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From integrability to weak chaos

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We analyze birational transformations obtained from very simple algebraic calculations, namely taking the inverse of $q \times q$ matrices and permuting some of the entries of these matrices. We concentrate on 4×4 matrices and elementary transpositions of two entries. This analysis brings out six classes of birational transformations. Three classes correspond to integrable mappings, their iteration yielding elliptic curves. The iterations corresponding to the three other classes are included in higher dimensional non-trivial algebraic varieties. For many initial conditions in the parameter space these orbits lie on (transcendental) curves, and finally explode in these higher dimensional varieties. These transformations act on fifteen (or $q^2 - 1$) variables, however one can associate to them remarkably simple non-linear recurrences bearing on a single variable. The study of these last recurrences gives a complementary understanding of these amazingly regular non-integrable mappings.

1. Introduction

In previous publications, the study of integrability of lattice models in statistical mechanics has brought out the existence of an infinite discrete symmetry group of the Yang–Baxter equations [1–6], which originates from the so-called inversion relations [7–10]. More generally this group corresponds to non-trivial symmetries of phase diagrams of lattice models in statistical mechanics [11–13]. The representations of this group are birational transformation groups, generated by involutions, acting on the parameter space of the model [1,2]. This analysis has been performed in detail for the sixteen vertex model associated to the two-dimensional square lattice and for a particular subcase of a sixty-four vertex model corresponding to the three-dimensional cubic lattice [4,5,14,15]. In both cases, the parameter space of the model can be represented by 4×4 matrices, one of the group generators I, coinciding with the matricial inverse and the other(s) being some permutations(s) of the entries of the 4×4 matrix, denoted t generically [1]. The study of this group brings to analyze these birational mappings and especially the (generically) infinite order transformation tI [1,2]. Remarkably, for the sixteen vertex model and some subcases of the sixty-four vertex model, the iterations of these

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birational transformations tI actually densify algebraic elliptic curves in the parameter space, $\mathbb{C}P_{15}$ [5] (generically for the sixty-four vertex model subcase detailed in [4,14], they densify algebraic surfaces). These algebraic curves (or algebraic surfaces) indeed define a foliation of the whole parameter space [5]. When elliptic curves occur in the whole parameter space, the mappings are integrable, though the associated lattice model may not be integrable itself (in the sense of Yang–Baxter equations, or of their higher dimensional generalizations, to be satisfied) [11]. Such lattice models have been denoted quasi-integrable [5]. Integrable mappings acting on fifteen inhomogeneous variables have thus emerged from this analysis of the symmetry of integrability for lattice models. This result is interesting in itself, as far as discrete dynamical systems are concerned, since the examples known in the literature were always systems bearing on few variables [16]. Besides, it suggests considering other birational mappings in $\mathbb{C}P_{15}$, generated by the matricial inverse and some (involutive) permutations of the entries of the matrix, not even related to any symmetry of lattice models of statistical mechanics [17,18]. The results will be reported here and, in parallel, in a series of papers [17–19], for the simplest examples of permutations: the transpositions of two entries. It is important to note that, though one mainly deals here with 4×4 matrices and therefore mappings of $\mathbb{C}P_{15}$, the results can be generalized to $q \times q$ matrices, the associated mappings acting in $\mathbb{C}P_{q^2-1}$ [17,18].

In fact, one can reduce the study to six classes of such mappings [20]. Their iterations often lie on curves, however this emergence of curves does not correspond to a unique situation, but on the contrary to two different ones. For 4×4 matrices three of these classes correspond to integrable mappings, their iterations actually yielding algebraic elliptic curves. The equations of these elliptic curves are given as intersections of fourteen quadrics in $\mathbb{C}P_{15}$. On the other hand, the three remaining classes correspond to another kind of mappings, which we will call "almost" integrable: they are not generically integrable, even if their iterations stay on curves in some regions of the projective space. Actually, these curves are not algebraic, but transcendental, though they may look very much like algebraic elliptic curves. This provides an illustration of a transition from integrable dynamical systems to weak chaos through highly stable curves. Indeed, for such "almost" integrable mappings, one can follow the evolution of an orbit from a transcendental curve close to an algebraic elliptic curve, up to an "explosion" into a spray of points. This is reminiscent of the KAM theorem [16]. More precisely, when iterating these mappings one gets orbits similar to the one described by Siegel's theorem [21], though there is no associated complex structure.

