

Higher dimensional mappings ☆

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We present a new class of mappings acting on many variables, and depending on many parameters. These mappings are non-linear (birational) representations of discrete groups generated by involutions, having their origin in the theory of integrable models in statistical mechanics. Various quantities of statistical mechanics present automorphy properties under the action of these groups, which appear thus as a generalization to several complex variables of the fundamental group for Riemann surface. They enjoy many remarkable properties, and we give a preliminary study of these mappings.

1. Introduction

In a previous publication (preceding article [1]), we have introduced a class of birational mappings in projective space, and initiated their study. These mappings are non-linear representations of infinite Coxeter groups. One of the interest of these transformations is that they are symmetries of the Yang–Baxter equations and their higher dimensional generalizations [2,3]. They are also symmetries of various physical quantities as, for example, the partition function: indeed these quantities have definite automorphy properties under these transformations [4,5]. We will not describe here the applications to integrability, but examine a number of aspects of our mappings.

Their main characteristics is that they are constructed as products of involutions. With a pair of non-commuting involutions I, J , we naturally have $f=IJ$ as a mapping to iterate, for it is generically of infinite order, and the group generated by I and J contains essentially the iterates of f (and its inverse f^{-1}).

This yields a natural generalization of iterated mappings: just consider the group generated by more than two involutions. This is what appears in ref. [3] for higher dimensional lattices.

The basis of our construction is thus to provide involutive transformations on the space of parameters we consider (projective space $\mathbb{CP}_n$). These involutions find their origin in the *inversion relations* of statistical mechanics [6–8]. *In all cases [1–3] they amount to taking the inverse of a matrix whose entries belong to the parameter space.*

From the construction method, and the characteristic features of these mappings, a number of properties arise, which will be sketched here. We deliberately use pictures, in order to give a synthetic presentation, and refer to refs. [1–3] for the more algebraic results.

We first exemplify our construction method by describing two classes of inversions, issued respectively from IRF [6] and vertex [9,10] models in statistical mechanics (section 2), and then proceed with their study.

Remarkably, the orbits of some of these mappings (called quasi-integrable in ref. [1]) are dense in smooth non-trivial algebraic subvarieties of the parameter space (curves in the simplest cases we show here in fig. 1). These subvarieties are of the same

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nature as the integrability tori appearing in the study “à la Poincaré” of dynamical systems.

We examine two types of deformations of our mappings:

- The first type is a continuous deformation preserving one of the characteristic features of our mappings: the (almost everywhere) invertibility (figs. 2a and 2b).

- The second type breaks the invertibility (fig. 3).

We next examine the problem of singularities. There are few true singularities: our mappings are defined on projective space and it is harmless to approach or reach the point at infinity. This may of course be quite dangerous numerically. We describe the distribution of the points going to infinity after a finite number of iterations (fig. 4).

We considered in ref. [1] a specific action of the mappings in a double copy of the parameter space motivated by the study of exactly solvable models. We have produced, in the quasi-integrable case, specific curves and we examine here their stability (fig. 5). We also give an example of this action in a non-quasi-integrable case, with the emergence of order from a rather chaotic background (fig. 6).

In all this the fundamental object is the representation Γ of some Coxeter group. In many cases, there is an attracting smooth subvariety (possibly points). It is then natural to look for a fundamental domain (and this will be useful in relation to automorphy properties of various quantities of statistical mechanics). We describe this fundamental domain for a mapping related to the symmetric Ashkin–Teller model [11] (fig. 7).

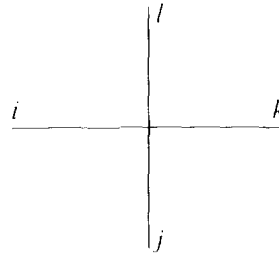
2. Construction of the mappings

We give here the construction in two specific examples, coming from statistical mechanics. This is of course not limitative [1,3].

2.1. Vertex models

Let us consider a vertex model on a two-dimensional square lattice of size $M \times M$ with periodic boundary conditions. To each bond is associated a variable with q possible states. A Boltzmann weight

$w(i, j, k, l)$ is assigned to each vertex configuration [10]



The q^4 homogeneous weights $w(i, j, k, l)$ are first arranged in a $q^2 \times q^2$ matrix R :

$$R_{kl}^j = w(i, j, k, l). \quad (1)$$

We introduce the inverse I by

$$\sum_{\alpha, \beta} R_{\alpha\beta}^j (IR)_{uv}^{\alpha\beta} = \delta_u^i \delta_v^j = \sum_{\alpha, \beta} (IR)_{\alpha\beta}^j R_{uv}^{\alpha\beta} \quad (2)$$

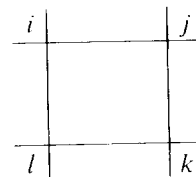
and another inverse J by

$$\sum_{\alpha, \beta} R_{\alpha\beta}^j (JR)_{\beta\beta}^{\alpha i} = \delta_u^i \delta_v^j = \sum_{\alpha, \beta} (JR)_{\alpha\beta}^j R_{\alpha\beta}^i. \quad (3)$$

I and J are two involutions. They are related by partial transposition of the indices (that is to say some rearrangement of the entries of the matrices one considers). They act as polynomial transformations with integer coefficients if we work with homogeneous coordinates, or equivalently rational transformations on the inhomogeneous coordinates.

