\Big(\frac{1}{u}\Big), \qquad (36)$$

the Atkin–Lehner-like involution $u\ \ \leftrightarrow\ \ 1/u$ suggesting a modular curve interpretation of (34).

Note that the two singular varieties $\tilde{S}_2(x,\ y)$ and $\mathcal{S}_2(x,\ y)$ (see (29)), are related by a simple involution:

$$\tilde{S}_2(x,\ y) = 2^{12} \cdot\ \ x^2\,y^2 \cdot\ \ \mathcal{S}_2\Big(\frac{1}{2^{10}\,x},\ \frac{1}{2^{10}\,y}\Big). \qquad (37)$$

We thus see that the various Picard–Fuchs systems 'above' a given Calabi–Yau ODE (i.e. reducing, when one takes the 'diagonal' $y\ =\ x$, to the same Calabi–Yau ODE), *do not necessarily have the same singular manifolds*, even if these various singular manifolds must reduce to the same singular points in the $y = x$ limit. Since the singular variety (34) contains the origin $(x,\ y) =\ (0,\ 0)$, it is easy to find, using the parametrization (35), a linear differential ODE satisfied by (30) when restricted[20] to the singular variety (34) (see (D.13) in appendix D). This cannot be done for (29), which does not contain the origin $(x,\ y) =\ (0,\ 0)$.

Breaking the $(x,\ y)$-symmetry in (30), by resumming the series as (31), corresponds to the viewpoint of seeing Kampé de Feriet functions of several complex variables as straight generalization[21] of hypergeometric functions [58–61]. The x-singularities in each of the (transcendental) ${}_4F_3$ coefficients of the y-expansion (31) are only the well-known $x\ =\ 0$, $x = 1$, $x = \infty$ singularities of hypergeometric functions (here $x = 1$ becomes $x = 1/64$), and are, of course, drastically different from the singular variety (34) for the double series (30).

[18] Such series are called *globally bounded* [62].
[19] Note that the limit $y =\ x$ of the Picard–Fuchs systems associated with (30) is *actually a singular limit*.
[20] See also the notion of the Fuchsian system of linear partial differential equations *along a submanifold* (see [63], in particular paragraph 6).
[21] The parameters of the hypergeometric functions become linear differential operators [60, 61].

The results for (30) can be generalized to more general (Kampé de Fériet) double series depending on several parameters:

$$K(x,\ y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(\alpha)_n^M \cdot (\beta)_m^M \cdot (\beta')_{m+n}}{(\gamma)_{m+n}^M\ n!\ m!} \cdot x^n \cdot y^m, \qquad (38)$$

where $(\alpha)_n$ is the usual Pochhammer symbol. The same calculations as before show that their singular curves *do not depend on the parameters*. These calculations for (38) are displayed in appendix D.

4.3. Picard–Fuchs systems with more than two variables 'above' the Calabi–Yau operator (28).

For heuristic reasons, we restricted ourselves to two variables, but one can find many Picard–Fuchs systems, with more than two complex variables, 'above' a given Calabi–Yau ODE like (28). For instance, the series (27) of the Calabi–Yau operator (28) can also be written as the $x = y = z = t$ subcase of the (hypergeometric) series of four complex variables [45]:

$$\sum_{j,k,l,m} \left[\binom{2(j+k+l+m)}{j+k+l+m} \cdot \left(\frac{(j+k+l+m)!}{j!\,k!\,l!\,m!} \right)^2 \right] \cdot x^j\, y^k\, z^l\, t^m. \qquad (39)$$

With the general term being hypergeometric, one obtains directly a system of four PDEs, from which we build a linear ODE in the variable x, with y, z and t being 'parameters'. Once one has series with four variables, and systems of PDEs with four variables, one can take many limits in order to reduce to two variables.

For instance, if one restricts the previous series to $y = z = t$, one gets a series of two variables (which will of course reduce, for $y = x$, to the series (27) of the Calabi–Yau operator (28)), but is no longer symmetric in x and y. The series can be written as

$$\sum_{N=0}^{\infty} \binom{2N}{N} \cdot {}_3F_2([-N, -N, 1/2], [1, 1]; 4) \cdot {}_2F_1([N+1,\ N+1/2],\ [1];\ 4x) \cdot y^N$$

$$= \sum_{N=0}^{\infty} \binom{2N}{N}^2 \cdot {}_3F_2([-N, -N, -N], [1, 1/2 - N]; 1/4) \times {}_2F_1([N+1,\ N+1/2],\ [1];\ 4x) \cdot y^N$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(2n+2m)!}{(n!\ m!)^2} \cdot {}_3F_2([-m, -m, 1/2],\ [1, 1],\ 4) \cdot x^n\, y^m. \qquad (40)$$

The corresponding system of PDEs reads

$$\Omega_x = \theta_x^2 - 2x \cdot (\theta_x + \theta_y + 1)\,(2\theta_x + 2\theta_y + 1),$$

$$\Omega_y = \theta_y^4 - 2y \cdot (10\theta_y^2 + 10\theta_y + 3)\,(\theta_x + \theta_y + 1)(2\theta_x + 2\theta_y + 1) + 36\,y^2 \cdot (2\theta_x + 2\theta_y + 3)\,(2\theta_x + 2\theta_y + 1)\,(\theta_x + \theta_y + 2)(\theta_x + \theta_y + 1). \qquad (41)$$

Again, one can recast this system into a form like (15), i.e. two linear differential operators $\tilde{\Omega}_x$ and $\tilde{\Omega}_y$ in the variable x (resp. y), both of *order* 8, each one with the same singular variety which is the union of the two genus-zero algebraic curves:

$$16x^2 - 8 \cdot (4y+1) \cdot x + (4y-1)^2 = 0, \quad \text{and}$$

$$16x^2 - 8 \cdot (36y+1) \cdot x + (36y-1)^2 = 0. \qquad (42)$$

These two order-8 operators both factorize in a similar way as (17) but, this time, in the product of *two* order- and one order-6 operators. These two order-6 operators right dividing respectively

$\tilde{\Omega}_x$ and $\tilde{\Omega}_y$ are not related by an (x, y)-symmetry, because the Picard–Fuchs system (41) is not (x, y)-symmetric. Again the exterior squares of these two order-6 operators are of order 14, instead of the order 15 one can expect for the exterior square of a generic irreducible order-6 operator. Furthermore one has, again, that these order-6 operators are homomorphic to their adjoints, with the intertwinner being of order 2 (see (22)).

More examples of the Picard–Fuchs system with two variables 'above' Calabi–Yau ODEs are sketched in appendix E, with their corresponding (simple) singular varieties being also given.

5. Singular manifolds for hypergeometric series of several complex variables

All these singular varieties (12), (24), (34) (as well as similar ones, (E.3), (E.7), given in appendix E) can, in fact, be easily obtained from very simple calculations when one remarks that the previous double series are the *hypergeometric series of several complex variables*. The calculations, corresponding to the Horn convergence theorem, are similar to the ones for Horn functions and Horn systems [64–67]. A very important property is the fact that the region of convergence for the hypergeometric series *does not depend on the parameters* [68].

Let us denote the coefficients of (1) by $c_{n,\,m}$:

$$c_{n,\,m} = \frac{(3m+3n)!}{n!^3\; m!^3}. \qquad (43)$$

The successive ratio of $c_{n,\,m}$ in the two 'directions' reads respectively

$$\frac{c_{n,\,m}}{c_{n+1,\,m}} = \frac{(n+1)^3}{b(n,\,m)}, \qquad\qquad \frac{c_{n,\,m}}{c_{n,\,m+1}} = \frac{(m+1)^3}{b(n,\,m)}, \qquad (44)$$

where the product $b(n,\,m)$ is given by (7). In the n and m large limits these two ratios behave respectively like

$$X(n,\,m) = \frac{n^3}{27\,(m+n)^3} \qquad \text{and} \qquad Y(n,\,m) = \frac{m^3}{27\,(m+n)^3}, \qquad (45)$$

where one remarks that $X(n,\,m)$ and $Y(n,\,m)$ depend only on the ratio n/m. The curve rationally parametrized by $(x,\,y) = (X(n,\,m),\,Y(n,\,m))$ can easily be obtained performing a resultant (elimination of m or n or the ratio n/m), and one recovers in a very simple way the singular manifold (12). One notes that (45) is nothing but the previous binary cubics (11) yielding (10), the discriminant of a two-parameter family of Calabi–Yau threefolds.