In other publications [17–19], it is shown that the iterations of these birational transformations present some remarkable factorization properties, and that the polynomial factors occurring in these factorizations satisfy for some classes non-trivial non-linear recurrences [17,18]. It is shown that these non-linear recurrences on one variable also describe algebraic elliptic curves or transcendental curves, and thus can also be classified in two categories: integrable recurrences and "almost" integrable recurrences. The equations of these elliptic curves are biquadratic relations, and the transcendental curves look like deformations of these algebraic elliptic curves.

2. Six equivalence classes

Let us consider the following 4×4 matrix:

$$R = \begin{pmatrix} a_1 & a_2 & b_1 & b_2 \\ a_3 & a_4 & b_3 & b_4 \\ c_1 & c_2 & d_1 & d_2 \\ c_3 & c_4 & d_3 & d_4 \end{pmatrix} . \tag{2.1}$$

Let us also introduce the homogeneous matrix inverse I:

$$I : R \rightarrow R^{-1} \cdot \det(R) . \tag{2.2}$$

The homogeneous inverse I is a polynomial transformation on each of the entries. It associates with each entry its corresponding cofactor. The homogeneous transformation I is an involution up to a multiplicative factor: it satisfies $I^2 = [\det(R)]^2 \cdot \mathrm{Id}$, where Id denotes the identity transformation.

One will also introduce the involution t, which denotes an arbitrary transposition of two entries of matrix R, and $K_t = t \cdot I$, the infinite order transformation associated with each transposition t. Transformation K_t is a polynomial transformation of the sixteen homogeneous entries of the matrix. Γ_t will denote the infinite discrete group generated by I and t [1,2,6,11]. Such groups Γ_t, generated by two involutions are isomorphic to $\mathbb{Z}$, up to a semi-direct product by a two element group (the infinite dihedral group). The "infinite part" of the group (which is isomorphic to $\mathbb{Z}$) is generated by K_t, that is the simplest infinite order generator of the group. Notice that Γ_t is a group of birational transformations [1,2].

Let us analyze these groups Γ_t of birational transformations. At first sight, one has to study as many groups of mappings as there are transpositions t of two elements among the sixteen entries of the matrix, that is $\binom{2}{16} = 120$.

In fact, the 120 corresponding groups Γ_t fall in only six different classes.

With the notation $[r_{ij} - r_{kl}]$ for transposition exchanging the two entries r_{ij} and r_{kl} of matrix R (2.1), seven classes emerge [20]:

Class $\mathscr{C}_1$ corresponds to all the 6 transpositions of the form $[r_{ij} - r_{ji}]$,

Class $\mathscr{C}_2$ corresponds to all the 6 transpositions of the form $[r_{ii} - r_{jj}]$,

Class $\mathscr{C}_3$ corresponds to all the 12 transpositions of the form $[r_{ij} - r_{kl}]$,

Class $\mathscr{C}_4$ corresponds to all the 24 transpositions of the form $[r_{ij} - r_{jk}]$ or $[r_{ji} - r_{kj}]$,

Class $\mathscr{C}_5$ corresponds to all the 24 transpositions of the form $[r_{ij} - r_{ik}]$ or $[r_{ji} - r_{ki}]$,

Class $\mathscr{C}_6$ corresponds to all the 24 transpositions of the form $[r_{ii} - r_{jk}]$,

Class $\mathscr{C}_7$ corresponds to all the 24 transpositions of the form $[r_{ii} - r_{ij}]$ or $[r_{ii} - r_{ji}]$, where all the indices i, j, k and l are different.