2.2. IRF models

Let us consider now an interaction around the face (IRF) model [6] on a two-dimensional square lattice of size $M \times M$ with periodic boundary conditions. To each site is associated a variable with q possible states and a Boltzmann weight $w(i, j, k, l)$ is assigned to each face



The q^4 homogeneous weights $w(i, j, k, l)$ are arranged in q^2 matrices $W(i, k)$ of size $q \times q$ with entries

$$W(i, k)_j^l = w(i, j, k, l). \quad (4)$$

We introduce an inverse (inversion by blocks) I by

$$\sum_j W(i, k)_j^l (IW)_j^m (i, k)_m^l = \delta_m^l, \quad (5)$$

that is to say

$$\sum_j (IW)_j^m (i, m, k, j) w(i, j, k, l) = \delta_m^l. \quad (6)$$

Similarly, we may define the inverse J corresponding to the arrangement of the weights obtained by exchanging the role of (i, k) and (j, l) :

$$\sum_j (Jw)(p, j, i, l) w(i, j, k, l) = \delta_k^p. \quad (7)$$

I and J yield birational transformations on the q^4 homogeneous weights $w(i, j, k, l)$.

One may reduce the number of independent variables as in refs. [1,12] by choosing admissible patterns for the action of I and J . In our construction, the number of homogeneous variables takes any value, depending on the model. We thus provide non-trivial (non-linear) mappings with arbitrary number of variables. There are very few such examples in the literature (see for instance the "trace mappings" [21,22]). The existence of singularities permits the multiplication of new examples.

Notice that there exists a difference between the transformations associated with vertex models and the ones associated to IRF models. For vertex models the determinant is a common factor we do not care about (homogeneity of the weights). For IRF models however, one has a priori q^2 independent determinants. The most general model will have transformations of degree $q(q-1)(q^2-1)$ in the IRF case and q^2-1 in the vertex case.

Note also that for both vertex and IRF models, the two inversions I and J do not commute. They generate an infinite discrete group Γ , the infinite dihedral group, isomorphic to the semi-direct product $\mathbb{Z} \ltimes \mathbb{Z}_2$. This group is represented on the matrix elements by *birational transformations* [1,13,14]. Remark that for *vertex models*, the birational transformations associated to the two involutions I and J are naturally related by collineations (which realize the abovementioned rearrangement). This should be compared with the situation for nearest neighbour interaction spin models. In the examples developed

in ref. [1], there is a collineation intertwining the two involutions (matrix inverse I and element by element inverse J). This collineation generalizes the Kramers–Wannier duality. This remarkable property is always verified for birational transformations in $\mathbb{CP}_2$ ^{#1}.

The situations where one defines and combines more than two involutions appear in the study of statistical mechanical models in three or more dimensions (see ref. [3]).

2.3. The Baxter model

As an example, let us consider the paradigm of the exactly solvable models: the symmetric eight-vertex model which can be seen both as a vertex and an IRF model. Seen as a vertex model, it has the following R -matrix:

$$R = \begin{pmatrix} a & 0 & 0 & d \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ d & 0 & 0 & a \end{pmatrix}. \quad (8)$$

Notice that this form is preserved by the operations I and J . The action of I is

$$a \rightarrow \frac{a}{a^2 - d^2}, \quad b \rightarrow \frac{b}{b^2 - c^2},$$

$$c \rightarrow \frac{-c}{b^2 - c^2}, \quad d \rightarrow \frac{-d}{a^2 - d^2},$$

and the action of J is

$$a \rightarrow \frac{a}{a^2 - c^2}, \quad b \rightarrow \frac{b}{b^2 - d^2},$$

$$c \rightarrow \frac{-c}{a^2 - c^2}, \quad d \rightarrow \frac{-d}{b^2 - d^2}.$$

One would obtain exactly the same formulae by considering the model as an IRF model.

We use the visualization method already employed [1] for spin models, i.e. just draw the orbits obtained by numerical iteration and look. Fig. 1

^{#1} In $\mathbb{CP}_2$, the Noether theorem [15] proves that every birational automorphism of the plane can be represented as a product of quadratic transformations and a projective transformation, but this is very specific of $\mathbb{CP}_2$; the birational transformations in $\mathbb{CP}_n$, $n > 2$, are much more complicated.