We can perform similar calculations for the hypergeometric series (25); the ratio of $c_{n,\,m}$s also reads (44), with the product $b(n,\,m)$ being now given by

$$b(n,\,m) = 2\cdot(2n+2m+1)\,(2n+2m+2)\,(2n+1). \qquad (46)$$

In the n and m large limits, this gives the rational parametrization of the singular variety (29), namely $(x,\,y) = (X(n,\,m),\,Y(n,\,m))$, with

$$X(n,\,m) = \frac{n^2}{16\,(m+n)^2} \qquad \text{and} \qquad Y(n,\,m) = \frac{m^2}{16\,(m+n)^2}. \qquad (47)$$

For the hypergeometric series (30), the ratio of the $c_{n,\,m}$ reads respectively

$$\frac{(n+m+1)^3\,(n+1)}{4\cdot(2n+1)^3\,(2n+2m+1)} \qquad \text{and} \qquad \frac{(n+m+1)^3\,(m+1)}{4\cdot(2m+1)^3\,(2n+2m+1)}.$$

In the n and m large limits, this gives the rational parametrization of the singular variety (34), namely $(x,\,y) = (X(n,\,m),\,Y(n,\,m))$, with

$$X(n,\,m) = \frac{(m+n)^2}{64\,n^2} \qquad \text{and} \qquad Y(n,\,m) = \frac{(m+n)^2}{64\,m^2}. \qquad (48)$$

Finally, for other hypergeometric series (E.2) and (E.6), given in appendix E, similar calculations also give rational parametrizations of the corresponding *genus-zero* singular curves (E.3) and (E.7).

For instance, the successive ratio of $c_{n,m}$ for (E.6) reads respectively

$$\frac{(n+1)^4}{b(n,\, m)}, \qquad \frac{(m+1)^4}{b(n,\, m)}, \qquad \text{where}$$
$$b(n,\, m) = (2n+m+1)\,(2n+m+2)\,(2m+n+1)\,(n+m+1). \qquad (49)$$

In the n and m large limits, this gives the rational parametrization of the singular variety (E.7), namely $(x,\, y) = (X(n,\, m),\, Y(n,\, m))$, with

$$X(n,\, m) = \frac{n^4}{(2n+m)^2\,(2m+n)\,(n+m)}, \qquad Y(n,\, m) = X(m,\, n).$$

Of course, all these calculations can be performed with a series of *any finite* number of complex variables. These (simple) calculations are only valid for the series of several complex variables, such that the ratios of the various consecutive coefficients (see (44)) are rational expressions (typically *the hypergeometric series*).

6. Toward singular manifolds of the Ising model D-finite system of PDEs

One thus sees, from the previous calculations, that one can actually define, and find without ambiguity, the singular manifolds of D-finite systems of PDEs. The singular manifolds are *fixed* and can (in principle) be obtained from (possibly tedious but well-defined) calculations from the D-finite system of PDEs. This is quite different from the case of generic (non-holonomic) systems of PDEs where singularities depend on initial boundary conditions. With the previous calculations, one can see that the singular manifolds can even be obtained from very simple calculations in the (selected) case of *the hypergeometric series*, with the singular varieties with *rational parametrization* being underlined.

For functions of several complex variables which are not known to be the solutions of D-finite systems of partial linear differential operators (or even partial nonlinear differential operators but with fixed critical points), the question of defining and finding the singular manifolds seems hopeless. There is, however, one category of functions of several complex variables that emerges quite naturally in physics, where some hope remains, thus partially justifying the 'guessing' approach often performed in lattice statistical mechanics [8–10, 18, 23, 69–71]. These functions of several complex variables are the ones which can be decomposed as infinite *sums of D-finite functions* (in a typical Feynman diagram approach). The best example is the full susceptibility of the anisotropic square Ising model which has such a decomposition [46]. Let us try to find the singularity manifolds of the anisotropic $\chi^{(n)}$s, trying in the second step, to understand the singularity manifolds of the *anisotropic* full susceptibility χ.

6.1. Landau approach for the singular manifolds of the anisotropic $\chi^{(n)}$

Finding the Fuchsian (and in fact globally nilpotent [72]) linear ODEs for the n-fold integrals $\chi^{(n)}$s of the decomposition of the full magnetic susceptibility of the square lattice Ising model is already a 'tour-de-force' in the isotropic case [44, 47, 73–75].

The anisotropic $\chi^{(2)}$ has a surprisingly nice factorized form (see equation (3.22) in [50]). It is the product of the isotropic $\chi^{(2)}$ and of a simple square-root algebraic function:

$$\chi^{(2)}(k,\, r) = \frac{((1+kr)\cdot(k+r))^{1/2}}{1+k}\cdot\chi^{(2)}(k,\, 1), \qquad (50)$$

where $k = s_1 s_2$ is the modulus of elliptic functions in the parametrization of the model, where the ratio $r = s_1/s_2$ is the anisotropy variable, with $s_1 = \sinh 2K_1$ and $s_2 = \sinh 2K_2$ (with notations $K_1 = E^v/k_BT$ and $K_2 = E^h/k_BT$, see (3.22) of [50]), and where $\chi^{(2)}(k, 1)$ is the isotropic $\chi^{(2)}$:

$$\chi^{(2)}(k, 1) = \frac{1}{3\pi} \cdot \frac{(1+k^2) \cdot E(k^2) - (1-k^2) \cdot K(k^2)}{(1-k)(1-k^2)}$$
$$= \frac{k^2}{4(1+k)^4} \cdot {}_2F_1\left(\left[\frac{3}{2}, \frac{5}{2}\right], [3]; \frac{4k}{(1+k)^4}\right). \qquad (51)$$

Beyond this surprisingly simple $\chi^{(2)}$ case, obtaining a D-finite (Picard–Fuchs) system for $\chi^{(3)}$, for the anisotropic square Ising model, would require too massive and extreme computer calculations. Furthermore, the simple 'Horn calculations' detailed in section (5) require some *closed asymptotic formula* (or some asymptotic formula of exact linear recursions) for the coefficients of the double series of the anisotropic $\chi^{(n)}$, and would require some assumption that $\chi^{(n)}$s are the hypergeometric series or, at least, that their singular part is dominated by the hypergeometric series.

However, if one is only interested in the singularities of such D-finite n-fold integrals, then the *Landau singularity approach*, which we have already used in the isotropic case to find [3, 49] these singularities, can again be worked out. We are not going to recall the details of this approach, which correspond in the anisotropic case to sometimes quite tedious (algebraic) calculations. The idea, which is specific to n-fold integrals of some algebraic integrands, amounts to saying that the singularities should, in principle, be deduced only from the algebraic integrands of these integrals from elementary algebraic calculations [1–3, 49, 48].

We will display in subsection 6.3 the results for the first $\chi^{(n)}$s after recalling in the following subsection the first set of fundamental singularities.

6.2. Nickelian singular manifolds for the anisotropic $\chi^{(n)}$s and zeros of the partition function

In contrast to the form factors [76, 77] $C^{(n)}(M, N)$, whose only singular points are $k = 0$, $k = 1$ and $k = \infty$, $\chi^{(n)}(k)$s have many further singularities. The first set of these singularities was found, by Nickel [51, 52], to be, for the isotropic case ($K_1 = K_2 = K$), located at

$$\cosh^2 2K - \sinh 2K \cdot (\cos(2\pi j/n) + \cos(2\pi l/n)) = 0, \qquad (52)$$

with ([x] being the integer part of x): $0 \leqslant j, \; l \leqslant [n/2]$, $j = l = 0$ excluded (for n even, $j + l = n/2$ is also excluded). Equivalently (52) reads

$$\sinh 2K_{j,l} = s_{j,l} = 1/2 \cdot (\cos(2\pi j/n) + \cos(2\pi l/n))$$
$$\pm i/2 \cdot [(4 - (\cos(2\pi j/n) + \cos(2\pi l/n))^2]^{1/2}. \qquad (53)$$

These Nickel's singularities are clearly on the unit circle $|s| = 1$ or $|k| = 1$. Do note that this is no longer the case for the anisotropic model where Nickel's singularities for the anisotropic $\chi^{(n)}$s become

$$\cosh 2K_1 \cdot \cosh 2K_2 - (\sinh 2K_1 \cdot \cos(2\pi j/n) + \sinh 2K_2 \cdot \cos(2\pi l/n)) = 0, \qquad (54)$$

with $j, l = 1, 2, \ldots, n$. These (complex) algebraic curves (54), in the two complex variables $s_1 = \sinh 2K_1$, $s_2 = \sinh 2K_2$, have to be singular loci (as will be suggested in the following section) for the D-finite system of PDEs satisfied by the anisotropic (holonomic) $\chi^{(n)}$s.

