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2012 J. Phys. A: Math. Theor. 45 075205

(http://iopscience.iop.org/1751-8121/45/7/075205)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **45** (2012) 075205 (32pp) doi:10.1088/1751-8113/45/7/075205

Diagonal Ising susceptibility: elliptic integrals, modular forms and Calabi–Yau equations

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Received 7 October 2011, in final form 11 December 2011
Published 1 February 2012
Online at stacks.iop.org/JPhysA/45/075205

Abstract
We give the exact expressions of the partial susceptibilities $\chi_d^{(3)}$ and $\chi_d^{(4)}$ for the diagonal susceptibility of the Ising model in terms of modular forms and Calabi–Yau ODEs, and more specifically, ${}_3F_2([1/3, 2/3, 3/2], [1, 1]; z)$ and ${}_4F_3([1/2, 1/2, 1/2, 1/2], [1, 1, 1]; z)$ hypergeometric functions. By solving the connection problems we analytically compute the behavior at all finite singular points for $\chi_d^{(3)}$ and $\chi_d^{(4)}$. We also give new results for $\chi_d^{(5)}$. We see, in particular, the emergence of a remarkable order-6 operator, which is such that its symmetric square has a rational solution. These new exact results indicate that the linear differential operators occurring in the n-fold integrals of the Ising model are not only 'derived from geometry' (globally nilpotent), but actually correspond to 'special geometry' (homomorphic to their formal adjoint). This raises the question of seeing if these 'special geometry' Ising operators are 'special' ones, reducing, in fact systematically, to (selected, k-balanced, ...) ${}_{q+1}F_q$ hypergeometric functions, or correspond to the more general solutions of Calabi–Yau equations.

PACS numbers: 05.50.+q, 05.10.−a, 02.30.Hq, 02.30.Gp, 02.40.Xx
Mathematics Subject Classification: 34M55, 47E05, 81Qxx, 32G34, 34Lxx, 34Mxx, 14Kxx

1. Introduction

The magnetic susceptibility of the Ising model is defined in terms of the two-point spin correlation function as

$$k_BT \cdot \chi = \sum_{M=-\infty}^{\infty} \sum_{N=-\infty}^{\infty} \{\langle \sigma_{0,0}\sigma_{M,N}\rangle - \mathcal{M}^2\}, \qquad (1)$$

where $\mathcal{M}$ is the spontaneous magnetization of the Ising model.

The exact analysis of the Ising model susceptibility is the most challenging and important open question in the study of the Ising model today. This study [1, 2] began in 1973–76 by means of summing the nth particle form factor contribution to the correlation function $\langle\sigma_{0,0}\sigma_{M,N}\rangle$. In these papers, it was shown that for $T < T_c$,

$$k_B T \cdot \chi(t) = (1-t)^{1/4} \cdot \left(1 + \sum_{n=1}^{\infty} \tilde{\chi}^{(2n)}(t)\right), \tag{2}$$

where[6] $t = (\sinh 2E^v/k_B T \sinh 2E^h/k_B T)^{-2}$ and for $T > T_c$ by

$$k_B T \cdot \chi(t) = (1-t)^{1/4} \cdot \sum_{n=0}^{\infty} \tilde{\chi}^{(2n+1)}(t), \tag{3}$$

where $t = (\sinh 2E^v/k_B T \sinh 2E^h/k_B T)^{2}$.

The $\tilde{\chi}^{(n)}$ are given by n-fold integrals. In [2], the integrals for $\tilde{\chi}^{(1)}$ and $\tilde{\chi}^{(2)}$ were evaluated, and since that time many important studies have been conducted [3–5] on the behavior, as $t \to 1$, of the singularities in the complex t-plane [4, 6, 7] and the analytic properties of $\tilde{\chi}^{(n)}$ as a function of t for the isotropic case [8–18] for $n = 3, 4, 5, 6$. These studies are still ongoing.

More recently, it was discovered [19] that if in (1) the sum is restricted to the spins on the diagonal

$$k_B T \cdot \chi_d(t) = \sum_{N=-\infty}^{\infty} \{\langle\sigma_{0,0}\sigma_{N,N}\rangle - \mathcal{M}^2\}, \tag{4}$$

the diagonal susceptibility reads

$$k_B T \cdot \chi_d(t) = (1-t)^{1/4} \cdot \left(1 + \sum_{n=1}^{\infty} \tilde{\chi}_d^{(2n)}(t)\right), \tag{5}$$

for $T < T_c$ and

$$k_B T \cdot \chi_d(t) = (1-t)^{1/4} \cdot \sum_{n=0}^{\infty} \tilde{\chi}_d^{(2n+1)}(t), \tag{6}$$

for $T > T_c$. The $\tilde{\chi}_d^{(n)}(t)$s are n-fold integrals which have a much simpler form than the integrals for $\tilde{\chi}^{(n)}(t)$ but retain all of the physically interesting properties of these integrals.

For $T < T_c$, the integrals $\tilde{\chi}_d^{(2n)}(t)$ read

$$\begin{aligned}\tilde{\chi}_d^{(2n)}(t) = \frac{t^{n^2}}{(n!)^2}\,\frac{1}{\pi^{2n}} \cdot \int_0^1 \cdots \int_0^1 \prod_{k=1}^{2n} \mathrm{d}x_k \cdot \frac{1 + t^n x_1 \cdots x_{2n}}{1 - t^n x_1 \cdots x_{2n}} \\ \times \prod_{j=1}^{n} \left(\frac{x_{2j-1}(1-x_{2j})(1-t x_{2j})}{x_{2j}(1-x_{2j-1})(1-t\,x_{2j-1})}\right)^{1/2} \prod_{1\leqslant j\leqslant n}\,\prod_{1\leqslant k\leqslant n} (1 - t\,x_{2j-1}\,x_{2k})^{-2} \\ \times \prod_{1\leqslant j<k\leqslant n} (x_{2j-1} - x_{2k-1})^2\,(x_{2j} - x_{2k})^2,\end{aligned} \tag{7}$$

where t is given by $t = (\sinh 2E^v/k_B T \sinh 2E^h/k_B T)^{-2}$.

[6] The classical interaction energy of the Ising model is $-\sum_{j,k}\left(E^v\sigma_{j,k}\sigma_{j+1,k} + E^h\sigma_{j,k}\sigma_{j,k+1}\right)$, where $j(k)$ specifies the row (column) of a square lattice and the sum is over all sites of the lattice.

For $T > T_c$, the integrals $\tilde{\chi}_d^{(2n+1)}(t)$ read

$$\tilde{\chi}_d^{(2n+1)}(t) = \frac{t^{n(n+1)}}{\pi^{2n+1} n!(n+1)!} \cdot \int_0^1 \cdots \int_0^1 \prod_{k=1}^{2n+1} \mathrm{d}x_k$$
$$\times \frac{1 + t^{n+1/2} x_1 \cdots x_{2n+1}}{1 - t^{n+1/2} x_1 \cdots x_{2n+1}} \cdot \prod_{j=1}^{n} ((1 - x_{2j})(1 - t\, x_{2j}) \cdot x_{2j})^{1/2}$$
$$\times \prod_{j=1}^{n+1} \Big((1 - x_{2j-1})(1 - t\, x_{2j-1}) \cdot x_{2j-1} \Big)^{-1/2}$$
$$\times \prod_{1 \leqslant j \leqslant n+1} \prod_{1 \leqslant k \leqslant n} (1 - t\, x_{2j-1}\, x_{2k})^{-2}$$
$$\times \prod_{1 \leqslant j < k \leqslant n+1} (x_{2j-1} - x_{2k-1})^2 \prod_{1 \leqslant j < k \leqslant n} (x_{2j} - x_{2k})^2, \qquad (8)$$

where $t = (\sinh 2E^v/k_BT \sinh 2E^h/k_BT)^2$. For these $\tilde{\chi}_d^{(2n+1)}$, it will be convenient, in the following, to use the variable $x = t^{1/2} = \sinh 2E^v/k_BT \sinh 2E^h/k_BT$.

In [19], we found that

$$\tilde{\chi}_d^{(1)}(x) = \frac{1}{1-x}, \qquad \text{and} \qquad \tilde{\chi}_d^{(2)}(t) = \frac{1}{4} \cdot \frac{t}{1-t}, \qquad (9)$$

and that $\tilde{\chi}_d^{(3)}(x)$ and $\tilde{\chi}_d^{(4)}(t)$ are the solutions of differential equations of orders 6 and 8. The corresponding linear differential operators of each are a direct sum of three factors. In both cases, there is a differential equation which was not solved in [19].

In this paper, we complete this study of $\tilde{\chi}_d^{(3)}(x)$ and $\tilde{\chi}_d^{(4)}(t)$ by solving all of the differential equations involved. We then use the solutions of these equations to analytically compute the singular behavior at all of the finite singular points. In this way, we are able to give analytic proofs of the results conjectured in appendix B of [19] by numerical means.

We split the presentation of our results into two parts: the solution of the differential equations and the use of the differential equations to compute the behavior of $\tilde{\chi}_d^{(3)}(x)$ and $\tilde{\chi}_d^{(4)}(t)$ at the singularities. The solution of the differential equations is presented in section 2 for $\tilde{\chi}_d^{(3)}(x)$ and in section 3 for $\tilde{\chi}_d^{(4)}(t)$. In particular, we focus on the difficult problem of solving a particular order-4 operator, to discover, finally, a surprisingly simple result. The linear differential equation for $\tilde{\chi}_d^{(5)}(x)$ is studied in section 4, yielding the emergence of a remarkable order-6 operator. The singular behaviors of $\tilde{\chi}_d^{(3)}(x)$ and $\tilde{\chi}_d^{(4)}(t)$ are given in sections 5 and 6, respectively. This analysis requires that the (global) connection problem to be solved. The details of these computations are given in appendices C and D. We conclude in section 7 with a discussion of the emergence of ${}_{q+1}F_q$ hypergeometric functions, with all these previous results underlying modularity [20, 21] in the Ising model through elliptic integrals, modular forms and Calabi–Yau ODEs [22, 23].

2. Computations for $\tilde{\chi}_d^{(3)}(x)$

It was shown in [19] that $\tilde{\chi}_d^{(3)}(x)$ is annihilated by an order-6 linear differential equation. The corresponding linear differential operator $\mathcal{L}_6^{(3)}$ is a direct sum of irreducible linear differential operators (the indices are the orders):

$$\mathcal{L}_6^{(3)} = L_1^{(3)} \oplus L_2^{(3)} \oplus L_3^{(3)}. \qquad (10)$$

The solution of $\mathcal{L}_6^{(3)}$, which is analytic at $x = 0$, is thus naturally decomposed as a sum

$$\mathrm{Sol}\big(\mathcal{L}_6^{(3)}\big) = a_1^{(3)} \cdot \tilde{\chi}_{d;1}^{(3)}(x) + a_2^{(3)} \cdot \tilde{\chi}_{d;2}^{(3)}(x) + a_3^{(3)} \cdot \tilde{\chi}_{d;3}^{(3)}(x), \qquad (11)$$

where $\tilde{\chi}^{(3)}_{d;j}$ are analytic at $x = 0$. The solutions $\tilde{\chi}^{(3)}_{d;1}(x)$ and $\tilde{\chi}^{(3)}_{d;2}(x)$ were explicitly found in [19] to be

$$\tilde{\chi}^{(3)}_{d;1}(x) = \frac{1}{1-x} \quad \text{and} \tag{12}$$

$$\tilde{\chi}^{(3)}_{d;2}(x) = \frac{1}{(1-x)^2} \cdot {}_2F_1([1/2, -1/2], [1]; x^2) - \frac{1}{1-x} \cdot {}_2F_1([1/2, 1/2], [1]; x^2), \tag{13}$$

where one notes the occurrence of $\tilde{\chi}^{(1)}_d(x) = \tilde{\chi}^{(3)}_{d;1}(x)$ in $\tilde{\chi}^{(3)}_d(x)$. The last term, $\tilde{\chi}^{(3)}_{d;3}(x)$, is annihilated by the order-3 linear differential operator

$$L_3^{(3)} = D_x^3 + \frac{3}{2}\frac{n_2(x)}{d(x)} \cdot D_x^2 + \frac{n_1(x)}{(x+1)(x-1)\cdot x \cdot d(x)} \cdot D_x + \frac{n_0(x)}{(x+1)\,(x-1)^2 \cdot x \cdot d(x)},$$

where

$$\begin{aligned} d(x) &= (x+2)\,(1+2x)\,(x+1)\,(x-1)\,(1+x+x^2)\cdot x, \\ n_0(x) &= 2x^8 + 8x^7 - 7x^6 - 13x^5 - 58x^4 - 88x^3 - 52x^2 - 13x + 5, \\ n_1(x) &= 14x^8 + 71x^7 + 146x^6 + 170x^5 + 38x^4 - 112x^3 - 94x^2 - 19x + 2, \\ n_2(x) &= 8x^6 + 36x^5 + 63x^4 + 62x^3 + 21x^2 - 6x - 4. \end{aligned} \tag{14}$$

The linear differential operator $L_3^{(3)}$ has the following regular singular points and exponents (z denotes the local variable $x - x_s$ of the expansion around a singular point x_s):

$$\begin{array}{lll} 1+x+x^2 = 0, & \rho = 0,\ 1,\ 7/2 \ \rightarrow & z^{7/2}, \\ x = 0, & \rho = 0,\ 0,\ 0 \ \rightarrow & \ln(z)^2 \ \text{terms}, \\ x = 1, & \rho = -2,\ -1,\ 1 \rightarrow & z^{-2},\ z^{-1}, \\ x = -1, & \rho = 0, 0, 0 \ \rightarrow & \ln(z)^2 \ \text{terms}, \\ x = \infty, & \rho = 1,\ 1,\ 1 \ \rightarrow & \ln(z)^2 \ \text{terms}. \end{array} \tag{15}$$

The last column shows the maximum $\ln(z)$-degree occurring in the formal solutions of $L_3^{(3)}$, with z being the local variable of the expansion.

The singularities at $x = -2, -1/2$ are apparent.

By the use of the command dsolve in Maple, we found in [13] that the solution of $L_3^{(3)}[\tilde{\chi}^{(3)}_{d;3}] = 0$, which is analytic at $x = 0$, is

$$\tilde{\chi}^{(3)}_{d;3}(x) = \frac{(1+2x)\cdot(x+2)}{(1-x)\cdot(x^2+x+1)} \cdot {}_3F_2([1/3, 2/3, 3/2], [1, 1]; Q), \tag{16}$$

where the pullback Q reads

$$Q = \frac{27}{4}\frac{(1+x)^2 \cdot x^2}{(x^2+x+1)^3}. \tag{17}$$

Now the coefficients $a_i^{(3)}$, in the sum decomposition (11) of $\tilde{\chi}^{(3)}_d(x)$, can be fixed by expanding and matching the rhs of (11) with the expansion of $\tilde{\chi}^{(3)}_d(x)$. This gives

$$\tilde{\chi}^{(3)}_d(x) = \tfrac{1}{3}\cdot\tilde{\chi}^{(3)}_{d;1}(x) + \tfrac{1}{2}\cdot\tilde{\chi}^{(3)}_{d;2}(x) - \tfrac{1}{6}\cdot\tilde{\chi}^{(3)}_{d;3}(x). \tag{18}$$

By the use of a family of identities on ${}_3F_2$ hypergeometric functions [24] (see equation (27) p 499) expression (16) of $\tilde{\chi}^{(3)}_{d;3}(x)$ reduces to

$$\begin{aligned} \tilde{\chi}^{(3)}_{d;3}(x) = \frac{(1+2x)\cdot(x+2)}{(1-x)\cdot(x^2+x+1)} \cdot \Big[& {}_2F_1([1/6, 1/3], [1]; Q)^2 \\ & + \frac{2Q}{9} \cdot {}_2F_1([1/6, 1/3], [1]; Q) \cdot {}_2F_1([7/6, 4/3], [2]; Q)\Big]. \end{aligned} \tag{19}$$

It is instructive, however, to discuss further the reason why $\tilde{\chi}^{(3)}_{d;3}(x)$ has this solution in terms of ${}_2F_1$ functions.