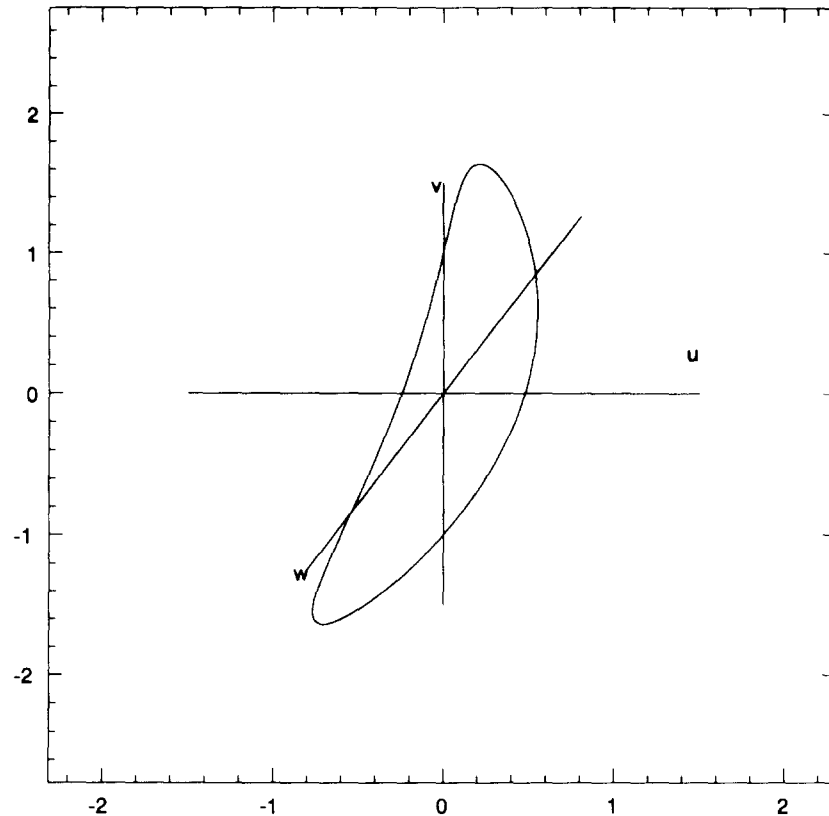


Fig. 1. 3d perspective of an orbit for the Baxter model.

shows in the variables $u=b/a$, $v=c/a$, $w=d/a$, the result of the iteration of IJ . The orbit is dense in a curve. The study of the invariants under the action of I and J shows that this curve is elliptic, as it should since it has an infinite number of automorphisms. It is given by the intersection of the two quadrics $\Delta_1=\text{const}$ and $\Delta_2=\text{const}$ (Clebsch's biquadratic), with Δ_1 and Δ_2 the Γ invariants [1],

$$\begin{aligned}\Delta_1 &= \frac{a^2 + b^2 - c^2 - d^2}{ab + cd}, \\ \Delta_2 &= \frac{ab - cd}{ab + cd}.\end{aligned}\quad (9)$$

There exists an elliptic parametrization of this curve in $\mathbb{CP}_3$ [16,17]. In terms of the elliptic parameter θ , IJ is merely a translation. This proves that the ge-

neric orbits of IJ are dense in curve similar to the one of fig. 1 and that $d\theta$ is the invariant density on this curve. We leave it to the reader to rewrite this density in the original variables.

3. Deformations

From the very construction of the two involutions I and J , they are birational transformations with *integer* coefficients. This means that the construction has a certain rigidity. It is nevertheless possible to introduce deformations of our mappings. In this section, we shall concentrate on the deformations of the mappings associated with the symmetric five-state chiral Potts model, denoted SP2 in ref. [1]. The original I and J read

$$I: \quad u \rightarrow \frac{-u+uv-u^2+v^2}{1+u+v-vu-v^2-u^2},$$

$$v \rightarrow \frac{-v+uv-v^2+u^2}{1+u+v-vu-v^2-u^2}, \quad (10)$$

$$J: \quad u \rightarrow 1/u, \quad v \rightarrow 1/v. \quad (11)$$

3.1. Deformations preserving invertibility

For the Baxter model (9) as well as for the symmetric five-state chiral Potts model (10) or various other models (see ref. [1]), there exists a foliation of the *whole* parameter space by a linear pencil of elliptic curves, and IJ is just a translation on each of these curves. It is represented by a shift of a “spec-

tral” parameter θ . This shift replaces the rotation number on a circle. The evaluation of this shift for each curve [18] should permit:

(1) to examine the location of periodic orbits, and probably show that they are dense in the parameter space;

(2) to have a notion of deformations with fixed rotation number.

In a first step, we consider a two parameter family of deformations preserving the involutive character of I and J . We choose to leave I unchanged and to deform J into $\tilde{J}_{a,b}$:

$$\tilde{J}_{a,b}: \quad u \rightarrow a/u, \quad v \rightarrow b/v. \quad (12)$$

Figs. 2a and 2b show how the initial invariant curve deforms with increasing values of a and b . For very

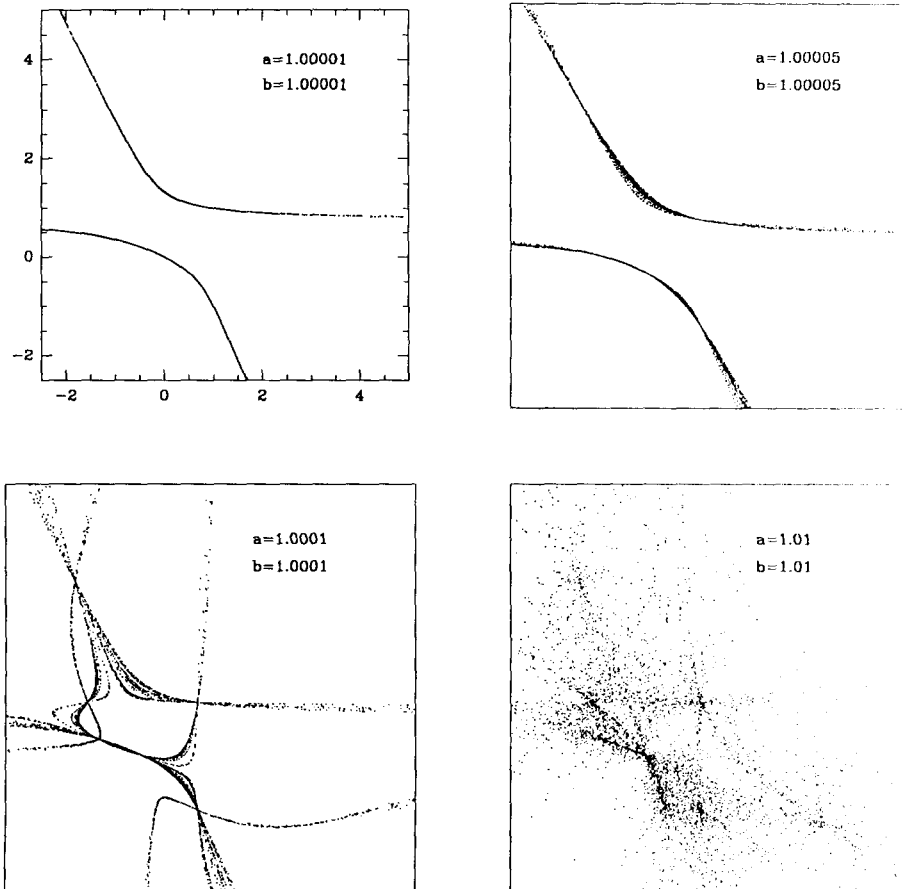


Fig. 2a. Blow up of the tori (I).