One can rewrite these algebraic curves in $k = s_1 \cdot s_2$ and $r = s_1/s_2$ as

$$(r+k) \cdot (kr+1) - k \cdot (rU \pm V)^2 = 0, \qquad (55)$$

where $U = \cos(2\pi j/n)$ and $V = \cos(2\pi l/n)$. Do remark that *these algebraic curves depend on the anisotropy variable* $r = s_1/s_2$. We will underline this important fact in subsection 6.4. Remarkably, these curves are *generically of genus* 1 [22], not only when $U = \cos(2\pi j/n)$ and $V = \cos(2\pi l/n)$, but for *any fixed value of* U and V. Their j-invariant [49, 78] reads[23]

$$j = 256 \cdot \frac{(U^4 + V^4 - V^2U^2 - U^2 - V^2 + 1)^3}{(V^2 - 1)^2\ (U^2 - 1)^2\ (U^2 - V^2)^2}. \tag{56}$$

We thus see that we have a *two-parameter family of elliptic curves*.

These elliptic (or rational) curves (54) accumulate with increasing values of n, in the same way Nickel's singularities (52) accumulate on the unit circle $|s| = 1$, in a certain (real) submanifold $\mathcal{S}$ of the two complex variables s_1, s_2 (four real variables). However, this 'singularity manifold' $\mathcal{S}$ is not a co-dimension-1 (real) submanifold (like the unit circle $|s| = 1$ in the s-complex plane), but actually a co-dimension zero submanifold, as can also be seen in various analyses of complex temperature zeroes (see[24] for instance [81–88] and more recently [89–91]). Note that this 'singularity manifold' becomes very 'slim' near the (critical) algebraic curve $k = s_1 s_2 = 1$. (See for instance the region near the real axis of figures 1–3 in [89].)

In the isotropic case, we actually obtained [44, 47, 48, 74, 75, 92] the linear ODEs satisfied by the first $\chi^{(n)}$s, for $n = 3, 4, 5, 6$ and, thus, of course, the corresponding ODE singularities. Furthermore, we also performed a *Landau singularity* approach that enabled us to obtain, and describe, the singularities for *all* [3, 49] $\chi^{(n)}$s. These exact results show, very clearly, that there are (non-Nickelian) singularities inside the unit circle and outside the unit circle (see figures 1–4 of [49]). From the figures of [49], it is easy to get convinced that the accumulation of these non-Nickelian singularities will probably be a *dense set of points inside the unit circle* and (by Kramers–Wannier duality) *outside the unit circle*. These non-Nickelian singularities are given in terms of Chebyshev polynomials of the first and second kinds (see equations (28) and (29) in [49]). Upgrading these slightly involved exact (Chebyshev) non-Nickelian results [49] for the isotropic model to the anisotropic model is, at the present moment, probably too ambitious.

Let us simply try, using the previous *Landau singularity* approach, to provide maybe not an exhaustive description of all the singularities for the anisotropic case, but at least the exact expression of all the singular manifolds (Nickelian or non-Nickelian) for the first anisotropic $\chi^{(n)}$s.

6.3. Singular manifolds for the first anisotropic $\chi^{(n)}$

The *Landau singularity* approach detailed in [3, 49] for the isotropic $\chi^{(n)}$s of the square Ising model can easily be generalized to the anisotropic $\chi^{(n)}$s. We are not going to explain here the details of these (slightly tedious) calculations, which are basically the same as in [3, 49] *mutatis mutandis*. With the calculations being slightly involved, we just give the results for the first $\chi^{(n)}$s.

[22] For $U = V$ (as well as $U = -V$, $U = \pm 1$, $V = \pm 1$) the curves are of *genus zero*. For instance, for $U = V$, they read $(r \pm 1)^2 k \cdot U - (r + k) \cdot (k r + 1) = 0$.

[23] This rational expression (56) of U and V is nothing but relation (36) in [78] with $J_x/J_z = U$, $J_y/J_z = V$. This rational expression remarkably factorizes for many Heegner numbers [79, 80] (complex multiplication cases): $j = 12^3, 20^3, (-15)^3, 2 \times 30^3, 66^3$ and selected quadratic values of j-invariant, like $j^2 + 191\,025\, j - 495^3 = 0$ or $j^2 - 1264\,000\, j - 880^3 = 0$. This (partially) explains the occurrence in (54) of several complex multiplication cases (for instance $U = \cos(2\pi 2/8)$, $V = \cos(2\pi/8)$, which give $j = 1728$).

[24] The first reference corresponds to the fact that zeroes can fill areas in the complex temperature plane. Some later papers contain results on the density of zeroes in the thermodynamic limit.

The singularities of $\chi^{(3)}$ and $\chi^{(4)}$ read respectively in k and r:

$$\mathrm{Sing}(\chi^{(3)}) = (k^2-1)\cdot(3\,kr+r+4\,k^2)\cdot(k^2r+3\,kr+4)\cdot(k^2r+r+k) \times(3\,r^2k-r-k-k^2r)\cdot(4+3\,kr+4\,k+4\,k^2)\,(r+k)\,(kr+1), \qquad (57)$$

$$\mathrm{Sing}(\chi^{(4)}) = (k^2-1)\cdot(kr+1+k^2)\cdot(3\,r^2k-r-k-k^2r). \qquad (58)$$

In order to compare these results with our previous exact results for the isotropic model, which were given [44, 48, 74] in the (quite natural for such n-fold integrals) variable [47, 74] w, let us rewrite these results in r and $w = s/(1+s^2)/2$, where, now, $s = (s_1\,s_2)^{1/2}$:

$$\mathrm{Sing}(\chi^{(3)}) = (w^2-1)\cdot w^2\cdot(r^2-4\,r+4+3\,w^2r^2-4\,w^2r+16\,w^4r)^2 \times(1+4\,w^2r-2\,r)^2\,(3\,r^2-1-4\,w^2r+2\,r)^2 \times(3\,r-4+16\,w^2)^2\cdot(1+4\,w^2r-2\,r+r^2)^2, \qquad (59)$$

$$\mathrm{Sing}(\chi^{(4)}) = w^2\cdot(w^2-1)\cdot\ (4\,w^2-2+r)^2\cdot(3\,r^2-1-4\,w^2r+2\,r)^2. \qquad (60)$$

Note that the complex multiplication points of the isotropic case [49], namely the roots of $1+3\,k+4\,k^2 = 0$ and $k^2+3\,k+4 = 0$, come from the $\mathrm{Sing}(\chi^{(3)})$ factor

$$r^2-4\,r+4+3\,w^2r^2-4\,w^2r+16\,w^4\,r, \qquad (61)$$

in (59), or equivalently with $(k,\ r)$, the two factors in (57):

$$(3\,kr+r+4\,k^2)\cdot\ (k^2r+3\,kr+4), \qquad (62)$$

The vanishing condition of (61) corresponds to a *genus-zero curve*, with its rational parametrization being

$$w = \frac{u^2+1}{2\,u}, \qquad r = \frac{-4}{u^2\cdot(u^2+3)}. \qquad (63)$$

Note that $\mathrm{Sing}(\chi^{(3)})$ and $\mathrm{Sing}(\chi^{(4)})$ have a non-trivial gcd (respectively in k, then w):

$$gcd(\mathrm{Sing}(\chi^{(3)}),\ \mathrm{Sing}(\chi^{(4)})) = (k^2-1)\cdot\ (3\,r^2k-r-k-k^2r),$$
$$gcd(\mathrm{Sing}(\chi^{(3)}),\ \mathrm{Sing}(\chi^{(4)})) = w^2\cdot(1-w)\,(1+w)\cdot(3\,r^2-1-4\,w^2r+2\,r)^2,$$

the last algebraic curve, $3\,r^2k-r-k-k^2r = 0$, is a *genus-one curve*. A way to understand, in the anisotropic case, the emergence of singular algebraic curves shared by several $\chi^{(n)}$s (n even and n odd) amounts to noting that these curves actually reduce, in the isotropic limit, to $k=1$, the singular variety of the partition function of the anisotropic model.

The fact that the singular curve $3\,r^2k-r-k-k^2r = 0$, together with the Nickelian algebraic curves (54), (55), are *not of genus-zero* (as all the genus-zero curves of section 4, like (29), (34), as well as the ones displayed in appendix E, see (E.3), (E.7), show that the series for the anisotropic $\chi^{(n)}$s *cannot be the hypergeometric series* in the variables k and r (see section (5))).

It would be interesting, before trying to generalize the Chebyshev polynomial formula [49] for the non-Nickelian singularities of the isotropic model to the anisotropic one, to accumulate, with this Landau singularity approach, more non-Nickelian algebraic curves in the anisotropic case. Recalling the systematic emergence of elliptic curves (see (55)) for the Nickelian algebraic curves, it would be interesting to *systematically look at the genus of these singular curves* to see if higher genus curves are also discarded for the non-Nickelian algebraic curves.

