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Singularities of n-fold integrals of the Ising class and the theory of elliptic curves

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Abstract

We introduce some multiple integrals that are expected to have the same singularities as the singularities of the n-particle contributions $\chi^{(n)}$ to the susceptibility of the square lattice Ising model. We find the Fuchsian linear differential equation satisfied by these multiple integrals for $n = 1, 2, 3, 4$ and only modulo some primes for $n = 5$ and 6, thus providing a large set of (possible) new singularities of $\chi^{(n)}$. We discuss the singularity structure for these multiple integrals by solving the Landau conditions. We find that the singularities of the associated ODEs identify (up to $n = 6$) with the leading pinch Landau singularities. The second remarkable obtained feature is that the singularities of the ODEs associated with the multiple integrals reduce to the singularities of the ODEs associated with a *finite number of one-dimensional integrals*. Among the singularities found, we underline the fact that the quadratic polynomial condition $1 + 3w + 4w^2 = 0$, that occurs in the linear differential equation of $\chi^{(3)}$, actually corresponds to a remarkable property of selected elliptic curves, namely the occurrence of complex multiplication. The interpretation of complex multiplication for elliptic curves as complex fixed points of the selected generators of the renormalization group, namely isogenies of elliptic curves, is sketched. Most of the other singularities occurring in our multiple integrals are not related to complex multiplication situations, suggesting an interpretation in terms of (motivic) mathematical structures beyond the theory of elliptic curves.

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1. Introduction

The susceptibility χ of the square lattice Ising model has been shown by Wu, McCoy, Tracy and Barouch [1] to be expressible as an infinite sum of holomorphic functions, given as multiple integrals, denoted by $\chi^{(n)}$, that is $kT\chi = \sum \chi^{(n)}$. Nickel found [2, 3] that each of these $\chi^{(n)}$'s is singular on a set of points located on the unit circle $|s| = |\sinh(2K)| = 1$, where $K = J/kT$ is the usual Ising model temperature variable.

These singularities are located at solution points of the following equations:

$$\frac{1}{w} = 2\left(s + \frac{1}{s}\right) = u^k + \frac{1}{u^k} + u^m + \frac{1}{u^m}$$
$$u^{2n+1} = 1, \qquad -n \leqslant m, k \leqslant n. \tag{1}$$

From now on, we will call these singularities of the 'Nickelian type' or simply 'Nickelian singularities'. The accumulation of this infinite set of singularities of the higher-particle components of $\chi(s)$ on the unit circle $|s| = 1$ leads, in the absence of mutual cancellation, to some consequences regarding the non-holonomic (non D-finite) character of the susceptibility, possibly building a natural boundary for $\chi(s)$. However, it should be noted that new singularities that are not of the 'Nickelian type' were discovered as singularities of the Fuchsian linear differential equation associated [4–6] with $\chi^{(3)}$ and as singularities of $\chi^{(3)}$ *itself* [7] but seen as a function of s. They correspond to the quadratic polynomial $1 + 3w + 4w^2$ where $2w = s/(1 + s^2)$. In contrast with this situation, the Fuchsian linear differential equation, associated [8] with $\chi^{(4)}$, does not provide any new singularities.

Some remarkable Russian-doll structure as well as direct sum decompositions were found for the corresponding linear differential operators for $\chi^{(3)}$ and $\chi^{(4)}$. In order to understand the 'true nature' of the susceptibility of the square lattice Ising model, it is of fundamental importance to have a better understanding of the singularity structure of the n-particle contributions $\chi^{(n)}$ and also of the mathematical structures associated with these $\chi^{(n)}$, namely the *infinite set* of (probably Fuchsian) linear differential equations associated with these holonomic functions. Finding more Fuchsian linear differential equations having $\chi^{(n)}$'s as solutions, beyond those already found [4, 8] for $\chi^{(3)}$ and $\chi^{(4)}$, probably requires the performance of a large set of analytical, mathematical and computer programming 'tours-de-force'.

As an alternative, and in order to bypass this 'temporary' obstruction, we have developed, in parallel, a new strategy.

We have introduced [7] some single (or multiple) 'model' integrals as an 'ersatz' for $\chi^{(n)}$'s as far as the locus of the singularities is concerned. $\chi^{(n)}$'s are defined by $(n-1)$-dimensional integrals [3, 9, 10] (omitting the prefactor[4])

$$\tilde{\chi}^{(n)} = \frac{(2w)^n}{n!}\prod_{j=1}^{n-1}\int_0^{2\pi}\frac{\mathrm{d}\phi_j}{2\pi}\left(\prod_{j=1}^{n} y_j\right)R^{(n)}(G^{(n)})^2 \tag{2}$$

where

$$G^{(n)} = \left(\prod_{j=1}^{n} x_j\right)^{(n-1)/2}\prod_{1\leqslant i<j\leqslant n}\frac{2\sin((\phi_i-\phi_j)/2)}{1-x_ix_j} \tag{3}$$

and

$$R^{(n)} = \frac{1+\prod_{i=1}^{n} x_i}{1-\prod_{i=1}^{n} x_i} \tag{4}$$

[4] The prefactor reads in the variable s, $(1-s^4)^{1/4}/s$ for $T > T_c$ and $(1-s^{-4})^{1/4}$ for $T < T_c$.

with

$$x_i = \frac{2w}{1 - 2w\cos(\phi_i) + \sqrt{(1 - 2w\cos(\phi_i))^2 - 4w^2}}, \qquad (5)$$

$$y_i = \frac{1}{\sqrt{(1 - 2w\cos(\phi_i))^2 - 4w^2}}, \qquad \sum_{j=1}^{n} \phi_j = 0. \qquad (6)$$

The two families of integrals we considered in [7] are very rough approximations of the integrals (2). For the first family[5], we considered the n-fold integrals corresponding to the product of (the square[6] of the) y_i's, integrated over the whole domain of integration of ϕ_i (thus getting rid of the factors $G^{(n)}$ and $R^{(n)}$). Here, we found a subset of singularities occurring in $\chi^{(n)}$ as well as the quadratic polynomial condition $1 + 3w + 4w^2 = 0$.

For the second family, we discarded the factor $G^{(n)}$ and the product of y_i's, and we restricted the domain of integration to the principal diagonal of the angles ϕ_i ($\phi_1 = \phi_2 = \cdots = \phi_{n-1}$). These simple integrals (over a *single* variable) were denoted [7] by $\Phi_D^{(n)}$:

$$\Phi_D^{(n)} = -\frac{1}{n!} + \frac{2}{n!}\int_0^{2\pi} \frac{\mathrm{d}\phi}{2\pi}\, \frac{1}{1 - x^{n-1}(\phi)x((n-1)\phi)} \qquad (7)$$

where $x(\phi)$ is given by (5).

Remarkably these very simple integrals both *reproduce all the singularities*, discussed by Nickel [2, 3], as well as the quadratic roots of $1 + 3w + 4w^2 = 0$ found [4, 5] for the linear ODE of $\chi^{(3)}$. One should however note that, in contrast with $\chi^{(n)}$, no Russian-doll or direct sum decomposition structure is found for the linear differential operators corresponding to these $\Phi_D^{(n)}$.

Another approach has been introduced as a simplification of the susceptibility of the Ising model by considering a magnetic field restricted to one diagonal of the square lattice [11]. For this 'diagonal susceptibility' model [11], we benefited from the *form factor decomposition* of the diagonal two-point correlations $C(N, N)$, that has been recently presented [12], and subsequently proved by Lyberg and McCoy [13]. The corresponding n-fold integrals $\chi_d^{(n)}$ were found to exhibit remarkable direct sum structures inherited from the direct sum structures of the form factor [11, 12]. The linear differential operators of the form factor [12] being closely linked to the second-order differential operator L_E (resp. L_K) of the complete elliptic integrals E (resp. K), this 'diagonal susceptibility' model [11] is closely linked to the elliptic curves of the two-dimensional Ising model. By way of contrast, we note that the singularities of the linear ODEs for these n-fold integrals [11] $\chi_d^{(n)}$ are quite elementary (consisting of only nth roots of unity) in comparison with the singularities we encounter for the integrals on a single variable (7).

These two approaches corresponding to two different sets of n-fold integrals of the Ising class [14] are complementary: (7) is more dedicated to reproduce the non-trivial head polynomials encoding the location of the singularities of $\chi^{(n)}$, but fails to reproduce some remarkable (Russian-doll, direct sum decomposition) algebraico-differential structures of the corresponding linear differential operators, while the other one [11] preserves these non-trivial structures of the corresponding linear differential operators but provides a poorer representation of the location of the singularities (nth roots of unity).

[5] Denoted by $Y^{(n)}(w)$ in [7].

[6] Surprisingly, the integrand with $(\prod_{j=1}^{n} y_j)^2$ yields second-order linear differential equations [7] and, consequently, we have been able to totally decipher the corresponding singularity structure. By way of contrast the integrand with the simple product $(\prod_{j=1}^{n} y_j)$ yields linear differential equations of higher order, but with identical singularities [7].

In this paper, we return to the integrals (2) where, this time, the natural next step is to consider the following family of n-fold integrals

$$\Phi_H^{(n)} = \frac{1}{n!} \prod_{j=1}^{n-1} \int_0^{2\pi} \frac{d\phi_j}{2\pi} \left(\prod_{j=1}^{n} y_j \right) \frac{1 + \prod_{i=1}^{n} x_i}{1 - \prod_{i=1}^{n} x_i} \qquad (8)$$

which amounts to getting rid of the (fermionic) factor $(G^{(n)})^2$ in the n-fold integral (2). This family is as close as possible to (2), for which we know that finding the corresponding linear differential ODEs is a huge task. The idea here is that the methods and techniques we have developed [4, 5] for series expansions calculations of $\chi^{(3)}$ and $\chi^{(4)}$ seem to indicate that the quite involved fermionic term $(G^{(n)})^2$ in the integrand of (2) should not impact greatly on the location of singularities of these n-fold integrals (2). With this simplification in the integrand of (2) we expect to retain much exact information about the location of the singularities of the original Ising problem. However, we certainly do not expect to recover from the n-fold integrals (8) the local singular behaviour (exponents, amplitudes of singularities). Getting rid of the (fermionic) factor $(G^{(n)})^2$ are we moving away from the elliptic curves of the two-dimensional Ising model? Could it be possible that we lose the strong (Russian-doll, direct sum decomposition) algebraico-differential structures of the corresponding linear differential operators inherited from the second-order differential operator L_E (resp. L_K) of the complete elliptic integrals E (resp. K) but keep some characterization of elliptic curves through more 'primitive' (universal) features of these n-fold integral like the location of their singularities?

In the following, we give the expressions of $\Phi_H^{(1)}$, $\Phi_H^{(2)}$ and the Fuchsian linear differential equations for $\Phi_H^{(n)}$ for $n = 3$ and $n = 4$. For $n = 5, 6$, the computation (linear ODE search of a series) becomes much harder. Consequently, we use a *modulo prime* method to obtain the form of the corresponding linear ODE with totally explicit singularity structure. These results provide a large set of 'candidate singularities' for $\chi^{(n)}$. From the resolution of the Landau conditions [7] for (8), we show that the singularities of (the linear ODEs of) these multiple integrals actually reduce to the union of the singularities of (the linear ODEs of) a set of one-dimensional integrals. We discuss the *mathematical, as well as physical, interpretation* of these new singularities. In particular, we will see that they correspond to *pinched Landau-like singularities* as previously noticed by Nickel [15]. Among all these polynomial singularities, the quadratic numbers $1 + 3w + 4w^2 = 0$ are very special. We will show that these selected quadratic numbers are related to *complex multiplication for the elliptic curves* parametrizing the square lattice Ising model.

The paper is organized as follows. Section 2 presents the multidimensional integrals $\Phi_H^{(n)}$ and the singularities of the corresponding linear ODE for $n = 3, \ldots, 6$ that we compare with the singularities obtained from the Landau conditions. We show that the set of singularities associated with the ODEs of the multiple integrals $\Phi_H^{(n)}$ reduces to the singularities of the ODEs associated with a *finite number of one-dimensional integrals*. Section 3 deals with the *complex multiplication for the elliptic curves* related to the singularities given by the zeros of the quadratic polynomial $1 + 3w + 4w^2 = 0$. Our conclusions are given in section 4.

In this paper, most of the results will be, for simplicity reasons, given in terms of the self-dual variable $w = s/(1 + s^2)/2$ which is a canonical one for the n-fold integrals (2), (5) and (6). Due to a possible natural boundary for the susceptibility on the unit circle $|s| = 1$ in the s-complex plane, and the Kramers–Wannier symmetry breaking between odd and even values of n in $\chi^{(n)}$, the figures showing the positions of the singularities are given in the s variable. The variable s is closely linked to the modulus k of the elliptic parametrization of the Ising model. We use the variable k in section 3 and appendix G to underline questions linked to the occurrence of elliptic curves.

2. The singularities of the linear ODE for $\Phi_H^{(n)}$

For the first two values of n, one obtains

$$\Phi_H^{(1)} = \frac{1}{1-4w} \tag{9}$$

and

$$\Phi_H^{(2)} = \frac{1}{2(1-16w^2)}\,{}_2F_1(1/2, -1/2; 1; 16w^2). \tag{10}$$

For $n \geqslant 3$, the series coefficients of the multiple integrals $\Phi_H^{(n)}$ are obtained by expanding in the variables x_i and performing the integration (see appendix A). One obtains

$$\Phi_H^{(n)} = \frac{1}{n!}\sum_{k=0}^{\infty}\sum_{p=0}^{\infty}(2-\delta_{k,0})(2-\delta_{p,0})w^{n(k+p)}a^n(k,p) \tag{11}$$

where $a(k, p)$ is a ${}_4F_3$ hypergeometric series dependent on w.

The advantage of using these simplified integrals (8) instead of the original ones (2) is twofold.

Using (11) the series generation is straightforward compared to the complexity related to $\chi^{(n)}$. As an illustration note that on a desk computer, $\Phi_H^{(n)}$ are generated up to w^{200} in less than 10 s CPU time for all values of n, while for the simplest case of $\chi^{(n)}$, namely $\chi^{(3)}$, it took 3 min to generate the series up to w^{200}. This difference between $\Phi_H^{(n)}$ and $\chi^{(n)}$ increases rapidly with increasing n and increasing number of generated terms. We note that for the $\Phi_H^{(n)}$ quantities and for a fixed order the CPU time is decreasing[7] with increasing n. For $\chi^{(n)}$ the opposite is the case.

The second point is that, for a given n, the linear ODE can be found with less terms in the series compared to the linear ODE for $\chi^{(n)}$. Indeed for $\chi^{(3)}$, 360 terms were needed while 150 terms were enough for $\Phi_H^{(3)}$. The same feature holds for $\chi^{(4)}$ and $\Phi_H^{(4)}$ (185 terms for $\chi^{(4)}$ and 56 terms[8] for $\Phi_H^{(4)}$).