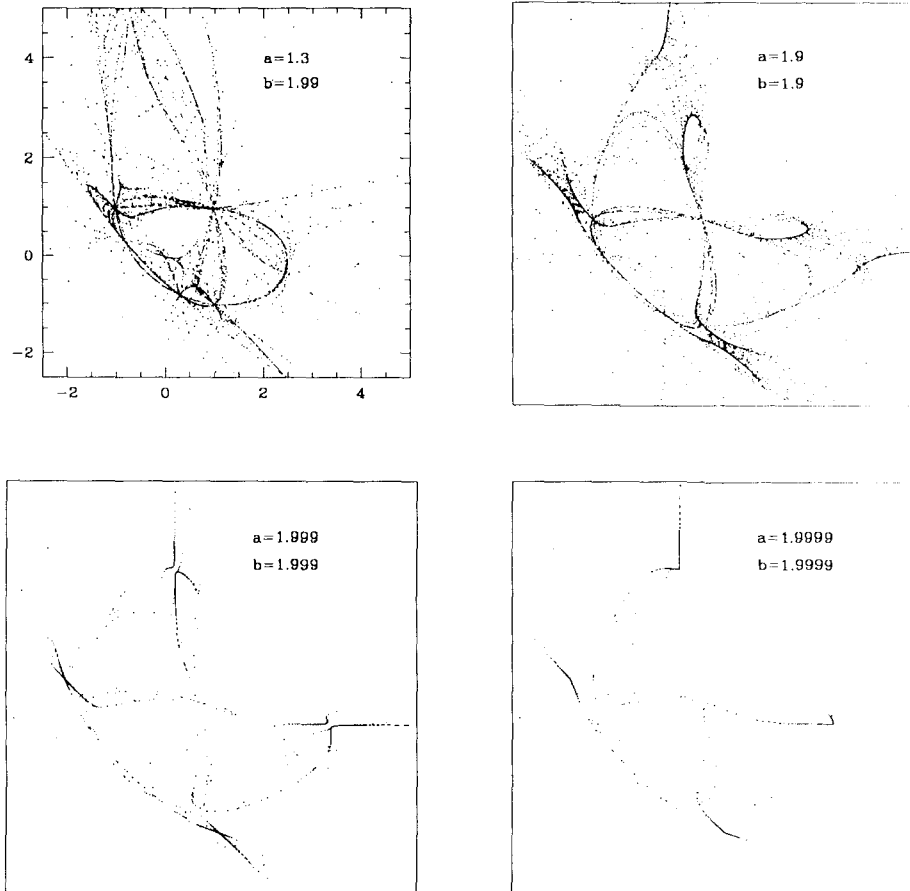


Fig. 2b. Blow up of the tori (II).

small deformations ($a=b=1+5\times 10^{-5}$), the curve is slightly deformed. For $a=b=1+10^{-4}$, the situation is drastically modified. The trajectories seem to wander between curves of the linear pencil described in ref. [1], with different values of the invariant. After a more or less chaotic crossover region, one recovers another particular situation in the limit where a and b are close to 2: the iterates rapidly contract towards a cycle of order four.

3.2. Deformations breaking invertibility

There are many other possible deformations. One can consider deformations breaking the birational character of these mappings. We may for instance replace I by $\tilde{I}$:

$$\begin{aligned} \tilde{I}: \quad u &\rightarrow \frac{-u+uv-u^2+v^2}{1+u+v-vu-v^2+u^2}, \\ v &\rightarrow \frac{-v+uv-v^2+u^2}{1+u+v-vu-v^2-u^2}. \end{aligned} \quad (13)$$

We have just switched the sign of the coefficient of u^2 in the denominator of $I(u)$, thus breaking the invertibility of I . Fig. 3 shows a typical orbit of $\tilde{I}$. The situation is now drastically different and any initial point leads to orbits trapped by this strange attractor [19].