Let us note that such a classification is still valid for $q \times q$-matrices instead of 4×4-matrices. Moreover, one can show [20] that classes $\mathscr{C}_1$ and $\mathscr{C}_2$ lead to the same behavior as far as iterations of their associated birational mappings are concerned. Therefore, classes $\mathscr{C}_1$ and $\mathscr{C}_2$ can be brought together in the same class, we will denote class I in the following. The five other classes ($\mathscr{C}_3, \ldots, \mathscr{C}_7$) can be relabelled classes (II, . . . , VI) in the same order.

One can now study a single mapping in each class and directly deduce the results concerning all the other transformations of the same class.

3. Numerical study

An efficient method to analyze such transformations, especially when there are many variables, is to iterate numerically the action of the (generically) infinite order transformation K_t on an arbitrary initial matrix and to visualize a two-dimensional projection of the orbit [1,2].

Fig. 1 shows a two-dimensional projection in the fifteen-dimensional space of a trajectory of the $K_{t_{\mathrm{I}}}$-iteration, where t_{I} is a transposition of class I. Similar figures can be obtained for the five other classes [20]. For instance, fig. 2 shows a set of iterations corresponding to many different initial points for a transposition of class IV.

All these figures exhibit curves [20]. This result is astonishing, if one takes into account the complexity of the birational transformation K_t^2. Moreover, this complexity, due to the degree of the homogeneous transformation and to the number of variables, does not yield any numerical instability of the curves.

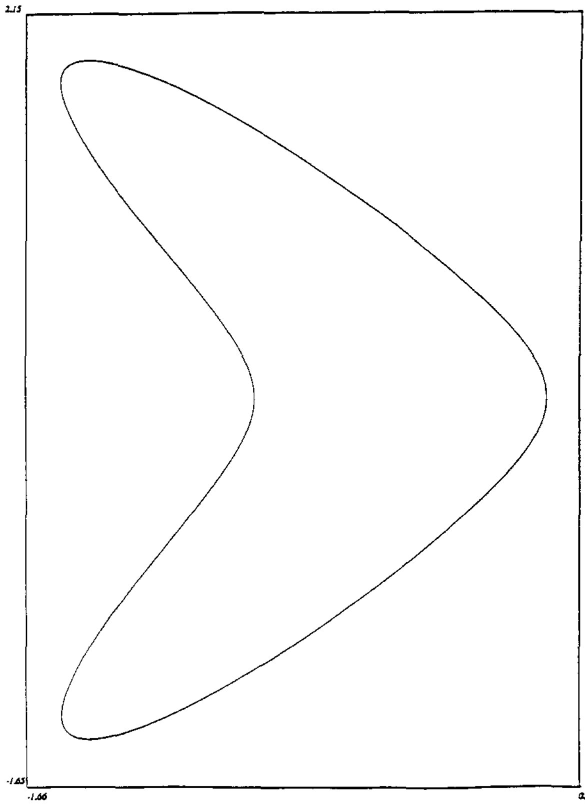

Fig. 1. Projection of the orbit of K_t for class I in a fifteen-dimensional space.

4. Algebraic invariants: almost integrable mappings

Such numerical studies tend to show that the orbits of the groups of birational transformations Γ_t associated with any transposition are curves (at least in some domain of the parameter space). One can thus try to get the equations of these curves in $\mathbb{C}P_{15}$, for instance from explicit algebraic expressions, invariant under Γ_t.

