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Fuchs versus Painlevé

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Abstract
We, briefly, recall the Fuchs–Painlevé elliptic representation of Painlevé VI. We then show that the polynomiality of the expressions of the correlation functions (and form factors) in terms of the complete elliptic integral of the first and second kinds, K and E, is a straight consequence of the fact that the differential operators corresponding to the entries of Toeplitz-like determinants are equivalent to the second-order operator L_E which has E as solution (or for off-diagonal correlations to the direct sum of L_E and $\mathrm{d}/\mathrm{d}t$). We show that this can be generalized, mutatis mutandis, to the anisotropic Ising model. The singled-out second-order linear differential operator L_E is replaced by an isomonodromic system of two third-order linear partial differential operators associated with Π_1, the Jacobi's form of the complete elliptic integral of the third kind (or equivalently two second-order linear partial differential operators associated with Appell functions, where one of these operators can be seen as a deformation of L_E). We finally explore the generalizations, to the anisotropic Ising models, of the links we made, in two previous papers, between Painlevé nonlinear ODEs, Fuchsian linear ODEs and elliptic curves. In particular, the elliptic representation of Painlevé VI has to be generalized to an 'Appellian' representation of Garnier systems.

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1. Introduction

In a previous paper [1] we have shown that the *diagonal* two-point correlation functions of the square Ising model are simultaneously solutions of a nonlinear differential equation associated with (the sigma form of[6]) Painlevé VI and solutions of Fuchsian linear differential equations. In a following paper [2] we have also shown that some one-parameter λ-extension of the diagonal two-point correlation functions (which also satisfy the same nonlinear differential equations associated with the sigma form of Painlevé VI) are such that their coefficients in the λ-series, the so-called form factors, also verify, in a rather unexpected way, Fuchsian linear differential equations [2]. More precisely, introducing the second-order differential operator L_E, associated [1, 2] with the complete elliptic integral of the second kind E, the linear differential operators forming these two sets of Fuchsian ODEs (for the two-point correlation functions and for the form factors) were seen to be equivalent[7] to direct sums of linear differential operators equivalent to symmetric powers of L_E or simply (for diagonal correlations) to symmetric powers of L_E. As a consequence, the two-point correlation functions, as well as the previously mentioned form factors [2], are polynomials expressions of K and E, the complete elliptic integral of the first and second kinds. These results underline the key role played by the second-order differential operator L_E, and this can be seen to be in perfect agreement with the *elliptic representation* of the Painlevé VI equations [2]. A surprisingly large amount of information on correlation functions and form factors is thus 'encapsulated' in this second-order differential operator L_E. This suggests two sets of work to be performed.

First, we revisit our two previous papers [1, 2] in order to show that the results displayed in these two papers are, in the case of the *isotropic*[8] Ising model, direct consequences of the natural occurrence of the second-order linear differential operator L_E.

Secondly, we try to generalize these calculations to the *anisotropic* Ising model, to see if a similar 'scheme' can be generalized, mutatis mutandis. In the *anisotropic* Ising model, the linear differential operators for the two-point correlation functions, and the form factors, could be, again, direct sums of linear differential operators equivalent to symmetric powers of new linear partial differential systems to be discovered. As a by-product, the finding of these new linear partial differential operators, generalizing L_E, would indicate the proper generalization of the elliptic representation of Painlevé VI, for the anisotropic Ising model (and, more generally, for integrable lattice models with a canonical elliptic parametrization of their Yang–Baxter equations, like the Baxter model). What is the 'natural' generalization of the elliptic representation of Painlevé VI for off-diagonal correlation functions and, their λ-extensions, for the *anisotropic* Ising model? Should we introduce higher order Painlevé ODEs (in analogy to the higher order KdV generalization of KdV)? Should we consider Garnier systems [5] or, even, more general Schlesinger systems [6, 7]?

This paper is organized as follows. We will, briefly, recall the Fuchs elliptic representation of Painlevé VI, and then show that correlation functions being polynomial expressions in the complete elliptic integrals E and K, is a straight consequence of the fact that the linear differential operators corresponding to the entries of some Toeplitz-like determinants, are equivalent to the second-order linear differential operator L_E (or for off-diagonal correlations are equivalent to the direct sum of L_E and $\mathrm{d}/\mathrm{d}t$). We will show that these previous calculations

[6] More precisely, the σ associated with the log-derivative of the diagonal two-point correlation function is solution of the sigma form of Painlevé VI.

[7] In the sense of the equivalence of linear differential operators [3, 4].

[8] Or the anisotropic Ising model but *only for diagonal* two-point correlation functions: the diagonal correlations are only a function of *only one variable*, the modulus k, and not of the anisotropy of the model.

can be generalized, mutatis mutandis, to the *anisotropic* Ising model, the singled-out second-order linear differential operator L_E being replaced by a system of two (isomonodromic) *third-order* linear partial differential operators corresponding to the elliptic integral of the third kind Π, or, equivalently, two (isomonodromic) *second-order* linear partial differential operators corresponding to *Appell functions*. We will finally explore the generalizations, to the anisotropic Ising models, of the links we made in the two previous papers, between Painlevé nonlinear ODEs, Fuchsian linear ODEs [1] and elliptic curves [2]. We will suggest that the elliptic representation of Painlevé VI has to be generalized to an 'Appellian' representation of *Garnier systems*.

2. About Painlevé VI

2.1. Ising model and the sigma form of Painlevé VI

For concreteness we first recall the specific sigma form of Painlevé VI obtained by Jimbo and Miwa [14] for the diagonal two-point Ising correlation $C(N, N)$:

$$(t(t-1)\sigma'')^2 = N^2 \cdot ((t-1)\sigma' - \sigma)^2 - 4\sigma' \cdot ((t-1)\sigma' - \sigma - 1/4) \cdot (t\sigma' - \sigma). \qquad (1)$$

The diagonal correlation $C_N = C(N, N)$ is related to σ, for $T > T_c$, by [1]

$$\sigma(t) = t \cdot (t-1) \cdot \frac{\mathrm{d}}{\mathrm{d}t} \log(C_N) - \frac{1}{4} \qquad \text{with} \quad t = (\sinh(2J_v/kT) \cdot \sinh(2J_h/kT))^2 < 1 \qquad (2)$$

and, for $T < T_c$, by

$$\sigma(t) = t \cdot (t-1) \cdot \frac{\mathrm{d}}{\mathrm{d}t} \log(C_N) - \frac{t}{4} \qquad \text{with} \quad t = (\sinh(2J_v/kT) \cdot \sinh(2J_h/kT))^{-2} < 1 \qquad (3)$$

where the variable J_v (J_h) is the Ising model vertical (horizontal) coupling constant.

2.2. Fuchs–Painlevé elliptic representation of Painlevé VI

Let us introduce K and E, the complete elliptic integral of the first kind and of the second kind, that we multiply[9] by $2/\pi$ in order to have series with integer coefficients:

$$K(t) = {}_2F_1(1/2, 1/2; 1; t), \qquad E(t) = {}_2F_1(1/2, -1/2; 1; t). \qquad (4)$$

Let us also introduce the second-order differential operator corresponding to E ($\mathrm{D}t$ denotes the derivative with respect to t: $\mathrm{D}t = \mathrm{d}/\mathrm{d}t$):

$$L_E = \mathrm{D}t^2 + \frac{\mathrm{D}t}{t} - \frac{1}{4(t-1)t}. \qquad (5)$$

In order to understand the key role played by L_E, let us first recall (see [8] or for a review [9]) the so-called Fuchs–Painlevé '*elliptic representation*' of Painlevé VI. This elliptic representation of Painlevé VI amounts to seeing Painlevé VI as a 'deformation' (see equation (33) in [9]) of the hypergeometric linear differential equation associated with the second-order linear differential operator:

$$\mathcal{L} = (1-t)t \cdot \mathrm{D}t^2 + (1-2t) \cdot \mathrm{D}t - \tfrac{1}{4}. \qquad (6)$$

One easily verifies that this linear differential operator has the complete elliptic integral of the first kind K as solution. We will denote by L_K the second-order operator defined by

[9] In maple's notations, for $t = k^2$ (k is the modulus), $K(t) = K(k^2)$ in (4) reads: *hypergeom* ([1/2, 1/2], [1], t) = $2/\pi \cdot$ *EllipticK*(k), but reads $2/\pi \cdot$ *EllipticK*$[k^2]$ in Mathematica.