4. The singularities of birational mappings

Our birational mappings are *non-linear represen-*

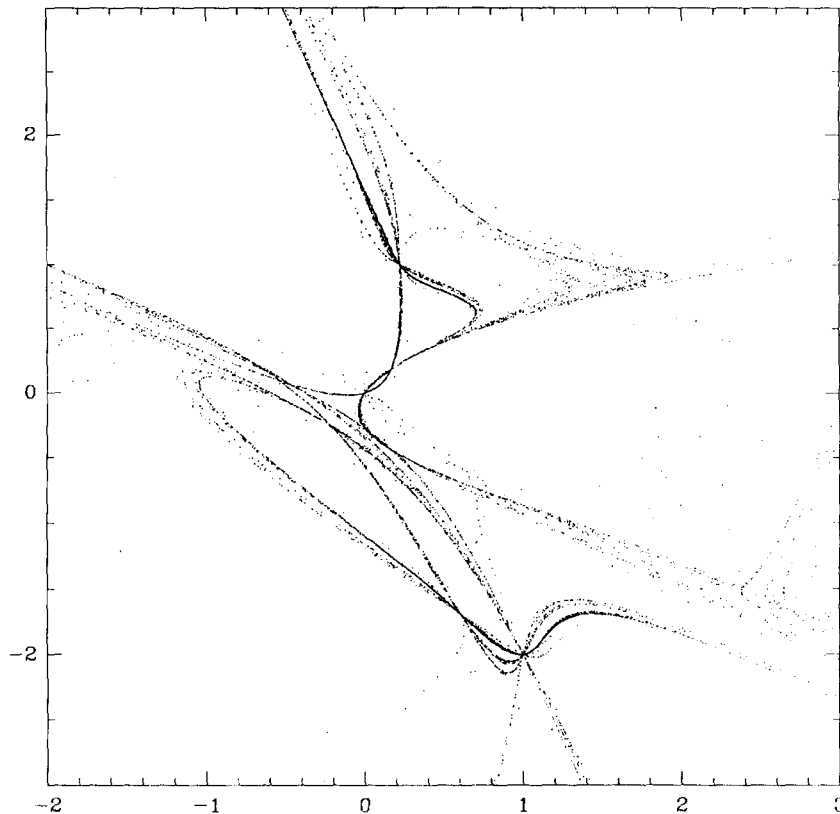


Fig. 3. Orbits of a non-invertible rational mapping.

tations of Coxeter groups. They renew the well-known linear representations of these groups [20]. The iteration may evidently send a point of the orbit to infinity. This is not a true singularity in $\mathbb{CP}_n$. It is merely an effect of the choice of coordinates. It may however be quite problematic for the numerical calculations. Fig. 4 shows the singularities of $(IJ)^p$ with p going from -17 to 17 for the symmetric five-state chiral Potts model (10, 11). In fact fig. 4 is representative of the orbits of an arbitrary curve under the action of the group Γ , which tend to fill the whole plane. This leads to the problem of the orbit of a curve (rather than of a point).

Remark. There might exist true singularities, i.e. points where the action of Γ is not defined. In the quasi-integrable case, these points are related to the intersection points of the pencils. There are also special points: the accumulation points of the action of Γ . The exegesis of these particular points will not be done here.

5. The representation $\rho \times \rho^{-1}$

We consider the action of Γ on a double copy of the parameter space introduced in ref. [1]. One acts with $g \in \Gamma$ on the first copy, and with its inverse g^{-1} on the second one, from which the notation $\rho \times \rho^{-1}$ (the variables corresponding to this second copy will be written with a bar). This seemingly odd construction can be justified from considerations on exactly solvable models [1,2]. For the sake of simplicity we examine the iterations in the plane $\mathbb{CP}_2$.

5.1. Stability of the quasi-integrable cases

When the iteration of IJ leads to a curve on one copy, there exist algebraic invariants Δ such that the curve reads $\Delta = \text{const}$ [1]. If Δ is the same for the initial points in the two copies, then the action of Γ on the double copy yields a curve. This can be easily understood in terms of the elliptic parameters θ and

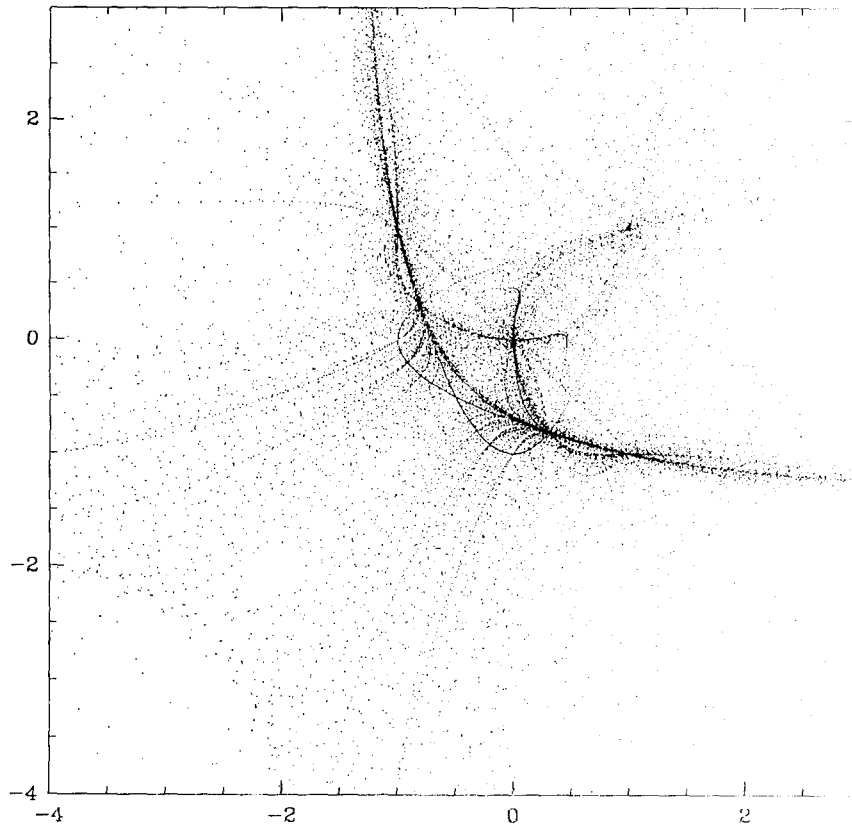


Fig. 4. A "few" singularity points of the iteration of IJ .

