

Heun functions and diagonals of rational functions (unabridged version).

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Abstract. We provide a set of diagonals of simple rational functions of four variables that are seen to be squares of Heun functions. Each time, these Heun functions, obtained by creative telescoping, turn out to be pullbacked ${}_2F_1$ hypergeometric functions and in fact classical modular forms. We even obtained Heun functions that are automorphic forms associated with Shimura curves as solutions of telescopers of rational functions.

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1. Introduction

Diagonals of rational functions naturally emerge in lattice statistical mechanics, enumerative combinatorics or more generally for n -fold integrals of theoretical physics [1, 2]. In previous papers [3, 4, 5] we have seen[†] that many diagonals of simple rational functions were pullbacked ${}_2F_1$ hypergeometric functions that turn out to be related to classical modular forms. Sticking with diagonals of simple rational functions that are solutions of linear differential operators of *order two*, it is natural to study diagonals of simple rational functions that are Heun functions.

Heun functions emerge in different areas of physics [1, 7, 8, 9] (see also page 60 of [2]) and enumerative combinatorics: the simple cubic lattice Green functions [10] can be written as a Heun function. Experimentally the Heun functions emerging in

[†] These calculations were performed using the creative telescoping program of C. Koutschan [6].

physics often† correspond to *globally bounded series* [1, 2], i.e. series that can be recast, after some rescaling, into series with *integer coefficients*. Most of the time they turn out to be pullbacked ${}_2F_1$ hypergeometric functions and in fact *classical modular forms*. This suggests to study the class of Heun functions that are diagonals of rational functions††, and thus, *globally bounded series* [2]. We will discard the case where the Heun functions are almost trivial, their order-two linear differential operators factorising into two order-one linear differential operators. Such rather trivial cases are recalled in Appendix A. In this paper we examine non trivial Heun functions, which happen to be *diagonals of simple rational functions* [2] of (mostly) four variables or solutions of telescopers of rational functions of (mostly) four variables. We see that they happen to fall into one of three categories:

- Heun functions that are diagonals of rational functions, having *globally bounded series expansions*, and can be rewritten as pullbacked hypergeometric functions that are *classical modular forms*.
- Heun functions that are diagonals of rational functions, having globally bounded series expansions, and can be rewritten as pullbacked hypergeometric functions that are *derivatives* of classical modular forms.
- Heun functions that are solutions of *telescopers*‡ of rational functions that have series expansions that are *not globally bounded*‡. They will be seen to correspond to *Shimura automorphic forms* or derivatives of automorphic forms.

The Heun function $Heun(a, q, \alpha, \beta, \gamma, \delta, x)$ is solution of the order-two Heun linear differential operator with four singularities (D_x denotes d/dx)

$$H_2 = D_x^2 + \left(\frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\epsilon}{x-a} \right) \cdot D_x + \frac{\alpha\beta x - q}{x \cdot (x-1) \cdot (x-a)}, \quad (1)$$

where one has the Fuchsian constraint $\epsilon = \alpha + \beta - \gamma - \delta + 1$, where $\alpha, \beta, \gamma, \delta$ need to be rational numbers, and a is an algebraic number. The parameter q is called the *accessory parameter* and the ratio $q/\alpha/\beta$ is called the *normalised accessory parameter*.

In the first two sections, we examine the Heun functions *emerging from diagonals of simple rational functions* that fall into the first or second category above, and show how they happen to be related to *classical modular forms*, or *derivatives of classical modular forms*, corresponding to pullbacked ${}_2F_1$ hypergeometric functions. These Heun functions have *integer coefficient series*, (or can be recast as series with *integer coefficients* [2] after a rescaling of the variable), and are solutions of globally nilpotent [11] linear differential operators: the critical exponents of all the singularities are *rational numbers*. This leads to define a criterion in Appendix F, that allows to draw up a list of parameters of the Gauss hypergeometric function ${}_2F_1([a, b], [c], x)$, for which it corresponds to a *classical modular form* (see section 2). Furthermore,

† This is not the case for the Heun functions in [9] which do not correspond to globally bounded series.

†† In [3] we found that diagonals of simple rational functions yield quite systematically classical modular forms when the corresponding telescoper is of order-two. Diagonals of rational functions are necessarily globally bounded [1, 2].

‡ By “telescoper” of a rational function, say $R(x, y, z)$, we here refer to the output of the creative telescoping program [6], applied to the *transformed* rational function $\hat{R} = R(x/y, y/z, z)/(yz)$. Such a telescoper is a differential operator T in x, D_x such that $T + D_y \cdot U + D_z \cdot V$ annihilates \hat{R} , where U, V are rational functions in x, y, z . In other words, the telescoper T represents a linear ODE that is satisfied by $Diag(R)$.

‡ And hence cannot be diagonals of rational functions.

we will find that some of these Heun functions turned out to be periods of *extremal rational surfaces* (see section 2.4). We do this while avoiding trivial cases¶.

In the third section, we examine first the solutions of the telescoper of a rational function, corresponding to a Heun function with a *non globally bounded* series expansion, and we show that this Heun function is related to a specific *Shimura curve* [12, 13, 14, 15, 16, 17]. We then examine a larger class of Heun functions listed in [18], and show that they are linked to *Shimura curves*. We are able to show the link between these Heun functions and *Shimura curves* thanks to a result by K.Takeuchi [19].

1.1. Recalls on lattice Green functions as diagonals of rational functions

The simple cubic lattice Green function [20]

$$\frac{1}{(2\pi)^3} \cdot \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \frac{d\theta_1 d\theta_2 d\theta_3}{1 - x \cdot (\cos(\theta_1) + \cos(\theta_2) + \cos(\theta_3))}, \quad (2)$$

is nothing but†† the diagonal of the rational function in four variables x, z_1, z_2, z_3 :

$$\begin{aligned} & \frac{1}{1 - x \cdot z_1 z_2 z_3 \cdot ((1 + z_1^2)/z_1/2 + (1 + z_2^2)/z_2/2 + (1 + z_3^2)/z_3/2)} \\ &= \frac{1}{2 - x \cdot z_1 z_2 z_3 \cdot (z_1 + 1/z_1 + z_2 + 1/z_2 + z_3 + 1/z_3)}. \end{aligned} \quad (3)$$

The linear differential operator annihilating the diagonal of this rational function in *four* variables has order three and is the *symmetric square* of a linear differential operator of order two (θ is the homogeneous derivative $x \cdot d/dx$):

$$9x^4 \cdot (2\theta + 3) \cdot (2\theta + 1) - 4x^2 \cdot (10\theta^2 + 10\theta + 3) + 4\theta^2, \quad (4)$$

having a Heun function as a solution. Consequently, the simple cubic lattice Green function (2), or equivalently the diagonal of (3) reads:

$$Heun\left(\frac{1}{9}, \frac{1}{12}, \frac{1}{4}, \frac{3}{4}, 1, \frac{1}{2}, x^2\right)^2 = Heun\left(9, \frac{3}{4}, \frac{1}{4}, \frac{3}{4}, 1, \frac{1}{2}, 9x^2\right)^2. \quad (5)$$

The Heun function on the RHS of (5) happens to be a period of an *extremal rational curve* as can be seen in the work of Doran and Malmendier [21]. The diagonal (5) can also be written as a Hadamard product† of a simple algebraic function and a Heun function:

$$\begin{aligned} & Heun\left(\frac{1}{9}, \frac{1}{12}, \frac{1}{4}, \frac{3}{4}, 1, \frac{1}{2}, x^2\right)^2 \\ &= (1 - 4x^2)^{-1/2} \star Heun\left(\frac{1}{9}, \frac{1}{3}, 1, 1, 1, 1, \frac{x^2}{4}\right). \end{aligned} \quad (6)$$

¶ (See Appendix A) corresponding to factorizations of the order-two linear differential operator of the Heun function (see Appendix A.1), or corresponding to situations where the fourth singularity is in fact an apparent singularity (see Appendix A.2), a situation which often corresponds to the previous factorization of the order-two linear differential operator.

††Cooking recipe: change $\cos(\theta_i) = (1 + z_i^2)/2/z_i$ (i.e. $z_i = \exp(i\theta_i)$) and $x \rightarrow x \cdot z_1 z_2 z_3$.

† Denoted here by a star (*).

Similarly, considering *pencils of K3-surfaces*, Peters and Stienstra introduced [22] the integral§

$$I(x) = -\left(\frac{1}{2i\pi}\right)^3 \cdot \int_{|z_1|=1} \int_{|z_2|=1} \int_{|z_3|=1} \frac{dz_1 dz_2 dz_3}{z_1 z_2 z_3} \times \frac{x}{1 - x \cdot (z_1 + 1/z_1 + z_2 + 1/z_2 + z_3 + 1/z_3)}, \quad (7)$$

is annihilated by the linear differential operator of order three that is the symmetric square of the order-two linear differential operator:

$$L_2 = 576 \cdot x^4 \cdot \theta \cdot (\theta + 1) - 8 \cdot x^2 \cdot (20\theta^2 + 1) + (2\theta - 1)^2, \quad (8)$$

where θ denotes the homogeneous derivative $\theta = x \cdot d/dx$. Its solution, analytic at $x = 0$, reads:

$$x^{1/2} \cdot Heun\left(\frac{1}{9}, \frac{1}{12}, \frac{1}{4}, \frac{3}{4}, 1, \frac{1}{2}, 4x^2\right), \quad (9)$$

the other solution having a formal series expansion with a logarithm. Note that this square of a Heun function can be recast into a series with *integer coefficients*:

$$\begin{aligned} Heun\left(9, \frac{3}{4}, \frac{1}{4}, \frac{3}{4}, 1, \frac{1}{2}, 36z\right)^2 &= Heun\left(\frac{1}{9}, \frac{1}{12}, \frac{1}{4}, \frac{3}{4}, 1, \frac{1}{2}, 4z\right)^2 \\ &= 1 + 6z + 90z^2 + 1860z^3 + 44730z^4 + 1172556z^5 + 32496156z^6 \\ &\quad + 936369720z^7 + 27770358330z^8 + \dots \end{aligned} \quad (10)$$

Alternative forms of Heun functions like

$$Heun\left(9, 3, 1, 1, 1, 1, x\right), \quad Heun\left(\frac{1}{9}, \frac{1}{3}, 1, 1, 1, 1, x\right), \quad Heun\left(4, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, x\right),$$

can be introduced for the simple cubic lattice Green function. They are displayed in Appendix B. They all reduce to pullbacked ${}_2F_1$ hypergeometric functions‡ which turn out to correspond to *classical modular forms*†.

2. Diagonals of rational functions of three and four variables yielding Heun functions corresponding to classical modular forms

We are going to provide a set of exact expressions for diagonals [23] of simple rational functions of three and four variables yielding Heun functions. These exact expressions are obtained using the creative telescoping approach and, more specifically, the program of C. Koutschan [6], these diagonals being analytic at $x = 0$ globally bounded series, solutions of a telescoper obtained with this creative telescoping program¶.

2.1. Diagonals of rational functions of four variables yielding Heun functions

- Example 1. The diagonal of the rational function

$$R(x, y, z, w) = \frac{1}{1 - (wxy + wxz + wyz + xyz + wx + yz)}, \quad (11)$$

§ In section 3 of [22], the variable x is denoted $t = 1/s$. The correspondence between this x and the x in the lattice Green function (2) which corresponds to $I(x)/x$, is $x \rightarrow x/2$. Thus (5) becomes, once divided by x , the square of the Heun function (9).

‡ See also Appendix D below for (9) or (10).

† The emergence, for this fibration into K3 surfaces, of modular functions, cusp forms of weight two, via Dedekind's η -functions, can be found in section 4 of [22].

¶ This program also provides other expressions called the “certificates” that we do not use here.

reads:

$$\text{Diag}\left(R(x, y, z, w)\right) = 1 + 2x + 18x^2 + 164x^3 + 1810x^4 + \dots \quad (12)$$

A creative telescoping program [6] gives the order-three linear differential operator annihilating the diagonal (12) of the previous rational function (11):

$$\begin{aligned} L_3 = & 2 + 60x - (1 - 40x - 444x^2) \cdot D_x - 3x \cdot (1 - 18x - 128x^2) \cdot D_x^2 \\ & - x^2 \cdot (1 + 4x) \cdot (1 - 16x) \cdot D_x^3. \end{aligned} \quad (13)$$

This order-three linear differential operator corresponds to the *symmetric square* of the order-two linear differential operator:

$$L_2 = x^2 \cdot (8\theta + 5) \cdot (8\theta + 3) + x \cdot (12\theta^2 + 6\theta + 1) - \theta^2. \quad (14)$$

Thus the solution corresponding to the diagonal of (11) is given by the square of a Heun function:

$$\begin{aligned} \text{Heun}\left(-\frac{1}{4}, \frac{1}{16}, \frac{3}{8}, \frac{5}{8}, 1, \frac{1}{2}, -4x\right)^2 \\ = 1 + 2x + 18x^2 + 164x^3 + 1810x^4 + 21252x^5 + 263844x^6 \\ + 3395016x^7 + 44916498x^8 + \dots \end{aligned} \quad (15)$$

This Heun function can be written as a pullbacked ${}_2F_1$ hypergeometric function :

$$\text{Heun}\left(-\frac{1}{4}, \frac{1}{16}, \frac{3}{8}, \frac{5}{8}, 1, \frac{1}{2}, -4x\right) = \mathcal{A} \cdot {}_2F_1\left(\left[\frac{1}{8}, \frac{3}{8}\right], [1], \mathcal{H}\right), \quad (16)$$

where \mathcal{A} and the Hauptmodul \mathcal{H} are algebraic functions expressed with square roots:

$$\begin{aligned} \mathcal{H}_\pm = & -128x \cdot \frac{1 - 20x + 50x^2 + 400x^3 - 224x^4 - 512x^5}{(1 - 88x - 112x^2 - 256x^3)^2} \\ & \pm 128x \cdot \frac{(1 + 2x)(1 - 12x)(1 - 4x) \cdot (1 + 4x)^{1/2} \cdot (1 - 16x)^{1/2}}{(1 - 88x - 112x^2 - 256x^3)^2}. \end{aligned} \quad (17)$$

These Hauptmoduls (17) are also given by the *genus-zero* quadratic relation

$$\begin{aligned} (256x^3 + 112x^2 + 88x - 1)^2 \cdot \mathcal{H}_\pm^2 \\ - 256 \cdot x \cdot (512x^5 + 224x^4 - 400x^3 - 50x^2 + 20x - 1) \cdot \mathcal{H}_\pm \\ + 65536x^6 = 0, \end{aligned} \quad (18)$$

and have the series expansions:

$$\begin{aligned} \mathcal{H}_- = & -256x - 39936x^2 - 5116416x^3 - 595357696x^4 - 65525931776x^5 \\ & - 6954923846656x^6 - 719583708750336x^7 + \dots \\ \mathcal{H}_+ = & -256x^5 - 5120x^6 - 89600x^7 - 1433600x^8 - 22201600x^9 \\ & - 337755136x^{10} - 5094679040x^{11} + \dots \end{aligned} \quad (19)$$

The relation between these two Hauptmoduls corresponds to a genus-zero $q \leftrightarrow q^5$ *modular equation* (q denotes the nome of the order-two operator).

This Heun function can also be written alternatively as:

$$\text{Heun}\left(-\frac{1}{4}, \frac{1}{16}, \frac{3}{8}, \frac{5}{8}, 1, \frac{1}{2}, -4x\right) = \mathcal{A}_1 \cdot {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], \mathcal{H}\right), \quad (20)$$

using the identity

$${}_2F_1\left(\left[\frac{1}{8}, \frac{3}{8}\right], [1], x\right) = \mathcal{A}_2 \cdot {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], H\right), \quad (21)$$

where $\mathcal{A}_1, \mathcal{A}_2$ denote some algebraic functions and where:

$$H = \frac{27 \cdot (27x^2 - 414x + 512) \cdot x}{(9x + 16)^3} - \frac{54 \cdot (81x - 256) \cdot x}{(9x + 16)^3} \cdot (1 - x)^{1/2}. \quad (22)$$

- Example 2. Let us consider the rational function in four variables:

$$R(x, y, z, w) = \frac{1}{1 - (wxy + wxz + wyz + xy + xz + y + z)}. \quad (23)$$

The diagonal of this rational function (23) reads:

$$\begin{aligned} \text{Diag}\left(R(x, y, z, w)\right) &= 1 + 4x + 48x^2 + 760x^3 + 13840x^4 + 273504x^5 \\ &\quad + 5703096x^6 + 123519792x^7 + \dots \end{aligned} \quad (24)$$

The linear differential operator annihilating the diagonal of this rational function is the third order linear differential operator:

$$\begin{aligned} x^2 \cdot (1 + x) \cdot (1 - 27x) \cdot D_x^3 + 3x \cdot (1 - 39x - 54x^2) \cdot D_x^2 \\ + (1 - 86x - 186x^2) \cdot D_x - 4 \cdot (1 + 6x) \end{aligned} \quad (25)$$

This third order linear differential operator (25) is the *symmetric square* of an order-two linear differential operator, having as solution a (square of a) Heun function given as series expansion with *integer coefficients*:

$$\begin{aligned} \text{Heun}\left(-\frac{1}{27}, \frac{2}{27}, \frac{1}{3}, \frac{2}{3}, 1, \frac{1}{2}, -x\right)^2 &= 1 + 4x + 48x^2 + 760x^3 + 13840x^4 \\ &\quad + 273504x^5 + 5703096x^6 + 123519792x^7 + \dots \end{aligned} \quad (26)$$

We also have the following series expansion with integer coefficients:

$$\begin{aligned} \text{Heun}\left(-\frac{1}{27}, \frac{2}{27}, \frac{1}{3}, \frac{2}{3}, 1, \frac{1}{2}, -x\right) &= 1 + 2x + 22x^2 + 336x^3 + 6006x^4 \\ &\quad + 117348x^5 + 2428272x^6 + 52303680x^7 + \dots \end{aligned} \quad (27)$$

This Heun function (27) can be written as a pullbacked ${}_2F_1$ hypergeometric function

$$\begin{aligned} \text{Heun}\left(-\frac{1}{27}, \frac{2}{27}, \frac{1}{3}, \frac{2}{3}, 1, \frac{1}{2}, -x\right) &= \\ &\quad \left(25 - 80x - 24 \cdot (1 + x)^{1/2} \cdot (1 - 27x)^{1/2}\right)^{-1/4} \cdot {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], \mathcal{H}_+\right), \end{aligned} \quad (28)$$

where the Hauptmodul \mathcal{H} reads:

$$\begin{aligned} \mathcal{H}_\pm &= 864 \cdot x \cdot \frac{(1 - 21x + 8x^2) \cdot (1 - 42x + 454x^2 - 1008x^3 - 1280x^4)}{(1 + 224x + 448x^2)^3} \\ &\quad \pm 864 \cdot x \cdot (1 - 8x) \cdot (1 - 2x) \cdot (1 - 24x) \cdot (1 - 16x - 8x^2) \\ &\quad \times \frac{(1 + x)^{1/2} \cdot (1 - 27x)^{1/2}}{(1 + 224x + 448x^2)^3}. \end{aligned} \quad (29)$$

The series expansions of these two Hauptmoduls (29) read respectively

$$\begin{aligned} \mathcal{H}_+ &= 1728x - 1270080x^2 + 593381376x^3 - 226343666304x^4 \\ &\quad + 76907095308288x^5 - 24246668175851520x^6 + 7253781581324351808x^7 \\ &\quad - 2087529169324932180288x^8 + \dots \end{aligned} \quad (30)$$

and:

$$\begin{aligned} \mathcal{H}_- = & 1728 x^7 + 108864 x^8 + 4536000 x^9 + 158251968 x^{10} \\ & + 5017070016 x^{11} + 150134378688 x^{12} + 4328271255168 x^{13} + \dots \end{aligned} \quad (31)$$

These two Hauptmoduls are the two solutions of the *quadratic* genus-zero relation:

$$\begin{aligned} 1728^2 \cdot x^8 + 1728 \cdot (1 - 21x + 8x^2)(1280x^4 + 1008x^3 - 454x^2 + 42x - 1) \cdot x \cdot \mathcal{H}_\pm \\ + (1 + 224x + 448x^2)^3 \cdot \mathcal{H}_\pm^2 = 0, \end{aligned} \quad (32)$$

and the two j -invariants ($\mathcal{H}_\pm = 1728/j_\pm$) are solution of the quadratic relation:

$$\begin{aligned} x^8 \cdot j_\pm^2 + (1 - 21x + 8x^2)(1280x^4 + 1008x^3 - 454x^2 + 42x - 1) \cdot x \cdot j_\pm \\ + (1 + 224x + 448x^2)^3 = 0. \end{aligned} \quad (33)$$

Denoting $A = \mathcal{H}_+$ and $B = \mathcal{H}_-$ and considering the two (identical) quadratic relations (32) $Q(x, A) = 0$ and $Q(x, B) = 0$, one easily gets by elimination of x (performing the resultant between $Q(x, A) = 0$ and $Q(x, B) = 0$ in x), the *modular equation* $P(A, B) = 0$. One gets a quite large *modular equation* (corresponding to $q \leftrightarrow q^7$ in the nome q , see (30) and (31)):

$$81600^9 \cdot A^6 B^6 \cdot (343 A^2 + 286 AB + 343 B^2) + \dots - 2^{36} 3^{18} \cdot AB = 0. \quad (34)$$

Note that this (symmetric) algebraic curve is a *genus-zero* curve.

Also note that the previous Heun function can be written alternatively with another algebraic Hauptmodul H (and another algebraic function \mathcal{A})

$$\text{Heun}\left(-\frac{1}{27}, \frac{2}{27}, \frac{1}{3}, \frac{2}{3}, 1, \frac{1}{2}, -x\right) = \mathcal{A} \cdot {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], H\right), \quad (35)$$

where this alternative Hauptmodul is solution of a degree six equation

$$\begin{aligned} p_6(x)^3 \cdot (1 - 2x)^6 \cdot H^6 + 3 \cdot 1728 \cdot x^4 \cdot p_{20}(x) \cdot (1 - 2x)^3 \cdot H^5 \\ - 1728^2 \cdot x \cdot p_{23}(x) \cdot H^4 + 1728^3 \cdot x^3 \cdot p_{21}(x) \cdot H^3 + 1728^4 \cdot x^8 \cdot p_{16}(x) \cdot H^2 \\ - 1728^5 \cdot x^{10} \cdot p_{14}(x) \cdot H + 1728^6 \cdot x^{24} = 0, \end{aligned} \quad (36)$$

where the polynomials $p_6(x)$, $p_{14}(x)$, $p_{16}(x)$, $p_{20}(x)$, $p_{21}(x)$, $p_{23}(x)$ are given in Appendix C. Note that the curve (36) is a *genus-one* curve. This degree six polynomial equation (36) in H , gives Hauptmoduls having the following series expansions:

$$\begin{aligned} 1728 x^2 + 31104 x^3 - 689472 x^4 - 34193664 x^5 - 431329536 x^6 \\ + 4925546496 x^7 + 262313555328 x^8 + 3508587850752 x^9 + \dots \end{aligned} \quad (37)$$

and

$$\begin{aligned} 1728 x^{14} + 217728 x^{15} + 15930432 x^{16} + 888039936 x^{17} + 41880888000 x^{18} \\ + 1763242411392 x^{19} + 68405965290432 x^{20} + \dots \end{aligned} \quad (38)$$

corresponding to $q \leftrightarrow q^7$ in the nome q .

Denoting A and B two Hauptmoduls solutions of the two identical degree six relations (36), $Q_6(x, A) = 0$ and $Q_6(x, B) = 0$, one easily gets† the modular equation $P(A, B) = 0$. This *modular curve* is also a *genus-one* curve.

- Example 3. The rational function in four variables

$$R(x, y, z, w) = \frac{1}{1 - (y + z + wz + xy + xz + wxy)}, \quad (39)$$

† By elimination of x performing a resultant of $Q_6(x, A)$ and $Q_6(x, B)$ in x .

has a diagonal whose series expansion with integer coefficients reads:

$$\begin{aligned} \text{Diag}\left(R(x, y, z, w)\right) &= 1 + 4x + 60x^2 + 1120x^3 + 24220x^4 + 567504x^5 \\ &\quad + 14030016x^6 + 360222720x^7 + \dots \end{aligned} \quad (40)$$

The linear differential operator annihilating the diagonal of this rational function (39) has order three:

$$\begin{aligned} 4 + 96 \cdot x - (1 - 92 \cdot x - 864 \cdot x^2) \cdot D_x - 3x \cdot (1 - 42 \cdot x - 256 \cdot x^2) \cdot D_x^2 \\ - x^2 \cdot (1 + 4x) \cdot (1 - 32x) \cdot D_x^3. \end{aligned} \quad (41)$$

This order-three linear differential operator is the *symmetric square* of an order-two linear differential operator:

$$L_2 = 8x^2 \cdot (4\theta + 3) \cdot (4\theta + 1) + 2 \cdot (14\theta^2 + 7\theta + 1) - \theta^2. \quad (42)$$

The solution of the linear differential operator (41), analytic at $x = 0$, is thus given by the square of a Heun function which has a series expansion with integer coefficients:

$$\begin{aligned} \text{Heun}\left(-\frac{1}{8}, \frac{1}{16}, \frac{1}{4}, \frac{3}{4}, 1, \frac{1}{2}, -4x\right)^2 &= 1 + 4x + 60x^2 + 1120x^3 \\ &\quad + 24220x^4 + 567504x^5 + \dots \end{aligned} \quad (43)$$

The linear differential operator operator (41) is the *symmetric square* of a linear differential operator of order two, such that one of its solutions can be written as a pullbacked ${}_2F_1$ hypergeometric function:

$$\begin{aligned} \text{Heun}\left(-\frac{1}{8}, \frac{1}{16}, \frac{1}{4}, \frac{3}{4}, 1, \frac{1}{2}, -4x\right) &= 1 + 2x + 28x^2 + 504x^3 + 10710x^4 \\ &\quad + 248220x^5 + 6091680x^6 + 155580000x^7 + 4092325500x^8 + \dots \\ &= \mathcal{A}_\pm^{(1)} \cdot {}_2F_1\left(\left[\frac{1}{6}, \frac{2}{3}\right], [1], \mathcal{H}_\pm^{(1)}\right) = \mathcal{A}_\pm^{(2)} \cdot {}_2F_1\left(\left[\frac{1}{8}, \frac{5}{8}\right], [1], \mathcal{H}_\pm^{(2)}\right), \end{aligned} \quad (44)$$

where $\mathcal{A}_\pm^{(1)}$, $\mathcal{A}_\pm^{(2)}$ and the two Hauptmoduls $\mathcal{H}_\pm^{(1)}$ are square root algebraic functions:

$$\begin{aligned} \mathcal{H}_\pm^{(1)} &= -54x \cdot \frac{1 - 19x - 200x^2}{(1 + 4x) \cdot (1 - 50x)^2} \\ &\quad \pm 54 \cdot x \cdot (1 - 32x)^{1/2} \cdot \frac{1 - 5x}{(1 + 4x) \cdot (1 - 50x)^2}. \end{aligned} \quad (45)$$

The two Hauptmoduls $\mathcal{H}_\pm^{(1)}$ are solutions of the quadratic relation:

$$\begin{aligned} (1 + 4x) \cdot (1 - 50x)^2 \cdot (\mathcal{H}_\pm^{(1)})^2 - 108x \cdot (200x^2 + 19x - 1) \cdot \mathcal{H}_\pm^{(1)} \\ + 11664x^3 = 0. \end{aligned} \quad (46)$$

The two Hauptmoduls $\mathcal{H}_\pm^{(2)}$ in (44) are also square root algebraic functions:

$$\begin{aligned} \mathcal{H}_\pm^{(2)} &= -28 \cdot x \cdot \frac{1 - 30x + 64x^2}{(1 - 96x)^2} \\ &\quad \pm 28 \cdot x \cdot (1 - 16x) \cdot \frac{(1 + 4x)^{1/2} \cdot (1 - 32x)^{1/2}}{(1 - 96x)^2}, \end{aligned} \quad (47)$$

solutions of the quadratic relation

$$\begin{aligned} (1 - 96x)^2 \cdot (\mathcal{H}_\pm^{(2)})^2 + 256x \cdot (64x^2 - 30x + 1) \cdot \mathcal{H}_\pm^{(2)} \\ + 65536x^4 = 0, \end{aligned} \quad (48)$$

the algebraic function $\mathcal{A}_{\pm}^{(1)}$ being solution of

$$512 - 27 \cdot (1 - 20x) \cdot (19 - 312x - 6000x^2 - 80000x^3) \cdot Y + (1 + 4x)^3 \cdot (1 - 50x)^6 \cdot Y^2 = 0, \quad (49)$$

where $Y = (\mathcal{A}_{\pm}^{(2)})^{18}$, the algebraic function $\mathcal{A}_{\pm}^{(2)}$ being solution of

$$1 + 2 \cdot q_8(x) \cdot Y + 3^{32} \cdot (1 - 96x)^{16} \cdot Y^2 = 0, \quad \text{where:} \quad (50)$$

$$\begin{aligned} q_8(x) = & 92393273930231100473344x^8 - 182396792383587915661312x^7 \\ & + 7442201965961886564352x^6 + 10564527655702470066176x^5 \\ & - 1994146206485388984320x^4 + 154408466296830427136x^3 \\ & - 6048257896412868608x^2 + 118593292086518528x - 926510094425921, \end{aligned} \quad (51)$$

where $Y = (\mathcal{A}_{\pm}^{(2)})^{64}$. The series expansions of the Hauptmoduls $\mathcal{H}_{\pm}^{(1)}$ read:

$$\begin{aligned} \mathcal{H}_{-}^{(1)} = & -108x - 8208x^2 - 547776x^3 - 34193664x^4 - 2048523264x^5 \\ & - 119335292928x^6 - 6811411267584x^7 - 382782182326272x^8 + \dots \end{aligned} \quad (52)$$

and:

$$\begin{aligned} \mathcal{H}_{+}^{(1)} = & -108x^2 - 2160x^3 - 56592x^4 - 1475712x^5 - 39711168x^6 \\ & - 1088716032x^7 - 30317739264x^8 - 854924599296x^9 + \dots \end{aligned} \quad (53)$$

The relation between these two Hauptmoduls corresponds to the genus-zero *modular equation*:

$$\begin{aligned} 625 A^3 B^3 - 525 A^2 B^2 \cdot (A + B) - 96 A B \cdot (A^2 + B^2) - 3 A^2 B^2 \\ - 4 \cdot (A^3 + B^3) + 528 \cdot A B \cdot (A + B) - 432 \cdot A B = 0, \end{aligned} \quad (54)$$

which can (for instance) be rationally parametrised as follows:

$$A(v) = \frac{108 \cdot v \cdot (1 + v)^2}{(16 + 15v) \cdot (2 + 3v)^2}, \quad B(v) = -\frac{108 \cdot (1 + v) \cdot v^2}{(4 + 3v) \cdot (32 + 33v)^2}, \quad (55)$$

where $A(v)$ and $B(v)$ are related by an involution:

$$B(v) = A\left(-\frac{64 \cdot (1 + v)}{63v + 64}\right), \quad A(v) = B\left(-\frac{64 \cdot (1 + v)}{63v + 64}\right). \quad (56)$$