2.1. Differential algebra structures and modular forms

From a differential algebra viewpoint, the linear differential operator $L_3^{(3)}$ can be seen to be homomorphic[7] to its formal adjoint:

$$L_3^{(3)} \cdot \mathrm{adjoint}(T_2) \quad = \quad T_2 \cdot \mathrm{adjoint}(L_3^{(3)}), \tag{20}$$

where

$$T_2 = \frac{(1+x+x^2)}{(1-x)^4} \cdot D_x^2 \;\; + \;\; \frac{m_1(x)}{(x+1)\,(x-1)^5\,(2\,x+1)\,(x+2)\cdot x} \cdot D_x$$
$$- \, \frac{1}{4} \cdot \frac{m_0(x)}{(2\,x+1)\,(x+2)\,(x+1)\,(1+x+x^2)\,(x-1)^6 \cdot x}, \tag{21}$$

and where

$$m_1(x) = 2\,x^6 \, - 6\,x^5 \, - 53\,x^4 \, - 92\,x^3 \, - 81\,x^2 \, - 34\,x \, - 6,$$
$$m_0(x) = 8\,x^8 \, - 4\,x^7 - 222\,x^6 \, - 769\,x^5 \, - 1153\,x^4 - 1341\,x^3 \, - 1129\,x^2 \, - 490\,x \, - 84.$$

Related to (20) is the property that the symmetric square[8] of $L_3^{(3)}$ actually has a (very simple) rational solution $R(x)$. It thus factorizes into an (involved) order-5 linear differential operator and an order-1 operator having the rational solution

$$R(x) = \frac{1+x+x^2}{(x-1)^4}, \qquad \mathrm{Sym}^2(L_3^{(3)}) = L_5 \cdot \Big(D_x - \frac{\mathrm{d}}{\mathrm{d}x}\,\ln(R(x))\Big). \tag{22}$$

In a forthcoming publication, we will show that the homomorphism of an operator with its adjoint naturally leads to a rational solution for its *symmetric square or exterior square* (according to the order of the operator).

Relation (20), or the fact that its symmetric square has a rational solution, means that this operator is not only a globally nilpotent operator [13], but it corresponds to 'special geometry'. In particular, it has a 'special' differential Galois group [27]. We will come back to this crucial point later, in section 3.1 (see (43)).

The operator $L_3^{(3)}$ is in fact homomorphic to the symmetric square of a second-order linear differential operator[9]

$$X_2 = D_x^2 \, + \frac{1}{2} \cdot \frac{(2\,x+1)\cdot(x^2+x+2)}{(1+x+x^2)\cdot(1+x)\cdot x} \cdot D_x \; - \frac{3}{2} \cdot \frac{1}{(1+x+x^2)^2}, \tag{23}$$

since one has the following simple operator equivalence [25] with two order-1 intertwinners:

$$L_3^{(3)} \cdot M_1 \;=\; N_1 \cdot \mathrm{Sym}^2(X_2), \quad \text{with}$$
$$M_1 = \frac{(1+x)\cdot x}{(1-x)^2} \cdot D_x \; + \; \frac{1}{2} \cdot \frac{(1+2\,x)\cdot(x+2)}{(1+x+x^2)\cdot(1-x)},$$
$$N_1 = \frac{(1+x)\cdot x}{(1-x)^2} \cdot D_x \; - \; \frac{1}{2} \cdot \frac{24\,x^5 + 15\,x^4 + 8\,x^6 - 10\,x^3 - 69\,x^2 - 60\,x - 16}{(1-x)^3(1+2\,x)\,(x+2)\,(1+x+x^2)}. \tag{24}$$

The second-order operator X_2 is not homomorphic to the second-order operators associated with the complete elliptic integrals of the first or second kind. However, from (20) and (22), we expect X_2 to be 'special'. This is confirmed by the fact that the solution $\mathrm{Sol}(X_2,\, x)$ of X_2, analytical at $x = 0$, has the integrality property[10]: if one performs a simple rescaling $x \, \rightarrow \, 4\,x$, the series expansion of this solution has *integer* coefficients:

$$\mathrm{Sol}(X_2,\, 4\,x) = \; 1 + 6\,x^2 \, - 24\,x^3 \, + 60\,x^4 \, - 96\,x^5$$
$$+ 120\,x^6 \, - 672\,x^7 \, + 5238\,x^8 \, - 25\,440\,x^9 \, + \, \cdots. \tag{25}$$

[7] For the notion of differential operator equivalence, see [25] and [26].

[8] In general, for an irreducible operator homomorphic to its adjoint, a rational solution occurs for the symmetric square (resp. exterior square) of that operator when it is of odd (resp. even) order.

[9] Finding X_2 (or an operator equivalent to it) can be done by downloading the implementation [28].

[10] See also the concept of 'globally bounded' solutions of linear differential equations by Christol [29].

From this *integrality* property [16, 32], we thus expect the solution of X_2 to be associated with a modular form, and thus, we expect this solution to be a ${}_2F_1$ up to not just one, *but two pullbacks*. Finding these pullbacks is a difficult task, except if the pullbacks are rational functions. Fortunately, we are in this simpler case of rational pullbacks, and consequently, we have been able to find the solution [13] to deduce that the third-order operator $L_3^{(3)}$ is ${}_3F_2$-solvable or ${}_2F_1$-solvable, up to a Hauptmodul [30] pullback (see (16), (19)).

We can make the modular form character of (16) and (19), which is already quite clear from the Hauptmodul form of (17), very explicit by introducing another rational expression, similar to (17)

$$Q_1(x) = \frac{27\,x \cdot (1+x)}{(1\,+\,2\,x)^6}. \tag{26}$$

The elimination of x between $Q = Q(x)$ and $Q_1 = Q_1(x)$ (see (17), (26)) gives a polynomial relation (with integer coefficients) $\Gamma(Q,\,Q_1) = 0$, where the algebraic curve $\Gamma(u,\,v) = 0$ which is, of course, a rational curve, is, in fact, a *modular curve* already encountered [16] with an order-3 operator F_3 which emerged in (the non-diagonal) $\tilde{\chi}^{(5)}$ (see [14]):

$$\begin{aligned} -\,4\,u^3\,v^3\,+12\,u^2\,v^2\cdot(v+u)\,-3\,u\,v\cdot(4\,v^2+4\,u^2\,-127\,u\,v)& \\ +4\,(v+u)\cdot(u^2+v^2\,+83\,u\,v)\,-432\,u\,v = 0.& \end{aligned} \tag{27}$$

The hypergeometric functions we encounter in (19), in the expression of the solution of $L_3^{(3)}$, have actually *two possible pullbacks* as a consequence of the remarkable identity on the *same* hypergeometric function[11]:

$$(1\,+2\,x)\cdot{}_2F_1\Big(\Big[\tfrac{1}{6},\,\tfrac{1}{3}\Big],\,[1];\,Q(x)\Big) = \,(1+x+x^2)^{1/2}\cdot{}_2F_1\Big(\Big[\tfrac{1}{6},\,\tfrac{1}{3}\Big],\,[1];\,Q_1(x)\Big). \tag{28}$$

Other rational parametrizations and pullbacks can also be introduced, as can be seen in appendix A. Relation (28) on ${}_2F_1$ yields other remarkable relations on the ${}_3F_2$ with the two pullbacks Q (see (16)) and Q_1: their corresponding order-3 linear differential operators are homomorphic. Consequently, one deduces, for instance, that ${}_3F_2([1/3, 2/3, 3/2], [1, 1];\,Q_1)$ is equal to the action of the second-order operator U_2 on ${}_3F_2([1/3, 2/3, 3/2], [1, 1];\,Q)$:

$$\begin{aligned} &(x^2\,+x\,+1)^3\cdot(1\,-8\,x\,-8\,x^2)\cdot{}_3F_2([1/3, 2/3, 3/2], [1, 1];\,Q_1) \\ &\qquad = -\,U_2\Big[{}_3F_2([1/3, 2/3, 3/2], [1, 1];\,Q)\Big], \quad \text{where} \end{aligned} \tag{29}$$

$$\begin{aligned} U_2 &= p(x)\cdot\Big(D_x^2\,-\,2\cdot\frac{\mathrm{d}}{\mathrm{d}x}\ln\Big(\frac{x^2\,+x\,+1}{(x\,-1)\,(x\,+2)\,(1+2\,x)}\Big)\cdot D_x\Big) \\ &\qquad +\Big(\frac{1+2\,x}{x^2\,+x\,+1}\Big)^2\cdot q(x), \quad \text{with} \\ p(x) &= x^2\cdot\,(1+x)^2\,(1+2\,x)^2\,(1+8\,x+12\,x^2+8\,x^3+4\,x^4), \\ q(x) &= 8\,x^{10}\,+40\,x^9\,+81\,x^8\,+84\,x^7\,+24\,x^6\,-54\,x^5\,-63\,x^4-18\,x^3\,-2\,x\,-1, \end{aligned} \tag{30}$$

thus generalizing the simple automorphic relation (28).

The *modularity* of these functions can also be seen from the fact that the series expansion of (16), (19) or (28) has the *integrality property* [16]. If one performs a simple rescaling $x \rightarrow\; 4\,x$, their series expansions actually have *integer coefficients* [16, 32]:

$$\begin{aligned} \tilde{\chi}^{(3)}_{d;3}(4\,x) \;=&\; 2\,+20\,x\,+104\,x^2\,+560\,x^3\,+2648\,x^4\,+12\,848\,x^5 \\ &+58\,112\,x^6\,+267\,776\,x^7\,+1181\,432\,x^8\,+5281\,328\,x^9\,+\,\cdots, \end{aligned} \tag{31}$$

[11] Along this line see for instance [31].

or

$$
{}_2F_1\left(\left[\tfrac{1}{6},\ \tfrac{1}{3}\right],\ [1];\ Q(x)\right)[x \rightarrow\ 4\cdot x]\ =\ 1\ + 6\,x^2\ - 24\,x^3\ + 60\,x^4\ - 96\,x^5 \\
+ 120\,x^6\ - 672\,x^7 + 5238\,x^8\ - 25\,440\,x^9\ + 81\,972\,x^{10} +\ \cdots, \tag{32}
$$

which can be turned into *positive integers* if we also change x into $-x$.

This provides more examples of the almost quite systematic occurrence [16] in the Ising model, of (globally nilpotent [33]) linear differential operators associated with *elliptic curves*, either because one gets straightforwardly elliptic integrals, or because one gets operators associated with *modular forms*. For the diagonal susceptibility of the Ising model, are we also going to see the emergence of *Calabi–Yau*-like operators [22, 23] as already discovered in (the non-diagonal) $\tilde{\chi}^{(6)}$ (see [16])?

3. Computations for $\tilde{\chi}_d^{(4)}(t)$

We now turn to the computation of $\tilde{\chi}_d^{(4)}(t)$, whose differential operator $\mathcal{L}_8^{(4)}$ is of order 8 and is a direct sum of three irreducible differential operators [19]:

$$
\mathcal{L}_8^{(4)} = L_1^{(4)}\ \oplus\ L_3^{(4)}\ \oplus\ L_4^{(4)}. \tag{33}
$$

The solution of $\mathcal{L}_8^{(4)}$, analytic at $t = 0$, is thus naturally decomposed as a sum:

$$
\mathrm{Sol}(\mathcal{L}_8^{(4)}) = a_1^{(4)}\cdot\tilde{\chi}_{d;1}^{(4)}(t)\ + a_2^{(4)}\cdot\tilde{\chi}_{d;2}^{(4)}(t)\ + a_3^{(4)}\cdot\tilde{\chi}_{d;3}^{(4)}(t). \tag{34}
$$

The solutions $\tilde{\chi}_{d;1}^{(4)}(t)$ and $\tilde{\chi}_{d;2}^{(4)}(t)$ were explicitly found[12] to be [19]

$$
\tilde{\chi}_{d;1}^{(4)}(t)\ =\ \frac{t}{1-t} \quad \text{and} \tag{35}
$$

$$
\begin{aligned}
\tilde{\chi}_{d;2}^{(4)}(t) &= \frac{9}{8}\cdot\frac{(1\ + t)\cdot t^2}{(1-t)^5}\cdot{}_3F_2\left(\left[\frac{3}{2},\frac{5}{2},\frac{5}{2}\right],[3,\ 3];\ \frac{-4t}{(1-t)^2}\right)\\
&= \frac{1+t}{(1-t)^2}\cdot{}_2F_1([1/2,-1/2],[1];\ t)^2 -\ {}_2F_1([1/2,\ 1/2],[1];\ t)^2\\
&\quad - \frac{2t}{1-t}\cdot{}_2F_1([1/2,1/2],[1];\ t)\ \cdot\ {}_2F_1([1/2,-1/2],[1];\ t).
\end{aligned} \tag{36}
$$

Here, again, one notes the occurrence of $\tilde{\chi}_{d;1}^{(4)}(t)$ which is $\tilde{\chi}_d^{(2)}(t)$ up to a normalization factor. One should be careful that the ${}_3F_2$ closed form (36) for $\tilde{\chi}_{d;2}^{(4)}(t)$, together with the previous exact result (16), may yield a ${}_{q+1}F_q$ with a rational pullback prejudice which has no justification for the moment.

Similar to $L_3^{(3)}$, the order-3 operator for $\tilde{\chi}_{d;2}^{(4)}(t)$ is homomorphic to its adjoint and its symmetric square has a simple rational function solution. The exact expressions (36) for $\tilde{\chi}_{d;2}^{(4)}(t)$ are obtained in a similar way to solution (16) and (19) of $L_3^{(3)}$ in the previous section. We first find [28] that the corresponding linear differential operator is homomorphic to the symmetric square of a second-order operator, which turns out to have complete elliptic integral solutions. The emergence in (36) of an ${}_3F_2$ hypergeometric function with the selected[13] rational pullback $-4t/(1-t)^2$ is totally reminiscent (even if it is not exactly of the same form) of

[12] The first line in (36) can, for instance, be found by directly using the command dsolve in Maple, and the second line follows by the use of identity 520 on page 526 of [24]. This result is also easily obtained by using Maple to directly compute the homomorphisms between an order-3 operator and the operator which annihilates ${}_2F_1([1/2, 1/2], [1]; t)$.
[13] The fundamental role played by such specific pullbacks as *isogenies of elliptic curves* has been underlined in [34].