Let us recall that the set of $p \times p$ minors of $2p \times 2p$-matrices is globally invariant, up to a multiplicative factor, under the matricial inverse I. Let us for instance restrict to 4×4 matrices, that is $p = 2$ and consider the following 2×2 minors of the 4×4 matrix R:

$$m = a_1 b_3 - a_3 b_1 , \qquad m' = a_1 b_4 - a_3 b_2 , \qquad m'' = b_3 d_4 - b_4 d_3 ,$$
$$m''' = b_3 d_2 - d_1 b_4 . \tag{4.1}$$

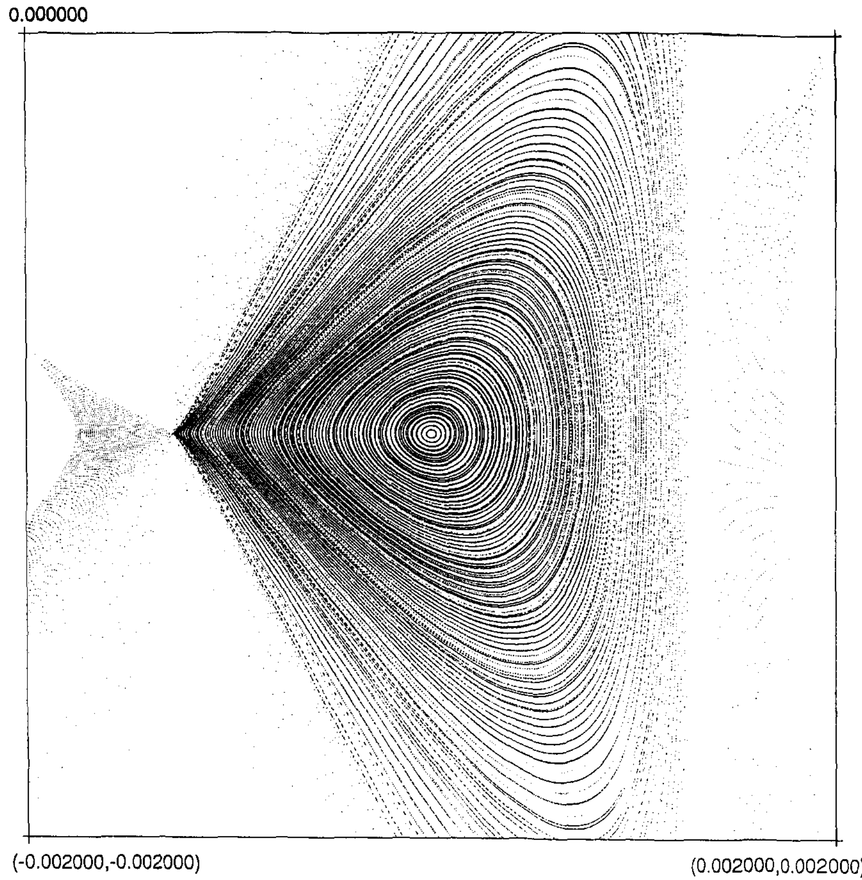

Fig. 2. One hundred different orbits of K_t for class IV.

These minors transform very simply under I:

$$I(m) = -\eta \cdot m'' , \qquad I(m') = \eta \cdot m''', \ldots , \tag{4.2}$$

where $\eta = \det(R)$.

The non-linear transformation I thus has a linear representation in terms of the 2×2 minors.

One can thus barter the sixteen homogeneous variables for these thirty six quadratic minors (4.1). Of course the number of homogeneous variables being sixteen (the number of entries of R), there must exist many relations between these minors (in fact such relations are closely related to the so-called Plücker relations [22]).

Let us now take a representative in each of the six previous classes. Let us consider for instance for class I transposition t_{I} exchanging a_2 and a_3. It is straightforward to see that by taking well-suited linear combinations (mostly sums or differences) of the abovementioned minors, one can actually get homogeneous polynomials invariant under t_{I} and covariant under I (see (4.2) and similarly for the other classes). For instance, one gets sixteen such homogeneous t_{I}-invariants polynomials, twenty-one t_{II}-invariants polynomials, twenty-two t_{IV}-invariants polynomials, ... We have thus obtained in each class a set of homogeneous polynomials denoted p_i, invariant under the transposition t and covariant under the homogeneous matricial inverse I, all with the same cofactor (the determinant of the matrix R) up to a sign. Therefore, one gets algebraic invariants (up to a sign) [20] under the group of transformation Γ_t as ratios of these covariants. For example, one can take in each case the ratios p_i/p_1.