The series expansions of the Hauptmoduls $\mathcal{H}_{\pm}^{(2)}$ read:

$$\begin{aligned} \mathcal{H}_{-}^{(2)} = & -56x - 9072x^2 - 1229256x^3 - 152418672x^4 - 17935321320x^5 \\ & - 2038883437584x^6 - 226173478925520x^7 + \dots \end{aligned} \quad (57)$$

and

$$\begin{aligned} \mathcal{H}_{+}^{(2)} = & -56x^3 - 1680x^4 - 46872x^5 - 1291248x^6 - 35752752x^7 \\ & - 998627616x^8 - 28151491032x^9 - 800518405680x^{10} + \dots \end{aligned} \quad (58)$$

The relation between these last two Hauptmoduls $\mathcal{H}_{\pm}^{(2)}$ corresponds to the (genus-zero) *modular equation*:

$$\begin{aligned} 640000 \cdot A^2 B^2 \cdot (9A^2 + 14AB + 9B^2) \\ + 4800AB \cdot (A + B) \cdot (A^2 - 1954AB + B^2) \\ + A^4 + B^4 - 56196AB \cdot (A^2 + B^2) + 3512070A^2B^2 \\ + 116736 \cdot AB \cdot (A + B) - 65536 \cdot AB = 0. \end{aligned} \quad (59)$$

- Example 4. The rational function in four variables

$$R(x, y, z, w) = \frac{1}{1 - (wxz + wy + wz + xy + xz + y + z)}, \quad (60)$$

has a diagonal that reads:

$$\text{Diag}(R(x, y, z, w)) = 1 + 6x + 114x^2 + 2940x^3 + 87570x^4 + \dots \quad (61)$$

The telescoper[†] of the diagonal (61) of this rational function of four variables (60) reads:

$$6 + 12 \cdot x - (1 - 144 \cdot x - 108 \cdot x^2) \cdot D_x - x \cdot (3 - 198 \cdot x - 96 \cdot x^2) \cdot D_x^2 - x^2 \cdot (1 - 44 \cdot x - 16 \cdot x^2) \cdot D_x^3, \quad (62)$$

It is the *symmetric square* of an order-two linear differential operator which has a Heun solution analytic at $x = 0$. Consequently the order-three telescoper (62) has a square of a Heun solution. It has a series expansion with *integer coefficients*:

$$\begin{aligned} \text{Heun}\left(-\frac{123}{2} + \frac{55}{2} \cdot 5^{1/2}, -\frac{33}{8} + \frac{15}{8} \cdot 5^{1/2}, \frac{1}{4}, \frac{3}{4}, 1, \frac{1}{2}, 2 \cdot (11 - 5 \cdot 5^{1/2}) \cdot x\right)^2 \\ = 1 + 6x + 114x^2 + 2940x^3 + 87570x^4 + 2835756x^5 + 96982116x^6 \\ + 3446781624x^7 + 126047377170x^8 + \dots \end{aligned} \quad (63)$$

The square of the Heun function (63), solution of (62), can be rewritten as a pullbacked ${}_2F_1$ hypergeometric function:

$$\mathcal{A} \cdot {}_2F_1\left(\left[\frac{1}{12}, \frac{7}{12}\right], [1], \mathcal{H}\right)^2 \quad (64)$$

where \mathcal{A} is an algebraic function and where the Hauptmodul \mathcal{H} reads:

$$\begin{aligned} \mathcal{H} = -864 \cdot \frac{7776x^4 - 12600x^3 + 1890x^2 - 80x + 1}{(6480x^2 + 540x - 1)^2} \cdot x \\ + 864 \cdot (1 - 4x) \cdot (1 - 18x) \cdot (1 - 36x) \cdot \frac{(1 - 44x - 16x^2)^{1/2}}{(1 - 540x - 6480x^2)^2} \cdot x \\ = -1728x^5 - 138240x^6 - 7793280x^7 - 383961600x^8 - 17716017600x^9 + \dots \end{aligned} \quad (65)$$

The pullback \mathcal{H} is solution of the *genus-zero* quadratic relation:

$$\begin{aligned} (6480x^2 + 540x - 1)^2 \cdot \mathcal{H}^2 \\ + 1728 \cdot (7776x^4 - 12600x^3 + 1890x^2 - 80x + 1) \cdot x \cdot \mathcal{H} \\ + 2985984x^6 = 0. \end{aligned} \quad (66)$$

Note that changing the sign of the square root in (65) (Galois conjugate) yields the alternative expansion:

$$\begin{aligned} -1728x - 1728000x^2 - 1388016000x^3 - 1005452352000x^4 \\ - 686965980744000x^5 - 451977565258368000x^6 + \dots \end{aligned} \quad (67)$$

[†] By abuse of terminology we will call, everywhere in this paper, “telescoper” of a rational function $R(x, y, z)$ the output of the creative telescoping program [6]. For instance, for a rational function of three variables $R(x, y, z)$, we will call “telescoper” of a rational function $R(x, y, z)$, what is strictly sensu, the telescoper of the rational function $R(x/y, y/z, z)/(yz)$.

These two Hauptmoduls series (65) and (67) are related by the *genus-zero* modular equation:

$$\begin{aligned}
& 383093207587837762627239936 \cdot A^4 B^4 \cdot (25 A^2 + 14 A B + 25 B^2) \\
& - 331453065290799513600 \cdot A^3 B^3 \cdot (A + B) \cdot (15047 A^2 + 31514658 A B + 15047 B^2) \\
& + 4480842240 A^2 B^2 \cdot \left(144903770079 \cdot (A^4 + B^4) \right. \\
& \quad \left. - 7730345599747300 \cdot A B \cdot (A^2 + B^2) + 401951713284567050 \cdot A^2 B^2 \right) \\
& + 3386880 \cdot A B \cdot (A + B) \cdot \left(15047 \cdot (A^4 + B^4) \right. \\
& \quad \left. - 419175723722072 \cdot A B \cdot (A^2 + B^2) - 4206296569303686878 \cdot A^2 B^2 \right) \\
& + A^6 + B^6 - 25 A B \cdot \left(72243325686 \cdot (A^4 + B^4) \right. \\
& \quad \left. - 38887039753371909735 \cdot A B \cdot (A^2 + B^2) + 17585442099134941585204 \cdot A^2 B^2 \right) \\
& + 1382400 \cdot A B \cdot (A + B) \cdot \left(8142703 (A^2 + B^2) - 149242947792862 \cdot A B \right) \\
& - 1373552640 A B \cdot \left(18909 (A^2 + B^2) - 3621715210 \cdot A B \right) \\
& + 25386119331840 \cdot A B \cdot (A + B) - 8916100448256 \cdot A B = 0. \tag{68}
\end{aligned}$$

The square of the Heun solution (63) can also be rewritten as a pullbacked ${}_2F_1$ hypergeometric function:

$$\mathcal{A}_{5/12}(x) \cdot {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], \mathcal{H}\right)^2, \tag{69}$$

where $\mathcal{A}_{5/12}(x)$ is an algebraic function and where the Hauptmodul \mathcal{H} in (69) is solution of the quadratic relation:

$$\begin{aligned}
& (144 x^2 + 216 x + 1)^3 \cdot \mathcal{H}^2 \\
& - 1728 x \cdot (3456 x^5 + 7776 x^4 - 12600 x^3 + 1890 x^2 - 80 x + 1) \cdot \mathcal{H} \\
& + 2985984 x^6 = 0. \tag{70}
\end{aligned}$$

The two Hauptmoduls read

$$\begin{aligned}
\mathcal{H}_\pm &= \frac{864 x \cdot (3456 x^5 + 7776 x^4 - 12600 x^3 + 1890 x^2 - 80 x + 1)}{(144 x^2 + 216 x + 1)^3} \\
&\pm \frac{864 (1 - 36 x) \cdot (1 - 18 x) (1 - 4 x) x}{(144 x^2 + 216 x + 1)^3} \cdot (1 - 44 x - 16 x^2)^{1/2}, \tag{71}
\end{aligned}$$

which expands respectively as:

$$\begin{aligned}
\mathcal{H}_+ &= 1728 x - 1257984 x^2 + 575828352 x^3 - 214274336256 x^4 + \dots \\
\mathcal{H}_- &= 1728 x^5 + 138240 x^6 + 7793280 x^7 + 383961600 x^8 + \dots \tag{72}
\end{aligned}$$

These two Hauptmoduls series (71) are related by a *genus-zero* modular equation which can be parametrized rationally \ddagger as:

$$\mathcal{H}_+ = \frac{1728 z}{(z^2 + 10 z + 5)^3}, \quad \mathcal{H}_- = \frac{728 z^5}{(z^2 + 250 z + 3125)^3}. \tag{73}$$

\ddagger It corresponds to $N = 5$ in Table 4 and Table 5 of [25].

Note: The Heun function (63) resembles the Heun function associated with *extreme rational surfaces* [21, 24] (ϕ denotes the golden number $(1 + \sqrt{5})/2$):

$$\begin{aligned} & \text{Heun}\left(\frac{8 - 5\phi}{3 + 5\phi}, \frac{816 + 165\phi}{(3 + 5\phi)^3}, 1, 1, 1, 1, t\right) \\ &= \text{Heun}\left(-\frac{123}{2} + \frac{55}{2} \cdot 5^{1/2}, \frac{3}{2} \cdot (145185 \cdot 5^{1/2} - 324643), 1, 1, 1, 1, t\right). \end{aligned} \quad (74)$$

This Heun function (63) is also *reminiscent* of the Heun function solution of the Apéry operator [1, 2]:

$$\begin{aligned} & \text{Heun}\left(-\frac{123}{2} + \frac{55}{2} \cdot 5^{1/2}, -\frac{33}{2} + \frac{15}{2} \cdot 5^{1/2}, 1, 1, 1, 1, \frac{1}{2} \cdot (11 - 5 \cdot 5^{1/2}) \cdot x\right) \\ &= 1 + 3x + 19x^2 + 147x^3 + 1251x^4 + 11253x^5 + 104959x^6 + \dots \end{aligned} \quad (75)$$

This Heun function (75) can be rewritten as a pullbacked hypergeometric function:

$$\begin{aligned} & \frac{1}{(1 - 12x + 14x^2 + 12x^3 + x^4)^{1/4}} \\ & \times {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], \frac{1728 \cdot x^5 \cdot (1 - 11x - x^2)}{(1 - 12x + 14x^2 + 12x^3 + x^4)^3}\right), \end{aligned} \quad (76)$$

or

$$\begin{aligned} & \frac{1}{(1 + 228x + 494x^2 - 228x^3 + x^4)^{1/4}} \\ & \times {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], \frac{1728 \cdot x \cdot (1 - 11x - x^2)^5}{(1 + 228x + 494x^2 - 228x^3 + x^4)^3}\right), \end{aligned} \quad (77)$$

where the Hauptmodul in (76) can be written

$$\begin{aligned} & \frac{1728x}{(x^2 + 10x + 5)^3} \circ \frac{1 - 11x - x^2}{x} \\ &= \frac{1728x^5}{(x^2 + 250x + 3125)^3} \circ \frac{125x}{1 - 11x - x^2}, \end{aligned} \quad (78)$$

when the Hauptmodul in (77) can be written:

$$\begin{aligned} & \frac{1728x^5}{(x^2 + 250x + 3125)^3} \circ \frac{1 - 11x - x^2}{x} \\ &= \frac{1728x}{(x^2 + 10x + 5)^3} \circ \frac{125x}{1 - 11x - x^2}. \end{aligned} \quad (79)$$

- Example 5. The rational function in four variables

$$R(x, y, z, w) = \frac{1}{1 - (y + z + wz + xy + xz + wxy + wxyz)}, \quad (80)$$

has a diagonal that reads:

$$\text{Diag}\left(R(x, y, z, w)\right) = 1 + 5x + 73x^2 + 1445x^3 + 33001x^4 + \dots \quad (81)$$

The telescoper of the diagonal of this rational function (80) of four variables reads:

$$\begin{aligned} L_3 &= x^2 \cdot (1 - 34x + x^2) \cdot D_x^3 + 3x \cdot (1 - 51x + 2x^2) \cdot D_x^2 \\ &+ (1 - 112x + 7x^2) \cdot D_x + x - 5. \end{aligned} \quad (82)$$

It is the *symmetric square* of an order-two linear differential operator with a Heun solution, analytic at $x = 0$. Consequently the diagonal of (80), solution of (82), can

be written in terms of the square of two (Galois conjugate) Heun functions which have a series expansion with *integer coefficients*:

$$\begin{aligned}
& (1 - 34x + x^2) \times \\
& \quad \text{Heun}\left(577 + 408 \cdot 2^{1/2}, \frac{663}{2} + 234 \cdot 2^{1/2}, \frac{3}{2}, \frac{3}{2}, 1, \frac{3}{2}, (17 + 12 \cdot 2^{1/2}) \cdot x\right)^2 \\
= & (1 - 34x + x^2) \times \\
& \quad \text{Heun}\left(577 - 408 \cdot 2^{1/2}, \frac{663}{2} - 234 \cdot 2^{1/2}, \frac{3}{2}, \frac{3}{2}, 1, \frac{3}{2}, (17 - 12 \cdot 2^{1/2}) \cdot x\right)^2 \\
& = 1 + 5x + 73x^2 + 1445x^3 + 33001x^4 + 819005x^5 + 21460825x^6 \\
& \quad + 584307365x^7 + \dots \tag{83}
\end{aligned}$$

It can also be written as a pullbacked ${}_2F_1$ hypergeometric function

$$\mathcal{A}_- \cdot {}_2F_1\left(\left[\frac{1}{3}, \frac{2}{3}\right], [1], \mathcal{H}_-\right)^2, \tag{84}$$

where the Hauptmodul \mathcal{H}_\pm reads

$$\mathcal{H}_\pm = \frac{1 - 24x + 30x^2 + x^3}{2 \cdot (1+x)^3} \pm \frac{1 - 7x + x^2}{2 \cdot (1+x)^3} \cdot (1 - 34x + x^2)^{1/2}, \tag{85}$$

with the expansions:

$$\begin{aligned}
\mathcal{H}_- = & 27x^2 + 648x^3 + 15471x^4 + 389016x^5 + 10234107x^6 + 278861616x^7 \\
& + 7808397759x^8 + 223397228880x^9 + \dots \tag{86}
\end{aligned}$$

$$1 - \mathcal{H}_+ = 27x - 81x^2 + 891x^3 + 15039x^4 + 389691x^5 + \dots \tag{87}$$

and where the algebraic factor \mathcal{A}_- reads:

$$\begin{aligned}
\mathcal{A}_- = & \frac{3}{2} \cdot \frac{1-x}{(1+x)^2} - \frac{(1-34x+x^2)^{1/2}}{2 \cdot (1+x)^2} \\
= & 1 + 5x + 61x^2 + 1097x^3 + 23737x^4 + 569549x^5 + \dots \tag{88}
\end{aligned}$$

The two series (86) and (87) are related by the (symmetric) modular equation

$$\begin{aligned}
& 8A^3B^3 - 12B^2A^2 \cdot (A+B) + 3AB \cdot (2A^2 + 13AB + 2B^2) \\
& - (A+B) \cdot (A^2 + 29AB + B^2) + 27AB = 0. \tag{89}
\end{aligned}$$

• Example 6. Let us consider the following rational function in four variables x , y , z and w

$$R(x, y, z, w) = \frac{1}{1 - (y+z + wy + xz + wxy + wxz)}, \tag{90}$$

or the rational function:

$$R(x, y, z, w) = \frac{1}{1 + xy + yz + zw + wx + yw + xz}. \tag{91}$$

The diagonals of these two rational functions (90), (91) give the *same* series expansion with *integer coefficients*:

$$\begin{aligned}
\text{Diag}\left(R(x, y, z, w)\right) = & 1 + 6x + 90x^2 + 1860x^3 + 44730x^4 + 1172556x^5 \\
& + 32496156x^6 + 936369720x^7 + \dots \tag{92}
\end{aligned}$$

The order-three linear differential operator annihilating this series (92) is the *symmetric square* of the linear differential operator of order two, and is given by:

$$L_3 = x^2 \cdot (1 - 36x) \cdot (1 - 4x) \cdot D_x^3 + 3x \cdot (1 - 60x + 288x^2) \cdot D_x^2 + (1 - 132x + 972x^2) \cdot D_x - 6 \cdot (1 - 18x). \quad (93)$$

The solution of this order-three telescoper L_3 reads:

$$\begin{aligned} \text{Heun}\left(\frac{1}{9}, \frac{1}{12}, \frac{1}{4}, \frac{3}{4}, 1, \frac{1}{2}, 4x\right)^2 &= (1 - 4x) \cdot \text{Heun}\left(\frac{1}{9}, \frac{5}{36}, \frac{3}{4}, \frac{5}{4}, 1, \frac{3}{2}, 4x\right)^2 \\ &= 1 + 6x + 90x^2 + 1860x^3 + 44730x^4 + 1172556x^5 + \dots \end{aligned} \quad (94)$$

This series expansion (94) already occurred in (10) for the simple cubic lattice Green function. This Heun function (94) is quite simply related[‡] to the Heun function of example 3:

$$\begin{aligned} \text{Heun}\left(-\frac{1}{8}, \frac{1}{16}, \frac{1}{4}, \frac{3}{4}, 1, \frac{1}{2}, -4x\right)^2 \\ = (1 + 4x)^{-1/2} \cdot \text{Heun}\left(\frac{1}{9}, \frac{1}{12}, \frac{1}{4}, \frac{3}{4}, 1, \frac{1}{2}, \frac{4x}{1 + 4x}\right)^2. \end{aligned} \quad (95)$$

This Heun function in (94) can also be written as a pullbacked ${}_2F_1$ hypergeometric function

$$\begin{aligned} \text{Heun}\left(\frac{1}{9}, \frac{1}{12}, \frac{1}{4}, \frac{3}{4}, 1, \frac{1}{2}, 4x\right) \\ = 1 + 3x + \frac{81}{2}x^2 + \frac{1617}{2}x^3 + \frac{152955}{8}x^4 + \frac{3969405}{8}x^5 + \dots \\ = \mathcal{A}_\pm^{(1)} \cdot {}_2F_1\left(\left[\frac{1}{6}, \frac{2}{3}\right], [1], \mathcal{H}_\pm^{(1)}\right) = \mathcal{A}_\pm^{(2)} \cdot {}_2F_1\left(\left[\frac{1}{8}, \frac{5}{8}\right], [1], \mathcal{H}_\pm^{(2)}\right), \end{aligned} \quad (96)$$

where $\mathcal{H}_\pm^{(1)}$, $\mathcal{H}_\pm^{(2)}$ are algebraic functions expressed in terms of square roots. The detailed calculations which are similar to the ones of example 3 are given in Appendix D. As a consequence of identity (95), there is a close relation between example 3 and example 6 and of course the simple cubic lattice Green functions (see (10)).

2.2. A few comments on example 6

Considering the second form for the rational function of example 6, namely (91), it is straightforward to see that the *one-parameter family* of rational functions

$$R_\lambda(x, y, z, w) = \frac{1}{1 + \lambda \cdot (xy + yz + zw + wx + yw + xz)}, \quad (98)$$

has a diagonal[†] deduced from (94)

$$\text{Heun}\left(\frac{1}{9}, \frac{1}{12}, \frac{1}{4}, \frac{3}{4}, 1, \frac{1}{2}, 4\lambda^2 x\right)^2 = 1 + 6\lambda^2 x + 90\lambda^4 x^2 + \dots, \quad (99)$$

the telescoper of (98) being the pullback of the order-three linear differential operator (93) by $x \rightarrow \lambda^2 \cdot x$.

[‡] This corresponds to the fact that the order-three linear differential operator of example 3 by (41) is equal to $(1 + 4x)^{-1/2} \cdot \text{pullback}(L_3, x/(1 + 4x)) \cdot (1 + 4x)^{1/2}$ where L_3 is the order-three linear differential operator given by (93).

[†] Deduced, without any calculation, for the scaling transformation $(x, y, z, u) \rightarrow (\lambda^{1/2} \cdot x, \lambda^{1/2} \cdot y, \lambda^{1/2} \cdot z, \lambda^{1/2} \cdot u)$.

Let us now consider the rational function

$$R(x, y, z, w) = \frac{1}{1 + xyzw \cdot (xy + yz + zw + wx + yw + xz)}, \quad (100)$$

deduced from (91) by the following monomial \ddagger transformation:

$$(x, y, z, w) \longrightarrow (yzw, xzw, xyw, xyz). \quad (101)$$

It is straightforward to see, from its definition, that the diagonal of the rational function (100), actually corresponds to the diagonal of the rational function (98) where the parameter λ is taken to be equal to the product $\lambda = xyzw$, thus reading the formula (99) with $\lambda = x$:

$$\text{Heun}\left(\frac{1}{9}, \frac{1}{12}, \frac{1}{4}, \frac{3}{4}, 1, \frac{1}{2}, 4x^3\right)^2 = 1 + 6x^3 + 90x^6 + 1860x^9 + \dots \quad (102)$$

One verifies that the telescoper of the rational function (100) is actually the pullback \blacklozenge of the order-three linear differential operator (93) by $x \rightarrow x^3$. In contrast let us now consider the rational function

$$R(x, y, z, w) = \frac{x}{wx^3y + x^3yz + wxz + x^2y + x + z + w}, \quad (103)$$

deduced from (91) by the following (involutive) *birational* monomial transformation:

$$(x, y, z, w) \longrightarrow \left(\frac{1}{x}, x^2y, z, w\right). \quad (104)$$

The telescoper of (103) is actually *the same telescoper as the one for (91)*, namely the order-three linear differential operator L_3 given by (93). This telescoper has, thus, the Heun function (94) as a solution. However (94) does not coincide with the diagonal $\dagger\dagger$ of the rational function (103). The solutions of the telescoper of a rational function and the diagonal of a rational function *are two concepts which do not necessarily coincide*. We will underline this point several times in this paper (see section 3, Appendix L.1 and Appendix L.2).

2.3. Comment on the simplicity of the rational functions yielding Heun functions

Let us consider the rational function (39) of example 3, and perform the simple transformation $(x, y, z, w) \rightarrow (1+x, 1+y, 1+z, 1+w)$ on (39). One obtains that way the following new rational function:

$$-\frac{1}{wxy + wx + wy + wz + 2xy + xz + 2w + 3x + 3y + 3z + 5}. \quad (105)$$

The telescoper of this quite simple rational function (105) is a *very large* order-nine linear differential operator of degree 48 in x . Along this line let us consider the rational function

$$R(x, y, z, w) = \frac{1}{1 - (y + 2z + wz + xy + xz + wxy)}, \quad (106)$$

\ddagger Note a typo in the footnote of section 4.2 in [4, 5]. A determinant equal ± 1 condition is missing to have a birational transformation. A monomial transformation like (101) is not birational (the determinant is -3 here).

\blacklozenge Performing the same monomial change of variable (101) on the rational function (90) instead of (91), one gets, as it should, the same telescoper pullback of the order-three linear differential operator (93) by $x \rightarrow x^3$, with the same diagonal series (102).

$\dagger\dagger$ Which is not well-defined (depending on the ordering of the four variables).

which corresponds to a very simple modification[†] of the rational function (39) of example 3. The telescoper of this new simple rational function (106) is an irreducible order-five linear differential operator[¶]. Again we are far from having diagonals of rational functions and solutions of telescopers that can be simply expressed as Heun functions (or even solutions of order-two linear differential operators). Considering simple rational functions of the form $1/P(x, y, z, w)$ with a polynomial $P(x, y, z, w)$ of degree at most one in x, y, z, w is far from being sufficient to get Heun functions. Conversely let us consider the slightly more involved rational function

$$R(x, y, z, w) = \frac{1 + x}{1 + x - 2y - 2z - 3xy - 3xz - uxy - uxz - x^2y - x^2z - ux^2y} \quad (107)$$

The diagonal of the rational function (107) reads:

$${}_2F_1\left(\left[\frac{1}{2}, \frac{1}{2}\right], [1], 32x\right) = 1 + 8x + 144x^2 + 3200x^3 + 78400x^4 + \dots \quad (108)$$

The telescoper of the rational function (107) is an order-six linear differential operator which is the *direct sum* of two order-three linear differential operators $L_6 = L_3 \oplus M_3$, where L_3 factorises into the product of an order-one and an order-two linear differential operator $L_3 = L_1 \cdot L_2$, and where M_3 is *exactly*[‡] the order-three linear differential operator (41) which has the Heun solution (43), namely:

$$\text{Heun}\left(-\frac{1}{8}, \frac{1}{16}, \frac{1}{4}, \frac{3}{4}, 1, \frac{1}{2}, -4x\right)^2 = \text{Heun}\left(-8, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, 1, \frac{1}{2}, -32x\right)^2. \quad (109)$$

The diagonal of the rational function (107) is, in fact, solution of the order-two linear differential operator L_2 , which is thus, the minimal order linear differential operator annihilating the diagonal (108). The creative telescoping method yields a higher order telescoper which provides a “companion” order-three linear differential operator M_3 with the same square of Heun solution^{††} (109) as example 3.

2.4. Periods of extremal rational surfaces

Let us now introduce the rational function in just *three* variables:

$$R(x, y, z) = \frac{1}{1 + x + y + z + xy + yz - x^3yz}. \quad (110)$$

The diagonal of this rational function (110) has the following series expansion:

$$\begin{aligned} \text{Diag}(R(x, y, z)) = & 1 - 2x + 6x^2 - 11x^3 - 10x^4 + 273x^5 - 1875x^6 \\ & + 9210x^7 - 34218x^8 + 78721x^9 + 108581x^{10} + \dots \end{aligned} \quad (111)$$

In order to find the diagonal of this rational function of three variables, one gets the telescoper annihilating this diagonal using creative telescoping [6]. This telescoper is actually an order-four linear differential operator L_4 which, not only factorizes into two order-two linear differential operators, but is actually the *direct sum* (LCLM) of

[†] We have just changed one coefficient in the denominator of (39): the z term becomes a $2z$ term.

[¶] This order-five linear differential operator is homomorphic to its adjoint, its differential Galois group being $SO(5, \mathbb{C})$.

[‡] It is thus is the symmetric square of the order-two linear differential operator (42) which has the Heun solution (44).

^{††} Which corresponds to a “Period” over another cycle than the evanescent cycle of the diagonal.

two† order-two linear differential operators $L_4 = L_2 \oplus M_2$. These two order-two linear differential operators read respectively

$$\begin{aligned} L_2 &= 27x^2 \cdot (\theta + 1)^2 + 3x \cdot (3\theta^2 + 3\theta + 1) + \theta^2 \\ &= (1 + 9x + 27x^2) \cdot x^2 \cdot D_x^2 + (1 + 9x)^2 \cdot x \cdot D_x + 3x \cdot (1 + 9x), \end{aligned} \quad (112)$$

and:

$$\begin{aligned} M_2 &= (1 + 9x + 27x^2) \cdot (5 + 18x) \cdot (1 - 2x) \cdot x^2 \cdot D_x^2 \\ &\quad + (5 + 70x + 261x^2 - 756x^3 - 2916x^4) \cdot x \cdot D_x \\ &\quad + x \cdot (1 - 9x) \cdot (5 + 60x + 108x^2). \end{aligned} \quad (113)$$

Note that L_2 and M_2 share exactly the same singularities $x = 0, \infty$ and $1 + 9x + 27x^2 = 0$. In contrast, the factor $(5 + 18x)$ in (113) corresponds to an *apparent* singularity, when the factor $(1 - 2x)$ corresponds to a true singularity. One can get rid of the $5 + 18x = 0$ apparent singularity performing the following desingularization $L_2 \rightarrow L_3 = L_1 \cdot L_2$, changing the order-two operator L_2 into an order-three linear differential operator L_3 , the order-one operator L_1 reading:

$$L_1 = D_x + \frac{A'(x)}{A(x)} \quad \text{where:} \quad (114)$$

$$A(x) = (5 + 18x) \cdot (1 - 2x) \cdot (1 + 9x + 27x^2) \cdot x^{2/7}.$$

The solution of the order-two linear differential operator L_2 has the following Heun function‡ solution, analytic at $x = 0$:

$$\begin{aligned} \mathcal{S}_1 &= Heun\left(\frac{1}{2} - \frac{i\sqrt{3}}{2}, \frac{1}{2} - \frac{i\sqrt{3}}{6}, 1, 1, 1, 1, \frac{3}{2} \cdot (-3 + i\sqrt{3}) \cdot x\right) \\ &= 1 - 3x + 9x^2 - 21x^3 + 9x^4 + 297x^5 - 2421x^6 + 12933x^7 \\ &\quad - 52407x^8 + 145293x^9 - 35091x^{10} - 2954097x^{11} + \dots \end{aligned} \quad (115)$$

This Heun function (115) can also be written alternatively in terms of other ${}_2F_1$ hypergeometric functions:

$$\begin{aligned} &Heun\left(\frac{1}{2} - \frac{i\sqrt{3}}{2}, \frac{1}{2} - \frac{i\sqrt{3}}{6}, 1, 1, 1, 1, \frac{3}{2} \cdot (-3 + i\sqrt{3}) \cdot x\right) \\ &= \frac{1}{1 + 3x} \cdot {}_2F_1\left(\left[\frac{1}{3}, \frac{2}{3}\right], [1], \frac{27 \cdot x^3}{(1 + 3x)^3}\right) \\ &= \left(\frac{1}{1 + 9x + 27x^2 - 27x^3}\right)^{1/3} \cdot {}_2F_1\left(\left[\frac{1}{6}, \frac{2}{3}\right], [1], -\frac{108 \cdot x^3 \cdot (1 + 9x + 27x^2)}{(1 + 9x + 27x^2 - 27x^3)^2}\right) \\ &= \left(\frac{1}{1 + 3x}\right)^{1/4} \cdot \left(\frac{1}{1 + 9x + 27x^2 + 3x^3}\right)^{1/4} \\ &\quad \times {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], \frac{1728 \cdot x^9 \cdot (1 + 9x + 27x^2)}{(1 + 3x)^3 \cdot (1 + 9x + 27x^2 + 3x^3)^3}\right) \\ &= (1 + 9x)^{-1/4} \cdot (1 + 3x)^{-1/4} \cdot (1 + 27x^2)^{-1/4} \\ &\quad \times {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], \frac{1728 \cdot x^3 \cdot (1 + 9x + 27x^2)^3}{(1 + 3x)^3 \cdot (1 + 9x)^3 \cdot (1 + 27x^2)^3}\right), \end{aligned} \quad (117)$$

† These two order-two linear differential operators L_2 and M_2 are not homomorphic.