Kummers's quadratic relation [31, 34], and its generalization to ${}_3F_2$ hypergeometric functions (see relations (4.12) and (4.13) in [35], and (7.1) and (7.4) in [36]), for example,

$$ {}_3F_2\left(\left[1+\alpha-\beta-\gamma,\, \frac{\alpha}{2},\, \frac{\alpha+1}{2}\right],\, [1+\alpha-\beta, 1+\alpha-\gamma];\, \frac{-4t}{(1-t)^2}\right) $$
$$ = (1-t)^\alpha \cdot {}_3F_2([\alpha,\,\beta,\,\gamma],\,[1+\alpha-\beta, 1+\alpha-\gamma];\, t), \qquad (37) $$

which relates *different*[14] ${}_3F_2$ hypergeometric functions. In fact, and similar to (29), we do have an equality between the ${}_3F_2$ hypergeometric function with the pullback $u = -4t/(1-t)^2$ and the *same* ${}_3F_2$ hypergeometric, where the pullback has been changed[15] into $v = -4(1-t)/t^2$,

$$ {}_3F_2\left(\left[\frac{3}{2},\frac{5}{2},\frac{5}{2}\right], [3,\,3];\, \frac{-4(1-t)}{t^2}\right) = V_2\left[{}_3F_2\left(\left[\frac{3}{2},\frac{5}{2},\frac{5}{2}\right],\,[3,\,3];\,\frac{-4t}{(1-t)^2}\right)\right], \qquad (38) $$

where V_2 is a second-order operator similar to the one in (29). The elimination of t in these two pullbacks gives the simple genus zero curve

$$ u^2\, v^2 - 48\, v\, u + 64 \cdot (u + v) = 0, \qquad (39) $$

reminiscent of the simplest *modular equations* [40]. This genus zero curve can also be simply parametrized with $u = -4t/(1-t)^2$ and[16] $v = 4t\cdot(1-t)$. Again, one gets an identity, similar to (38), with another order-2 intertwiner $\mathcal{V}_2$:

$$ {}_3F_2\left(\left[\frac{3}{2},\frac{5}{2},\frac{5}{2}\right], [3,\,3];\, 4t\cdot(1-t)\right) = \mathcal{V}_2\left[{}_3F_2\left(\left[\frac{3}{2},\frac{5}{2},\frac{5}{2}\right],\,[3,\,3];\,\frac{-4t}{(1-t)^2}\right)\right]. \qquad (40) $$

3.1. Computation of $\tilde{\chi}^{(4)}_{d,3}(t)$

The third-term $\tilde{\chi}^{(4)}_{d;3}(t)$ in the sum (64) is the solution analytic at $x = 0$ of the order-4 linear differential operator

$$ L_4^{(4)} = D_t^4 + \frac{n_3(t)}{(t+1)\cdot d_4(t)}\cdot D_t^3 + 2\frac{n_2(t)}{(t^2-1)\cdot t\cdot d_4(t)}\cdot D_t^2 $$
$$ + 2\frac{n_1(t)}{(t^2-1)\cdot t\cdot d_4(t)}\cdot D_t - 3\frac{(t+1)^2}{(t-1)\cdot t^2\cdot d_4(t)}, \qquad (41) $$

where

$d_4(t) = (t^2-10t+1)\cdot(t-1)\cdot t, \qquad n_1(t) = t^4 - 13t^3 - 129t^2 + 49t - 4,$

$n_2(t) = 5t^5 - 55t^4 - 169t^3 + 149t^2 - 28t + 2,$

$n_3(t) = 7t^4 - 68t^3 - 114t^2 + 52t - 5.$

The operator $L_4^{(4)}$ has the following regular singular points and exponents:

$$ \begin{array}{llcl} t = 0, & \rho = 0,0,0,1 & \rightarrow & \ln(z)^3 \quad \text{terms}, \\ t = 1, & \rho = -2,-1,0,1 & \rightarrow & z^{-2},\; z^{-1},\; \ln(z) \quad \text{terms}, \\ t = -1, & \rho = 0,1,2,7 & \rightarrow & z^7\,\ln(z) \quad \text{terms}, \\ t = \infty, & \rho = 0,0,0,1 & \rightarrow & \ln(z)^3 \quad \text{terms}. \end{array} \qquad (42) $$

The last column shows the maximum $\ln(z)$-degree occurring in the formal solutions of $L_4^{(4)}$, with z being the local variable of the expansion. The singularities at the roots of

[14] Note that the Saalschützian difference (54) (see below) of the ${}_3F_2$ on the lhs of (37) is independent of α, β and γ and is equal to $1/2$, in contrast with the rhs.

[15] This amounts to changing t into $1-t$.

[16] This amounts to changing t into $-t/(1-t)$ or $-1/(1-t)$.

$t^2 - 10t + 1 = 0$ are apparent. This order-4 operator (41) is actually homomorphic to its (formal) adjoint:

$$\mathrm{adjoint}(L_2) \cdot L_4^{(4)} = \mathrm{adjoint}\big(L_4^{(4)}\big) \cdot L_2, \tag{43}$$

where L_2 is the order-2 intertwinner:

$$L_2 = \left(D_t - \frac{\mathrm{d}}{\mathrm{d}t}\ln(r(t))\right) \cdot D_t, \quad \text{where}$$
$$r(t) = \frac{(t^2 - 10t + 1)(t + 1)}{t \cdot (t-1)^3}. \tag{44}$$

The remarkable equivalence (43) of operator (41) with its adjoint is related to the fact that the exterior square of (41) has a rational function solution, that is, this exterior square factors into an order-5 operator L_5 and an order-1 operator with a rational function solution (which coincides with $r(t)$ in (44)):

$$\mathrm{Ext}^2\big(L_4^{(4)}\big) = L_5 \cdot \left(D_t - \frac{\mathrm{d}}{\mathrm{d}t}\ln(r(t))\right). \tag{45}$$

In other words, the (irreducible) order-4 operator (41) is not only globally nilpotent (*derived from geometry* [13]), but also a 'special' G-operator [33] (special geometry): its *differential Galois group* becomes 'special' (symplectic or orthogonal groups, see for instance [27]).

This highly selected character of the order-4 operator (41) is further confirmed by the *'integrality property'* [16] of the series expansion of its analytical solution at $t = 0$:

$$\mathrm{Sol}\big(L_4^{(4)}\big) = t + 11/8\, t^2 + 27/16\, t^3 + 2027/1024\, t^4$$
$$+ 9269/4096\, t^5 + 83\,297/32\,768\, t^6 + \cdots, \tag{46}$$

which, after rescaling $t = 16u$, becomes a series with *integer* coefficients:

$$\begin{aligned}\mathrm{Sol}\big(L_4^{(4)}\big) = {} & 16u + 352u^2 + 6912u^3 + 129\,728u^4 + 2372\,864u^5 \\ & + 42\,648\,064u^6 + 756\,609\,024u^7 + 13\,286\,784\,384u^8 \\ & + 231\,412\,390\,144u^9 + 4002\,962\,189\,824u^{10} + 68\,843\,688\,570\,880u^{11} \\ & + 1178\,125\,203\,260\,416u^{12} + 20\,074\,611\,461\,902\,336u^{13} \\ & + 340\,769\,765\,322\,760\,192u^{14} + 5765\,304\,623\,564\,259\,328u^{15} \\ & + 97\,249\,731\,220\,784\,896\,768u^{16} + 1636\,034\,439\,292\,348\,588\,288u^{17} + \cdots.\end{aligned} \tag{47}$$

This *integrality property* [32] suggests a *modularity* [16, 20, 21] of this order-4 operator (41). The simplest scenario would correspond to (46) being elliptic integrals or, beyond, modular forms that would typically be (up to differential equivalence) an ${}_2F_1$ hypergeometric function with *not one, but two pullbacks* (the relation between these two pullbacks being a modular curve). More involved scenarios would correspond to *Calabi–Yau ODEs* [22, 23] and some other *mirror maps* (see [16]). We have first explored the simplest scenarios (elliptic integrals, *modular forms*), which, as far as differential algebra is concerned, amounts to seeing if this order-4 operator (41) can be reduced, up to differential operator equivalence, to symmetric powers of a second-order operator. This simple scenario is ruled out[17]. We are now forced to explore the, much more complex, Calabi–Yau framework, with two possible scenarios: a general Calabi–Yau order-4 ODE [22, 23], or a Calabi–Yau order-4 ODE that is ${}_4F_3$ solvable, the solution like (46) being expressed, up to operator equivalence, in terms of an ${}_4F_3$ hypergeometric function *up to a pullback that remains to be discovered*. This last situation would correspond to the ${}_4F_3$ Calabi–Yau situation we already encountered in $\tilde{\chi}^{(6)}$

[17] See for instance, van Hoeij's program [28] from ISSAC'2007.

(see [16]). The ${}_4F_3$ solvability is clearly a desirable situation because everything can be much more explicit.

In constrast with the (globally nilpotent) order-2 operators, finding that a given order-4 operator corresponds to a given ${}_4F_3$ operator up to a pullback (and up to homomorphisms) is an extremely difficult task, because the necessary techniques have not yet been developed. Quite often, it goes the other way (no go result): assuming a rational pullback, one can rule out a given order-4 operator being an ${}_4F_3$ operator with a rational pullback (up to differential operator equivalence).

In fact, and fortunately, operator (41) turns out to be, a nice example. It has singularities at $0,\ 1,\ -1,\ \infty$, and these points have to be mapped to $0,\ 1,\ \infty$ (i.e. the singularities of ${}_4F_3$ hypergeometric functions) by the pullback. *Assuming a rational pullback of degree 2*, there is a systematic algorithm to find all of the rational pullbacks of degree 2 mapping $0,\ 1,\ -1,\ \infty$ onto $0,\ 1,\ \infty$. This systematic algorithm is described in [37] for order-2 operators, but the same approach works (with little change) for fourth-order operators as well. The rational pullback function can actually be obtained[18] (with some trial and error) from this mapping of singularities constraint and from the exponent differences, in the same way as in section 2.6 in [39]. The reader who is just interested in the surprisingly simple final result and not in the mathematical structures, in particular, the interesting relations between some Calabi–Yau ODEs and selected ${}_4F_3$, can skip the next three subsections[19] and jump directly to the solution of (41) given by (63) with (61).

3.2. Simplification of $L_4^{(4)}$

As a ‘warm up’, let us, for the moment, try to simplify the order-4 operator (41), getting rid of the apparent singularities $t^2 - 10t + 1 = 0$, and trying to take into account all the symmetries of (41): for instance, one easily remarks that (41) is actually invariant by the involutive symmetry $t \leftrightarrow 1/t$.

Let us introduce the order-4 operator

$$\mathcal{L}_4 = D_x^4 + \frac{10x^2 - 2x - 5}{(x-1)(1+2x)x} \cdot D_x^3 + \frac{1}{4} \cdot \frac{(5x+4)\cdot(6x^2 - 13x + 4)}{(x-1)^2(1+2x)x^2} \cdot D_x^2 + \frac{1}{4} \cdot \frac{x+8}{(x-1)^2(1+2x)x^2} \cdot D_x - \frac{3}{4\cdot(x-1)(1+2x)x^3}, \qquad (48)$$

where $1 + 2x = 0$ is an apparent singularity. One can easily verify that the order-4 operator (41) is the previous operator (48), where we have performed the $t \leftrightarrow 1/t$ invariant pullback:

$$x = -\frac{4t}{(1-t)^2}, \qquad L_4^{(4)} = \mathcal{L}_4\left[x \to -\frac{4t}{(1-t)^2}\right]. \qquad (49)$$

The operator (48) is homomorphic to another order-4 operator with *no apparent singularities*

$$\mathcal{M}_4 = D_x^4 + 2\cdot\frac{5x-4}{(x-1)\cdot x} \cdot D_x^3 + \frac{1}{4} \cdot \frac{(95x^2 - 160x + 56)}{(x-1)^2\cdot x^2} \cdot D_x^2 + \frac{1}{4} \cdot \frac{45x^3 - 124x^2 + 104x - 16}{(x-1)^3\cdot x^3} \cdot D_x - \frac{2x-5}{4\cdot(x-1)^3\cdot x^3}, \qquad (50)$$

[18] For order-2 equations with four singularities (HeunG ...), there are already hundreds of cases (now all found), see [38]. Looking at the size of that table [38] it is clear that providing an algorithm for finding pullbacks will be quite hard.

[19] Which correspond, in fact, to the way we originally found the result.

as can be seen by the (very simple) intertwining relation

$$\mathcal{M}_4 \cdot D_x = \left(D_x + \frac{10\,x^2 \,-4\,x\,-3}{(x-1)\,(1+2\,x)\cdot x}\right) \cdot \mathcal{L}_4. \tag{51}$$

This last operator with no apparent singularities is homomorphic to its adjoint in a very simple way:

$$\mathrm{adjoint}(\mathcal{M}_4) \cdot x^4 \cdot (1-x) = x^4 \cdot (1-x) \cdot \mathcal{M}_4. \tag{52}$$

Do note that, remarkably, the exterior square of $\mathcal{M}_4$ *is an order-5 operator* and not the order-6 operator, one could expect generically from an intertwining relation like (51) (the exterior square of the order-4 operator (48) is order 6 with a rational function solution $(1\,+2\,x)/x$). Taking into account all these last results (no apparent singularities, the singularities being the standard 0, 1, ∞ singularities, the intertwining relation (51), the fact that the exterior square is of order 5), the order-4 operator (50) looks like a *much simpler* operator to study than the original operator (41).

3.3. k-balanced $_4F_3$ hypergeometric function

Let us make here an important preliminary remark on the $_4F_3$ linear differential operators. Let us consider an $_4F_3$ hypergeometric function

$$_4F_3([a_1, a_2, a_3, a_4],\ [b_1, b_2, b_3];\, t), \tag{53}$$

with *rational values* of the parameters a_i and b_j. Its exponents at $x=\ 0$ are 0, $1\,-\,b_1$, $1\,-\,b_2$ and $1\,-\,b_3$, its exponents at $x=\ \infty$ are a_1, a_2, a_3 and a_4, and its exponents at $x\,=1$ are 0, 1, 2 and $\mathcal{S}$, where $\mathcal{S}$ is the *Saalschützian difference*:

$$\mathcal{S} = (b_1 + b_2 + b_3) - (a_1 + a_2 + a_3 + a_4). \tag{54}$$

The Saalschützian condition [42–44] $\mathcal{S}\,=\,1$ is thus a condition of confluence of two exponents at $x = 1$.

The linear differential order-4 operators annihilating the $_4F_3$ hypergeometric function (53) are necessarily globally nilpotent, and *they will remain globally nilpotent up to pullbacks and up to differential operator equivalence*[20]. In contrast, the corresponding order-4 operators are not, for generic (rational) values of the parameters a_i and b_j, such that they are homomorphic to their (formal) adjoint (special geometry), or such that their exterior square of order-6 has a rational function solution (a degenerate case corresponding to the exterior square being of order 5).

These last 'special geometry' conditions (see (43) and (45)) correspond to selected algebraic subvarieties in the parameters a_i and b_j. In the particular case of the *exterior square of the order-4 operator being*[21] *of order 5*, we will show, in forthcoming publications, that the parameters a_i and b_j of the hypergeometric functions are necessarily restricted to three sets of algebraic varieties: a *codimension-3 algebraic variety* included in the Saalschützian condition [42–44] $\mathcal{S}\,=\,1$ and two (self-dual for the adjoint) *codimension-4 algebraic varieties*, respectively, included in the two hyperplanes $\mathcal{S}\,=\,-1$ and $\mathcal{S}\,=\,3$.

Imagine that one is lucky enough to see the order-4 operator (50) (which is such that its exterior square is of order 5) as a $_4F_3$ solvable Calabi–Yau situation; one is, thus, exploring particular $_4F_3$ hypergeometric functions corresponding to these (narrow sets of) algebraic varieties which single out particularly $(k\,=\,-1,\ 1,\ 3)$ *k-balanced* hypergeometric functions[22]

[20] Global nilpotence is preserved by pullbacks (change of variables) and by homomorphisms (operator equivalence).
[21] This condition is seen by some authors, see (11) in [41], as a condition for the ODE to be a *Picard–Fuchs equation* of a Calabi–Yau manifold. These conditions, namely (11) in [41], are preserved by pullbacks, not operator equivalence.
[22] *k*-balanced hypergeometric functions correspond to the Saalschützian difference (54) being an integer k: $\mathcal{S} = k$.

(rather than the well-poised hypergeometric functions or very well poised[23] hypergeometric functions [46, 47] one could have imagined[24]). We are actually working up to operator equivalence (which amounts to performing derivatives of these hypergeometric functions). It is straightforward to see that the nth derivative of a hypergeometric function shifts the Saalschützian difference (54) by an integer, and that this does not preserve the condition for the exterior square of the corresponding order-4 operator to be order 5: it becomes an order-6 operator (homomorphic to its formal adjoint) with a rational function solution. The natural framework for seeking ${}_4F_3$ hypergeometric functions (if any) for our order-4 operators (41), (48) and (50) is thus (selected) k-balanced hypergeometric functions (rather than the well-poised, or very well poised, hypergeometric functions [46] ...).

3.4. $L_4^{(4)}$ is ${}_4F_3$ solvable, up to a pullback

Let us restrict ourselves to the, at first sight simpler, order-4 operators (48) and (50): even if we know exactly the rational values of the parameters a_i and b_j, finding that a given order-4 operator corresponds to this given ${}_4F_3$ operator, up to a pullback (and up to homomorphisms), remains a quite difficult task. We have first studied the case where the pullback in our selected ${}_4F_3$ hypergeometric functions is a rational function. This first scenario has been ruled out on arguments based on the matching of the singularities and of the exponents of the singularities.

We thus need to start exploring pullbacks that are *algebraic functions*. Algebraic functions can branch at certain points (this can, for instance, turn a regular point into a singular point). The set of algebraic functions is a very large one, so we started[25] with the simplest algebraic functions situation, namely *square-root singularities*. The first examination of the matching of the singularities, and of the exponents of the singularities, indicates that we should have square roots at $x = 1$ only.