With the fully integrated sum (11), a sufficient number of terms is generated to obtain the linear differential equations. We succeeded in obtaining the linear differential equations, respectively of minimal order five and six, corresponding to $\Phi_H^{(3)}$ and $\Phi_H^{(4)}$. These linear ODEs are given in appendix B.

For $\Phi_H^{(n)}$ ($n \geqslant 5$), the calculations, in order to get the linear ODEs, become really huge[9]. For this reason, we introduce a modular strategy which amounts to generating long series *modulo a prime* and then deducing the ODE modulo that prime. Note that the ODE of minimal order is *not necessarily the optimal one*, i.e., an ODE of order higher than the minimal order may require fewer terms in the series expansion to be found. We have already encountered such a situation [8, 11]. For $\Phi_H^{(5)}$ (resp. $\Phi_H^{(6)}$), the linear ODE of minimal order is of order 17 (resp. 27) and needs 8471 (resp. 9272) terms in the series expansion to be found.

Actually, for $\Phi_H^{(5)}$ (resp. $\Phi_H^{(6)}$), we have found the corresponding linear ODEs of order 28 (resp. 42) with *only* 2208 (resp. 1838) terms from which we have deduced the minimal ones.

The form of these two minimal order linear ODEs obtained modulo a prime is sketched in appendix B. In particular, the singularities (given by the roots of the head polynomial in front of the highest order derivative) are given with the corresponding multiplicity in appendix B. Some details about the ODE search are also given in appendix B.

[7] This can be seen from the series expansion (11). Denoting R_0 by the fixed order, one has $n(p+k) \leqslant R_0$, while the CPU time for the series generation of $a^n(k, p)$ is not strongly dependent on n.

[8] From now on, for even n, the number of terms stands for the number of terms in the variable $x = w^2$.

[9] Except the generation of long series which remains reasonable.

We have also obtained very long series (20 000 coefficients) modulo a prime for $\Phi_H^{(7)}$, but, unfortunately, this has not been sufficient to identify the linear ODE (mod prime) up to order 100.

The singularities of the linear ODE for the first $\Phi_H^{(n)}$ are respectively zeros of the following polynomials (besides $w = \infty$):

$$\begin{aligned}
n &= 3, \qquad && w(1-16w^2)(1-w)(1+2w)(1+3w+4w^2),\\
n &= 4, && w(1-16w^2)(1-4w^2),\\
n &= 5, && w(1-16w^2)(1-w^2)(1+2w)(1+3w+4w^2)\\
& && (1-3w+w^2)(1+2w-4w^2)(1+4w+8w^2)\\
& && (1-7w+5w^2-4w^3)(1-w-3w^2+4w^3)\\
& && (1+8w+20w^2+15w^3+4w^4),
\end{aligned} \tag{12}$$

$$\begin{aligned}
n &= 6, \qquad && w(1-16w^2)(1-4w^2)(1-w^2)(1-25w^2)\\
& && (1-9w^2)(1+3w+4w^2)(1-3w+4w^2)\\
& && (1-10w^2+29w^4).
\end{aligned} \tag{13}$$

For $n = 7$ and $n = 8$, besides modulo primes series calculations described above, we also generated very long series from which we obtained in floating point form the polynomials given in appendix C (using generalized differential Padé methods).

If we compare the singularities for $\Phi_H^{(n)}$ to those obtained with the 'diagonal model[10]' presented in [7], i.e. $\Phi_D^{(n)}$, one sees that the singularities of the linear ODE for the 'diagonal model' are identical to those of the linear ODE of $\Phi_H^{(n)}$ for $n = 3, 4$ (and are a proper subset to those of $\Phi_H^{(n)}$ for $n = 5, 6$). The additional singularities for $n = 5, 6$ are zeros of the polynomials:

$$\begin{aligned}
n &= 5, \qquad && (1+3w+4w^2)(1+4w+8w^2)(1-7w+5w^2-4w^3),\\
n &= 6, && (1+3w+4w^2)(1-3w+4w^2)(1-25w^2).
\end{aligned}$$

For $n = 7$, the zeros of the following polynomials (among others) are singularities which are not of Nickel's type (1) and do not occur for $\Phi_D^{(n)}$:

$$\begin{aligned}
&1+8w+15w^2-21w^3-60w^4+16w^5+96w^6+64w^7,\\
&1-4w-16w^2-48w^3+32w^4-128w^5.
\end{aligned}$$

The linear ODEs of the multiple integrals $\Phi_H^{(n)}$ thus display *additional singularities* for $n = 5, 6$ and $n = 7$ ($n = 8$ see below) compared to the linear ODE of the single integrals $\Phi_D^{(n)}$.

We found it remarkable that the linear ODEs for the integrals $\Phi_D^{(n)}$ display all the Nickelian singularities, as well as the new quadratic numbers $1+3w+4w^2 = 0$ found for $\chi^{(3)}$. It is thus interesting to see how the singularities for $\Phi_D^{(n)}$ are included in the singularities for $\Phi_H^{(n)}$ and whether the new (with respect to $\Phi_D^{(n)}$) singularities can be given by one-dimensional integrals similar to $\Phi_D^{(n)}$. Let us mention that the singularities of the linear ODE for $\Phi_H^{(3)}$ (respectively $\Phi_H^{(4)}$) *are remarkably also the singularities* of the linear ODE for $\Phi_H^{(5)}$ (respectively $\Phi_H^{(6)}$). In the following, we will show how this comes about and how it generalizes. For this, we solve in the following the Landau conditions for the n-fold integrals (8).

[10] Not to be confused with the 'diagonal susceptibility' and the corresponding [11] n-fold integrals $\chi_d^{(n)}$.

2.1. Landau conditions for $\Phi_H^{(n)}$

The Landau conditions [7] amount to carrying out algebraic calculations [7] on the integrand (8) to get singularities of these n-fold integrals.

We remind the reader that the Landau conditions [7] are *necessary* conditions for singularities to be singularities of the *integral representation itself*. In a previous paper [7], we have shown for particular integral representations belonging to the Ising class integrals [14] that in fact the solutions of the Landau conditions identify for specific[11] configurations (see below) with the singularities of the ODE associated with the quantity under consideration. As we will see in the following, here also, the singularities obtained by the Landau conditions for (8) are the singularities of the corresponding linear ODEs.

In what follows we use the following integral representation [1, 2]:

$$y_j x_j^n = \int_0^{2\pi} \frac{\mathrm{d}\psi_j}{2\pi} \frac{\exp(\mathrm{i}n\psi_j)}{1 - 2w(\cos(\phi_j) + \cos(\psi_j))}. \tag{14}$$

Defining

$$D(\phi_j, \psi_j) = 1 - 2w(\cos(\phi_j) + \cos(\psi_j)), \tag{15}$$

the integral $\Phi_H^{(n)}$ (see its expansion (A.1) in appendix A) becomes

$$\Phi_H^{(n)} = \frac{1}{n!} \int_0^{2\pi} \prod_{j=1}^n \frac{\mathrm{d}\phi_j}{2\pi} \frac{\mathrm{d}\psi_j}{2\pi} D^{-1}(\phi_j, \psi_j) \delta\left(\sum_{j=1}^n \phi_j\right) \delta\left(\sum_{j=1}^n \psi_j\right), \tag{16}$$

where the Dirac delta's are introduced to take care of the conditions

$$\sum_{j=1}^n \phi_j = 0, \qquad \sum_{j=1}^n \psi_j = 0 \qquad \mathrm{mod}\ 2\pi \tag{17}$$

on both the angles ϕ_j and the auxiliary angles ψ_j.

The Landau conditions [16, 17] can easily be written [7] as

$$\alpha_j D(\phi_j, \psi_j) = 0, \qquad j = 1, \ldots, n, \tag{18}$$

$$\beta_j \phi_j = 0, \qquad \gamma_j \psi_j = 0, \qquad j = 1, \ldots, n-1, \tag{19}$$

$$\alpha_j \sin(\phi_j) - \alpha_n \sin(\phi_n) + \beta_j = 0, \qquad j = 1, \ldots n-1, \tag{20}$$

$$\alpha_j \sin(\psi_j) - \alpha_n \sin(\psi_n) + \gamma_j = 0, \qquad j = 1, \ldots, n-1 \tag{21}$$

together with (17). The Landau singularities are obtained by solving these equations[12] in all the unknowns, where the Lagrange multipliers $\alpha_j, \beta_j, \gamma_j$ should not all be equal to zero.

In this paper, our aim is not to find all the solutions of the above equations but to show that the singularities of the linear ODE for $\Phi_H^{(n)}$ are the solutions of the Landau conditions. Furthermore, in working out various Ising class integrals [14] and the two models of [7] (see appendix D), we remarked that the singularities of the linear ODE are, in fact, included in a particular 'configuration'. What we mean by 'configuration' is the set of values (equal to zero or not) of the parameters $\alpha_j, \beta_j, \gamma_j$.

The 'configuration' we consider

$$\alpha_j \neq 0, \qquad \beta_j = \gamma_j = 0, \tag{22}$$

[11] In that respect one must recall the notion of leading singularities in contrast with the subleading singularities (see p 54 in [16]).

[12] Note that conditions (19), $\beta_j\phi_j = 0, \gamma_j\psi_j = 0, j = 1, \ldots, n-1$, have to be considered in the general Landau conditions. They do not occur if one restricts oneself to pinch singularities.

corresponds to *pinch singularities* on the manifolds $D(\phi_j, \psi_j) = 0$. One may also be convinced to take $\beta_j = \gamma_j = 0$, since the integrand is periodic [54] in ϕ_j and ψ_j.

Let us stress that the configuration with all the Lagrange multipliers of the singularity manifolds $D(\phi, \psi)$ different from zero ($\alpha_j \neq 0$, for any j) leads to the so-called *leading Landau singularities* following the terminology of p 54 of [16].

The Landau conditions become

$$1 - 2w(\cos(\phi_j) + \cos(\psi_j)) = 0, \qquad j = 1, \ldots, n, \tag{23}$$
$$\alpha_j \sin(\phi_j) - \alpha_n \sin(\phi_n) = 0, \qquad j = 1, \ldots, n-1, \tag{24}$$
$$\alpha_j \sin(\psi_j) - \alpha_n \sin(\psi_n) = 0, \qquad j = 1, \ldots, n-1. \tag{25}$$

and

$$\sum_{j=1}^{n} \phi_j = 0, \qquad \sum_{j=1}^{n} \psi_j = 0 \qquad \mod 2\pi. \tag{26}$$

The Landau singularities are the solutions of these conditions (see appendix E for details). Note that the first three conditions (23)–(25) are invariant under transformation

$$w \longrightarrow -w, \qquad \phi_j \longrightarrow \phi_j + \pi, \qquad \psi_j \longrightarrow \psi_j + \pi, \tag{27}$$

but the Landau conditions (23)–(25) together with (26) are invariant under transformation (27) *if and only if* n *is even*. This distinction between even and odd n (corresponding to the symmetry breaking of $w \leftrightarrow -w$) is reminiscent of the distinction between even and odd n for $\chi^{(n)}$ associated with the distinction between low and high temperature regimes.

The Landau conditions yield two families of singularities expressed in terms of Chebyshev polynomials of the first and second kinds. The first family reads

$$\begin{aligned} &T_{2p_1}(1/2w + 1) = T_{n-2p_1-2p_2}(1/2w - 1), \\ &0 \leqslant p_1 \leqslant [n/2], \qquad 0 \leqslant p_2 \leqslant [n/2] - p_1. \end{aligned} \tag{28}$$

The second family is given by the elimination of z from

$$\begin{aligned} &T_{n_1}(z) - T_{n_2}\left(\frac{4w - z}{1 - 4wz}\right) = 0, \\ &T_{n_1}\left(\frac{1}{2w} - z\right) - T_{n_2}\left(\frac{1}{2w} - \frac{4w - z}{1 - 4wz}\right) = 0, \\ &U_{n_2-1}(z)U_{n_1-1}\left(\frac{1}{2w} - \frac{4w - z}{1 - 4wz}\right) - U_{n_2-1}\left(\frac{1}{2w} - z\right)U_{n_1-1}\left(\frac{4w - z}{1 - 4wz}\right) = 0 \end{aligned} \tag{29}$$

with

$$n_1 = p_1, \qquad n_2 = n - p_1 - 2p_2, \tag{30}$$
$$0 \leqslant p_1 \leqslant n, \qquad 0 \leqslant p_2 \leqslant [(n - p_1)/2]. \tag{31}$$

One recognizes in the first set of equation (28) a generalization of the singularities given by Nickel [15] for the pinch singularities coming from the product of y_j's and also derived for our multiple integral denoted by $Y^{(n)}$ in [7]. These have been written as [7, 15]

$$T_k(1/2w + 1) = T_{n-k}(1/2w - 1). \tag{32}$$

Note that, comparatively to (28), the integer k should be even[13].

[13] This is a consequence of (23)–(26) yielding $k\pi = 0 \mod 2\pi$ (see appendix E.1).

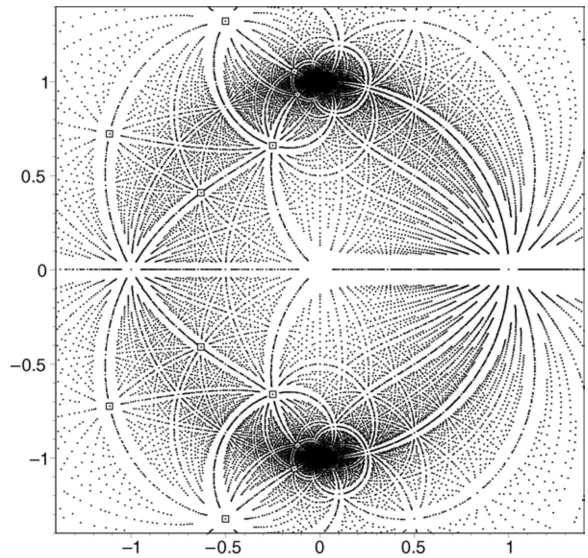

Figure 1. First family of singularities (28) in the complex s plane ($n \leqslant 51$).

The second set of equation (29) is a generalization of the singularities we derived for $\Phi_D^{(n)}$ in [7]. In both formulae, one notes the occurrence of a second varying integer p_2, leading to a better understanding of the singularities of these integrals. Indeed with p_2 running, the linear ODE for $\Phi_H^{(n)}$ will automatically contain all the singularities of the linear ODEs for $\Phi_H^{(n-2)}, \Phi_H^{(n-4)}, \ldots, \Phi_H^{(n-2q)}$.

For $n = 7$, we have checked that the singularities specific to $n = 7$ ($p_2 = 0$ in (28) and (29)) also appear as the singularities of the linear ODE in floating point form (see appendix D for details). For $p_2 = 1$, part of the singularities appears in floating point form, while for $p_2 = 2$ (i.e. singularities of $\Phi_H^{(3)}$) no singularities appear in floating point form. Similarly, for $n = 8$, we have checked that the singularities specific to $n = 8$ ($p_2 = 0$ in (28) and (29)) also appear as the singularities of the linear ODE in floating point form (see appendix D for details). For $p_2 \geqslant 1$, no singularities appear in floating point form.