‡ This Heun function $Heun(a, q, \alpha, \beta, \gamma, \delta, \rho x)$ is such that $q = a/(1 + a)$, $q/\rho = -1/9$, $a/\rho^2 = 1/27$, $1/\rho$ and a/ρ being complex conjugate.

$$= (1 + 9x)^{-1/4} \cdot (1 + 243x + 2187x^2 + 6561x^3)^{-1/4} \\ \times {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], \frac{1728 \cdot x \cdot (1 + 9x + 27x^2)}{(1 + 9x)^3 \cdot (1 + 243x + 2187x^2 + 6561x^3)^3}\right).$$

Note that the Hauptmoduls in (117) can be rewritten as the composition of two pullbacks:

$$\frac{1728 \cdot x^9 \cdot (1 + 9x + 27x^2)}{(1 + 3x)^3 \cdot (1 + 9x + 27x^2 + 3x^3)^3} \\ = \left(\frac{1728z^3}{(z + 27)(z + 243)^3}\right) \circ \left(\frac{729x^3}{1 + 9x + 27x^2}\right), \quad (118)$$

$$\frac{1728 \cdot x^3 \cdot (1 + 9x + 27x^2)^3}{(1 + 3x)^3 \cdot (1 + 9x)^3 \cdot (1 + 27x^2)^3} \\ = \left(\frac{1728z}{(z + 27)(z + 3)^3}\right) \circ \left(\frac{729x^3}{1 + 9x + 27x^2}\right), \quad (119)$$

$$\frac{1728 \cdot x \cdot (1 + 9x + 27x^2)}{(1 + 9x)^3 \cdot (1 + 243x + 2187x^2 + 6561x^3)^3} \\ = \left(\frac{1728z}{(z + 27)(z + 3)^3}\right) \circ \left(729x \cdot (1 + 9x + 27x^2)\right). \quad (120)$$

The modular equation relating the Hauptmodul (118) with the Hauptmodul (120) corresponds to $q \leftrightarrow q^9$ in the nome q (see also Table 4 and Table 5 in [25]).

This Heun function (117) is in fact the *period* of an *extremal rational surface* [21], and was shown to be related \P to *classical modular forms* in table 15 in [25] for $N = 9$:

$$\text{Heun}\left(\frac{-9 \mp 3\sqrt{3}i}{-9 \pm 3\sqrt{3}i}, \frac{9 \pm 3\sqrt{3}i}{18}, 1, 1, 1, 1, \frac{2x}{-9 \pm 3\sqrt{3}i}\right) \\ = \text{Heun}\left(\frac{1 \pm \sqrt{3}i}{2}, \frac{3 \pm \sqrt{3}i}{6}, 1, 1, 1, 1, \frac{-3 \mp \sqrt{3}i}{18} \cdot x\right). \quad (121)$$

The other order-two linear differential operator M_2 has the following (classical modular form, see Appendix G) pullbacked ${}_2F_1$ hypergeometric solution \ddagger analytic at $x = 0$:

$$\mathcal{S}_2 = \frac{1}{(1 + 4x - 2x^2 - 36x^3 + 81x^4)^{1/4}} \cdot \\ \times {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], \frac{1728 \cdot x^5 \cdot (1 + 9x + 27x^2) \cdot (1 - 2x)^2}{(1 + 4x - 2x^2 - 36x^3 + 81x^4)^3}\right) \\ = 1 - x + 3x^2 - x^3 - 29x^4 + 249x^5 - 1329x^6 + 5487x^7 - 16029x^8 \\ + 12149x^9 + 252253x^{10} + \dots \quad (122)$$

This second order-two linear differential operator M_2 is *not homomorphic* to the previous one $\ddagger\ddagger$. The Hauptmoduls \mathcal{H}_1 for L_2 (see (118)) and \mathcal{H}_2 for M_2 (see (122))

$$\mathcal{H}_1 = \frac{1728 \cdot x \cdot (1 + 9x + 27x^2)}{(1 + 9x)^3 \cdot (1 + 243x + 2187x^2 + 6561x^3)^3}, \quad (123)$$

\P Change $x \rightarrow x/27$ to match \mathcal{S}_1 , given by (115), with (121).

\ddagger It seems that this pullbacked ${}_2F_1$ hypergeometric (122) cannot be seen as a (simple) Heun function: see Appendix G.

$\ddagger\ddagger$ They cannot be homomorphic: they do not have exactly the same singularities. The order-two linear differential operator M_2 has the extra $x = 1/2$ singularity.

$$\mathcal{H}_2 = \frac{1728 \cdot x^5 \cdot (1 + 9x + 27x^2) \cdot (1 - 2x)^2}{(1 + 4x - 2x^2 - 36x^3 + 81x^4)^3}, \quad (124)$$

are *not simply related*†. They just both vanish at $1 + 9x + 27x^2 = 0$. A rational parametrisation is introduced in Appendix G for these two order-two linear differential operators making clear the differences and similarities of these two linear differential operators.

One finds that the diagonal of (110) *is actually the half-sum of the two series* (115) and (122):

$$\text{Diag}(R(x, y, z)) = \frac{\mathcal{S}_1 + \mathcal{S}_2}{2}. \quad (125)$$

The order-four linear differential operator $L_4 = L_2 \oplus M_2$ is thus the *minimal order telescoper*. The diagonal of the rational function (110) is the *sum of two classical modular forms*.

Remark 1: The previous results can also be understood as follows. The telescoper of the rational function

$$R(x, y, z) = \frac{x}{1 + x + y + z + xy + yz - x^3 yz}, \quad (126)$$

similar to (110) (where the numerator of the rational function has been changed from 1 to x) is actually the *same as the one for* (110), namely $L_4 = L_2 \oplus M_2$. The diagonal of (127) reads:

$$\begin{aligned} \text{Diag}(R(x, y, z)) &= \frac{\mathcal{S}_2 - \mathcal{S}_1}{2} = x - 3x^2 + 10x^3 - 19x^4 - 24x^5 \\ &+ 546x^6 - 3723x^7 + 18189x^8 - 66572x^9 + 143672x^{10} + \dots \end{aligned} \quad (127)$$

The telescoper of the rational function of three variables

$$R(x, y, z) = \frac{1 - x}{1 + x + y + z + xy + yz - x^3 yz}, \quad (128)$$

is the order-two linear differential operator L_2 with the hypergeometric solution \mathcal{S}_1 . The telescoper of the rational function of three variables

$$R(x, y, z) = \frac{1 + x}{1 + x + y + z + xy + yz - x^3 yz}, \quad (129)$$

is the order-two linear differential operator M_2 with the hypergeometric solution \mathcal{S}_2 . Note however that the telescoper of the rational function

$$R(x, y, z) = \frac{xy}{1 + x + y + z + xy + yz - x^3 yz}, \quad (130)$$

is an order-five linear differential operator L_5 which is the direct sum of the order-two operator L_2 and an order-three linear differential operator $L_3 = N_2 \cdot D_x$, namely $L_5 = L_2 \oplus (N_2 \cdot D_x)$ where N_2 *is non-trivially homomorphic to* M_2 (with new apparent singularities $36x^2 - 6x + 1 = 0$ and $(1 + 3x) = 0$). The series expansion of the diagonal of this last rational function is non trivial and reads:

$$\begin{aligned} \text{Diag}(R(x, y, z)) &= -x + 3x^2 - 10x^3 + 23x^4 - 6x^5 - 378x^6 \\ &+ 3009x^7 - 15993x^8 + 64394x^9 - 175102x^{10} + \dots \end{aligned} \quad (131)$$

† Their associated nomes are not simply related. However, one can imagine that these Hauptmoduls (123), (124) are Igusa invariants of an algebraic surface (for instance a split Jacobian of a genus-two algebraic curve).

Remark 2: All the previous Heun functions occurring as diagonals of simple rational functions can all be rewritten in terms of pullbacked ${}_2F_1$ hypergeometric functions which turn out to correspond to *classical modular curves*. These ${}_2F_1$ hypergeometric functions are not arbitrary, they are “special” ${}_2F_1$ ’s corresponding to selected parameters, namely ${}_2F_1$ ’s related to *classical modular curves*. Appendix F gives a simple condition *on the nome* of these ${}_2F_1$ ’s to be related to *classical modular curves*. In Appendix H we give the exhaustive list of these 28 hypergeometric ${}_2F_1$ ’s related to *classical modular curves*.

2.5. Derivatives of classical modular forms

Let us recall example 6, and let us consider, instead of the rational function (91), its *homogeneous partial derivative* with respect to one of its four variables:

$$x \cdot \frac{\partial R(x, y, z, w)}{\partial x} = \frac{x \cdot (y + z + w)}{(1 + xy + yz + zw + wx + yw + xz)^2}. \quad (132)$$

The telescoper of this rational function (132) is an order-three linear differential operator M_3 which is homomorphic to the order-three operator L_3 given by (93) which was the telescoper of the rational function (91). This homomorphism reads:

$$M_3 \cdot \theta = L_1 \cdot L_3 \quad \text{where:} \quad L_1 = (1 - 18x) \cdot \theta + 18x, \quad (133)$$

where θ is the *homogeneous derivative* $\theta = x \cdot D_x$. Consequently the solutions of the order-three linear differential operator M_3 are simply obtained by taking the homogeneous derivative $\theta = x \cdot D_x$ of the solutions of the order-three linear differential operator L_3 . In particular, the diagonal of the rational function (132) is the *homogeneous derivative* of the diagonal of the rational function (91):

$$\text{Diag}\left(x \cdot \frac{\partial R(x, y, z, w)}{\partial x}\right) = x \cdot \frac{d}{dx}\left(\text{Diag}\left(R(x, y, z, w)\right)\right), \quad (134)$$

The diagonal of (132) will thus be the *homogeneous derivative of the classical modular form* (94). This is a general result on diagonals of rational functions. We have the following identity valid for any order- N linear differential operator L

$$\text{Diag}\left(\mathcal{L}\left(R(x, y, z, w)\right)\right) = L\left(\text{Diag}\left(R(x, y, z, w)\right)\right), \quad (135)$$

$$\text{where:} \quad L = \sum_{n=0}^N P_n(x) \cdot \theta^n, \quad \mathcal{L} = \sum_{n=0}^N P_n(xyzw) \cdot \Theta^n, \quad (136)$$

$$\text{with:} \quad \theta = x \cdot \frac{d}{dx}, \quad \dots \quad \Theta = w \cdot \frac{\partial}{\partial w}, \quad (137)$$

where the P_n ’s are polynomials. This identity can, of course be generalised to the diagonal of rational functions of an arbitrary number of variables. For any Heun function or classical modular form of this paper, obtained as a diagonal of a rational function, we can use these identities (134), (135) to get other rational functions that will not be Heun functions or classical modular forms, but *derivatives of Heun functions or classical modular forms*.

Note that the derivative of a classical modular form, or more generally an order-one linear differential operator like (136) acting on a classical modular form, is *no longer a classical modular form*. With this example we see that a Heun function which has a series expansion with *integer coefficients* (or more generally is globally

bounded series), is *not necessarily a classical modular form*, but can be *an order-one linear differential operator acting on a classical modular form*.

A simple example of diagonals of rational functions of three variables, corresponding to derivatives of Heun functions, is given in Appendix I. Along this line see also Appendix H.2.

3. Heun function solutions of telescopers of rational functions related to Shimura curves

The rational function of four variables

$$R(x, y, z, u) = \frac{xyz}{1 - xyz u + xyz \cdot (x + y + z) + xy + yz + xz}, \quad (138)$$

has a telescoper that is a linear differential operator of order three:

$$\begin{aligned} L_3 = & 8x \cdot (1 - x) \cdot (1 - 4x) \cdot D_x^3 + 12 \cdot (1 - 10x + 12x^2) \cdot D_x^2 \\ & - 6 \cdot (7 - 17 \cdot x) \cdot D_x + 3, \end{aligned} \quad (139)$$

which corresponds to the *symmetric square* of an order-two linear differential operator reading in terms of the homogeneous derivative θ :

$$x^2 \cdot (8\theta + 3) \cdot (8\theta + 1) - x \cdot (80\theta^2 + 1) + 8 \cdot \theta \cdot (2\theta - 1). \quad (140)$$

The solutions of order-three linear differential operator L_3 are, thus, expressed in terms of the following Heun functions

$$\text{Heun}\left(\frac{1}{4}, \frac{1}{64}, \frac{1}{8}, \frac{3}{8}, \frac{1}{2}, \frac{1}{2}, x\right)^2, \quad x \cdot \text{Heun}\left(\frac{1}{4}, \frac{21}{64}, \frac{5}{8}, \frac{7}{8}, \frac{3}{2}, \frac{1}{2}, x\right)^2, \quad (141)$$

or:

$$x^{1/2} \cdot \text{Heun}\left(\frac{1}{4}, \frac{1}{64}, \frac{1}{8}, \frac{3}{8}, \frac{1}{2}, \frac{1}{2}, x\right) \cdot \text{Heun}\left(\frac{1}{4}, \frac{21}{64}, \frac{5}{8}, \frac{7}{8}, \frac{3}{2}, \frac{1}{2}, x\right). \quad (142)$$

The series expansion of the first expression in (141) reads:

$$\begin{aligned} 1 + \frac{1}{4}x + \frac{5}{16}x^2 + \frac{5}{8}x^3 + \frac{2795}{1792}x^4 + \frac{15691}{3584}x^5 + \frac{1039363}{78848}x^6 + \frac{1872975}{45056}x^7 \\ + \frac{4786080975}{35323904}x^8 + \frac{1142244025}{2523136}x^9 + \frac{1182929670845}{767033344}x^{10} + \dots \end{aligned} \quad (143)$$

Do note that the diagonal of this rational function (138) *is actually equal to zero: it is different from this solution of the telescoper* (139). The two concepts, namely being the diagonal of a rational function and being the solution of the telescoper of that rational function *do not necessarily identify*. The solutions of the telescoper are n -fold integrals of that rational function integrand *over all possible cycles*: a solution like (141) is thus a “Period” of an algebraic variety corresponding to a particular *non-evanescent* cycle. It is different from the diagonal of that rational function which is a “Period” over *evanescent cycles*.

In contrast with all the other (square of) Heun functions of this paper, which are associated with *classical modular forms*, the series expansion (143) is *not globally bounded: it cannot be recast¶ into a series with integer coefficients. It cannot be a diagonal of a rational function: it is only a solution of the telescoper of a rational*

¶ After a rescaling of the variable.

function. The order-three linear differential operator (139), is the *symmetric square* of the linear differential operator of order two L_2 :

$$L_2 = D_x^2 + \frac{1 - 10x + 12x^2}{2x \cdot (1 - 4x) \cdot (1 - x)} \cdot D_x - \frac{1 - 3x}{16 \cdot x \cdot (1 - 4x) \cdot (1 - x)}, \quad (144)$$

whose (formal) series expansions at 0, 1, and ∞ do not contain[‡] *logarithms*. This order-two linear differential operator L_2 admits the solutions:

$$\begin{aligned} & x^{1/2} \cdot (1 - x)^{-7/8} \cdot {}_2F_1\left(\left[\frac{7}{24}, \frac{11}{24}\right], \left[\frac{5}{4}\right], \frac{27}{4} \cdot \frac{x^2}{(1 - x)^3}\right), \\ & (1 - x)^{-1/8} \cdot {}_2F_1\left(\left[\frac{1}{24}, \frac{5}{24}\right], \left[\frac{3}{4}\right], \frac{27}{4} \cdot \frac{x^2}{(1 - x)^3}\right). \end{aligned} \quad (145)$$

The precise correspondence with the Heun functions in (141) reads:

$$\begin{aligned} & \text{Heun}\left(\frac{1}{4}, \frac{1}{64}, \frac{1}{8}, \frac{3}{8}, \frac{1}{2}, \frac{1}{2}, x\right) \\ & = (1 - x)^{-1/8} \cdot {}_2F_1\left(\left[\frac{1}{24}, \frac{5}{24}\right], \left[\frac{3}{4}\right], \frac{27}{4} \cdot \frac{x^2}{(1 - x)^3}\right), \end{aligned} \quad (146)$$

$$\begin{aligned} & \text{Heun}\left(\frac{1}{4}, \frac{21}{64}, \frac{5}{8}, \frac{7}{8}, \frac{3}{2}, \frac{1}{2}, x\right) \\ & = (1 - x)^{-7/8} \cdot {}_2F_1\left(\left[\frac{7}{24}, \frac{11}{24}\right], \left[\frac{5}{4}\right], \frac{27}{4} \cdot \frac{x^2}{(1 - x)^3}\right). \end{aligned} \quad (147)$$

The pullbacks in all the ${}_2F_1$ hypergeometric functions of this paper are special rational (or algebraic) functions: they correspond to the concept of *Belyi maps*^{††}. In this case which *does not correspond to a classical modular form* but a (Shimura) *automorphic form*[†], the pullback $\frac{27}{4} \cdot \frac{x^2}{(1-x)^3}$ in (145) being also “special” (see Appendix J.1).

The two solutions of the linear differential operator (144) can be used to construct a basis for space of *automorphic forms*, which can then be used to construct Hecke operators relative to this basis[¶]. The second solution in (145) corresponds to an *automorphic form associated with a Shimura curve* with signature (0, 4, 2, 6) which appears in Table 1 in [19]. More details on Heun, or ${}_2F_1$, *automorphic forms* associated with *Shimura curves* [29, 30, 31] are given in Appendix J. Note in particular the fact that there exists an *algebraic series* $y(x)$ corresponding to a *modular equation*, such that the two ${}_2F_1$ hypergeometric functions (146), (147) *actually verify the following identity/symmetry*:

$$\begin{aligned} & w^{3/8} \cdot \rho \cdot y'(x)^{1/2} \cdot x^{3/8} \cdot (1 - x)^{1/4} \cdot {}_2F_1\left(\left[\frac{1}{24}, \frac{5}{24}\right], \left[\frac{3}{4}\right], x\right) \\ & = y(x)^{3/8} \cdot (1 - y(x))^{1/4} \cdot {}_2F_1\left(\left[\frac{1}{24}, \frac{5}{24}\right], \left[\frac{3}{4}\right], y(x)\right), \end{aligned} \quad (148)$$

where the two complex numbers w and ρ are on the unit circle. More details are given in Appendix J. Such an identity is reminiscent of the hypergeometric identities we studied in [32] for classical modular forms.

[‡] We have three elliptic points.

^{††} An important area where Belyi functions [18, 26, 27] appear is *precisely Shimura curves*. Any Belyi covering gives a modular curve with respect to some (not necessarily congruence) subgroup.

[†] They lie at the crossroads of many areas of mathematics. They have played an important role in the proof of Fermat’s last theorem. A Shimura curve is simply a Riemann surface which is uniformized by an arithmetic Fuchsian group.

[¶] See example 9 in [28] for more details.

This ‘‘Shimura’’ ${}_2F_1$ hypergeometric function can be seen to correspond to other Heun functions than (146) or (147). Using the general identity [8, 33]

$$\begin{aligned} & \text{Heun}(2, 16ab, 4a, 4b, a+b+1/2, 2(a+b), x) \\ &= {}_2F_1\left([a, b], [a+b+\frac{1}{2}], 4 \cdot x \cdot (2-x) \cdot (1-x)^2\right), \end{aligned} \quad (149)$$

one deduces the identity

$$\begin{aligned} & \text{Heun}\left(2, \frac{5}{36}, \frac{1}{6}, \frac{5}{6}, \frac{3}{4}, \frac{1}{2}, x\right) \\ &= {}_2F_1\left([\frac{1}{24}, \frac{5}{24}], [\frac{3}{4}], 4 \cdot x \cdot (2-x) \cdot (1-x)^2\right), \end{aligned} \quad (150)$$

as well as the identity:

$$\begin{aligned} & \text{Heun}\left(2, \frac{77}{36}, \frac{7}{6}, \frac{11}{6}, \frac{5}{4}, \frac{3}{2}, x\right) \\ &= {}_2F_1\left([\frac{7}{24}, \frac{11}{24}], [\frac{5}{4}], 4 \cdot x \cdot (2-x) \cdot (1-x)^2\right). \end{aligned} \quad (151)$$

More generally, Heun functions, related to Shimura curves, often emerge in the context of *Belyi maps* where Heun functions with four singularities, are expressed as pullbacked ${}_2F_1$ hypergeometric functions. For example Table 3.4.4 of [18] (see also [34]), corresponds to the ${}_2F_1$ hypergeometric function ${}_2F_1([\frac{1}{3}, \frac{1}{12}], [\frac{3}{4}], x)$ with three different pullbacks

$$\begin{aligned} & {}_2F_1\left([\frac{1}{3}, \frac{1}{12}], [\frac{3}{4}], \frac{x^4 \cdot (x^2 - 3)}{1 - 3x^2}\right) \\ &= (1 - 3x^2)^{1/3} \cdot \text{Heun}\left(\frac{1}{9}, \frac{1}{6}, \frac{1}{2}, 1, \frac{1}{2}, \frac{3}{4}, \frac{x^2}{3}\right), \end{aligned} \quad (152)$$

namely to the pullback B3 in the Table 3.4.4 in [18]. In the paper [18] most of the ${}_2F_1$ hypergeometric functions are in fact associated with *Shimura curves*: all tables‡ except 2.3.13, 2.3.14, 2.5.7, correspond to Heun functions corresponding to pullbacked ${}_2F_1$ hypergeometric functions, that are *automorphic forms associated with Shimura curves*†.

On a related issue we found the transformation‡:

$${}_2F_1\left([\frac{1}{3}, \frac{1}{12}], [\frac{3}{4}], x\right) = (1-x)^{-1/12} \cdot {}_2F_1\left([\frac{1}{24}, \frac{5}{24}], [\frac{3}{4}], -\frac{4x}{(1-x)^2}\right). \quad (153)$$

Relations (152) and (153) show that there is a relation between several Heun functions corresponding to *automorphic forms associated with Shimura curves*, namely $\text{Heun}(\frac{1}{9}, \frac{1}{6}, \frac{1}{2}, 1, \frac{1}{2}, \frac{3}{4}, x)$ in (152), and the Heun functions (141) which emerged as solutions of telescoper of the rational function (138). We actually have the relation

$$\begin{aligned} & (1 - 3x^2)^{3/8} \cdot \left(1 - \frac{x^2}{3}\right)^{1/8} \cdot \text{Heun}\left(\frac{1}{9}, \frac{1}{6}, \frac{1}{2}, 1, \frac{1}{2}, \frac{3}{4}, \frac{x^2}{3}\right) \\ &= \text{Heun}\left(\frac{1}{4}, \frac{1}{64}, \frac{1}{8}, \frac{3}{8}, \frac{1}{2}, \frac{1}{2}, \frac{4x^2}{(1-3x^2) \cdot (x^2-3)}\right), \end{aligned} \quad (154)$$

‡ In [18] the table number, e.g. 3.4.4, means that the elliptic points are 3, 4 and 4. For an example on how to obtain the hypergeometric function associated to these elliptic points, see for example, paragraph 2.5 in [29].

† Because they do not appear in Takeuchi’s [19] table 1, which gives a complete list of hypergeometric functions that are associated with Shimura curves.

‡ Which might be in the literature, yet we have not seen it.

which is a consequence of the identity on Belyi maps:

$$\frac{-4x}{(1-x)^2} \circ \frac{x^4 \cdot (x^2 - 3)}{1 - 3x^2} = \frac{27}{4} \cdot \frac{x^2}{(1-x)^3} \circ \frac{4x^2}{(1-3x^2) \cdot (x^2 - 3)}. \quad (155)$$

3.1. Other Heun functions solutions of telescopers of rational functions related to Shimura curves

The rational function of four variables

$$R(x, y, z, u) = \frac{xyz u}{ux^2y^2z^2 + ux^2yz + uxy^2z + uxyz^2 + uxy + uxz + uyz - xyz}, \quad (156)$$

has a telescoper that is a linear differential operator of order three:

$$M_3 = -3 + 6 \cdot (1 - 7x + 4x^2) \cdot x \cdot D_x + 12 \cdot (4 - 10x + 3x^2) \cdot x^2 \cdot D_x^2 + 8 \cdot (x - 1) \cdot (x - 4) \cdot x^3 \cdot D_x^3, \quad (157)$$

which actually corresponds to the *symmetric square* of an order-two linear differential operator readings in terms of the homogeneous derivative θ :

$$M_2 = x^2 \cdot \theta^2 - 2x \cdot (14\theta^2 - 7\theta + 1) - 8 \cdot (4\theta - 1) \cdot (4\theta - 3). \quad (158)$$

The solutions of order three operator M_3 are, thus, expressed in terms of the following (square and product of) Heun functions:

$$x^{1/4} \cdot \text{Heun}\left(4, \frac{9}{64}, \frac{1}{8}, \frac{5}{8}, \frac{3}{4}, \frac{1}{2}, x\right)^2, \quad x^{3/4} \cdot \text{Heun}\left(4, \frac{49}{64}, \frac{3}{8}, \frac{7}{8}, \frac{5}{4}, \frac{1}{2}, x\right)^2,$$

and

$$x^{1/2} \cdot \text{Heun}\left(4, \frac{9}{64}, \frac{1}{8}, \frac{5}{8}, \frac{3}{4}, \frac{1}{2}, x\right) \cdot \text{Heun}\left(4, \frac{49}{64}, \frac{3}{8}, \frac{7}{8}, \frac{5}{4}, \frac{1}{2}, x\right). \quad (159)$$

This order-two linear differential operator M_2 has the pullbacked ${}_2F_1$ solutions:

$$x^{3/8} \cdot (1-x)^{-7/8} \cdot {}_2F_1\left(\left[\frac{7}{24}, \frac{11}{24}\right], \left[\frac{5}{4}\right], \frac{-27}{4} \cdot \frac{x}{(1-x)^3}\right), \quad (160)$$

$$x^{1/8} \cdot (1-x)^{-1/8} \cdot {}_2F_1\left(\left[\frac{1}{24}, \frac{5}{24}\right], \left[\frac{3}{4}\right], \frac{-27}{4} \cdot \frac{x}{(1-x)^3}\right).$$

reminiscent of (145). One recovers the same (Shimura) ${}_2F_1$ hypergeometric function as the one in (145), *but with another selected pullback*. Similarly to the pullback in (145), this last pullback $\frac{-27}{4} \cdot \frac{x}{(1-x)^3}$ is also “special” as can be seen in Appendix J.1 with equations (J.4) and (J.5).

These ${}_2F_1$ solutions (160) can also be rewritten[†] as

$$\mathcal{A}_1(x) \cdot {}_2F_1\left(\left[\frac{1}{3}, \frac{2}{3}\right], \left[\frac{5}{4}\right], \mathcal{H}_1(x)\right), \quad \mathcal{A}_2(x) \cdot {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], \left[\frac{3}{4}\right], \mathcal{H}_2(x)\right), \quad (161)$$

or

$$\mathcal{A}_3(x) \cdot {}_2F_1\left(\left[\frac{1}{12}, \frac{1}{4}\right], \left[\frac{1}{2}\right], \mathcal{H}_3(x)\right), \quad \mathcal{A}_4(x) \cdot {}_2F_1\left(\left[\frac{7}{12}, \frac{3}{4}\right], \left[\frac{3}{2}\right], \mathcal{H}_4(x)\right), \quad (162)$$

where the $\mathcal{A}_i(x)$'s are algebraic functions, and where the $\mathcal{H}_i(x)$'s are algebraic functions that can be simply expressed with square roots.

[†] All these ${}_2F_1$ hypergeometric functions are Shimura hypergeometric functions [13, 19] corresponding to (4, 4, 3) and (6, 6, 2) difference of exponents in Table (1) of [13, 19].

As far as, not Heun, but ${}_2F_1$ hypergeometric functions related with *Shimura curves* are concerned, several identities also appear in the literature (see also Appendix J.6). Note that the set of Gauss hypergeometric functions, or Heun functions, that are associated with Shimura curves *is a finite set* [13, 19].

Remark 1: There exist “true” Heun functions (that *cannot be reduced to pullbacked ${}_2F_1$ hypergeometric functions*) which correspond to *automorphic forms associated with a Shimura curve*. One example comes from the order-two linear differential operator

$$L_2 = D_x^2 + \frac{12x^4 - 238x^3 + 3157x^2 - 3648x + 2592}{16 \cdot x^2 \cdot (x-1)^2 \cdot (2x-27)^2}, \quad (163)$$

which has the two Heun solutions

$$\begin{aligned} & x^{1/3} \cdot (1-x)^{1/4} \cdot (27-2x)^{1/4} \cdot \text{Heun}\left(\frac{27}{2}, \frac{7}{36}, \frac{1}{12}, \frac{7}{12}, \frac{2}{3}, \frac{1}{2}, x\right), \\ & x^{2/3} \cdot (1-x)^{1/4} \cdot (27-2x)^{1/4} \cdot \text{Heun}\left(\frac{27}{2}, \frac{47}{18}, \frac{5}{12}, \frac{11}{12}, \frac{4}{3}, \frac{1}{2}, x\right), \end{aligned} \quad (164)$$

We have not (yet ...) been able to see such “true” Heun functions as solutions of telescopers of rational functions. This Heun example (164) has a genus-zero level three modular equation given in [35, 28] and in Appendix J.7.

Remark 2: The Heun functions we found as diagonals of rational functions or solutions of telescopers of rational functions, were pullbacked ${}_2F_1$ hypergeometric functions which turn out to correspond to *classical modular forms or (Shimura) automorphic forms*. In both cases this means that the Heun function can be rewritten as a ${}_2F_1$ hypergeometric function with, *not just one pullback*, but an *infinite number of pullbacks* (generated by the *modular equations*, see for instance (148)). Is it possible for a Heun function to correspond to a *globally bounded series* and to reduce to pullbacked ${}_2F_1$ hypergeometric function with a rational or algebraic pullback, without being automatically a classical modular forms? In Appendix K we show that a Heun function can actually correspond to a globally bounded series, being reducible to a pullbacked ${}_2F_1$ hypergeometric function, without necessarily corresponding to a classical modular form. We have not yet been able to find such Heun functions as diagonal of rational functions, or even, as solutions of telescopers of rational functions.

Remark 3: More rational functions yielding Heun functions for their diagonals can be obtained using[‡] the $(x, y, z, u) \rightarrow (x^n, y^n, z^n, u^n)$ transformation for *positive* integers n . The case where the integer n is negative, in particular $n = -1$, is different and sketched in Appendix L.