It would also be interesting to confirm these Landau singularity calculations with differential algebra calculations. Even with the last progress performed by Koutschan on the *creative telescopic* method [93, 94], getting the (Picard–Fuchs) system of PDEs satisfied by the several complex variables series of the anisotropic $\chi^{(n)}$s corresponds, at the present moment, to too large calculations (even for the anisotropic $\chi^{(3)}$). However, if one considers particular anisotropic subcases ($s_2 = 3\,s_1$, $s_2 = 5\,s_1^2$, ...), obtaining the corresponding ODEs for the anisotropic $\chi^{(3)}$, in the unique complex variable, could be imagined using the creative telescopic method [93, 94], or even from series expansion as we did in the isotropic case [44].

6.4. Singular manifolds and the anisotropy variable

For experts of Yang–Baxter integrability, the fact that the singularity varieties, namely the Nickelian elliptic curves (55), or the non-Nickelian rational curves (62), *do depend on the anisotropy* of the model may come as a surprise. Indeed, within the Yang–Baxter integrable framework, and as a consequence of the existence of families of commuting transfer matrices (row-to-row, diagonal or corner transfer matrices), one used to have many quantities like the order parameter, the eigenvectors of row-to-row or corner transfer matrices, ..., which are *independent* of the so-called spectral parameter (the parameter that enables us to move along each elliptic curve). The selected quantities *depend only on the modulus* k of the elliptic functions. Along this line, one certainly expects the singular manifolds, which are highly symmetric, 'invariant' and 'universal' manifolds [11, 16, 17], to be *also independent of the spectral variables*. With the previous variables k and r, the singular manifolds should just depend on the modulus k, and *not on the anisotropy variable* r (related to the spectral parameter). The surprise is that the singular manifolds do depend also on the anisotropy variable r, and thus on the spectral variable.

$\chi^{(n)}$s are known [76] to be an infinite sum of form factors $C^{(n)}(N,\, M)$:

$$\chi^{(n)} = \sum_M \sum_N C^{(n)}(N,\, M), \qquad (64)$$

with this relation being inherited from the fact that the full susceptibility is the sum of all the two-point correlation functions [76].

Recalling the simplest (nearest-neighbor) correlation function $C(0, 1)$, it reads [95] in the anisotropic case[25]:

$$C(0,\, 1) = \frac{2}{\pi\, r} \cdot \left(\frac{k+r}{k}\right)^{1/2} \cdot \left((1 + k\,r) \cdot \Pi(-k\,r,\, k) - K(k)\right),$$

where, again, $k = s_1\, s_2$ is the modulus of the elliptic functions parametrizing the model and r is the ratio $r = s_1/s_2$, and where $\Pi(x,\, y)$ is the complete elliptic integral of the third kind.

The singular manifolds correspond to the singular points of the complete elliptic integrals of the first and third kinds, namely $k = 0$, $k = 1$ and $k = \infty$. Therefore, they depend only on the modulus k in the elliptic parametrization of the model.

The form factors have been seen to be solutions of linear differential equations associated with elliptic functions [76, 77]. Consequently, their singular points correspond to the singular points of the complete elliptic integrals of the first or second kind E or K, namely $k = 0$, $k = 1$ and $k = \infty$. The generalization to the anisotropic case has been sketched in [95]. One expects the results to be polynomial expressions of the complete elliptic integrals of the first (or second) and third kinds, yielding again singular manifolds which depend only on the modulus k, and are actually $k = 0$, $k = 1$ or $k = \infty$.

Finite sums of correlation functions or form factors certainly have $k = 0$, $k = 1$ or $k = \infty$ as singularities, even for the anisotropic model. However, the anisotropic $\chi^{(n)}$s are sums of an *infinite* number of form factors. One cannot try to deduce the singular points of

[25] We use the Maple notations for Π and K.

these *infinite* sums $\chi^{(n)}$s from the singular points of the form factors. $\chi^{(n)}$s are, in fact, quite involved 'composite' quantities with no simple combinatorics interpretation (like being the sum over graphs of a certain type). It is worth noting that exploring all the algebraic singular curves for all $\chi^{(n)}$s, condition $k = 1$ always occurs for all $\chi^{(n)}$s.

The previous results provide quite an interesting insight into the 'true mathematical and physical' nature of $\chi^{(n)}$s: they are quite involved 'composite' quantities, with their singularities being drastically different from the ones of the $C^{(n)}(N, M)$ form factors [76, 77].

In the isotropic case, strong evidence has been given [48–52] that the full susceptibility χ has a *natural boundary* corresponding to the accumulation of singular points on the $|k| = 1$ unit circle, thus *discarding* common wisdom that 'of course' the singularities of the partition function are the *same as the singularities of the full susceptibility*.

By analogy with the situation encountered in the isotropic case, we are going to have an accumulation of singular curves densifying the whole parameter space (two complex variables s_1 and s_2, i.e. four real variables). The equivalent of the unit circle is now a co-dimension-zero manifold in the four real variables parameter space, which disentangles two co-dimension-zero domains in the parameter space. Is it the singular locus for the full anisotropic susceptibility χ? Do we have here a generalization of the concept of natural boundary for several complex variables ? If the answer to the question of the location of the singularities of non-holonomic functions seems to be *dependent on the decomposition of the non-holonomic function in infinite sums of holonomic functions*, is it simply well defined?

All we can reasonably say is that, probably, and in the same way as in the isotropic case, the double series for $\chi^{(n)}$s are not singular in one domain (the equivalent of the inside of the unit circle), and one probably has the same result for the full anisotropic susceptibility χ.

6.5. Anisotropic models: n-fold integrals of several complex variables

In the anisotropic case, $\chi^{(n)}$s are n-fold integrals of several complex variables. After [39], we do know that these 'functions' of several complex variables are *holonomic*. Let us restrict ourselves to the case we often encounter in physics, where the integrand is an *algebraic function* of these several complex variables (and of the integration variables). In contrast with the one complex variable case, the holonomic character here corresponds to an extremely rich structure: the solutions of the *overdetermined* system of linear PDEs correspond to a *finite* set of solutions (for one complex variable this is obvious), and the singularities, which are no longer points but manifolds, are *fixed algebraic varieties* (for one complex variable this is obvious). Furthermore, these operators are globally nilpotent (the holonomic functions can, in this 'derived from geometry' framework [96, 97], be interpreted as 'periods' of an algebraic variety closely related to the integrand). We have many other remarkable properties. For instance, the operators are often (always?) homomorphic to their formal adjoint. (This is related to the occurrence of selected differential Galois groups.) All these remarkable properties correspond to a differential algebra description of these structures. Finally, we have also other properties of more *arithmetic* and *algebraic geometry* nature. The series expansions of these holonomic functions are often *globally bounded* [62], which means that they can be recast (after rescaling) into series expansions of several variables with *integer* coefficients. This raises the question of the '*modularity*' in these problems [98, 99]. Along this 'modularity' line, beyond the occurrence of many *modular forms* [96, 100], we also see the emergence of *Calabi–Yau ODEs*. From a differential algebra perspective, the emergence of Calabi–Yau structures [101] is not clear. In some integrability framework, the argument that Calabi–Yau manifolds are, after K3 surfaces, the 'next' generalization of elliptic curves, remains an insufficient and much too general argument.

Let us inject, beyond the differential algebra description of these structures, some *birational* algebraic geometry ideas. In lattice statistical mechanics, the models defined by local Boltzmann weights depending on several complex variables are known to have, generically, an *infinite set of birational symmetries* generated by the combination of the so-called *inversion relations* [102, 103].

It has been shown that n-fold integrals like $\chi^{(n)}$s of the Ising model present some nice inversion relation functional equations in the *anisotropic case* [104] (several complex variables):

$$\chi^{(n)}(K_1, K_2) = \chi^{(n)}\Big(K_1, K_2 + i\frac{\pi}{2}\Big), \qquad (65)$$

inherited from the same inversion relation functional equation on the full anisotropic susceptibility.

Since the previous ideas underline the crucial role of the integrand of the n-fold integrals as the algebraic variety from which 'everything', in principle, can be deduced [49, 96, 97], it is interesting to see if this integrand, itself, is not going to be invariant (resp. covariant) by these birational involutions (and, thus, by the composition of these birational involutions) when we keep the integration variables fixed. One can verify that this is actually the case for the integrand of the anisotropic $\chi^{(n)}$s of the Ising model.

Unfortunately, the group of *birational transformations* of the Ising model is a finite set of transformations. However, for generic models, one can easily imagine being in a situation where the integrand of the n-fold integrals of *several complex variables* emerging in these models will be invariant (resp. covariant) by an *infinite set of birational transformations* [5].