Let us remark that some singularities may not appear in the floating point analysis when the series is not long enough. For instance, for $\Phi_H^{(7)}$ (resp. $\Phi_H^{(8)}$) we have used 1250 (resp. 1200 terms) while the $\Phi_H^{(7)}$ and $\Phi_H^{(8)}$ linear ODEs need more than 20 000 terms. In these cases, the number of terms used is not sufficient to encode the location of all the singularities. Note however that this information can be obtained with less terms than the number required to get the exact ODEs (i.e., the 20 000 terms).

Figure 1 shows the first family of singularities (28) displayed in the complex s plane close to the unit circle. This figure clearly shows a quite rich structure for these set of points. This figure looks like a network of nodal points linked together by (cardioid-like) curves that can, at first sight, hardly be distinguished from arcs of circles.

Figure 1 shows some selected points (open squares) that can be seen to occur quite clearly as some of these nodal points. The four open squares nearer to the axis $\mathrm{Re}(s) = 0$ correspond to the roots in the variable s of $1 + 3w + 4w^2 = 0$. The four open squares at the left of the figure correspond to the roots in the variable s of $1 + 8w + 20w^2 + 15w^3 + 4w^4 = 0$ and corresponding to $\Phi_H^{(5)}$. The four other singularities are hidden in the spray of points near $s = \pm\mathrm{i}$.

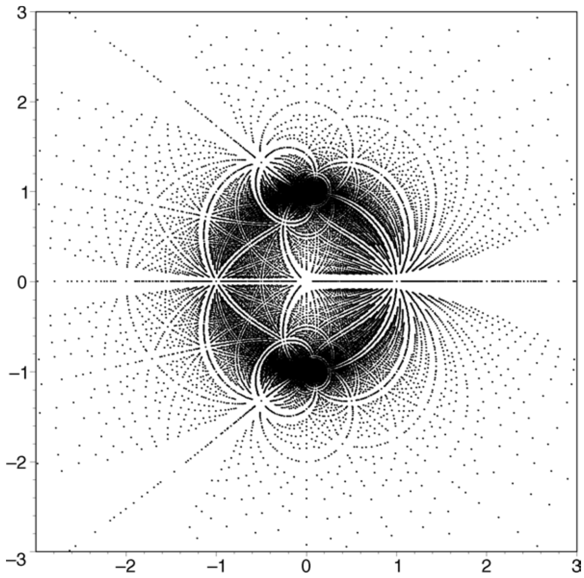

Figure 2. First family of singularities (28) in the complex s plane far from the unit circle ($n \leqslant 51$).

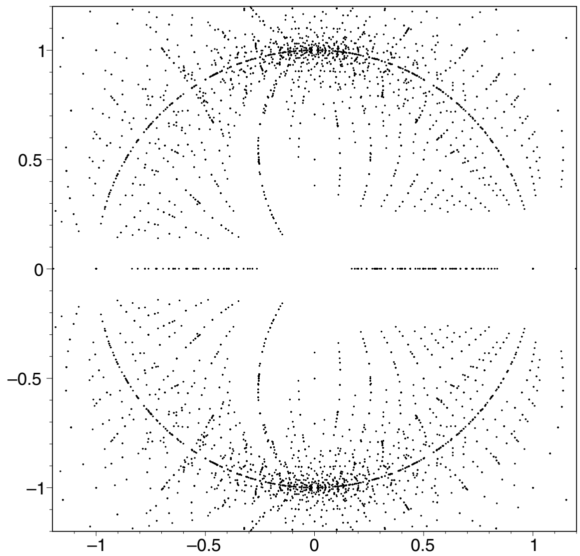

Figure 3. First and second families of singularities (28) and (29) in the complex s plane ($n \leqslant 16$).

Figure 2 shows the first family of singularities (28) far from the unit circle. Figure 3 shows all the singularities altogether (first and second families) close to the unit s-circle. Finally, figure 4 shows all the singularities together with (28) and (29) that are not so close to the unit s-circle.

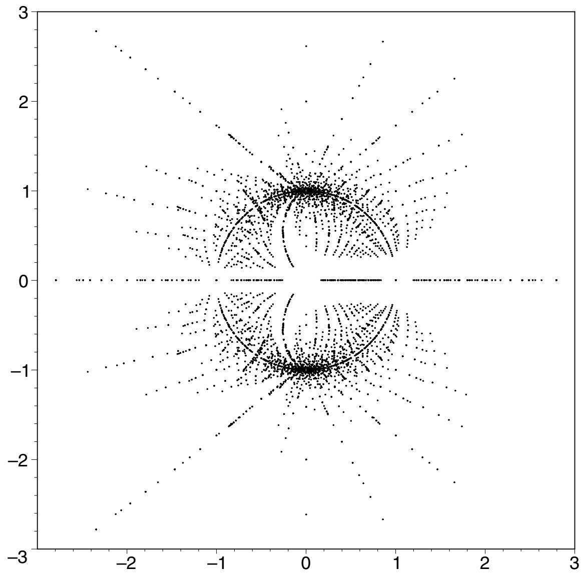

Figure 4. First and second families of singularities (28) and (29) in the complex s plane far from the unit circle ($n \leqslant 16$).

The accumulation of singularities one can see in figure 1 near $s = \mathrm{i}$ and $s = -\mathrm{i}$ seems to confirm the statement made in Orrick *et al* [18] that these two points are two quite unpleasant points for the susceptibility of the Ising model for which the series expansions are not even asymptotically convergent.

Besides reproducing exactly the singularities of the linear ODE for $\Phi_H^{(n)}$, it is remarkable to see from formulae (28) and (29) how to track where each singularity polynomial comes from. This allows one to understand how the singularities of the Ising like integrals $Y^{(n)}$ and $\Phi_D^{(n)}$ (see [7]) and even the Nickelian singularities (1) emerge in these multiple integrals (8). This comes simply from the partition (30) and the equivalent one in (28).

2.2. *Singularities: from n-fold integrals to one-dimensional integrals*

Consider for instance the singularities $1 - 7w + 5w^2 - 4w^3 = 0$ occurring in $\Phi_H^{(5)}$, which are given by (28) for $n = 5$, $p_1 = 1$ and $p_2 = 0$. As far as conditions on the integration angles (see (33)), this arises from a situation where two angles are equal and the three others are equal. Recall that the $\Phi_D^{(n)}$ integrals are constructed with the following restrictions on the angles:

$$\phi_1 = \phi_2 = \cdots = \phi_{n-1} = \phi, \qquad \phi_n = -(n-1)\phi. \tag{33}$$

One sees that a generalization of this model (33) is simply

$$\begin{aligned} &\phi_1 = \phi_2 = \cdots = \phi_k, \\ &\phi_{k+1} = \phi_{k+2} = \cdots = \phi_n, \qquad k = 0, 1, \ldots, [n/2]. \end{aligned} \tag{34}$$

By the condition (6) on the angles, this writes $k\phi_1 + (n-k)\phi_n = 0$ or, equivalently, $k\phi_n + (n-k)\phi_1 = 0$. *This case is indeed one dimensional*, with (denoting $\phi_1 = \phi$)

$$\phi_n = -\frac{(n-k)}{k}\phi + \frac{2j\pi}{k}, \qquad j \text{ integer.} \tag{35}$$

The model (33) is obviously given by (34) for $k = 1$. The Nickelian singularities are also given by (34) for $k = 0$, but this time the underlying model is 'zero dimensional'. The model constructed along the same lines as in [7] corresponds to an integrand:

$$\sum_{j=0}^{n-1} \frac{1}{1 - x^n\left(\frac{2\pi j}{n}\right)}. \tag{36}$$

The Nickelian singularities arise as poles.

For $k \geqslant 2$, the singularities given by the model (34), which appear in (8), are thus given neither by (1) nor by $\Phi_D^{(n)}$. Consider one variable of integration such as (7), where the integrand is

$$\frac{1}{1 - x^{n-1}(\phi)x((n-1)\phi)} \quad \longrightarrow \quad \frac{1}{1 - x^{n-k}(\phi)x^k(\phi_n)} \tag{37}$$

and denote by $\Phi_k^{(n)}$ such integrals (one then has $\Phi_1^{(n)} = \Phi_D^{(n)}$).

Fix $n = 5$ and $k = 2$. The constraint (35) on the angles reads

$$\phi_5 = -\frac{3}{2}\phi + j\pi, \qquad j \text{ integer} \tag{38}$$

with one integration variable. The series of coefficients of $\Phi_2^{(5)}$ is generated along the same lines as for $\Phi_D^{(n)}$ (see appendix A). The Fuchsian linear differential equation is of order six and this order is independent of the value of j in (38). The singularities of the linear ODE are zeros of the following polynomials:

$$w(1-16w^2)(1+w)(1-3w+w^2)(1+2w-4w^2)(1+4w+8w^2)(1-7w+5w^2-4w^3). \tag{39}$$

We obtain singularities (from the last two polynomials) appearing for $\Phi_H^{(5)}$ and not occurring for $\Phi_D^{(5)}$.

The occurrence of the singularities $1 + 3w + 4w^2 = 0$ for (the linear ODE of) $\Phi_H^{(5)}$ *but not for* (the linear ODE of) $\Phi_D^{(5)}$ is explained along similar lines. Note that these singularities are common to (the linear ODE of) $\Phi_H^{(3)}$, $\Phi_H^{(5)}$ and $\Phi_H^{(6)}$. The polynomial $1 + 3w + 4w^2$ appears for (the linear ODE of) $\Phi_H^{(5)}$ from (28), namely,

$$T_{2p_1}(1/2w + 1) = T_{n-2p_2-2p_1}(1/2w - 1). \tag{40}$$

The singularities $1 + 3w + 4w^2 = 0$ occur for $\Phi_H^{(5)}$ with $n = 5$, $p_1 = 1$ and $p_2 = 1$, but this polynomial pops out also from (40) for $n = 3$, $p_1 = 1$ and $p_2 = 0$ which shows a situation with three angles, two of them being equal. This is precisely the integrand in (7), i.e., in $\Phi_D^{(3)}$.

In general, the polynomial that arises from (40) for given (n, p_1, p_2) *will also be given* by (40) for $(n - 2p_2, p_1, 0)$.

Consider now the case $n = 6$ and $k = 2$. This amounts to considering the n-fold integral $\Phi_2^{(6)}$ with

$$\phi_6 = -2\phi + j\pi, \qquad j \text{ integer}. \tag{41}$$

The results are dependent on the parity of the integer j. The series around $w = 0$ reads

$$\Phi_2^{(6)} = 1 + w^6 + 32w^8 \pm w^9 + 659w^{10} \pm 1296w^{11} + 11\,691w^{12} + \cdots.$$

With the + sign in the series (42) and corresponding to $j = 0$, the linear differential equation is of order five and the singularities are given by the zeros of the polynomials:

$$w(1-16w^2)(1-w)(1+2w)(1-9w^2)(1-25w^2)(1+3w+4w^2). \tag{43}$$

The results corresponding to the choice of a minus sign in the series (42), and corresponding to $j = 1$ in (41), are obviously obtained by[14] $w \to -w$. We obtain the singularities $1 - 25w^2$ and $1 \pm 3w + 4w^2 = 0$ occurring for (the linear ODE of) $\Phi_H^{(6)}$ but not for (the linear ODE of) $\Phi_D^{(6)}$.

Similarly, for $n = 7$ (k goes to 3), one obtains for $k = 2$ the singularities as zeros of the following polynomial $1+8w+15w^2-21w^3-60w^4+16w^5+96w^6+64w^7$, which has indeed been found numerically in the linear ODE search on a long series corresponding to $\Phi_H^{(7)}$ (see appendix C).

We have the remarkable fact that the singularities of the linear ODE for the multiple integral $\Phi_H^{(n)}$ are given by a finite set of singularities of linear ODEs of a set of one-dimensional integrals, namely, $N(N+1)/2$ one-dimensional integrals, with $N = [n/2]$. For instance, the singularities of the four-dimensional integral $\Phi_H^{(5)}$ identify with those of, at most, three one-dimensional integrals. This appears, simply, from the pair of integers in (28) which read $(2p_1, n - 2p_2 - 2p_1)$. For fixed n, when p_2 varies, one sees that we are in fact considering all the lower integer values $n - 2p_2$. The same situation holds for (29). This identification leads, obviously, to particular structures in the singularities for different n. This is what we show in the following.

2.3. Singularity structures of n-fold integrals and particular sets of one-dimensional integrals

The Landau singularities given in appendix E are checked against the singularities of the linear ODE for $\Phi_H^{(n)}$ ($n = 3, \ldots, 6$) and *are found to be identical*. Assume that these formulae do indeed reproduce all the singularities of the linear ODE for $\Phi_H^{(n)}$, for any n. In this case, we can check whether the singularities appearing at $n = m$ also occur for $n = m+1, n = m+2, \ldots$. With the singularities obtained from these formulae up to $n = 16$, we infer the following findings.

We have found that the singularities at order $2n$ will *also* be singularities at order $2n+2p$, where p is a positive integer. Similarly, the singularities at order $2n+1$ will also be present at the following odd orders.

What is remarkable is the fact that the singularities at odd order also appear at even orders. The rule is: *all the singularities at odd order n also appear in the higher orders (odd and even) except for the first* $(n-1)/2$ *even orders*. For instance, the singularities appearing at $n = 3$ will occur for all n, except the first even order, i.e. 4. The singularities appearing at $n = 5$ will occur for all n, except the first two even orders, i.e. 6 and 8.

The consequence of this *embedding* of the singularities is the occurrence of some singularities at *predefined orders*. The singularity $1 + 2w = 0$ is present *at any order n*. The singularity $1 - 2w = 0$ is present for any even order $2n$. The singularity $1 + w = 0$ occurs at any order $n \geqslant 5$. The singularity $1 - w = 0$ occurs at any order n, except for $n = 4$. All these singularities are Nickelian. The first non-Nickelian singularity $1 + 3w + 4w^2 = 0$ *appears at all orders n*, except for $n = 4$.

Moreover, we have given in [7] the Landau singularities for the (linear ODEs of the) integrals $\Phi_D^{(n)}$. These singularities have been found to be identical with the singularities of the linear ODE for $\Phi_D^{(n)}$ obtained exactly up to $n = 8$ and modulo a prime up to $n = 14$. We have seen that all the singularities of the linear ODE of $\Phi_D^{(n)}$ in the variable s lie in the annulus defined by two concentric circles of radius $\sqrt{2}$ and $1/\sqrt{2}$. The radii of the two concentric circles are the roots, in the variable s, of the polynomial $1+3w+4w^2 = 0$, that is $s^2+s+2 = 0$

[14] The last case for $n = 6$, i.e. $k = 3$, does not provide singularities other than Nickel's.

and $1 + s + 2s^2 = 0$. With the multiple integrals $\Phi_H^{(n)}$, one sees that some of the singularities *are not confined* to this annulus anymore.