4. Conclusion

The examples of diagonals of rational functions in three or four variables, that we presented here, illustrate cases where the diagonal of the rational functions are given by Heun functions with *integer coefficients series*, and can be expressed either in terms of pullbacked hypergeometric functions that are *classical modular forms*, or *derivatives* of classical modular forms. Furthermore, we constructed in subsection 2.4, a rational function whose diagonal is given by a Heun function that has already

[‡] Or, more generally monomial transformations.

been identified as a “Period” of an *extremal rational elliptic surface* [21], and that has also emerged in the context of pullbacked ${}_2F_1$ hypergeometric functions [25]. The emergence of squares of Heun functions for most of the diagonals of rational functions of this paper, suggests a “*Period of algebraic surfaces (possibly product of elliptic curves) interpretation*”. The exact expressions of the diagonal of rational functions in this paper, or in previous papers [3, 4, 5], are always obtained using the creative telescoping approach, being globally bounded series, solutions of the telescopers of these rational functions. Finally we have also seen a case where the rational function has a telescoper with Heun function solutions, that can be expressed as pullbacked ${}_2F_1$ hypergeometric functions that are *not* globally bounded, and happen to be associated with one of the 77 cases of *Shimura curves* [19]. Such remarkable ${}_2F_1$ hypergeometric functions solutions of a telescoper of a rational function are *not* diagonals of that rational function (the series are not globally bounded). They can be interpreted as “Periods” [36, 37] of an algebraic variety over some non-evanescent† cycles. With these ${}_2F_1$ Shimura examples one sees clearly that solutions, analytic at $x = 0$, of telescopers of rational functions are *not* diagonals of these rational functions.

All these examples seem to suggest an *algebraic geometrical* link between the diagonals/solutions of the telescopers, and the original rational functions, and this link should be investigated. This study should help shed light on the geometrical nature of the algebraic varieties associated with the denominators of the rational functions (K3, Calabi-Yau threefolds, extremal rational elliptic surfaces, Shimura varieties). In a forthcoming paper which is a work in progress at the current stage, we intend to introduce an *algebraic geometry approach* that proves to be efficient in explaining this link, in the cases where the order-two linear differential telescopers of the rational functions or the diagonals of rational functions are related to *classical modular forms*.

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† Diagonals are periods over evanescent cycles.

Appendix A. Trivialization cases of Heun functions

We already encountered in [32] (see section 3 equations (47), (48) and (C.12) in [32]) an interesting example of Heun function solution of an order-two linear differential operator which *factorises into two order-one operators*. The Heun function

$$\Phi(x) = x^{1/2} \cdot \text{Heun}\left(M, \frac{M+1}{4}, \frac{1}{2}, 1, \frac{3}{2}, \frac{1}{2}, x\right), \quad (\text{A.1})$$

satisfies the identity:

$$\Phi\left(\frac{4 \cdot x \cdot (1-x) \cdot (1-x/M)}{(1-x^2/M)^2}\right) = 2 \cdot \Phi(x). \quad (\text{A.2})$$

For $M = 2, 1/2, -1$, the Heun function (A.1) can be written as pullbacked ${}_2F_1$ hypergeometric functions. Let us recall, in the next subsections, the more general results of Ronveaux [38, 39].

Appendix A.1. Factorization cases of the order-two operator for Heun functions

Table 1 page 181 in [38] gives a set of six cases for which the order-two linear differential operator factors into the product of two order-one linear differential operators. Let us recall the conditions of Table 1 in [38] to get a factorization of the order-two Heun linear differential operator. The order-two Heun linear differential operator reads:

$$H_2 = D_x^2 + \left(\frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\epsilon}{x-a}\right) \cdot D_x + \frac{\alpha \beta x - q}{x \cdot (x-1) \cdot (x-a)}. \quad (\text{A.3})$$

where one has the Fuchsian constraint $\epsilon = \alpha + \beta - \gamma - \delta + 1$. One can easily verify that the order-two Heun linear differential operator (A.3) factorizes into two order-one linear differential operators when

$$\begin{aligned} q &= a \delta \cdot (\gamma - 1) + \epsilon + \gamma - 2 \\ \text{and:} \quad & (\gamma + \epsilon - 2) \cdot (\delta + 1) = \alpha \beta \end{aligned} \quad (\text{A.4})$$

i.e. after using the Fuchsian constraint:

$$\begin{aligned} q &= a \delta \cdot (\gamma - 1) + \alpha + \beta - \delta - 1, \\ \text{and:} \quad & (\alpha + \beta - \delta - 1) \cdot (\delta + 1) = \alpha \beta. \end{aligned} \quad (\text{A.5})$$

One has the following factorization $H_2 = L_1 \cdot M_1$ where:

$$\begin{aligned} L_1 &= (x-1) \cdot D_x + \delta, \\ M_1 &= x \cdot (x-a) \cdot D_x + (\gamma + \epsilon - 2) \cdot x + a \cdot (1-\gamma) \\ &= x \cdot (x-a) \cdot D_x + (\alpha + \beta - \delta - 1) \cdot x + a \cdot (\gamma - 1). \end{aligned} \quad (\text{A.6})$$

The case VI in Table 1 of Ronveaux [38], $\alpha = \delta + 1$, and thus $\beta = \gamma + \epsilon - 2$ (since one has the Fuchsian constraint $\epsilon = \alpha + \beta - \gamma - \delta + 1$), corresponds to this case. Along this line, let us consider the Heun function[†]:

$$\text{Heun}\left(a, a \delta \cdot (\gamma - 1) + \beta, \delta + 1, \beta, \gamma, \delta, x\right). \quad (\text{A.7})$$

The previous factorisation (A.6) yields

$$M_1 \left[\text{Heun}\left(a, a \delta \cdot (\gamma - 1) + \beta, \delta + 1, \beta, \gamma, \delta, x\right) \right] = \lambda \cdot (1-x)^{-\delta}, \quad (\text{A.8})$$

[†] For $a = 9, \beta = \gamma = \delta = 1$, the Heun function (A.7) is the simple rational function $\text{Heun}(9, 1, 2, 1, 1, 1, 27x) = 1/(1-3x)$.

the RHS of (A.8) being an algebraic function when the parameter δ is a rational number. The LHS of (A.8) can thus be written as the diagonal of a rational function [40, 41]. The Heun function (A.7) is not necessarily the diagonal of a rational function but the order-one operator M_1 acting on that Heun function is a diagonal¶ of a rational function [40, 41]. This Heun function (A.7) is *locally bounded*. For instance $Heun(9, 92, 6, 2, 3, 5, 9x)$ is *not globally bounded*, but the series expansion of $(\theta + 2)[Heun(9, 92, 6, 2, 3, 5, 9x)]$ is actually globally bounded (it is a series with *integer coefficients* corresponding to the expansion of an algebraic function):

$$\begin{aligned} x \cdot \frac{dHeun(9, 92, 6, 2, 3, 5, 9x)}{dx} + 2 \cdot Heun(9, 92, 6, 2, 3, 5, 9x) &= 2 + 92x + 2522x^2 \\ &+ 53552x^3 + 972092x^4 + 15852440x^5 + 239057660x^6 + \dots \\ &= \frac{2}{(1-x) \cdot (1-9x)^5}. \end{aligned} \quad (A.9)$$

• For the parameters $a = 9$, $q = 3$, $\alpha = 1/2$, $\beta = 1$, γ and δ being deduced from (A.19), (A.21), one finds that

$$\begin{aligned} Heun\left(9, 3, 1/2, 1, 40, -\frac{73}{2}, x\right) &= 1 + \frac{1}{120} \cdot x + \frac{1}{6560} \cdot x^2 - \frac{1}{1353984} \cdot x^4 \\ &- \frac{3}{19858432} \cdot x^5 - \frac{1}{36106240} \cdot x^6 - \frac{13}{2491330560} \cdot x^7 + \dots \end{aligned} \quad (A.10)$$

is an *algebraic function*. Let us consider the Laurent series expansion of the algebraic function $\mathcal{A}(x)$

$$\begin{aligned} \mathcal{A}(x) &= \frac{2^{73}}{729027183996402643275} \cdot \frac{(1-x)^{75/2} \cdot (x-12)}{x^{39}} \\ &= -\frac{2^{75}}{243009061332134214425 x^{39}} + \dots - \frac{292448}{5x^3} + \frac{9880}{3x^2} - \frac{104}{x} \\ &\quad + 1 + \frac{1}{120} \cdot x + \frac{1}{6560} \cdot x^2 - \frac{1}{1353984} \cdot x^4 + \dots \\ &= PP(\mathcal{A}(x)) + Heun\left(9, 3, 1/2, 1, 40, -\frac{73}{2}, x\right), \end{aligned} \quad (A.11)$$

where $PP(\mathcal{A}(x))$ denotes the principal part (negative powers) of the Laurent expansion of the algebraic function $\mathcal{A}(x)$. Note that the Heun function $Heun\left(9, 3, 1/2, 1, 40, -73/2, x\right)$ is solution of an order-two linear differential operator L_2 , which is the direct-sum (LCLM) $L_2 = L_1 \oplus M_1$ of two order-one linear differential operators. One order-one linear differential operator

$$L_1 = D_x + \frac{3}{2} \frac{x-26}{(x-1) \cdot x} - \frac{1}{x-12}, \quad (A.12)$$

has the algebraic function $\mathcal{A}(x)$ as solution, and the other one M_1 is a quite large order-one linear differential operator having the rational function $PP(\mathcal{A}(x)) = P(x)/x^{39}$ given in (A.11) as solution.

• For other values of the parameters $a = 9$, $q = 3$, $\alpha = 1/2$, $\beta = 1$, γ and δ being deduced from (A.19), (A.21), one finds that the Heun function

$$\begin{aligned} Heun(9, 3, 1, 1, 37, -33, x) &= 1 + \frac{x}{111} - \frac{x^2}{2109} - \frac{x^3}{9139} - \frac{x^4}{54834} \\ &- \frac{7x^5}{2248194} - \frac{3x^6}{5245786} - \frac{11x^7}{96672342} - \frac{x^8}{40899837} + \dots \end{aligned} \quad (A.13)$$

¶ We have, of course, a similar result for the previous Heun function (A.1).

is solution of an order-two linear differential operator L_2 which factorises in the product of two order-one operators $L_2 = N_1 \cdot P_1$ where:

$$\begin{aligned} N_1 &= (x-9) \cdot D_x - 1, \\ P_1 &= x \cdot (x-1) \cdot (x-15) \cdot D_x + x^2 - 65x + 540. \end{aligned} \quad (\text{A.14})$$

where the order-one linear differential operator P_1 has the rational function solution:

$$\mathcal{R}(x) = \frac{(x-15) \cdot (x-1)^{34}}{x^{36}}. \quad (\text{A.15})$$

One deduces:

$$\begin{aligned} &x \cdot (x-1) \cdot (x-15) \cdot \frac{d}{dx} \left(\text{Heun}(9, 3, 1, 1, 37, -33, x) \right) \\ &+ (x^2 - 65x + 540) \cdot \text{Heun}(9, 3, 1, 1, 37, -33, x) + 60 \cdot (x-9) \\ &= P_1 \left(\text{Heun}(9, 3, 1, 1, 37, -33, x) \right) + 60 \cdot (x-9) = 0. \end{aligned} \quad (\text{A.16})$$

Note that the series (A.13) is *not globally bounded* but $P_1(\text{Heun}(9, 3, 1, 1, 37, -33, x))$ is *globally bounded*††.

• For other values of the parameters $a = 10$, $q = 3$, $\alpha = 1/2$, $\beta = 3/2$, γ and δ being deduced from (A.19), (A.21), one finds that the corresponding Heun function $\text{Heun}(10, 3, 1/2, 3/2, 81/2, -73/2, x)$ is solution of an order-two linear differential operator

$$\begin{aligned} L_2 &= 4x \cdot (x-1) \cdot (x-10) \cdot D_x^2 \\ &+ 6 \cdot (2x^2 - 53x + 270) \cdot D_x + 3 \cdot (x-4), \end{aligned} \quad (\text{A.17})$$

which *factorises*† in the product of two order-one linear differential operators $L_2 = N_1 \cdot P_1$, where the order-one linear differential operator P_1 has an algebraic solution of the form $x^{-79/2} \cdot P_{38}(x)$, where $P_{38}(x)$ is a polynomial of degree 38, when the order-one linear differential operator N_1 has an algebraic solution of the form $(1-x)^{73/2} \cdot (x-10)/x/P_{38}(x)$, with the *same* polynomial $P_{38}(x)$. This polynomial of degree 38 is solution of the non-linear ODE:

$$\begin{aligned} &2 \cdot (152x^2 - 1579x + 770) \cdot \frac{P'(x)}{P(x)} - 4 \cdot x \cdot (x-1) \cdot (x-10) \cdot \frac{P''(x)}{P(x)} \\ &= 24 \cdot (247x - 2410). \end{aligned} \quad (\text{A.18})$$

Besides this selected polynomial solution $P_{38}(x)$, the solutions of (A.18) can be expressed in terms of Heun functions like $\text{Heun}(10, 84, 5/2, 7/2, 81/2, -73/2, x)$ or $\text{Heun}(10, 14383, -37, -36, -77/2, -73/2, x)$, which are *not globally bounded series*¶.

Appendix A.2. Heun functions where the fourth singularity is an apparent singularity

Let us recall some results of [42]. Let us consider a Heun function $\text{Heun}(a, q, \alpha, \beta, \gamma, \delta, x)$ where‡:

$$\delta = \alpha + \beta - \gamma + 2. \quad (\text{A.19})$$

†† A situation we already encountered [2].

† The fact that the second order linear differential operator associated with the Heun function factorises is noticed in the Remark of page 15 of [42].

¶ These Heun functions are not diagonals of rational functions, they are just solutions of the non-linear ODE (A.18).

‡ i.e. $\epsilon = -1$ see the definition of ϵ in (1) or (A.3).

The fourth singularity a will be an apparent singularity when

$$q^2 + \left((\gamma - 1) - (2\alpha\beta + \alpha + \beta) \cdot a \right) \cdot q + \alpha\beta a \cdot \left((\alpha + 1) \cdot (\beta + 1) \cdot a - \gamma \right) = 0, \quad (\text{A.20})$$

or:

$$\gamma = \frac{a^2 \alpha \beta \cdot (\alpha + 1) \cdot (\beta + 1) - a q \cdot (2\alpha\beta + \alpha + \beta) + q \cdot (q - 1)}{\alpha \beta a - q}. \quad (\text{A.21})$$

This condition (A.20) can also be rationally parametrized as:

$$a = \frac{e \cdot (e - \gamma + 1)}{(e - \alpha)(e - \beta)}, \quad q = \alpha\beta \cdot \frac{(e + 1) \cdot (e - \gamma + 1)}{(e - \alpha)(e - \beta)}. \quad (\text{A.22})$$

Introducing the order-three linear differential operator $L_3 = x^3 \cdot (x - 1) \cdot L_1 \cdot L_2$ where

$$L_1 = D_x + \frac{e+1}{x} + \frac{1}{x-1} + \frac{1}{x-a}, \quad (\text{A.23})$$

and where L_2 is the previous Heun operator (A.3) for $\epsilon = -1$

$$L_2 = D_x^2 + \left(\frac{\gamma}{x} + \frac{\delta}{x-1} - \frac{1}{x-a} \right) \cdot D_x + \frac{\alpha\beta x - q}{x \cdot (x-1) \cdot (x-a)}, \quad (\text{A.24})$$

one finds that the order-three linear differential operator L_3 reads:

$$L_3 = (x-1) \cdot x^3 \cdot D_x^3 + \left((\alpha + \beta + e + 4) \cdot x - e - \gamma - 1 \right) \cdot x^2 \cdot D_x^2 + \left((\alpha\beta + (\alpha + \beta + 1) \cdot (e + 2)) \cdot x - e\gamma \right) \cdot x \cdot D_x + \alpha\beta(e+1) \cdot x, \quad (\text{A.25})$$

which, in term of the homogeneous derivative $\theta = x D_x$, can be written in a more compact way:

$$L_3 = (\theta + e + 1) \cdot (\theta + \beta) \cdot (\theta + \alpha) \cdot x - \theta \cdot (\theta + \gamma - 1) \cdot (\theta + e - 1). \quad (\text{A.26})$$

This last expression (A.26) shows, very clearly, that L_3 corresponds to a ${}_3F_2$ hypergeometric function (see page 16 of [42]):

$${}_3F_2\left([\alpha, \beta, e + 1], [\gamma, e], x\right). \quad (\text{A.27})$$

The singularity $x = a$ in the Heun linear differential operator (A.24) is an *apparent singularity*. The head polynomial of (A.25) does not have this apparent singularity $x = a$. This apparent singularity can thus be removed introducing a (non minimal) *higher order* linear differential operator L_3 : this is called [43, 44, 45] the *desingularization*[†] of the linear differential operator L_2 . Introducing the order-two linear differential operator M_2 having the ${}_2F_1$ hypergeometric solution ${}_2F_1([\alpha, \beta], [\gamma], x)$, one finds that the Heun linear differential operator (A.24), and M_2 are actually homomorphic:

$$L_2 \cdot (\theta + e) = \left(\theta + e + 2 + \frac{1}{x-1} - \frac{a}{x-a} \right) \cdot M_2. \quad (\text{A.28})$$

[†] Often the desingularization removes the apparent singularities but creates unpleasant irregular singularities (for instance at $x = \infty$). Such desingularization destroys the Fuchsian character of the original linear differential operators. In physics one is interested in *desingularization preserving the Fuchsian character of the linear differential operators*. This is precisely the case here.

Consequently, as far as series expansions at $x = 0$ are concerned, the Heun function[‡] such that its parameters verify (A.19) and (A.22), can be written in several ways:

$$\begin{aligned} Heun(a, q, \alpha, \beta, \gamma, \delta, x) &= {}_3F_2\left([\alpha, \beta, e + 1], [\gamma, e], x\right) \\ &= \frac{1}{e} \cdot (\theta + e) \left[{}_2F_1([\alpha, \beta], [\gamma], x) \right]. \end{aligned} \quad (\text{A.29})$$

Appendix B. Alternative Heun functions for the simple cubic lattice Green function.

Note that such a Heun function, like (10), can also be written as a $Heun(9, 3, 1, 1, 1, 1, \mathcal{A}(x))$ function, where $\mathcal{A}(x)$ is an algebraic function, using the identity [42]

$$\begin{aligned} Heun\left(9, 3, 1, 1, 1, 1, x\right) &= \\ &\left(1 - \frac{x^2}{9}\right)^{-1/2} \cdot Heun\left(9, \frac{3}{4}, \frac{1}{4}, \frac{3}{4}, 1, \frac{1}{2}, \frac{36 \cdot x \cdot (1-x) \cdot (9-x)}{(9-x^2)^2}\right), \end{aligned} \quad (\text{B.1})$$

or

$$\begin{aligned} Heun\left(\frac{1}{9}, \frac{1}{3}, 1, 1, 1, 1, x\right) &= \\ &\left(1 - 9x^2\right)^{-1/2} \cdot Heun\left(\frac{1}{9}, \frac{1}{12}, \frac{1}{4}, \frac{3}{4}, 1, \frac{1}{2}, \frac{4 \cdot x \cdot (1-x) \cdot (1-9x)}{(1-9x^2)^2}\right), \end{aligned}$$

which is a special case of:

$$\begin{aligned} Heun(a, q, 1, 1, 1, 1, x) &= \\ &\left(1 - \frac{x^2}{a}\right)^{-1/2} \cdot Heun\left(a, \frac{q}{4}, \frac{1}{4}, \frac{3}{4}, 1, \frac{1}{2}, \frac{4 a \cdot x \cdot (1-x) \cdot (a-x)}{(a-x^2)^2}\right). \end{aligned} \quad (\text{B.2})$$

Appendix B.1. Other Heun functions for the simple cubic lattice Green function.

One has the identity

$$\begin{aligned} Heun\left(9, \frac{3}{4}, \frac{1}{4}, \frac{3}{4}, 1, \frac{1}{2}, x\right) &= \\ &= \frac{3}{8x} \cdot \left(72 - 40x - 72 \cdot \left(1 - \frac{x}{9}\right)^{1/2} \cdot \left(1 - x\right)^{1/2}\right)^{1/2} \\ &\quad \times Heun\left(4, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, \frac{5}{2} - \frac{9}{2x} + \frac{9}{2x} \cdot \left(1 - \frac{x}{9}\right)^{1/2} \cdot (1-x)^{1/2}\right), \end{aligned} \quad (\text{B.3})$$

which can be written using the parametrization (B.19):

$$\begin{aligned} Heun\left(9, \frac{3}{4}, \frac{1}{4}, \frac{3}{4}, 1, \frac{1}{2}, -\frac{9y}{(y-1) \cdot (y-4)}\right) &= \\ &= \left(\frac{(y-1) \cdot (y-4)}{4}\right)^{1/4} \cdot Heun\left(4, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, y\right). \end{aligned} \quad (\text{B.4})$$

[‡] When γ is not an integer, the series expansions of these Heun functions (A.29) are not generically globally bounded.

One also has the identity:

$$\begin{aligned} \text{Heun}\left(9, \frac{3}{4}, \frac{1}{4}, \frac{3}{4}, 1, \frac{1}{2}, \frac{36 \cdot u \cdot (u-1)^2 \cdot (u-4) \cdot (u^2-4)}{(u^2-2u-2)^2 \cdot (u^2-2u+4)^2}\right) \\ = \left(\frac{(2+2u-u^2) \cdot (4-2u+u^2)}{8 \cdot (1-u)^3}\right)^{1/2} \cdot {}_2F_1\left(\left[\frac{1}{2}, \frac{1}{2}\right], [1], -\frac{u \cdot (4-u)^3}{16 \cdot (1-u)^3}\right). \end{aligned} \quad (\text{B.5})$$

Appendix B.2. Simple cubic lattice Green function: focus on
 $\text{Heun}(4, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, x)$

Also note that the square of this Heun function can be written, in a different way, as a *product* of pullbacked ${}_2F_1$ hypergeometric functions. Let us recall the identity [20]

$$\begin{aligned} \text{Heun}\left(4, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, x\right)^2 \\ = {}_2F_1\left(\left[\frac{1}{2}, \frac{1}{2}\right], [1], \mathcal{H}_+\right) \cdot {}_2F_1\left(\left[\frac{1}{2}, \frac{1}{2}\right], [1], \mathcal{H}_-\right), \end{aligned} \quad (\text{B.6})$$

where:

$$\mathcal{H}_\pm = \frac{1}{2} \pm \frac{x}{2} \cdot \left(1 - \frac{x}{4}\right)^{1/2} - \frac{1}{2} \cdot \left(1 - \frac{x}{2}\right) \cdot \left(1 - x\right)^{1/2}. \quad (\text{B.7})$$

Their series expansion reads:

$$\begin{aligned} \mathcal{H}_+ &= x - \frac{1}{8}x^2 - \frac{1}{256}x^3 + \frac{7}{2048}x^4 + \frac{251}{65536}x^5 + \frac{1785}{524288}x^6 \\ &\quad + \frac{24555}{8388608}x^7 + \frac{168927}{67108864}x^8 + \dots \\ \mathcal{H}_- &= \frac{1}{256}x^3 + \frac{9}{2048}x^4 + \frac{261}{65536}x^5 + \frac{1799}{524288}x^6 + \frac{24597}{8388608}x^7 \\ &\quad + \frac{168993}{67108864}x^8 + \frac{9372077}{4294967296}x^9 + \frac{65602251}{34359738368}x^{10} + \dots \end{aligned} \quad (\text{B.8})$$

The relation between x and these two Hauptmoduls gives the *genus-zero* quartic relation[†]:

$$\begin{aligned} 256 \cdot \mathcal{H}_\pm^2 \cdot (1 - \mathcal{H}_\pm)^2 \\ - 32 \cdot x \cdot (2x^2 - 9x + 8) \cdot \mathcal{H}_\pm \cdot (1 - \mathcal{H}_\pm) + x^4 = 0, \end{aligned} \quad (\text{B.9})$$

the relation between these two Hauptmoduls reading the *genus-zero modular equation*:

$$\begin{aligned} -256 \cdot A^3 B^3 + 384 A^2 B^2 \cdot (A + B) + A^4 + B^4 - 132 \cdot A B \cdot (A^2 + B^2) \\ - 762 \cdot A^2 B^2 + 384 \cdot A B \cdot (A + B) - 256 \cdot A B = 0, \end{aligned} \quad (\text{B.10})$$

corresponding to $q \leftrightarrow q^3$ in the nome q .

One can rewrite the two ${}_2F_1$ hypergeometric functions in the RHS of (B.6)

$${}_2F_1\left(\left[\frac{1}{2}, \frac{1}{2}\right], [1], \mathcal{H}_\pm\right) = {}_2F_1\left(\left[\frac{1}{4}, \frac{1}{4}\right], [1], 4\mathcal{H}_\pm \cdot (1 - \mathcal{H}_\pm)\right) = {}_2F_1\left(\left[\frac{1}{4}, \frac{1}{4}\right], [1], H_\pm\right),$$

where the two pullbacks H_\pm

$$H_\pm = \frac{1}{4} \cdot (2x^2 - 9x + 8) \cdot x \pm (x-2) \cdot (1-x)^{1/2} \cdot \left(1 - \frac{x}{4}\right)^{1/2} \cdot x, \quad (\text{B.11})$$

are solutions of the genus-zero quadratic relation

$$16 \cdot H_\pm^2 - 8 \cdot x \cdot (2x^2 - 9x + 8) \cdot H_\pm + x^4 = 0, \quad (\text{B.12})$$

[†] Note that this quartic relation (B.9) is invariant by $\mathcal{H}_\pm \rightarrow 1 - \mathcal{H}_\pm$.

their expansions reading:

$$\begin{aligned} H_- &= 4x - \frac{9}{2}x^2 + \frac{63}{64}x^3 - \frac{9}{512}x^4 - \frac{261}{16384}x^5 - \frac{1791}{131072}x^6 + \dots, \\ H_+ &= \frac{1}{64}x^3 + \frac{9}{512}x^4 + \frac{261}{16384}x^5 + \frac{1791}{131072}x^6 + \dots \end{aligned} \quad (\text{B.13})$$

Thus (B.6) can also be rewritten as:

$$\begin{aligned} &Heun\left(4, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, x\right)^2 \\ &= {}_2F_1\left(\left[\frac{1}{4}, \frac{1}{4}\right], [1], H_+\right) \cdot {}_2F_1\left(\left[\frac{1}{4}, \frac{1}{4}\right], [1], H_-\right). \end{aligned} \quad (\text{B.14})$$

Besides (B.7), let us introduce the two other Hauptmoduls:

$$\mathcal{H}_\pm^{(+)} = \frac{1}{2} \pm \frac{x}{2} \cdot \left(1 - \frac{x}{4}\right)^{1/2} + \frac{1}{2} \cdot \left(1 - \frac{x}{2}\right) \cdot \left(1 - x\right)^{1/2}. \quad (\text{B.15})$$

One can easily see that the four pullbacked hypergeometric functions, corresponding to ${}_2F_1([1/2, 1/2], [1], \mathcal{H}_\pm^{(+)})$ and ${}_2F_1([1/2, 1/2], [1], \mathcal{H}_\pm)$, are all solutions of the order-four linear differential operator H_4 :

$$\begin{aligned} H_4 &= 256 \cdot (x-1)^2 \cdot (x-4)^2 \cdot x^2 \cdot D_x^4 \\ &\quad + 256 \cdot (x-1)(x-4) \cdot (8x^2 - 33x + 16) \cdot x \cdot D_x^3 \\ &\quad + 32 \cdot (115x^4 - 991x^3 + 2370x^2 - 1696x + 256) \cdot D_x^2 \\ &\quad + 16 \cdot (76x^3 - 693x^2 + 1128x - 448) \cdot D_x + 9 \cdot x \cdot (x-16). \end{aligned} \quad (\text{B.16})$$

Note that this order-four linear differential operator is homomorphic to its adjoint with an order-one linear differential intertwiner

$$\begin{aligned} H_4 \cdot L_1 &= \text{adjoint}(L_1) \cdot \text{adjoint}(H_4) \quad \text{where:} \\ L_1 &= 2 \cdot (x-1) \cdot (x-4) \cdot x \cdot D_x + 5x^2 - 4x - 4, \end{aligned} \quad (\text{B.17})$$

and with an order-three intertwiner

$$\text{adjoint}(H_4) \cdot L_3 = \text{adjoint}(L_3) \cdot H_4, \quad (\text{B.18})$$

suggesting that the differential Galois group of this order-four linear differential operator could be $SO(3, \mathbb{C})$. This is not the case. In fact this order-four linear differential operator is *not irreducible*[†]: it has an *absolute* factorization (see (B.24) below). Introducing the parametrization

$$x = -\frac{u \cdot (4-u) \cdot (4-u^2)}{4 \cdot (1-u)^2}, \quad (\text{B.19})$$

the product identity (B.6) becomes:

$$\begin{aligned} &Heun\left(4, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, -\frac{u \cdot (4-u) \cdot (4-u^2)}{4 \cdot (1-u)^2}\right)^2 \\ &= {}_2F_1\left(\left[\frac{1}{2}, \frac{1}{2}\right], [1], -\frac{u \cdot (4-u)^3}{16 \cdot (1-u)^3}\right) \cdot {}_2F_1\left(\left[\frac{1}{2}, \frac{1}{2}\right], [1], -\frac{u^3 \cdot (4-u)^3}{16 \cdot (1-u)^3}\right). \end{aligned} \quad (\text{B.20})$$

[†] At first sight using the command DFactor of Maple, one could imagine that this linear differential operator is irreducible.

