

# Diagonals of rational functions: from differential algebra to effective algebraic geometry (unabridged version)

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## Abstract.

We show that the results we had obtained on diagonals of nine and ten parameters families of rational functions using creative telescoping, yielding modular forms expressed as pullbacked  ${}_2F_1$  hypergeometric functions, can be obtained, much more efficiently, calculating the  $j$ -invariant of an elliptic curve canonically associated with the denominator of the rational functions. In the case where creative telescoping yields pullbacked  ${}_2F_1$  hypergeometric functions, we generalise this result to other families of rational functions in three, and even more than three, variables. We also generalise this result to rational functions in more than three variables when the denominator can be associated to an algebraic variety corresponding to products of elliptic curves, foliation in elliptic curves. We also extend these results to rational functions in three variables when the denominator is associated with a *genus-two curve such that its Jacobian is a split Jacobian* corresponding to the product of two elliptic curves. We sketch the situation where the denominator of the rational function is associated with algebraic varieties that are not of the general type, having an infinite set of birational automorphisms. We finally provide some examples of rational functions in more than three variables, where the telescopers have pullbacked  ${}_2F_1$  hypergeometric solutions, the denominator corresponding to an algebraic variety being not simply foliated in elliptic curves, but having a selected elliptic curve in the variety explaining the pullbacked  ${}_2F_1$  hypergeometric solution.

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## 1. Introduction

In a previous paper [1, 2], using creative telescoping [3], we have obtained *diagonals*‡ of nine and ten parameters families of rational functions, given by (classical) *modular forms* expressed as pullbacked  ${}_2F_1$  hypergeometric functions††. The natural emergence of diagonal of rational functions in lattice statistical mechanics is explained in [18, 19]. This can be seen as the reason of the frequent occurrence of modular forms, Calabi-Yau operators in lattice statistical mechanics [20, 21, 22, 23, 24, 25]. In another previous paper [16, 17], dedicated to Heun functions that are diagonals of simple rational functions, or only solutions of *telescopers* [26, 27] of simple rational functions of three variables, but most of the time four variables, we have obtained many solutions of order-three telescopers having squares of Heun functions as solutions that turn out to be squares of pullbacked  ${}_2F_1$  hypergeometric solutions corresponding to *classical modular forms* and even *Shimura automorphic forms* [28, 29], strongly reminiscent of periods of *extremal rational surfaces* [30, 31], and other foliation of K3 surfaces in elliptic curves. In other words one finds experimentally that the  ${}_2F_1$  hypergeometric functions emerging in the calculation of diagonal of rational functions, or of solutions of the telescopers of rational functions, seem to be only special  ${}_2F_1([a, b], [c], x)$  hypergeometric functions with a selected set of parameters  $[a, b], [c]$  (see the list (B.1) in Appendix B of [16], corresponding to classical modular forms†, together with a finite set of parameters, like  $[7/24, 11/24], [5/4]$ , corresponding to Shimura automorphic forms [28, 29]), pullbacked by selected pullbacks. This last paper [16] also underlined the difference between the diagonal of a rational function  $Diag(R)$ , and the solutions of the telescoper of the same rational function. These results strongly suggested to find an algebraic geometry interpretation for all these results, and, more generally, suggested to provide an *alternative algebraic geometry approach of the results emerging from the creative telescoping*¶. This is the purpose of the present paper. In particular, we are going to show that most of these pullbacked  ${}_2F_1$  hypergeometric functions can be obtained efficiently through algebraic geometry calculations, thus providing a more intrinsic algebraic geometry interpretation of the creative telescoping calculations which are typically *differential algebra calculations* [26, 27, 33, 34].

Creative telescoping [26, 27, 33, 35] is a methodology to deal with parametrized symbolic sums and integrals that yields differential/recurrence equations for such expressions. This methodology became popular in computer algebra in the past twenty five years. By “telescoper” of a rational function, say  $R(x, y, z)$ , we here refer to the output of the creative telescoping program [3], applied to the *transformed* rational function  $\tilde{R} = R(x/y, y/z, z)/(yz)$ . Such a telescoper is a linear differential operator  $T$  in  $x$  and  $\frac{\partial}{\partial x}$ , such that  $T + \frac{\partial U}{\partial y} + \frac{\partial V}{\partial z}$  annihilates  $\tilde{R}$ , where  $U, V$  are rational functions in  $x, y, z$ . In other words, the telescoper  $T$  represents a linear ODE that is satisfied by  $Diag(R)$ .