$\bar{\theta}$ on the two copies of the curve: $\theta + \bar{\theta}$ is invariant under this action of Γ .

It is natural to wonder if the existence of this curve is stable with respect to small differences between the $\mathcal{A}$'s of the two initial points. Let us consider again the mapping associated to the symmetric five-state chiral Potts model (10, 11). We take initial points belonging to the hyperbola $2uv + 3u + 3v + 2 = 0$, i.e. a genus zero component of the curve $\mathcal{A} = \frac{5}{2}$. If we introduce a mismatch of the invariants $\mathcal{A}$ of the order 10^{-5} , the curve in the $u, \bar{u}$ plane give way to the trajectories shown in fig. 5.

This situation is representative of most of the studies we have performed in the quasi-integrable cases. Although the orbits in one copy are astonishingly stable with respect to the numerical errors (they "survive" a few thousand iterations in single precision mode) and they still appear in the perturbed mapping (see fig. 2a), the precise location of the

points on the curve diverges in no more than a hundred points with numerical drift and no trace of the curve in the $(u, \bar{u})$ plane subsists after perturbation.

5.2. A non-integrable case

We examine this action $\rho \times \rho^{-1}$ for the model linked to $\mathbb{Z}_7$ introduced in ref. [1]. For this model the orbits of IJ are chaotic (see fig. 4 of ref. [1]). Fig. 6 shows a remarkable structure: when one observes the accumulation of the points, one sees that the hyperbolae appear one after the other over an already constituted background. A very heuristic explanation of this emergence of order from disorder is the following: with the same notations as in ref. [1], the two lines $u=1$ and $v=1$ are exchanged under the action of IJ , and contain an infinite

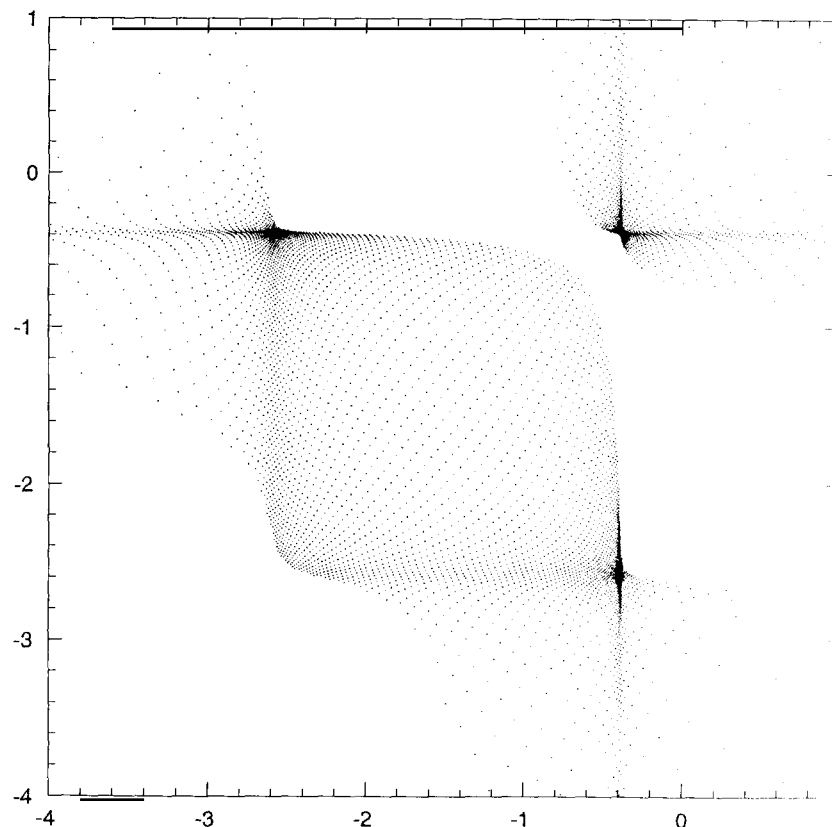


Fig. 5. $(u, \bar{u})$ trajectories with a mismatch of invariants.

set of remarkable attractive–repulsive points $\sin[(n+1)\alpha]/\sin(n\alpha)$, with $\tan(\alpha)=\sqrt{7}$. When the point (u, v) roves about the plane, it eventually passes nearby one of these highly unstable points. This stabilizes for a while the trajectories near their exact values. When restricted to the two lines $u=1$ and $v=1$, the transformation IJ is an homography of infinite order. Orbits in the $(u, \bar{u})$ plane are then hyperbolae as shows as calculation similar to the one of ref. [4].