Note that these two pullbacked ${}_2F_1$ hypergeometric functions are simply related:

$$\begin{aligned} & {}_2F_1\left(\left[\frac{1}{2}, \frac{1}{2}\right], [1], -\frac{u \cdot (4-u)^3}{16 \cdot (1-u)^3}\right) \\ &= (1-u) \cdot {}_2F_1\left(\left[\frac{1}{2}, \frac{1}{2}\right], [1], -\frac{u^3 \cdot (4-u)^3}{16 \cdot (1-u)^3}\right). \end{aligned} \quad (\text{B.21})$$

Therefore the identity (B.20), in fact, relates the square of a Heun function with the square of a pullbacked ${}_2F_1$ hypergeometric function, or more simply, gives this Heun function as a function of a pullbacked ${}_2F_1$ hypergeometric function:

$$\begin{aligned} & \text{Heun}\left(4, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, -\frac{u \cdot (4-u) \cdot (4-u^2)}{4 \cdot (1-u)^2}\right) \\ &= (1-u)^{-1/2} \cdot {}_2F_1\left(\left[\frac{1}{2}, \frac{1}{2}\right], [1], -\frac{u \cdot (4-u)^3}{16 \cdot (1-u)^3}\right), \end{aligned} \quad (\text{B.22})$$

or:

$$\begin{aligned} & \text{Heun}\left(4, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, -\frac{t \cdot (t+2) \cdot (3t+2) \cdot (3t+4)}{4 \cdot (t+1)^2}\right) \\ &= (1+t)^{-1/2} \cdot {}_2F_1\left(\left[\frac{1}{2}, \frac{1}{2}\right], [1], -\frac{t \cdot (3t+4)^3}{16 \cdot (t+1)^3}\right). \end{aligned} \quad (\text{B.23})$$

Let us note that the order-four linear differential operator H_4 , given by (B.16), pullbacked by the $x \rightarrow x(u)$ pullback (B.19), gives an order-four linear differential operator which, not only factorizes, but has a *direct-sum* factorization into two order-two linear differential operators that are simply conjugated:

$$\text{pullback}\left(H_4, -\frac{u \cdot (4-u) \cdot (4-u^2)}{4 \cdot (1-u)^2}\right) = L_2 \oplus M_2, \quad M_2 = u \cdot L_2 \cdot \frac{1}{u}. \quad (\text{B.24})$$

These two linear differential operators L_2 and M_2 have respectively the pullbacked ${}_2F_1$ hypergeometric solutions:

$${}_2F_1\left(\left[\frac{1}{2}, \frac{1}{2}\right], [1], -\frac{u^3 \cdot (4-u)^3}{16 \cdot (1-u)^3}\right), \quad u \cdot {}_2F_1\left(\left[\frac{1}{2}, \frac{1}{2}\right], [1], -\frac{u^3 \cdot (4-u)^3}{16 \cdot (1-u)^3}\right). \quad (\text{B.25})$$

Recalling the identity

$$\begin{aligned} & {}_2F_1\left(\left[\frac{1}{2}, \frac{1}{2}\right], [1], x\right) \\ &= (1-x+x^2)^{-1/4} \cdot {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], \frac{27x^2 \cdot (1-x)^2}{4 \cdot (1-x+x^2)^3}\right), \end{aligned} \quad (\text{B.26})$$

one finds that the ${}_2F_1$ hypergeometric functions in (B.6) can be rewritten as:

$$\begin{aligned} & {}_2F_1\left(\left[\frac{1}{2}, \frac{1}{2}\right], [1], \mathcal{H}_\pm\right) \\ &= (1 - \mathcal{H}_\pm + \mathcal{H}_\pm^2)^{-1/4} \cdot {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], \frac{27 \cdot \mathcal{H}_\pm^2 \cdot (1 - \mathcal{H}_\pm)^2}{4 \cdot (1 - \mathcal{H}_\pm + \mathcal{H}_\pm^2)^3}\right) \\ &= (1 - \mathcal{H}_\pm + \mathcal{H}_\pm^2)^{-1/4} \cdot {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], H_\pm\right), \end{aligned} \quad (\text{B.27})$$

where H_\pm is no longer solution of a quartic equation, but of a *quadratic genus-zero* equation:

$$\begin{aligned} & (x-4)^3 \cdot (x^3 - 60x^2 + 48x - 64)^3 \cdot H_\pm^2 \\ &+ 216 \cdot x^2 \cdot (2x^6 + 15x^5 + 92x^4 + 464x^3 + 3648x^2 - 12288x + 8192) \cdot H_\pm \\ &+ 11664 \cdot x^8 = 0. \end{aligned} \quad (\text{B.28})$$

The relation between the two Hauptmoduls H_{\pm} corresponds to the *genus-zero* modular equation[†]

$$\begin{aligned} & 26214400000000 \cdot A^3 B^3 \cdot (A + B) \\ & + 4096000000 \cdot A^2 B^2 \cdot \left(27 \cdot (A^2 + B^2) - 45946 \cdot AB \right) \\ & + 15552000 \cdot AB \cdot (A + B) \cdot (A^2 + B^2 + 241433 \cdot AB) \\ & + 729 \cdot \left(A^4 + B^4 - 1069956 \cdot AB(A^2 + B^2) + 2587918086 \cdot A^2 B^2 \right) \\ & + 2811677184 \cdot AB \cdot (A + B) - 2176782336 \cdot AB = 0, \end{aligned} \quad (\text{B.29})$$

The series expansion of the two Hauptmoduls H_{\pm} reads respectively:

$$\frac{27}{4} \cdot x^2 + \frac{81}{16} \cdot x^3 - \frac{8181}{512} \cdot x^4 - \frac{27351}{1024} \cdot x^5 + \dots, \quad \text{and:} \quad (\text{B.30})$$

$$\frac{27}{262144} \cdot x^6 + \frac{243}{1048576} \cdot x^7 + \frac{11421}{33554432} \cdot x^8 + \frac{14013}{33554432} \cdot x^9 + \dots \quad (\text{B.31})$$

Anecdotal remark: The order-three intertwiner L_3 in (B.18) is, in fact[‡], the *symmetric square* of an order-two linear differential operator

$$L_2 = D_x^2 + \frac{3x^2 - 10x + 4}{2x \cdot (x-4) \cdot (x-1)} \cdot D_x + \frac{(x+4) \cdot (x-2)}{32 \cdot (x-1) \cdot (x-4) \cdot x^2}, \quad (\text{B.32})$$

which has the two Heun solutions $S_{\pm} = x^{(1 \pm \sqrt{2})/4} \cdot h_{\pm}$ where:

$$h_{\pm} = \text{Heun}\left(4, \frac{11}{8} \pm \frac{7 \cdot 2^{1/2}}{8}, 1 \pm \frac{3 \cdot 2^{1/2}}{8}, 1 \pm \frac{2^{1/2}}{8}, 1 \pm \frac{2^{1/2}}{2}, x\right). \quad (\text{B.33})$$

It is clear, from the $x^{(1 \pm 2^{1/2})/4}$ factors, that the order-two operator L_2 is not even[¶] *globally nilpotent* [11]. Furthermore, the two series $h_+ + h_-$ and $2^{-1/2} \cdot (h_+ - h_-)$ are series with rational number coefficients that are *not globally bounded* series. Such Heun functions (B.33) cannot be written as pullbacked ${}_2F_1$ hypergeometric functions.

Appendix C. Polynomials of the degree six equation for the Hauptmodul in (35).

The polynomials of the degree six equation for the Hauptmodul in (35) read respectively:

$$\begin{aligned} p_6(x) = & 1 - 53324x + 3340572x^2 - 47158880x^3 + 453452848x^4 \\ & + 867240000x^5 + 729000000x^6, \end{aligned} \quad (\text{C.1})$$

$$\begin{aligned} p_{14}(x) = & 1 - 126x + 6657x^2 - 191100x^3 + 3224004x^4 - 32165952x^5 \\ & + 179161346x^6 - 459836304x^7 + 116094384x^8 + 1082203136x^9 \\ & - 247538592x^{10} - 690095616x^{11} - 102971392x^{12} \\ & + 15237120x^{13} + 324000x^{14}, \end{aligned} \quad (\text{C.2})$$

[†] Corresponding to $q \leftrightarrow q^3$ in the nome q .

[‡] When written in a unitary way $D_x^3 + \dots$

[¶] The intertwiners of Fuchsian (resp. globally nilpotent) linear differential operators have no reason to be Fuchsian (resp. globally nilpotent) linear differential operators as well.

$$\begin{aligned}
p_{16}(x) = & 1 - 144x + 12624x^2 + 42210112x^3 + 35493701376x^4 + 4373215830144x^5 \\
& + 146527536091776x^6 + 1709973141608448x^7 + 8301405990184512x^8 \\
& + 19700334651209215x^9 + 25456135068016191x^{10} + 18571208108112576x^{11} \\
& + 7732095471912574x^{12} + 1556868770685984x^{13} + 183059891926656x^{14} \\
& + 2050894080000x^{15} + 43740000000x^{16}, \tag{C.3}
\end{aligned}$$

$$\begin{aligned}
p_{20}(x) = & 496 + 14229477x + 10755437160x^2 + 607313973993x^3 \\
& + 21837165846834x^4 - 8741350651741356x^5 + 602696000526139688x^6 \\
& - 18362650954659075270x^7 + 237729206666512798092x^8 \\
& - 755131861209486545984x^9 - 4078730236814710350912x^{10} \\
& + 445545555487369556416x^{11} + 8298505398959353031040x^{12} \\
& - 501211331403375909060096x^{13} + 32930923081507234916352x^{14} \\
& - 7365760252808436401159680x^{15} - 14299198145937514719360000x^{16} \\
& - 5550618706232520960000000x^{17} - 2323303457201280000000000x^{18} \\
& - 2534505901920000000000000x^{19} + 114791256000000000000000x^{20}, \tag{C.4}
\end{aligned}$$

$$\begin{aligned}
p_{21}(x) = & 1 - 189x + 15939x^2 - 790713x^3 + 25604460x^4 - 567479130x^5 \\
& + 8729096106x^6 - 129136524678x^7 - 3128791781472x^8 \\
& - 301592422936140x^9 + 9302223231205632x^{10} - 898829709897155904x^{11} \\
& - 3001729628561501184x^{12} - 14056123657998705984x^{13} \\
& - 75146837553583537200x^{14} - 220865053128551921712x^{15} \\
& - 233016707230759517184x^{16} - 25485724878879707232x^{17} \\
& + 57908320494660830720x^{18} + 11705232438547200000x^{19} \\
& - 65745768960000000x^{20} - 3149280000000000x^{21}, \tag{C.5}
\end{aligned}$$

$$\begin{aligned}
p_{23}(x) = & 1 - 207x + 21552x^2 + 41491618x^3 + 32829303696x^4 \\
& + 2194878922992x^5 - 81778493396032x^6 - 2027922617204064x^7 \\
& + 50756763414324000x^8 + 304451170309086240x^9 - 5117266473854922240x^{10} \\
& - 23872757678772284352x^{11} + 92761784722387529728x^{12} \\
& - 1131857205540040786944x^{13} + 4168576271341671432192x^{14} \\
& - 35184687910528656122881x^{15} + 2169420555967834017888x^{16} \\
& - 270473856235208160230976x^{17} - 471011087555724046299136x^{18} \\
& - 105170018593490449009152x^{19} - 71294201738328407040000x^{20} \\
& - 141314879220788736000000x^{21} + 1238649615360000000000x^{22} \\
& - 12754584000000000000000x^{23}, \tag{C.6}
\end{aligned}$$

Remark: Taking the resultant (eliminating x) between equation (32), in the example 2, with the following genus-zero curve

$$\begin{aligned}
& 8z^3x^3 - 12z^2x^2 \cdot (x+z) + xz \cdot (6x^2 - 83xz + 6z^2) \\
& - (x+z) \cdot (x^2 + 17xz + z^2) + xz = 0, \tag{C.7}
\end{aligned}$$

one obtains immediately equation (36) and polynomials (C.1)-(C.6) of this appendix.

Appendix D. Pullbacked ${}_2F_1$ representation of $Heun(\frac{1}{9}, \frac{1}{12}, \frac{1}{4}, \frac{3}{4}, 1, \frac{1}{2}, 4x)$.

Let us consider example 6 of section (2.1), which diagonal is the square of the Heun function (96), which can be written as a pullbacked ${}_2F_1$ hypergeometric function

$$\begin{aligned} Heun\left(\frac{1}{9}, \frac{1}{12}, \frac{1}{4}, \frac{3}{4}, 1, \frac{1}{2}, 4x\right) \\ = \mathcal{A}_{\pm}^{(1)} \cdot {}_2F_1\left(\left[\frac{1}{6}, \frac{2}{3}\right], [1], \mathcal{H}_{\pm}^{(1)}\right) = \mathcal{A}_{\pm}^{(2)} \cdot {}_2F_1\left(\left[\frac{1}{8}, \frac{5}{8}\right], [1], \mathcal{H}_{\pm}^{(2)}\right), \end{aligned} \quad (D.1)$$

where the two pullbacks $\mathcal{H}_{\pm}^{(1)}$, $\mathcal{H}_{\pm}^{(2)}$ are square root algebraic functions

$$\begin{aligned} \mathcal{H}_{\pm}^{(1)} = -54 \cdot x \cdot \frac{1 - 27x - 108x^2}{(1 - 54x)^2} \\ \pm 54 \cdot x \cdot (1 - 9x) \cdot \frac{(1 - 4x)^{1/2} \cdot (1 - 36x)^{1/2}}{(1 - 54x)^2}, \end{aligned} \quad (D.2)$$

$$\begin{aligned} \mathcal{H}_{\pm}^{(2)} = -128 \cdot x \cdot \frac{1 - 38x + 200x^2}{(1 - 100x)^2 \cdot (1 - 4x)} \\ \pm 128 \cdot x \cdot (1 - 120x) \cdot \frac{(1 - 36x)^{1/2}}{(1 - 100x)^2 \cdot (1 - 4x)}, \end{aligned} \quad (D.3)$$

where $Y_{\pm} = (\mathcal{A}_{\pm}^{(1)})^{12}$ are simple algebraic functions respectively solutions of

$$64 + p_3(x) \cdot Y_+ + (1 - 54x)^4 \cdot Y_+^2 = 0, \quad (D.4)$$

$$1 + p_3(x) \cdot Y_- + 64 \cdot (1 - 54x)^4 \cdot Y_-^2 = 0, \quad (D.5)$$

where

$$p_3(x) = 186624x^3 - 15552x^2 + 2484x - 65, \quad (D.6)$$

and where $Y_{\pm} = (\mathcal{A}_{\pm}^{(2)})^8$ are simple algebraic functions, respectively solutions of:

$$81 - 2 \cdot (41 - 900x) \cdot (1 - 4x) \cdot Y_+ + (1 - 100x)^2 \cdot (1 - 4x)^2 \cdot Y_+^2 = 0, \quad (D.7)$$

$$1 - 2 \cdot (41 - 900x) \cdot (1 - 4x) \cdot Y_- + 81 \cdot (1 - 100x)^2 \cdot (1 - 4x)^2 \cdot Y_-^2 = 0. \quad (D.8)$$

The two Hauptmoduls $\mathcal{H}_{\pm}^{(1)}$ have the following series expansions

$$\begin{aligned} \mathcal{H}_{\pm}^{(1)} = -108x - 8640x^2 - 615168x^3 - 41167872x^4 - 2650337280x^5 \\ - 166137937920x^6 - 10213026103296x^7 - 618505440067584x^8 + \dots \end{aligned} \quad (D.9)$$

and

$$\begin{aligned} \mathcal{H}_{\pm}^{(1)} = -108x^2 - 3024x^3 - 87696x^4 - 2616192x^5 - 79800768x^6 \\ - 2477350656x^7 - 78006945024x^8 - 2485113716736x^9 + \dots \end{aligned} \quad (D.10)$$

and are related by the *genus-zero† modular equation*:

$$\begin{aligned} 625 A^3 B^3 - 525 A^2 B^2 \cdot (A + B) - 96 AB \cdot (A^2 + B^2) - 3 A^2 B^2 \\ - 4 \cdot (A^3 + B^3) + 528 \cdot AB \cdot (A + B) - 432 \cdot AB = 0, \end{aligned} \quad (D.11)$$

which is the same *modular equation* as (54). Note that one can get rid of these square root algebraic expressions for the Hauptmoduls introducing some rational parametrisation such that all the arguments of the miscellaneous ${}_2F_1$ hypergeometric functions (94) introduced to rewrite the Heun solution, are rational functions. Such

† Its parametrisation is in agreement with the rational parametrisation given below in Appendix E.

a rational parametrisation is given in Appendix E. The previous genus-zero modular equation (D.11) has the following rational parametrisation (see (E.4) and (E.5) in Appendix E):

$$\begin{aligned} A &= -\frac{108 z \cdot (1 + 3z) \cdot (1 - 4z - 12z^2)^2}{(1 - 36z - 108z^2)^2}, \\ B &= -\frac{108 z^2 \cdot (1 + 3z)^2 \cdot (1 - 6z) \cdot (1 + 2z)}{(1 + 6z)^2 \cdot (1 - 6z - 18z^2)}. \end{aligned} \quad (\text{D.12})$$

The two Hauptmoduls $\mathcal{H}_{\pm}^{(2)}$ have the following series expansions:

$$\begin{aligned} \mathcal{H}_{-}^{(2)} &= -256x - 42496x^2 - 5955328x^3 - 766211584x^4 - 93688527616x^5 \\ &\quad - 11075543976448x^6 - 1278221854881280x^7 + \dots \end{aligned} \quad (\text{D.13})$$

$$\begin{aligned} \mathcal{H}_{+}^{(2)} &= -256x^3 - 10752x^4 - 361728x^5 - 11580928x^6 - 367348224x^7 \\ &\quad - 11679151104x^8 - 373464444160x^9 - 12021474516480x^{10} + \dots \end{aligned} \quad (\text{D.14})$$

and are related by the *genus-zero modular equation*:

$$\begin{aligned} &640000 \cdot A^2 B^2 \cdot (9A^2 + 14AB + 9B^2) \\ &\quad + 4800 \cdot AB \cdot (A + B) \cdot (A^2 - 1954AB + B^2) \\ &\quad + A^4 + B^4 - 56196 \cdot AB \cdot (A^2 + B^2) + 3512070 \cdot A^2 B^2 \\ &\quad + 116736 \cdot AB \cdot (A + B) - 65536 \cdot AB = 0. \end{aligned} \quad (\text{D.15})$$

Remark: The modular equation (D.15) is actually the same as the modular equation (59) of example 3. This can be seen as a consequence of identity (95), showing a relation between example 3 and example 6.

Appendix E. A rational parametrisation for $Heun(\frac{1}{9}, \frac{1}{12}, \frac{1}{4}, \frac{3}{4}, 1, \frac{1}{2}, 4x)$.

The diagonal of the following rational function of the four variables x , y , z and w :

$$R(x, y, z) = \frac{1}{1 + xy + yz + zw + wx + yw + xz}. \quad (\text{E.1})$$

reads:

$$(1 - 4x) \cdot Heun\left(\frac{1}{9}, \frac{5}{36}, \frac{3}{4}, \frac{5}{4}, 1, \frac{3}{2}, 4x\right)^2. \quad (\text{E.2})$$

This Heun function can be rewritten as miscellaneous ${}_2F_1$ with rational pullbacks. Let us introduce the following rational parametrisation:

$$x = \frac{z \cdot (1 + 2z) \cdot (1 - 6z) \cdot (1 + 3z)}{(1 + 6z)^2}. \quad (\text{E.3})$$

With this rational parametrisation (E.3) the diagonal of the rational function (E.1) can be rewritten in the following ways:

$$\begin{aligned} &Heun\left(\frac{1}{9}, \frac{1}{12}, \frac{1}{4}, \frac{3}{4}, 1, \frac{1}{2}, \frac{4z \cdot (1 + 2z) \cdot (1 - 6z) \cdot (1 + 3z)}{(1 + 6z)^2}\right)^2 \\ &= \frac{(1 + 4z + 12z^2)^2}{(1 + 6z)^2} \\ &\quad \times Heun\left(\frac{1}{9}, \frac{5}{36}, \frac{3}{4}, \frac{5}{4}, 1, \frac{3}{2}, \frac{4z \cdot (1 + 2z) \cdot (1 - 6z) \cdot (1 + 3z)}{(1 + 6z)^2}\right)^2 \end{aligned}$$

$$= \frac{(1+6z)}{(1-36z-108z^2)^{2/3}} \quad (\text{E.4})$$

$$\times {}_2F_1\left(\left[\frac{1}{6}, \frac{2}{3}\right], [1], -\frac{108z \cdot (1+3z) \cdot (1-4z-12z^2)^2}{(1-36z-108z^2)^2}\right)^2$$

$$= \frac{(1+6z)^{1/3}}{(1-6z-18z^2)^{2/3}} \quad (\text{E.5})$$

$$\times {}_2F_1\left(\left[\frac{1}{6}, \frac{2}{3}\right], [1], -\frac{108z^2 \cdot (1+3z)^2 \cdot (1-6z) \cdot (1+2z)^2}{(1+6z)^2 \cdot (1-6z-18z^2)}\right)^2$$

$$= \frac{(1+6z)^{1/2}}{(1+234z+972z^2+1080z^3)^{1/2}}$$

$$\times {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], \frac{1728z \cdot (1+3z)^3 \cdot (1-6z)^6 \cdot (1+2z)^2}{(1+6z)^3 \cdot (1+234z+972z^2+1080z^3)^3}\right)^2$$

$$= \frac{(1+6z)^{1/2}}{(1-6z+12z^2+120z^3)^{1/2}}$$

$$\times {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], \frac{1728z^3 \cdot (1+3z) \cdot (1-6z)^2 \cdot (1+2z)^6}{(1+6z)^3 \cdot (1-6z+12z^2+120z^3)^3}\right)^2.$$

$$= 1 + 6z + 12z^2 + 96z^3 + 360z^4 + 2160z^5 + 10488z^6 + 58464z^7 + \dots$$

**Appendix F. A nome necessary condition to be a classical modular form:
why ${}_2F_1([1/5, 1/5], [1], x)$ is not a classical modular form.**

Consider the identity:

$${}_2F_1\left(\left[\frac{1}{3}, \frac{2}{3}\right], [1], x\right) \quad (\text{F.1})$$

$$= (1+8x)^{-1/4} \cdot {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], 64 \cdot \frac{x \cdot (1-x)^3}{(1+8x)^3}\right).$$

The nome associated to the linear differential operator of order two having ${}_2F_1([1/3, 2/3], [1], x)$ as a solution is given by:

$$Q(x) = x + \frac{5}{9}x^2 + \frac{31}{81}x^3 + \frac{5729}{19683}x^4 + \frac{41518}{177147}x^5 + \frac{312302}{1594323}x^6 + \dots \quad (\text{F.2})$$

and the nome associated to the operator of order two having ${}_2F_1([1/12, 5/12], [1], x)$ as a solution expands as follows:

$$q(x) = x + \frac{31}{72}x^2 + \frac{20845}{82944}x^3 + \frac{27274051}{161243136}x^4 + \frac{183775457147}{1486016741376}x^5 + \dots \quad (\text{F.3})$$

The two ${}_2F_1$ hypergeometric series are *globally bounded*, the series of the corresponding nomes (F.2) and (F.3) are *also globally bounded*, as one expects for a classical modular form. The identity (F.1) on the other solutions of the linear differential operators annihilating ${}_2F_1([1/3, 2/3], [1], x)$ and ${}_2F_1([1/12, 5/12], [1], p(x))$, gives the following identity on their respective ratio τ

$$\tau\left(\left[\frac{1}{3}, \frac{2}{3}\right], [1], x\right) = \mu \cdot \tau\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], 64 \cdot \frac{x \cdot (1-x)^3}{(1+8x)^3}\right), \quad (\text{F.4})$$

where μ is a constant, which gives after exponentiation:

$$64 \cdot Q(x) = q \left(64 \cdot \frac{x \cdot (1-x)^3}{(1+8x)^3} \right). \quad (\text{F.5})$$

Now, the RHS of (F.5) is *necessarily globally bounded*, which agrees with the globally bounded character of the nome (F.2).

In contrast, let us consider ${}_2F_1([1/5, 1/5], [1], x)$. The corresponding series is *globally bounded*[¶], however the corresponding nome which reads

$$\begin{aligned} Q_{[1/5, 1/5]}(x) &= x + \frac{8}{25}x^2 + \frac{102}{625}x^3 + \frac{4744}{46875}x^4 + \frac{81914}{1171875}x^5 \\ &+ \frac{63094248}{1220703125}x^6 + \frac{11003093386}{274658203125}x^7 + \dots \end{aligned} \quad (\text{F.6})$$

is *not* globally bounded. Therefore, it is not possible to find any *algebraic (or rational) pullback* $p(x)$ such that

$$\mu \cdot Q_{[1/5, 1/5]}(x) = q(p(x)), \quad (\text{F.7})$$

since the RHS of (F.7) is *necessarily globally bounded* when $\mu \cdot Q_{[1/5, 1/5]}(x)$ *cannot be globally bounded* regardless of the constant μ .

Appendix F.1. The globally bounded nome condition to be a classical modular form for Heun functions.

Let us recall the order-two linear differential operator (1), which has $Heun(a, q, \alpha, \beta, \gamma, \delta, x)$ as a solution. Since we are interested in diagonals of rational functions, we focus on series expansions at $x = 0$. To have a nome requires the other solution to have a formal series expansion with a logarithm which corresponds to impose that the parameter[†] γ is an integer. To have a logarithm is a necessary but not sufficient condition to have a nome which is analytic at $x = 0$ (mirror map). When[‡] $\gamma = 1$ we have a nome Q , analytic at $x = 0$, which series expansion reads:

$$Q = x + \left(\delta - 2 + \frac{\alpha + \beta - \delta}{a} \right) \cdot x^2 + \dots \quad (\text{F.8})$$

The next terms become more and more involved rational expressions of the parameters of the Heun function. For a given Heun function with $\gamma = 1$ one can easily use the globally bounded nome condition. In practice this is an efficient way to discard the $\gamma = 1$ Heun functions which are not classical modular forms. However, finding exhaustively the Heun functions with $\gamma = 1$ corresponding to classical modular forms, remains a quite involved task (see [33] and Appendix K below).

[¶] Any ${}_2F_1(a, b, [c], x)$ with $c = 1$ is globally bounded since it is of *weight zero*: it is of the form ${}_nF_{n-1}$, and has c given by an integer and not a fractional number.

[†] The condition to have a logarithm for the formal series of (1) at $x = 1$ is that the parameter δ is an integer. The condition to have a logarithm at $x = a$ is $\epsilon = \alpha + \beta - \gamma - \delta + 1$ being an integer.

[‡] This actually corresponds to what we had called “premodular condition” in [32]. The premodular condition $W(x) = -1/2/x^2 + \dots$ yields exactly $\gamma = 1$. In contrast with ${}_2F_1$ hypergeometric functions, this premodular condition is far from imposing that the corresponding Heun series is globally bounded: $Heun(3, 5, 7, 11, 1, 13, x)$, for instance, is not globally bounded.

Appendix G. Periods of an extremal rational surface: a rational parametrisation.

Let us introduce a rational parametrisation for the linear differential operator M_2 and its solution (122):

$$x = \frac{z^2}{(1-z) \cdot (2+z)}. \quad (\text{G.1})$$

With that rational change of variable (G.1) the solution (122) of M_2 becomes:

$$\begin{aligned} \mathcal{S}_2 &= \frac{(z+2)^2 \cdot (1-z)}{2 \cdot p_{12}(z)^{1/4}} \cdot {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], H_1\right) \\ &= \frac{(z+2)^2 \cdot (1-z)}{q_{12}(z)^{1/4}} \cdot {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], H_2\right) \\ &= 1 - \frac{1}{4}z^2 - \frac{1}{16}z^5 + \frac{1}{8}z^6 + \frac{3}{64}z^8 + \frac{3}{32}z^9 + \frac{45}{512}z^{10} + \dots \end{aligned} \quad (\text{G.2})$$

where

$$H_1 = \frac{1728 \cdot z^5 \cdot (1-z^2)^4 (1-z^3)^2 (z+2)^3 (z^2+4z-4) (z^2-2z+4)}{p_{12}(z)^3}, \quad (\text{G.3})$$

$$H_2 = \frac{1728 \cdot z^{10} \cdot (1-z^3) (1-z^2)^2 (z+2)^6 (z^2+4z-4)^2 (z^2-2z+4)^2}{q_{12}(z)^3}, \quad (\text{G.4})$$

with:

$$\begin{aligned} p_{12}(z) &= z^{12} + 8z^{11} + 16z^{10} + 8z^9 + 48z^8 + 112z^7 - 96z^5 \\ &\quad + 16z^4 - 16z^3 - 32z^2 + 16, \end{aligned} \quad (\text{G.5})$$

$$\begin{aligned} q_{12}(z) &= z^{12} + 8z^{11} + 16z^{10} + 8z^9 + 48z^8 - 128z^7 + 384z^5 \\ &\quad + 256z^4 - 256z^3 - 512z^2 + 256. \end{aligned} \quad (\text{G.6})$$

Relation (G.2) makes crystal clear that the solution (122) of M_2 corresponds to a classical modular form, associated with the well-known genus-zero fundamental modular equation†:

$$\begin{aligned} &1953125 \cdot A^3 B^3 - 187500 \cdot A^2 B^2 \cdot (A+B) \\ &\quad + 375 \cdot AB \cdot (16A^2 - 4027AB + 16B^2) \\ &\quad - 64 \cdot (A+B) \cdot (A^2 + 1487AB + B^2) + 110592 \cdot AB = 0. \end{aligned} \quad (\text{G.7})$$

With the same rational change of variable (G.1), the solution (117) of the order-two linear differential operator L_2 can be written in term of a *very simple* Heun function associated with the following remarkable Heun identity:

$$\begin{aligned} \mathcal{S}_1 &= \text{Heun}\left(\frac{1}{2} - \frac{i\sqrt{3}}{2}, \frac{1}{2} - \frac{i\sqrt{3}}{6}, 1, 1, 1, 1, \frac{3}{2} \cdot (-3 + i\sqrt{3}) \cdot \frac{z^2}{(1-z) \cdot (2+z)}\right) \\ &= \frac{1}{4} \cdot (1-z) \cdot (2+z) \cdot \text{Heun}\left(-\frac{1}{8}, \frac{1}{4}, 1, 1, 1, 1, -\frac{z^3}{8}\right). \end{aligned} \quad (\text{G.8})$$

The solution \mathcal{S}_1 can also be written:

$$\mathcal{S}_1 = \frac{(z+2)^2 \cdot (1-z)}{2 \cdot (z^3+2)} \cdot {}_2F_1\left(\left[\frac{1}{3}, \frac{2}{3}\right], [1], \left(\frac{3z}{z^3+2}\right)^3\right). \quad (\text{G.9})$$

† Corresponding to $q \leftrightarrow q^2$ in the nome, see equation (4) in [32].