Along this square-root line, let us recall the well-known *inverse Landen transformation* in terms of k, the *modulus of the elliptic functions parametrizing the Ising model*:

$$k \quad \longrightarrow \quad \frac{1-\sqrt{1-k^2}}{1+\sqrt{1-k^2}}. \tag{55}$$

In terms of the variable $x = k^2$, this inverse Landen transformation reads

$$x \quad \longrightarrow \quad P(x) = \left(\frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}\right)^2 = \frac{x^2 - 8x + 8}{x^2} - 4\cdot(2-x)\cdot\frac{(1-x)^{1/2}}{x^2}. \tag{56}$$

Using this pullback $P(x)$, we have actually been able to obtain the solution of the order-4 differential operator (50) in terms of four terms like

$${}_4F_3\left(\left[\tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{1}{2}\right],\ [1,\ 1,\ 1];\ P(x)\right). \tag{57}$$

This slightly involved solution is given in appendix B.

We can now get the solution of (41), the original operator $L_4^{(4)}$, from this slightly involved result, since (41) is (50) up to a simple pullback, namely the change of variable (49). Coming

[23] Note that very well poised hypergeometric series are known [45] to be related to $\zeta(2)$, $\zeta(3)$, ..., which are constants known to occur in the Ising model [11].

[24] Note that the conditions to be well-poised hypergeometric series are actually preserved by the transformation $a_i \to 1 - a_i$, $b_j \to 2 - b_j$, which corresponds to changing the linear differential operator, associated with hypergeometric functions, into its formal adjoint.

[25] And also because we had an Ising model prejudice in favor of square roots [18] ...

back with (49) to the original variable t in $L_4^{(4)}$, the previous pullback (56) simplifies remarkably to

$$P\left(-\frac{4\,t}{(1-t)^2}\right) = \frac{1+t^4}{2\cdot t^2} - \frac{1-t^4}{2\cdot t^2} = t^2, \tag{58}$$

the Galois conjugate of (56) giving $1/t^2$. Of course, once this key result is known, namely that a t^2 pullback works, it is easy to justify, *a posteriori*, this simple monomial result: after all, $L_4^{(4)}$ has singularities at $0,\ 1,\ -1,\ \infty$, and these points can be mapped (under t^2) to $0,\ 1,\ \infty$ (i.e. the singularities of ${}_4F_3$ hypergeometric functions).

Pullbacks have a natural structure with respect to *composition of functions*[26]. It is worth noting that (58) describes the *composition of two well-known isogenies of elliptic curves*, the *inverse Landen transformation* (56), and the *rational isogeny* $-4t/(1-t)^2$ underlined by Vidunas [31] and in [34], giving the simple quadratic transformation $t\ \to\ t^2$.

All this means that the solution of $L_4^{(4)}$ can be expressed in terms of the hypergeometric function

$${}_4F_3\left(\left[\tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{1}{2}\right],\ [1,\ 1,\ 1];\ t^2\right), \tag{59}$$

and its derivatives. Actually, considering the hypergeometric operator $\mathcal{H}$ having (59) as a solution, it can be seen to be homomorphic to (41)

$$\mathcal{A}_3\cdot\mathcal{H}\ =\ L_4^{(4)}\cdot A_3, \tag{60}$$

where the order-3 intertwinners A_3 and $\mathcal{A}_3$ read respectively (with $d_3(t) = t\cdot(t+1)\cdot(t-1)^2\cdot(t^2-10\,t+1)$):

$$A_3\ = 2\cdot(1+t)\cdot t^3\cdot D_t^3\ + \frac{2}{3}\cdot\frac{16t^2-t-11}{t-1}\cdot t^2\cdot D_t^2 + \frac{1}{3}\cdot\frac{31t^2-4t-11}{t-1}\cdot t\cdot D_t + t, \tag{61}$$

$$\begin{aligned}\mathcal{A}_3 = \frac{2}{t-1}\cdot D_t^3\ &+ \frac{2}{3}\cdot\frac{1}{d_3(t)}\cdot(10\,t^4-107\,t^3-225\,t^2+163\,t-17)\cdot D_t^2\\ &+\frac{1}{3}\cdot\frac{1}{t\cdot d_3(t)}\cdot\ (5\,t^4-66\,t^3-900\,t^2+290\,t-33)\cdot D_t\\ &+\frac{1}{3}\cdot\frac{1}{t\cdot d_3(t)}\cdot\ (t^3-21\,t^2\ +99\,t-23).\end{aligned} \tag{62}$$

From the intertwinning relation (60), one easily finds that the solution of $L_4^{(4)}$, which is analytic at $t\ = 0$, is A_3 acting on (59):

$$\tilde{\chi}^{(4)}_{d;3}(t)\ =\ A_3[{}_4F_3([1/2,\ 1/2,\ 1/2,\ 1/2],[1,\ 1,\ 1];\ t^2)]. \tag{63}$$

Having with (63), a normalization for $\tilde{\chi}^{(4)}_{d;3}$, we can now fix the values of the coefficients a_i in the sum (34) for $\tilde{\chi}^{(4)}_d(t)$. They can be fixed by expanding and matching the rhs of (34) and $\tilde{\chi}^{(4)}_d(t)$. This gives

$$\tilde{\chi}^{(4)}_d(t)\ =\ \ \tfrac{1}{2^3}\cdot\tilde{\chi}^{(4)}_{d;1}(t)\ \ +\tfrac{1}{3\cdot 2^3}\cdot\tilde{\chi}^{(4)}_{d;2}(t)\ \ -\tfrac{1}{2^3}\cdot\tilde{\chi}^{(4)}_{d;3}(t). \tag{64}$$

Remark. It is quite surprising to find exactly the same ${}_4F_3$ hypergeometric function (59) with the exact *same* remarkably simple pullback t^2, as the one we already found [16] in the order-4 Calabi–Yau operator L_4 in $\tilde{\chi}^{(6)}$.

[26] Suppose that an operator O_2 is a pullback of an operator O_1, where the pullback f is a rational function and that O_3 is also a pullback of O_1, where the pullback is a rational function g. Then O_3 is also a pullback of O_2. To compute this pullback function, one has to compose g and the inverse of f.

Comment. Of course, from a mathematical viewpoint, when looking for a pullback, one can in principle always ignore all apparent singularities. These calculations displayed here look a bit paradoxical: the calculations performed with the (no apparent singularities) operator (50), which looks *simpler* (it has an exterior square of order 5, and is very simply homomorphic to its adjoint, ...), turn out to have a more complicated pullback (56) than the amazingly simple pullback (namely t^2) we finally discover for the original operator (41) (see (63)). The 'complexity' of the original operator (41) is mostly encapsulated in the order-3 intertwinner A_3 (see (61)). The '${}_4F_3$-solving' of the operator amounts to reducing the operator, up to operator equivalence (60), to an ${}_4F_3$ hypergeometric operator up to a pullback. Finding the pullback is the difficult step: as far as '${}_4F_3$-solving' of an operator is concerned, what matters is the *complexity of the pullback*, not the complexity of the operator equivalence.

Ansatz. Of course, knowing the key ingredient in the final result (63), namely that the pullback is just t^2, it would have been much easier to get this result. Along this line, one may recall the conjectured existence of a natural boundary at a unit circle $|t| = 1$ for the full susceptibility of the Ising model, and, more specifically for the diagonal susceptibility n-fold integrals we study here, the fact that the singularities are all Nth roots of unity (N integer). Consequently, one may have, for the Ising model, a t^N (N integer) prejudice for pullbacks.

In forthcoming studies of linear differential operators occurring in the next (bulk) $\tilde{\chi}^{(n)}$s or (diagonal) $\tilde{\chi}_d^{(n)}$s, when trying to see if these new (Calabi–Yau like, special geometry) operators are ${}_{q+1}F_q$ reducible up to a pullback, we may save some large amount of work by assuming that the corresponding pullbacks are of the simple form t^N, where N is an integer.

4. The linear differential equation of $\tilde{\chi}_d^{(5)}$ in mod. prime and exact arithmetics

The first terms of the series expansion of $\tilde{\chi}_d^{(5)}(x)$ read

$$\tilde{\chi}_d^{(5)}(x) = \frac{3}{262\,144}\cdot x^{12} + \frac{39}{1048\,576}\cdot x^{14} + \frac{5085}{67\,108\,864}\cdot x^{16} + \frac{9}{67\,108\,864}\cdot x^{17} + \frac{33\,405}{268\,435\,456}\cdot x^{18} + \frac{315}{536\,870\,912}\cdot x^{19} + \cdots, \tag{65}$$

where $x = t^{1/2} = \sinh 2E_v/kT \, \sinh 2E_h/kT$ is our independent variable. In order to obtain the linear differential equation for $\tilde{\chi}_d^{(5)}(x)$, we have used in [19] a 'mod. prime' calculation which amounts to generating large series *modulo a given prime*, and then deduce, the linear differential operator for $\tilde{\chi}_d^{(5)}(x)$ *modulo that prime*. With 3000 coefficients for the series expansion of $\tilde{\chi}_d^{(5)}(x)$ modulo a prime, we have obtained linear differential equations of order 25, 26, ... The smallest order we have reached is 19, and we have *assumed* that the linear differential equation of $\tilde{\chi}_d^{(5)}(x)$ is of minimal order 19. In [18], we have introduced a method to obtain the minimal order of the ODE by producing some ($\geqslant 4$) nonminimal order ODEs and then using the 'ODE formula' (see [14, 15, 18] for details and how to read the ODE formula). The ODE formula for $\tilde{\chi}_d^{(5)}(x)$ reads

$$31\,Q + 19\,D - 302 = (Q+1)\cdot(D+1) - f, \tag{66}$$

confirming that the minimal order of the ODE for $\tilde{\chi}_d^{(5)}(x)$ is 19. Note that the degree of the polynomial carrying apparent singularities should be 237 (see appendix B in [14]). Call $\mathcal{L}_{19}^{(5)}$ the differential operator (known mod. prime) for $\tilde{\chi}_d^{(5)}(x)$. The singularities and local exponents of $\mathcal{L}_{19}^{(5)}$ are[27]

$$x = 0, \qquad \rho = 0^5, 1/2, 1^4, 2^3, 4^3, 3, 7, 12,$$

[27] The local exponents are given as e.g. 2^3 meaning 2, 2, 2.

$$\begin{aligned}
&x = \infty, \qquad && \rho = 1^5, 3/2, 2^4, 3^3, 4, 5^3, 8, 13,\\
&x = 1, && \rho = -3, -2, -1, 0^4, 2^3, 4^2, \ldots,\\
&x = -1, && \rho = 0^5, 2^4, 4^3, 6^2, 8^2, 10^2, \ldots,\\
&x = x_0, && \rho = 5/2, 7/2, 7/2, \ldots,\\
&x = x_1, && \rho = 23/2, \ldots,
\end{aligned} \tag{67}$$

where x_0 (resp. x_1) is any root of $1+x+x^2 = 0$ (resp. $1+x+x^2+x^3+x^4 = 0$), and the trailing $\cdots$ denotes integers not in the list. Note that, in practice, we do not deal with the minimal order differential operator $\mathcal{L}_{19}^{(5)}$ but with an operator of order 30 (that $\mathcal{L}_{19}^{(5)}$ rightdivides): order 30 is what we have called in [14, 15, 18] the 'optimal order', namely the order for which finding the differential operator annihilating the series requires the minimum number of terms in the series. With the tools and methods developed in [14, 15, 17], we are now able to factorize the differential operator and *recognize some factors in exact arithmetic*. This way, we may see whether some factors occurring $\mathcal{L}_{19}^{(5)}$ follow the 'special geometry' line we encountered for $\tilde{\chi}_d^{(3)}(x)$ and $\tilde{\chi}_d^{(4)}(t)$. Our first step in the factorization of $\mathcal{L}_{19}^{(5)}$ is to check whether $\mathcal{L}_6^{(3)}$ (the differential operator for $\tilde{\chi}_d^{(3)}$) is a right factor of $\mathcal{L}_{19}^{(5)}$, meaning that the solutions of $\mathcal{L}_6^{(3)}$ (and in particular the integral $\tilde{\chi}_d^{(3)}(x)$) are also solutions of $\mathcal{L}_{19}^{(5)}$. This is indeed the case. Using the methods developed in [14, 15, 17], we find that the series for the difference $\tilde{\chi}_d^{(5)}(x) - \alpha\, \tilde{\chi}_d^{(3)}(x)$ requires an ODE of minimal order 17 for the value[28] $\alpha = 8$. This confirms that $\mathcal{L}_6^{(3)}$ is in the direct sum in $\mathcal{L}_{19}^{(5)}$, and that some (order-4) factors of $\mathcal{L}_6^{(3)}$ are *still* in $\mathcal{L}_{17}^{(5)}$:

$$\mathcal{L}_{19}^{(5)} \; = \; \mathcal{L}_6^{(3)} \oplus \mathcal{L}_{17}^{(5)}. \tag{68}$$

This order-4 factor is obviously $L_1^{(3)} \oplus L_3^{(3)}$. Since these factors are in the direct sum in $\mathcal{L}_6^{(3)}$, the order-17 operator $\mathcal{L}_{17}^{(5)}$ is also the annihilator of $\tilde{\chi}_d^{(5)}(x) - \beta\, \tilde{\chi}_{d,2}^{(3)}(x)$ for $\beta = 4$, meaning that we also have

$$\mathcal{L}_{19}^{(5)} \; = \; L_2^{(3)} \oplus \mathcal{L}_{17}^{(5)}. \tag{69}$$

At this step, the differential operator $\mathcal{L}_{17}^{(5)}$ is known in prime. To go further in the factorization, we use the method developed in [14, 15] along with various singularities and local exponents of $\mathcal{L}_{17}^{(5)}$ which read[29]

$$\begin{aligned}
&x = 0, \qquad && \rho = 0^5, 1/2, 1^4, 2^3, 4^2, 3, 7, \qquad && \ln(z)^4 \text{ terms},\; z^{1/2},\\
&x = \infty, && \rho = 1^5, 3/2, 2^4, 3^3, 4, 5^2, 8, && \ln(z)^4 \text{ terms},\; z^{3/2},\\
&x = 1, && \rho = -3, -2, -1, 0^4, 2^3, \ldots, && \ln(z)^3 \text{ terms},\; z^{-3}, z^{-2}, z^{-1},\\
&x = -1, && \rho = 0^5, 2^4, 4^3, 6^2, 8^2, \ldots, && \ln(z)^4 \text{ terms},\\
&x = x_0, && \rho = 5/2, 7/2, 7/2, \ldots, && z^{5/2}, z^{7/2}, z^{7/2}\ln(z) \text{ terms},\\
&x = x_1, && \rho = 23/2, \ldots, && z^{23/2},
\end{aligned} \tag{70}$$

where x_0 and x_1 are again the roots of $1 + x + x^2 = 0$ and $1 + x + x^2 + x^3 + x^4 = 0$, and the trailing $\cdots$ denotes integers not in the list. The last column shows the maximum $\ln(z)$-degree occurring in the formal solutions of $\mathcal{L}_{17}^{(5)}$, with z being the local variable of the expansion.

Section 5 of [19] is used to recognize exactly some factors. This is completed by a usual *rational reconstruction* [17]. We are now able to give new results completing what was given in section 5 of [19]. The linear differential operator $\mathcal{L}_{17}^{(5)}$ has the factorization

$$\mathcal{L}_{17}^{(5)} = L_6^{(5)} \cdot \mathcal{L}_{11}^{(5)}. \tag{71}$$

[28] Comparing with equation (58) in [19], one should not expect a $1/2$ contribution, since the sum on the $g^5(N, t)$s still contains $\tilde{\chi}_d^{(3)}$.

[29] There are solutions analytic at $x = 0$ with exponents $0, 1, 2, 4, 7$.