Thanks to the Landau conditions, one can now understand this structure from the reduction of the multiple integrals $\Phi_H^{(n)}$ to a set of one-dimensional integrals $\Phi_k^{(n)}$ as far as the location of singularities is concerned. For $k = 0$, which corresponds to the Nickelian singularities, the 'annulus' is the unit circle. For $k = 1$ corresponding to the integrals $\Phi_D^{(n)}$, one has the annulus of radii $\sqrt{2}$ and its inverse. For each k, one expects the singularities to lie in an annulus with a concentric structure. For these annuli the larger radius increases (smaller radius decreases) as k increases. From the reduction of the singularities of $\Phi_H^{(n)}$ to these $\Phi_k^{(n)}$, all the singularities for fixed $p_1 = k$ in (28) and for fixed $p_1 = k$ in (29) will be confined to one annulus. For instance for $k = 2$, all the singularities occurring in the linear ODE for $\Phi_k^{(n)}$ (i.e. for all n) or, equivalently, all the singularities obtained by (28) for $p_1 = 1$ and by (29) for $p_1 = 2$ will be confined to the annulus of radii $2.79\ldots$ and its inverse. This value is the root, in the variable s, of $1 - 7w + 5w^2 - 4w^3 = 0$ occurring for $\Phi_H^{(5)}$. For $k = 3$, one remarks that the annulus will not be obtained from (28) which is restricted by $2p_1$, an even integer. In fact this is general. The radii of the annuli are given by (28) for k even and by (29) for k odd. The root in the variable s that will define the annulus occurs at odd order n given by $2k + 1$. The radii of the annulus are determined by the singularities which are furthest from the unit circle, among all singularities obtained from (28) or (29) for a given p_1.

The picture now is as follows. The singularities of the linear ODE for the integrals $\Phi_H^{(n)}$ are partitioned into 'families' indexed by the integer k. The singularities for $k = 0$ are Nickelian and lie on the unit circle, say, $r_0 = 1$. The singularities for $k = 1$ lie in the annulus $r_1 = \sqrt{2}, 1/\sqrt{2}$ (we discard from now on the smaller radius). The singularities for $k = 2$ will be confined in the annulus r_2. The singularities for $k = N$ will be in the annulus r_N. These concentric annuli are such that $r_0 < r_2 < \cdots < r_{2N}$ and $r_1 < r_3 < \cdots < r_{2N+1}$ (with $r_{2k} < r_{2k+1}$). As k grows, the radii of two neighbouring circles behave as $r_{2k+2} - r_{2k} \to 0$ and $r_{2k+3} - r_{2k+1} \to 0$. This decrease is not enough to create an accumulation of circles. We checked with $k = 300$ circles that the decrease goes as $k^{-\alpha}$ with $\alpha < 1$ preventing any convergence. For n large these radii diverge: $r_N \to \infty$ when $N \to \infty$.

Note that these families (i.e. the index k) come from the resolution of the Landau conditions and from the reduction of the singularities for $\Phi_H^{(n)}$ to those of $\Phi_k^{(n)}$ $(k = 0, 1, \ldots, [n/2])$. We have no idea as to how these families can be seen directly from the multiple integrals $\Phi_H^{(n)}$. If the singularities for $\Phi_H^{(n)}$ happen to be identical with those occurring in the linear ODE for $\chi^{(n)}$, it may become important to see whether this picture persists and whether this picture is showing another partition of the susceptibility χ instead of the known sum on $\chi^{(n)}$.

Figures 5–7 show how the first family of singularities (28) in the s-complex plane is decomposed according to the integer k in (34). Figure 5 shows singularities (28) for a *given odd value* of k, namely $k = 5$ *for any odd values* of n up to 91. Figure 6 shows singularities (28) for a given even value of k, namely $k = 2$ for any odd values of n up to 71. Figure 7 shows singularities (28) for a given even value of k, namely $k = 6$ for any even values of n up to 80. The figures corresponding to the filtration of the singularities of the first family (28) in terms of the integer k (previously displayed altogether with figures 1 and 2) deserve some comments. First, one sees that the various 'crescent' corresponding to different values of k are very similar. Secondly, one sees from figure 5 that the odd n, odd k 'crescent' break the $s \leftrightarrow -s$ symmetry (for even n, even k, the equations for the set of singularities are functions of s^2, see figure 7) in a quite dramatic way: the singularities in the 'crescent' of figure 5 all lie *only in the left half* s-complex plane. Similarly, the singularities in the 'crescent' of figure 6 all lie in the right half s-complex plane.

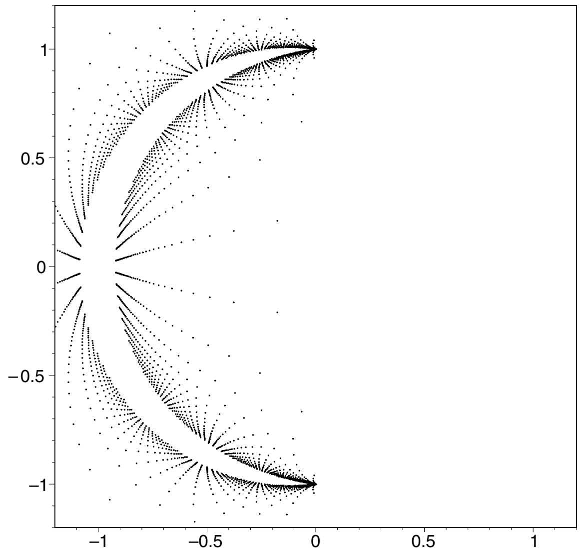

Figure 5. Crescent in the complex s plane given by (28): $k = 5, n \leqslant 91, n$ odd.

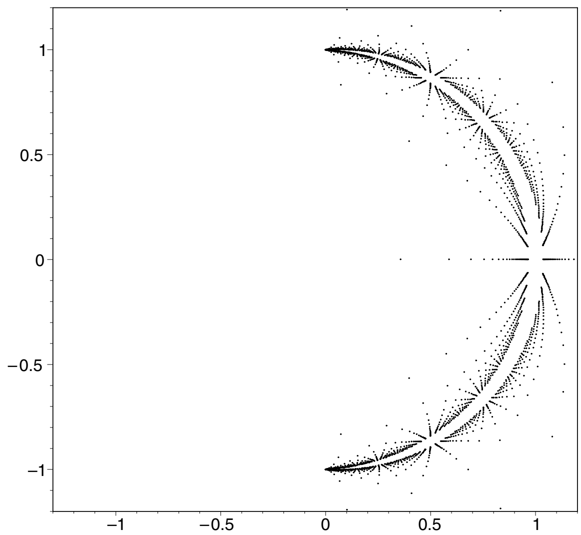

Figure 6. Crescent in the complex s plane given by (28): $k = 2, n \leqslant 71, n$ odd.

Along this $s \leftrightarrow -s$ symmetry line it is worth recalling that the low-temperature susceptibility of the Ising model has this $s \leftrightarrow -s$ symmetry (the low-temperature susceptibility is a function of s^2 or w^2) but the high-temperature susceptibility breaks that $s \leftrightarrow -s$ symmetry, and this is also the case for the n-fold integral $\chi^{(n)}$ with n odd. Our n-fold integrals (8) are introduced to provide an educated guess as to the location of the singularities of $\chi^{(n)}$. As far as the location of singularities of $\chi^{(n)}$ are concerned, it is not totally clear for n odd if the $s \leftrightarrow -s$ (resp. $w \leftrightarrow -w$) symmetry will not be partially restored on the global set of singularities with

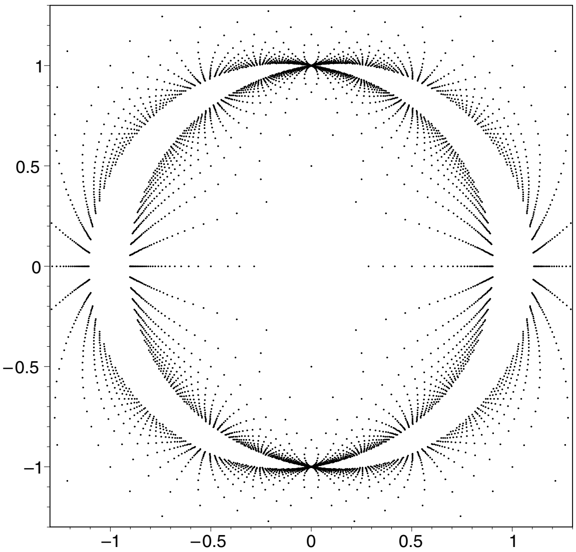

Figure 7. Crescent in the complex s plane given by (28): $k = 6$, $n \leqslant 80$, n even.

the occurrence for a singularity $P_n(w) = 0$ for a given value of n, of the opposite value for, perhaps, a different value of n : $P_m(-w) = 0$.

Remark. Quite often, in this paper, we use (by abuse of language) the words 'singularities of an n-fold integral' to describe a larger set of singularities, namely the singularities of the linear ODEs that the n-fold integral satisfies. A rigorous study would require, for any 'singularity', to perform the (differential Galois group and connection matrix) analysis we have performed in [6]. It amounts to getting extremely long series, deduced from the obtained linear ODE, that coincide with the series expansion of the n-fold integral we are interested in, and find out if these series actually exhibit these singularities. With this tedious, but straightforward, procedure we can extract the singularities of a specific n-fold integral among the larger set of singularities of the corresponding linear differential equation. In view of the large number of singularities we display in this paper, we have not performed such a systematic analysis that would have been quite huge. Furthermore, it is important to note that this 'connection matrix' approach [6] requires one to have the linear ODE of the n-fold integral. A knowledge of the linear ODE modulo a prime *is not sufficient*. We could have performed this analysis for $\Phi_H^{(3)}$ and $\Phi_H^{(4)}$, but, in that case, we already have a deeper result [6] namely the connection matrix analysis for $\chi^{(3)}$ and $\chi^{(4)}$, providing an understanding of the singularities of these n-fold integrals themselves (in w and *also* in s).

Right now, the only singularities found for $\chi^{(n)}$, other than Nickelian, are the quadratic roots of $1 + 3w + 4w^2 = 0$ (i.e. the first annulus) which appear at all orders (except $n = 4$) for $\Phi_H^{(n)}$. Let us show, in the following, how this polynomial is 'special'.

3. Towards a mathematical interpretation of the singularities

In a set of papers [19, 20], we have underlined the central role played by the *elliptic parametrization* of the Ising model, in particular the role played by the second-order linear differential operators corresponding to the complete elliptic integral E (or K), and the occurrence of an infinite number of *modular curves* [12], canonically associated with *elliptic curves*. The deep link between the theory of elliptic curves and the theory of modular forms is now well established [21].

Consequently, it may be interesting to seek 'special values' of the modulus k (singularities of $\chi^{(n)}$) that might have a 'physical meaning', as well as a 'mathematical interpretation'.

For that purpose, recall that the modular group requires one to introduce the elliptic nome, defined in terms of the periods of the elliptic functions,

$$q = \exp\left(-\pi \frac{K(1-k^2)}{K(k^2)}\right) = \exp(\mathrm{i}\pi\tau) \tag{44}$$

and the half-period ratio[15] τ. We write the complete elliptic integral K as

$$K(k) = {}_2F_1(1/2, 1/2; 1; k). \tag{45}$$

Relations between $K(k)$ evaluated at two different moduli can be found in, e.g., [22].

3.1. Some isogenies of elliptic curves seen as generators of the renormalization group

The arguments in K in these identities are related by the so-called, respectively, descending Landen and ascending Landen (or Gauss) transformations:

$$k \quad\longrightarrow\quad k_{-1} = \frac{1-\sqrt{1-k^2}}{1+\sqrt{1-k^2}} \tag{46}$$

$$k \quad\longrightarrow\quad k_1 = \frac{2\sqrt{k}}{1+k}. \tag{47}$$

These transformations (or correspondences [23, 24]) decrease or increase the modulus, respectively. Iterating (46) or (47), one converges to $k = 0$ or $k = 1$, respectively. The half-period ratio transforms through (46) and (47) as

$$\tau \rightarrow 2\tau, \qquad \tau \rightarrow \tfrac{1}{2}\tau \tag{48}$$

respectively. The *real* fixed points of the transformations (46) and (47) are $k = 0$ (the trivial infinite or zero temperature points) and $k = 1$ (the ferromagnetic and antiferromagnetic critical point of the square Ising model). In terms of the half-period ratio, this reads $\tau = \infty$ and $\tau = 0$, respectively, which *correspond to a degeneration of the elliptic parametrization into a rational parametrization*. In view of these fixed points, it is natural to identify the transformations (46) or (47), and more generally any transformation[16] $\tau \rightarrow n\tau$ or $\tau \rightarrow \tau/n$ (n integer), as *exact generators of the renormalization group* of the two-dimensional Ising model[17].

One does not need to restrict the analysis to the real fixed points of the transformations. If one considers the Landen transformation (47) as an algebraic transformation of the *complex variable* k and if one solves $k_1^2 - k^2 = 0$, one obtains

$$k(1-k)(k^2+3k+4) = 0. \tag{49}$$

The quadratic roots

$$k^2 + 3k + 4 = 0 \tag{50}$$

[15] In the theory of modular forms q^2 is also sometimes used instead of q. In number theory literature the half-period ratio is taken as $-\mathrm{i}\tau$.
[16] See relation (1.3) in [25].
[17] A similar identification of these isogenies $\tau \rightarrow n\tau$ with exact generators of the renormalization group can be introduced for any lattice model with an elliptic parametrization (Baxter model, ...).

are (up to a sign) *fixed points* of (47). We thus see the occurrence of *additional non-trivial complex selected values* of the modulus k, beyond the well-known values $k = 1, 0, \infty$ (corresponding to degeneration of the elliptic curve into a *rational curve*). Physically, these well-known values $k = 1, 0, \infty$ correspond to the *critical* Ising model ($k = 1$) and to (high–low temperature) trivializations of the model ($k = 0, \infty$).

3.2. Complex multiplication for elliptic curves as (complex) fixed points of the renormalization group

We come now to our point. The first 'unexpected' singularities $1 + 3w + 4w^2 = 0$ found [4, 5] for the Fuchsian linear differential equation of $\chi^{(3)}$, and also in other n-fold integrals of the Ising class [7], read in the variable $k = s^2$

$$(k^2 + 3k + 4)(4k^2 + 3k + 1) = 0. \tag{51}$$

The first polynomial[18] corresponds to *fixed points* of the Landen transformation (see (49)). In other words we see that the selected quadratic values $1 + 3w + 4w^2 = 0$, occurring in the (high-temperature) susceptibility of the Ising model as the singularities of the three-particle term $\chi^{(3)}$, can be seen *as fixed points of the renormalization group when extended to complex values of the modulus* k.