Appendix H. Special ${}_2F_1$ hypergeometric functions associated with classical modular forms

The Heun functions of this paper can all be rewritten in terms of pullbacked ${}_2F_1$ hypergeometric functions which turn out to correspond to *classical modular curves* (with the exception of the ‘‘Shimura’’ Heun functions of section (3), see also Appendix J below). These ${}_2F_1$ hypergeometric functions are not arbitrary, they are ‘‘special’’ ${}_2F_1$ ’s corresponding to selected parameters, namely ${}_2F_1$ ’s related to *classical modular curves*. These various ${}_2F_1$ are often simply related

$$\begin{aligned} {}_2F_1\left(\left[\frac{1}{2}, \frac{1}{2}\right], [1], x\right) & \quad (H.1) \\ &= (1 + \omega^2 x)^{-1/2} \cdot {}_2F_1\left(\left[\frac{1}{6}, \frac{1}{2}\right], [1], \frac{3 \cdot \omega \cdot (\omega - 1) \cdot x \cdot (1 - x)}{(1 + \omega^2 \cdot x)^3}\right), \end{aligned}$$

where $1 + \omega + \omega^2 = 0$ (ω is a third root of unity), or:

$$\begin{aligned} {}_2F_1\left(\left[\frac{1}{4}, \frac{1}{4}\right], [1], 64x\right) &= \frac{1}{(1 - 64x)^{1/4}} \cdot {}_2F_1\left(\left[\frac{1}{4}, \frac{3}{4}\right], [1], -\frac{64x}{1 - 64x}\right) \\ &= \frac{1}{(1 + 64x)^{1/4}} \cdot {}_2F_1\left(\left[\frac{1}{8}, \frac{5}{8}\right], [1], \frac{256x}{(1 + 64x)^2}\right) \quad (H.2) \\ &= \frac{1}{(1 - 64x)^{1/12} \cdot (1 + 8x)^{1/6}} \\ &\quad \times {}_2F_1\left(\left[\frac{1}{12}, \frac{7}{12}\right], [1], -\frac{1728x^2}{(1 - 64x) \cdot (1 + 8x)^2}\right) \end{aligned}$$

$$\begin{aligned} {}_2F_1\left(\left[\frac{1}{2}, \frac{5}{6}\right], [1], 144x\right) &= (1 - 144x)^{-1/3} \cdot {}_2F_1\left(\left[\frac{1}{2}, \frac{1}{6}\right], [1], 144x\right) \\ &= (1 - 144x)^{-5/12} \cdot {}_2F_1\left(\left[\frac{1}{6}, \frac{5}{6}\right], [1], \frac{1}{2} - \frac{1}{2} \cdot \frac{1 - 72x}{(1 - 144x)^{1/2}}\right) \\ &= (1 - 72x)^{-1/6} \cdot (1 - 144x)^{-1/3} \cdot {}_2F_1\left(\left[\frac{1}{12}, \frac{7}{12}\right], [1], \frac{5184x^2}{(1 - 72x)^2}\right) \\ &= (1 - 144x)^{-5/12} \cdot {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], -\frac{5184x^2}{(1 - 144x)}\right), \quad (H.3) \end{aligned}$$

$$\begin{aligned} {}_2F_1\left(\left[\frac{2}{3}, \frac{2}{3}\right], [1], 27x\right) & \quad (H.4) \\ &= (1 - 729x^2)^{-1/3} \cdot {}_2F_1\left(\left[\frac{1}{6}, \frac{2}{3}\right], [1], \frac{108x}{(1 + 27x)^2}\right) \\ &= \frac{1}{(1 - 27x)^{5/12} \cdot (1 - 3x)^{1/4}} \\ &\quad \times {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], -\frac{1728x^3}{(1 - 27x) \cdot (1 - 3x)^3}\right) \\ &= \frac{1}{(1 - 27x)^{5/12} \cdot (1 - 243x)^{1/4}} \\ &\quad \times {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], -\frac{1728x}{(1 - 27x) \cdot (1 - 243x)^3}\right). \end{aligned}$$

$${}_2F_1\left(\left[\frac{3}{8}, \frac{7}{8}\right], [1], 256x\right) = (1 - 256x)^{-1/4} \cdot {}_2F_1\left(\left[\frac{1}{8}, \frac{5}{8}\right], [1], 256x\right), \quad (\text{H.5})$$

$${}_2F_1\left(\left[\frac{1}{3}, \frac{5}{6}\right], [1], 108x\right) = (1 - 108x)^{-1/6} \cdot {}_2F_1\left(\left[\frac{1}{6}, \frac{2}{3}\right], [1], 108x\right), \quad (\text{H.6})$$

$${}_2F_1\left(\left[\frac{2}{3}, \frac{5}{6}\right], [1], 108x\right) = (1 - 108x)^{-2/3} \cdot {}_2F_1\left(\left[\frac{1}{6}, \frac{2}{3}\right], [1], -\frac{108x}{1 - 108x}\right), \quad (\text{H.7})$$

$$\begin{aligned} {}_2F_1\left(\left[\frac{1}{2}, \frac{3}{4}\right], [1], 32x\right) &= (1 - 32x)^{-1/4} \cdot {}_2F_1\left(\left[\frac{1}{4}, \frac{1}{2}\right], [1], 32x\right) \\ &= (1 - 32x)^{-1/4} \cdot (1 - 16x)^{-1/4} \cdot {}_2F_1\left(\left[\frac{1}{8}, \frac{5}{8}\right], [1], \frac{256x^2}{(1 - 16x)^2}\right), \end{aligned} \quad (\text{H.8})$$

$$\begin{aligned} {}_2F_1\left(\left[\frac{5}{6}, \frac{5}{6}\right], [1], 432x\right) &= (1 - 432x)^{-5/6} \cdot {}_2F_1\left(\left[\frac{1}{6}, \frac{5}{6}\right], [1], -\frac{432x}{1 - 432x}\right) \\ &= (1 - 432x)^{-2/3} \cdot (1 + 432x)^{-1/6} \cdot {}_2F_1\left(\left[\frac{1}{12}, \frac{7}{12}\right], [1], \frac{1728x}{(1 + 432x)^2}\right). \end{aligned} \quad (\text{H.9})$$

$$\begin{aligned} {}_2F_1\left(\left[\frac{3}{4}, \frac{3}{4}\right], [1], 64x\right) &= (1 - 64x)^{-3/4} \cdot {}_2F_1\left(\left[\frac{1}{4}, \frac{3}{4}\right], [1], -\frac{64x}{1 - 64x}\right) \\ &= (1 - 64x)^{-1/2} \cdot (1 + 64x)^{-1/4} \cdot {}_2F_1\left(\left[\frac{1}{8}, \frac{5}{8}\right], [1], \frac{256x}{(1 + 64x)^2}\right), \end{aligned} \quad (\text{H.10})$$

$$\begin{aligned} {}_2F_1\left(\left[\frac{1}{2}, \frac{2}{3}\right], [1], 36x\right) &= (1 - 36x)^{-1/6} \cdot {}_2F_1\left(\left[\frac{1}{3}, \frac{1}{2}\right], [1], 36x\right) \\ &= (1 - 36x)^{-1/6} \cdot (1 - 18x)^{-1/3} \cdot {}_2F_1\left(\left[\frac{1}{6}, \frac{2}{3}\right], [1], \frac{324x^2}{(1 - 18x)^2}\right), \end{aligned} \quad (\text{H.11})$$

$${}_2F_1\left(\left[\frac{1}{3}, \frac{2}{3}\right], [1], x\right) = (1 + 8x)^{-1/4} \cdot {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], \frac{64 \cdot x \cdot (1 - x)^3}{(1 + 8x)^3}\right), \quad (\text{H.12})$$

$${}_2F_1\left(\left[\frac{1}{4}, \frac{3}{4}\right], [1], x\right) = (1 + 3x)^{-1/4} \cdot {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], \frac{27 \cdot x \cdot (1 - x)^2}{(1 + 3x)^3}\right). \quad (\text{H.13})$$

In fact, all these “special” ${}_2F_1$ ’s hypergeometric functions correspond to *classical modular forms* because they can be rewritten [32] as $\mathcal{A} \cdot {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], p(x)\right)$ where the pullback $p(x)$ is in general more involved than simple rational pullbacks like (H.12) or (H.13), being often *algebraic* functions. For instance one has the following identities

$$\begin{aligned} {}_2F_1\left(\left[\frac{1}{8}, \frac{5}{8}\right], [1], x\right) &= \\ &= \left(\frac{8}{(5 \cdot (1 - x)^{1/2} + 3)^{1/4}}\right)^{1/4} \cdot {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], 27 \cdot x \cdot \frac{1 - (1 - x)^{1/2}}{3 + 5 \cdot (1 - x)^{1/2}}\right) \\ &= \left(\frac{2}{(5 \cdot (1 - x)^{1/2} - 3)^{1/4}}\right)^{1/4} \cdot {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], 27 \cdot x \cdot \frac{1 + (1 - x)^{1/2}}{3 - 5 \cdot (1 - x)^{1/2}}\right), \end{aligned} \quad (\text{H.14})$$

or

$${}_2F_1\left(\left[\frac{1}{6}, \frac{2}{3}\right], [1], x\right) = \mathcal{A}(x) \cdot {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], \mathcal{H}(x)\right), \quad (\text{H.15})$$

where $\mathcal{H}(x)$ reads

$$\mathcal{H}(x) = 4 \cdot x \cdot \frac{1458 - 1215x + 125x^2}{(25x - 9)^3} + 8 \cdot x \cdot \frac{(27 - 11x) \cdot (27 - 25x)}{\sqrt{1 - x} \cdot (9 - 25x)^3}, \quad (\text{H.16})$$

and where $\mathcal{A}(x)$ reads:

$$\mathcal{A}(x) = (1-x)^{-1/24} \cdot \left(\frac{81}{9-25x}\right)^{1/8} \cdot \left(\frac{5 \cdot (1-x)^{1/2} - 4}{5 \cdot (1-x)^{1/2} + 4}\right)^{1/8}, \quad (\text{H.17})$$

which is solution of

$$531441 - 7290 \cdot (x-1) \cdot (25x-73) \cdot Z + (x-1) \cdot (25x-9)^3 \cdot Z^2 = 0, \quad (\text{H.18})$$

where $Z = \mathcal{A}(x)^{12}$.

Generalizing the globally bounded nome condition approach of the previous Appendix F, we looked for all possible ${}_2F_1$ hypergeometric functions related§ to pullbacked ${}_2F_1([1/12, 5/12], [1], x)$, *looking at the globally bounded condition of their nome* (see (F.7)). We give here a *finite list* of only 28 hypergeometric functions that have *integer coefficient series*, that are related† to *modular forms*.

$$\begin{aligned} & {}_2F_1\left(\left[\frac{1}{2}, \frac{1}{2}\right], [1], 16x\right), \quad {}_2F_1\left(\left[\frac{1}{2}, \frac{1}{3}\right], [1], 36x\right), \quad {}_2F_1\left(\left[\frac{1}{3}, \frac{1}{3}\right], [1], 27x\right), \\ & {}_2F_1\left(\left[\frac{1}{3}, \frac{2}{3}\right], [1], 27x\right), \quad {}_2F_1\left(\left[\frac{1}{6}, \frac{1}{2}\right], [1], 108x\right), \quad {}_2F_1\left(\left[\frac{1}{6}, \frac{1}{3}\right], [1], 108x\right), \\ & {}_2F_1\left(\left[\frac{1}{6}, \frac{2}{3}\right], [1], 432x\right), \quad {}_2F_1\left(\left[\frac{1}{6}, \frac{1}{6}\right], [1], 432x\right), \quad {}_2F_1\left(\left[\frac{1}{6}, \frac{5}{6}\right], [1], 432x\right), \\ & {}_2F_1\left(\left[\frac{1}{4}, \frac{1}{4}\right], [1], 64x\right), \quad {}_2F_1\left(\left[\frac{1}{4}, \frac{1}{2}\right], [1], 32x\right), \quad {}_2F_1\left(\left[\frac{1}{4}, \frac{3}{4}\right], [1], 64x\right), \\ & {}_2F_1\left(\left[\frac{1}{8}, \frac{3}{8}\right], [1], 256x\right), \quad {}_2F_1\left(\left[\frac{1}{8}, \frac{5}{8}\right], [1], 256x\right), \quad {}_2F_1\left(\left[\frac{3}{8}, \frac{7}{8}\right], [1], 256x\right), \\ & {}_2F_1\left(\left[\frac{2}{3}, \frac{5}{6}\right], [1], 108x\right), \quad {}_2F_1\left(\left[\frac{1}{3}, \frac{5}{6}\right], [1], 108x\right), \quad {}_2F_1\left(\left[\frac{1}{2}, \frac{3}{4}\right], [1], 32x\right), \\ & {}_2F_1\left(\left[\frac{3}{4}, \frac{3}{4}\right], [1], 64x\right), \quad {}_2F_1\left(\left[\frac{5}{8}, \frac{7}{8}\right], [1], 256x\right), \quad {}_2F_1\left(\left[\frac{2}{3}, \frac{2}{3}\right], [1], 27x\right), \\ & {}_2F_1\left(\left[\frac{5}{6}, \frac{5}{6}\right], [1], 432x\right), \quad {}_2F_1\left(\left[\frac{1}{2}, \frac{5}{6}\right], [1], 144x\right), \quad {}_2F_1\left(\left[\frac{1}{2}, \frac{2}{3}\right], [1], 36x\right) \\ & {}_2F_1\left(\left[\frac{1}{12}, \frac{7}{12}\right], [1], 1728x\right), \quad {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], 1728x\right), \quad (\text{H.19}) \\ & {}_2F_1\left(\left[\frac{5}{12}, \frac{11}{12}\right], [1], 1728x\right), \quad {}_2F_1\left(\left[\frac{7}{12}, \frac{11}{12}\right], [1], 1728x\right). \end{aligned}$$

Using this globally bounded condition of the nome criterion, we wrote a program that went through all the values of a and b in $[-1, 1]$ (with small increments like $1/200$), with $c = 1$, singling out the ${}_2F_1$ hypergeometric functions that have integer coefficients (or more generally globally bounded) series expansions, both for the ${}_2F_1$ hypergeometric functions, and *for the nome*. Running this program returned to us exactly the ${}_2F_1$ hypergeometric functions in the above list (H.19), and *only this list of twenty eight hypergeometric functions*.

Appendix H.1. Derivatives of classical modular forms

Recalling identities (A.29), where the parameters in (A.29) verify (A.19) and (A.22), one can also deduce, for instance for $e = 1/2$, some identities on some (homogeneous)

§ See [1, 2], and the hypergeometric functions in the previous sections in this paper.

† By related to classical modular forms, we mean, from now, that any of the hypergeometric functions below can be rewritten as a pullbacked ${}_2F_1([1/12, 5/12], [1], x)$ function, and hence is necessarily a classical modular form.

derivatives of the classical modular form ${}_2F_1([1/6, 1/6], [1], x)$ of the previous list (H.19):

$$\begin{aligned} \text{Heun}\left(\frac{9}{4}, \frac{3}{16}, \frac{1}{6}, \frac{1}{6}, 1, \frac{4}{3}, x\right) &= (2\theta + 1) \left[{}_2F_1\left(\left[\frac{1}{6}, \frac{1}{6}\right], [1], x\right) \right] \\ &= {}_3F_2\left(\left[\frac{1}{6}, \frac{1}{6}, \frac{3}{2}\right], \left[\frac{1}{2}, 1\right], x\right). \end{aligned} \quad (\text{H.20})$$

Such homogeneous derivatives of classical modular forms are not classical modular forms. We have however seen, in section (2.5), that when a diagonal of a rational function can be expressed as a classical modular form, the homogeneous derivatives of that classical modular form are also diagonals of other rational functions simply deduced from the first rational function.

Appendix H.2. Hypergeometric functions with negative values related to classical modular forms

One remarks that *no negative values* of a and b occur in the previous list of ${}_2F_1$ classical modular forms (H.19). But what about negative values of the parameters of the ${}_2F_1$ functions? Can we have ${}_2F_1$ hypergeometric functions with negative values of the parameters a and b such that it is not a classical modular form, but a derivative of a classical modular form?

Denoting $\theta = x \cdot D_x$ the homogeneous derivative, one has the following identities:

$$\begin{aligned} {}_2F_1\left(\left[-\frac{1}{6}, \frac{5}{6}\right], [1], x\right) &= (1-x)^{1/3} \cdot (1+6\cdot\theta) \left[{}_2F_1\left(\left[\frac{1}{6}, \frac{1}{6}\right], [1], x\right) \right], \\ {}_2F_1\left(\left[\frac{1}{6}, \frac{1}{6}\right], [1], x\right) &= (1-x)^{2/3} \cdot (1-6\cdot\theta) \left[{}_2F_1\left(\left[-\frac{1}{6}, \frac{5}{6}\right], [1], x\right) \right], \end{aligned} \quad (\text{H.21})$$

and thus, by elimination, one finds that the order-two operators annihilating respectively ${}_2F_1(\left[-\frac{1}{6}, \frac{5}{6}\right], [1], x)$ and ${}_2F_1(\left[\frac{1}{6}, \frac{1}{6}\right], [1], x)$ read:

$$\begin{aligned} L_2 &= (1-x)^{1/3} \cdot (1+6\cdot\theta) \cdot (1-x)^{2/3} \cdot (1-6\cdot\theta) - 1 \\ &= x \cdot (6\theta + 5) \cdot (6\theta - 1) - 36\theta^2, \end{aligned} \quad (\text{H.22})$$

$$\begin{aligned} M_2 &= (1-x)^{2/3} \cdot (1-6\cdot\theta) \cdot (1-x)^{1/3} \cdot (1+6\cdot\theta) - 1 \\ &= x \cdot (6\theta + 1) \cdot (6\theta + 1) - 36\theta^2. \end{aligned} \quad (\text{H.23})$$

These two order-two linear differential operators L_2 and M_2 are *equivalent*[†] with order-one intertwiners (with algebraic coefficients):

$$M_2 \cdot (1-x)^{2/3} \cdot (6\theta - 1) = (1-x)^{2/3} \cdot (6\theta - 1) \cdot L_2, \quad (\text{H.24})$$

and:

$$L_2 \cdot (1-x)^{1/3} \cdot (6\theta + 1) = (1-x)^{1/3} \cdot (6\theta + 1) \cdot M_2, \quad (\text{H.25})$$

Furthermore we also have the identities

$${}_2F_1\left(\left[\frac{5}{6}, \frac{7}{6}\right], [1], x\right) = (1-x)^{-5/6} \cdot {}_2F_1\left(\left[-\frac{1}{6}, \frac{5}{6}\right], [1], -\frac{x}{1-x}\right), \quad (\text{H.26})$$

or:

$${}_2F_1\left(\left[-\frac{1}{6}, \frac{5}{6}\right], [1], x\right) = (1-x)^{-5/6} \cdot {}_2F_1\left(\left[\frac{5}{6}, \frac{7}{6}\right], [1], -\frac{x}{1-x}\right). \quad (\text{H.27})$$

[†] Use the command “equiv” of Mark van Hoeij and not the command “Homomorphisms” of DEtools in Maple. This command is the algebraic extension of the command “Homomorphisms”.

Therefore, we see that hypergeometric functions like

$${}_2F_1\left(\left[-\frac{1}{6}, \frac{5}{6}\right], [1], x\right), \quad {}_2F_1\left(\left[\frac{5}{6}, \frac{7}{6}\right], [1], x\right), \quad (\text{H.28})$$

are *not* classical modular forms, but are *related* to classical modular forms, *being an order-one operator acting on a classical classical modular form.*

Remark 1: The order-two linear differential operators for ${}_2F_1\left(\left[\frac{5}{6}, \frac{7}{6}\right], [1], x\right)$, and for ${}_2F_1\left(\left[\frac{1}{6}, \frac{5}{6}\right], [1], x\right)$ which corresponds to a classical modular form, are equivalent, up to a $x \rightarrow -x/(1-x)$ pullback. The nome of the order-two linear differential operators for ${}_2F_1\left(\left[\frac{5}{6}, \frac{7}{6}\right], [1], x\right)$ reads:

$$q = x + \frac{x^2}{18} + \frac{49x^3}{5184} + \frac{7201x^4}{2519424} + \frac{6889571x^5}{5804752896} + \frac{311739307x^6}{522427760640} + \dots \quad (\text{H.29})$$

which is *not* a globally bounded series. The equivalence of linear differential operators *does not preserve the globally bounded character of the nome.*

Remark 2: For any integers n_1, n_2, n_3 the order-two linear differential operator annihilating ${}_2F_1([a, b], [c], x)$ and the order-two linear differential operator annihilating ${}_2F_1([a + n_1, b + n_2], [c + n_3], x)$, are homomorphic \ddagger . Denoting N_2 the order-two linear differential operator annihilating ${}_2F_1\left(\left[\frac{1}{6}, \frac{5}{6}\right], [1], x\right)$, and P_2 the order-two linear differential operator annihilating ${}_2F_1\left(\left[\frac{7}{6}, \frac{5}{6}\right], [1], x\right)$, one has the homomorphism

$$P_2 \cdot (6\theta + 1) = (6\theta + 7) \cdot N_2, \quad (\text{H.30})$$

yielding the relation:

$${}_2F_1\left(\left[\frac{7}{6}, \frac{5}{6}\right], [1], x\right) = (6\theta + 1) \left[{}_2F_1\left(\left[\frac{1}{6}, \frac{5}{6}\right], [1], x\right) \right]. \quad (\text{H.31})$$

From this last identity (H.31) we see that ${}_2F_1\left(\left[\frac{5}{6}, \frac{7}{6}\right], [1], x\right)$ is just a *homogeneous derivative of a classical modular form* ${}_2F_1\left(\left[\frac{1}{6}, \frac{5}{6}\right], [1], x\right)$ (see (H.19)).

Appendix H.2.1. A non minimal order telescoper associated with ${}_2F_1\left(-\frac{1}{6}, \frac{5}{6}\right], [1], z$.

Using Koutschan’s creative telescoping program we have obtained the telescoper of the rational function $1/(1 + xy + zx + 3(x^2 + y^2))$. It is an order-four linear differential operator W_4 which factors \P as $W_4 = U_2 \cdot V_2$ where V_2 is a linear differential operator of order two with two *algebraic solutions*. A well-suited linear combination of these two algebraic solutions gives the integer coefficient series expansion

$$1 - 9x^2 + 135x^4 - 2268x^6 + 40095x^8 - 729729x^{10} + 13533156x^{12} + \dots \quad (\text{H.32})$$

which is actually the diagonal of the rational function $1/(1 + xy + zx + 3(x^2 + y^2))$. Therefore the order-two linear differential operator V_2 is the *minimal order linear differential operator* for the diagonal (H.32). The creative telescoping method, however, provides a *larger order \ddagger telescoper* W_4 and, consequently, a “companion”

\ddagger Note that this is not the case for Heun functions. Recalling (27) and changing $\gamma \rightarrow \gamma + 1$, one sees easily that the linear differential operators for $Heun(-1/27, 2/27, 1/3, 2/3, 1, 1/2, -x)$ and $Heun(-1/27, 2/27, 1/3, 2/3, 2, 1/2, -x)$ are not equivalent.

\P But no direct-sum factorisation.

\ddagger Non minimal order linear differential operator as far as the diagonal of the rational function is concerned.

to this minimal order operator V_2 . The linear differential operator “companion” of order two, U_2 , admits the solution:

$$\left(1 + \frac{81}{4}x^2\right)^{-1} \cdot \frac{dH(x)}{dx} \quad \text{where:} \quad H(x) = {}_2F_1\left(\left[\frac{5}{6}, \frac{7}{6}\right], [1], -\frac{315}{16} \cdot x^2\right). \quad (\text{H.33})$$

The series expansion of (H.33) is a globally bounded series[†]. From (H.26) this last hypergeometric function is related to

$${}_2F_1\left(\left[-\frac{1}{6}, \frac{5}{6}\right], [1], \frac{315x^2}{16 + 315x^2}\right), \quad (\text{H.34})$$

which is related, using (H.21), to:

$$\begin{aligned} & {}_2F_1\left(\left[\frac{1}{6}, \frac{1}{6}\right], [1], \frac{315x^2}{16 + 315x^2}\right) \\ &= \left(1 + \frac{315}{16}x^2\right)^{1/6} \cdot {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], -\frac{315}{64} \cdot x^2 \cdot (16 + 315x^2)\right). \end{aligned} \quad (\text{H.35})$$

Hence, if a ${}_2F_1$ hypergeometric function appears as the diagonal of a rational function, or a (globally bounded) solution of a factor of a (non minimal order) telescoper, it seems often to be still related to a *classical modular form*: in this case here, it is the *derivative of a classical modular form*.

Appendix I. Diagonals of rational functions in three variables: derivatives of modular forms

Let us give a simple example of diagonal of rational function of three variables yielding a derivative of a classical modular form (or a derivative of a Heun function). Let us consider the following rational function of three variables:

$$R(x, y, z) = \frac{3x^3y}{1 + x + y + z}. \quad (\text{I.1})$$

The diagonal of (I.1) has the following series expansion with *integer coefficients*:

$$\begin{aligned} & -30x^3 + 840x^4 - 20790x^5 + 504504x^6 - 12252240x^7 + 299304720x^8 \\ & - 7362064710x^9 + 182298745200x^{10} + \dots \end{aligned} \quad (\text{I.2})$$

The telescoper of this rational function of three variables (I.1) gives an order-three linear differential operator $L_3 = L_1 \oplus L_2$ which is the direct sum (LCLM) of an order-one linear differential operator L_1 and an order-two linear differential operator L_2 , where:

$$\begin{aligned} L_1 &= x \cdot D_x - 1, \\ L_2 &= (1 + 27x) \cdot (1 + 30x) \cdot x \cdot D_x^2 - 3 \cdot x \cdot D_x + 180x + 3. \end{aligned} \quad (\text{I.3})$$

[†] The series (H.33) becomes a series with integer coefficients with $x \rightarrow 96x$.

The order-one linear differential operator L_1 has the simple solution $y(x) = x$, and the order-two linear differential operator has the following Heun solution:

$$\begin{aligned} x \cdot \text{Heun}\left(\frac{9}{10}, 0, \frac{1}{3}, \frac{2}{3}, 2, 1, -27 \cdot x\right) &= x - 30x^3 + 840x^4 - 20790x^5 \\ &\quad + 504504x^6 - 12252240x^7 + 299304720x^8 + \dots \\ &= -x \cdot {}_2F_1\left(\left[\frac{1}{3}, \frac{2}{3}\right], [1], -27 \cdot x\right) \end{aligned} \quad (\text{I.4})$$

$$\begin{aligned} &\quad + 2 \cdot x \cdot (1 + 27x) \cdot {}_2F_1\left(\left[\frac{4}{3}, \frac{5}{3}\right], [2], -27 \cdot x\right) \\ &= \mathcal{L}_1\left({}_2F_1\left(\left[\frac{1}{3}, \frac{2}{3}\right], [1], -27 \cdot x\right)\right) \end{aligned} \quad (\text{I.5})$$

$$\text{where:} \quad \mathcal{L}_1 = -x - \frac{1 + 27x}{3} \cdot x \cdot \frac{d}{dx}. \quad (\text{I.6})$$

It can also be written alternatively as

$$x \cdot \text{Heun}\left(\frac{9}{10}, 0, \frac{1}{3}, \frac{2}{3}, 2, 1, -27 \cdot x\right) \quad (\text{I.7})$$

$$\begin{aligned} &= x \cdot \frac{5 + 108x}{(1 + 54x)^{4/3}} \cdot {}_2F_1\left(\left[\frac{1}{6}, \frac{2}{3}\right], [1], \frac{108 \cdot x \cdot (1 + 27x)}{(1 + 54x)^2}\right) \\ &\quad - 4x \cdot \frac{1 + 27x}{(1 + 54x)^{10/3}} \cdot {}_2F_1\left(\left[\frac{7}{6}, \frac{5}{3}\right], [2], \frac{108 \cdot x \cdot (1 + 27x)}{(1 + 54x)^2}\right) \\ &= x \cdot \frac{(1 + 18x)(1 + 24x)}{(1 + 36x + 216x^2)^{7/6}} \cdot {}_2F_1\left(\left[\frac{1}{12}, \frac{7}{12}\right], [1], \frac{1728 \cdot x^3 \cdot (1 + 27x)}{(1 + 36x + 216x^2)^2}\right) \\ &\quad - 84x^3 \cdot \frac{(1 + 27x)(1 + 24x)^2}{(1 + 36x + 216x^2)^{19/6}} \cdot {}_2F_1\left(\left[\frac{13}{12}, \frac{19}{12}\right], [2], \frac{1728 \cdot x^3 \cdot (1 + 27x)}{(1 + 36x + 216x^2)^2}\right) \\ &= \mathcal{L}_1\left({}_2F_1\left(\left[\frac{1}{12}, \frac{7}{12}\right], [1], \frac{1728 \cdot x^3 \cdot (1 + 27x)}{(1 + 36x + 216x^2)^2}\right)\right) \\ &= \mathcal{M}_1\left({}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], -\frac{1728x^3(1 + 27x)}{(1 + 24x)^3}\right)\right), \end{aligned} \quad (\text{I.8})$$

where:

$$\begin{aligned} \mathcal{L}_1 &= x \cdot \frac{(1 + 18x) \cdot (1 + 24x)}{(1 + 36x + 216x^2)^{7/6}} - \frac{1}{3} \cdot \frac{(1 + 27x)}{(1 + 36x + 216x^2)^{1/6}} \cdot x \cdot D_x, \\ \mathcal{M}_1 &= \frac{1 + 30x}{(1 + 24x)^{5/4}} \cdot x - \frac{1}{3} \cdot \frac{(1 + 27x)}{(1 + 24x)^{1/4}} \cdot x \cdot D_x. \end{aligned} \quad (\text{I.9})$$

Again we see that the derivative of a classical modular form, or more generally an order-one linear differential operator like (I.4) acting on a classical modular form, is *no longer a classical modular form*. With this example we see, one more time (recall section (2.5)), that a Heun function which has a series expansion with integer coefficients, is not necessarily a classical modular form but can be *an order-one linear differential operator acting on a classical modular form*.

Appendix J. Automorphic forms associated with a Shimura curve

Appendix J.1. The pullback in ${}_2F_1([\frac{1}{24}, \frac{7}{24}], [\frac{5}{6}], x)$ and ${}_2F_1([\frac{5}{24}, \frac{11}{24}], [\frac{7}{6}], x)$ is special.