We will thus have a natural emergence (in lattice statistical mechanics) of *algebraic varieties with an infinite set of birational symmetries* [5]. These algebraic varieties have zero canonical class, *Kodaira dimension zero*. We, now, *understand the emergence of Calabi–Yau manifolds in these problems*: Abelian varieties and Calabi–Yau manifolds (in dimension 1, elliptic curves; in dimension 2, complex tori and K3 surfaces) have *Kodaira dimension zero*[26].

One can expect that the singular varieties (like (9) or (12)) will have to be invariant by the (generically infinite) set of birational transformations generated by the inversion relations. When the singular manifolds are algebraic curves, the existence of a (generically infinite) set of birational automorphisms for the algebraic curves implies that the curves are, necessarily, genus 0 or 1 [5]. This enables us to understand[27] the emergence of remarkable structures like the two-parameter family of elliptic curves (55). Actually, this is the way many singular varieties have been discovered on many lattice statistical mechanics models (see [16, 17, 19, 20]). This birational invariance fits quite well with the interpretation of the singular variety (12), as the discriminant of a two-parameter family of Calabi–Yau threefolds.

7. Conclusion

In the theory of critical phenomena (renormalization group, etc), singularities are often seen as fixed points of a 'dynamical system' called renormalization [105], and one takes for granted, with a (*lex parsimoniae*) simplicity prejudice, that these singularities are isolated points or smooth manifolds (hopefully algebraic varieties [8–10, 106–108] if one has an integrability prejudice as well). In the theory of discrete dynamical systems, a totally opposite prejudice

[26] Zero canonical class, corresponding to admitting flat metrics and Ricci flat metrics, respectively.

[27] *Cum grano salis*: in the (free-fermion) Ising case the birational transformations generated by the two inversion relations form a *finite set* [109, 110], which allows, in principle, higher genus curves. One must imagine the Ising model as a subcase of a larger model with n-fold integrals, where one would recover a (generic) infinite set of birational transformations.

exists like the belief in a frequent occurrence of strange attractors for the set of fixed points of many 'dynamical systems'. Singularity theory in mathematics, and in particular Arnolds's viewpoint [38], is a perfect illustration that the set of singular points should actually correspond to much more involved manifolds than what is expected in the mainstream doxa of critical phenomena.

We have performed some kind of 'deconstruction'[28] of the concept of singularities in lattice statistical mechanics. The sets of singularities are much more complex sets of points than physicists tend to believe (see figures 1–4 of [49]).

The mathematician's viewpoint that singularities are much more complex than what physicists believe with their (*lex parsimoniae*) simplicity optimism is the correct viewpoint. On the other side, the mathematician's viewpoint that nothing serious and/or rigorous can be done with several complex variables is too pessimistic: within that viewpoint, singularities are seen as too involved to analyze, impossible to localize (of course outside the hypergeometric series framework), or simply, not a well-defined concept. Even in the case of several complex variables, many singular manifolds conjectured by physicists, in particular Wu [8–10], turned out to be true singular varieties of lattice models, because physicists are (sometimes without being fully conscious) often working with *holonomic* (D-finite) functions of several complex variables.

Focusing on the full susceptibility χ of the (anisotropic) Ising model and on the holonomic $\chi^{(n)}$s, we have obtained singular manifolds of the linear partial differential systems of $\chi^{(n)}$s. The fact that these singular manifolds *depend on the spectral parameter* of this Yang–Baxter integrable model is a strong indication that these $\chi^{(n)}$s are *highly composite* objects (even if the exact expression of these singular varieties remains simple enough for the first $\chi^{(n)}$s). Furthermore, the fact that most of these singular manifolds are *not genus-zero curves* shows that the series of the anisotropic $\chi^{(n)}$s, despite all their remarkable properties, *cannot be reduced to the hypergeometric series*.

In the case of the full susceptibility χ of the (anisotropic) Ising model, we seem to have the following situation: among the quite large, and rich, set of singular varieties of the linear ODEs of $\chi^{(n)}$s, there is a restricted set (see (54), (55)) of singular varieties which actually correspond to zeros of the (anisotropic) partition function, and, at the same time, correspond to singularities of the linear PDEs of $\chi^{(n)}$s. This set could correspond (by analogy with the isotropic case) to *singularities of the series expansions* of $\chi^{(n)}$s. A fundamental idea to keep in mind is that it is crucial to make a difference between the singularities of the (series expansions of the) D-finite functions and the singularities[29] of the linear partial differential systems for these functions.

It would be interesting to see if, inside some reasonable theoretical physics framework, similar results[30] can also be obtained for other *non-holonomic* functions of *several* complex variables that decompose into an infinite set of holonomic (D-finite) functions.

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[28] Using Derrida's wording.

[29] The singular manifolds seem to have, in the case of *n*-fold integrals of algebraic integrand, a projective invariant interpretation as a discriminant of the algebraic varieties associated with the integrand.

[30] With the problem that the results seem, at first sight, to depend on the decomposition in an infinite sum of holonomic functions.

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Appendix A. The nine formal solutions of the Picard–Fuchs system 'above' the Calabi–Yau ODE (4)

Let us find the 'formal solutions' around $(x, y) = (0, 0)$ of the PDE system (5) 'above' the Calabi–Yau ODE (4). One plugs, in (5), the series

$$\sum_{j=0} \sum_{k=0}^{j} \mathcal{H}_{j,k}(x, y) \cdot \ln(x)^k \ln(y)^{j-k}, \qquad (A.1)$$

where $\mathcal{H}_{j,k}(x, y)$ are the series in x and y, and solves the system term by term. Collecting on the non-fixed coefficients, one finds $S_0 = H_0(x, y)$ and

$$
\begin{aligned}
S_1 &= H_0(x, y) \cdot \ln(x) + H_1(x, y), \qquad S_2 = H_0(x, y) \cdot \ln(y) + H_1(y, x),\\
S_3 &= H_0(x, y) \cdot \ln(x)^2 + 2H_1(x, y) \cdot \ln(x) + H_2(x, y),\\
S_4 &= H_0(x, y) \cdot \ln(y)^2 + 2H_1(y, x) \cdot \ln(y) + H_2(y, x),\\
S_5 &= H_0(x, y) \cdot \ln(x) \cdot \ln(y) + H_1(y, x) \cdot \ln(x) + H_1(x, y) \cdot \ln(y) + H_3(x, y),\\
S_6 &= H_0(x, y) \cdot \ln(x)^2 \cdot \ln(y) + 2H_1(x, y) \cdot \ln(x) \cdot \ln(y) + H_1(y, x) \cdot \ln(x)^2\\
&\quad + 2H_3(x, y) \cdot \ln(x) + H_2(x, y) \cdot \ln(y) + H_4(x, y),\\
S_7 &= H_0(x, y) \cdot \ln(x) \cdot \ln(y)^2 + 2H_1(y, x) \cdot \ln(x) \cdot \ln(y) + H_1(x, y) \cdot \ln(y)^2\\
&\quad + 2H_3(x, y) \cdot \ln(y) + H_2(y, x) \cdot \ln(x) + H_4(y, x),\\
S_8 &= H_0(x, y) \cdot \ln(x)^2 \cdot \ln(y)^2 + 2H_1(y, x) \cdot \ln(x)^2 \cdot \ln(y) + 2H_1(x, y) \cdot \ln(x) \cdot \ln(y)^2\\
&\quad + 4H_3(x, y) \cdot \ln(x) \cdot \ln(y) + H_2(y, x) \cdot \ln(x)^2 + H_2(x, y) \cdot \ln(y)^2\\
&\quad + 2H_4(y, x) \cdot \ln(x) + 2H_4(x, y) \cdot \ln(y) + H_5(x, y),
\end{aligned} \qquad (A.2)
$$

where (only the first terms of the series are given)

$$
\begin{aligned}
H_0(x, y) &= 1 + 6\,(x + y) + (90\,(x^2 + y^2) + 720\,xy) + \cdots,\\
H_1(x, y) &= (15\,x + 33\,y) + \left(\frac{513}{2}x^2 + 3132\,xy + \frac{1323}{2}\,y^2\right) + \cdots,\\
H_2(x, y) &= (108\,y - 18\,x) - \left(\frac{279}{2}x^2 - 6120xy - 3654y^2\right) + \cdots,\\
H_3(x, y) &= 9 \cdot (x + y) + \left(\frac{2709}{4}x^2 + 3960\,xy + \frac{2709}{4}\,y^2\right) + \cdots,\\
H_4(x, y) &= -(90\,x + 162\,y) - \left(\frac{8505}{4}\,x^2 + 11\,178xy + \frac{6237}{4}\,y^2\right) + \cdots,\\
H_5(x, y) &= 324 \cdot (x + y) - \left(\frac{14\,931}{4}(x^2 + y^2) - 6912xy\right) + \cdots.
\end{aligned}
$$

There are *nine solutions* for the system (5). One notes that H_0, H_3 and H_5 are symmetric in x, y, while H_1, H_2 and H_4 are not symmetric in x, y. For the formal solutions, S_0, S_5 and S_8 are symmetric in x, y, and the six others are pairwise symmetric. These nine independent formal solutions are the solutions of the PDE system (5), and thus of the order-9 differential operator $\tilde{\Omega}_x$ and its (x, y)-symmetric $\tilde{\Omega}_y$.