$\mathcal{L} = (1-t)t \cdot L_K$. The operator $\mathcal{L}$ is actually equivalent (in the sense of the equivalence of linear differential operators [3, 4]) with L_E:

$$L_E \cdot (2(t-1)t \cdot \mathrm{D}t + t - 1) = \left(-2 \cdot \mathrm{D}t - \frac{3}{t}\right) \cdot \mathcal{L}. \tag{7}$$

This deep relation between *elliptic curves and Painlevé VI* explains the occurrence of Painlevé VI on the isotropic Ising model and on other lattice Yang–Baxter integrable models which are canonically parametrized in terms of *elliptic functions* (like the eight-vertex Baxter model, the RSOS models, see for instance [10]).

3. Fuchsian linear ODEs for Ising two-point correlations

In this section, we prove the polynomiality of the two-point correlation functions in E and K, in a way that underlies differential algebra and the equivalence of linear differential operators (see (12)), since this approach can easily (but tediously) be generalized to the anisotropic Ising model (see section (5.2)).

For the isotropic square Ising model we consider the regime $T > T_c$, and we use the same notations[10] as in [1, 2], namely $s = \sinh(2K)$ and $t = k^2 = s^4$ (k is the modulus of the elliptic functions). We will use, alternatively, the two variables t and s (according to the quantity we study: for off-diagonal two-point correlations the s variable is better suited). The diagonal two-point correlation functions of the square Ising model $C(N, N)$, and its dual $C^*(N, N)$, can be calculated from Toeplitz determinants [11–13]:

$$C(N, N) = \det(a_{i-j}), \qquad 1 \leqslant i, j \leqslant N \tag{8}$$

$$C^*(N, N) = (-1)^N \cdot \det(a_{i-j-1}), \qquad 1 \leqslant i, j \leqslant N \tag{9}$$

where a_n's read in terms of ${}_2F_1$ hypergeometric function for $n \geqslant -1$

$$a_n = -\frac{(-1/2)_{n+1}}{(n+1)!} \cdot t^{n/2+1/2} \cdot {}_2F_1(1/2, n+1/2; n+2; t), \tag{10}$$

and for $n \leqslant -1$

$$a_n = -\frac{(1/2)_{-n-1}}{(-n-1)!} \cdot t^{-n/2-1/2} \cdot {}_2F_1(-1/2, -n-1/2; -n; t),$$

where $(\alpha)_n$ is the usual Pochhammer symbol.

Introducing the second-order linear differential operator,

$$H_n = \mathrm{D}t^2 + \frac{1}{t} \cdot \mathrm{D}t - \frac{n^2 t - (n+1)^2}{4(t-1)t^2}, \tag{11}$$

one can verify that $H_n(a_n) = 0$. One sees that these second-order linear differential operators H_n are all equivalent (in the sense of the equivalence of linear differential operators [3, 4]) over $C(\sqrt{t})$. Indeed, for consecutive H_n, we have

$$H_n \cdot Z_n = R_n \cdot H_{n-1} \tag{12}$$

with

$$Z_n = \sqrt{t} \cdot \left((t-1) \cdot \mathrm{D}t + \frac{(n-1) \cdot t + n}{2t}\right). \tag{13}$$

[10] We apologize for possible repetition of material appearing in this section and some relevant parts of [1]. We consider that the reader may not be familiar with differential algebra concepts, in particular the notion of equivalence of linear differential operators.

We then find an intertwinning relation between H_n and H_{n-2}. Letting $\tilde{Z}_2$ denote the remainder of the right division of $Z_n \cdot Z_{n-1}$ by H_{n-2}, we find that $H_n \cdot \tilde{Z}_2 = \tilde{R}_2 \cdot H_{n-2}$. Iteratively, we find an intertwinner between H_n and H_0 that way (the same process is easily achieved the same way for negative values of n). The degree in $\sqrt{t}$ of this intertwinner grows linearly.

It follows that all H_n are equivalent (over $C(\sqrt{t})$) to the second-order differential operator L_E. Actually, the second-order differential operator L_E can be seen to be *nothing else but* H_n for $n = -1$. The equivalence (12) remains valid between H_n for $n = 1$ and $n = -1$, and, furthermore, the equivalence between $L_E = H_n(n = -1)$ and $H_n(n = 0) = L_{11}$ had been seen in [1] (L_{11} is the linear differential operator corresponding to $C(1, 1)$).

These equivalences of linear differential operators can be expressed on the entries a_n (solutions of H_n):

$$a_n = \frac{((n-1)t + n)}{2\sqrt{t}} \cdot a_{n-1}(t) + \sqrt{t} \cdot (t-1) \cdot a'_{n-1}(t). \qquad (14)$$

Considering the fact that all a_{n-i} satisfy a second-order linear differential equation (namely $H_{n-i}(a_{n-i}) = 0$), we see that the above equivalences also imply that

$$a_n(t) = t^{-n/2} \cdot (p_n(t) \cdot E(t) + q_n(t) \cdot K(t)) \qquad (15)$$

with p_n, q_n being the polynomials in t. Now, we have seen that the correlation functions could be seen as Toeplitz determinants in a_n; so we recover the fact that $C(N, N)$, and $C^*(N, N)$, are (homogeneous) polynomials in E and K. This proof will be generalized in later sections.

Remark. The fact that the diagonal correlations $C(N, N)$ are homogeneous polynomials of the first and second complete elliptic functions K and E is seen, here, as a simple consequence of the Toeplitz determinant representation and the contiguity relations for hypergeometric functions a_n. Note that it can probably also be seen as obvious for some specialists of Painlevé, from the recurrence relations $N \mapsto N + 1$ given in Jimbo and Miwa [14] and from the work of Forrester and Witte [15].

4. The isotropic Ising model

In [1] it was shown that the diagonal two-point correlation functions $C(N, N)$ satisfy Fuchsian linear differential equations of order $N+1$. Recalling the $\sigma(t)$ variables defined by (2) and (3), the compatibility between these $N+1$ order Fuchsian linear differential equations and (1), the sigma form of Painlevé VI, actually corresponds to polynomial relations [1], $P(\sigma', \sigma, t) = 0$, which, seen as functions of σ' and σ (seeing t as a parameter), are *algebraic curves of genus zero*.

The fact that there are algebraic relations between $\sigma(t)$ and $\sigma'(t)$ for some classical solutions of the sixth Painlevé system can be seen as a consequence of the fact that classical solutions[11] are related by birational Bäcklund transformations (in some Hamiltonian variables q, p) to a seed solution which is itself determined by a solution to a specific Riccati equation: such algebraic relations are implied for the nth iterate of the Bäcklund transformation.

Let us recall the $N = 2$ case detailed in [1]. The elimination of the variable $S_2 = \sigma''(t)$ between the 'generalized Riccati form' of the Fuchsian ODE and (1), but seen as a polynomial relation between the three variables[12] S_0, S_1 and S_2, yields an algebraic relation between

[11] Classical solutions are functions obtained by finite numbers of differentiations, arithmetic calculations, substitution into Abelian functions, as well as solving homogeneous linear differential equations [16].

[12] In the spirit of the 'differential algebra' [17, 18], one performs as much algebraic geometry calculations as possible in the nth derivative $S_n = \sigma^{(n)}(t)$ considered as *independent variables*. It is only at the last step that one recalls that there is some differential structure by imposing, for instance, that the variable S_1 is actually the derivative with respect to t of the variable S_0.

$S_0 = \sigma(t)$ and $S_1 = \sigma'(t)$ which reads the *rational curve*

$$(4S_0 - 3)\big(64S_0^3 - 16(16t+1)S_0^2 + 4(64t^2 - 16t - 21)\cdot S_0 + 45\big)$$
$$- 32t(4S_0 - 3)(t-1)(8t - 1 - 4S_0)\cdot S_1 + 256t^2(t-1)^2\cdot S_1^2 = 0 \qquad (16)$$

which is, actually, the compatibility condition between the Fuchsian linear differential equation for $C(2, 2)$ and the nonlinear differential equation (1). This can be checked by eliminating S_2 between the derivative of (16) and the Fuchsian linear differential equation for $C(2, 2)$, or (1), to get again (16). This can also be checked directly by plugging a series expansion or an exact expression of $C(2, 2)$ into (16).