For elliptic curves in fields of characteristic zero, the only well-known selected set of values for k corresponds to the values for which the elliptic curve has *complex multiplication* [26]. Complex multiplication for elliptic curves corresponds to algebraic integer values (integers in the case of the Heegner numbers, see appendix F) of the modular j-function, which corresponds to Klein's absolute invariant multiplied by $(12)^3 = 1728$:

$$j(k) = 256\frac{(1 - k^2 + k^4)^3}{k^4(1 - k^2)^2}. \tag{52}$$

A straightforward calculation of the elliptic nome (44) gives, for the polynomials (51), respectively, an exact value for τ, the half-period ratio, as very simple *quadratic numbers*:

$$\tau_1 = \frac{\pm 3 + \mathrm{i}\sqrt{7}}{4}, \qquad \tau_2 = \frac{\pm 1 + \mathrm{i}\sqrt{7}}{2} \tag{53}$$

These quadratic numbers actually correspond to *complex multiplication* of the elliptic curve and for both one has $j = (-15)^3$. These two quadratic numbers are such that $2\tau_1 \mp 1 = \tau_2$. Let us focus on τ_2 for which we can write

$$\tau = 1 - \frac{2}{\tau}. \tag{54}$$

Taking into account the two modular group involutions $\tau \to 1 - \tau$ and $\tau \to 1/\tau$, we find that $1-2/\tau$ is, *up to the modular group*, equivalent to $\tau/2$. The quadratic relation $\tau^2-\tau+2 = 0$ thus amounts to looking at the fixed points of the Landen transformation $\tau \to 2\tau$ *up to the modular group*. This is, in fact, a quite general statement. The *complex multiplication* values can all be seen as fixed points, *up to the modular group*, of the generalizations of Landen transformation, namely $\tau \to n\tau$ for n integer, $\tau^2 - \tau + n = 0$ or $\tau = 1 - \frac{n}{\tau} \simeq n\tau$, where $\simeq$ denotes the equivalence *up to the modular group*. Appendix G presents an alternative view by considering the solutions as fixed points under Landen transformations of the modular j-function.

In view of the remarkable mathematical (and physical) interpretation of the quadratic values $1 + 3w + 4w^2 = 0$ in terms of *complex multiplication for elliptic curves, or fixed points*

[18] Note that the two polynomials in (51) are related by the Kramers–Wannier duality $k \to 1/k$.

of the renormalization group, it is natural to see if such a 'complex multiplication of elliptic curves' interpretation also exists for other singularities of $\chi^{(n)}$, and as a first step for the singularities of the linear differential equations of our n-fold integrals (8), that we expect to be identical, or at least have some overlap, with the singularities of $\chi^{(n)}$.

Noting that the modular j-function is a function of s^2 or w^2 (see (F.2) in appendix F) the occurrence of $1 + 3w + 4w^2 = 0$ as a selected quadratic polynomial condition means, at the same time, the occurrence of the other quadratic polynomial condition $1 - 3w + 4w^2 = 0$ (see appendix F and appendix G.2).

Besides $1 - 3w + 4w^2 = 0$, we have found two other polynomial conditions which correspond to remarkable integer values of the modular j-function. The singularities $1 - 8w^2 = 0$ correspond to $j = (12)^3$ and $\tau = \pm 1 + \mathrm{i}$ (see appendix F). They correspond to 'Nickelian singularities' for $\chi^{(8)}$ (and thus $\Phi_H^{(8)}$) and to 'non-Nickelian singularities' for $\Phi_H^{(10)}$. Another polynomial condition is $1 - 32w^2 = 0$, which gives 'non-Nickelian singularities' that begin to appear at $n = 10$ for $\Phi_H^{(10)}$. These singularities correspond to the integer value of the modular j-function, $j = (66)^3$ and to $\tau = 2\mathrm{i}$ or $\tau = -4/5 + 2\mathrm{i}/5$.

3.3. *Beyond elliptic curves*

Among the singularities of the linear ODE for $\Phi_H^{(n)}$ given in (12) and (13) or obtained from the formula given in appendix E up to $n = 15$, we have found no other singularity identified with selected algebraic values of the modular j-function corresponding to complex multiplication for elliptic curves. Could it be that the (non-Nickelian) singularities (12) and (13), which do not match with complex multiplication for elliptic curves, are actually remarkable selected situations for mathematical structures more complex than elliptic curves? With these new singularities, are we possibly exploring some remarkable 'selected situations' of some *moduli space of curves corresponding to pointed (marked) curves* [27], instead of simple elliptic curves [28]? In practice this just corresponds to considering a product of n times a rational, or elliptic, curve minus some sets of remarkable codimension-one algebraic varieties [11], $x_i x_j = 1$, $x_i x_j x_k = 1$, hyperplanes $x_i = x_j, \ldots$.

We try to fully understand the singularities of the n-fold integrals corresponding to $\chi^{(n)}$, that is to say particular n-fold integrals linked to the theory of elliptic curves. These n-fold integrals are more involved than the (simpler) n-fold integrals introduced by Beukers, Vasilyev [29, 30] and Sorokin [31, 32] or the Goncharov–Manin integrals [33] which occur in some *moduli space of curves* [34, 35] simply corresponding to a *product of rational curves* $(\mathrm{CP}_1 \times \mathrm{CP}_1 \times \cdots \times \mathrm{CP}_1)$. An example of such integrals, linked[19] to $\zeta(3)$, is displayed[20] in appendix H.

Let us close this section by noting that Heegner numbers and, more generally, *complex multiplication* have already occurred in other contexts, even if the statement was not explicit. In the framework of the construction of Liouville field theory, Gervais and Neveu have suggested [41] new classes of critical statistical models, where, besides the well-known N th root of unity situation, they found the following selected values of the *multiplicative crossing t* [42]:

$$t = \mathrm{e}^{\mathrm{i}\pi(1+\mathrm{i}\sqrt{3})/2} = \mathrm{i}\,\mathrm{e}^{-\pi\sqrt{3}/2}, \tag{55}$$

$$t = \mathrm{e}^{\mathrm{i}\pi(1+\mathrm{i})} = -\mathrm{e}^{-\pi}. \tag{56}$$

[19] Note that ζ (or the polyzeta) function evaluated at integer values ($\zeta(3), \zeta(5), \ldots$) do occur in our more involved n-fold integrals, in particular in the representation of the connection matrices [6] of the differential Galois group of the Fuchsian linear ODEs of $\chi^{(n)}$.

[20] These n-fold integrals [36–40] look almost the same as those we have introduced and analysed in the study of the diagonal susceptibility of the Ising model [11] for which n th root of unity singularities occur.

If one wants to see this multiplicative crossing as a modular nome, the two previous situations actually correspond to selected values of the modular j-function namely $j((1+i\sqrt{3})/2) = (0)^3$ for (55), and $j(1+i) = (12)^3$ for (56), which actually correspond to *Heegner numbers and, more generally, complex multiplication* [26]. It is however important not to feed the confusion already too prevalent in the literature, between a *'temperature-like' nome* such as (44) and a *multiplicative crossing modular nome*. In the Baxter model [43, 44], the first is denoted by q and the second one by x. In fact one probably has *not one but two modular groups* taking place, one acting on the 'temperature-like' nome q and the other acting on the multiplicative crossing x. We will not go further along this quite speculative line which amounts to introducing *elliptic quantum groups* [45] and *elliptic gamma functions*[21] (generalization of theta functions[22]).

4. Conclusion

The ultimate goal of our 'Ising class' integrals is to get some insight into $\chi^{(n)}$ and, hopefully, into the susceptibility of the Ising model. For that purpose we have introduced n-fold integrals (8) for which we expect the singularities of the corresponding linear ODE to overlap, as much as possible, with the singularities of the linear ODE for $\chi^{(n)}$. We have obtained the linear differential equations for these n-fold integrals $\Phi_H^{(n)}$, up to $n = 4$ and up to $n = 6$ modulo a prime. From these exact results together with an exhaustive Landau singularity analysis, we provided a quite complete description of the singularities of these linear ODEs.

From the Landau conditions, the singularity structures are explained. The singularities corresponding to $\Phi_H^{(n)}$ are found to also occur at a higher predefined order $p > n$. With these multiple integrals and the associated Landau conditions, we have been able to understand why the simple integrals $\Phi_D^{(n)}$ have succeeded in reproducing the Nickelian singularities and the new quadratic $1 + 3w + 4w^2 = 0$. These simple integrals appear to be 'a first approximation' to $\Phi_H^{(n)}$. Other one-dimensional integrals pop up to account for the additional singularities not occurring for $\Phi_D^{(n)}$.

We have then a remarkable finding that the singularities for the multiple integrals can be associated with the singularities for a finite number of one-dimensional integrals. If the singularities, associated with these n-fold integrals (8), happen to be identical with (or to overlap) the singularities associated with $\chi^{(n)}$, it becomes important to understand this mechanism for $\chi^{(n)}$ themselves. If this mechanism of singularity embedding occurs for $\chi^{(n)}$, it might be explained by a Russian-doll structure for the same linear differential operators. We know that the linear differential operator for $\chi^{(1)}$ (respectively $\chi^{(2)}$) is 'contained' in (rightdivides) the linear differential operator for $\chi^{(3)}$ (respectively $\chi^{(4)}$), and furthermore we even have direct sum decomposition properties. For $\Phi_H^{(n)}$, it is not these mechanisms which are at work.

Our primary goal in this study is to identify as many singularities as possible for $\chi^{(n)}$. The singularities of the ODEs associated with the $\Phi_H^{(n)}$ quantities correspond, in the Landau equations framework, to *leading pinch singularities* (relatively to the singularities' manifolds $D(\phi, \psi) = 0$). For the other quantities previously studied [7] which belong to the Ising class integrals, the same feature holds.

[21] Which can be seen [46] as 'automorphic forms of degree 1' when the Jacobi modular forms are 'automorphic forms of degree 0' and are associated (up to simple semi-direct products) with $SL(3, Z)$ instead of $SL(2, Z)$

[22] The partition function of the Baxter model can be seen as a ratio and product of elliptic gamma functions and theta functions. It is thus naturally expressed as a double infinite product. Similar double, and even triple, products appear in correlation functions of the eight vertex model [47, 48].

At this stage, the natural questions arising are: whether the scheme, from the Landau singularities point of view, which holds for $\Phi_H^{(n)}$, still holds for $\chi^{(n)}$ and whether the singularities of $\Phi_H^{(n)}$ can be considered as singularities of $\chi^{(n)}$?

From the Landau singularities viewpoint, the Fermionic determinant $G(n)^2$ is going to introduce new manifolds of singularities. When the Lagrange multipliers relative to the singularities' manifolds introduced by the Fermionic determinant are all set equal to zero, one deals with the Landau equations of the $\Phi_H^{(n)}$ quantities. Thus, the singularities obtained for the $\Phi_H^{(n)}$ quantities are also the solutions of the Landau equations of $\chi^{(n)}$. However, this feature does not mean that the singularities of the $\Phi_H^{(n)}$ quantities will necessarily appear as the singularities of the $\chi^{(n)}$ ODEs. Indeed some selection rules may take place and may reject some of them. For instance, one expects singularities linked to $\prod y_i$ to occur for the Landau singularities of $\Phi_H^{(n)}$. One finds that some selection rules exclude them. Our 'educated guess' is that all the Landau singularities of $\Phi_H^{(n)}$ will be in the Landau singularities of $\chi^{(n)}$, however we do not exclude the possibility that $\chi^{(n)}$ will have more Landau singularities than $\Phi_H^{(n)}$. Another 'educated guess' is that the Landau singularities of $\chi^{(n)}$ will exhibit a similar embedding to the one we found for $\Phi_H^{(n)}$. This naturally raises the question already considered in [8], of a 'strong' Russian-doll structure for the linear differential operators of the $\chi^{(n)}$, namely that the linear differential operator of $\chi^{(3)}$ (resp. $\chi^{(4)}$) could right divide the linear differential operator of $\chi^{(5)}$ (resp. $\chi^{(6)}$) and so on.

Knowledge of the singularities will help in the search for the corresponding linear ODE. For instance, we have 24 head polynomial 'candidates' for $\chi^{(5)}$ and 19 'candidates' for $\chi^{(6)}$ that can, from the outset, be put in front of the highest order derivative of the unknown linear ODE. Furthermore, as shown for the linear ODE for $\Phi_H^{(5)}$ and $\Phi_H^{(6)}$ (and also from previous ODEs), we know that the 'cost' (in terms of the number of series coefficients) will be much less for a non-minimal order linear ODE than for the minimal order one.

Concerning the non-Nickelian singularities that the multiple integrals $\Phi_H^{(n)}$ have given, we focused on $1 + 3w + 4w^2 = 0$ which actually occurs for the linear ODE of $\chi^{(3)}$ or for $\chi^{(3)}$ seen as a function of s. As far as a *mathematical interpretation* is concerned, we have shown that this quadratic polynomial condition corresponds to a selected situation for elliptic curves namely the *occurrence of complex multiplication*. The other non-Nickelian (candidate) singularities, (12) and (13), *do not correspond to* complex multiplication of elliptic curves.

Assuming that the non-Nickelian singularities obtained in the linear ODE for the integrals (8) will be, at least, included in those for $\chi^{(n)}$, various lines of thought are possible.

One may imagine that the decomposition of the susceptibility of the Ising model in terms of an infinite sum of $\chi^{(n)}$ is quite an artificial one with no deep mathematical meaning, i.e. $\chi^{(n)}$ are quite arbitrary n-fold integrals. In this case, no interpretation within the theory of elliptic curves has to be looked for and the occurrence for $1+3w+4w^2 = 0$ of complex multiplication for elliptic curves would be just a coincidence.

Another option amounts to saying that one needs to introduce (motivic) mathematical structures [36–40] *beyond the theory of elliptic curves* (moduli spaces, marked curves, ...), and beyond the elliptic curves of the Ising (or Baxter) model, to get a *mathematical interpretation of these singularities*. We tend to favour the latter option.

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Appendix A. Series expansions of $\Phi_H^{(n)}$ and of single integrals $\Phi_k^{(n)}$

We give in this appendix the series expansion that has been used for $\Phi_H^{(n)}$. Expanding the integrand of (8) in the variables x_j, one obtains

$$\Phi_H^{(n)} = \frac{1}{n!}\prod_{j=1}^{n-1}\int_0^{2\pi}\frac{\mathrm{d}\phi_j}{2\pi}\sum_{p=0}^{\infty}(2-\delta_{p,0})\prod_{j=1}^{n} y_j x_j^p. \tag{A.1}$$

We make use of the $y_j x_j^p$ Fourier expansion [4, 5, 8]

$$y_j x_j^p = w^p \sum_{k=-\infty}^{\infty} w^{|k|} a(p,|k|) Z_j^k, \qquad Z_j = \exp(i\phi_j) \tag{A.2}$$

where $a(k,p)$ is a *non-terminating* hypergeometric function that reads (with $m = k + p$)

$$a(k,p) = \binom{m}{k}\, {}_4F_3\left(\frac{1+m}{2}, \frac{1+m}{2}, \frac{2+m}{2}, \frac{2+m}{2}; 1+k, 1+p, 1+m; 16w^2\right). \tag{A.3}$$

We define $\langle\rho\rangle$ by

$$\langle\rho\rangle = \left(\prod_{j=1}^{n}\int_0^{2\pi}\frac{\mathrm{d}\phi_j}{2\pi}\right) 2\pi\,\delta\left(\sum_{j=1}^{n}\phi_j\right)\rho \tag{A.4}$$

where the angular constraint is introduced through the delta function that has the Fourier expansion:

$$2\pi\,\delta\left(\sum_{j=1}^{n}\phi_j\right) = \sum_{k=-\infty}^{\infty}(Z_1 Z_2 \cdots Z_n)^k. \tag{A.5}$$

The integrals (A.1) become

$$\Phi_H^{(n)} = \frac{1}{n!}\sum_{k=-\infty}^{\infty}\sum_{p=0}^{\infty}(2-\delta_{p,0})\left\langle\prod_{j=1}^{n} y_j x_j^p Z_j^k\right\rangle \tag{A.6}$$

where the integration is over independent angles.