The linear differential operator $\mathcal{L}_{11}^{(5)}$ has been fully factorized and the factors *are known in exact arithmetic* (the indices are the orders)

$$\mathcal{L}_{11}^{(5)} = L_1^{(3)} \oplus L_3^{(3)} \oplus \left(W_1^{(5)} \cdot U_1^{(5)}\right) \oplus \left(L_4^{(5)} \cdot V_1^{(5)} \cdot U_1^{(5)}\right), \qquad (72)$$

and are given in appendix C. The factor $L_6^{(5)}$ is the only one which is known in primes[30], and it is irreducible. The irreducibility has been proven with the method presented in section 4 of [14]. This is technically tractable since there are only two free coefficients (see (77) below) that survive in the expansion of the analytical series at $x = 0$ of $L_6^{(5)}$. In the factorization of (71) and (72) of $\mathcal{L}_{17}^{(5)}$ and $\mathcal{L}_{11}^{(5)}$, respectively, the factors are either known and occur elsewhere ($L_1^{(3)}, L_3^{(3)}$) or simple order-1 linear differential operators ($U_1^{(5)}, V_1^{(5)}, W_1^{(5)}$), *except* the order-4 operator $L_4^{(5)}$ and the order-6 operator $L_6^{(5)}$. It is then for these specific operators that we examine whether they are 'special geometry'.

4.1. The linear differential operator $L_4^{(5)}$

The order-4 linear differential operator $L_4^{(5)}$ has the following local exponents:

$$\begin{aligned}
&x = 0, && \rho = -2, -2, -1, 0, \\
&x = \infty, && \rho = 3, 3, 4, 5, \\
&x = 1, && \rho = -2, -2, -2, -2, \\
&x = -1, && \rho = -2, -2, 0, 0, \\
&1 + x + x^2 = 0, && \rho = -1, 0, 1, 2.
\end{aligned} \qquad (73)$$

At all these singularities x_s, the solutions have the maximum allowed degree of logs (i.e. $\ln(x - x_s)^3$), except at the singularities roots of $1 + x + x^2 = 0$, where the solutions carry no logs. In view of the negative local exponents in (73), we introduce

$$\mu(x) = (8 \cdot (x-1)^2 \cdot (x+1)^2 \cdot (x^2 + x + 1))^{-1}. \qquad (74)$$

Then, if we consider the linear differential operator $\mu(x)^{-1} \cdot L_4^{(5)} \cdot \mu(x)$, nothing prevents us (as far as the ρ's and logs are concerned) from checking whether this conjugated operator is homomorphic to a symmetric cube of the order-2 linear differential operator of an elliptic integral. We find the solution of $L_4^{(5)}$, which is analytic at $x = 0$, as a cubic expression of ${}_2F_1$ hypergeometric functions, with *palindromic* polynomials

$$\begin{aligned}
\mathrm{Sol}(L_4^{(5)}) \;=\; & \mu(x) \cdot (3x^5 \cdot F_1^3 + 8 \cdot (2 + 2x + x^2 + 2x^3 + 2x^4) \cdot F_0^3 \\
& + 4x \cdot (5 + x - 3x^2 + x^3 + 5x^4) \cdot F_1 \cdot F_0^2 \\
& + 2x^2 \cdot (2 - 10x - 17x^2 - 10x^3 + 2x^4) \cdot F_1^2 \cdot F_0),
\end{aligned} \qquad (75)$$

where F_0 and F_1 are, respectively, ${}_2F_1([1/2, 1/2], [1]; x^2)$ and ${}_2F_1([1/2, 3/2], [2]; x^2)$ (closely related, up to a $\pi/2$ factor, to the usual complete elliptic integrals). Again the occurrence of (very simple) elliptic integrals is underlined. Note that $L_4^{(5)}$ contributes to the solutions of $\mathcal{L}_{17}^{(5)}$ in the block $L_4^{(5)} \cdot V_1^{(5)} \cdot U_1^{(5)}$ which has the local exponents

$$\begin{aligned}
&x = 0, && \rho = 0^3, 1^2, 2, \\
&x = \infty, && \rho = 1^3, 2^2, 3, \\
&x = 1, && \rho = -3, -2, 0^4, \\
&x = -1, && \rho = 0^4, 2^2, \\
&1 + x + x^2 = 0, && \rho = 0, 1, 2, 3, 4, 5/2.
\end{aligned} \qquad (76)$$

There are two solutions analytic at $x = 0$ with exponents 1 and 2.

[30] This factor known only in primes does not allow the computation of the singular behavior of $\tilde{\chi}_d^{(5)}$.

4.2. On the order-6 linear differential operator $L_6^{(5)}$: 'special geometry'

Let us write the formal solutions $L_6^{(5)}$ at $x = 0$, where the notation $[x^p]$ means that the series begins as x^p (const. $+ \cdots$). There is one set of five solutions and one extra solution analytical at $x = 0$ (i.e. four solutions with a log, and two solutions analytical at $x = 0$):

$$\begin{aligned} S_1 &= [x^7]\,\ln(x)^4 + [x^4]\,\ln(x)^3 + [x^2]\,\ln(x)^2 + [x^0]\,\ln(x) + [x^0], \\ S_2 &= [x^7]\,\ln(x)^3 + [x^4]\,\ln(x)^2 + [x^2]\,\ln(x) + [x^0], \\ S_3 &= [x^7]\,\ln(x)^2 + [x^4]\,\ln(x) + [x^3], \\ S_4 &= [x^7]\,\ln(x) + [x^4], \qquad S_5 = [x^7], \qquad \text{and} \\ S_6 &= [x^2]. \end{aligned} \tag{77}$$

In view of this structure, the linear differential operator $L_6^{(5)}$ cannot be (homomorphic to) a symmetric fifth power of the linear differential operator corresponding to the elliptic integral.

The next step is to see whether the exterior square of $L_6^{(5)}$ has a rational solution, which means that $L_6^{(5)}$ corresponds to 'special geometry'. With the six solutions (77), seen as series obtained mod. primes, one can easily build the general solution of $\mathrm{Ext}^2(L_6^{(5)})$ as

$$\sum_{k,p} d_{k,p} \cdot \left(S_k\,\frac{\mathrm{d}S_p}{\mathrm{d}x} - S_p\,\frac{\mathrm{d}S_k}{\mathrm{d}x} \right), \qquad k \neq p = 1, \ldots, 6, \tag{78}$$

which should not contain logs, fixing then some of the coefficients $d_{k,p}$.

For a rational solution of $\mathrm{Ext}^2(L_6^{(5)})$ to exist, the form (free of logs)

$$D(x) \cdot \sum_{k,p} d_{k,p} \cdot \left(S_k\,\frac{\mathrm{d}S_p}{\mathrm{d}x} - S_p\,\frac{\mathrm{d}S_k}{\mathrm{d}x} \right), \tag{79}$$

should be a polynomial, where the denominator $D(x)$ reads

$$D(x) = x^{n_1} \cdot (x+1)^{n_2} \cdot (x-1)^{n_3} \cdot (1+x+x^2)^{n_4} \cdot (1+x+x^2+x^3+x^4)^{n_5},$$

with the order of magnitude of the exponents n_j being obtained from the local exponents of the singularities. With a series of length 700, we have found no rational solution for $\mathrm{Ext}^2(L_6^{(5)})$.

Even if $L_6^{(5)}$ is an irreducible operator of *even* order, we have also looked for a rational solution for its *symmetric square*. The general solution of $\mathrm{Sym}^2(L_6^{(5)})$ is built from (77) as

$$\sum_{k,p} f_{k,p} \cdot S_k\,S_p, \qquad k \geqslant p = 1, \ldots, 6, \tag{80}$$

and the same calculations are performed. With some 300 terms, we actually found that $\mathrm{Sym}^2(L_6^{(5)})$ *has a rational solution* of the form[31] (with $P_{196}(x)$ a polynomial of degree 196):

$$\frac{x^4 \cdot P_{196}(x)}{(x+1)^{10} \cdot (x-1)^{14} \cdot (1+x+x^2)^{21} \cdot (1+x+x^2+x^3+x^4)^9}, \tag{81}$$

thus showing that $L_6^{(5)}$ does correspond to 'special geometry'.

Note that the occurrence (77) of *two analytic* solutions at $x = 0$, for $L_6^{(5)}$, which is irreducible, is a situation we have encountered in Ising integrals [14, 17]. The order-12 differential operator (called L_{12}^{left} in [17]) has four analytical solutions at $x = 0$, and it has been demonstrated that it is irreducible [17].

[31] Note that this form occurs, for a nonminimal representative of $L_6^{(5)}$, in the factorization (71). On this point, see the details around (43) and (44) in [14].

5. Singular behavior of $\tilde{\chi}_d^{(3)}(x)$

Now that we have obtained all the solutions of the linear differential equations of $\tilde{\chi}_d^{(3)}(x)$ and $\tilde{\chi}_d^{(4)}(t)$, analytic at the origin, we turn to the exact computation of their singular behavior at the finite singular points.

Obtaining the singular behavior of $\tilde{\chi}_d^{(3)}(x)$ amounts to calculating the singular behavior of each term in (18). The details are given in appendix D.

5.1. The behavior of $\tilde{\chi}_d^{(3)}(x)$ as $x \to 1$

The evaluation of the singular behavior as $x \to 1$ corresponds to straightforward calculations that are given by (12) (see (D.7) and (D.8)):

$$\mathrm{Sol}(L_6^{(3)})(\mathrm{Singular}, x=1) = \frac{2}{\pi}\cdot\frac{3\,a_3^{(3)} + a_2^{(3)}}{(1-x)^2} + \left(a_1^{(3)} - \left(\frac{3a_3^{(3)} + a_2^{(3)}}{\pi}\right) + 3\,a_2^{(3)}\right.$$
$$\left.\times\left(\frac{5\pi}{9\Gamma^2(5/6)\Gamma^2(2/3)} - \frac{8\pi}{\Gamma^2(1/6)\Gamma^2(1/3)}\right)\right)\cdot\frac{1}{1-x} + \frac{a_2^{(3)}}{2\pi}\cdot\ln(1-x). \tag{82}$$

When specialized to the combination (18) defining $\tilde{\chi}_d^{(3)}(x)$, the singular behavior reads

$$\tilde{\chi}_d^{(3)}(x)(\mathrm{Singular}, x=1) = \left(\frac{1}{3} - \frac{5\pi}{18\Gamma^2(5/6)\Gamma^2(2/3)} + \frac{4\pi}{\Gamma^2(1/6)\Gamma^2(1/3)}\right)\frac{1}{1-x}$$
$$+\frac{1}{4\pi}\cdot\ln(1-x). \tag{83}$$

This result agrees with the result determined numerically in appendix B of [19].

One remarks, for the particular combination (18) giving $\tilde{\chi}_d^{(3)}(x)$, that the *most divergent term disappears*. Note that this is what has been obtained [12] for the susceptibility $\tilde{\chi}^{(3)}$, where the singularity $(1-4w)^{-3/2}$ of the ODE is not present in $\tilde{\chi}^{(3)}$.

5.2. The behavior of $\tilde{\chi}_d^{(3)}(x)$ as $x \to -1$

The calculations of the singular behavior as $x \to -1$ rely mostly on connection formulae of ${}_2F_1$ hypergeometric functions, and the results are given below in (D.9) and (D.15). For the combination (18), the singular behavior reads

$$\tilde{\chi}_d^{(3)}(x)(\mathrm{Singular}, x=-1) = \frac{1}{4\pi^2}\cdot\ln(1+x)^2 + \left(\frac{1}{4\pi} - \frac{2\ln(2)-1}{2\pi^2}\right)\cdot\ln(1+x),$$

which agrees with the result of appendix B of [19].

5.3. The behavior of $\tilde{\chi}_d^{(3)}(x)$ as $x \to \mathrm{e}^{\pm 2\pi\mathrm{i}/3}$

The result for the singular behavior $\tilde{\chi}_d^{(3)}(x)$ as $x \to x_0 = \mathrm{e}^{\pm 2\pi\mathrm{i}/3}$ reads

$$\tilde{\chi}_d^{(3)}(\mathrm{Singular}, x=x_0) = -\frac{8\cdot 3^{1/4}}{35\pi}\,\mathrm{e}^{\pi\mathrm{i}/12}\cdot(x-x_0)^{7/2}$$
$$= -0.095\,7529\,\cdots\,\mathrm{e}^{\pi\mathrm{i}/12}\cdot(x-x_0)^{7/2}. \tag{84}$$

This result is in agreement with the numerical result of appendix B of [19], namely $-\sqrt{2/3}\,\mathrm{e}^{\pi\mathrm{i}/12}\cdot b\cdot(x-x_0)^{7/2}$, with $b = 0.203\,122\,784\cdots$.

6. Singular behavior of $\tilde{\chi}_d^{(4)}(x)$

Obtaining the singular behavior of $\tilde{\chi}_d^{(4)}(x)$ amounts to obtaining the singular behavior of each term in (34).

6.1. Behavior of $\tilde{\chi}_d^{(4)}(t)$ as $t \rightarrow 1$

The calculations of the singular behavior of $\tilde{\chi}_{d;2}^{(4)}(t)$ as $t \rightarrow 1$ are displayed in appendix E, and read

$$\tilde{\chi}_{d;2}^{(4)}(t)(\mathrm{Singular}, t = 1) = \frac{8}{\pi^2(1-t)^2} - \frac{8}{\pi^2(1-t)} + \frac{5}{2\pi^2} \cdot \ln \frac{16}{1-t} - \frac{3}{2\pi^2} \cdot \ln^2 \frac{16}{1-t}. \tag{85}$$

To compute the singular behavior of $\tilde{\chi}_{d;3}^{(4)}(t)$ as $t \rightarrow 1$, we need the expression of the hypergeometric function ${}_4F_3([1/2, 1/2, 1/2, 1/2], [1, 1, 1]; z)$ as $z \rightarrow 1$. This hypergeometric function is an example of solution of a Calabi–Yau ODE, and explicit computations of its monodromy matrices have been given [48].

The differential equation for ${}_4F_3([1/2, 1/2, 1/2, 1/2], [1, 1, 1]; z)$ is *Saalschützian and well poised* (but not very well poised). At $z = 1$, it has one logarithmic solution and three analytic solutions of the form

$$\sum_{n=0}^{\infty} c_n \cdot (1-z)^n. \tag{86}$$

The c_n satisfy the fourth-order recursion relation

$$16\, n \cdot (n-1)^2\, (n-2) \cdot c_n - 24\, (n-1)(n-2)(2n^2 - 6n + 5) \cdot c_{n-1} + 16\, (n-2)^2\, (3n^2 - 12n + 13) \cdot c_{n-2} - (2n-5)^4 \cdot c_{n-3} = 0, \tag{87}$$

where $c_n = 0$ for $n \leqslant -1$. The vanishing of the coefficient c_n at $n = 0, 1, 2$, of c_{n-1} at $n = 1, 2$ and c_{n-2} at $n = 2$ guarantees that c_0, c_1 and c_2 may be chosen arbitrarily.