Let us now consider the $N = 3$ case and the corresponding compatibility condition between the Fuchsian linear differential equation for $C(3, 3)$ and equation (1), the sigma form of Painlevé VI. The compatibility condition also corresponds to a polynomial relation between S_0 and S_1, and has been written in [1].

Seen as a relation between S_0 and S_1 (considering t as a parameter), the corresponding algebraic curve is again a *rational curve*. It can thus be parametrized in terms of two rational functions of a parameter u:

$$S_0 = \frac{N_S}{D_S}, \qquad (17)$$

where

$$D_S = 4u^3 + 8192(26t+11)(t-1)(t-9)t^2\cdot u^2$$
$$+ 16\,777\,216(19 + 68t + 220t^2)(t-1)^2(t-9)^2t^4\cdot u$$
$$+ 103\,079\,215\,104(3 - 178t - 140t^2 + 200t^3)(t-1)^3(t-9)^3t^6,$$
$$N_S = (5+6t)\cdot u^3 + 2048(t-1)(t-9)(148t^2 + 268t + 55)t^2\cdot u^2$$
$$+ 4194\,304(10t+19)(116t^2 + 44t + 5)(t-1)^2(t-9)^2t^4\cdot u$$
$$+ 8589\,934\,592\cdot(45 - 4560t - 8192t^2 + 4640t^3$$
$$+ 2800t^4)\cdot(t-1)^3(t-9)^3t^6,$$
$$S_1 = \frac{4\cdot W_1\cdot W_2}{(t-1)\cdot D_S^2}, \qquad (18)$$

where

$$W_1 = u^2 + 4096t^2(5+6t)(t-1)(t-9)\cdot u + 4\,194\,304t^4(9 - 52t + 20t^2)(t-1)^2(t-9)^2,$$
$$W_2 = (3+5t)\cdot u^4 + 8192(t-1)(t-9)(46t^2 + 51t - 9)t^2\cdot u^3$$
$$- 25\,165\,824\cdot(165 + 239t - 568t^2 - 420t^3)(t-1)^2(t-9)^2t^4\cdot u^2$$
$$- 34\,359\,738\,368\cdot(1041 + 2881t + 6642t^2$$
$$- 4740t^3 - 3800t^4)(t-1)^3(t-9)^3t^6\cdot u$$
$$- 17\,592\,186\,044\,416\cdot(3213 - 70\,749t - 38\,176t^2$$
$$+ 158\,280t^3 - 22\,800t^4 - 34\,000t^5)(t-1)^4(t-9)^4t^8.$$

Recalling that S_1 is the derivative of S_0 with respect to t, one finds the following *Riccati relation* on the parameter u:

$$\frac{du}{dt} = -\frac{1}{16384}\frac{u^2}{(t-9)(t-1)^2t^3} + \frac{(5t-13)(2t-9)}{4(t-1)(t-9)t}\cdot u$$
$$- 256(t-9)(100t^2 - 132t + 9)t. \qquad (19)$$

We have similar results for any value of N, with, again, Riccati relations on the corresponding rational parameter u.

These results can be simply understood, and generalized, as follows. The diagonal correlation function $C(N, N)$ is a *homogeneous* polynomial [1] of E and K (or E and E'). The variable σ amounts to calculating the log-derivative of $C(N, N)$. Recalling that the derivative of monomials of degree N in E and E', like $E^n \cdot (E')^{N-n}$, also yields monomials of degree N like $E^{n-1} \cdot (E')^{N+1-n}$, one easily sees that the log-derivative of $C(N, N)$ is the ratio of two homogeneous polynomials of the same degree N or, equivalently, rational functions of the ratio $\tau = E'/E$ (or E/K). Using the fact that E is solution of a second-order linear differential equation, one can rewrite its second derivative with respect to t, namely E'', into a linear combination of E and E' (or, equivalently, E and K). One immediately deduces that σ', the first-order derivative of σ with respect to t, is also the ratio of two homogeneous polynomial of the same degree N or, equivalently, a *rational function* of the ratio $\tau = E'/E$ (or E/K). This means that any polynomial relation $P(\sigma, \sigma') = 0$ corresponding to the existence of a common solution $C(N, N)$ of (1) and of a $(N + 1)$th order Fuchsian linear differential equation is necessarily parametrized rationally, and *is therefore of genus zero*. The 'rational' parameter of this rational curve $P(\sigma, \sigma') = 0$ is, for instance, the ratio τ. Do note that this ratio satisfies a Riccati equation, in t, inherited from the second-order differential equation satisfied by E:

$$\frac{d\tau}{dt} = A + B \cdot \tau + C \cdot \tau^2.$$

The emergence of a rational *curve* is, thus, a straight consequence of the *diagonal* two-point correlation functions $C(N, N)$ being *homogeneous* polynomials in E and K.

For the *off-diagonal isotropic* two-point correlation functions we have the following generalization: σ and σ' are both *rational expressions*[13] of the two variables E and K (or equivalently E and E'). Therefore, we are naturally led to consider *rational surfaces* [19], instead of rational curves. Recalling, for instance, the polynomial expression [1] of the off-diagonal two-point $C(1, 3)$, the two σ and $\sigma'_s = d\sigma/ds$ variables read, respectively,

$$\frac{P_1(E, K) + P_3(E, K)}{Q_1(E, K) + Q_3(E, K)}, \qquad \frac{P_2(E, K) + P_4(E, K) + P_6(E, K)}{(Q_1(E, K) + Q_3(E, K))^2} \tag{20}$$

where P_n and Q_n denote homogeneous polynomials of degree n in E and K.

4.1. The μ-extension of $C(N, N)$s

Let us consider $C(2, 2)$: the three solutions of the corresponding Fuchsian differential operator L_{22} are respectively $C(2, 2)$, a solution with a log term for the $t \simeq 0$ expansions, that we will denote by $\mathcal{S}_1$, and a solution with a $\log^2$ term, that we will denote by $\mathcal{S}_2$. Consider, now, a general linear combination of these three solutions of L_{22}, namely $C(2, 2) + c_1 \cdot \mathcal{S}_1 + c_2 \cdot \mathcal{S}_2$. Such a general solution of the Fuchsian ODE of order three is also a solution of the sigma form of Painlevé VI (1) *if* (and only if for non-singular solutions) it is a solution of (16). A straightforward calculation (using formal series in maple) gives the following one-parameter family of solutions of (1) *as well as* L_{22}:

$$C_\mu(2, 2) = C(2, 2) + c_1 \cdot \mathcal{S}_1 + c_2 \cdot \mathcal{S}_2, \tag{21}$$

with

$$c_1 = \frac{648\mu}{162 - 2851\mu + 14\,255\mu^2}, \qquad c_2 = \mu \cdot c_1$$

[13] Not birational: E and K are *not* rational expressions of σ and σ'.

and

$$\begin{aligned} \mathcal{S}_1 &= C(2,2)\cdot \ln(t) + A_1(t), \\ \mathcal{S}_2 &= C(2,2)\cdot \ln^2(t) + \left(2A_1(t) - \frac{2851}{324}\cdot C(2,2)\right)\cdot \ln(t) + A_2(t) \end{aligned} \tag{22}$$

where the two (holonomic) functions $A_1(t)$ and $A_2(t)$ have the following Laurent series expansions:

$$A_1(t) = \frac{2}{3t} + \frac{1}{3} + \frac{151}{1728}\cdot t + \cdots, \qquad A_2(t) = \frac{2}{3t} - 7 - \frac{1961}{1728}\cdot t + \cdots.$$

In fact, as far as solutions of the linear operator L_{22} compatible with (1) are concerned, since (1) bears on log-derivatives, a rescaling of $C_\mu(2,2)$ is harmless: we can also introduce $C(2,2;\mu) = (162 - 2851\mu + 14\,255\mu^2)\cdot C_\mu(2,2)/162$ which reads

$$C(2,2;\mu) = C(2,2) + \mu\cdot \mathcal{S}_1^{(\mathrm{norm})} + \mu^2\cdot \mathcal{S}_2^{(\mathrm{norm})} \tag{23}$$

where the two new normalized solutions $\mathcal{S}_1^{(\mathrm{norm})}$ and $\mathcal{S}_2^{(\mathrm{norm})}$ read, respectively,

$$4\cdot \mathcal{S}_1 - \frac{2851}{162}\cdot C(2,2), \qquad 4\cdot \mathcal{S}_2 + \frac{14\,255}{162}\cdot C(2,2).$$