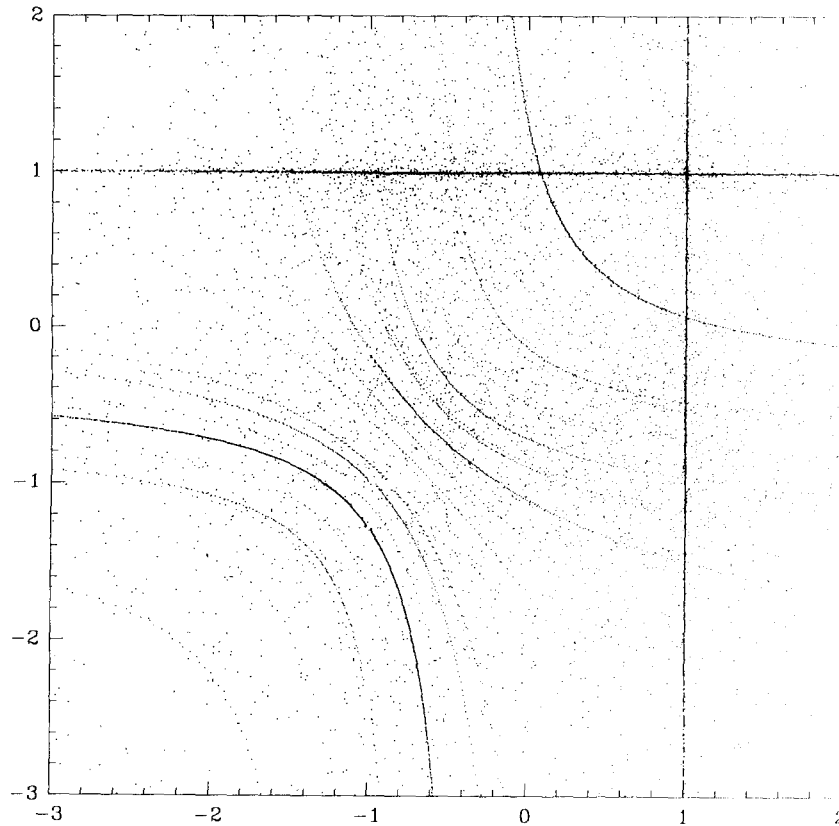
6. Γ as a fundamental group

One should emphasize again that the group Γ is a symmetry group in the whole parameter space, and not only to possible integrability varieties. The partition function is a multivalued automorphic function of several complex variables for this group. A

rough glimpse of the involved but remarkable structure of the covering for the standard scalar checkerboard Potts model can be found in ref. [5]. We want to find the fundamental domain of the group Γ , in the same spirit as was done by Poincaré for automorphic functions (Fuchsian groups, Poincaré groups, ...). We deal here with the group associated to the symmetric Ashkin–Teller model. This is model SPI in ref. [1]. The interest of this model is that there exists an algebraic invariant

$$\Delta = \frac{1-uv}{u-v}, \quad (14)$$

with u and v the inhomogeneous coordinates given in ref. [1], and that there exists moreover a partition of the parameter space in two regions with a different behaviour of the orbits under the group. The two regions are delimited by the value of the invariant. If $|\Delta| \leq 1$, the generic orbits of Γ are dense in the

Fig. 6. Saturn's rings for a $\mathbb{Z}_7$ model.

curve $\Delta = \text{const}$. If $|\Delta| \geq 1$, the points of the orbits accumulate towards the diagonal $u+v=0$ (and of course remain on the curve $\Delta = \text{const}$). Looking for a fundamental domain has a meaning only in this last region. Indeed the action of Γ is not discontinuous in the other region.

The model is particularly simple since there is a rational parametrization [11]. There exists again a parameter θ on the curves $\Delta = \text{const}$, for which the action of I and J reduces to

$$I: \theta \rightarrow -\theta, \quad (15)$$

$$J: \theta \rightarrow \lambda - \theta. \quad (16)$$

Remark that the action of IJ is the translation of θ by λ .

From the action of the symmetry group, we may deduce the expression (or more precisely the ana-

lytic continuation [4]) of (e.g.) the partition function for all θ from its value on the interval $\theta \in [0, \lambda/2]$. The points $\theta = \lambda/2$ are the blowing up of the singular points $(u=1, v=1)$, $(u=-1, v=-1)$ (fixed points of J) in the direction tangent to the curve $\Delta = \text{const}$. We can give the localization of the points $\theta = \lambda$ for each curves $\Delta = \text{const}$. They all lie on the hyperbola

$$2uv + u + v = 0, \quad (17)$$

or its symmetric with respect to the diagonal $u+v=0$. These hyperbolae, together with the straight lines $u = \pm 1$, $v = \pm 1$, delimit the fundamental domain of Γ . The straight lines are globally invariant by Γ . The images of the two hyperbolae gives the boundaries of the successive images of the fundamental domain. Fig. 7 shows the different regions and the funda-

