SYMMETRIES OF MODELS WITH GENUS >1

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The new solutions of the star–triangle relation for the chiral Potts model obtained recently by Au-Yang et al. are revisited. Their symmetries are studied with a particular emphasis on their group of automorphisms.

Last year, a whole set of solutions of the star–triangle relation have been obtained for the chiral Potts models by Au-Yang, Baxter, McCoy and Perk [1–5]. A remarkable feature is that they are the very first examples of models in two-dimensional statistical mechanics uniformized with curves of genus greater than one. In a more recent paper [5] new exact results for this nearest neighbor two spin interaction model given in terms of two "rapidities" restricted to lie on a curve which is the intersection of two Fermat surfaces have been presented. These solutions present many remarkable properties.

In this note, we consider the symmetries of the model of the chiral N state Potts model with special emphasis on the duality and the inversion relation. In general, the set of transformations generated by the inversion relation and the other symmetries is an infinite discrete group. However, this situation is incompatible with a parametrization with curves of genus greater than one or with Fermat surfaces [6]. Here we analyze these symmetries and show that these integrable models correspond precisely to the cases where this group degenerates and becomes finite. The symmetries of the model are then closely related to the automorphisms of the algebraic varieties that occur in the parametrization of the model.

In ref. [5] a general solution of the star–triangle relation was discovered. This solution was written in terms of two sets of rapidities namely two four vectors (a_p, b_p, c_p, d_p) and (a_q, b_q, c_q, d_q). These four vectors occur in the model in particular combinations:

$$x_1=b_qd_p\,,\quad x_2=a_pc_q\,,\quad x_3=b_pd_q\,,\quad x_4=c_pa_q\,,$$
$$x_5=d_qa_p\,,\quad x_6=d_pa_q\,,\quad x_7=c_pb_q\,,\quad x_8=b_pc_q\,. \tag{1}$$

In order to point out the central role played by these variables x_i we consider the following problem. Given the Boltzmann weights $w(n)$ and $\bar{w}(n)$ we ask whether a set of x_i exists such that:

$$w(n)x_1-w(n)\omega^{n+1}x_2-w(n+1)x_3$$
$$+w(n+1)\omega^{n+1}x_4=0\,,$$
$$\bar{w}(n)\omega x_5-\bar{w}(n)\omega^{n+1}x_6-\bar{w}(n+1)x_7$$
$$+\bar{w}(n+1)\omega^{n+1}x_8=0\,, \tag{2}$$

ω is an Nth root of unity. Saying that the x_i are products (such as (1)) amounts to imposing the two conditions

$$x_2x_3=x_5x_8\quad\text{and}\quad x_1x_4=x_6x_7\,. \tag{3}$$

The periodicity of w and $\bar{w}$ is equivalent to

$$x_1^N+x_4^N=x_2^N+x_3^N \quad \text{and} \quad x_5^N+x_8^N=x_6^N+x_7^N \tag{4}$$

(when the x_i are products this corresponds to eq. (6) of ref. [5]).

Clearly, $N=3$ and $N>3$ are two cases to be distinguished as for $N>3$ the homogeneous linear system (2) is overdetermined.

(1) $N=3$. In this case eqs. (2) are always compatible. The integrability condition of ref. [5] is written as

$$F(w(0), w(1), w(2))F(\bar{w}(0), \bar{w}(1), \bar{w}(2))=\omega\,, \tag{5}$$

where $F(w(0), w(1), w(2))=x_1x_4/x_2x_3$, with

$$x_1=w(2)w(0)^2+\omega w(1)w(2)^2+\omega^2w(0)w(1)^2\,,$$

$$x_2=w(0)w(1)^2+\omega w(1)w(2)^2+\omega^2w(2)w(0)^2\,,$$

$$-x_3=w(2)w(1)^2+\omega w(1)w(0)^2+\omega^2w(0)w(2)^2\,,$$

$$-x_4=w(0)w(2)^2+\omega w(1)w(0)^2+\omega^2w(2)w(1)^2\,. \tag{6}$$

Eq. (5) is nothing but eq. (3) which means that the x_i are products of two sets of homogeneous parameters (the "rapidities"). In the limit of the three state scalar Potts model ($w(1)=w(2)$, $\bar{w}(1)=\bar{w}(2)$), (5) degenerates into the two conditions for which the model also satisfies the star–triangle relations namely the ferromagnetic and the antiferromagnetic critical conditions [7,8]:

$$AB+A+B=0 \tag{7a}$$

or

$$AB-A-B=2\,, \tag{7b}$$

where $A=w(0)/w(1)$ and $B=\bar{w}(0)/\bar{w}(1)$.

One also remarks that, when $w(n)$ *and* $\bar{w}(n)$ are real for all n, (5) leads to only one condition.