The orbits of a particular group of birational transformations Γ_t are included into the intersection of quadrics defined by the invariants ($p_i/p_1 = k_i$, k_i being arbitrary constants). One has to calculate the dimension of this intersection, to confirm, or not, if they actually are curves, as the numerical study previously suggested. The results are as follows:

(i) the orbits of the groups of classes I, II and III are actually included into algebraic curves;

(ii) in contrast, for class IV, one invariant is missing, the orbits are only included into algebraic surfaces. It is shown in [17], that these surfaces are actually planes;

(iii) finally for classes V and VI, only thirteen algebraically independent polynomials are covariant, the orbits of the corresponding mappings are only assigned to lie on three dimensional algebraic varieties.

In all these cases, the different algebraic varieties foliate the whole parameter space.

The orbits corresponding to classes I, II and III are actually elliptic curves since they are algebraic curves stable under an infinite number of automorphisms [6] (or even they may degenerate into rational curves) [1,2]. Thus the corresponding mappings are integrable. This situation is very similar to the one encountered with the birational mappings associated to the sixteen-vertex model [5]. In terms of discrete dynamical systems, these three classes provide new interesting examples of integrable mappings, since the parameter space is a fifteen-dimensional one.

In contrast, since some algebraic invariants are missing for classes IV, V and VI, one does not understand straightforwardly how one can get curves from the iterations of the corresponding birational transformations: either the curves are not really curves (but for instance fractal-like set of point with the Hausdorff dimension very close to 1), either one does have algebraic curves and some covariants are missing, and have to be hinted among polynomials of higher degrees, or the curves are not algebraic anymore but transcendental. In fact, iteration calculations with high precision rule out some subtle fractal-like difference with curves: one actually gets curves. These numerical figures are highly stable under a very large number of iterations (more than 10^9), moreover, they also remain stable under perturbations of the initial matrix. Nevertheless strong enough perturbations can make some of them pop out in Julia-set like set of points [20]. This rules out the existence of some additional algebraic invariant, and shows that, in some domain of $\mathbb{C}P_{15}$, one actually has non-algebraic (transcendental) curves. Such a situation, where one gets transcendental curves in some regions of the parameter space, has been called "almost" integrable [20].

The situation one encounters here is visually similar to the one described by Siegel's theorem [21]. This theorem corresponds to describing the iteration of a quadratic transformation on one complex variable, namely

$$z \rightarrow \lambda z + z^2 , \tag{4.3}$$

where $\lambda = e^{2i\pi\theta}$, θ being diophantian. Siegel's theorem shows that, in some neighbourhood of $z = 0$, these iterations yield curves holomorphically conjugated to circles. These curves are $\mathbb{R}$-analytic transcendental curves and are included in some domain with an involved Julia set-like frontier [23]. The situation encountered here with these birational mappings seems, as far as the visualization of the orbits is concerned, more related to Siegel's theorem than to the KAM theorem. This is well illustrated by fig. 2. One does not see any rapid succession of ordered and disordered regions like in the KAM dynamics. In fact it will be shown in [17] that, at least for class IV, one does not have any hidden complex structure enabling to introduce a unique complex variable z, but that transformation $K^2_{t_{\mathrm{IV}}}$ actually reduces to a birational transformation in some (a, b)-plane reading as follows (with origin taken at some fixed point of $K^2_{t_{\mathrm{IV}}}$):

$$\begin{pmatrix} a \\ b \end{pmatrix} \rightarrow \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} R_1(a,b) \\ R_2(a,b) \end{pmatrix} , \tag{4.4}$$

where $R_1(a, b)$ and $R_2(a, b)$ are simple rational expressions of a and b, the lowest degree of their numerators being greater than two.