Using the Fourier expansion (A.2), one obtains the integration rule

$$\left\langle y_j x_j^p Z_j^k\right\rangle = w^{p+|k|} a(p,|k|) \tag{A.7}$$

and finally

$$\Phi_H^{(n)} = \frac{1}{n!}\sum_{k=0}^{\infty}\sum_{p=0}^{\infty}(2-\delta_{k,0})(2-\delta_{p,0}) w^{n(k+p)} a^n(k,p). \tag{A.8}$$

The derivation of the series expansions for the one-dimensional integrals (37) proceeds along similar lines. The integrand of the integrals (37) is expanded in x

$$\frac{1}{1 - x^{n-k}(\phi)x^k(\phi_n)} = \sum_{p=0}^{\infty} x^{p(n-k)}(\phi) x^{pk}(\phi_n) \tag{A.9}$$

with

$$\phi_n = -\frac{n-k}{k}\phi + \frac{2\pi j}{k}. \tag{A.10}$$

Here, we use the Fourier expansion

$$x^m = w^m \sum_{p=0}^{\infty} (2 - \delta_{p,0}) w^p b(p, m) \cos(p\phi) \tag{A.11}$$

where $b(k, p)$ is a *non-terminating* hypergeometric function that reads (with $m = k + p$)

$$b(k, p) = \binom{m-1}{k} {}_4F_3\left(\frac{1+m}{2}, \frac{1+m}{2}, \frac{2+m}{2}, \frac{m}{2}; 1+k, 1+p, 1+m; 16w^2\right). \tag{A.12}$$

The integration of the one-dimensional integrals (A.9) gives

$$\Phi_k^{(n)} = \left\langle \frac{1}{1 - x^{n-k}(\phi) x^k(\phi_n)} \right\rangle = \sum_{p=0}^{\infty} \sum_{p_1=0}^{\infty} \sum_{p_2=0}^{\infty} (2 - \delta_{p_1,0})(2 - \delta_{p_2,0})$$
$$\times\, w^{pn+p_1+p_2} b(p_1, p(n-k)) b(p_2, pk) I(p_1, p_2) \tag{A.13}$$

with

$$I(p_1, p_2) = \tfrac{1}{2}(1 + \delta_{p_1,0}) \cos(c), \qquad \text{for} \quad p_2(n-k) = kp_1,$$

and

$$I(p_1, p_2) = \frac{1}{\pi} \frac{b^2}{b^2 - p_1^2} \sin(b\pi) \cos(b\pi - c), \qquad \text{for} \quad p_2(n-k) \neq kp_1,$$

where

$$b = \frac{n-k}{k} p_2, \qquad c = \frac{2\pi j}{k} p_2. \tag{A.14}$$

Appendix B. Linear differential equations of some $\Phi_H^{(n)}$

B.1. Linear ODE for $\Phi_H^{(3)}$

The minimal order linear differential equation satisfied by $\Phi_H^{(3)}$ reads

$$\sum_{n=0}^{5} a_n(w) \frac{d^n}{dw^n} F(w) = 0, \tag{B.1}$$

where

$$\begin{aligned}
a_5(w) &= (1-w)(1-4w)^4(1+4w)^2(1+2w)(1+3w+4w^2)w^3 P_5(w),\\
a_4(w) &= (1-4w)^3(1+4w)w^2 P_4(w),\\
a_3(w) &= -2(1-4w)^2 w P_3(w), \qquad a_2(w) = 2(1-4w)P_2(w),\\
a_1(w) &= -8P_1(w), \qquad a_0(w) = -96P_0(w),
\end{aligned} \tag{B.2}$$

with

$$\begin{aligned}
P_5(w) = {} & -5 + 21w + 428w^2 + 5364w^3 - 82\,416w^4 - 299\,504w^5 + 714\,944w^6 + 3127\,872w^7\\
& - 8220\,672w^8 - 25\,858\,048w^9 - 7077\,888w^{10} + 31\,424\,512w^{11} - 42\,467\,328w^{12}\\
& - 31\,457\,280w^{13} - 4194\,304w^{14} + 4194\,304w^{15},
\end{aligned}$$

$$
\begin{aligned}
P_4(w) = {} & -40 + 7w + 5232w^2 + 37\,159w^3 - 447\,778w^4 - 4947\,500w^5 + 19\,493\,448w^6 \\
& + 25\,846\,4112w^7 + 49\,920\,5984w^8 - 16\,127\,518\,08w^9 - 46\,678\,178\,56w^{10} \\
& + 13\,827\,459\,072w^{11} + 67\,078\,416\,384w^{12} + 62\,392\,041\,472w^{13} \\
& - 81\,535\,369\,216w^{14} - 116\,835\,483\,648w^{15} + 124\,662\,054\,912w^{16} \\
& + 146\,016\,305\,152w^{17} - 197\,258\,117\,120w^{18} - 131\,667\,591\,168w^{19} \\
& - 11\,676\,942\,336w^{20} + 15\,032\,385\,536w^{21}, \\
P_3(w) = {} & 35 - 25w - 8683w^2 - 10\,149w^3 + 619\,246w^4 + 5273\,820w^5 - 52\,472\,072w^6 \\
& - 588\,147\,792w^7 + 491\,073\,248w^8 + 18\,721\,819\,584w^9 + 47\,622\,771\,584w^{10} \\
& - 97\,459\,630\,592w^{11} - 441\,418\,588\,160w^{12} + 65\,1003\,559\,936w^{13} \\
& + 4694\,018\,588\,672w^{14} + 4729\,946\,636\,288w^{15} - 7193\,770\,328\,064w^{16} \\
& - 11\,814\,519\,701\,504w^{17} + 7399\,599\,505\,408w^{18} + 10\,981\,996\,494\,848w^{19} \\
& - 16\,439\,524\,196\,352w^{20} - 10\,434\,623\,045\,632w^{21} \\
& - 916\,975\,517\,696w^{22} + 1125\,281\,431\,552w^{23}, \\
P_2(w) = {} & -10 + 101w + 11\,088w^2 - 42\,855w^3 - 1117\,278w^4 - 1918\,516w^5 \\
& + 72\,221\,464w^6 + 460\,080\,656w^7 - 4999\,186\,016w^8 - 33\,474\,428\,224w^9 \\
& + 67\,440\,200\,320w^{10} + 808\,560\,558\,592w^{11} + 535\,166\,693\,376w^{12} \\
& - 6771\,457\,933\,312w^{13} - 7468\,556\,451\,840w^{14} + 46\,143\,514\,476\,544w^{15} \\
& + 91\,488\,863\,125\,504w^{16} - 75\,107\,733\,078\,016w^{17} - 239\,438\,663\,778\,304w^{18} \\
& + 31\,904\,728\,350\,720w^{19} + 234\,058\,806\,198\,272w^{20} - 237\,446\,193\,217\,536w^{21} \\
& - 164\,181\,567\,340\,544w^{22} - 18\,975\,165\,513\,728w^{23} + 16\,973\,710\,753\,792w^{24}, \\
P_1(w) = {} & -5 - 1142w + 8106w^2 + 210\,846w^3 - 1070\,376w^4 - 7771\,160w^5 \\
& - 22\,029\,952w^6 + 833\,894\,752w^7 + 3334\,510\,976w^8 - 39\,736\,449\,920w^9 \\
& - 156\,101\,859\,328w^{10} + 663\,306\,718\,208w^{11} + 2995\,615\,555\,584w^{12} \\
& - 5033\,154\,314\,240w^{13} - 26\,250\,785\,980\,416w^{14} + 28\,618\,066\,755\,584w^{15} \\
& + 158\,047\,775\,227\,904w^{16} - 42\,836\,217\,036\,800w^{17} - 410\,317\,620\,248\,576w^{18} \\
& - 95\,925\,074\,657\,280w^{19} + 462\,245\,318\,361\,088w^{20} - 328\,990\,199\,906\,304w^{21} \\
& - 249\,443\,110\,617\,088w^{22} - 35\,270\,271\,434\,752w^{23} + 24\,464\,133\,718\,016w^{24}, \\
P_0(w) = {} & -5 + 58w + 3234w^2 - 18\,994w^3 - 229\,330w^4 + 1516w^5 + 7017\,504w^6 \\
& + 74\,689\,472w^7 - 647\,069\,792w^8 - 4260\,373\,952w^9 + 15\,887\,163\,648w^{10} \\
& + 96\,789\,618\,688w^{11} - 136\,120\,508\,416w^{12} - 917\,765\,144\,576w^{13} \\
& + 877\,996\,605\,440w^{14} + 5646\,695\,006\,208w^{15} - 2888\,887\,697\,408w^{16} \\
& - 16\,785\,155\,817\,472w^{17} - 5241\,017\,729\,024w^{18} + 17\,952\,426\,426\,368w^{19} \\
& - 13\,058\,311\,192\,576w^{20} - 9329\,742\,708\,736w^{21} - 1275\,605\,286\,912w^{22} \\
& + 824\,633\,720\,832w^{23}.
\end{aligned}
\tag{B.3}
$$

B.2. Linear ODE for $\Phi_H^{(4)}$

The minimal order linear differential equation satisfied by $\Phi_H^{(4)}$ reads (with $x = 16w^2$)

$$\sum_{n=0}^{6} a_n(x)\frac{\mathrm{d}^n}{\mathrm{d}x^n}F(x) = 0, \tag{B.4}$$

where

$$a_6(x) = 64(x-4)(1-x)^4x^4P_6(x), \qquad a_5(x) = -128(1-x)^3x^3P_5(x),$$
$$a_4(x) = 16(1-x)^2x^2P_4(x), \qquad a_3(x) = -64(1-x)xP_3(x),$$
$$a_2(x) = -4P_2(x), \qquad a_1(x) = -8P_1(x), \qquad a_0(x) = -3(1-x)P_0(x),$$

with

$$\begin{aligned}
P_6(x) &= 128 + 2233x - 2847x^2 + 3143x^3 - 3601x^4 + 144x^5 - 64x^6,\\
P_5(x) &= 3712 + 51\,523x - 216\,377x^2 + 289\,918x^3 - 312\,896x^4 + 262\,111x^5\\
&\quad - 63\,167x^6 + 5512x^7 - 896x^8,\\
P_4(x) &= -121\,856 - 1102\,304x + 11\,038\,289x^2 - 26\,106\,487x^3 + 31\,515\,802x^4\\
&\quad - 31\,027\,694x^5 + 21\,291\,429x^6 - 5166\,011x^7 + 410\,160x^8 - 67\,776x^9,\\
P_3(x) &= 38\,144 + 10\,604x - 4644\,281x^2 + 20\,909\,702x^3 - 37\,890\,772x^4 + 42\,011\,874x^5\\
&\quad - 37\,552\,559x^6 + 22\,474\,036x^7 - 5465572x^8 + 392\,536x^9 - 65\,984x^{10},\\
P_2(x) &= 163\,840 - 4162\,688x - 18\,120\,152x^2 + 277\,110\,610x^3 - 880\,048\,289x^4\\
&\quad + 1357\,147\,519x^5 - 1395\,938\,590x^6 + 1141\,353\,668x^7 - 621\,323\,833x^8\\
&\quad + 150\,842\,795x^9 - 9676\,720x^{10} + 1656\,512x^{11},\\
P_1(x) &= -366\,592 + 3113\,752x + 17\,465\,700x^2 - 120\,658\,444x^3 + 240\,321\,805x^4\\
&\quad - 259\,277\,988x^5 + 219\,951\,814x^6 - 142\,314\,304x^7 + 42\,534\,921x^8\\
&\quad - 2056\,040x^9 + 435\,200x^{10},\\
P_0(x) &= 561\,152 - 1496\,400x - 13\,171\,575x^2 + 30\,840\,556x^3 - 24\,381\,198x^4\\
&\quad + 20\,352\,948x^5 - 13\,268\,091x^6 + 309\,360x^7 - 120\,000x^8.
\end{aligned}$$

B.3. Linear ODE modulo a prime for $\Phi_H^{(5)}$

The linear differential equation of minimal order 17 satisfied by $\Phi_H^{(5)}$ is of the form

$$\sum_{n=0}^{17} a_n(w)\frac{\mathrm{d}^n}{\mathrm{d}w^n}F(w) = 0, \tag{B.5}$$

with

$$\begin{aligned}
a_{17}(w) &= (1-4w)^{12}(1+4w)^9(1-w)^2(1+w)(1+2w)(1+3w+4w^2)^2(1-3w+w^2)\\
&\quad \times (1+2w-4w^2)(1+4w+8w^2)(1-7w+5w^2-4w^3)(1-w-3w^2+4w^3)\\
&\quad \times (1+8w+20w^2+15w^3+4w^4)w^{12}P_{17}(w),\\
a_{16}(w) &= w^{11}(1-4w)^{11}(1+4w)^8(1-w)(1+3w+4w^2)P_{16}(w),\\
a_{15}(w) &= w^{10}(1-4w)^{10}(1+4w)^7P_{15}(w),\\
a_{14}(w) &= w^9(1-4w)^9(1+4w)^6P_{14}(w),\\
&\ldots
\end{aligned}$$

where the 430 roots of $P_{17}(w)$ are *apparent singularities*. The degrees of these polynomials $P_n(w)$ are such that the degrees of $a_i(w)$ are decreasing as $\deg(a_{i+1}(w)) = \deg(a_i(w)) + 1$. In fact, with 2208 terms we have found the ODE of $\Phi_H^{((5)}$ at order $q = 28$ using the following ansatz for the linear ODE search (Dw denotes $\mathrm{d}/\mathrm{d}w$):

$$\sum_{i=0}^{q} s(i)p(i)Dw^i \tag{B.6}$$

with

$$s(i) = w^{\alpha(-1+i)}(1-16w^2)^{\alpha(-1+i)} s_0^{\alpha(1+i-q)} \tag{B.7}$$

where $\alpha(n) = \min(0, n)$ and

$$s_0 = (1+w)(1-w)(1+2w)(1-3w+w^2)(1+2w-4w^2)(1+3w+4w^2)(1+4w+8w^2)$$
$$\times (1-7w+5w^2-4w^3)(1-w-3w^2+4w^3)(1+8w+20w^2+15w^3+4w^4)$$

with $p(i)$ being the unknown polynomials.