All these calculations are not specific of $N = 2$ and can be generalized straightforwardly for *any value* of N, the only difference being that one will have to consider N solutions $\mathcal{S}_r$ with their $\ln^r$ term [1]. For instance, for $N = 3$, with the formal solutions of L_{33}, around $t = 0$, written as

$$\begin{aligned} \mathcal{S}_0 &= C(3,3), \\ \mathcal{S}_1 &= C(3,3)\cdot \ln(t) + \mathcal{S}_{10}, \\ \mathcal{S}_2 &= C(3,3)\cdot \ln^2(t) + \mathcal{S}_{21}\cdot \ln(t) + \mathcal{S}_{20}, \\ \mathcal{S}_3 &= C(3,3)\cdot \ln^3(t) + \mathcal{S}_{32}\cdot \ln^2(t) + \mathcal{S}_{31}\cdot \ln(t) + \mathcal{S}_{30}, \end{aligned} \tag{24}$$

the linear combination

$$C_\mu(3,3) = C(3,3) + c_1\cdot \mathcal{S}_1 + c_2\cdot \mathcal{S}_2 + c_3\cdot \mathcal{S}_3 \tag{25}$$

satisfies the nonlinear differential equation (1) with

$$\begin{aligned} c_1 &= \frac{810\,000\cdot (648 - 684\mu + 11\,615\mu^2)\cdot \mu}{58\,320\,000 - 1835\,320\,680\mu + 22\,002\,037\,020\mu^2 - 99\,370\,573\,271\mu^3} \\ c_2 &= \frac{1944\mu}{648 - 684\mu + 11\,615\mu^2}\cdot c_1, \qquad c_3 = \mu\cdot c_2. \end{aligned} \tag{26}$$

Similarly, multiplying $C_\mu(3,3)$ by the denominator of c_1 (divided by 58 320 000), and introducing well-suited normalized solutions, one can write a μ-dependent solution of L_{33}, *also compatible* with (1), the sigma-form of Painlevé, as

$$C(3,3;\mu) = C(3,3) + \mu\cdot \mathcal{S}_1^{(\mathrm{norm})} + \mu^2\cdot \mathcal{S}_2^{(\mathrm{norm})} + \mu^3\cdot \mathcal{S}_3^{(\mathrm{norm})}$$

with

$$\begin{aligned} \mathcal{S}_1^{(\mathrm{norm})} &= 9\cdot \mathcal{S}_1 - \frac{188\,819}{6000}\cdot C(3,3), \\ \mathcal{S}_2^{(\mathrm{norm})} &= 27\cdot \mathcal{S}_2 - \frac{19}{2}\cdot \mathcal{S}_1 + \frac{40\,744\,513}{108\,000}\cdot C(3,3), \\ \mathcal{S}_3^{(\mathrm{norm})} &= 27\cdot \mathcal{S}_3 + \frac{11\,615}{72}\cdot \mathcal{S}_1 - \frac{99\,370\,573\,271}{58\,320\,000}\cdot C(3,3). \end{aligned} \tag{27}$$

This scheme will continue for *any value* of N. The formal solutions of L_{NN} which also satisfy the nonlinear differential equation (1) can be written as

$$C(N, N, \mu) = C(N, N) + \sum_{j=1}^{N} \mu^j \cdot \mathcal{S}_j^{(\mathrm{norm})}$$

where $\mathcal{S}_j^{(\mathrm{norm})}$ are sum of holonomic expressions with $\ln^k(t)$ terms

$$\mathcal{S}_j^{(\mathrm{norm})} = \sum_{k=0}^{j} \ln^k(t) \cdot \mathcal{S}_{jk}^{(\mathrm{norm})}, \qquad j = 1, \ldots, N$$

where $\mathcal{S}_{jk}^{(\mathrm{norm})}$ have Laurent expansions in t, around $t = 0$.

Do note that such μ-series with $\ln^k t$ terms do not appear in the Ising correlations (i.e. $\mu = 0$). These μ-extensions of the two-point correlation functions of the Ising model are, like the λ-extension of the next section, mathematical extensions of the Ising correlations $C(N, N)$: we do not try to give a physical content to the parameter μ as a μ-deformation of the Ising model.

4.2. Towards (λ, μ)-extensions of $C(N, N)$ s

For a (nonlinear) second-order differential equation like (1), the sigma form of Painlevé VI, corresponds to a *two-parameters* family of solutions (the 'boundary conditions'). In a previous paper [2] we underlined a particular one-parameter family of solutions of (1), the so-called λ-extensions $C(N, N; \lambda)$ that were such that their 'regular' (low or high temperature) series expansions, analytical in $t^{1/2} = k = s^2$, were *actually* solutions[14] of the sigma form of Painlevé VI (1). Note that, generically (when $\lambda \neq \cos(\pi m/n)$), the λ-extensions $C(N, N; \lambda)$ *are not D-finite (not holonomic) anymore*. With these λ-extensions we are performing another kind of 'deformation' of $C(N, N)$: we are exploring the *analytical* (at $s = 0$) *deformations* of $C(N, N)$. In contrast, with the μ-extensions of $C(N, N)$s of the previous subsection, we were exploring (in the restricted framework of solutions of Fuchsian linear differential equations) 'deformations' corresponding to *formal series* (series which are *not* analytic in s or t, but are *formal series* in t and $\ln(t)$). The μ-extensions, $C(N, N; \lambda = 1, \mu)$, are analytic at $s = 0$, *only when* $\mu = 0$.

Of course, one can 'dream' of (λ, μ)-extensions, $C(N, N; \lambda, \mu)$, of the diagonal two-point correlation function $C(N, N)$, still solutions of the sigma form of Painlevé VI (1). These (λ, μ)-extensions would be defined by *formal series* that verify (1), the sigma form of Painlevé VI, but *do not verify* any finite-order linear differential equations. This, more or less, amounts to considering the 'formal series' of Jimbo [20] that we recalled in equation (5) of our paper [1].

5. The anisotropic Ising model

The previous calculations can be modified, mutatis mutandis, in the case of the *anisotropic* Ising model. In this section, we will denote $s_1 = \sinh(2K_1)$, $s_2 = \sinh(2K_2)$, $c_1 = \cosh(2K_1)$ and $c_2 = \cosh(2K_2)$. We will also introduce the modulus of the elliptic functions parametrizing the model $k = \sinh(2K_1)\sinh(2K_2)$ and the 'anisotropy variable' $\nu = \sinh(2K_1)/\sinh(2K_2)$. Let

[14] For singled-out values of λ (like $\lambda = \cos(\pi m/n)$, with m, n being integers), we found [2] that these λ-extensions are actually solutions of Fuchsian linear differential equations, and we even found that these λ-extensions $C(N, N; \lambda)$ are *algebraic expressions* in t and, more specifically, *modular functions*!

us recall Montroll *et al* paper [12]. The *off-diagonal* two-point correlation functions $C(N, M)$ are given, in the anisotropic case, by determinants generalizing the Toeplitz determinants of section (3). For the off-diagonal two-point correlation functions the entries a_n, in the corresponding determinants, read (see (57) p 314 in [12]) for instance for the row correlation functions

$$a_n = \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}n\omega} \left(\frac{(z_1 z_2^* \mathrm{e}^{\mathrm{i}\omega} - 1)(z_1 \mathrm{e}^{\mathrm{i}\omega} - z_2^*)}{(\mathrm{e}^{\mathrm{i}\omega} - z_1 z_2^*)(z_2^* \mathrm{e}^{\mathrm{i}\omega} - z_1)} \right)^{1/2} \mathrm{d}\omega, \qquad \text{with} \quad z_2^* = \frac{1 - z_2}{1 + z_2}, \qquad (28)$$

where z_i denotes the well-known high-temperature variables $\tanh(K_i)$ $\left(s_i = \sinh(2K_i) = 2z_i\big/\left(1 - z_i^2\right)\right)$. These are clearly *holonomic functions* of z_1 and z_2. One can try to write the two partial differential equations satisfied by (28) in terms of z_1 and z_2.