Note however that the linear differential operator $\tilde{\Omega}_x$ has been constructed from the PDE system (5) and factorizes as written in (17); it then might be that $H_0(x, y)$ is a solution of only the right factor operator $L_6(x, y)$. Indeed, plugging a series

$$\sum_{n,m} c_{n,m} \cdot x^n\, y^m, \qquad\qquad c_{n,m} = \; c_{m,n}, \tag{A.3}$$

into $L_6(x, y)$ and solving term by term, one obtains (up to the overall $c_{0,0}$) the double hypergeometric series $H_0(x,\, y)$. The solutions of $L_6(x, y)$ can be expressed in terms of the previous formal solutions (A.2):

$$S_0, \quad S_1, \quad S_2, \quad S_3 - S_4, \quad S_5 + \frac{S_4}{2}, \quad S_6 + S_7. \tag{A.4}$$

Appendix B. Factorization (17) of the order-9 operator $\tilde{\Omega}_x$

The order-9 operator $\tilde{\Omega}_x$ of subsection 3.2 factorizes (see (17)) into three order-1 operators and the order-6 operator $L_6(x,\, y)$:

$$L_6(x,\, y) = \frac{1}{p_6(x,\, y)} \cdot \sum_{n=0}^{6} p_n(x,\, y) \cdot D_x^n. \tag{B.1}$$

The three order-1 operators are encoded by three rational functions of x and y, namely $\tilde{r}_1(x,\, y)$, $\tilde{r}_2(x,\, y)$ and $\tilde{r}_3(x,\, y)$. These polynomials factorize (see (19)) and thus $\tilde{r}_i(x,\, y)$s reduce to the expressions of four polynomials with integer coefficients $\mathcal{P}_9(x,\, y)$, $\mathcal{P}_6(x,\, y)$, q_1 and q_2, where $\mathcal{P}_9(x,\, y)$ is the polynomial of the apparent singularities of the order-9 operator $\tilde{\Omega}_x$ and where $\mathcal{P}_6(x,\, y)$ is the polynomial of the apparent singularities of the order-6 operator $L_6(x,\, y)$.

These polynomials read

$$\begin{aligned}\mathcal{P}_9(x,\, y) = {} & 2^4 \times 3^{18} \times x^6 - 2 \times 3^{16} \times (31\,951 + 1602\,072\, y) \times x^5 \\ & + 3^{13} \times (14\,397\,329 + 913\,784\,868\, y + 17\,712\,588\,816\, y^2) \times x^4 \\ & + 3^9 \times (2986\,814\,425 + 60\,616\,383\,939\, y - 1350\,750\,590\,172\, y^2 \\ & - 24\,695\,209\,500\,192\, y^3) \times x^3 + 3^7 \times (5310\,925\,151 \\ & - 333\,452\,529\,387\, y - 14\,254\,789\,072\,275\, y^2 \\ & + 241\,096\,254\,564\,492\, y^3 + 7702\,353\,325\,801\,296\, y^4) \times x^2 \\ & - 81 \times (27\, y - 1) \times (39\,319\,888\,296\,092\,688\, y^4 + 122\,020\,942\,792\,986\, y^3 \\ & - 111\,685\,613\,173\,821\, y^2 + 22\,118\,310\,900\, y + 86\,524\,357\,339) \times x \\ & + 2^4 \times 5^3 \times (10\,827\, y + 364)^3 \times (27\, y - 1)^3,\end{aligned} \tag{B.2}$$

$$\begin{aligned}\mathcal{P}_6(x,\, y) = {} & 387\,420\,489 \times (x^2 - 142\, xy + 343\, y^2) \times (x + y)^4 - 43\,046\,721 \\ & \times (x + y) \times (89\, x^4 - 196\, y^4 - 823\, xy^3 + 13\,287\, x^2y^2 - 3493\, x^3y) \\ & + 1594\,323 \times (3482\, x^4 + 662\, xy^3 + 2972\, x^3y - 427\, y^4 + 25\,365\, x^2y^2) \\ & + 19\,683 \times (33\,307\, x^3 - 1784\, y^3 - 14\,487\, xy^2 + 44\,904\, x^2y) - 2187 \\ & \times (27\,394\, x^2 - 88\, xy - 671\, y^2) \quad + 162 \times (1325\, x + 242\, y) \; - 1331,\end{aligned} \tag{B.3}$$

$$\begin{aligned}q_1 = {} & 4 \times 3^{18} \times (x^6 + 113\,061\,462\, xy^5 + 4560\, x^5y - 8876\,482\, x^3y^3 + 284\,847\, x^4y^2 \\ & - 52\,726\,107\, x^2y^4 + 28\,140\,175\, y^6) \\ & + 3^{16} \times (4108\, x^5 - 11\,112\,875\, x^3y^2 + 587\,276\, x^4y - 105\,291\,883\, xy^4 \\ & + 14\,516\,200\, y^5 - 4491\,914\, x^2y^3) \\ & + 3^{13} \times (198\,311\, x^4 \; - 370\,624\,786\, xy^3 \; + 6765\,614\, x^3y\end{aligned}$$

$$- 130\,714\,000\, y^4 + 116\,112\,144\, x^2 y^2)$$
$$+ 3^9 \times (18\,879\,841\, x^3 - 64\,727\,000\, y^3 + 773\,936\,148\, xy^2 + 17\,519\,934\, x^2 y)$$
$$- 3^7 \times (45\,403\,057\, x^2 - 221\,205\,178\, xy - 141\,045\,500\, y^2)$$
$$- 567 \times (22\,002\,263\, x - 1112\,800\, y) - 145\,745\,600,$$

$$q_2 = 774\,840\,978 \times (x^6 - 841\,926\, xy^5 - 462\, x^5 y - 341\,728\, x^3 y^3 + 32\,721\, x^4 y^2$$
$$+ 810\,681\, x^2 y^4 + 98\,245\, y^6)$$
$$- 43\,046\,721 \times (223\, x^5 + 54\,121\, x^3 y^2 - 47\,245\, x^4 y - 613\,336\, xy^4$$
$$- 68\,810\, y^5 + 20\,707\, x^2 y^3)$$
$$+ 1594\,323 \times (22\,489\, x^4 + 1358\,236\, xy^3 + 304\,861\, x^3 y$$
$$- 250\,820\, y^4 - 1645\,923\, x^2 y^2)$$
$$+ 19\,683\, (415\,049\, x^3 - 505\,660\, y^3 - 4725\,138\, xy^2 + 65\,103\, x^2 y)$$
$$+ 10\,935 \times (229\,157\, x^2 - 163\,880\, xy + 68\,006\, y^2)$$
$$+ 162 \times (492\,079\, x + 45\,925\, y) - 440\,440.$$

Appendix C. Alternative linear differential operator for the double hypergeometric series

Recalling the double hypergeometric series (1), $H_0(x,\, c\,x)$ is a solution of an order-6 c-dependent linear differential operator

$$W_6 = (1 + 162 \cdot (c+1) \cdot x)\,(1 - 81 \cdot (c+1) \cdot x + 2187 \cdot (c^2 - 7\,c + 1) \cdot x^2 - 19\,683 \cdot (c+1)^3 \cdot x^3\,) \cdot x^4 \cdot D_x^6 + \cdots. \qquad \text{(C.1)}$$

In the $c = 1$ limit, this order-6 operator becomes the direct sum of the order-2 linear differential operator

$$\theta^2 - 3\,x \cdot (3\,\theta + 1) \cdot (3\,\theta + 2),$$

with the hypergeometric function solution

$${}_2F_1\left(\left[\frac{1}{3}, \frac{2}{3}\right], [1];\; -27\,x\right), \qquad \text{(C.2)}$$

and of the order-4 Calabi–Yau ODE (4), with the analytic solution (3), which can be written as the Hadamard product [111]:

$${}_2F_1\left(\left[\frac{1}{3}, \frac{2}{3}\right], [1];\; -27\,x\right) \star \left(\frac{1}{1-4\,x} \cdot {}_2F_1\left(\left[\frac{1}{3}, \frac{2}{3}\right], [1];\; -\frac{27 \cdot x}{(1-4\,x)^3}\right)\right).$$