Like all the *Belyi coverings* [27], the pullback $\frac{27}{4} \cdot \frac{x^2}{(1-x)^3}$ in (145) is “special”. It is such that:

$$\begin{aligned} & \left(\frac{27}{4} \cdot \frac{x^2}{(1-x)^3} \right) \circ \left(\frac{480 \cdot (1-x) \cdot x}{(17x-32)^2} \right) \\ &= \left(\frac{27}{4} \cdot \frac{x^2}{(1-x)^3} \right) \circ \left(\frac{15 \cdot (17x-32) \cdot x}{1024 \cdot (1-x)^2} \right). \end{aligned} \quad (\text{J.1})$$

It has already been seen to occur in another framework [46], namely† the walk in a Weyl chamber of the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$. It actually occurs in the well-known “Kernel equation” for that particular walk described in [46]

$$G(x, y) + G(0, 0) = G(x, 0) + G(0, y), \quad (\text{J.2})$$

$$\text{where:} \quad G(x, y) = L(x, y) \cdot H(x, y), \quad (\text{J.3})$$

and where the generating function $H(x, y)$ of the walk, and the Kernel of the walk $L(x, y)$, read respectively:

$$H(x, y) = \frac{1-xy}{(1-x)^3 \cdot (1-y)^3}, \quad L(x, y) = \frac{27}{4} \cdot (y + xy^2 + x^2 - 3xy).$$

Noticeably, $G(x, y)$ is the sum of the particular rational function pullback $w(x) = \frac{27}{4} \cdot \frac{x^2}{(1-x)^3}$ and of another rational function of y :

$$G(x, y) = \frac{27}{4} \cdot \frac{x^2}{(1-x)^3} + \frac{27}{4} \cdot \frac{y}{(1-y)^3}. \quad (\text{J.4})$$

Note that this additional rational function of y corresponds to the duality $x \leftrightarrow 1/x$:

$$G(x, y) = L(x, y) \cdot H(x, y) = w(x) - w\left(\frac{1}{y}\right). \quad (\text{J.5})$$

The genus-zero curve $L(x, y) = 0$ has the rational parametrisation

$$\begin{aligned} y &= \frac{225 t^2}{32 \cdot (1-t) \cdot (17t-32)}, & \text{and:} \\ x &= \frac{480 \cdot (1-t) \cdot t}{(17t-32)^2} & \text{or:} \quad x' = \frac{15 \cdot (17t-32) \cdot t}{1024 \cdot (1-t)^2}, \end{aligned} \quad (\text{J.6})$$

where one recovers, in the last two rational parametrisations (J.6) for x , the two rational functions in the identity (J.1). The two rational functions are the rational parametrisation of the (symmetric) genus-zero curve $x^2 x'^2 - 3x x' + x + x' = 0$ (corresponding to the elimination of y in $L(x, y) = 0$ and $L(x', y) = 0$). These two rational functions (J.6) are simply related by an involution:

$$x'(t) = x\left(\frac{32t}{49t-32}\right), \quad x(t) = x'\left(\frac{32t}{49t-32}\right). \quad (\text{J.7})$$

† See equation $w(x) = \frac{27}{4} \cdot \frac{x^2}{(1-x)^3}$, page 3165 in [46].

Appendix J.2. ${}_2F_1\left(\left[\frac{1}{24}, \frac{7}{24}\right], \left[\frac{5}{6}\right], x\right)$, ${}_2F_1\left(\left[\frac{5}{24}, \frac{11}{24}\right], \left[\frac{7}{6}\right], x\right)$ and ${}_2F_1\left(\left[\frac{1}{24}, \frac{5}{24}\right], \left[\frac{3}{4}\right], x\right)$, ${}_2F_1\left(\left[\frac{7}{24}, \frac{11}{24}\right], \left[\frac{5}{4}\right], x\right)$ Shimura examples.

The order-two linear differential operator

$$L_2 = \theta \cdot \left(\theta - \frac{1}{6}\right) - x \cdot \left(\theta + \frac{1}{24}\right) \cdot \left(\theta + \frac{7}{24}\right), \quad (\text{J.8})$$

has the two ${}_2F_1$ hypergeometric solutions:

$${}_2F_1\left(\left[\frac{1}{24}, \frac{7}{24}\right], \left[\frac{5}{6}\right], x\right), \quad x^{1/6} \cdot {}_2F_1\left(\left[\frac{5}{24}, \frac{11}{24}\right], \left[\frac{7}{6}\right], x\right), \quad (\text{J.9})$$

A modular equation of level five has been given in [13] on this example of ${}_2F_1$ hypergeometric function corresponding to automorphic forms associated with Shimura curves. This modular equation of level five corresponds to the elimination of u between

$$A(u) = -\frac{1350000 \cdot u^6}{225 u^2 + 18 u + 1},$$

$$B(u) = A\left(\frac{11 u + 2}{252 u - 11}\right) = -\frac{2160 \cdot (11 u + 2)^6}{(225 u^2 + 18 u + 1)(252 u - 11)^4}, \quad (\text{J.10})$$

Changing x into $1/x$, the order-two linear differential operator (J.8) becomes the order-two linear differential operator

$$L_2\left(x \rightarrow \frac{1}{x}\right) = \left(\theta - \frac{1}{24}\right) \cdot \left(\theta - \frac{7}{24}\right) - x \cdot \theta \cdot \left(\theta + \frac{1}{6}\right), \quad (\text{J.11})$$

which has the two ${}_2F_1$ hypergeometric solutions:

$$x^{1/24} \cdot {}_2F_1\left(\left[\frac{1}{24}, \frac{5}{24}\right], \left[\frac{3}{4}\right], x\right), \quad x^{7/24} \cdot {}_2F_1\left(\left[\frac{7}{24}, \frac{11}{24}\right], \left[\frac{5}{4}\right], x\right). \quad (\text{J.12})$$

Instead of (J.11) one can also introduce the conjugate of (J.11) by $x^{1/24}$

$$\theta \cdot \left(\theta - \frac{1}{4}\right) - x \cdot \left(\theta + \frac{1}{24}\right) \cdot \left(\theta + \frac{5}{24}\right), \quad (\text{J.13})$$

which has the two ${}_2F_1$ hypergeometric solutions:

$${}_2F_1\left(\left[\frac{1}{24}, \frac{5}{24}\right], \left[\frac{3}{4}\right], x\right), \quad x^{1/4} \cdot {}_2F_1\left(\left[\frac{7}{24}, \frac{11}{24}\right], \left[\frac{5}{4}\right], x\right). \quad (\text{J.14})$$

One can also introduce the order-two linear differential operator

$$\mathcal{L}_2 = D_x^2 + \frac{135 x^2 - 167 x + 140}{576 \cdot x^2 \cdot (x-1)^2}, \quad (\text{J.15})$$

which has the two hypergeometric solutions:

$$x^{5/12} \cdot (1-x)^{1/4} \cdot {}_2F_1\left(\left[\frac{1}{24}, \frac{7}{24}\right], \left[\frac{5}{6}\right], x\right),$$

$$x^{7/12} \cdot (1-x)^{1/4} \cdot {}_2F_1\left(\left[\frac{5}{24}, \frac{11}{24}\right], \left[\frac{7}{6}\right], x\right), \quad (\text{J.16})$$

and

$$\mathcal{M}_2 = D_x^2 + \frac{140 x^2 - 167 x + 135}{576 \cdot x^2 \cdot (x-1)^2}, \quad (\text{J.17})$$

which has the two hypergeometric solutions:

$$S_1 = x^{3/8} \cdot (1-x)^{1/4} \cdot {}_2F_1\left(\left[\frac{1}{24}, \frac{5}{24}\right], \left[\frac{3}{4}\right], x\right), \quad (\text{J.18})$$

$$S_2 = x^{5/8} \cdot (1-x)^{1/4} \cdot {}_2F_1\left(\left[\frac{7}{24}, \frac{11}{24}\right], \left[\frac{5}{4}\right], x\right). \quad (\text{J.19})$$

Appendix J.3. Identities on Shimura ${}_2F_1$ hypergeometric functions and modular equations.

There exists an algebraic series $y(x)$ such that the two hypergeometric (J.18), (J.19) actually verify the two following identities:

$$\begin{aligned} w^{3/8} \cdot \rho \cdot y'(x)^{1/2} \cdot x^{3/8} \cdot (1-x)^{1/4} \cdot {}_2F_1\left(\left[\frac{1}{24}, \frac{5}{24}\right], \left[\frac{3}{4}\right], x\right) \\ = y(x)^{3/8} \cdot (1-y(x))^{1/4} \cdot {}_2F_1\left(\left[\frac{1}{24}, \frac{5}{24}\right], \left[\frac{3}{4}\right], y(x)\right), \end{aligned} \quad (\text{J.20})$$

and (with the same ρ and w)

$$\begin{aligned} w^{5/8} \cdot \rho \cdot y'(x)^{1/2} \cdot x^{5/8} \cdot (1-x)^{1/4} \cdot {}_2F_1\left(\left[\frac{7}{24}, \frac{11}{24}\right], \left[\frac{5}{4}\right], x\right) \\ = y(x)^{5/8} \cdot (1-y(x))^{1/4} \cdot {}_2F_1\left(\left[\frac{7}{24}, \frac{11}{24}\right], \left[\frac{5}{4}\right], y(x)\right), \end{aligned} \quad (\text{J.21})$$

where the two complex constants ρ and w are given by:

$$\rho = \frac{7}{25} - \frac{24i}{25}, \quad w = \frac{1}{\rho^2} = -\frac{527}{625} + \frac{336i}{625}. \quad (\text{J.22})$$

These two complex numbers w and ρ are on the unit circle $|w| = |\rho| = 1$ but are not N -th root of unity. The algebraic series $y(x)$ is given by the (symmetric genus-zero) modular equation of level† five $P(x, y) = 0$ where the symmetric polynomial $P(a, b)$ reads

$$\begin{aligned} P(a, b) = & 6979147079584381377970176000000 \cdot a^6 b^6 \\ & + 4434969287855682628249190400000 \cdot a^5 b^5 \cdot (a + b) \\ & + 1752976676930715648 \cdot a^4 b^4 \cdot \left(669871503125 \cdot (a^2 + b^2) - 862324029349392 ab\right) \\ & + 25572696989368320 \cdot a^3 b^3 \cdot (a + b) \cdot \left(6484356150250 \cdot (a^2 + b^2) \right. \\ & \quad \left. + 170835586907964203 ab\right) \\ & + 57330892800 \cdot a^2 b^2 \cdot \left(229748649317860805 (a^4 + b^4) \right. \\ & \quad \left. - 55981841121913535287822 (a^3 b + ab^3) + 170909095105030243444933 a^2 b^2\right) \\ & + 829440 \cdot ab \cdot (a + b) \cdot \left(672749994932560009201 (a^4 + b^4) \right. \\ & \quad + 561918211777371330718156199 \cdot (a^3 b + ab^3) \\ & \quad \left. - 52602925387952898241194053424 a^2 b^2\right) \end{aligned}$$

† The level of the (quaternionic) modular equation is the reduced norm of α such that $\Phi_L(j(\tau), j(\alpha\tau)) = 0$, see section 2.1 in [47]. For the definition of modular polynomials for these quaternionic cases see section 3.1 page 8 of [48]. It depends only on the integer index $n := [\Lambda : \beta \Lambda \beta^{-1} \cap \Lambda]$.

$$\begin{aligned}
& + 9849732675807611094711841 \cdot (a^6 + b^6) \\
& \quad - 10462859500072645481855465070150 \cdot (a^5b + ab^5) \\
& \quad \quad + 26604718024918444951713833439099375 \cdot (a^4b^2 + a^2b^4) \\
& \quad \quad \quad + 10825060619732076783684974180027500 a^3b^3 \\
& + 1680315840 \cdot (a + b) \cdot \left(5559917313492231481 (a^4 + b^4) \right. \\
& \quad \quad \quad \left. + 2234383287388481537541244 \cdot (a^3b + ab^3) \right. \\
& \quad \quad \quad \left. + 62713512471894189372026006 a^2b^2 \right) \\
& + 3322961675259430805968122201600 \cdot (a^4 + b^4) \\
& \quad - 70573817430654328766881842418944000 \cdot (a^3b + ab^3) \\
& - 104957798363467459886332890685209600 a^2b^2 \tag{J.23} \\
& + 11860766958110577033461760 \cdot (a + b) (44289025 a^2 - 3617524994 ab + 44289025 b^2) \\
& \quad + 49824586485654547652165013405696 \cdot (625 a^2 + 1054 ab + 625 b^2) = 0,
\end{aligned}$$

which is parametrised by:

$$\begin{aligned}
a(v) &= -\frac{225v^2 + 18v + 1}{1350000 \cdot v^6}, \tag{J.24} \\
b(v) &= a\left(\frac{11v + 2}{252v - 11}\right) = -\frac{(225v^2 + 18v + 1) \cdot (252v - 11)^4}{2160 \cdot (11v + 2)^6}.
\end{aligned}$$

The algebraic series $y(x)$ in (J.21) or (J.20), given by the *modular equation* (J.23) of level five $P(x, y) = 0$ reads

$$\begin{aligned}
y(x) &= w \cdot x + \left(\frac{172937w}{168750} + \frac{103}{270}\right) \cdot x^2 + \left(\frac{338124694601w}{398671875000} + \frac{270081319}{637875000}\right) \cdot x^3 \\
& \quad + \left(\frac{46359287214498287w}{67275878906250000} + \frac{42068837566753}{107641406250000}\right) \cdot x^4 \tag{J.25} \\
& \quad + \left(\frac{25717676203788236624381w}{45411218261718750000000} + \frac{25116272080576089139}{72657949218750000000}\right) \cdot x^5 + \dots
\end{aligned}$$

where w was given previously in (J.22):

$$w = -\frac{527}{625} + \frac{336i}{625}, \quad 625w^2 + 1054w + 625 = 0. \tag{J.26}$$

Appendix J.4. Automorphic forms and Schwarzian equations: Schwarz map and Schwarz function.

It is straightforward to check that the Taylor expansion (J.25) is such that the two identities (J.21) or (J.20) are actually verified. This algebraic series (J.25) *is actually a solution of the Schwarzian equation*:

$$\begin{aligned}
\{y(x), x\} &+ \frac{140y(x)^2 - 167y(x) + 135}{288 \cdot (y(x) - 1)^2 y(x)^2} \cdot y'(x)^2 \\
&- \frac{140x^2 - 167x + 135}{288 \cdot (x - 1)^2 \cdot x^2} = 0. \tag{J.27}
\end{aligned}$$

This is a consequence of the fact that the *algebraic transformation*[‡] $x \rightarrow y(x)$, given by the modular equation of level five (J.23), is a *symmetry* of the order-two linear

[‡] Such a transformation is called a *modular correspondence*.

differential operator (J.17) (see for instance [32]):

$$y'(x)^{1/2} \cdot \mathcal{M}_2 \cdot y'(x)^{-1/2} = \text{pullback}(\mathcal{M}_2, y(x)). \quad (\text{J.28})$$

The (Shimura) *modular correspondences* $x \rightarrow y(x)$ are *algebraic* solutions of that Schwarzian equation (J.28) which thus “encapsulates” all the modular correspondences.

Let us introduce τ the ratio[†] of the two hypergeometric (J.18), (J.19) solutions of (J.17). One has the following identity:

$$\begin{aligned} \tau &= \frac{S_2}{S_1} = x^{1/4} \cdot \frac{{}_2F_1\left(\left[\frac{7}{24}, \frac{11}{24}\right], \left[\frac{5}{4}\right], x\right)}{{}_2F_1\left(\left[\frac{1}{24}, \frac{5}{24}\right], \left[\frac{3}{4}\right], x\right)} = \int \frac{dx}{4 \cdot S_1^2} \\ &= \frac{1}{4} \cdot \int \frac{dx}{x^{3/4} \cdot (1-x)^{1/2} \cdot {}_2F_1\left(\left[\frac{1}{24}, \frac{5}{24}\right], \left[\frac{3}{4}\right], x\right)^2}. \end{aligned} \quad (\text{J.29})$$

This identity corresponds to the (automorphy theory) relation that one of the two solutions of (J.17) can be written (up to an overall factor) as $f'(\tau)^{1/2}$ (the square root of the τ -derivative of an automorphic function $x = f(\tau)$):

$$\left(\frac{dx}{d\tau}\right)^{1/2} = 2 \cdot S_1, \quad \text{i.e.} \quad \frac{dx}{d\tau} = 4 \cdot S_1^2, \quad \text{i.e.} \quad d\tau = \frac{dx}{4 \cdot S_1^2}, \quad \text{i.e.} \quad \tau = \int \frac{dx}{4 \cdot S_1^2},$$

when the other solution S_2 of (J.17) can be written (up to an overall factor) as $\tau \cdot f'(\tau)^{1/2}$:

$$\tau \cdot \left(\frac{dx}{d\tau}\right)^{1/2} = \frac{S_2}{S_1} \cdot 2 \cdot S_1 = 2 \cdot S_2. \quad (\text{J.30})$$

The Schwarz map τ , given by (J.29), seen as a function of x , is a *differentially algebraic function*. It verifies the Schwarzian equation:

$$\{\tau(x), x\} - \frac{140x^2 - 167x + 135}{288 \cdot x^2 \cdot (x-1)^2} = 0. \quad (\text{J.31})$$

Relation (J.29) which can be rewritten

$$\tau^4 = x \cdot \frac{{}_2F_1\left(\left[\frac{7}{24}, \frac{11}{24}\right], \left[\frac{5}{4}\right], x\right)^4}{{}_2F_1\left(\left[\frac{1}{24}, \frac{5}{24}\right], \left[\frac{3}{4}\right], x\right)^4}, \quad (\text{J.32})$$

yields the following expansion of the *compositional inverse* of the Schwarz map called the *Schwarz function* (here z denotes $z = \tau^4$)

$$\begin{aligned} x(\tau) &= z - \frac{103}{270}z^2 + \frac{53399}{680400}z^3 - \frac{51411991}{4378374000}z^4 + \frac{4865571860153}{3376251758880000}z^5 \\ &\quad - \frac{534844409962319}{3464034304610880000}z^6 + \frac{19352805342627235628117}{1291989881079911901888000000}z^7 + \dots \end{aligned} \quad (\text{J.33})$$

which can be seen as an *automorphic function* of the variable τ . This automorphic function verifies the Schwarzian equation:

$$\{x(\tau), \tau\} + \frac{140x(\tau)^2 - 167x(\tau) + 135}{288 \cdot (x(\tau) - 1)^2 \cdot x(\tau)^2} \cdot \left(\frac{dx(\tau)}{d\tau}\right)^2 = 0. \quad (\text{J.34})$$

[†] Such a ratio is called the Schwarz map.

Recalling the modular equation (J.23) of level five $P(x, y) = 0$, one easily verifies that this modular equation of level five is such that

$$P\left(x(\tau), x\left(\mathcal{E} \cdot \tau\right)\right) = P\left(x(\tau), x\left(\left(-\frac{3}{5} + i \cdot \frac{4}{5}\right) \cdot \tau\right)\right) = 0, \quad (\text{J.35})$$

where $\mathcal{E} = w^{1/4}$ (see (J.38) below) but also

$$P\left(x(\tau), x\left(\frac{\tau}{\mathcal{E}}\right)\right) = P\left(x(\tau), x\left(\left(-\frac{3}{5} - i \cdot \frac{4}{5}\right) \cdot \tau\right)\right) = 0, \quad (\text{J.36})$$

corresponding to the $x \leftrightarrow y$ symmetry. In other words, the modular equation of level five is *also parametrised by the automorphic Schwarz function* (J.33). We thus have two parametrisations of the modular equation of level five: a rational parametrisation (J.24), and a parametrisation (uniformisation) by *automorphic functions*†.

Recalling the two ${}_2F_1$ hypergeometric identities (J.21) or (J.20), and taking their ratio, one finds the *following simple covariance of the ratio* $\tau(x)$ by the algebraic series (J.25):

$$\tau(y(x)) = w^{1/4} \cdot \tau(x), \quad \text{where:} \quad w = -\frac{527}{625} + \frac{336i}{625}. \quad (\text{J.37})$$

Do note that $w^{1/4}$ takes a simpler form‡:

$$\tau(y(x)) = \mathcal{E} \cdot \tau(x), \quad \text{where:} \quad \mathcal{E} = -\frac{3}{5} + i \cdot \frac{4}{5}. \quad (\text{J.38})$$

Recalling the chain rule relation for the Schwarzian derivative of composition of functions

$$\{\tau(y(x)), x\} = \{\tau(y), y\}_{y=y(x)} \cdot y'(x)^2 + \{y(x), x\}, \quad (\text{J.39})$$

one deduces from the Schwarzian equation (J.31) that:

$$\begin{aligned} \{\tau(y(x)), x\} &= \{\mathcal{E} \cdot \tau(x), x\} = \{\tau(x), x\} = \frac{140x^2 - 167x + 135}{288 \cdot x^2 \cdot (x-1)^2} \\ &= \frac{140y(x)^2 - 167y(x) + 135}{288 \cdot y(x)^2 \cdot (y(x)-1)^2} \cdot y'(x)^2 + \{y(x), x\}. \end{aligned} \quad (\text{J.40})$$

One thus recovers the Schwarzian equation (J.28) on the algebraic correspondence $x \rightarrow y(x)$. Conversely, and more generally, if the Schwarz map $\tau(x)$ verifies the Schwarzian equation

$$\{\tau(x), x\} - W(x) = 0, \quad (\text{J.41})$$

and the algebraic correspondence $x \rightarrow y(x)$ verifies the Schwarzian equation

$$\{y(x), x\} + W(y(x)) \cdot y'(x)^2 - W(x) = 0, \quad (\text{J.42})$$

with the same rational function $W(x)$, one deduces using the the chain rule relation (J.40) that:

$$\{\tau(y(x)), x\} = \{\tau(x), x\}. \quad (\text{J.43})$$

Consequently this means that $\tau(y(x))$ can only be a linear fractional transformation of $\tau(x)$:

$$\tau(y(x)) = \frac{a \cdot \tau(x) + b}{c \cdot \tau(x) + d}. \quad (\text{J.44})$$

† This is the well-known Poincaré result [49] that, *whatever the genus of an algebraic curve*, this algebraic curve is *uniformised by automorphic functions of a new variable* (here τ).

‡ See th 2.5 in J. Voight and J. Willis paper [50] for the simplicity of the complex numbers like \mathcal{E} .

If one of the modular correspondence $x \rightarrow y(x)$ is such that (see (J.38)) $\tau(y(x)) = \mathcal{E} \cdot \tau(x)$, the other *modular correspondences* $x \rightarrow y(x)$ of the form (J.44) commuting with that modular correspondence will also be such that $\tau(y(x)) = \alpha \cdot \tau(x)$.

The action of the modular correspondence on the Schwarz map τ , given by this simple relation (J.38), is an *infinite order* transformation. Such a simple relation (J.38) for *Shimura automorphic forms* has to be compared with the action of the modular correspondence on the Schwarz map τ for *classical modular forms* (cusp forms \ddagger with a nome q) where we have transformations like $q \rightarrow q^N$ where N is an integer.

Appendix J.5. Derivative of Shimura automorphic functions.

Let us introduce the order-two linear differential operator

$$\theta \cdot \left(\theta + \frac{3}{4}\right) - x \cdot \left(\theta + \frac{29}{24}\right) \cdot \left(\theta + \frac{25}{24}\right), \quad (\text{J.45})$$

which has the two ${}_2F_1$ hypergeometric solutions:

$${}_2F_1\left(\left[\frac{25}{24}, \frac{29}{24}\right], \left[\frac{7}{4}\right], x\right), \quad x^{-3/4} \cdot {}_2F_1\left(\left[\frac{7}{24}, \frac{11}{24}\right], \left[\frac{1}{4}\right], x\right). \quad (\text{J.46})$$

The linear differential operator (J.45) is homomorphic to (J.13). One deduces from that homomorphism the following identity

$$(4\theta + 1) \left[{}_2F_1\left(\left[\frac{7}{24}, \frac{11}{24}\right], \left[\frac{5}{4}\right], x\right) \right] = {}_2F_1\left(\left[\frac{7}{24}, \frac{11}{24}\right], \left[\frac{1}{4}\right], x\right). \quad (\text{J.47})$$

Hypergeometric functions like (J.46) are *not* Shimura ${}_2F_1$ hypergeometric functions: they do not correspond to automorphic forms, but *derivatives of automorphic forms*.

Along this line, it is tempting to revisit the arguments of section (2.5) for classical modular forms, to (*Shimura automorphic forms*). Let us consider the rational function (138) of section (3), which yields the Heun functions (141) and therefore the Shimura ${}_2F_1$ functions (145). Recalling the identity (132) of section (2.5) that the diagonal of the partial homogeneous derivative of a rational function is the homogeneous derivative of that diagonal

$$x \cdot \frac{d}{dx} \left(\text{Diag}(R(x, y, z, u)) \right) = \text{Diag} \left(u \cdot \frac{\partial R(x, y, z, u)}{\partial u} \right), \quad (\text{J.48})$$

it is tempting to calculate the telescoper of the homogeneous partial derivative with respect to u of the rational function (138). One remarks that the homogeneous partial derivative with respect to u of this rational function (138), is very simple: it is equal to u times the square of that rational function. The telescoper of

$$u \cdot \frac{\partial R(x, y, z, u)}{\partial u} = \frac{u \cdot x^2 y^2 z^2}{(1 - x y z u + x y z \cdot (x + y + z) + x y + y z + x z)^2}, \quad (\text{J.49})$$

is an order-three linear differential operator M_3 which is actually homomorphic to the order-three linear differential operator L_3 given by (139) and yielding the Heun functions (141) or equivalently squares of Shimura ${}_2F_1$ functions (145). This homomorphism reads:

$$M_3 \cdot \theta = x^2 \cdot \theta \cdot L_3. \quad (\text{J.50})$$

\ddagger This correspond to the fact that one of solutions has logarithmic terms in its formal series solution expansion at $x = 0$.

From (J.50) one finds that the solutions of M_3 (and thus the solutions of the telescoper of (J.49)) are simply obtained from the action of the homogeneous derivative θ on the Shimura solutions of the order-three linear differential operator S_3 , i.e. on the Heun functions (141) or the (square of) Shimura ${}_2F_1$ hypergeometric functions (145). This example of rational function (J.49) thus yields a telescoper which solutions are *not* Shimura ${}_2F_1$ automorphic forms, but *derivatives of Shimura ${}_2F_1$ automorphic forms*.

Anecdotal remark: recalling (A.29) where the parameters in (A.29) verify (A.19) and (A.22), one can also write (J.47) as ${}_3F_2([7/24, 11/24, 5/4], [5/4, 1/4], x)$ but also, since here $e = 1/4$, as the Heun function

$$\text{Heun}\left(0, 0, \frac{7}{24}, \frac{11}{24}, \frac{5}{4}, \frac{3}{2}, x\right). \quad (\text{J.51})$$

This simple form of the Heun function is a consequence of the fact that one has $\gamma = e + 1$ in this case. With $e = 1/2$ the identities (A.29) become:

$$\begin{aligned} \text{Heun}\left(\frac{72}{5}, \frac{231}{40}, \frac{7}{24}, \frac{11}{24}, \frac{5}{4}, \frac{3}{2}, x\right) &= (2\theta + 1) \left[{}_2F_1\left(\left[\frac{7}{24}, \frac{11}{24}\right], \left[\frac{5}{4}\right], x\right) \right] \\ &= {}_3F_2\left(\left[\frac{7}{24}, \frac{11}{24}, \frac{3}{2}\right], \left[\frac{1}{2}, \frac{5}{4}\right], x\right) \end{aligned} \quad (\text{J.52})$$

Appendix J.6. Identities linking hypergeometric functions that are related with Shimura curves.

In fact, several identities linking ${}_2F_1$ hypergeometric functions related with *Shimura curves*, appear in the litterature. For example we see in [51] (equation (4.8) page 14):

$$\begin{aligned} {}_2F_1\left(\left[\frac{5}{42}, \frac{19}{42}\right], \left[\frac{5}{7}\right], x\right) &= \left(\frac{6561 - 13851x - 9261x^2 + 16807x^3}{6561}\right)^{-1/28} \\ &\times {}_2F_1\left(\left[\frac{1}{84}, \frac{29}{84}\right], \left[\frac{6}{7}\right], \frac{x^2 \cdot (1-x) \cdot (49x-81)^7}{4 \cdot (6561 - 13851x - 9261x^2 + 16807x^3)^3}\right), \end{aligned} \quad (\text{J.53})$$

where the ${}_2F_1$ hypergeometric function on the RHS of the identity (J.53) corresponds to a *Shimura curve* with *elliptic points* (2, 3, 7). To see that the hypergeometric function on the LHS also corresponds to a Shimura curve is not obvious on the difference of exponents[†] of the hypergeometric function. However considering the order-two linear differential operator N_2 annihilating ${}_2F_1\left(\left[\frac{5}{42}, \frac{19}{42}\right], \left[\frac{5}{7}\right], x\right)$, one finds that the pullback of N_2 by one of the Euler's hypergeometric transformations, namely $x \rightarrow 1/x$, has the solution $x^{5/42} \cdot {}_2F_1\left(\left[\frac{11}{42}, \frac{23}{42}\right], \left[\frac{2}{3}\right], x\right)$ which is a Shimura hypergeometric function of the (3, 3, 7) type.

Note that the set of Gauss hypergeometric functions that are associated with Shimura curves *is a finite set* [13, 19].

Similarly in [30] we find on page 2, equation (3), the identity:

$$\begin{aligned} {}_2F_1\left(\left[\frac{1}{20}, \frac{1}{4}\right], \left[\frac{4}{5}\right], \frac{64 \cdot x \cdot (1-x-x^2)^5}{(1-x^2)(1+4x-x^2)^5}\right) \\ = (1-x^2)^{1/20} \cdot (1+4x-x^2)^{1/4} \cdot {}_2F_1\left(\left[\frac{3}{10}, \frac{2}{5}\right], \left[\frac{9}{10}\right], x^2\right), \end{aligned} \quad (\text{J.54})$$

[†] The difference of exponents of a ${}_2F_1([a, b], [c], x)$ hypergeometric function is the triplet $(c - a - b, b - a, 1 - c)$. These rational numbers $(c - a - b, b - a, 1 - c)$ have to be (up to a sign) reciprocals of integer, and furthermore in some table given by Takeuchi [19].

where the ${}_2F_1$ hypergeometric function on the RHS of the identity corresponds to a Shimura curve with elliptic points (5, 2, 5). The hypergeometric function on the LHS also corresponds to a Shimura curve. We also see the identity:

$$\begin{aligned} & \left(1 + (4 + 2b) \cdot x - (1 + 2b) \cdot x^2\right)^{-1/2} \\ & \times {}_2F_1\left(\left[\frac{1}{8}, \frac{3}{8}\right], \left[\frac{3}{4}\right], -\frac{4 \cdot (1+b)^4 \cdot x \cdot (1 - (7-4b)/3 \cdot x^2)^4}{(1+x) \cdot (1-3x) \cdot (1 + (4+2b) \cdot x - (1+2b) \cdot x^2)^4}\right) \\ & = \frac{(1+x)^{1/8} \cdot (1-3x)^{1/8}}{(1+a \cdot x)^{5/4}} \cdot {}_2F_1\left(\left[\frac{5}{24}, \frac{3}{8}\right], \left[\frac{3}{4}\right], \frac{12 \cdot a \cdot x \cdot (1-x^2) \cdot (1-9x^2)}{(1+a \cdot x)^6}\right), \end{aligned} \quad (\text{J.55})$$

for $a^2 + 3 = 0$ and $b^2 + 2 = 0$, where the ${}_2F_1$ hypergeometric function on the left of the identity, corresponds to an *automorphic form associated with a Shimura curve* with elliptic points (4, 4, 4).

Appendix J.7. A level three modular equation for Shimura Heun functions.