The simplest off-diagonal two-point correlation function, namely the nearest neighbour two-point correlation function $C(0, 1) = a_0$, reads for $T > T_c$ (see equation (4.3*a*) chapter 8 on p 200 of [13])

$$C(0, 1) = a_0 = 2z_1(1 + z_2)^2 F_{0,0} - z_1^2(1 - z_2)^2 F_{0,1} - (1 - z_2)^2 F_{0,-1}. \qquad (29)$$

Since, after subsection (2.2), we have a prejudice that the *elliptic function parametrization* of the Ising model plays a crucial role, it is tempting to rewrite the previous result (29) (expressed in terms of the two variables z_1 and z_2), in the variables s_1 and s_2 (and also c_1 and c_2), closer to the modulus of the elliptic functions of the Ising model. Recalling [13, 21] and the modulus $k = \sinh(2K_1)\sinh(2K_2)$, the nearest neighbour two-point correlation function $C(0, 1)$ reads (see equation (4.3*a*) chapter 8 on p 200 of [13]) in term of $\Pi_1(y, x)$, the *Jacobi form* of the complete elliptic integral of the third kind [22–26], and of $K(k)$, the complete elliptic integral of the first kind (multiplied by $2/\pi$):

$$C(0, 1) = \frac{c_1^2 c_2}{s_1} \cdot \frac{2}{\pi} \Pi_1\left(s_1^2, k^2\right) - \frac{c_2}{s_1} \cdot K(k^2) \qquad \text{with} \quad \Pi_1(y, x) = \Pi(-y, x) \qquad (30)$$

where the complete elliptic integral of the third kind $\Pi(y, x)$ reads

$$\Pi(y, x) = \int_0^1 \frac{\mathrm{d}u}{(1 - yu^2)\sqrt{(1 - u^2)(1 - xu^2)}}. \qquad (31)$$

Of course, for $C(1, 0)$, we have the same result as (30) where the indices 1 and 2 have been permuted. Recalling the identity

$$\frac{2}{\pi} \cdot \left(\Pi_1(k\nu, k^2) + \Pi_1\left(\frac{k}{\nu}, k^2\right)\right) = K(k^2) + \left((1 + k\nu)\left(1 + \frac{k}{\nu}\right)\right)^{-1/2} \qquad (32)$$

one deduces, for instance, that the following linear combination of $C(0, 1)$ and $C(1, 0)$ is a function depending only[15] on the modulus k:

$$\begin{aligned} c_1 c_2 - s_1 c_2 \cdot C(0, 1) - s_2 c_1 \cdot C(1, 0) &= (1 - k^2) \cdot K(k^2) \\ &= c_1 c_2 - c_1^2 c_2^2 \cdot \frac{2}{\pi} \cdot \left(\Pi_1\left(s_1^2, k^2\right) + \Pi_1\left(s_2^2, k^2\right)\right) + K(k^2) \cdot \left(c_1^2 + c_2^2\right). \qquad (33) \end{aligned}$$

In the isotropic limit, from (32) one easily gets $2/\pi \cdot \Pi_1(k, k^2) = 1/(1+k)/2 + K(k^2)/2$. The Jacobi form Π_1 of the complete elliptic integral of the third kind thus reduces to the complete elliptic integral of the first kind.

We see that the key role played, in the case of the isotropic Ising model, by the complete elliptic integral of the first, or second kind K, or E, and the second-order linear differential operator L_E (or L_K), is going to be played, in the anisotropic case, by the *complete elliptic*

[15] In maple's notations, identity (32) amounts to verifying that $2/\pi EllipticPi(-kn, k) + 2/\pi EllipticPi(-k/n, k) - 2/\pi EllipticK(k) - ((1 + kn)(1 + k/n))^{-1/2}$ equals zero.

integral of the third kind $\Pi(y, x)$ and its associated partial differential operators. Before going further in the generalizations of the calculations displayed in section (3), and in the search for the well-suited generalization of the sigma form of Painlevé VI to the *anisotropic* Ising model, let us analyse, in some details, what is going to generalize the second-order linear differential operator L_E (or the Fuchs operator $\mathcal{L}$ in subsection (2.2) or L_K), namely the partial differential operators corresponding to $\Pi(y, x)$ or $\Pi_1(y, x)$, and, as mathematicians say, their '$\mathcal{D}$-module' structure.

5.1. Revisiting the complete elliptic integral of the third kind

The complete elliptic integral of the third kind $\Pi(y, x)$ is solution of two partial differential equations (see for instance [27]) corresponding to two partial differential operators that, nicely, depend, respectively, only on the derivative D_x in x and the derivative D_y in y, separately. We will denote these two partial differential operators by $\mathcal{L}_x$ and $\mathcal{L}_y$. They read, respectively,

$$\mathcal{L}_x = Dx^3 + \frac{1}{2}\frac{(11x^2 - 6xy - 7x + 2y)}{(x-1)(x-y)x} \cdot Dx^2 + \frac{3}{4}\frac{(7x - y - 2)}{(x-1)(x-y)x} \cdot Dx + \frac{3}{8}\frac{1}{(x-1)(x-y)x}$$

and

$$\mathcal{L}_y = D_y^3 + \frac{(8xy + 8y - 3x - 13y^2)}{2\cdot(y-1)(x-y)y} \cdot D_y^2 + \frac{2(x - 4y + 1)}{(y-1)(x-y)y} \cdot D_y - \frac{1}{(y-1)(x-y)y}.$$

It is easy to see that $\mathcal{L}_x$ is always the product of an *order-two* differential operator, and an *order-one* differential operator

$$\mathcal{L}_x = \mathcal{L}_x^{(2)} \cdot \mathcal{L}_x^{(1)} \qquad (34)$$

where

$$\mathcal{L}_x^{(2)} = Dx^2 + \frac{(5x^2 + y - 3x(y+1))}{(x-1)(x-y)x} \cdot Dx + \frac{1}{4}\frac{15x - 3y - 4}{(x-1)(x-y)x},$$
$$\mathcal{L}_x^{(1)} = Dx + \frac{1}{2(x-y)},$$

and that $\mathcal{L}_y$ is actually the product of the square of an order-one operator with another order-one operator

$$\mathcal{L}_y = \left(\mathcal{L}_y^{(1)}\right)^2 \cdot \mathcal{L}_y^{(2)} \qquad (35)$$

where

$$\mathcal{L}_y^{(1)} = D_y + \frac{2y(x+1) - x - 3y^2}{(y-1)(x-y)y},$$
$$\mathcal{L}_y^{(2)} = D_y + \frac{1}{2}\frac{x - y^2}{(y-1)(x-y)y}.$$

Generically $\mathcal{L}_x$ or $\mathcal{L}_y$ are only factored in simple product such as (34) and (35). They *are not direct sums* of linear differential operators. They are direct sums of linear differential operators only for $x = 0, 1, \infty$ or $y = 0, 1, \infty$ (and, to some extent, $x = y$ which corresponds to the isotropic limit of the Ising model). For instance, for $y = 1$, the third-order partial differential operator $\mathcal{L}_x$ is the direct sum of $\mathcal{L}_x^{(2)}$ taken for $y = 1$, with a second-order operator L_2 (actually equivalent to L_E):

$$\mathcal{L}_x(y = 1) = \mathcal{L}_x^{(1)}(y = 1) \oplus L_2 \qquad \text{where} \quad L_2 = Dx^2 + \frac{2}{x-1} \cdot Dx + \frac{1}{4}\frac{1}{(x-1)x}.$$

Let us now analyse the differential module associated with $\mathcal{L}_x$ and $\mathcal{L}_y$, i.e. the minimal system of partial linear differential equations whose solutions are exactly the solutions that are common to $\mathcal{L}_x$ and $\mathcal{L}_y$. Let us introduce the two polynomials $P_A = 8\,(x-1)\,(x-y)\cdot x$ and $P_B = 8\,(y-1)\,(x-y)\cdot y$, and the two 3×3 matrices $\mathcal{A} = A/P_A$ and $\mathcal{B} = B/P_B$, where

$$A = \begin{bmatrix} 0 & 8(x-y)(x-1)x & 0 \\ 0 & 0 & 8x(x-1)(x-y) \\ -3 & 6(y+2-7x) & 4(7x-11x^2+6xy-2y) \end{bmatrix},$$

and where the 3×3 matrix B reads

$$\begin{bmatrix} 4(x-y)^2 & 16(2x-y-1)(x-y)x & 16(x-y)^2x(x-1) \\ 2(y-x) & 8(x-y)(1-2x-y) & -8(x-y)(x-1)x \\ 3 & 12(2x+y-1) & 12(x-1)x+8(y-1)y \end{bmatrix}.$$