Finally, setting

$$w(0)=1, \quad \bar{w}(0)=1\,,$$

$$w(1)=1+a_1(\omega-1)u+a_2(1-\omega)u+\ldots\,,$$

$$\bar{w}(1)=\bar{a}_1u+\ldots\,,$$

$$w(2)=1+a_1(1-\omega)u+a_2(\omega-1)u+\ldots\,,$$

$$\bar{w}(2)=\bar{a}_2u+\ldots\,, \tag{8}$$

the leading order in u of (5) is nothing else but eq. (7) of ref. [1].

(2) $N=4$ and $N>4$. For $N=4$, the homogeneous linear system is overdetermined. One has non-trivial solutions when the determinant of the system vanishes namely

$$\begin{aligned}&[w(0)^2+w(2)^2]w(1)w(3)\\&\quad+[w(3)^2+w(1)^2]w(0)w(2)\\&\quad-2w(0)^2w(2)^2-2w(1)^2w(3)^2=0\,.\end{aligned} \tag{9}$$

One recovers here eq. (19) of ref. [2]. The conditions obtained in refs. [1,2,5] for the star–triangle relation to be satisfied fall into two different classes: this determinantal condition which insures the existence of the x_i and condition (5) which amounts to saying that the x_i are products. As for $N=3$ one can expand the Boltzmann weights to obtain from (5) eq. (33b) of ref. [1] and from (9) the second factor of eqs. (33c) and (33d) of ref. [1].

For $N>4$ we have a similar situation with more than one determinantal equation.

Let us also recall the conjectured critical variety of the model described in ref. [5]. When written in our set of variables one finds

$$(x_1^N+x_4^N)^2=(x_6^N+x_7^N)^2\,. \tag{10a}$$

For $N=3$ this leads to

$$\begin{aligned}&[w(1)^3-w(2)^3][w(2)^3-w(0)^3][w(0)^3-w(1)^3]\\&\quad\pm[\bar{w}(1)^3-\bar{w}(2)^3][\bar{w}(2)^3-\bar{w}(0)^3]\\&\quad\times[\bar{w}(0)^3-\bar{w}(1)^3]=0\,.\end{aligned} \tag{10b}$$

We are now ready to discuss the symmetries of the general chiral N state Potts model.

A trivial symmetry S of the model permutes the horizontal and the vertical bonds:

$$S\colon w(n)\leftrightarrow\bar{w}(n)\,.$$

Another, but "non-trivial" symmetry is the so called inversion symmetry I which is defined as follows [9]: consider the $N\times N$ cyclic matrix $M(W)$ defined by $M_{0,n}=w(n)$ for $n=0, \ldots, N-1$. One can introduce another set of weights denoted W_I by the relation: $M(W_I)=M^{-1}(W)$ where M^{-1} is the cyclic matrix which is the inverse matrix of $M(W)$. Then, the inversion symmetry I acts on the Boltzmann weights of the model according to

$$I\colon w(n)\to 1/w(n) \quad\text{and}\quad \bar{w}(n)\to\bar{w}_I(n)\,. \tag{11}$$

As in the scalar Potts model, a functional equation is associated to I [10]:

$$T(w(0),...,w(N-1);\bar{w}(0),...,\bar{w}(N-1))$$
$$\times T(1/w(0),...,1/w(N-1);\bar{w}_I(0),...,\bar{w}_I(N-1))$$
$$=\mathbb{1}\,, \tag{12}$$

where T is the symmetrized transfer matrix of the model and $\mathbb{1}$ an identity matrix. With the notations of ref. [5] one can verify that the transformation $\bar{w}(n)\to\bar{w}_I(n)$ corresponds to the permutation of the two rapidities p and q provided that eq. (6) of ref. [5] is satisfied. This a consequence of the star–triangle relation in the limit $r\to p$. The inversion relation (12), implicit in ref. [5], has also been explicitly written out by Baxter and used in a recent paper on the solution of the free energy of this model [11].

The well known Kramers–Wannier duality [12] of the scalar Potts model may be generalized in the chiral Potts model by

$$D\colon w(n)\to w(0)+\omega^{-n}w(1)+\omega^{-2n}w(2)+...$$
$$+\omega^{-n(N-1)}w(N-1) \tag{13}$$

and by the similar expressions for $\bar{w}(n)$. D is no longer an involution. Note that it is actually a transformation of order 4 for any value of N.

This other symmetry of the model may be added to the previous ones and it is easy to see that the transformations $w(n)\to w_I(n)$ and $w(n)\to 1/w(n)$ are conjugated under the duality transformation D (diagonalization of a cyclic matrix).