The behavior at $z = 1$ of ${}_4F_3([1/2, 1/2, 1/2, 1/2], [1, 1, 1]; z)$, which is the solution of the ODE that is analytic at $z = 0$, is given in theorem 3 of Bühring [49] with the parameter

$$s = \sum_{j=1}^{3} b_j - \sum_{j=1}^{4} a_j = 1, \tag{88}$$

(i.e. the *Saalschützian condition* [42–44]). For completeness, we quote this theorem which is valid for all ${}_{p+1}F_p([a_1, \ldots, a_{p+1}], [b_1, \ldots, b_p]; z)$ when the parameter s of (88) is *any integer*[32] $s \geqslant 0$:

$$\frac{\Gamma(a_1) \cdots \Gamma(a_{p+1})}{\Gamma(b_1) \cdots \Gamma(b_p)} \cdot {}_{p+1}F_p([a_1, \ldots, a_{p+1}], [b_1, \ldots, b_p]; z) = \sum_{n=0}^{s-1} I_n^{<} \cdot (1-z)^n + \sum_{n=s}^{\infty} I_n^{>} \cdot (1-z)^n + (1-z)^s \cdot \sum_{n=0}^{\infty} [w_n + q_n \cdot \ln(1-z)] \cdot (1-z)^n, \tag{89}$$

[32] Again we emphasize the role of k-balanced hypergeometric functions.

for $|1 - z| < 1$, $-\pi < \arg(1 - z) < \pi$ and $p = 2, 3, \ldots$, where for $0 \leqslant n \leqslant s - 1$

$$I_n^< = (-1)^n \cdot \frac{\Gamma(a_1+n)\Gamma(a_2+n)(s-n-1)!}{\Gamma(a_1+s)\Gamma(a_2+s)n!} \cdot \sum_{k=0}^{\infty} \frac{(s-n)_k}{(a_1+s)_k\,(a_2+s)_k} \cdot A_k^{(p)}, \tag{90}$$

for $s \leqslant n$,

$$I_n^> = (-1)^n \cdot \frac{(a_1+s)_{n-s}\,(a_2+s)_{n-s}}{n!} \cdot \sum_{k=n-s+1}^{\infty} \frac{(k-n+s)!}{(a_1+s)_k\,(a_2+s)_k} \cdot A_k^{(p)}, \tag{91}$$

and w_n and q_n are such that

$$\begin{aligned} w_n + q_n \cdot \ln(1-z) = {} & (-1)^s \cdot \frac{(a_1+s)_n\,(a_2+s)_n}{(s+n)!\,n!} \\ & \times \left(\sum_{k=0}^{n} \frac{(-n)_k}{(a_1+s)_k(a_2+s)_k} \cdot A_k^{(p)} \cdot [\psi(1+n-k) + \psi(1+s+n) \right. \\ & \left. - \psi(a_1+s+n) - \psi(a_2+s+n) - \ln(1-z)] \right), \end{aligned} \tag{92}$$

where $(a)_n = a\,(a+1)\cdots(a+n-1)$ is Pochhammer's symbol. The $A_k^{(p)}$ are computed recursively in [49] as $p-1$ fold sums. In particular,

$$A_k^{(2)} = \frac{(b_2-a_3)_k\,(b_1-a_3)_k}{k!}, \tag{93}$$

and

$$\begin{aligned} A_k^{(3)} = {} & \sum_{k_2=0}^{k} \frac{(b_3+b_2-a_4-a_3+k_2)_{k-k_2}\,(b_1-a_3)_{k-k_2}\,(b_3-a_4)_{k_2}\,(b_2-a_4)_{k_2}}{(k-k_2)!\,k_2!} \\ = {} & \frac{(b_1+b_3-a_3-a_4)_k\,(b_2+b_3-a_3-a_4)_k}{k!} \\ & \times\ {}_3F_2([b_3-a_3,\ b_3-a_4,\ -k],\ [b_1+b_3-a_3-a_4,\ b_2+b_3-a_3-a_4];\ 1). \end{aligned} \tag{94}$$

For use in (63), we need to specialize to $a_j = 1/2, b_j = 1$, where

$$\begin{aligned} A_k^{(3)} = {} & \sum_{k_2=0}^{k} \frac{(1+k_2)_{k-k_2}\,(1/2)_{k-k_2}\,(1/2)_{k_2}^2}{(k-k_2)!\,k_2!} \\ = {} & k! \cdot {}_3F_2([1/2, 1/2, -k],\ [1, 1];\ 1), \end{aligned} \tag{95}$$

and for, respectively, $n = 0$ and $n \geqslant 1$

$$I_0^< = 4\sum_{k=0}^{\infty} \frac{k!}{(3/2)_k^2} \cdot A_k^{(3)}, \qquad I_n^> = (-1)^n \frac{(3/2)_{n-1}^2}{(n)!} \cdot \sum_{k=n}^{\infty} \frac{(k-n)!}{(3/2)_k^2} \cdot A_k^{(3)}. \tag{96}$$

We note, in particular, the terms

$$A_0^{(3)} = 1, \qquad A_1^{(3)} = 3/4, \qquad A_2^{(3)} = 41/32. \tag{97}$$

Using these specializations in (89), we compute the terms in $\tilde{\chi}_{d;3}^{(4)}(t)$ which diverge as $t \to 1$. The term $(1-t)^{-1} \cdot \ln(1-t^2)$ cancels and we are left with

$$\begin{aligned} \tilde{\chi}_{d;3}^{(4)}(t)(\mathrm{Singular}, t=1) = {} & \frac{1}{\pi^2} \cdot \left(\frac{8}{3\,(1-t)^2} + \frac{56}{3\,(1-t)} + \frac{16}{3\cdot(1-t)} \cdot (3\,I_1^> - 4\,I_2^>) \right) \\ & + \frac{8}{3\,\pi^2} \cdot \ln\frac{1-t^2}{16}. \end{aligned} \tag{98}$$

Thus, using (35), (85) and (98), we find that the terms in $\mathrm{Sol}(L_8^{(4)})$, which diverge as $t \rightarrow 1$, are

$$\mathrm{Sol}\big(L_8^{(4)}\big)(\mathrm{Singular}, t = 1) = \frac{8\left(a_3^{(4)} + 3a_2^{(4)}\right)}{8\pi^2} \cdot \frac{1}{(1-t)^2} + \left(a_1^{(4)} - \frac{8\big(3a_2^{(4)} - 7a_3^{(4)}\big)}{3\pi^2} + \frac{16\, a_3^{(4)}}{3\pi^2} \cdot \big(3I_1^{>} - 4I_2^{>}\big)\right) \cdot \frac{1}{1-t} + \frac{15a_2^{(4)} - 16a_3^{(4)}}{\pi^2} \cdot \ln\left(\frac{16}{1-t}\right) - \frac{3a_2^{(4)}}{2\pi^2} \cdot \ln^2\left(\frac{16}{1-t}\right), \qquad (99)$$

where the constant $3I_1^{>} - 4I_2^{>}$ reads (with 200 digits)

$$3I_1^{>} - 4I_2^{>} = -2.212\,812\,128\,930\,821\,923\,547\,976\,814\,986\,050\,021\,481\,359\,293\,357\,467\,766\,171\,630\,847\,360\,232\,164\,854\,964\,985\,815\,375\,185\,842\,526\,324\,049\,358\,792\,616\,932\,061\,331\,297\,671\,076\,950\,376\,704\,358\,248\,264\,961\,101\,007\,730\,925\,578\,212\,714\,241\,825\,520\,532\,318\,171\,192\,313\,5264\,\cdots. \qquad (100)$$

When specializing to the particular combination (64), the singular behavior of the integral $\tilde{\chi}_d^{(4)}(t)$ reads

$$\tilde{\chi}_d^{(4)}(t)(\mathrm{Singular}, t = 1) = \frac{1}{8\,(1-t)} \cdot \left(1 - \frac{1}{3\pi^2}[64 + 16 \cdot (3I_1^{>} - 4I_2^{>})]\right) + \frac{7}{16\pi^2} \cdot \ln\frac{16}{1-t} - \frac{1}{16\pi^2} \cdot \ln^2\frac{16}{1-t}. \qquad (101)$$

This agrees[33] with the result determined numerically in appendix B of [19].

We find again and similar to $\tilde{\chi}_d^{(3)}(t)$ that the most divergent term disappears for the particular combination giving $\tilde{\chi}_d^{(4)}(t)$. And here again, this is what has been observed [12] for the susceptibility $\tilde{\chi}^{(4)}$ at the singularity $x = 16w^2 = 1$ which occurs in the ODE as $z^{-3/2}$ and cancels in the integral $\tilde{\chi}^{(4)}$.

Remark. It is worth recalling that similar calculations for $\tilde{\chi}^{(4)}$, also based on the evaluation of a connection matrix (see section 9 of [11]), require the evaluation of a constant I_4^{-} that is actually expressed in terms of $\zeta(3)$:

$$I_4^{-} = \frac{1}{16\pi^3} \cdot \left(\frac{4\pi^2}{9} - \frac{1}{6} - \frac{7}{2} \cdot \zeta(3)\right), \qquad (102)$$

when the bulk $\tilde{\chi}^{(3)}$ requires a Clausen constant [11] that can be written as

$$Cl_2(\pi/3) = \frac{3^{1/2}}{108} \cdot (3 \cdot \psi(1,\ 1/3) + 3 \cdot \psi(1,\ 1/6) - 8\pi^2). \qquad (103)$$

It is quite natural to see if the constant $3I_1^{>} - 4I_2^{>}$ given with 200 digits in (100) can also be obtained exactly in terms of known transcendental constants ($\zeta(3), \ldots$), or evaluations of hypergeometric functions that naturally occur in connection matrices [11] (see (F.2) in appendix F). This question is sketched in appendix F.

[33] Note that there is an overall factor of 2 between this result and the results given in appendix B of [19] which comes from a multiplicative factor of 2 in the series (around $t = 0$) of $\tilde{\chi}_d^{(4)}(t)$ used in [19]. This applies also to the result of the singular behavior at $t = -1$.

6.2. Behavior of $\tilde{\chi}_d^{(4)}(t)$ as $t \rightarrow -1$

When $t \rightarrow -1$, the only singular terms come from $\tilde{\chi}_{d;3}^{(4)}(t)$. Furthermore, the operator A_3 of (61) is non-singular at $t = -1$. Therefore, the only singularities in $\tilde{\chi}_d^{(4)}(t)$ come from the terms with $\ln(1-t^2)$ in expansion (89) of ${}_4F_3([1/2, 1/2, 1/2, 1/2], [1, 1, 1]; t^2)$ at $t \rightarrow -1$. Thus, from (64) we find that the singular part of $\tilde{\chi}_d^{(4)}(t)$ at $t = -1$ reads

$$\tilde{\chi}_{d;\mathrm{sing}}^{(4)}(t) = -\frac{1}{8}\,\tilde{\chi}_{d;3;\mathrm{sing}}^{(4)}(t) = -\frac{1}{8}\,\ln(1-t^2)\cdot A_3\left(\sum_{n=0}^{\infty} q_n\cdot(1-t^2)^{n+1}\right), \tag{104}$$

with q_n obtained from (92) as

$$q_n = \frac{(3/2)_n^2}{(n+1)!\,n!}\cdot\sum_{k=0}^{n}\frac{(-n)_k}{(3/2)_k^2}\cdot A_k^{(3)}, \tag{105}$$

where $A_k^{(3)}$ is given by (94). We know from the exponents of $L_4^{(4)}$ at $t = -1$ that the result has the form $(t+1)^7\cdot\ln(t+1)$. Therefore, to obtain this term in a straightforward way we need to expand the coefficient of $\ln(1-t^2)$ to order $(1+t)^9$ in order that the term from $(1+t)\cdot D_t^3$ be of order $(1+t)^7$. This is tedious by hand but is easily done on Maple and we find that the leading singularity in $\tilde{\chi}_d^{(4)}(t)$ at $t = -1$ is

$$\tilde{\chi}_d^{(4)}(\mathrm{Singular}, t = -1) = \frac{1}{26\,880\pi^2}\cdot(1+t)^7\cdot\ln(1+t), \tag{106}$$

which agrees with appendix B of [19].

7. Conclusion: Is the Ising model 'modularity' reducible to selected ${}_{(q+1)}F_q$ hypergeometric functions?

In this paper, we have derived the exact analytic expressions for $\tilde{\chi}_d^{(3)}(x)$ and $\tilde{\chi}_d^{(4)}(t)$ for the diagonal susceptibility of the Ising model and from them have computed the behavior at all singular points. We have also obtained some additional exact results for $\tilde{\chi}_d^{(5)}(x)$ (see section 4). This completes the program initiated in [19], where the singularities were studied by means of formal solutions found on Maple and numerical studies of the connection problem [11]. In this sense, we have a complete solution to the problem. However, in another sense, there are still most interesting open questions.

In section 6.1, we used the solution of the hypergeometric connection problem [49] which gave the connection constants $I_n^<$ and $I_n^>$ as multiple sums. However, there are special cases, as mentioned in [52], where it is known by indirect means that the series can be simplified, but for which a direct simplification of the series has not been found. One example is given by the computation in section 5.2 of the singularity of $\tilde{\chi}_d^{(3)}(x)$ at $t = 1$, which we accomplished by means of the reduction (19) of an ${}_3F_2$ function to a product of ${}_2F_1$ functions. This produced the gamma function evaluation of the singularity at $x = 1$ of (D.8). This singularity could also have been computed directly from the ${}_3F_2$ function in (16) by the use of the Bühring formula (89), but a reduction of the sums for the required I_n to the gamma function form is lacking. There are two suggestions that such a reduction may exist for $\tilde{\chi}_d^{(4)}(t)$ at $t = 1$. The first is that, by analogy with the corresponding calculation for $\tilde{\chi}^{(4)}(t)$ in the bulk [3], the amplitude could be evaluated in terms of $\zeta(3)$. The second is that evaluations of Calabi–Yau [48] hypergeometric functions like ${}_4F_3([1/2, 1/2, 1/2, 1/2], [1, 1, 1], z)$ take place. The larger question, of course, is how much the structure seen in $\tilde{\chi}_d^{(n)}$ and $\tilde{\chi}^{(n)}$ for $n = 1, 2, 3, 4$ can be expected to generalize to higher values of n. It is the opinion of the authors that there is a great deal of mathematical structure of deep significance remaining to be discovered.

These new exact results for the diagonal susceptibility of the Ising model confirm that the linear differential operators that emerge in the study of these Ising n-fold integrals, are *not only* 'derived from geometry' [13], but actually correspond to 'special geometries' (they are homomorphic to their adjoints, which means [27] that their differential Galois groups are 'special', their symmetric square or exterior square has rational function solutions, ...). More specifically, when we are able to get the exact expressions of these linear differential operators, we find out that they are associated with elliptic function theory (*elliptic functions* [53] *or modular forms*), and, in more complicated cases, *Calabi–Yau* ODEs [22, 23]. This totally confirms what we already saw [13] on $\tilde{\chi}^{(5)}$ and $\tilde{\chi}^{(6)}$. We see, in particular, with $\chi_d^{(5)}(x)$, the emergence of a remarkable order-6 operator which is such that *its symmetric square has a rational solution*.

Let us recall that it is, generically, extremely difficult to see that a linear differential operator, corresponding to a Calabi–Yau ODE [22, 23], is homomorphic to a ${}_{q+1}F_q$ hypergeometric linear differential operator up to an algebraic pullback. In the worst case, it is not impossible that many of the Calabi–Yau ODEs are actually reducible (up to operator equivalence) to ${}_{q+1}F_q$ hypergeometric functions up to algebraic pullbacks that have not been found yet. Let us assume that this is not the case and that the Calabi–Yau world is not reducible to the hypergeometric world (up to involved algebraic pullback), we still have to see if the 'special geometry' operators that occur for the Ising model are 'hypergeometric' ones, reducing, in fact, systematically to (selected k-balanced) ${}_{q+1}F_q$ hypergeometric functions, or correspond to the more general solutions of Calabi–Yau equations.

Acknowledgments

This work was supported in part by the National Science Foundation grant PHY-0969739. We thank D Bertrand, A Bostan, J Morgan, A Okunkov and J-A Weil for fruitful discussions. M van Hoeij is supported by NSF grant 1017880. This work has been performed without any support of the ANR, the ERC or the MAE.

Appendix A. Miscellaneous comments on the modular curve (27)

Let us introduce other rational expressions, similar to (17) and (26),

$$Q_2(x) = \frac{27\,x^4 \cdot (1+x)}{(x\,+\,2)^6}, \qquad Q_3(x) = -\,\frac{27\,x \cdot (1+x)^4}{(x-1)^6},$$

where recalling expression (26) one has (for instance)

$$Q_2(x) = Q_1\Big(\frac{1}{x}\Big) = Q_1\Big(-\,\frac{1\,+x}{x}\Big),$$
$$Q_3(x) = Q_1\Big(-\,\frac{1}{1\,+\,x}\Big) = Q_1\Big(-\,\frac{x}{1\,+\,x}\Big) = Q_2(-1\,-x) = Q_2\Big(-\,\frac{1\,+x}{x}\Big).$$

Remarkably, the elimination of x between the Hauptmodul $Q = Q(x)$ and $Q_2 = Q_2(x)$ (or $Q = Q(x)$ and $Q_3 = Q_3(x)$) also gives the *same* modular curve (27).