The paper is essentially dedicated to *solutions of telescopers* of rational functions which are *not necessarily diagonals* of rational functions. These solutions correspond to periods [36] of algebraic varieties over some cycles which are not necessarily

‡ For the introduction of the concept of diagonals of rational functions see [4, 5, 6, 7, 8, 9, 10, 11].

†† The lattice Green functions are the simplest examples of such diagonal of rational functions [12, 13, 14, 15, 16, 17].

† See for instance Felix Klein’s connection of the  ${}_2F_1([1/12, 5/12], [1], x)$  Gauss hypergeometric function with modular forms, for instance in the very pedagogical and heuristic paper [32].

¶ The reader may refer to [33] for an extensive survey of “creative telescoping” approaches.

vanishing [37] cycles¶ like in the case of diagonals of rational functions. The reader interested in the connection between the process of taking diagonals, calculating telescopers, and the notion of Periods, deRham cohomology (i.e. differential forms) and other Picard-Fuchs equations can read in detail the thesis of Pierre Lairez [34] (see also [40]). We just sketch some of these ideas in Appendix A.

The purpose of this paper is not to give an introduction on creative telescoping [26, 27], but to provide many pedagogical (non-trivial) examples of telescopers using†† extensively the “*HolonomicFunctions*” Mathematica package [3].

Let us first recall the exact results of [1, 2].

## 2. Classical modular forms and diagonals of nine and ten parameters family of rational functions

In a previous paper [1, 2], using creative telescoping [3], we have obtained diagonals of nine and ten parameters families of rational functions, given by (classical) modular forms expressed as pullbacked  ${}_2F_1$  hypergeometric functions. Let us recall these results.

### 2.1. Nine-parameters rational functions giving pullbacked ${}_2F_1$ hypergeometric functions for their diagonals

Let us recall the *nine-parameters* rational function in three variables  $x$ ,  $y$  and  $z$ :

$$\frac{1}{a + b_1 x + b_2 y + b_3 z + c_1 y z + c_2 x z + c_3 x y + d y^2 z + e z x^2}. \quad (1)$$

Calculating† the telescoper of this rational function (1), one gets an *order-two* linear differential operator annihilating the diagonal of the rational function (1). The diagonal of the rational function (1) can be written [1, 2] as a pullbacked hypergeometric function

$$\frac{1}{P_4(x)^{1/4}} \cdot {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], 1 - \frac{P_6(x)^2}{P_4(x)^3}\right), \quad (2)$$

where  $P_4(x)$  and  $P_6(x)$  are two polynomials of degree four and six in  $x$  respectively. The Hauptmodul pullback in (2) has the form

$$\mathcal{H} = \frac{1728}{j} = 1 - \frac{P_6(x)^2}{P_4(x)^3} = \frac{1728 \cdot x^3 P_8(x)}{P_4(x)^3}, \quad (3)$$

where  $P_8(x)$  is a polynomial of degree eight in  $x$ . Such a pullbacked  ${}_2F_1$  hypergeometric function (2) corresponds to a *classical modular forms* [1, 2].

### 2.2. Ten-parameters rational functions giving pullbacked ${}_2F_1$ hypergeometric functions for their diagonals.

Let us recall the *ten-parameters* rational function in three variables  $x$ ,  $y$  and  $z$ :

$$R(x, y, z) = \frac{1}{a + b_1 x + b_2 y + b_3 z + c_1 y z + c_2 x z + c_3 x y + d_1 x^2 y + d_2 y^2 z + d_3 z^2 x}. \quad (4)$$

¶ In french “cycles évanescents” [38, 39].

†† One can obtain these telescopers using Chyzak’s algorithm [41] or Koutschan’s semi-algorithm [3, 42] (the termination is not proven). For the examples displayed in this paper, Koutschan’s package [3] is more efficient.

† Using the “*HolonomicFunctions*” Mathematica package [3].

Calculating<sup>‡</sup> the telescoper of this rational function (4), one gets an order-two linear differential operator annihilating the diagonal of the rational function (4). The diagonal of the rational function (4) can be written [1, 2] as a pullbacked hypergeometric function

$$\frac{1}{P_3(x)^{1/4}} \cdot {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], 1 - \frac{P_6(x)^2}{P_3(x)^3}\right), \quad (5)$$

where  $P_3(x)$  and  $P_6(x)$  are two polynomials of degree three and six in  $x$  respectively. Furthermore, the Hauptmodul pullback in (5) is seen to be of the form:

$$\mathcal{H} = \frac{1728}{j} = 1 - \frac{P_6(x)^2}{P_3(x)^3} = \frac{1728 \cdot x^3 \cdot P_9(x)}{P_3(x)^3}. \quad (6)$$

where  $P_9(x)$  is a polynomial of degree nine in  $x$ . Again, (5) corresponds to a *classical modular form* [1, 2].