S and I are two involutions that generate a discrete group of transformations in the parameter space of the model ($\mathbb{P}_{N-1}\times\mathbb{P}_{N-1}$). On the other hand it has been shown that the star–triangle relations are compatible with this discrete group of transformations [13]. More precisely, the algebraic varieties related to the parametrization of the star–triangle relations must possess some invariance properties under the action of this group: this group is a set of automorphisms of these algebraic varieties [6].

In general, for $N>3$, this set of automorphisms is an infinite discrete set of birational transformations (for $N=3$ this group is finite and (5) is globally invariant under its action). At first sight this would be in contradiction with the occurrence of curves of genus greater than one or Fermat surfaces [6]. It is in fact possible to show that for $N=4$ the algebraic varieties defined by (5) and (9) are invariant under the group of automorphisms and that the restriction of the group to this variety is finite. Moreover, the intersection of (5) with the condition (9) or with the conjectured critical condition (10) is also invariant under the group of automorphisms. To prove this invariance it is sufficient to verify the invariance under T: $w(n)\to 1/w(n)$ and under D. However, in terms of the w and $\bar{w}$, the calculations are involved.

It is possible to give a more suggestive proof of the degeneracy of the automorphy group by using the x_i variables. Indeed, it is a straightforward matter to see, when the determinantal conditions (such as (9)) are satisfied, that the transformation T reads:

$$T\colon x_1\to x_3\,,\quad x_2\to x_4\,,\quad x_3\to x_1\,,\quad x_4\to x_2\,. \tag{14}$$

The duality acts as:

$$D\colon x_1\to x_2\omega\,,\quad x_2\to x_4\,,\quad x_3\to x_1\,,\quad x_4\to x_3\,, \tag{15}$$

and of course similar transformations for x_5, x_6, x_7, x_8. Note that a relation exists between T and D: $D^2T=TD^2$. For $N>3$ the transformation $\bar{w}(n)\to\bar{w}_I(n)$ reads:

$$x_5\to x_6\,,\quad x_6\to x_5\,,\quad x_7\to x_8\,,\quad x_8\to x_7\,. \tag{16}$$

Now it is obvious that (3)–(5) and (10) are invariant under this set of transformations which is finite.

As mentioned before, the group generated by S and I is in general infinite. The linear system (2) defines a set of variables which appears to be well suited to the analysis of this automorphy group and which, at the same time, corresponds to saying that this group is finite even if (5), the condition for the existence of the star–triangle relaton, is not satisfied (when the star–triangle relation is satisfied, at least $4N^2$ of the automorphisms are explicitly written out in ref. [5]). On the example of the scalar N state Potts model, there also exist homogeneous variables well suited to the description of the action of the automorphy group defined by [10]

$$w(0)x_1-w(0)x_2-w(1)q_-x_1+w(1)q_+x_2=0\,, \tag{17}$$

where $q_+ = 1 - \frac{1}{2}N + \frac{1}{2}\sqrt{N(N-4)}$. Despite the fact that here q_+ is not in general an Nth root of unity, (17) is also compatible with the duality. Eq. (17) is reminiscent of the linear system (2). One can even verify that for $q=3$, for which q_+ is a root of unity, the linear system (2) degenerates into (17). One has also a similar but more involved situation in another subcase of the $N=4$ chiral Potts model, that is, the anisotropic symmetric Ashkin–Teller model [14]. It is thus tempting to suppose that such linear systems could play a key role in the analysis of the automorphic properties of the models.

For the chiral Potts model, one remarks that two different kinds of varieties may be considered in the parameter space:

- One for which it is possible to define well suited variables x_i and for which the automorphy group becomes finite.
- A subvariety of the previous one (5) for which the model satisfies the star–triangle equation.

One can look for generalizations of these ideas to the general case where the Boltzmann weights associated to two nearest neighbor spins s_i and s_j depend on the two values of these spins and no longer on their differences. The inversion relation again corresponds to:

$$w(s_i, s_j) \to 1/w(s_i, s_j)$$

$$\text{and} \quad W \to W_I \quad \text{with} \quad M(W_I) = M^{-1}(W) .$$

Here M is an $N \times N$ matrix (not necessearily cyclic). In this very general class of models there is a particular subclass for which the matrices $M(W)$ and $M(W_I)$ belong to the same family of commuting matrices and which has also to be closed under the transformation $w(s_i, s_j) \to 1/w(s_i, s_j)$. The basis in which these matrices are simultaneously diagonal defines a generalization of the Fourier transform (which is the keypoint in refs. [1–5]). Is it possible in such a general case to get linear systems similar to (2) and to define well suited variables? Do non-trivial solutions of the star–triangle relations which generalize refs. [1–5] exist? What then would be the symmetries of these models? These questions will be considered in the future.

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