The minimal order ODE is deduced from the set of linearly independent ODEs found at order 28.

B.4. Linear ODE modulo a prime for $\Phi_H^{(6)}$

The linear differential equation of minimal order 27 satisfied by $\Phi_H^{(6)}$ reads (with $x = w^2$)

$$\sum_{n=0}^{27} a_n(x) \frac{\mathrm{d}^n}{\mathrm{d}x^n} F(x) = 0, \tag{B.8}$$

with

$$\begin{aligned}
a_{27}(x) &= (1-16x)^{16}(1-4x)^3(1-x)(1-25x)(1-9x)x^{21}(1-x+16x^2) \\
&\quad \times (1-10x+29x^2)P_{27}(x), \\
a_{26}(x) &= (1-16x)^{15}(1-4x)^2x^{20}P_{26}(x), \\
a_{25}(x) &= (1-16x)^{14}(1-4x)x^{19}P_{25}(x), \\
a_{24}(x) &= (1-16x)^{13}x^{18}P_{24}(x), \\
&\ldots
\end{aligned} \tag{B.9}$$

where the 307 roots of $P_{27}(x)$ are *apparent singularities*. The degrees of the $P_n(w)$ polynomials are such that the degrees of $a_i(w)$ are decreasing as $\deg(a_{i+1}(w)) = \deg(a_i(w)) + 1$.

In fact, with 1838 terms we have found the linear ODE of $\Phi_H^{(6)}$ at order $q = 42$ using the following ansatz for the linear ODE search ($\mathrm{D}x$ denotes $\mathrm{d}/\mathrm{d}x$):

$$\sum_{i=0}^{q} s(i)p(i)\mathrm{D}x^i \tag{B.10}$$

with

$$s(i) = x^{\alpha(-1+i)}(1-16x)^{\alpha(-1+i)} s_0^{\alpha(1+i-q)} \tag{B.11}$$

where $\alpha(n) = \min(0, n)$ and

$$s_0 = (1-25x)(1-9x)(1-4x)(1-x)(1-x+16x^2)(1-10x+29x^2) \tag{B.12}$$

with $p(i)$ being the unknown polynomials.

The minimal order ODE is deduced from the set of linearly independent ODEs found at order 42.

Appendix C. Singularities in the linear ODE for $\Phi_H^{(7)}$ and $\Phi_H^{(8)}$

For $\Phi_H^{(7)}$, we generated long series (1250 coefficients and 20 000 coefficients modulo primes), unfortunately insufficient to obtain the corresponding linear ODE. However, by steadily increasing the order q of the ODE (and consequently decreasing the degrees n of the polynomials in front of the derivatives), one may recognize, in floating point form, the singularities of the ODE as the roots of the polynomial in front of the highest derivative. A root is considered a singularity of the still unknown linear ODE, if as q increases (and consequently decreasing n), it persists with more stabilized digits.

Using 1250 terms in the series for $\Phi_H^{(7)}$, the following singularities are recognized:

$$\begin{aligned}
&(1-4w)(1-5w+6w^2-w^3)(1+2w-8w^2-8w^3)(1+4w)w\\
&(1+2w-w^2-w^3)(1-3w+w^2)(1+2w-4w^2)(1+w)\\
&(1-3w-10w^2+35w^3+5w^4-62w^5+17w^6+32w^7-16w^8)\\
&(1+8w+15w^2-21w^3-60w^4+16w^5+96w^6+64w^7)\\
&(1-4w-16w^2-48w^3+32w^4-128w^5)\\
&(1-10w+35w^2-51w^3+21w^4-4w^5)\\
&(1-7w+5w^2-4w^3)(1+7w+26w^2+7w^3+4w^4)\\
&(1+8w+20w^2+15w^3+4w^4)\\
&(1+12w+54w^2+112w^3+105w^4+35w^5+4w^6)=0.
\end{aligned}$$

We will see in appendix E.3 that we missed the polynomials:

$$(1+3w+4w^2)(1+4w+8w^2)(1-w)(1+2w)(1-w-3w^2+4w^3). \quad \text{(C.1)}$$

Note that we have not seen with the precision of these calculations the occurrence of the singularities of $\Phi_H^{(3)}$.

With similar calculations using 1200 terms for $\Phi_H^{(8)}$, the following singularities are recognized:

$$\begin{aligned}
&(1-2w)(1+2w)(1-2w^2)(1-4w)(1-4w+2w^2)(1+4w)\\
&(1+4w+2w^2)(1-8w^2)(1-3w)(1-w)(1+w)(1+3w)w\\
&(1-26w^2+242w^4-960w^6+1685w^8-1138w^{10})\\
&(1-10w^2+32w^4)(1-30w^2+56w^4-1312w^6)\\
&(1-6w+10w^2)(1-6w+8w^2-4w^3)\\
&(1-5w)(1+2w^2)(1+5w)\\
&(1+6w+10w^2)(1+6w+8w^2+4w^3)=0.
\end{aligned}$$

We will see in appendix E.3 that we missed the polynomials:

$$(1-3w+4w^2)(1+3w+4w^2)(1-10w^2+29w^4).$$

Note that the stabilized digits in these singularities can be as low as two digits.

Appendix D. Landau conditions and pinch singularities for $\Phi_D^{(n)}$ and integrals of $\prod y_j$

Similarly to the integral representation (16) of $\Phi_H^{(n)}$, one has

$$\Phi_D^{(n)} = \int_0^{2\pi} \frac{\mathrm{d}\phi}{2\pi} \int_0^{2\pi} \frac{\mathrm{d}\psi}{2\pi} \frac{\sqrt{(1-2w\cos\phi)^2-4w^2}}{D(\phi,\psi)} \frac{\sqrt{(1-2w\cos((n-1)\phi))^2-4w^2}}{D((n-1)\phi,(n-1)\psi)}, \quad \text{(D.1)}$$

and

$$\prod_{i=1}^{n} y_i = \int_0^{2\pi} \prod_{i=1}^{n} \frac{\mathrm{d}\phi_i}{2\pi} \frac{\mathrm{d}\psi_i}{2\pi} \frac{1}{D(\phi_i,\psi_i)} \delta\left(\sum_{i=1}^{n} \phi_i\right). \quad \text{(D.2)}$$

For $\Phi_D^{(n)}$ the singularities of the associated ODEs are given as the solutions of

$$\begin{aligned} & D(\phi, \psi) = 0, \\ & D((n-1)\phi, (n-1)\psi) = 0, \\ & \alpha_1 \sin(\phi) + \alpha_2 \sin((n-1)\phi) = 0, \qquad \text{with} \qquad \alpha_1, \alpha_2 \neq 0, \\ & \alpha_1 \sin(\psi) + \alpha_2 \sin((n-1)\psi) = 0 \end{aligned} \tag{D.3}$$

which are nothing but the Landau conditions restricted to pinch singularities of the singularity manifolds $D(\phi_i, \psi_i) = 0$. For[23] $\prod_{i=1}^n y_i$, the singularities of the associated ODEs can be written as the solutions of

$$\begin{aligned} & D(\phi_i, \psi_i) = 0, \\ & \alpha_i \sin(\phi_i) - \alpha_n \sin(\phi_n) = 0, \quad i = 1, \ldots, n-1, \qquad \text{with} \quad \alpha_i \neq 0 \\ & \alpha_i \sin(\psi_i) = 0, \qquad i = 1, \ldots, N \end{aligned} \tag{D.4}$$

which are also Landau conditions restricted to pinch singularities of the singularity manifolds $D(\phi_i, \psi_i) = 0$.

Appendix E. The singularities from Landau conditions

In this appendix, we give further details corresponding to (28) and (29) obtained from the Landau conditions:

$$1 - 2w(\cos(\phi_j) + \cos(\psi_j)) = 0, \qquad j = 1, \ldots, n, \tag{E.1}$$

$$\alpha_j \sin(\phi_j) - \alpha_n \sin(\phi_n) = 0, \qquad j = 1, \ldots, n-1, \tag{E.2}$$

$$\alpha_j \sin(\psi_j) - \alpha_n \sin(\psi_n) = 0, \qquad j = 1, \ldots, n-1. \tag{E.3}$$

and

$$\sum_{j=1}^n \phi_j = 0, \qquad \sum_{j=1}^n \psi_j = 0 \qquad \text{mod } 2\pi. \tag{E.4}$$

We solve these equations for the values (zero or not) of $\sin(\phi_n)$ and $\sin(\psi_n)$. For $\sin(\phi_n) = \sin(\psi_n) = 0$, the case is simple and gives $w = \pm 1/4$.

E.1. The case $\sin(\phi_n) \neq 0, \sin(\psi_n) = 0$

In this case, there are k angles $\psi_j = \pi$ and the remaining ones are $\psi_j = 0$. By (17), the integer k should be even, $k = 2p$. From (E.1), we obtain and define[24]

$$\cos(\phi^+) = \frac{1}{2w} + 1, \qquad \cos(\phi^-) = \frac{1}{2w} - 1. \tag{E.5}$$

One obtains $2p$ angles $\phi_j = \pm\phi^+$ and $n - 2p$ angles $\phi_j = \pm\phi^-$. The angles ϕ_j are then partitioned in sets of p_1 angles $+\phi^+$, $(2p - p_1)$ angles $-\phi^+$, $(n - 2p - p_2)$ angles $+\phi^-$ and p_2 angles $-\phi^-$. By (E.4), one gets $(2p - 2p_1)\phi^+ = (n - 2p - 2p_2)\phi^-$. Note that

[23] The $\prod y_i$ or $\prod y_i^2$ integrands are similar as far as the location of the singularities of the corresponding ODEs is concerned.

[24] Note that ϕ^+ and ϕ^- (which correspond to $\psi_j = \pi$ and $\psi_j = 0$, respectively) are not on the same footing: indeed, the number of ϕ^+ angles must be even, while the number of ϕ^- angles depends on the parity of n.

some manipulations on the indices lead to $\cos(|2p|\phi^+) = \cos(|n-2p-2k|\phi^-)$ and thus $|2p|\phi^+ = \pm|n-2p-2k|\phi^-$, allowing us to write

$$\begin{aligned} &T_{2p}(1/2w+1) = T_{n-2p-2k}(1/2w-1), \\ &0 \leqslant p \leqslant [n/2], \qquad 0 \leqslant k \leqslant [n/2]-p, \end{aligned} \tag{E.6}$$

where $T_n(x)$ is the Chebyshev polynomial of the first kind.

One obtains the same results for the case $\sin(\phi_n) = 0$ and $\sin(\psi_n) \neq 0$.

E.2. The case $\sin(\phi_n) \neq 0, \sin(\psi_n) \neq 0$

In this case, by (E.2) and (E.3), we have $\sin(\phi_j) \neq 0$ and $\sin(\psi_j) \neq 0$. Equations (E.1) and (E.3) become

$$\cos(\psi_j) = 1 - 2w\cos(\phi_j), \qquad j = 1, \ldots, n, \tag{E.7}$$

$$\sin(\psi_j) = \sin(\phi_j)\frac{\sin(\psi_n)}{\sin(\phi_n)}, \qquad j = 1, \ldots, n. \tag{E.8}$$

Squaring both sides of both equations and summing, one obtains

$$(\cos(\phi_j) - \cos(\phi_n))(\cos(\phi_j) - \cos(\phi_0)) = 0, \tag{E.9}$$

where we have defined

$$\cos(\phi_0) = \frac{4w - \cos(\phi_n)}{1 - 4w\cos(\phi_n)}. \tag{E.10}$$

The angles ϕ_j are then partitioned into four sets $\pm\phi_0$ and $\pm\phi_n$. Note that a similar condition (E.9) occurs for the angles ψ_j which are partitioned likewise. Writing (E.7) and (E.8) for $j = 0$ and $j = n$ and with the conditions (17), the equations become in terms of Chebyshev polynomials[25]:

$$\begin{aligned} &T_{n_1}(z) - T_{n_2}\left(\frac{4w-z}{1-4wz}\right) = 0, \\ &T_{n_1}\left(\frac{1}{2w} - z\right) - T_{n_2}\left(\frac{1}{2w} - \frac{4w-z}{1-4wz}\right) = 0, \\ &U_{n_2-1}(z)U_{n_1-1}\left(\frac{1}{2w} - \frac{4w-z}{1-4wz}\right) - U_{n_2-1}\left(\frac{1}{2w} - z\right)U_{n_1-1}\left(\frac{4w-z}{1-4wz}\right) = 0 \end{aligned} \tag{E.11}$$

with

$$n_1 = p, \qquad n_2 = n - p - 2k, \tag{E.12}$$

$$0 \leqslant p \leqslant n, \qquad 0 \leqslant k \leqslant [(n-p)/2]. \tag{E.13}$$

At this step, some computational remarks are in order. In the course of deriving (E.11), some manipulations such as dividing by a term have been done. Rigorously, the solutions that come from (E.11) have to be checked against this point. We have found that as they are written, the formulae are 'safe' from this perspective, except of the following. For $n = p/2$ (fixing $k = 0$ for convenience), thus for n even, the formulae (E.11) give a common curve which reads

$$w = \frac{1}{2}\frac{z}{1+z^2}. \tag{E.14}$$

[25] Note that in equation (E.11) one must realize that one takes the numerator of these rational expressions.

This relation comes from the condition $\cos(\phi_0) = \cos(\phi_n)$ in (E.10) which makes (E.9) a perfect square. We have checked that considering this condition at the outset, i.e. (E.7) and (E.8), yields no solution.

E.3. Landau singularities

We can write the singularities obtained from (E.6) as

$$\begin{aligned}
n &= 3, \qquad (1-4w)(1-w)(1+3w+4w^2) = 0,\\
n &= 4, \qquad (1-16w^2)(1-4w^2) = 0,\\
n &= 5, \qquad (1-4w)(1-w)(1+3w+4w^2)(1-3w+w^2)(1-7w+5w^2-4w^3)\\
&\qquad\quad \times (1+8w+20w^2+15w^3+4w^4) = 0,\\
n &= 6, \qquad (1-16w^2)(1-4w^2)(1-w^2)(1-25w^2)(1-9w^2)\\
&\qquad\quad \times (1+3w+4w^2)(1-3w+4w^2) = 0.
\end{aligned}$$

The solutions of (E.11) include some of the solutions of (E.6). We give in the following only those not occurring in (E.6):

$$\begin{aligned}
n &= 3, \qquad w(1+4w)(1+2w) = 0,\\
n &= 4, \qquad w = 0,\\
n &= 5, \qquad w(1+4w)(1+w)(1+2w)(1+2w-4w^2)(1+4w+8w^2)\\
&\qquad\quad \times (1-w-3w^2+4w^3) = 0,\\
n &= 6, \qquad w(1-10w^2+29w^4) = 0.
\end{aligned}$$

All these singularities can be identified with the singularities occurring in the linear ODE for $\Phi_H^{(n)}$ $(n = 3, \ldots, 6)$.