### 3. Deducing creative telescoping results from effective algebraic geometry

Obtaining the previous pullbacked hypergeometric results (2) and (5) required [1, 2] an accumulation of creative telescoping calculations, and a lot of “guessing” using all the symmetries of the diagonals of these rational functions (1) and (4). We are looking for a more efficient and intrinsic way of obtaining these exact results. These two pullbacked hypergeometric results (2) and (5), are essentially “encoded” by their *Hauptmodul* pullbacks (3) and (6), or, equivalently, their corresponding *j-invariants*. The interesting question, which will be addressed in this paper, is whether it is possible to canonically associate an elliptic curve with precisely *j-invariants* corresponding to these Hauptmoduls  $\mathcal{H} = \frac{1728}{j}$ .

#### 3.1. Revisiting the pullbacked hypergeometric results in an algebraic geometry perspective.

One expects such an elliptic curve to correspond to the singular part of the rational function, namely the *denominator* of the rational function. Let us recall that the diagonal of a rational function is obtained through the multi-Taylor expansion of the rational function [18, 19]

$$R(x, y, z) = \sum_m \sum_n \sum_l a_{m,n,l} \cdot x^m y^n z^l, \quad (7)$$

by extracting the “diagonal” terms, i.e. the powers of the product  $p = xyz$ :

$$Diag(R(x, y, z)) = \sum_m a_{m,m,m} \cdot x^m. \quad (8)$$

Consequently, it is natural to consider the algebraic curve corresponding to the intersection of the surface corresponding to the vanishing condition  $D(x, y, z) = 0$  of the denominator  $D(x, y, z)$  of these rational functions (1) and (4), with the hyperbola  $p = xyz$  (where  $p$  is seen, here, as a constant). This amounts, for instance, to eliminating the variable  $z$ , substituting  $z = p/x/y$  in  $D(x, y, z) = 0$ .

<sup>‡</sup> Using the “*HolonomicFunctions*” program [3].

3.1.1. *Nine-parameters case:* In the case of the rational functions (1) this corresponds to the (planar) algebraic curve

$$\begin{aligned} a + b_1 x + b_2 y + b_3 \frac{p}{xy} + c_1 y \frac{p}{xy} + c_2 x \frac{p}{xy} + c_3 xy \\ + d y^2 \frac{p}{xy} + e \frac{p}{xy} x^2 = 0, \end{aligned} \quad (9)$$

which can be rewritten as a (general, nine-parameters) *biquadratic*:

$$\begin{aligned} a x y + b_1 x^2 y + b_2 x y^2 + b_3 p + c_1 p y + c_2 p x + c_3 x^2 y^2 \\ + d p y^2 + e p x^2 = 0. \end{aligned} \quad (10)$$

Using formal calculations<sup>†</sup> one can easily calculate the genus of the planar algebraic curve (10), and find that this planar algebraic curve *is actually an elliptic curve* (genus-one). Furthermore, one can (almost instantaneously) find the exact expression of the  $j$ -invariant of this elliptic curve as a rational function of the nine parameters  $a, b_1, b_2, \dots, e$  in (1). One actually finds that this  $j$ -invariant *is precisely the  $j$ -invariant  $j$  such that the Hauptmodul  $\mathcal{H} = \frac{1728}{j}$  is the exact expression* (3). In other words, the classical modular form result (2) could have been obtained, almost instantaneously, calculating the  $j$ -invariant of an elliptic curve canonically associated with the denominator of the rational function (1). The algebraic planar curve (10) *corresponds to the most general biquadratic of two variables*, which depends on nine homogeneous parameters. Such general biquadratic is well-known to be an elliptic curve for *generic values* of the *nine parameters*<sup>‡</sup>.