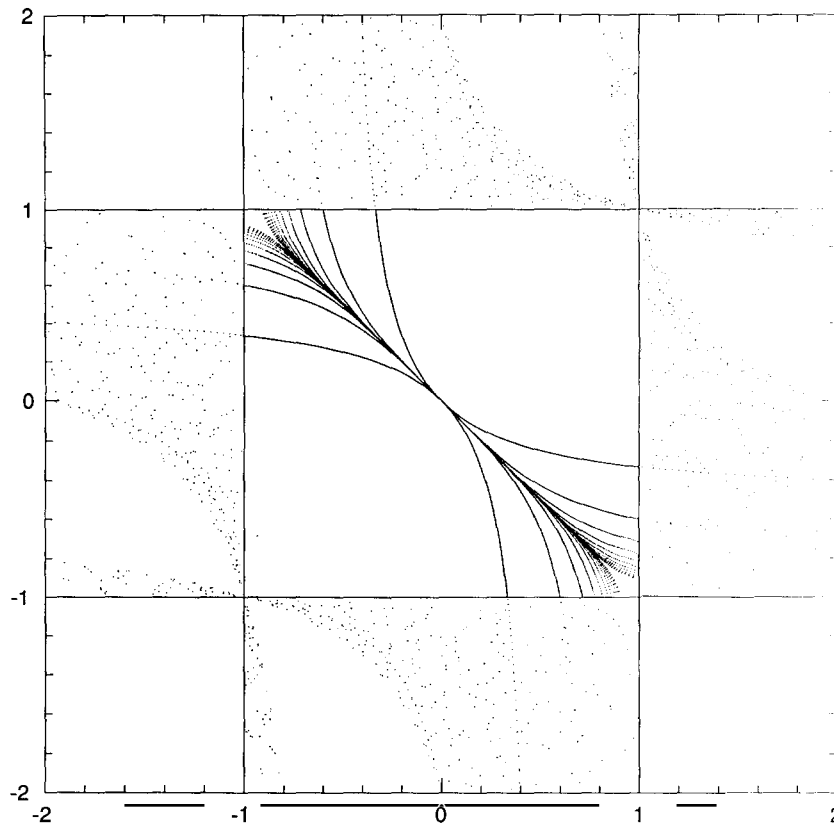


Fig. 7. Fundamental domains for the symmetric Ashkin-Teller model.

mental domain together with its images under the iteration of IJ .

It is worthwhile saying that any other symmetry acting on the model, such as for example the renormalization group has to be compatible with the group Γ [23].

7. Conclusion

We would like to emphasize that the construction we present here could not have easily been found ex nihilo. Considerations about statistical mechanical models were essential in their building up. Indeed the “inversion relations” have been a long standing tool in statistical mechanics on lattices, and they strongly motivate our construction. What is noticeable is that, in return, it yields a powerful tool for the exploration of lattice models, integrable or not, to-

gether with new openings in many more directions of research. Among them are of course the investigations related to the mappings per se. Beyond that, and more in the direction of statistical mechanics and field theory, is the understanding of the role of the dimension of the lattice on both the algebraic aspects and the topological aspects.

All this touches various fields of mathematics and physics: algebraic geometry, algebraic topology, quantum algebra. Indeed the Coxeter groups we use are at the same time groups of automorphisms of algebraic varieties, generalized fundamental groups of algebraic varieties, symmetries of quantum Yang-Baxter equations (and their higher dimensional avatars).

We believe moreover that the space of parameters in the appropriate place to look at, if one wants to substantiate yet more the deep topological notion embodied in the notion of $\mathbb{Z}$ -invariance [19] and free

the models from the details of the lattice shape.

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References

- [1] M.P. Bellon, J.-M. Maillard and C.-M. Viallet, *Phys. Lett. A* 159 (1991) 221.
- [2] M.P. Bellon, J.-M. Maillard and C.-M. Viallet, Infinite discrete symmetry group for the Yang–Baxter equations: spin models, preprint LP THE 91-6 and Helsinki HU TFT 91-6.
- [3] M.P. Bellon, J.-M. Maillard and C.-M. Viallet, Infinite discrete symmetry group for the Yang–Baxter equations: vertex models, preprint LP THE 91-7 and Helsinki HU TFT 91-7.
- [4] M.T. Jackel and J.-M. Maillard, *J. Phys. A* 15 (1982) 2241.
- [5] M.T. Jackel and J.-M. Maillard, *J. Phys. A* 17 (1984) 2079.
- [6] R.J. Baxter, *Exactly solved models in statistical mechanics* (Academic Press, New York, 1981).
- [7] R.J. Baxter, *J. Stat. Phys.* 28 (1982) 1.
- [8] Y.G. Stroganov, *Phys. Lett. A* 74 (1979) 116.
- [9] P.W. Kasteleyn, Exactly solvable lattice models, in: *Proc. 1974 Wageningen Summer School: Fundamental problems in statistical mechanics III* (North-Holland, Amsterdam, 1974).
- [10] E.H. Lieb and F.Y. Wu, Two dimensional ferroelectric models, in: *Phase transitions and critical phenomena*, Vol. 1, eds. C. Domb and M.S. Green (Academic Press, New York, 1972) pp. 331–490.
- [11] J.-M. Maillard, P. Rujan and T. Truong, *J. Phys. A* 18 (1985) 339.
- [12] M.P. Bellon, J.-M. Maillard and C.-M. Viallet, Matrix patterns for integrability, in preparation.
- [13] M.P. Bellon, J.-M. Maillard and C.-M. Viallet, Higher dimensional mappings with exact properties, in preparation.
- [14] M.P. Bellon, J.-M. Maillard and C.-M. Viallet, Mappings for the $\mathbb{Z}_N$ chiral Potts model, in preparation.
- [15] I.R. Shafarevich, *Basic algebra geometry* (Springer, Berlin, 1977) 216.
- [16] R.J. Baxter, *Ann. Phys.* 70 (1972) 193.
- [17] R.J. Baxter, *Philos. Trans. R. Soc.* 289 (1978) 315.
- [18] M.P. Bellon, J.-M. Maillard and C.-M. Viallet, Elliptic parametrization for a six-state spin model, in preparation.
- [19] A.S. Wightman, The mechanics of stochasticity in classical dynamical systems, *Perspectives in statistical physics* (1981) pp. 343–363, reprinted in: *Hamiltonian dynamical systems*, eds. R.S. MacKay and J.D. Meiss (Hilger, Bristol, 1987).
- [20] J.E. Humphreys, *Reflection groups and Coxeter groups* (Cambridge Univ. Press, Cambridge, 1990).
- [21] M. Kohmoto, L.P. Kadanoff and C. Tang, *Phys. Rev. Lett.* 50 (1983) 1870.
- [22] M. Kohmoto and Y. Oono, *Phys. Lett. A* 102 (1984) 145.
- [23] J.-M. Maillard, *J. Math. Phys.* 27 (1986) 2776.