Let us also introduce Z, the vector of entries $z(x,y)$, $\partial z/\partial x$, $\partial^2 z/\partial x^2$, and the system

$$\frac{\partial Z}{\partial x} = \mathcal{A}\cdot Z, \qquad \frac{\partial Z}{\partial y} = \mathcal{B}\cdot Z. \tag{36}$$

One easily verifies that the compatibility condition of this system (36), namely

$$\mathcal{A}\cdot\mathcal{B} - \mathcal{B}\cdot\mathcal{A} + \frac{\partial\mathcal{A}}{\partial y} - \frac{\partial\mathcal{B}}{\partial x} = 0, \tag{37}$$

is actually satisfied. This can be seen as a Schlesinger system [6]. As a consequence, the two third-order operators $\mathcal{L}_x$ and $\mathcal{L}_y$ *are isomonodromic* (see this result of Malgrange for instance in Singer and Cassidy [28]).

The Schlesinger system (37) just means that the two operators $\mathcal{L}_x$ and $\mathcal{L}_y$ are compatible and have a common solution, namely $\Pi(y,x)$, the complete elliptic of the third kind. Instead of introducing Z, the vector of entries $z(y,x)$, $\partial z/\partial x$, $\partial^2 z/\partial x^2$, one could have performed an equivalent calculation with the vector of entries $z(y,x)$, $\partial z/\partial x$, $\partial z/\partial y$. This last calculation just corresponds to a change of basis for the two 3×3 matrices A and B.

Along this line we can recall the fact that the complete elliptic integral of the third kind $\Pi(y,x)$ verifies the following differential formula (see [29, 30] and also (3.107) and (3.112) in chapter 5 of [13]):

$$\begin{aligned}
\frac{2}{\pi}\cdot\frac{\partial\Pi(y,x)}{\partial y} &= \frac{1}{2(x-y)(y-1)}\cdot\left(E(x) + \frac{x-y}{y}\cdot K(x) + \frac{y^2-x}{y}\cdot\frac{2}{\pi}\cdot\Pi(y,x)\right),\\
\frac{2}{\pi}\cdot\frac{\partial^2\Pi(y,x)}{\partial y^2} &= \frac{(5y-2)y+(1-4y)x}{4(x-y)(y-1)^2y^2}\cdot K(x) - \frac{(2y+1)x+(2-5y)y}{4(x-y)^2(y-1)^2y}\cdot E(x)\\
&\quad + \frac{3y^4+2(2-5y)xy+(4y-1)x^2}{4(x-y)^2(y-1)^2y^2}\cdot\frac{2}{\pi}\cdot\Pi(y,x),\\
\frac{2}{\pi}\cdot\frac{\partial\Pi(y,x)}{\partial x} &= \frac{1}{2(y-x)}\cdot\left(\frac{E(x)}{x-1} + \frac{2}{\pi}\cdot\Pi(y,x)\right),\\
\frac{2}{\pi}\cdot\frac{\partial^2\Pi(y,x)}{\partial x^2} &= \frac{4x^2-(y+2)x-y}{4(x-1)^2(x-y)^2x}\cdot E(x) + \frac{1}{4(x-1)(x-y)x}\cdot K(x)\\
&\quad + \frac{3}{4(x-y)^2}\cdot\frac{2}{\pi}\cdot\Pi(y,x).
\end{aligned} \tag{38}$$

These relations show that the vector space spanned by $\Pi(y,x)$, $\partial\Pi(y,x)/\partial x$, $\partial^2\Pi(y,x)/\partial x^2$, the vector space spanned by $\Pi_1(y,x)$, $\partial\Pi(y,x)/\partial x$, $\partial\Pi(y,x)/\partial y$ and the vector space spanned by $\Pi(y, x)$, $E(x)$, $K(x)$ *actually identify*. In particular, one can write a Schlesinger system (37) of compatibility (isomonodromy) of $\mathcal{L}_x$ and $\mathcal{L}_y$ in the $\Pi_1(y, x)$, $E(x)$, $K(x)$ basis.

For fixed y the complete elliptic integral of the third kind $\Pi(y, x)$ has two branch points $x = 1$ and $x = \infty$. For fixed x the complete elliptic integral of the third kind $\Pi(y, x)$ has two branch points $y = 1$ and $y = \infty$. The branch cut locations are complicated. The complete elliptic integral of the third kind $\Pi(y, x)$ has no poles and essential singularities with respect to y, and similarly no poles and essential singularities with respect to x. Less known is the fact that the complete elliptic integral of the third kind $\Pi(y, x)$ *can be represented through Appell functions*[16] *or hypergeometric functions of two variables* [31]:

$$\frac{2}{\pi} \cdot \Pi(y, x) = F_1(1/2; 1/2, 1; 1; x, y). \tag{39}$$

Appell showed that these functions satisfy two simultaneous partial differential equations (see [32]). Let us write these two partial differential operators [33, 34]:

$$(1-y)y \cdot \frac{\partial^2}{\partial y^2} + (1-y)x \cdot \frac{\partial^2}{\partial y \partial x} + (c - (1+a+b')y) \cdot \frac{\partial}{\partial y} - b'x\frac{\partial}{\partial x} - ab',$$

$$\begin{aligned}(1-x)x \cdot \frac{\partial^2}{\partial x^2} &+ (1-x)y \cdot \frac{\partial^2}{\partial x \partial y} + (c - (1+a+b)y) \cdot \frac{\partial}{\partial x} - by\frac{\partial}{\partial y} - ab \\ &= \mathcal{L} + (1-x) \cdot y \cdot \frac{\partial^2}{\partial x \partial y} - \frac{y}{2}\frac{\partial}{\partial y}\end{aligned} \tag{40}$$

where $\mathcal{L}$ is the Fuchs second-order linear differential operator (6) of subsection (2.2), where the variable t has been changed into x.

In the particular case $c = 1$, $a = 1/2$, $b = 1/2$, $b' = 1$, these equations[17] read

$$\mathcal{A}_y = (1-y)y \cdot \frac{\partial^2}{\partial y^2} + (1-y)x \cdot \frac{\partial^2}{\partial y \partial x} + \left(1 - \frac{5}{2}y\right) \cdot \frac{\partial}{\partial y} - x\frac{\partial}{\partial x} - \frac{1}{2}, \tag{41}$$

$$\begin{aligned}\mathcal{A}_x &= (1-x)x \cdot \frac{\partial^2}{\partial x^2} + (1-x)y \cdot \frac{\partial^2}{\partial x \partial y} + (1-2x) \cdot \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial y} - \frac{1}{4} \\ &= \mathcal{L} + (1-x) \cdot y \cdot \frac{\partial^2}{\partial x \partial y} - \frac{y}{2}\frac{\partial}{\partial y}.\end{aligned} \tag{42}$$

The complete elliptic integral of the third kind (31), $\Pi(y, x)$, is *a solution of that system* of partial differential operators (41) and (42). The complete elliptic integral of the third kind $\Pi(y, x) = \pi/2 \cdot F_1(1/2; 1/2, 1; 1; x, y)$ is thus an *Appell function* associated with a system (41), (42), closely linked to *del Pezzo surfaces* [33] and *Garnier systems* [5, 33, 34] (along this line see also a set of very nice papers [36–42]).

Note that x and y *are not on the same footing*: (42) can be seen as a deformation of $\mathcal{L}$ (or L_K or L_E), when, in contrast, (41) can be seen as a deformation of the operator (which factorizes into two *order-one* operators):

$$(1-y)y \cdot \frac{\partial^2}{\partial y^2} + \left(1 - \frac{5}{2}y\right) \cdot \frac{\partial}{\partial y} - \frac{1}{2} = \left((1-y) \cdot y \cdot \frac{\partial}{\partial y} + (1-2y)\right) \cdot \left(\frac{\partial}{\partial y} - \frac{1}{2(1-y)}\right). \tag{43}$$

[16] Appell defined the functions in 1880, and Picard showed in 1881 that they may all be expressed by integrals of the form $\int_0^1 u^\alpha \cdot (1-u)^\beta \cdot (1-xu)^\gamma \cdot (1-yu)^\delta \cdot du$.