In the (less natural) $c = 0$ limit, this order-6 linear differential operator is the product of homomorphic operators:

$$W_6(c=0) = N_2 \cdot M_2 \cdot L_2, \qquad \text{(C.3)}$$

where L_2 has the hypergeometric function solution

$${}_2F_1\left(\left[\frac{1}{3}, \frac{2}{3}\right], [1];\; 27\,x\right). \qquad \text{(C.4)}$$

In the $c \to \infty$ limit, this order-6 operator degenerates into the direct sum:

$$(3 \cdot \theta + 1) \oplus (3 \cdot \theta + 2) \oplus (3 \cdot \theta + 4) \oplus (3 \cdot \theta + 5) \oplus (3 \cdot \theta + 7) \oplus (3 \cdot \theta + 8).$$

Appendix D. Another series of two complex variables

D.1. Double hypergeometric series

Without the factor 64, the results for (30) in subsection (4.2) correspond to the double hypergeometric series

$$K(x,\, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\alpha)_n^3 \cdot (\beta)_m^3 \cdot (\beta')_{m+n}}{(\gamma)_{m+n}^3 \; n! \; m!} \cdot x^n \cdot y^m,$$

where $(\alpha)_n$ is the usual Pochhammer symbol. The double hypergeometric series $K(x,\, y)$ is a *Kampé de Fériet* function [58–61]

$$F^{1,3,3}_{3,0,0}([\beta'],\, [\alpha,\, \alpha,\, \alpha],\, [\beta,\, \beta,\, \beta];\, [\gamma,\, \gamma,\, \gamma],\, -,\, -;\, x,\, y). \qquad \text{(D.1)}$$

The singularity varieties of (D.1) are *independent* of the parameters α, β, β', γ, and are $x \cdot (1-x) \cdot (1-y) \cdot (y-x) = 0$, together with

$$y^2\, x^2 \;\; - 2\, x\, y \cdot (y+x) \;\; + (x-y)^2 \; = \; 0, \qquad \text{(D.2)}$$

in agreement, in the $\alpha = \beta = \beta' = 1/2$, $\gamma = 1$ limit, with (34), taking into account the rescaling $(x,\, y) \,\rightarrow\, (64\, x,\, 64\, y)$.

D.2. Other double hypergeometric series

Introducing the other double hypergeometric series

$$K_2(x,\, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\alpha)_n^M \cdot (\beta)_m^M \cdot (\beta')_{m+n}}{(\gamma)_{m+n}^M \; n! \; m!} \cdot x^n \cdot y^m. \qquad \text{(D.3)}$$

It is also a *Kampé de Fériet* function [58–61]

$$F^{1,M,M}_{M,0,0}([\beta'],\, [\alpha,\, \ldots,\, \alpha],\, [\beta, \ldots, \beta];\, [\gamma,\, \ldots,\, \gamma],\, -,\, -;\, x,\, y). \qquad \text{(D.4)}$$

Let us restrict, in the following, to $\alpha = \beta = \beta' = 1/2$ and $\gamma = 1$.

The singularity varieties of the PDE system are actually different from (D.2) and depend on M. For $M = 2$ and $M = 4$, they read respectively

$$(x+y)^2 \, - \, x^2\, y^2 \, = \, 0, \qquad\quad (x+y - \, x\,y)^3 \, + \, 27\, x^2\, y^2 \, = \, 0. \qquad \text{(D.5)}$$

More generally, for M an even integer, besides the conditions $x \cdot (1-x) \cdot (1-y) = 0$, the singular manifold reads an algebraic curve of parametrization

$$x = t^{M-1}, \qquad\quad y = \left(\frac{-\,t}{1\,-t}\right)^{M-1}, \qquad \text{(D.6)}$$

or equivalently

$$x = \left(\frac{1}{2} \,+ v\right)^{1-M}, \qquad\quad y = \left(\frac{1}{2} \,- v\right)^{1-M}, \qquad \text{(D.7)}$$

that can be thought of as a 'Fermat-like' curve:

$$x^{\frac{1}{1-M}} \, + \, y^{\frac{1}{1-M}} \, = 1. \qquad \text{(D.8)}$$

For $M = 3$, we have (D.2) and for $M = 5$, we have (besides the conditions $x \cdot (1-x) \cdot (1-y) \cdot (y-x) = 0$) the singular variety

$$(x+y\, + x\, y)^4 - 136\, x^2\, y^2 \cdot (x+y\, + x\,y) - 8\, x\, y \cdot (x+1+y)\, (x^2+y^2)$$
$$- 8\, x^2\, y^2 \cdot (x+y)\, (x\,y - 1) = 0. \qquad \text{(D.9)}$$

More generally, for M an odd integer, besides the conditions $x \cdot (1-x) \cdot (1-y) \cdot (y-x) = 0$, the singular manifold reads as an algebraic curve of parametrization

$$x = t^{M-1}, \qquad y = \left(\frac{-t}{1-t}\right)^{M-1}, \tag{D.10}$$

or equivalently

$$x = \left(-\frac{1}{2} + v\right)^{1-M}, \qquad y = \left(-\frac{1}{2} - v\right)^{1-M}, \tag{D.11}$$

that can be thought of as a 'Fermat-like' curve:

$$x^{\frac{1}{1-M}} + y^{\frac{1}{1-M}} + 1 = 0. \tag{D.12}$$

D.3. Differential operators restricted to singular varieties

Let us restrict ourselves to the *singular variety* (D.2) for $M = 3$, using the rational parametrization (D.10), that is $(x, y) = (t^2, (t/(1-t))^2)$. The double series expansion (D.3) becomes a series expansion in the t variable which is a solution of the order-4 linear differential operator ($D_t = d/dt$):

$$\begin{aligned}\mathcal{C}_4 = t^3 \cdot (t-1)(2t+1)(t+2)(t^2+t+1)^2 (t+1)^4 \cdot D_t^4 \\ +2t^2 \cdot (t^2+t+1) \cdot (t+1)^3 \cdot c_3(t) \cdot D_t^3 + t \cdot (t+1)^2 \cdot c_2(t) \cdot D_t^2 \\ +2(t+1) \cdot c_1(t) \cdot D_t + 2t \cdot (t+2)(t^2+t+1)^4,\end{aligned} \tag{D.13}$$

where

$$c_3(t) = 10t^6 + 32t^5 + 39t^4 + 20t^3 - 17t^2 - 24t - 6,$$
$$c_2(t) = 50t^9 + 243t^8 + 588t^7 + 903t^6 + 885t^5 + 501t^4 + 33t^3 - 174t^2 - 99t - 14,$$
$$c_1(t) = 15t^{10} + 82t^9 + 228t^8 + 411t^7 + 531t^6 + 513t^5 + 333t^4 + 99t^3 - 12t^2 - 12t - 1.$$

This 'critical' order-4 operator $\mathcal{C}_4$ is such that its *exterior square* is a linear differential operator of order 5 (and not 6 as it should be for a generic order-4 operator). This condition that the exterior square is of order 5 is called the 'Calabi–Yau condition': it is one of the conditions defining Calabi–Yau ODEs [53, 112–114]. Related to this exterior square condition one also has the property that this order-4 operator $\mathcal{C}_4$ is homomorphic to its adjoint, up to a conjugation by the polynomial $(x+1)^3 (x^2+x+1)^3$.

Note that the limit $y = x$, yielding to the Calabi–Yau operator (28) (also such that its *exterior square* is a linear differential operator of order *five*), is *actually a singular limit* of the Picard–Fuchs system.

Similarly, let us restrict ourselves to the *singular variety* (D.5) for $M = 2$, using the rational parametrization (D.6), namely $(x, y) = (t, -t/(1-t))$. The double series expansion (D.3) becomes a series expansion in the t variable which is the solution of the order-3 linear differential operator ($D_t = d/dt$):

$$\mathcal{C}_3 = D_t^3 + \frac{3}{2} \cdot \frac{(3t-2)}{t(t-1)} \cdot D_t^2 + \frac{1}{4} \cdot \frac{13t^2 - 16t + 4}{(t-1)^2 \cdot t^2} \cdot D_t + \frac{1}{8} \cdot \frac{t-2}{t \cdot (t-1)^3}.$$

This 'critical' order-3 operator $\mathcal{C}_3$ is such that its *symmetric square* is a linear differential operator of order 5 (and not 6 as it should be for a generic order-3 operator). Related to this last property, one also has the property that this order-3 operator $\mathcal{C}_3$ is homomorphic to its adjoint, up to a conjugation by the rational function $1/x^2/(x-1)$.