We also have remarkable identities on the *same* hypergeometric function with these new Hauptmodul pullbacks (A.1):

$$(x\,+2) \cdot {}_2F_1\left(\left[\tfrac{1}{6},\,\tfrac{1}{3}\right],\,[1];\;Q(x)\right) = 2 \cdot (1+x+x^2)^{1/2} \cdot {}_2F_1\left(\left[\tfrac{1}{6},\,\tfrac{1}{3}\right],\,[1];\;Q_2(x)\right) \qquad (A.1)$$

and

$$(1\,-x) \cdot {}_2F_1\left(\left[\tfrac{1}{6},\,\tfrac{1}{3}\right],\,[1];\;Q(x)\right) = (1+x+x^2)^{1/2} \cdot {}_2F_1\left(\left[\tfrac{1}{6},\,\tfrac{1}{3}\right],\,[1];\;Q_3(x)\right). \qquad (A.2)$$

The well-known fundamental modular curve [16]

$$5^9\, v^3\, u^3 \;-\; 12 \cdot 5^6\, u^2\, v^2 \cdot (u+v) \;+\; 375\; u\, v \cdot (16\, u^2 \;+\; 16\, v^2 \;-\; 4027\, v\, u) \\ -\, 64\, (v+u) \cdot (v^2 + 1487\, v\, u \;+\; u^2) \;+\; 2^{12} \cdot 3^3 \cdot u\, v = 0, \qquad \text{(A.3)}$$

corresponding to the elimination of the variable x between the previous Hauptmodul (17) and another Hauptmodul[34] $Q_L(x)$:

$$Q_L(x) = -108 \cdot \frac{(1\;+x)^4 \cdot x}{(x^2\;-\;14\,x\;+\;1)^3}, \qquad \text{(A.4)}$$

should not be confused with the (modular) curve [16] (27).

The new modular curve (27) also has a rational parametrization $(u,\ v) = (Q_L(x),\ Q_4(x))$ between this last new Hauptmodul (A.4) and a new simple Hauptmodul:

$$Q_4(x) = 108 \cdot \frac{(1\;+x)^2 \cdot x^2}{(1\;-x)^6}. \qquad \text{(A.5)}$$

Appendix B. Solution of $\mathcal{M}_4$ analytical at $x\;=0$

The solution of $\mathcal{M}_4$ (see (50)), analytical at $x\;=0$, reads

$$\mathrm{Sol}(x) =\; (1\;-x)^{3/4} \cdot \rho(x) \cdot S(x), \qquad \text{(B.1)}$$

where $S(x)$ reads

$$Z_1 \cdot {}_4F_3\left(\left[\tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{1}{2}\right],\ [1,\ 1,\ 1];\ P(x)\right) \;+ Z_2 \cdot {}_4F_3\Big(\left[\tfrac{3}{2},\ \tfrac{3}{2},\ \tfrac{3}{2},\ \tfrac{3}{2}\right],\ [2,\ 2,\ 2];\ P(x)\Big) \\ + Z_3 \cdot {}_4F_3\left(\left[\tfrac{5}{2},\ \tfrac{5}{2},\ \tfrac{5}{2},\ \tfrac{5}{2}\right],\ [3,\ 3,\ 3];\ P(x)\right) \\ + Z_4 \cdot {}_4F_3\left(\left[\tfrac{7}{2},\ \tfrac{7}{2},\ \tfrac{7}{2},\ \tfrac{7}{2}\right],\ [4,\ 4,\ 4];\ P(x)\right),$$

with

$$Z_1 = -512 \cdot \frac{n_1}{d_1}, \quad Z_2 = 128 \cdot \frac{n_2}{d_2}, \quad Z_3 = -54 \cdot \frac{n_3}{d_3}, \quad Z_4 = -625 \cdot \frac{n_4}{d_4},$$

and

$$n_1 = (7\,x^3\;-56\,x^2\;+112\,x\;-64)\;\cdot\;(1\;-x)^{1/2} + (x-1)\,(x^3\;-24\,x^2\;+80\,x\;-64),$$
$$n_2 = (2352\,x^2\;-472\,x^3\;-3904\,x\;+2048\;+19\,x^4)\;\cdot\;(1\;-x)^{1/2} \\ +\,x^5\;-125\,x^4\;+1288\,x^3 - 4048\,x^2\;+4928\,x\;-2048,$$
$$n_3 = (x^6\;-28\,080\,x^3\;-355\,x^5\;-52\,992\,x\;+17\,920 + 5750\,x^4\;+57\,760\,x^2)\cdot\;(1\;-x)^{1/2} \\ -\,2\,(x-1)\,(20\,x^5\;-855\,x^4\;+6736\,x^3\;-18\,992\,x^2\;+22\,016\,x\;-8960),$$
$$n_4 = (x^2\;-8\,x\;+8)\,(x^4\;-64\,x^3\;+320\,x^2\;-512\,x\;+256)\cdot\;(1\;-x)^{1/2} \\ -\,4\,(x-1)\,(x-2)\,(3\,x\;-4)\,(x\;-4)\,(x^2\;-16\,x\;+16),$$

and

$$d_1 = (1-x)\cdot x^2 \cdot ((x-2)\cdot(x^2\;-16\,x\;+16)\cdot(1\;-x)^{1/2} - 2\,(x-1)\,(3\,x-4)\,(x-4)),$$
$$d_2 = (1-x)\cdot x^4 \cdot (4\cdot(x-2)\cdot(1\;-x)^{1/2}\;+\;x^2\;-8\,x\;+8),$$
$$d_3 = (1-x)\cdot x^6 \cdot (2\,(x-1)\;-\;(x-2)\cdot(1\;-x)^{1/2}),$$
$$d_4 = (1-x)\cdot x^8,$$

[34] Related by a Landen transformation on $x^{1/2}$ see [16].

and

$$\rho(x) = \Big((2-x)\cdot(1-x)^{1/2} + 2\cdot(x-1) \Big)^{1/2},$$

and where $P(x)$ denotes the pullback (56):

$$P(x) = \frac{x^2 - 8x + 8}{x^2} - 4\cdot(2-x)\cdot\frac{(1-x)^{1/2}}{x^2}. \tag{B.2}$$

This solution has the integrality property [32]. Changing x into $64x$, the series expansion of the previous solution (B.1) has *integer coefficients*:

$$\begin{aligned}\mathrm{Sol}(64x) = \; & 128 + 2560\,x + 116\,736\,x^2 + 6072\,320\,x^3 + 335\,104\,000\,x^4 \\ & + 19\,117\,744\,128\,x^5 + 1114\,027\,622\,400\,x^6 + 65\,874\,638\,708\,736\,x^7 \\ & + 3937\,277\,209\,282\,560\,x^8 + \cdots.\end{aligned}$$

Appendix C. The linear differential operator $\mathcal{L}_{11}^{(5)}$ in exact arithmetic

The factors occurring in the differential operator $\mathcal{L}_{11}^{(5)}$ read

$$U_1^{(5)} = D_x - \frac{d}{dx}\ln\Big(\frac{x}{(1-x)^3}\Big), \tag{C.1}$$

$$V_1^{(5)} = D_x - \frac{1}{2}\cdot\frac{d}{dx}\ln\Big(\frac{(1+x+x^2)^3}{(1+x)^2\cdot(1-x)^6\cdot x^2}\Big), \tag{C.2}$$

$$W_1^{(5)} = D_x - \frac{1}{2}\cdot\frac{d}{dx}\ln\Big(\frac{(x^2+1)^2}{(1+x)^2\cdot(1-x)^6\cdot x}\Big), \tag{C.3}$$

$$L_4^{(5)} = D_x^4 + \frac{p_3}{p_4}\cdot D_x^3 + \frac{p_2}{p_4}\cdot D_x^2 + \frac{p_1}{p_4}\cdot D_x + \frac{p_0}{p_4}, \tag{C.4}$$

with

$$\begin{aligned}p_4 = \; & x^3\cdot(1+x+x^2)\cdot(x+1)^3\cdot(x-1)^4\cdot(160 + 3148\,x + 24\,988\,x^2 + 86\,008\,x^3 \\ & +141698\,x^4 + 69\,707\,x^5 - 141\,750\,x^6 - 358\,707\,x^7 - 356\,606\,x^8 - 1071\,x^9 + 347\,302\,x^{10} \\ & +510\,214\,x^{11} + 347\,302\,x^{12} - 1071\,x^{13} - 356\,606\,x^{14} - 358\,707\,x^{15} - 141\,750\,x^{16} \\ & +69\,707\,x^{17} + 141\,698\,x^{18} + 86\,008\,x^{19} + 24\,988\,x^{20} + 3148\,x^{21} + 160\,x^{22}),\end{aligned}$$

$$\begin{aligned}p_3 = \; & 2\,x^2\cdot(x+1)^2\cdot(x-1)^3\cdot(-880 - 16\,620\,x - 126\,586\,x^2 - 421\,558\,x^3 - 520\,547\,x^4 \\ & +733\,378\,x^5 + 3794\,648\,x^6 + 6252\,130\,x^7 + 3922\,367\,x^8 - 4349\,032\,x^9 - 12\,817\,741\,x^{10} \\ & -12\,881\,692\,x^{11} - 2612\,141\,x^{12} + 10\,986\,996\,x^{13} + 16\,830\,947\,x^{14} + 12\,283\,572\,x^{15} \\ & +729\,267\,x^{16} - 8919\,176\,x^{17} - 10\,905\,121\,x^{18} - 5398\,478\,x^{19} + 866\,024\,x^{20} \\ & +3665\,682\,x^{21} + 3069\,821\,x^{22} + 1351\,818\,x^{23} + 323\,590\,x^{24} + 36\,308\,x^{25} + 1680\,x^{26}),\end{aligned}$$

$$\begin{aligned}p_2 = \; & 2\,x\cdot(x-1)^2\cdot(2400 + 38\,692\,x + 228\,422\,x^2 + 366\,806\,x^3 - 1591\,741\,x^4 - 8948\,446\,x^5 \\ & -18\,137\,183\,x^6 - 10\,301\,088\,x^7 + 31\,576\,074\,x^8 + 82\,978\,356\,x^9 + 80\,098\,415\,x^{10} \\ & -8308\,172\,x^{11} - 123\,518\,048\,x^{12} - 158\,759\,046\,x^{13} - 65\,285\,821\,x^{14} + 78\,248\,130\,x^{15} \\ & +152\,708\,392\,x^{16} + 124\,727\,752\,x^{17} + 26\,488\,355\,x^{18} - 65\,301\,174\,x^{19} - 90\,679\,899\,x^{20} \\ & -47\,527\,872\,x^{21} + 4032\,496\,x^{22} + 27\,473\,954\,x^{23} + 23\,107\,094\,x^{24} + 9927\,812\,x^{25} \\ & +2288\,564\,x^{26} + 245\,416\,x^{27} + 10\,800\,x^{28}),\end{aligned}$$

$$p_1 = 2\ (x-1)\cdot(-1440 - 15\,176\,x - 3552\,x^2 + 632\,252\,x^3 + 3988\,986\,x^4 + 11\,012\,538\,x^5 + 10\,122\,851\,x^6 - 31\,358\,640\,x^7 - 125\,311\,964\,x^8 - 166\,380\,144\,x^9 + 20\,063\,039\,x^{10} + 375\,202\,188\,x^{11} + 523\,233\,277\,x^{12} + 189\,830\,162\,x^{13} - 422\,078\,559\,x^{14} - 747\,281\,488\,x^{15} - 440\,223\,099\,x^{16} + 161\,161\,298\,x^{17} + 530\,901\,457\,x^{18} + 491\,902\,752\,x^{19} + 168\,466\,049\,x^{20} - 168\,274\,188\,x^{21} - 282\,329\,480\,x^{22} - 158\,906\,808\,x^{23} - 754\,525\,x^{24} + 72\,189\,798\,x^{25} + 61\,435\,092\,x^{26} + 25\,677\,392\,x^{27} + 5672\,988\,x^{28} + 577\,984\,x^{29} + 24\,000\,x^{30}),$$

$$p_0 = -3600 - 52\,880\,x - 324\,108\,x^2 - 1147\,996\,x^3 - 1575\,180\,x^4 + 8228\,874\,x^5 + 52\,977\,905\,x^6 + 108\,476\,130\,x^7 - 739\,178\,x^8 - 371\,064\,711\,x^9 - 563\,202\,298\,x^{10} - 29\,824\,206\,x^{11} + 842\,725\,375\,x^{12} + 1075\,242\,362\,x^{13} + 273\,493\,047\,x^{14} - 909\,934\,423\,x^{15} - 1189\,246\,308\,x^{16} - 414\,338\,515\,x^{17} + 420\,114\,304\,x^{18} + 702\,981\,552\,x^{19} + 447\,865\,799\,x^{20} + 30\,467\,322\,x^{21} - 270\,639\,170\,x^{22} - 233\,990\,685\,x^{23} - 67\,035\,676\,x^{24} + 45\,089\,100\,x^{25} + 61\,580\,064\,x^{26} + 29\,851\,532\,x^{27} + 7030\,080\,x^{28} + 714\,400\,x^{29} + 28\,800\,x^{30}.$$

Appendix D. Analysis of the singular behavior of $\tilde{\chi}_d^{(3)}(x)$

Let us give some detailed analysis of the singular behavior of $\tilde{\chi}^{(3)}_{d;2}(x)$ and $\tilde{\chi}^{(3)}_{d;3}(x)$ around the three singularities: $x = +1,\ -1$ and $e^{\pm 2\pi i/3}$.

D.1. Limit of the connection matrix

The hypergeometric operator $D_z^2 + ((1+a+b)\,z - c)\,D_z + a\,b$ has two solutions, $u_1 = {}_2F_1([a,b],[c];\ z)$ and $u_2 = z^{1-c}\cdot {}_2F_1([a+1-c, b+1-c],\ [2-c];\ z)$, and they connect to $z = 1$ (see, for instance, (1) on page 108 of [51]) according to the connection matrix valid for $c \neq 1$:

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = C\cdot \begin{bmatrix} {}_2F_1([a,b],[a+b-c+1];\ 1-z) \\ (1-z)^{c-a-b}\cdot {}_2F_1([c-a, c-b],\ [c-a-b+1];\ 1-z) \end{bmatrix}, \qquad \text{(D.1)}$$

with

$$C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}, \quad \text{where} \qquad \text{(D.2)}$$

$$C_{11} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \qquad C_{12} = \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)},$$
$$C_{21} = \frac{\Gamma(2-c)\Gamma(c-a-b)}{\Gamma(1-a)\Gamma(1-b)}, \qquad C_{22} = \frac{\Gamma(2-c)\Gamma(a+b-c)}{\Gamma(a+1-c)\Gamma(b+1-c)}. \qquad \text{(D.3)}$$

If we now take the limit $c \to 1$, which is the case of interest in our problem, the connection matrix becomes singular. In this limit, solutions with $\ln(z)$ occur, and we write $u_2 = u_1 + (1-c)\cdot\tilde{u}_2$, where $u_1 = {}_2F_1([a,b],\ [1];\ z)$ and

$$\tilde{u}_2 = \ln z\cdot {}_2F_1([a,b],[1];\ z) - \frac{\partial}{\partial c}\,{}_2F_1([a+1-c, b+1-c],\ [2-c];\ z)|_{c=1}, \qquad \text{(D.4)}$$

yielding for the connection of the solutions u_1 and $\tilde{u}_2$:

$$\begin{bmatrix} u_1 \\ \tilde{u}_2 \end{bmatrix} = \tilde{C} \cdot \begin{bmatrix} {}_2F_1([a, b], [a+b];\ 1-z) \\ (1-z)^{1-a-b} \cdot {}_2F_1([1-a, 1-b], [2-a-b];\ 1-z) \end{bmatrix}, \tag{D.5}$$

with

$$\tilde{C} = \begin{bmatrix} C_{11} & C_{12} \\ \tilde{C}_{21} & \tilde{C}_{22} \end{bmatrix}, \quad \text{where}$$

$$\tilde{C}_{21} = \lim_{c\to 1} \frac{C_{21} - C_{11}}{1-c} = \frac{\Gamma(1-a-b)}{\Gamma(1-a)\Gamma(1-b)} \cdot (2\psi(1) - \psi(1-a) - \psi(1-b)),$$

$$\tilde{C}_{22} = \lim_{c\to 1} \frac{C_{22} - C_{12}}{1-c} = \frac{\Gamma(a+b-1)}{\Gamma(a)\Gamma(b)} \cdot (2\psi(1) - \psi(a) - \psi(b)), \tag{D.6}$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$.