[17] Do note that Okamoto and Kimura have given [35] the linear partial differential equations for the classical seed solutions of the two-variable Garnier system and its confluent degenerations, and the integral representations of their solutions for general parameters. The Appell functions are discussed there, and the general forms of equations (41) and (42) are given.

These two new 'Appellian' partial differential operators $\mathcal{A}_x$, $\mathcal{A}_y$ are slightly different from those we previously introduced, namely $\mathcal{L}_x$, $\mathcal{L}_y$. The order-three partial differential operators $\mathcal{L}_x$, $\mathcal{L}_y$ are more 'decoupled' (just derivatives with respect to x, resp. y) than the 'Appellian' partial differential operators of order two $\mathcal{A}_x$, $\mathcal{A}_y$ which present a 'mixed' $\partial^2/\partial x \partial y$ derivative. Again, all these partial differential operators have to be compatible. It is a straightforward, but slightly tedious, exercise to see that the compatibility of any choice of two partial differential operators among these four partial differential operators ($\mathcal{A}_x$, $\mathcal{A}_y$, $\mathcal{L}_x$, $\mathcal{L}_y$) yields Schlesinger systems like (37) and that these Schlesinger systems can be written in (at least) three different basis ($\Pi_1(y, x)$, $\partial\Pi_1(y, x)/\partial x$, $\partial^2\Pi_1(y, x)/\partial x^2$ or $\Pi_1(y, x)$, $\partial\Pi_1(y, x)/\partial x$, $\partial\Pi_1(y, x)/\partial y$ or $\Pi_1(y, x)$, $E(x)$, $K(x)$).

5.2. The Fuchsian PDEs of the anisotropic Ising model

The calculations performed in section (3) can now be generalized, mutatis mutandis, replacing the central role played by the second-order linear differential operator L_E (or $\mathcal{L}$ or L_K) by two of the partial linear differential operators $\mathcal{A}_x$, $\mathcal{A}_y$, $\mathcal{L}_x$, $\mathcal{L}_y$. Similarly to (15), one should find that a_n occurring in the entries of the determinants associated with $C(N, M)$ are linear combinations (with rational coefficients in x) of $\Pi_1(y, x)$, $E(x)$, $K(x)$. The straight generalization of the calculations of section (3) would correspond to write the partial linear differential operators in z_1 and z_2 corresponding to the entries a_n, given by the holonomic expressions (28), and find that these partial linear differential operators are actually equivalent, and thus equivalent (in the sense of equivalence of partial differential operators [3, 4]) to the partial linear differential operators corresponding to a_0 given by (29) or (30). Relation (30) means that the partial linear differential operator, corresponding to a_0, can be expressed as the direct sum of the partial linear differential operators in s_1 and s_2 corresponding to $\Pi_1\left(s_1^2, s_1^2 s_2^2\right)$ and $K\left(s_1^2 s_2^2\right)$, that is, up to some change of variables (to get partial linear differential operator in s_1 and s_2), to direct sum of $\mathcal{L}_x$ and L_E. These calculations are straightforward, but tedious: for instance, the two partial linear differential operators in z_1 and z_2, corresponding to a_0, are quite large: we have actually found, and checked, the Schlesinger relation like (37) in z_1 and z_2 for these two partial linear differential operators. It is probably easier to see, directly, that the relations generalizing (15) for off-diagonal two-point correlations are actually verified. For instance, a_n in (28), corresponding to the row correlation functions (and beyond off-diagonal correlations), may be written as

$$a_n = 2/\pi \cdot p_n \cdot \Pi_1\left(s_1^2, k^2\right) + t_n + q_n \cdot E(k^2) + r_n \cdot K(k^2), \qquad k = s_1 s_2 \tag{44}$$

where p_n, q_n, r_n and t_n are rational expressions in s_1 and s_2.

Relation (44), or similar relations for the general *off-diagonal anisotropic* correlation functions, implies that the *off-diagonal anisotropic* correlation functions are *non-homogeneous polynomials* in[18] the complete elliptic integral of the third kind, $\Pi_1(y, x)$, and of the first, and second, complete elliptic integral[19] $K(x)$, $E(x)$.

Most of the results that we found in [1] for $C(N, N)$, or $C(N, M)$, in particular the fact that their corresponding linear differential operators are actually equivalent to the symmetric power of L_E, or direct sums of operators equivalent to the symmetric power of L_E, generalize mutatis mutandis to direct sums of operators equivalent to least common left multiple (LCLM)

[18] Or, equivalently, of $\Pi_1(y, x)$, its first and second derivatives $\Pi_1(y, x)'$ and $\Pi_1(y, x)''$, with respect to the variable x or, equivalently, of $\Pi_1(y, x)$ and its first derivatives with respect to x and y.

[19] To be rigorous in this polynomiality demonstration, denoting by $\Phi_1(y, x)$, $\Phi_2(y, x)$ the two other solutions of the third-order linear differential operator $\mathcal{L}_x$, we can show that one does not have an algebraic relation between $\Pi_1(y, x)$, $\Phi_1(y, x)$, $\Phi_2(y, x)$ and their first-order derivatives with respect to x, hence the one-to-one identification between homogeneous polynomials in $\Pi_1(y, x)$, $\Phi_1(y, x)$, $\Phi_2(y, x)$ and solutions of symmetric powers of $\mathcal{L}_x$.

of L_E and $\mathcal{A}_x$, $\mathcal{A}_y$ or $\mathcal{L}_x$, $\mathcal{L}_y$. Most of the examples of such relations are, even in the simplest cases, too tedious and too large to be displayed in this paper (due to length constraints of this special issue), so we will display them elsewhere.

5.3. Generalization of Painlevé VI, Garnier system

The sigma functions, associated with the diagonal two-point correlations $C(N, N)$ of the isotropic Ising model, are solutions of (1), the sigma form of Painlevé VI. This is even true for their λ-extensions [2]. In a previous paper [2], we used the Ising model to give crystal clear examples of the deep relations that exist between the Painlevé VI equations, the theory of elliptic curves, the modular curves and an infinite number of Fuchsian linear differential equation of order $N + 1$. These deep relations, together with the results, displayed in the previous sections, give a strong motivation to find the structures that generalize the Painlevé VI equations in the case of the *off-diagonal* two-point correlations $C(N, M)$ for the isotropic Ising model and, beyond, for the *anisotropic* Ising model.

For Painlevé specialists, the results we display in [1] for the isotropic Ising model can, at first sight, probably be seen as very special cases of affine Weyl group symmetries and Riccati-type solutions of Painlevé equations. On the 'Painlevé side', it is natural to try to generalize a one-parameter sigma form (1) to the most general four-parameter case or, even, to more general Garnier, or Schlesinger, systems. On the lattice statistical mechanics side, it is tempting to generalize the isotropic Ising model to a more general Yang–Baxter integrable model with an elliptic parametrization (since we saw that the occurrence of elliptic curves was a crucial point), namely the Baxter model, which can be seen as two copies of the anisotropic Ising model with a four spin coupling. In such a move to a broader framework, the 'dictionary' between the 'Painlevé language' and the 'Yang–Baxter integrable models' remains to be done in a clean way. For instance, is there a correspondence between the μ and λ parameters of our μ- and λ-extensions (see (4.1) and (4.2)), some of the four parameters of the most general Painlevé equation, and the anisotropy, or the four spin coupling, of the Baxter model? The results we have obtained in [2] for singled-out values of λ, that the λ-extensions of the two-point correlation functions of the Ising model actually become algebraic functions (corresponding to modular curves), seem to indicate that the parameter λ identifies with the cosinus of the crossing parameter (denoted by η in the Baxter model).

Though it is probably too early to see the full 'global picture', the generalizations of [1], that we addressed here, seem to be a first, and necessary, step paving the way to a deeper understanding of the anisotropic Ising model.