This order-3 operator $\mathcal{C}_3$ is, in fact, exactly the symmetric square of

$$16t \cdot (t-1)^2 \cdot D_t^2 + 8 \cdot (3t-2) \cdot (t-1) \cdot D_t + t, \tag{D.14}$$

which has $(1-t)^{1/4} \cdot K(t^{1/2})$ as a solution (K is the complete elliptic integral of the first kind).

Let us now restrict ourselves to the *singular variety* (D.5) for $M = 4$, using the (alternative) rational parametrization

$$x = 8\,t, \qquad y = -\frac{8\,t}{1 - 8\,t}. \tag{D.15}$$

With this parametrization the double series expansion (D.3) becomes a series expansion in the t variable with *integer coefficients*. It is a solution of an order-8 operator, its symmetric square is of order 35 (and not 36 as it should be generically[31]).

For $M = 4$, the double series can also be resummed in one variable and rewritten as

$$\sum_{m=0}^{\infty} \frac{(2m)!^5}{4^{5m} \cdot m!^{10}} \cdot {}_5F_4\left(\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, m + \frac{1}{2}\right], [m+1, m+1, m+1, m+1]; x\right) \cdot y^m,$$

corresponding to the identity

$$\frac{(2m)!^5}{4^{5m} \cdot m!^{10}} \cdot \left(\frac{(1/2)_n^4 \cdot (m+1/2)_n}{n! \cdot (m+1)_n^4}\right) = \frac{(1/2)_n^4 \cdot (1/2)_m^4 \cdot (1/2)_{m+n}}{n! \cdot m! \cdot (1)_{m+n}^4}.$$

More generally, one has the identities

$$\frac{(2m)!^M}{4^{Mm} \cdot m!^{2M}} = \frac{(1/2)_m^M}{m!^M}, \tag{D.16}$$

and

$$\frac{(2m)!^{M+1}}{4^{(M+1)m} \cdot m!^{2(M+1)}} \cdot \left(\frac{(1/2)_n^M \cdot (m+1/2)_n}{n! \cdot (m+1)_n^M}\right) = \frac{(1/2)_n^M \cdot (1/2)_m^M \cdot (1/2)_{m+n}}{n! \cdot m! \cdot (1)_{m+n}^M},$$

and the alternative writing of the double series (D.3), for $\alpha = \beta = \beta' = 1/2$ and $\gamma = 1$, as

$$\sum_{m=0}^{\infty} \frac{(2m)!^{M+1}}{4^{(M+1)m} \cdot m!^{2(M+1)}} \times_{M+1} F_M\left(\left[\frac{1}{2}, \cdots, \frac{1}{2}, m + \frac{1}{2}\right], [m+1, \cdots, m+1]; x\right) \cdot y^m, \tag{D.17}$$

Let us now restrict ourselves to the *singular variety* $y = x$. For $M = 4$ and $M = 5$, the double series expansion (D.3) becomes a series expansion in x that is the solution of an order-6 linear differential operator. For $M = 4$, this order-6 operator is such that its symmetric square is of order 20 (instead of the order 21 one could expect generically). For $M = 5$, this order-6 operator is such that its exterior square is of order 14 (instead of the order 15 one could expect generically).

Appendix E. More Picard–Fuchs systems above Calabi–Yau ODEs

E.1. More Picard–Fuchs system with two variables

Another example is the two-variable Picard–Fuchs system 'above' the order-4 Calabi–Yau operator (see ODE number 18 in [53]))

$$\begin{aligned} \theta^4 \; - 4x \cdot (3\theta^2 + 3\theta + 1) \cdot (2\theta + 1)^2 \\ - 4x^2 \cdot (4\theta + 5) \cdot (4\theta + 6) \cdot (4\theta + 2) \cdot (4\theta + 3) \\ = (1 - 64x) \cdot (1 + 16x) \cdot x^4 \cdot D_x^4 \; + \; \cdots . \end{aligned} \tag{E.1}$$

[31] Its exterior square is of order 28 as it should be for a generic order-8 operator.

The Picard–Fuchs system corresponds to the double series [45]

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(n+m)!^2\ (2m+2n)!}{n!^4\ m!^4} \cdot x^n y^m$$
$$= \sum_{m=0}^{\infty} \frac{(2m)!}{m!^2} \cdot {}_4F_3\left(\left[m+1, m+1, m+1,\ m+\frac{1}{2}\right], [1,\ 1,\ 1];\ 4y\right) \cdot x^n$$
$$= 1\ +2\cdot (x+y)\ +6\cdot (x^2+y^2+16xy)\ +20\,(y+x)\,(x^2+y^2+80xy)$$
$$+\,70\cdot (x^4+y^4+256x^3y+256xy^3+1296x^2y^2+x^4)+\cdots. \tag{E.2}$$

Note that all the coefficients of odd orders in x and y factor $(x+y)$.

The singular variety is the union of $xy\cdot (x-y) = 0$ together with the $(x,\ y)$-symmetric *genus-zero* algebraic curve which reads

$$2^8\cdot (x-y)^4 - 2^8\cdot (x+y)\cdot (x^2+y^2+30xy)$$
$$+\,2^5\cdot (3x^2+3y^2-62xy) - 2^4\cdot (x+y)+1 = 0. \tag{E.3}$$

This genus-zero curve has the following polynomial parametrization:

$$x = \frac{(t-1)^4}{64}, \qquad y = \frac{(t+1)^4}{64}. \tag{E.4}$$

In the $y = x$ limit, the singular variety (E.3) gives $(1-64x)\cdot (1+16x)^2 = 0$, in agreement with the singularities of the order-4 Calabi–Yau operator (E.1).

E.2. Last Picard–Fuchs system with two variables

The last example is the two-variables Picard–Fuchs system 'above' the order-4 Calabi–Yau operator (see ODE number 19 in [53])):

$$529\,\theta^4 - 23\,x\cdot (921\,\theta^4+2046\,\theta^3+1644\,\theta^2+621\,\theta+92)$$
$$-\,x^2\cdot (380\,851\,\theta^4+1328\,584\,\theta^3+1772\,673\,\theta^2+1033\,528\,\theta+221\,168)$$
$$-\,2x^3\cdot (475\,861\,\theta^4+1310\,172\,\theta^3+1028\,791\,\theta^2+208\,932\,\theta-27\,232)$$
$$-\,68\,x^4\cdot (8873\,\theta^4+14\,020\,\theta^3+5139\,\theta^2-1664\,\theta-976)$$
$$+\,6936\,x^5\cdot (3\theta+4)\cdot (3\theta+2)\cdot (\theta+1)^2. \tag{E.5}$$

The Picard–Fuchs system corresponds to the double series [45]

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(n+m)!\ (2n+m)!\ (2m+n)!}{n!^4\ m!^4} \cdot x^n y^m$$
$$= \sum_{m=0}^{\infty} \frac{(2m)!}{m!^2} \cdot {}_4F_3\left(\left[m+1,\ m+\frac{1}{2},\ 2m+1,\ \frac{m+1}{2}\right], [1,\ 1,\ 1];\ 4y\right) \cdot x^n$$
$$= 1\ +2\cdot (x+y)\ +(6x^2+6y^2+72xy)+20\cdot (x+y)\cdot (x^2+y^2+53xy)$$
$$+\ 10\cdot (1120xy^3+1120x^3y+7x^4+7y^4+4860x^2y^2)\ +\ \cdots. \tag{E.6}$$

Note that all the coefficients of odd orders in x and y factor $(x+y)$.

The singular variety is the union of $xy\cdot (x+y) = 0$ together with the $(x,\ y)$-symmetric *genus-zero* algebraic curve which reads

$$27\cdot x^2y^2\cdot (y+x)\ -\ [256\,(x^4+y^4)+304\,xy\cdot (x^2+y^2)+69\,x^2y^2]$$
$$+\,8\cdot (y+x)\cdot [32\,(x^2+y^2)+339\,xy]$$
$$-\,[96\,(x^2+y^2)-1261\,xy]\ +16\cdot (y+x)\ -1 = 0, \tag{E.7}$$

with the simple rational parametrization (see section (5)):

$$(x,\, y) = \Big(\frac{t^4}{(t+1)\,(t+2)\,(2t\,+1)^2},\, \frac{1}{(t+1)\,(t+2)^2\,(2t\,+1)}\Big). \qquad \text{(E.8)}$$

In the $y = x$ limit, this singular variety reduces to $(1 - 54x) \cdot (1 + 11x - x^2)^2 = 0$ in agreement with the singular points of (E.5).

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