For $n = 7$ and $n = 8$, the solutions of (E.6) and (E.11) can be identified with the singularities given in appendix C and obtained in floating point form. They also give

$$\begin{aligned}
n &= 7, \qquad (1+3w+4w^2)(1+4w+8w^2)(1-w)(1+2w)(1-w-3w^2+4w^3) = 0,\\
n &= 8, \qquad (1-3w+4w^2)(1+3w+4w^2)(1-10w^2+29w^4) = 0,
\end{aligned}$$

which have not been found in the series with the currently available number of terms.

Appendix F. Heegner numbers and other selected values of the modular j-function

The nine Heegner numbers [51] and their associated modular j-function $j(\tau)$ yield the following conditions in the variable w:

$$\begin{aligned}
&j(1+i) = (12)^3, \qquad (1-8w^2)(1-16w^2-8w^4) = 0,\\
&j(1+i\sqrt{2}) = (20)^3, \qquad (1-16w^2-64w^4)\\
&\qquad\qquad \times (1-32w^2+368w^4-1792w^6-64w^8) = 0,\\
&j\left(\frac{1+i\sqrt{3}}{2}\right) = (0)^3, \qquad 1-16w^2+16w^4 = 0,\\
&j\left(\frac{1+i\sqrt{7}}{2}\right) = (-15)^3, \qquad (1-31w^2+256w^4)\left(1-16w^2+w^4\right)\\
&\qquad\qquad \times (1+3w+4w^2)(1-3w+4w^2) = 0,
\end{aligned}$$

$$j\left(\frac{1+i\sqrt{11}}{2}\right) = (-32)^3, \qquad P_3 = 1 - 48w^2 + 816w^4 - 5632w^6$$
$$+45\,824w^8 - 536\,576w^{10} + 4096w^{12} = 0,$$

and

$$j\left(\frac{1+i\sqrt{d}}{2}\right) = (-m)^3, \qquad P_d = 0 \qquad \text{with} \qquad P_d = P_3 + N(1-16w^2)w^8,$$

with the following values for the triplet (d, m, N):

$$(19, 96, 851\,968), \qquad (43, 960, 884\,703\,232),$$
$$(67, 5280, 147\,197\,919\,232), \qquad (163, 640\,320, 262\,537\,412\,640\,735\,232).$$

Beyond Heegner numbers there are many other selected quadratic values [52, 53] of j, for instance

$$j = -4096(15+7\sqrt{5})^3 = j\left(\frac{1+\mathrm{i}\sqrt{35}}{2}\right) \tag{F.1}$$

which is known [51] to be one of the 18 numbers having class number $h(-d) = 2$, and which corresponds to the quadratic relation $-134\,217\,728\,000 + 117\,964\,800j + j^2 = 0$. Recalling the expression of the modular j-function in term of the variable w,

$$j = \frac{(1-16w^2+16w^4)^3}{(1-16w^2)w^8}, \tag{F.2}$$

this quadratic relation in j becomes a quite involved polynomial expression that we have not seen emerging as the singularities of (the linear ODEs of) our n-fold integrals.

Appendix G. Landen transformations and the modular j-function

In this appendix the modular j-function (52) will be seen, alternatively, as a function of the modulus k, and thus denoted by $j[k]$, or as a function of the half-period ratio τ, and thus denoted by $j(\tau)$. The modular function called the j-function when seen as a function of the modulus k reads

$$j[k] = 256\frac{(1-k^2+k^4)^3}{k^4(1-k^2)^2}. \tag{G.1}$$

Increasing the modulus by (47), the modular function $j(k)$ becomes

$$j[k_1] = j_1[k] = 16\frac{(1+14k^2+k^4)^3}{k^2(1-k^2)^4}. \tag{G.2}$$

Iterating this procedure once more one obtains

$$j_1[k_1] = j_2[k] = 4\frac{(1+60k+134k^2+60k^3+k^4)^3}{k(1+k)^2(1-k)^8}. \tag{G.3}$$

The decrease of the modulus by (46) gives

$$j[k_{-1}] = j_{-1}[k] = 16\frac{(16-16k^2+k^4)^3}{k^8(1-k^2)}. \tag{G.4}$$

The next iterations (the cube of (47) and the square of (46)) gives algebraic expressions for $j[k]$.

It is easy to get a *representation of the Landen transformation on the modular j-functions* by the elimination of the modulus k between (52) and (G.2). One obtains the well-known fundamental *modular curve* [49, 50]

$$\Gamma_1(j, j_1) = j^2 j_1^2 - (j + j_1)\left(j^2 + 1487\, j j_1 + j_1^2\right) + 315^3\left(16 j^2 - 4027\, j j_1 + 16 j_1^2\right) - 1230^6 (j + j_1) + 830^9 = 0. \tag{G.5}$$

This algebraic curve is *symmetric* in j and j_1. We will obtain the same modular curve (G.5) by the elimination of the modulus k between (G.2) and (G.3) or between (G.1) and (G.4). The two modular functions j and j_1 are invariant by the $SL(2, Z)$ modular group and, in particular, transformation $\tau \rightarrow 1/\tau$. As a consequence, the transformation $\tau \rightarrow 2\tau$, and its inverse $\tau \rightarrow \tau/2$, *has to be on the same footing* in the modular curve representation (G.5) for the Landen and Gauss transformations.

Similarly, one can easily find the (genus zero) modular curve Γ_2 obtained by the elimination of the modulus k between (G.1) and (G.3) (or between (G.4) and (G.4)), which corresponds to the transformation $\tau \rightarrow 4\tau$ and, *at the same time*, to its inverse $\tau \rightarrow \tau/4$. This last algebraic curve is, of course, *also a modular curve*.

G.1. Fixed points of these modular representations in terms of j-function

Transformations such as $j \rightarrow j_1$, or $j \rightarrow j_2$, corresponding to the previous modular curves, are not (one-to-one) mappings, they are called ‘correspondence’ by Veselov [23, 24]. In order to look at the fixed points of the Landen, Gauss transformations (or their iterates) seen as transformations on *complex variables*, within the framework of (modular) representations on the modular j-functions, we write, respectively, $\Gamma_1(j, j_1 = j) = 0$ and $\Gamma_2(j, j_2 = j) = 0$

The ‘fixed points’ $\Gamma_1(j, j_1 = j) = 0$ of the (modular) ‘correspondence’ (G.5) are $j = j_1 = (12)^3$ or $(20)^3$ or $(-15)^3$.

The ‘fixed points’ $\Gamma_2(j, j_2 = j) = 0$ of modular curve corresponding to the square of the Landen transformation are $j = j_2 = (66)^3$ or $2(30)^3$ or $(-15)^3$ or the solutions[26] of $j^2 + 191\,025 j - 121\,287\,375 = 0$, namely,

$$j = -3^3\left(\frac{1+\sqrt{5}}{2}\right)^2 (5 + 4\sqrt{5})^3 = j\left(\tau = \frac{1 + i\sqrt{15}}{2}\right) \tag{G.6}$$

and its Galois conjugate (change $\sqrt{5}$ into $-\sqrt{5}$).

G.2. Alternative approach to fixed points of the Landen transformation and its iterates

In order to get the ‘fixed points’ of the Landen transformation, let us impose that (G.1) and (G.2) are actually equal, thus $j[k] = j[k_1]$. This yields the condition (already seen to correspond to the $\chi^{(3)}$-singularities $1 + 3w + 4w^2 = 0$)

$$(4k^2 + 3k + 1)(k^2 + 3k + 4) = 0 \tag{G.7}$$

together with

$$\begin{aligned} &(4k^2 - 3k + 1)(k^2 - 3k + 4) \\ &(k^2 + 2k - 1)(k^2 - 2k - 1)(k^2 + 1) = 0. \end{aligned} \tag{G.8}$$

The first two polynomial conditions in (G.8), $(4k^2-3k+1)(k^2-3k+4) = 0$, correspond to the Heegner number associated with the integer value $j = (-15)^3$. The next two polynomial

[26] This corresponds to a value of j of class number $h(-d) = 2$, see (58) in [51].

conditions in (G.8), $k^2 \pm 2k - 1 = 0$, correspond to the Heegner number associated with the integer value $j = (20)^3$. The last polynomial condition in (G.8), $1 + k^2 = 0$, corresponds to the Heegner number associated with the integer value $j = (12)^3$.

Similarly, in order to get the 'fixed points' of the square of the Landen transformation, let us require that (G.1) and (G.3) are actually equal: $j[k] = j[k_2]$. This yields the conditions (G.7) (fixed points of the Landen transformation) together with

$$(k^2 - 6k + 1)(k^4 + 14k^2 + 1) = 0 \tag{G.9}$$

$$(k^4 - 6k^3 + 17k^2 + 36k + 16)(16k^4 + 36k^3 + 17k^2 - 6k + 1) = 0. \tag{G.10}$$

In (G.9) the condition $1+14k^2+k^4 = 0$ (or $1-16w^2+256w^4 = 0$) corresponds to $j = 2(30)^3$ which is not a Heegner number but actually corresponds to complex multiplication. The condition $k^2 - 6k + 1 = 0$ in (G.9) (or $1 - 32w^2 = 0$) corresponds to $j = (66)^3$ which is not a Heegner number either but actually corresponds to complex multiplication. Note that both polynomials under the Landen transformation (47) give respectively $j = (0)^3$ and $j = (12)^3$, i.e. Heegner numbers. The last two (self-dual) conditions in (G.10) read in w

$$\begin{aligned} &1 - 9w + 17w^2 + 24w^3 + 6w^4 = 0, \\ &1 + 9w + 17w^2 - 24w^3 + 6w^4 = 0 \end{aligned} \tag{G.11}$$

and yield as selected value [52, 53] of j, the quadratic roots $-121\,287\,375 + 191\,025\,j + j^2 = 0$, already given in (G.6).

One more step can be performed writing the condition $j[k_{-1}] = j[k_2]$. One gets the conditions

$$(k^2 + 3k + 4)^2(4k^2 - 3k + 1)(k^2 + 2k - 1)(k^2 + 1) = 0$$

previously obtained and corresponding to $j = (-15)^3, 20^3, 12^3$, together with

$$\begin{aligned} &k^6 - 27k^5 + 363k^4 + 423k^3 - 168k^2 - 144k + 64 = 0, \\ &k^6 + 17k^5 + 143k^4 + 203k^3 + 52k^2 + 32k + 64 = 0 \end{aligned} \tag{G.12}$$

corresponding, respectively, to the two cubic relations on j:

$$\begin{aligned} &1566\,028\,350\,940\,383 - 58\,682\,638\,134\,j + 39\,491\,307\,j^2 + j^3 = 0, \\ &12\,771\,880\,859\,375 - 5151\,296\,875\,j + 3491\,750\,j^2 + j^3 = 0. \end{aligned} \tag{G.13}$$

These conditions (G.13) yield quite involved polynomial expressions in the variable w that we have not seen emerging as the singularities of (the linear ODEs of) our n-fold integrals (or $Y^{(n)}$ or $\Phi^{(n)}$ either).

Appendix H. Linear differential operators for the Sorokin integrals

Recall the occurrence of zeta functions evaluated at integer values in many n-fold integrals corresponding to particle physics, field theory, ... For instance, the following integral [33, 36] is associated with $\zeta(3)$:

$$I_n(z) = \int_0^1 \mathrm{d}u\, \mathrm{d}v\, \mathrm{d}w\, \frac{(1-u)^n u^n (1-v)^n v^n (1-w)^n w^n}{(1-uv)^{n+1}(z-uvw)^{n+1}}. \tag{H.1}$$

From the series expansion of this holonomic n-fold integral, we have obtained the corresponding order four Fuchsian linear differential equation. On these linear differential operators the 'logarithmic' nature of these integrals becomes clear.

The fully integrated series expansion of the triple integral (H.1) is given by (where x denotes $1/z$)

$$I_n(x) = \sum_{i=0}^{\infty} x^{n+i+1} \frac{\Gamma^2(n+1)\Gamma^4(n+i+1)}{\Gamma(i+1)\Gamma^3(2+2n+i)} \times {}_3F_2(n+1, n+i+1, n+i+1; 2n+i+2, 2n+i+2; 1).$$

The triple integral $I_n(x)$ is a solution of the order four Fuchsian linear differential operator ($\mathrm{D}x$ denotes $\mathrm{d}/\mathrm{d}x$)

$$L_n = \mathrm{D}x^4 + \frac{2(3x-1)}{(x-1)x}\mathrm{D}x^3 + \frac{(7x^2 + (n^2+n-5)x - 2n(n+1))}{(x-1)^2x^2}\mathrm{D}x^2 + \frac{(x^2 + 2n(n+1))}{(x-1)^2x^3}\mathrm{D}x + \frac{n(n+1)((n^2+n+1)x + (n-1)(n+2))}{(x-1)^2x^4}$$

which has the following factorization:

$$L_n = \left(\mathrm{D}x + \frac{\mathrm{d}\ln(A_1)}{\mathrm{d}x}\right)\left(\mathrm{D}x + \frac{\mathrm{d}\ln(A_2)}{\mathrm{d}x}\right)\left(\mathrm{D}x + \frac{\mathrm{d}\ln(A_3)}{\mathrm{d}x}\right)\left(\mathrm{D}x + \frac{\mathrm{d}\ln(A_4)}{\mathrm{d}x}\right) \qquad \text{(H.2)}$$

where

$$\begin{aligned}
A_1 &= -(n-1)\ln(x) + 2\ln(x-1) + \ln(P_n),\\
A_2 &= (n+1)\ln(x) - (n-1)\ln(x-1) - \ln(P_n) + \ln(Q_n),\\
A_3 &= -n\ln(x) + (n+1)\ln(x-1) + \ln(P_n) - \ln(Q_n),\\
A_4 &= n\ln(x) - \ln(P_n),
\end{aligned}$$

and where P_n and Q_n are the polynomials in x of degree n. They are the polynomial solutions behaving as $\cdots + x^n$ for a system of coupled differential equations ($P_n^{(m)}$ (resp. $Q_n^{(m)}$) denotes the m th derivative of $P_n(x)$ (resp. $Q_n(x)$) with respect to x):

$$\begin{aligned}
&(x-1)^2x^2P_n^{(4)} - 2(2(x-1)n - 3x + 1)(x-1)xP_n^{(3)}\\
&\quad + ((2x-1)(3x-4)n^2 - (12x^2 - 13x + 2)n + (7x-5)x)P_n^{(2)}\\
&\quad - (2(2x-3)n^3 - 2(3x-1)n^2 + 2(2x-1)n - x)P_n^{(1)} + n^4P_n = 0,\\
&-(x-1)xP_nQ_n^{(2)} + (2(x-1)xP_n^{(1)} + (1 - x + 2n)P_n)Q_n^{(1)}\\
&\quad - (2(x-1)xP_n^{(2)} - 2((x-2)n - x)P_n^{(1)} + n^2P_n)Q_n = 0.
\end{aligned}$$

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