Thus, the nine-parameters exact result (2) *can be seen as a simple consequence of the fact that the most general nine-parameters biquadratic is an elliptic curve.*

3.1.2. *Ten-parameters case:* In the case of the rational function (4), one must consider the (planar) algebraic curve

$$\begin{aligned} a + b_1 x + b_2 y + b_3 \frac{p}{xy} + c_1 y \frac{p}{xy} + c_2 x \frac{p}{xy} + c_3 xy \\ + d_1 x^2 y + d_2 y^2 \frac{p}{xy} + d_3 \frac{p^2}{x^2 y^2} x = 0, \end{aligned} \quad (11)$$

i.e. the *ten-parameters bicubic*:

$$\begin{aligned} a x y^2 + b_1 x^2 y^2 + b_2 x y^3 + b_3 p y + c_1 p y^2 + c_2 p x y + c_3 x^2 y^3 \\ + d_1 x^3 y^3 + d_2 y^3 + d_3 p^2 = 0. \end{aligned} \quad (12)$$

Using formal calculations, one can easily calculate the genus of this selected planar algebraic curve (12), and find that this planar algebraic curve *is actually an elliptic curve*<sup>§</sup> (genus-one). Again one can find<sup>¶</sup> the exact expression of the  $j$ -invariant of this elliptic curve as a rational function of the ten parameters  $a, b_1, b_2, \dots, d_3$  in (4). One actually finds that this  $j$ -invariant *is precisely the  $j$ -invariant  $j$  such that the*

<sup>†</sup> Namely using `with(algcurves)` in Maple, and, in particular, the command `j_invariant`.

<sup>‡</sup> So many results in integrable models correspond to this most general biquadratic: the Bethe ansatz of the Baxter model [43, 44], the elliptic curve foliating the sixteen-vertex model [44], so many QRT birational maps [45], etc ...

<sup>§</sup> Generically, the most general planar bicubic is *not* a genus-one algebraic curve. It is a genus-four curve.

<sup>¶</sup> For the bicubic (12) the calculation of the  $j$ -invariant using the command `j_invariant` using `with(algcurves)` in Maple, requires much more computing time.

Hauptmodul  $\mathcal{H} = \frac{1728}{j}$  is the exact expression (6). In other words, the *classical modular form* result (5) could have been obtained, much more simply, calculating the  $j$ -invariant of an elliptic curve canonically associated with the denominator of the rational function (4).

Thus, this ten-parameters result (5) can again be seen as a simple consequence of the fact that *there exists a family of ten-parameters bicubics* (see (12)) *which are elliptic curves for generic values of the ten parameters*.

These preliminary calculations are a strong incentive to try to replace the differential algebra calculations of the *creative telescoping*, by more intrinsic algebraic geometry calculations, or, at least, perform effective algebraic geometry calculations to provide an algebraic geometry interpretation of the exact results obtained from creative telescoping.

### 3.2. Finding creative telescoping results from $j$ -invariant calculations.

One might think that these results are a consequence of the simplicity of the denominators of the rational functions (1) or (4), being associated with biquadratics or selected bicubics.

Let us consider a nine-parameters family of planar algebraic curves that are not biquadratics or (selected) bicubics:

$$a_1 x^4 + a_2 x^3 + a_3 x^2 + a_4 x + a_5 + a_6 x^2 y + a_7 y^2 + a_8 y + a_9 x y = 0. \quad (13)$$

One can easily calculate the genus of this planar curve and see that this genus is actually one for arbitrary values of the  $a_n$ 's. Thus the planar curve (13) *is an elliptic curve for generic values of the nine parameters*  $a_1, \dots, a_9$ . It is straightforward to see that the algebraic surface  $S(x, y, z) = 0$ , corresponding to

$$a_1 x^4 + a_2 x^3 + a_3 x^2 + a_4 x + a_5 + a_6 x^2 y + a_7 y^2 + a_8 y + a_9 \frac{p}{z} = 0, \quad (14)$$

or

$$z \cdot (a_1 x^4 + a_2 x^3 + a_3 x^2 + a_4 x + a_5 + a_6 x^2 y + a_7 y^2 + a_8 y) + a_9 p = 0, \quad (15)$$

will automatically be such that its intersection with the hyperbola  $p = xyz$  gives back the elliptic curve (13).

Using this kind of “reverse engineering” yields to consider the rational function in three variables  $x$ ,  $y$  and  $z$

$$R(x, y, z) = \frac{1}{1 + z \cdot (a_1 x^4 + a_2 x^3 + a_3 x^2 + a_4 x + a_5 + a_6 x^2 y + a_7 y^2 + a_8 y)}, \quad (16)$$

which will be such that *its denominator is canonically associated with an elliptic curve*. Again we can immediately calculate the  $j$ -invariant of that elliptic curve. If one calculates the telescoper of this eight-parameters family of rational functions (16), one finds that this telescoper is an order-two linear differential operator with pullbacked hypergeometric solutions of the form

$$\mathcal{A}(x) \cdot {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], \mathcal{H}\right), \quad (17)$$

where  $\mathcal{A}(x)$  is an algebraic function and, where again, the pullback-Hauptmodul  $\mathcal{H} = 1728/j$ , *precisely corresponds to the  $j$ -invariant of the elliptic curve*. In Appendix B, we give another example of a (planar) elliptic curve corresponding to

the *intersection of two quadrics*‡ where, again, one can get the (creative telescoping) pullbacked  ${}_2F_1$  result from a simple calculation of a  $j$ -invariant.