In order to study from another point of view this distinction between integrable and "almost" integrable mappings, let us now mention in the next section some results obtained in [17–19]: one can associate to these birational mappings new recurrences in a single variable. This makes the numerical iterations simpler and numerically much more controlled.

5. From birational mappings in $\mathbb{CP}_{q^2-1}$ to recurrences in one variable

With the method described in [17,18], one can exactly study the iteration of $K_t = t \cdot I$ on an arbitrary $q \times q$ matrix. This exact study brings out remarkable factorization properties for these $(q^2 - 1)$-dimensional mappings [17]. In terms of homogeneous variables, K_t is an homogeneous polynomial transformation of degree $(q - 1)$, thus $(K_t)^n(R)$ is a priori a matrix which entries are homogeneous polynomials of degree $(q - 1)^n$. In the cases considered here, the entries of $(K_t)^n(R)$ matrices happen to factorize, and since they are homogeneous, they can all be divided by their greatest common polynomial divisor. These homogeneous polynomials which factorize in all entries can be expressed in term of some elementary polynomials (related to determinants), we will denote the f_n's (see [17–19]). Thus, the degree of transformations $(K_t)^n$ falls sometimes to the point of being polynomial in term of the variable n [17,18]. In other publications [17–19,24,25], the link between this polynomial growth, instead of the generic exponential growth, and integrability is detailed [17–19,24,25]. Indeed, if one can easily imagine that the integrable mappings do have a polynomial growth of the "complexity", the reciprocal statement is far from being obvious [24]. In fact, the degree of transformation $(K_t)^n$ becomes polynomial for all the mappings of classes I, and II and III, but only for $q = 4$ [17], and even for the mapping associated with the sixteen vertex model [19], that is, for all cases corresponding to integrable mappings. This degree is exponential for the three remaining classes IV, V and VI, which do not correspond to integrable mappings [17][#1].

Let us illustrate these factorization properties on a lattice model of statistical mechanics. For instance, let us consider the symmetry group of the sixteen vertex model [5]. This infinite discrete group is generated by the matricial inverse I and a permutation of entries denoted t_1 [5,19]. Namely t_1 permutes the two off-diagonal 2×2 blocks of the 4×4 matrix [5,19].

When dealing with birational transformations associated to the sixteen vertex model [5] the polynomials which factorize are given in terms of some elementary homogeneous polynomials, denoted F_n (instead of f_n) [17,18]. Let us denote M_n the successive "reduced" matrices equal to $(K_{t_1})^n(R)$ divided by the greatest polynomial which factorizes in all entries of $(K_{t_1})^n(R)$. M_0 denotes

[#1] However, this exponential growth is independent of q and in fact strictly bounded by 3^n (in comparison with $(q - 1)^n$ generically).

the initial matrix R. One does not have any factorization for the first two iterates:

$$M_1 = K_{t_1}(M_0)\,, \qquad M_2 = K_{t_1}(M_1)\,, \tag{5.1}$$

but one does have factorizations for the next iterations [17,18]:

$$K_{t_1}(M_{n+1}) = F_n^2 \cdot M_{n+2}\,. \tag{5.2}$$

Moreover the F_n's do satisfy the following relation [17,18]:

$$F_{n+2} = \frac{\det(M_{n+1})}{F_n^3}\,. \tag{5.3}$$

Factorizations of the $K^n(R)$ matrices, of course, yield factorizations of their determinants. However, these determinants factorize even more: one also has similar factorization equations as eq. (5.3) for all the six classes I, . . . , VI [17–19]. For class I, and surprisingly for the generically non-integrable mappings of class IV, the determinants of the $K^n(R)$ matrices actually satisfy recurrences of a very simple form [17,18]. Let us give here some results obtained in [17] concerning these recurrences.