D.2. The behavior as $x \to 1$

To evaluate $\tilde{\chi}^{(3)}_{d;2}(x)$ for $x \to 1$, we use the evaluation of ${}_2F_1([1/2, \pm 1/2], [1];\ x^2)$ for $x \to 1$, and find, from (13), that its singular part reads

$$\tilde{\chi}^{(3)}_{d;2}(\text{Singular}, x = 1) = \frac{2}{\pi(1-x)^2} - \frac{1}{\pi(1-x)} + \frac{1}{2\pi} \ln(1-x). \tag{D.7}$$

To evaluate $\tilde{\chi}^{(3)}_{d;3}(x)$ as $x \to 1$, we specialize (D.5) to $a = 1/6$, $b = 1/3$ and $z = Q$, where Q is defined by (17). Then, as $x \to 1$ one has $(1-Q)^{-1/2} \to 2/\sqrt{3}/(1-x) - 1/\sqrt{3}$, and, thus, one deduces the singular part of $\tilde{\chi}^{(3)}_{d;3}(x)$ using (19), or rather[35] $\tilde{\chi}^{(3)}_{d;3}(x) = M_1({}_2F_1([1/3,\ 1/6],\ [1];\ Q)^2)$, where the linear differential operator M_1 is defined by (24):

$$\tilde{\chi}^{(3)}_{d;3}(\text{Singular}, x = 1) = \frac{6}{\pi} \cdot \frac{1}{(1-x)^2} + \frac{3}{(1-x)} \cdot \left[-\frac{1}{\pi} + \frac{5\pi}{9\Gamma^2(5/6)\Gamma^2(2/3)} - \frac{8\pi}{\Gamma^2(1/6)\Gamma^2(1/3)} \right]. \tag{D.8}$$

D.3. The behavior as $x \to -1$

When $x \to -1$, it is straightforward from (13) to obtain

$$\tilde{\chi}^{(3)}_{d;2}(\text{Singular}, x = -1) = \frac{1}{2\pi} \cdot \ln(1+x). \tag{D.9}$$

To evaluate $\tilde{\chi}^{(3)}_{d;3}$ we note, when $x \to -1$, that Q vanishes as $Q \sim \frac{27}{4}(1+x)^2$. However, we cannot directly set $Q = 0$ in (16) or (19) because we must analytically connect the solution analytic at $x = 0$ to the proper solution at $x = -1$. To do this, we use the results of appendix D.1. Using the fact that there is no singularity at $x = -1/2$, we see that we must choose near $x = -1/2$,

$$(1-Q)^{1/2} = \frac{(1-x)\cdot(1+2x)\cdot(2+x)}{2\cdot(1+x+x^2)^{3/2}}, \tag{D.10}$$

which is positive for $-1/2 < x < 0$ and negative for $-1 < x < -1/2$. Therefore, for $-1 < x < -1/2$, we see that

$$u_1 = {}_2F_1([1/6, 1/3],\ [1];\ Q) \longrightarrow \frac{\Gamma(1/2)}{\Gamma(5/6)\Gamma(2/3)} \cdot {}_2F_1([1/6, 1/3], [1/2];\ 1-Q) - \frac{\Gamma(-1/2)}{\Gamma(1/6)\Gamma(1/3)} \cdot (1-Q)^{1/2} \cdot {}_2F_1([5/6, 2/3], [3/2];\ 1-Q). \tag{D.11}$$

[35] Using (24) and the fact that ${}_2F_1([1/3,\ 1/6],\ [1];\ Q)$ is a solution of X_2.

Furthermore, for $-1 < x < -1/2$

$$
\begin{aligned}
{}_2F_1([1/6,1/3],\,[1];\,Q) &\rightarrow \frac{\sqrt{3}}{2\pi}\,\tilde{u}_2 + \frac{1}{2}\left(\frac{\Gamma(1/2)}{\Gamma(5/6)\Gamma(2/3)}\cdot\tilde{C}_{22} + \frac{\Gamma(-1/2)}{\Gamma(1/6)\Gamma(1/3)}\,\tilde{C}_{21}\right)\cdot u_1 \\
&= \frac{\sqrt{3}}{2\pi}\left({}_2F_1([1/6,1/3],\,[1];\,Q)\,\ln Q - \frac{\partial}{\partial c}\,{}_2F_1([7/6-c,4/3-c],\,[2-c];\,Q)|_{c=1}\right) \\
&\quad + \frac{1}{2}\left(\frac{\Gamma(1/2)}{\Gamma(5/6)\Gamma(2/3)}\cdot\tilde{C}_{22} + \frac{\Gamma(-1/2)}{\Gamma(1/6)\Gamma(1/3)}\cdot\tilde{C}_{21}\right)\cdot{}_2F_1([1/6,1/3],\,[1];\,Q). \qquad \text{(D.12)}
\end{aligned}
$$

By using relations (D.6), we note that

$$\frac{\Gamma(1/2)}{\Gamma(5/6)\Gamma(2/3)}\cdot\tilde{C}_{22} + \frac{\Gamma(-1/2)}{\Gamma(1/6)\Gamma(1/3)}\cdot\tilde{C}_{21} = -\frac{\sqrt{3}}{2\pi}\cdot(6\ln 3 + 4\ln 2), \qquad \text{(D.13)}$$

and, thus, as $x \rightarrow -1$

$${}_2F_1([1/6,1/3],\,[1];\,Q) \rightarrow \frac{\sqrt{3}}{\pi}\,(\ln(1+x) - 2\ln 2)\,. \qquad \text{(D.14)}$$

Again using (19) we find, as $x \rightarrow -1$, that

$$\tilde{\chi}^{(3)}_{d;3}\,(\text{Singular}, x=-1) = -\frac{3}{2\pi^2}\cdot\ln^2(1+x) \;+ 3\,\frac{2\ln 2 - 1}{\pi^2}\cdot\ln(1+x). \qquad \text{(D.15)}$$

D.4. The behavior as $x \rightarrow \mathrm{e}^{\pm 2\pi \mathrm{i}/3}$

When $x \rightarrow \mathrm{e}^{\pm 2\pi\mathrm{i}/3}$, then $Q \rightarrow \infty$ and $\tilde{\chi}^{(3)}_{d;3}$ becomes singular. Thus, to extract this singularity we have to connect that solution analytic at $x = 0$ to the singularity at $x = \mathrm{e}^{\pm 2\pi\mathrm{i}/3}$. To do this, it is convenient to note that Q is symmetric about $x = -1/2$. This is seen by letting $x = -1/2 + \mathrm{i}y$, to obtain $Q(y) = (1+4y^2)^2/(1-\frac{4}{3}y^2)^3$, and defining z by

$$z = (1 - Q(y))^{1/2} = \frac{\mathrm{i}\,y\cdot(9/4+y^2)}{(3/4-y^2)^{3/2}}. \qquad \text{(D.16)}$$

Furthermore, as y goes from 0 to $\sqrt{3}/2$, $Q(y)$ goes from 1 to ∞. In the previous section we have already connected the solution analytic at $x = 0$ with the solution analytic at $x = -1/2$.

We rewrite the solutions using (D.16) as

$$
\begin{aligned}
{}_2F_1([1/6,1/3],\,[1];\,Q) &= \frac{\Gamma(1/2)}{\Gamma(5/6)\Gamma(2/3)}\cdot{}_2F_1([1/6,1/3],\,[1/2];\,z^2) \\
&\quad + \frac{\Gamma(-1/2)}{\Gamma(1/6)\Gamma(1/3)}\cdot z\cdot{}_2F_1([5/6,2/3],\,[3/2];\,z^2), \qquad \text{(D.17)} \\
F([7/6,4/3],[2];Q) &= 18\cdot\frac{\partial}{\partial Q}\,{}_2F_1([1/6,1/3],\,[1];\,Q).
\end{aligned}
$$

These solutions must be connected from $y = 0$ to $y = \sqrt{3}/2$ along the straight line path $x = -1/2 + \mathrm{i}y$. On this path, z^2 is on the negative real axis and, hence, we may use the connection formula (2) on page 109 of [51]

$$
\begin{aligned}
{}_2F_1([a,b],\,[c];\,z^2) &= \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}\cdot(-z^2)^{-a}\cdot{}_2F_1([a,1-c+a],[1-b+a];\,z^{-2}) \\
&\quad + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}\cdot(-z^2)^{-b}\cdot{}_2F_1([b,1-c+b],[1-a+b];\,z^{-2}). \qquad \text{(D.18)}
\end{aligned}
$$

Setting $z = \mathrm{i}\,\bar{z}$, with $\bar{z}$ being real and non-negative, we obtain

$$
\begin{aligned}
{}_2F_1([1/6,1/3],\,[1];\,Q) &= \frac{\sqrt{3}}{2}(\sqrt{3}-\mathrm{i})\frac{\Gamma(2/3)}{\Gamma(5/6)^2}\cdot\bar{z}^{-1/3}\cdot{}_2F_1([1/6,2/3],\,[5/6];\,-\bar{z}^{-2}) \\
&\quad + \frac{3}{2}\,(\mathrm{i}\sqrt{3}-1)\,\frac{\Gamma(5/6)^2}{\pi\,\Gamma(2/3)}\cdot\bar{z}^{-2/3}\cdot{}_2F_1([1/3,5/6],\,[7/6];\,-\bar{z}^{-2}), \qquad \text{(D.19)}
\end{aligned}
$$

and a similar relation for ${}_2F_1([7/6, 4/3], [2]; Q)$. Now we note that $Q = 1 - z^2 = 1 + \bar{z}^2$, and thus (see (19))

$$2F_1([1/6, 1/3], [1]; Q)^2 + \frac{2Q}{9}\, {}_2F_1([1/6, 1/3], [1]; Q) \cdot {}_2F_1([7/6, 4/3], [2]; Q),$$

can be rewritten in terms of ${}_2F_1$ hypergeometric functions of argument $-\bar{z}^{-2}$; this last expression goes as $-54\sqrt{3} \cdot \mathrm{i}\, \bar{z}^{-3}/\pi/35$ when $\bar{z} \to \infty$. Noting, as $x \to \mathrm{e}^{2\pi \mathrm{i}/3}$, that

$$\left(\bar{z}, \frac{(1+2x)\cdot(x+2)}{(1-x)\cdot(x^2+x+1)}\right) \longrightarrow \left(\frac{3^{3/4}}{2}\,\mathrm{e}^{-3\pi \mathrm{i}/4} \cdot (x-x_0)^{-3/2}, \frac{\mathrm{e}^{\pi \mathrm{i}/3}}{x-x_0}\right), \qquad \text{(D.20)}$$

we find that the leading singularity at $x_0 = \mathrm{e}^{2\pi \mathrm{i}/3}$ in $\tilde{\chi}^{(3)}_{d,3}$ is

$$\tilde{\chi}^{(3)}_{d;3}(\text{Singular}, x = x_0) = \frac{16 \cdot 3^{5/4}}{35\pi} \cdot \mathrm{e}^{\pi \mathrm{i}/12} \cdot (x-x_0)^{7/2}. \qquad \text{(D.21)}$$

Appendix E. Analysis of the singular behavior of $\tilde{\chi}^{(4)}_{d;2}(t)$ as $t \to 1$

To get the singular behavior of $\tilde{\chi}^{(4)}_{d;2}(t)$ as $t \to 1$, we use (12) of page 110 of [51]

$${}_2F_1([1/2, -1/2], [1]; t) \to \frac{2}{\pi} \cdot \left(1 + \frac{1-t}{4} \cdot \left(\ln\frac{16}{1-t} - 1\right)\right) + \cdots \qquad \text{(E.1)}$$

and

$${}_2F_1([1/2, 1/2], [1]; t) \longrightarrow \frac{1}{\pi} \cdot \left(\ln\frac{16}{1-t} + \frac{1-t}{4} \cdot \ln\frac{16}{1-t} - 2\right) + \cdots. \qquad \text{(E.2)}$$

Using these in (13), we find the result quoted in the text in (85).

Appendix F. Toward an exact expression for $3I_1^{>} - 4I_2^{>}$

The constant $3I_1^{>} - 4I_2^{>}$ occurs for $\tilde{\chi}^{(4)}_d(t)$ through $\tilde{\chi}^{(4)}_{d;3}(t)$ which is obtained in (63) by the action of the linear differential operator A_3 on ${}_4F_3([1/2, 1/2, 1/2, 1/2], [1, 1, 1]; t^2)$.

The constant $3I_1^{>} - 4I_2^{>}$ can then be deduced from the 4×4 connection matrix for ${}_4F_3([1/2, 1/2, 1/2, 1/2], [1, 1, 1]; t^2)$. The line of the connection matrix giving the solution analytic at $t = 0$ in terms of the solutions at $t = 1$ is

$$[A_{4,1}, -1/2 \cdot A_{4,1} + 2/\pi^2, A_{4,3} - 2\,\mathrm{i}/\pi, A_{4,4} + \mathrm{i}/\pi],$$

and the constant $3I_1^{>} - 4I_2^{>}$ reads

$$\frac{4}{3\pi^2} \cdot (3I_1^{>} - 4I_2^{>}) = -\frac{16}{\pi^2} + \frac{17}{108} \cdot A_{4,1} - \frac{2}{3} \cdot A_{4,3} - \frac{4}{3} \cdot A_{4,4}. \qquad \text{(F.1)}$$

The entry $A_{4,1}$ of the connection matrix is actually the *evaluation of the hypergeometric function* (59) at $t = 1$:

$$A_{4,1} = -2 \cdot {}_4F_3\left(\left[\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right], [1, 1, 1]; 1\right). \qquad \text{(F.2)}$$

There is, at first sight, a '$\ln(2)$' coming from the terms in the Bühring formula [49] involving the ψ function. This is the 'same' $\ln(2)$ which appears in the connection formulas for $E(k)$ and $K(k)$. However, numerically, these $\ln(2)$ contributions in $A_{4,3}$ and $A_{4,4}$ read, respectively, ($\alpha = -1.945\,304\,0783\ldots$, $\gamma = 0.527\,449\,5683\ldots$):

$$A_{4,3} = \alpha + 2 \cdot \beta \cdot \ln(2), \qquad A_{4,4} = \gamma - \beta \cdot \ln(2), \qquad \text{(F.3)}$$
$$\beta = 0.101\,321\,183\,\cdots.$$

The fact that these two entries occur through the linear combination, $A_{4,3} + 2 \cdot A_{4,4}$ actually cancels a $\ln(2)$ contribution in the expression of the constant $3I_1^{>} - 4I_2^{>}$.

Similar constants (see (102) for the bulk $\tilde{\chi}^{(4)}$, (103) for the bulk $\tilde{\chi}^{(3)}$) can be deduced from entries of the connection matrices (occurring in the exact calculation of the differential Galois group [11]), such entries being often closely related to the evaluation at selected singular points of the holonomic solutions we are looking at. When hypergeometric functions like (59) pop out, it is not a surprise to have entries that can be simply expressed as these hypergeometric functions at $x = 1$ (see (F.2)). Along this line, it is worth recalling that $\zeta(3)$ (or $\zeta(5)$, ...) can simply be expressed in terms of a simple evaluation at $x = 1$ of an ${}_{q+1}F_q$ hypergeometric function [50] (see also [54]):

$$\zeta(3) = {}_4F_3([1, 1, 1, 1], [2, 2, 2];\ 1),$$
$$\zeta(5) = \frac{32}{31} \cdot {}_6F_5\left(\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1\right], \left[\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}\right];\ 1\right). \tag{F.4}$$

It is, thus, quite natural to ask if the sums in $I_1^{>}$ and $I_2^{>}$ can be evaluated in terms of known constants such as $\zeta(3)$ or evaluations (for instance at $t = 1$) of hypergeometric functions.

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