Recalling the elliptic representation of Painlevé VI, which amounts to seeing Painlevé VI as a deformation of the Fuchs–Painlevé second-order linear differential operator $\mathcal{L}$ (or equivalently L_E), it is clear, for the *anisotropic* Ising model, that we are seeking for *a deformation of the 'Appellian system' corresponding to two of the four partial differential operators* $\mathcal{A}_x$, $\mathcal{A}_y$ or $\mathcal{L}_x$, $\mathcal{L}_y$.

The *Garnier systems* [5] are isomonodromic systems providing the simplest, the most canonical, and natural, generalization of Painlevé VI. The Garnier system depends on an integer n, giving Painlevé VI for $n = 1$. In an inspired note [43], where the authors look for natural canonical generalizations of Painlevé VI, Enolskii *et al* indicate that the $n = 2$ Garnier system yields an *order four nonlinear ODE* that can rightly be considered as the *higher order Painlevé VI equation* [43]. This fourth-order ODE is, however, a 'rather huge' one. In general, Garnier systems and Schlesinger systems yield *systems of nonlinear partial differential equations* rather than ODEs.

With the Appell–Picard hypergeometric differential operators in one variable, we are moving from the theory of elliptic curves to *hyperelliptic curves*. *But is it really hyperelliptic curves or rational surfaces that should be considered*? In the case of the *off-diagonal* two-point correlations functions for the *isotropic* Ising model we saw that the complete elliptic integral of the third kind, Π_1, actually reduces to the complete elliptic integral of the first kind and that the price to pay to move from diagonal to *off-diagonal* two-point correlations is a move from *curves to surfaces* (rational curve to rational surfaces: see (20) in section (4)). At the moment, it is still not clear what is the proper generalization of the sigma form of Painlevé VI for the *off-diagonal* two-point correlations of the *isotropic* Ising model: should we seek for a sigma form of the fourth-order ODE previously mentioned [43] or should we look, even for the isotropic model, for system of PDEs associated with surfaces?

We have a probably 'cleaner' situation with the *off-diagonal* two-point correlations of the *anisotropic* Ising model: to generalize the elliptic representation of Painlevé VI we should probably look for some 'Appellian representation' of Garnier systems, namely a 'deformation theory' of the partial differential operators $\mathcal{A}_x$, $\mathcal{A}_y$.

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References

[1] Boukraa S, Hassani S, Maillard J M, McCoy B M, Weil J-A and Zenine N 2006 *J. Phys. A: Math. Gen.* **39** 12245–63 (*Preprint* math-ph 0602010)
[2] Boukraa S, Hassani S, Maillard J M, McCoy B M, Orrick W and Zenine N 2007 *J. Phys. A: Math. Gen.* **40** 75–111
[3] Singer M F 1996 *Appl. Alg. Eng. Commun. Comput.* **7** 77–104
[4] van der Put M and Singer M F 2003 *Galois Theory of Linear Differential Equations* (*Grundlehren der Mathematischen Wissenschaften* vol 328) (Berlin: Springer)
[5] Garnier R 1912 *Ann. Ecole Norm. Sup.* **29** 1–126
[6] Schlesinger L 1912 *J. Reine Angew. Math.* **141** 96–145
[7] Chernyakov Y, Levin A M, Olshanetsky M and Zotov A 2006 *J. Phys. A: Math. Gen.* **39** 12083–101
[8] Manin Yu I 1998 *AMS Transl. (2)* **186** 131–51
[9] Guzzetti D 2001 The elliptic representation of the general Painlevé 6 equation *Preprint* Math.CV/0108073
[10] Bazhanov V V and Mangazeev V V 2006 *J. Phys. A: Math. Gen.* **39** 12235–43 (*Preprint* hep-th 0602122)
[11] Kaufman B and Onsager L 1949 *Phys. Rev.* **76** 1244
[12] Montrol E W, Potts R B and Ward J C 1963 *J. Math. Phys.* **4** 308–22
[13] McCoy B M and Wu T T 1973 *The Two-Dimensional Ising Model* (Cambridge: Harvard University Press)
[14] Jimbo M and Miwa T 1980 *Proc. Japan Acad.* A **56** 405
Jimbo M and Miwa T 1981 *Proc. Japan Acad.* A **57** 347 (erratum)
[15] Forrester P and Witte N 2005 *Nonlinearity* **18** 2061–88
[16] Umemura H and Watanabe H 1997 *Nagoya Math. J.* **148** 151
[17] Ritt J F 1966 *Differential Algebra* (New York: Dover)
[18] Ritt J F 1932 *Differential Equations from the Algebraic Viewpoint* (Washington, DC: American Mathematical Society)
[19] Sakai H 2001 *Commun. Math. Phys.* **220** 165–229
[20] Jimbo M 1982 *Publ. RIMS* **18** 1137–61
[21] Maillard J-M 1984 *J. Physique* **46** 329–41

[22] Legendre A M 1825–1828 *Traité des Fonctions Elliptiques et des intǵrales eulériennes* vol 3 (Paris: Huzard-Courcier)
[23] Houzel C 1978 *Abrégé d' histoire des mathématiques* vol 2, ed J Dieudonné (Paris: Hermann) chapter VII
[24] *Encyclopedic Dictionary of Mathematics* 1993 (appendix A, table 16) (Cambridge, MA: MIT Press)
[25] Weisstein E W 1999 *CRC Concise Encyclopedia of Mathematics* (Boca Raton, FL: CRC Press) p 540
[26] Abramovicz M and Stegun C A 1972 *Elliptic Integrals of the Third Kind* (Chapter 17 and paragraph 17.7 in Handbooks of Mathematics Functions with Formulas, Graphs, and Mathematical Tables, 9th printing) (New York: Dover) pp 587–607
[27] http://functions.wolfram.com/EllipticIntegrals/EllipticPi/13/01/01
[28] Cassidy P J and Singer M F 2005 Galois theory of parametrized differential equations and linear differential algebraic groups *Andrey Bolibrukh* (special issue on) (*IRMA Lectures in Mathematics and Theoretical Physics*) vol 9 ed D Bertrand, B Enriquec, C Mitschi, C Sabbah and R Schaefke (Zurich: EMS Publishing House) pp 113–57
[29] Cayley A 1961 *An Elementary Treatise on Elliptic Functions* (New York: Dover)
[30] http://functions.wolfram.com/EllipticIntegrals/EllipticPi/20/01/01
[31] http://functions.wolfram.com/EllipticIntegrals/EllipticPi/26/03/02
[32] Appell P and Kampé de Fériet J 1926 *Fonctions hypergéométriques et hypersphériques, polynômes d' Hermite* (Paris: Gauthier-Villars)
[33] Koike K 2000 On the family of pentagonal curves of genus 6 and associated modular curves on the ball *Preprint* math.AG/0009200
[34] Yang Lei 2004 Geometry and arithmetic associated to Appell hypergeometric partial differential equations *Preprint* math.NT/0309415 v2
[35] Okamoto K and Kimura H On particular solutions of the Garnier systems and the hypergeometric functions of several variables *Q. J. Math.* **37** 61–80
[36] Picard E 1883 *Acta Math.* **2** 114–35
[37] Cohen P B and Wolfart J 1993 *Ann. Sci. Ecol. Norm. Sup.* **26** 665–90
[38] Matsumoto K and Wolfart J 1989 *Ann. Scuola Norm. Sup. Pisa Cl. Sci.(4)* **16** 557–78
[39] Deligne P and Mostow G D 1986 On monodromy of hypergeometric functions and non-lattice integral monodromy *Publ. Math. IHES* **63** 5–88
[40] Shimura G 1964 *Osaka J. Math.* **1** 1–14
[41] Terada T 1983 Fonctions hypergéométriques F_1 et fonctions automorphes I–II *J. Math. Soc. Japan* **35** 451–75
Terada T 1985 Fonctions hypergéométriques F_1 et fonctions automorphes I–II *J. Math. Soc. Japan* **37** 173–85
[42] Yamazaki T and Yashida M 1984 *Math. Ann.* **266** 421–31
[43] Enolskii V, Nijhoff F and Previato E Notes on isomonodromic Garnier systems: Lagrangian structure and higher-genus analogues of the Painlevé VI equation *Report No* 28 2005-2006 (Institut Mittag-Leffler)
http://www.mittag-leffler.se/preprints/0506f/files/IML-0506f-28.pdf