Actually the recurrence obtained for class I is exactly the same as the one corresponding to the sixteen vertex model up to a simple change of variables [18] and can be written in a more compact form in terms of some homogeneous variables q_n (simply related to the f_n's) [18], namely

$$\frac{q_n - q_{n+1}}{q_{n-1} - q_{n+2}} \cdot \frac{1}{q_n q_{n+1}} = \frac{q_{n+1} - q_{n+2}}{q_n - q_{n+3}} \cdot \frac{1}{q_{n+1} q_{n+2}}\,. \tag{5.4}$$

The recurrence obtained for class IV is actually different and reads [17]

$$\frac{q_{n+3} - q_{n+1}}{q_{n+4} - q_n} \cdot \frac{1}{q_{n+3} q_{n+1}} = \frac{q_{n+5} - q_{n+3}}{q_{n+6} - q_{n+2}} \cdot \frac{1}{q_{n+5} q_{n+3}}\,. \tag{5.5}$$

In this last case, the relation between the f_n's and q_n's is not as simple as for class I [17].

Note that recurrences (5.4) and (5.5) are still valid for classes I and IV generalized to $q \times q$ matrices. One thus has universal recurrences independent of q [17]. Let us now study the possible integrability of these recurrences.

In fact, eq. (5.4) can be "integrated". A first step introduces a constant λ as follows:

$$q_{n+2} - q_{n-1} = -\lambda\left(\frac{1}{q_{n+1}} - \frac{1}{q_n}\right) \tag{5.6}$$

and after two other "integrations" introducing two other constants ρ and μ, one finally gets a biquadratic equation relating q_n and q_{n+1}:

$$(\rho - q_n - q_{n+1})(q_n q_{n+1} + \lambda) = \mu \ . \tag{5.7}$$

It is well known that biquadratic equations are associated with elliptic curves [5]. Hence this recurrence on one variable has an elliptic parametrization, corresponding to the biquadratic relation (5.7). In contrast, the left-hand side and the right-hand side of eq. (5.5) are the same up to a shift of two. Therefore introducing the two constants of integration λ_1 and λ_2, one can see that eq. (5.5) is equivalent to equation

$$q_{n+4} - q_n = \lambda_n\left(\frac{1}{q_{n+1}} - \frac{1}{q_{n+3}}\right), \tag{5.8}$$

where $\lambda_{2n+1} = \lambda_1$ and $\lambda_{2n} = \lambda_2$. Unfortunately the $\lambda_1 \neq \lambda_2$ case does not yield generically integrable mappings. However, one can study the restricted recurrence, corresponding to $\lambda_1 = \lambda_2 = \lambda$. In that subcase recurrence (5.5) is actually integrable. The integration in fact yields two biquadratic equations:

$$\begin{aligned}&(\rho_2 - q_{2n})(\rho_1 - q_{2n+1})(q_{2n} q_{2n+1} + \lambda) - \mu = 0\ ,\\ &(\rho_2 - q_{2n+2})(\rho_1 - q_{2n+1})(q_{2n+2} q_{2n+1} + \lambda) - \mu = 0\ .\end{aligned} \tag{5.9}$$

One can also perform iterations of transformation $K_{t_{\mathrm{IV}}}$ in the (q_n, q_{n+1})-plane. For $\lambda_1 \neq \lambda_2$, we have systematically considered the iterations of transformation $K_{t_{\mathrm{IV}}}$ in the (q_n, q_{n+1})-plane. Remarkably, for a large set of initial conditions for the iterations, one still gets curves: these curves are highly stable even after more than 10^9 iterations!! This study of a one variable recurrence associated with class IV is complementary of the one performed in fifteen variables and actually confirms the "almost" integrability of this class of mappings.

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