Similar calculations can be performed on the Heun example (164). This Heun example has a *level three modular equation* [13]. Introducing the following ratio (Schwarz map):

$$\tau = x^{1/3} \cdot \frac{\text{Heun}\left(\frac{27}{2}, \frac{47}{18}, \frac{5}{12}, \frac{11}{12}, \frac{4}{3}, \frac{1}{2}, x\right)}{\text{Heun}\left(\frac{27}{2}, \frac{7}{36}, \frac{1}{12}, \frac{7}{12}, \frac{2}{3}, \frac{1}{2}, x\right)}. \quad (\text{J.56})$$

Introducing

$$\begin{aligned} \tau(z) &= \eta(t(z)) = \frac{108 \cdot (t-1)^3}{(t+1)^2(9t^2 - 10t + 17)}, \\ \tilde{\tau}(z) &= \eta\left(\frac{10}{9} - t(z)\right) = \frac{12 \cdot (1-9t)^3}{(19-9t)^2 \cdot (9t^2 - 10t + 17)}, \end{aligned} \quad (\text{J.57})$$

the elimination of the variable t yields the (genus-zero) *level three modular equation* [13] which reads:

$$\begin{aligned} & 9604 \cdot A^2 B^2 \cdot (9 A^2 + 14 A B + 9 B^2) \\ & - 2940 \cdot A B \cdot (A + B) (25 A^2 + 518 A B + 25 B^2) \\ & + 5 \cdot \left(3125 (A^4 + B^4) + 334944 (A^3 B + A B^3) + 1605078 A^2 B^2\right) \\ & \quad + 480 \cdot (A + B) \cdot (375 A^2 - 13513 A B + 375 B^2) \\ & + 5120 \cdot (135 A^2 + 142 A B + 135 B^2) + 884736 \cdot (A + B) = 0. \end{aligned} \quad (\text{J.58})$$

Appendix K. Heun functions that are pullbacked ${}_2F_1$ hypergeometric functions but are not related to classical modular forms or Shimura automorphic forms.

Let us recall Maier's identity (see Theorem 3.8 in [33])

$$\begin{aligned} & \text{Heun}\left(\omega, \alpha\beta \cdot \eta, \alpha, \beta, \frac{\alpha + \beta + 1}{3}, \frac{\alpha + \beta + 1}{3}, \eta \cdot x\right) \\ & = {}_2F_1\left(\left[\frac{\alpha}{3}, \frac{\beta}{3}\right], \left[\frac{\alpha + \beta + 1}{3}\right], x \cdot (3 - 3x + x^2)\right), \end{aligned} \quad (\text{K.1})$$

where ω is a sixth root of unity and η read†:

$$\omega = \frac{1}{2} + i \cdot \frac{3^{1/2}}{2}, \quad \eta = \frac{1}{2} + i \cdot \frac{3^{1/2}}{6}. \quad (\text{K.2})$$

Such a Heun function is thus a pullbacked ${}_2F_1$ hypergeometric function for all the values of the parameters α, β . This two parameters (α, β) space is large enough to encapsulate a set of interesting subcases.

• For selected values of the two parameters α, β , the Heun function (K.1) actually reduces to Shimura ${}_2F_1$ hypergeometric functions. For instance for $\alpha = 1$ and $\beta = 1/4$, we recover the Shimura ${}_2F_1$ hypergeometric function (153) related to the telescope of the rational function (138). The relation (K.1) becomes:

$$\begin{aligned} & \text{Heun}\left(\omega, \frac{\eta}{4}, 1, \frac{1}{4}, \frac{3}{4}, \frac{3}{4}, \eta \cdot x\right) \\ &= {}_2F_1\left(\left[\frac{1}{3}, \frac{1}{12}\right], \left[\frac{3}{4}\right], x \cdot (3 - 3x + x^2)\right). \end{aligned} \quad (\text{K.3})$$

Many other values of (α, β) yield Shimura ${}_2F_1$ hypergeometric functions:

$$\begin{aligned} (\alpha, \beta) = & \left(\frac{1}{8}, \frac{9}{8}\right), \left(\frac{1}{3}, \frac{4}{3}\right), \left(\frac{3}{8}, \frac{11}{8}\right), \left(\frac{2}{5}, \frac{7}{5}\right), \left(\frac{1}{4}, \frac{5}{4}\right), \left(\frac{2}{3}, \frac{7}{6}\right), \\ & \left(\frac{1}{12}, \frac{19}{12}\right), \left(\frac{5}{32}, \frac{53}{32}\right), \left(\frac{3}{4}, \frac{9}{8}\right), \left(\frac{1}{5}, \frac{17}{10}\right), \left(\frac{1}{2}, \frac{5}{4}\right), \left(\frac{1}{2}, \frac{5}{4}\right), \\ & \left(\frac{1}{8}, \frac{13}{8}\right), \left(\frac{1}{2}, 1\right), \left(\frac{5}{8}, 1\right), \left(\frac{1}{6}, \frac{5}{13}\right), \left(\frac{1}{4}, \frac{7}{4}\right). \end{aligned} \quad (\text{K.4})$$

• In Appendix F.1 we have seen that one can introduce a nome, for that linear differential Heun operator (1), when $\gamma = 1$. This condition is, of course, *necessary but not sufficient* to reduce to a classical modular form. In that $\gamma = 1$ subcase where one can introduce a nome, the previous Heun-to- ${}_2F_1$ reduction (K.1) reads (since we have $\gamma = (\alpha + \beta + 1)/3 = 1$):

$$\begin{aligned} & \text{Heun}\left(\omega, \alpha(2 - \alpha) \cdot \eta, \alpha, 2 - \alpha, 1, 1, \eta \cdot x\right) \\ &= {}_2F_1\left(\left[\frac{\alpha}{3}, \frac{2 - \alpha}{3}\right], [1], x \cdot (3 - 3x + x^2)\right), \end{aligned} \quad (\text{K.5})$$

For selected values of α we actually get ${}_2F_1$ hypergeometric functions corresponding to *classical modular curves* (see (H.19)). The selected values are $\alpha = 1, 1/2, 1/4$, corresponding respectively to:

$$\begin{aligned} & {}_2F_1\left(\left[\frac{1}{3}, \frac{1}{3}\right], [1], x \cdot (3 - 3x + x^2)\right), \quad {}_2F_1\left(\left[\frac{1}{6}, \frac{1}{2}\right], [1], x \cdot (3 - 3x + x^2)\right), \\ & {}_2F_1\left(\left[\frac{1}{12}, \frac{7}{12}\right], [1], x \cdot (3 - 3x + x^2)\right). \end{aligned} \quad (\text{K.6})$$

Let us now, consider a rational value of α , different from $1, 1/2, 1/4$, for instance $\alpha = 5/8$. The identity (K.5) becomes

$$\begin{aligned} & \text{Heun}\left(\omega, \frac{55}{64} \cdot \eta, \frac{5}{8}, \frac{11}{8}, 1, 1, \eta \cdot x\right) \\ &= {}_2F_1\left(\left[\frac{5}{24}, \frac{11}{24}\right], [1], x \cdot (3 - 3x + x^2)\right). \end{aligned} \quad (\text{K.7})$$

† Note that the *normalised accessory parameter* $q/\alpha/\beta$ is equal to the constant η .

The series expansion of (K.7) gives a ${}_2F_1$ globally bounded series[†]. The Heun function (K.7) is annihilated by the order-two linear differential operator

$$L_2 = 64 \cdot x \cdot (x^2 - 3x + 3) \cdot D_x^2 + 192 \cdot (x - 1)^2 \cdot D_x + 55 \cdot (x - 1), \quad (\text{K.8})$$

which yields the following series expansion for the nome of L_2 :

$$q = x + \frac{41x^2}{96} + \frac{24925x^3}{147456} + \frac{21593429x^4}{382205952} + \frac{57805761947x^5}{4696546738176} \\ - \frac{4091796188773x^6}{2254342434324480} - \frac{400657105197062971x^7}{93492089436304834560} + \dots \quad (\text{K.9})$$

This series is *not* globally bounded. Therefore (K.7) *does not correspond to a classical modular form*, as expected since the ${}_2F_1$ hypergeometric function in the RHS of (K.7) is not in the list of the twenty-eight ${}_2F_1$ hypergeometric functions (H.19). With this example (K.7) we see that a Heun function can actually correspond to a globally bounded series, being reducible to a pullbacked ${}_2F_1$ hypergeometric function, without necessarily corresponding to a classical modular form[‡]. We have not been able to find such Heun functions as diagonal of rational functions or even as solutions of telescopers of rational functions.

Appendix L. The $x \leftrightarrow 1/x$ and $x \leftrightarrow A/x$ dualities.

Using the (multi-Taylor) definition of the diagonal of a rational function, it is straightforward to show, for any *positive* integer n , that the diagonal of a rational function $R(x, y, z, w)$ and the diagonal of a rational function $R(x^n, y^n, z^n, w^n)$ are simply related. Denoting these two diagonals respectively $D_1(x) = \text{Diag}(R(x, y, z, w))$ and $D_n(x) = \text{Diag}(R(x^n, y^n, z^n, w^n))$, one has the simple relation: $D_n(x) = D_1(x^n)$. Of course this demonstration cannot be extended to negative integers n , in particular $n = -1$.

Let us revisit example 5 where the diagonal of the rational function (80) is the Heun series expansion (83). Let us consider, instead of the rational function (80), the rational function where the four variables (x, y, z, w) have been changed into their reciprocal $(1/x, 1/y, 1/z, 1/w)$ in the rational function $R(x, y, z, w)$ given by (80):

$$\mathcal{R}(x, y, z, w) = R\left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}, \frac{1}{w}\right) \\ = \frac{xyzw}{xyzw - xyw - xzw - yw - zw - xy - z - 1}. \quad (\text{L.1})$$

The diagonal of this “reciprocal” rational function (L.1) reads:

$$\text{Diag}\left(\mathcal{R}(x, y, z, w)\right) = -x - 5x^2 - 73x^3 - 1445x^4 - 33001x^5 + \dots \quad (\text{L.2})$$

The telescoper of the diagonal of this rational function (L.1) of four variables reads:

$$M_3 = 5x - 1 + (1 - 10x + x^2) \cdot x \cdot D_x + 3 \cdot (x - 17) \cdot x^3 \cdot D_x^2 \\ + (1 - 34x + x^2) \cdot x^3 \cdot D_x^3. \quad (\text{L.3})$$

This order-three linear differential operator M_3 given by (L.3) *actually corresponds to the telescoper L_3 given by (82) pullbacked by $x \rightarrow 1/x$* . Recalling the solution (83)

[†] This series can be recast into a series with integer coefficients after the $x \rightarrow 2304 \cdot x$ rescaling.

[‡] We cannot totally exclude the fact that a nome series like (K.9) could correspond to an order-one linear differential operator acting on a classical modular form (see the second ${}_2F_1$ hypergeometric function of (H.28) and equation (H.29) in Appendix H.2).

of the telescoper L_3 given by (82), one finds easily that the diagonal series expansion (L.2), solution of the order-three linear differential operator M_3 , can be written as

$$\begin{aligned}
& -x \cdot (1 - 34x + x^2) \times \\
& \quad Heun\left(577 + 408 \cdot 2^{1/2}, \frac{663}{2} + 234 \cdot 2^{1/2}, \frac{3}{2}, \frac{3}{2}, 1, \frac{3}{2}, (17 + 12 \cdot 2^{1/2}) \cdot x\right)^2 \\
= & -x \cdot (1 - 34x + x^2) \times \\
& \quad Heun\left(577 - 408 \cdot 2^{1/2}, \frac{663}{2} - 234 \cdot 2^{1/2}, \frac{3}{2}, \frac{3}{2}, 1, \frac{3}{2}, (17 - 12 \cdot 2^{1/2}) \cdot x\right)^2 \\
& = -x - 5x^2 - 73x^3 - 1445x^4 - 33001x^5 - 819005x^6 + \dots \quad (\text{L.4})
\end{aligned}$$

In other words there is a simple relation between the diagonal of (80) and the diagonal of its “reciprocal” (L.1):

$$\begin{aligned}
\text{Diag}\left(\mathcal{R}(x, y, z, w)\right) &= \text{Diag}\left(R\left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}, \frac{1}{w}\right)\right) \\
&= -x \cdot \text{Diag}\left(R(x, y, z, w)\right). \quad (\text{L.5})
\end{aligned}$$

This is confirmed by the homomorphism between the two telescopers L_3 and M_3 :

$$M_3 \cdot x = x^2 \cdot L_3. \quad (\text{L.6})$$

Recalling the fact that the order-three linear differential operator M_3 corresponds to the telescoper L_3 pullbacked by $x \rightarrow 1/x$, this homomorphism means a slightly puzzling homomorphism between the telescoper L_3 and itself pullbacked by $x \rightarrow 1/x$.

Appendix L.1. The $x \longleftrightarrow A/x$ dualities: dualities on the telescopers.

Let us revisit example 1, considering, similarly, the diagonal of the rational function of four variables (11), its telescoper L_3 given by (13), and the corresponding Heun solutions (15). We introduce, as previously, a new rational function (for the rational function (11) of four variables $R(x, y, z, w)$) corresponding to simple involutions of the form $t \rightarrow A/t$ on the four variables:

$$\begin{aligned}
\mathcal{R}(x, y, z, w) &= R\left(-\frac{1}{4x}, -\frac{1}{4y}, -\frac{1}{4z}, \frac{1}{w}\right) \\
&= \frac{64xyzw}{64xyzw - 4xw + 16yz + w - 4x - 4y - 4z}. \quad (\text{L.7})
\end{aligned}$$

The telescoper of this rational function (L.7) reads:

$$\begin{aligned}
M_3 = & -15 + 32x + (3 - 32x) \cdot (5 + 32x) \cdot x \cdot D_x - 96 \cdot (3 + 32x) \cdot x^3 \cdot D_x^2 \\
& + (1 + 4x) \cdot (1 - 16x) \cdot x^3 \cdot D_x^3. \quad (\text{L.8})
\end{aligned}$$

This order-three linear differential operator (L.8) is *exactly the telescoper* L_3 given by (13), *pullbacked by* $x \rightarrow -1/64x$. The telescoper M_3 has the following Heun solutions (to be compared with (15))

$$x^{3/4} \cdot Heun\left(-\frac{1}{4}, \frac{7}{256}, \frac{3}{8}, \frac{3}{8}, \frac{3}{4}, \frac{1}{2}, -4x\right)^2, \quad (\text{L.9})$$

$$x^{5/4} \cdot Heun\left(-\frac{1}{4}, \frac{31}{256}, \frac{5}{8}, \frac{5}{8}, \frac{5}{4}, \frac{1}{2}, -4x\right)^2, \quad (\text{L.10})$$

and:

$$x \cdot Heun\left(-\frac{1}{4}, \frac{7}{256}, \frac{3}{8}, \frac{3}{8}, \frac{3}{4}, \frac{1}{2}, -4x\right) \cdot Heun\left(-\frac{1}{4}, \frac{31}{256}, \frac{5}{8}, \frac{5}{8}, \frac{5}{4}, \frac{1}{2}, -4x\right). \quad (\text{L.11})$$

These (non globally bounded) solutions (L.9), (L.10), and (L.11) are different from the diagonal of the rational function (L.7) which is trivial in that case. We are in the situation where the solutions of the telescoper are different from the diagonal of the rational function: they are “Periods” [36] over *non-evanescent cycles* of the algebraic variety corresponding to the rational function. Example 1 was seen to correspond to a classical modular form with an integer series and a formal series with a logarithm at $x = 0$. The point at infinity $x = \infty$ is an elliptic point with no logarithm, and series that are *not* globally bounded. Let us note that this order-three linear differential operator (L.8) has the *same singularities* as the order-three operator L_3 given by (13). Consequently the first parameter of these Heun functions solutions and of the Heun function (15) solution of L_3 is the same.

More generally the telescoper T_R of a rational function $R(x, y, z, w)$ and the telescoper $T_{\mathcal{R}}$ of the rational function

$$\mathcal{R}(x, y, z, w) = R\left(\frac{A_1}{x}, \frac{A_2}{y}, \frac{A_3}{z}, \frac{A_4}{w}\right), \quad (\text{L.12})$$

are simply related. The telescoper $T_{\mathcal{R}}$ is the telescoper T_R pullbacked by the involution $x \rightarrow A/x$ where $A = A_1 A_2 A_3 A_4$.

One can easily perform the same calculations for examples 2, 3, 4, 6 of section (2.1), and find exactly similar results†, namely the fact that the telescoper $T_{\mathcal{R}}$ of the new rational function (L.12) is exactly the telescoper T_R of the rational function, pullbacked by $x \rightarrow A/x$. Note that, for these examples, there exists a choice of the constant A such that the singularities of T_R and $T_{\mathcal{R}}$ *remain the same*. For examples 2, 3, 4, 6 one must take respectively $A = -1/27, -1/128, -1/16, 1/144$. For the first lattice Green example of section (1.1) take $A = 1/4$.

Remark: Do note that the other Shimura example of section (3.1) has been build that way: the rational function (156) is actually the rational function of (138) where (x, y, z, w) has been changed into $(1/x, 1/y, 1/z, 1/w)$. Again we have that the telescoper M_3 of the “recipocal” rational function is exactly the telescoper L_3 of the rational function, pullbacked by $x \rightarrow 1/x$.

Appendix L.2. Periods of extremal rational surfaces: the $x \leftrightarrow 1/x$ duality on the telescoper.

Let us recall the rational function (110) which diagonal was the sum of two classical modular forms. Let us consider the rational function where the three variables (x, y, z) have been changed into their reciprocal $(1/x, 1/y, 1/z)$ in the rational function $R(x, y, z)$ given by (110):

$$\begin{aligned} \mathcal{R}(x, y, z) &= R\left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right) \\ &= \frac{x^3 y z}{x^3 y z + x^3 y + x^3 z + x^2 y z + x^3 + x^2 z - 1}. \end{aligned} \quad (\text{L.13})$$

The telescoper of this rational function (L.13) is an order-four linear differential operator which is the direct sum of two order-two operators $L_2^{(p)} \oplus M_2^{(p)}$ which are precisely the two order-two linear differential operators occurring of section (2.4) pullbacked by $x \rightarrow 1/x$. The solutions (117) and (122) where one changes $x \rightarrow 1/x$ are solutions of these new linear pullbacked differential operators $L_2^{(p)}$ and $M_2^{(p)}$. Such

† With a set of new (non globally bounded) Heun functions like (L.11).

(classical modular forms) solutions are actually expandable at $x = 0$ and correspond to *globally bounded* series expansions. The solution of $L_2^{(p)}$, analytic at $x = 0$, can be written as a Heun function

$$\begin{aligned} x \cdot \text{Heun}\left(\frac{1}{2} - \frac{i\sqrt{3}}{2}, \frac{1}{2} - \frac{i\sqrt{3}}{6}, 1, 1, 1, 1, \left(\frac{-3 + i\sqrt{3}}{18}\right) \cdot x\right) \\ = x - \frac{1}{9}x^2 + \frac{1}{81}x^3 - \frac{7}{6561}x^4 + \frac{1}{59049}x^5 + \frac{11}{531441}x^6 + \dots \quad (\text{L.14}) \end{aligned}$$

to be compared with (117). Again these nice classical modular forms globally bounded series expansions are different from the diagonal of (L.13) which is equal to zero.

References

- [1] A. Bostan, S. Boukraa, G. Christol, S. Hassani, J-M. Maillard, *Ising n -fold integrals as diagonals of rational functions and integrality of series expansions*, (2013), J. Phys. **A 46**: Math. Theor. 185202 (44 pages), <http://arxiv.org/abs/1211.6645v2>
- [2] Bostan A, Boukraa S, Christol G, Hassani S and Maillard J-M, *Ising n -fold integrals as diagonals of rational functions and integrality of series expansions: integrality versus modularity* Preprint, <http://arxiv.org/abs/1211.6031>
- [3] A. Bostan, S. Boukraa, J-M. Maillard, J-A. Weil, *Diagonal of rational functions and selected differential Galois groups*, (2015), J. Phys. **A 48**: Math. Theor. 504001 (29 pages) arXiv:1507.03227v2 [math-ph]
- [4] Y. Abdelaziz, S. Boukraa, C. Koutschan, J-M. Maillard, *Diagonals of rational functions, pullbacked ${}_2F_1$ hypergeometric functions and modular forms (unabridged version)*, arXiv:1805.04711v1 [math-ph] (2018)
- [5] Y. Abdelaziz, S. Boukraa, C. Koutschan, J-M. Maillard, *Diagonals of rational functions, pullbacked ${}_2F_1$ hypergeometric functions and modular forms*, J. Phys. **A 51**: Math. Theor. 455201 (30 pages),
- [6] HolonomicFunctions Package version 1.7.1 (09-Oct-2013) written by Christoph Koutschan, Copyright 2007-2013, Research Institute for Symbolic Computation (RISC), Johannes Kepler University, Linz, Austria
- [7] K. Takemura, *The Heun equation and the Calogero-Moser-Sutherland system I: the Bethe Ansatz method*, Comm. Math. Phys. **258**, Issue 2, pp 367-403 and arXiv:math/0103077v2 [math.CA]
- [8] G. Valent, *Heun functions versus elliptic functions*, in “Difference Equations, Special Functions and Orthogonal Polynomials”, Proceedings of the International Conference, Munich, Germany, 25-30 July 2005, World Scientific and arXiv:math-ph/0512006v1 [math-ph] and <https://hal.archives-ouvertes.fr/hal-00015063>
- [9] A.O. Smirnov, *Elliptic solitons and Heun equations*, in “The Kowalewski Property”, Vadim B. Kuznetsov editor, CRM Proceedings and Lecture Notes, (2002) and arXiv:math/0109149v2 [math.CA]
- [10] Glasser M L and Guttman A J, 1994 Lattice Green function (at 0) for the 4D hypercubic lattice *J. Phys. A* **27** 7011–7014, <http://arxiv.org/abs/cond-mat/9408097>
- [11] A. Bostan, S. Boukraa, S. Hassani, J-M. Maillard, J-A. Weil and N. Zenine, *Globally nilpotent differential operators and the square Ising model*, 2009, J. Phys. **A 42**, p.125206 (50pp) and arXiv:0812.4931v1 [math-ph]
- [12] N. D. Elkies, *Shimura curve computations via $K3$ surfaces of Néron-Severi rank at least 19*, Algorithmic Number Theory: 8th International Symposium, ANTS-VIII Banff, May 2008, Proceedings (pp. 196-2111) and arXiv:0802.1301v1 [math-ph]
- [13] J. Voight, *Shimura curves of genus at most two*, Math. Comp. 78 pp 1155-1172, (2009).
- [14] J. Voight, *Three lectures on Shimura curves*, 16th april (2006).
- [15] T. Shaska, *Genus 2 fields with degree 3 elliptic subfields*, Forum Math. **16**, (2004) pp. 263-280, arXiv:math/0109155 [math.AG]
- [16] E. Hallouin, *Computation of a cover of Shimura curves using a Hurwitz space*, November 16 (2013), Journal of Algebra, Elsevier, 2009, 351 (2), pp.558-566.
- [17] A. Kurihara, *On some examples of equations defining Shimura curves and the Mumford uniformization*, J. Fac. Sci. Univ. Tokyo **25**, (1979) pp. 277-301.

- [18] M. van Hoeij and R. Vidunas, *Belyi functions for hyperbolic Hypergeometric-to-Heun transformations*, Journal of Algebra, Volume **441**, 1 November 2015, Pages 609-659, (2015) [arXiv:1212.3803v3](https://arxiv.org/abs/1212.3803v3)
- [19] K. Takeuchi, *Commensurability classes of arithmetic triangle groups*, J. Fac. Science Univ. Tokyo Sec. IA. Math. **24**, (1977) pp. 201-212.
- [20] G. S. Joyce, *On the Simple Cubic Lattice Green Function*, Phil. Trans. of the Royal Soc. of London, Series A, Math. and Phys. Sciences, **273**, (1973) pp. 583-610.
- [21] C. F. Doran and A. Malmendier *Calabi-Yau Manifolds Realizing Symplectically Rigid Monodromy Triples*, <https://arxiv.org/pdf/1503.07500.pdf>
- [22] C. Peters and J. Stienstra, *A pencil of K3-surfaces related to Apéry's recurrence for $\zeta(3)$ and Fermi surfaces for potential zero*, 1989, In book: Arithmetic of Complex Manifolds, Proc. Erlangen 1988, Edition: Lect. Notes Math. 1399, Publisher: Springer Verlag, Editors: Barth, Lange, pp.110-127
- [23] http://www.unilim.fr/pages_perso/jacques-arthur.weil/diagonals/ and http://www.unilim.fr/pages_perso/jacques-arthur.weil/diagonals/4var/4var_ORDER_3.txt
- [24] A. Malmendier, T. Shaska, editors *Higher Genus Curves in Mathematical Physics and Arithmetic Geometry*, Contemporary Mathematics, AMS Special Session Higher Genus Curves and Fibrations in Mathematical Physics and Arithmetic Geometry January 2016
- [25] Maier R S 2009 On rationally parametrized modular equations *J. Ramanujan Math. Soc.* **24** pp. 1-73 [arXiv:math/0611041v4](https://arxiv.org/abs/math/0611041v4) [math.NT]
- [26] J. Slisling and J. Voight, *On computing Belyi maps*, [arXiv:1311.2529](https://arxiv.org/abs/1311.2529) [math.NT] 2014.
- [27] M. Musty, S. Shiovone, J. Slisling and J. Voight, *A database of Belyi maps*, [arXiv:1805.07751v3](https://arxiv.org/abs/1805.07751v3) [math.NT] 2018.
- [28] Yifan Yang, *Schwarzian differential equations and Hecke eigenforms on Shimura curves* [arXiv:1110.6284v1](https://arxiv.org/abs/1110.6284v1) [math.NT]
- [29] Fang-Ting Tu, *Algebraic Transformations of hypergeometric functions arising from theory of Shimura curves*, RIMS Kôkyûroku Bessatsu, **B 44**, pp.223-245 (2013).
- [30] Fang-Ting Tu and Yifan Yang, *Algebraic Transformations of Hypergeometric Functions and automorphic Forms on Shimura Curves*, Transactions of the American Mathematical Society, **365**, Number 12, (2013), pp. 6697-6729 and [arXiv:1112.1001v1](https://arxiv.org/abs/1112.1001v1) [math.NT]
- [31] R. Vidunas and G. Filipuk, *A Classification of coverings yielding Heun-to-hypergeometric Reductions*, [arXiv: 1204.2730v1](https://arxiv.org/abs/1204.2730v1) [math.CA] 2012.
- [32] Y. Abdelaziz, J-M. Maillard, *Modular forms, Schwarzian conditions, and symmetries of differential equations in physics*, (2016), J. Phys. **A 49**: Math. Theor. 215203 (44 pages), [arXiv:1611.08493v3](https://arxiv.org/abs/1611.08493v3) [math-ph]
- [33] Maier R S On reducing the Heun equation to the hypergeometric equation, Journal of Differential Equations **213**, Issue 1, 1 June 2005, pp. 171-203 [arXiv:math/0203264v4](https://arxiv.org/abs/math/0203264v4) [math.CA]
- [34] M. van Hoeij, <https://www.math.fsu.edu/~hoeij/math.fsu.edu/~hoeij/Heun/BelyiMaps> and <https://www.math.fsu.edu/~hoeij/Heun/>
- [35] N. D. Elkies, *Shimura curve computations*, [arXiv:math/0005160v1](https://arxiv.org/abs/math/0005160v1) [math.NT] 2000
- [36] M. Kontsevich and D. Zagier, *Periods*, IHES/M/01/22 2001, <https://www.maths.ed.ac.uk/~v1ranick/papers/kontzagi.pdf>
- [37] D. Zagier, *Arithmetic and Topology of Differential Equations*, <https://people.mpim-bonn.mpg.de/zagier/files/tex/ECMHirzebruchLecture/ECMHirzebruchLectureCorrected.pdf>
- [38] A. Ronveaux (1995) *Factorization of Heun's differential operator*, Applied Mathematics and Computation **141**, pp. 177-184 (2003).
- [39] M. N. Hounkonnou and A. Ronveaux (1995) *Generalized Heun and Lamé equations: factorization*, Commun. Math. Anal. Volume **11**, Number 1 (2011), pp. 121-136, and [arXiv: 0902.2991v1](https://arxiv.org/abs/0902.2991v1) [math-ph] (2009).
- [40] Furstenberg H 1967 Algebraic functions over finite fields *J. Algebra* **7** pp. 271-277 [http://dx.doi.org/10.1016/0021-8693\(67\)90061-0](https://dx.doi.org/10.1016/0021-8693(67)90061-0)
- [41] Denev J and Lipshitz L 1987 Algebraic power series and diagonals *J. Number Theory* **26** 46-67 [http://dx.doi.org/10.1016/0022-314X\(87\)90095-3](https://dx.doi.org/10.1016/0022-314X(87)90095-3)
- [42] R. S. Maier, *P-symbols, Heun identities, and ${}_3F_2$ identities*, (2007) [arXiv: 0712.4299v2](https://arxiv.org/abs/0712.4299v2) [math.CA]
- [43] S. Chen, M. Kauers and M. F. Singer, *Desingularization of Ore Operators*, [arXiv:1408.5512](https://arxiv.org/abs/1408.5512) [cs.SC], <https://arxiv.org/pdf/1408.5512.pdf>
- [44] M. A. Barkatou, *Apparent Singularities of Differential Systems with Rational Function Coefficients*, (2017), Workshop on Computer Algebra in Enumerative Combinatorics, <https://www.mat.univie.ac.at/~kratt/esi4/barkatou.pdf>
- [45] M. A. Barkatou, *Removing Apparent Singularities of Linear Differential Systems with*

- Rational Function Coefficients*, (2015), ISSAC, University of Bath, <https://www.issac-conference.org/2015/Slides/Barkatou.pdf>
- [46] Kilian Raschel, *Random walks in the quarter plane, discrete harmonic functions and conformal mappings*, Stochastic Processes and their Applications **124** (2014) pp. 3147-3178.
- [47] S. Baba and H. Granath, *Differential equations and expansions for quaternionic modular forms in the discriminant 6 case*, LMS Journal of Computation and Mathematics, Vol **15**, pp. 385-399.
- [48] S. Baba and H. Granath, *Quaternionic Modular Forms and Exceptional Sets of Hypergeometric Functions*, International Journal of Number Theory Vol. 11, No. 02, pp. 631-643 (2015) <http://www.diva-portal.org/smash/record.jsf?pid=diva2>
- [49] E.T. Whittaker, *On the Connexion of Algebraic Functions with Automorphic Functions*, Philosophical Transactions of the Royal Society of London, Vol CXCII. A, 1898, <https://www.maths.ed.ac.uk/~v1ranick/whittaker1898.pdf>
- [50] J. Voight and J. Willis, *Computations with Modular Forms*, chapter Computing Power Series Expansions of Modular Forms, in Computations with Modular Forms, Contributions in Mathematical and Computational Sciences, Vol. 6. Springer Int. Pub. 2014, B#ockle G and Wiese G. eds.
- [51] R. Vidunas and G. Filipuk, *Parametric transformations between the Heun and Gauss hypergeometric functions*, arXiv:0910.3087v2 [math.CA] 2009