More generally, seeking for planar elliptic curves, one can look for planar algebraic curves

$$\sum_{n=0}^{n=N} \sum_{m=0}^{m=M} a_{m,n} \cdot x^n y^m = 0, \quad (18)$$

defined by the set of  $a_{m,n}$ 's which are equal to zero, apart of  $\mathcal{N}$  homogeneous parameters  $a_{m,n}$  being, as in (10) or (12) or (15), *independent parameters*. Finding such an  $\mathcal{N}$ -parameters family of (planar) elliptic curves automatically provides an  $\mathcal{N}$ -parameters family of rational functions such that their telescopers have a pullbacked  ${}_2F_1$  hypergeometric solution we can simply deduce from the  $j$ -invariant of that elliptic curve.

**Question:** Recalling section 2.2, is it possible to find families of such (planar) elliptic curves *which depend on more than ten independent parameters*?

Before addressing this question, let us recall the concept of *birationally equivalent elliptic curves*. Let us consider for example the following monomial transformation:

$$(x, y) \longrightarrow (x^{12} y^{11}, x^{205} y^{188}). \quad (19)$$

Its compositional inverse is the monomial transformation:

$$(x, y) \longrightarrow \left( \frac{x^{188}}{y^{11}}, \frac{y^{12}}{x^{205}} \right). \quad (20)$$

This monomial transformation (19) is thus a *birational*† transformation. A birational transformation transforms an elliptic curve, like (13), into another elliptic curve *with the same  $j$ -invariant*: these two elliptic curves are called *birationally equivalent*. In the case of the birational and monomial transformation (19), the elliptic curve (13) is changed into††:

$$\begin{aligned} a_1 x^{48} y^{44} + a_2 x^{36} y^{33} + a_3 x^{24} y^{22} + a_4 x^{12} y^{11} + a_5 \\ + a_6 x^{229} y^{210} + a_7 x^{410} y^{376} + a_8 x^{205} y^{108} + a_9 x^{217} y^{199} = 0. \end{aligned} \quad (21)$$

With this kind of birational monomial transformation (19), we see that one can find *families of elliptic curves* (21) *of arbitrary large degrees* in  $x$  and  $y$ . Consequently one can find nine or ten parameters families of rational functions of *arbitrary large degrees* yielding pullbacked  ${}_2F_1$  hypergeometric functions. There is no constraint on the degree of the planar algebraic curves (21): *the only relevant question is the question of the maximum number of (linearly) independent parameters of families of planar elliptic curves*. In fact, *it is possible to show that the maximum number of independent parameters is actually ten*. We sketch the demonstration¶ in Appendix C.

### 3.3. Pullbacked ${}_2F_1$ functions for higher genus curves: monomial transformations.

We have already remarked in [1, 2] that once we have an exact result for a diagonal of a rational function  $R(x, y, z)$ , we immediately get another exact result for the diagonal of the rational function  $R(x^n, y^n, z^n)$  for any positive integer  $n$ . As a result

‡ Intersections of quadrics are well-known to give elliptic curves [46, 44].

† This transformation is rational and its compositional inverse is also rational (here monomial).

†† One can easily verify for particular values of the  $a_k$ 's, using with(algcurves) in Maple, that the  $j$ -invariants of (13) and (21) are actually equal.

¶ We thank Josef Schicho for providing this demonstration.

we obtain a new expression for the diagonal changing  $x$  into  $x^n$ . In fact, this is also a result on the telescoper of the rational function  $R(x, y, z)$ : the telescoper of the rational function  $R(x^n, y^n, z^n)$  is the  $x \rightarrow x^n$  pullback of the telescoper of the rational function  $R(x, y, z)$ . Having a pullbacked  ${}_2F_1$  solution for the telescoper of the rational function  $R(x, y, z)$  (resp. the diagonal of the rational function  $R(x, y, z)$ ), we will immediately deduce a pullbacked  ${}_2F_1$  solution for the telescoper of the rational function  $R(x^n, y^n, z^n)$  (resp. the diagonal of the rational function  $R(x^n, y^n, z^n)$ ).

Along this line, let us change in the rational function (1),  $(x, y, z)$  into  $(x^2, y^2, z^2)$ :

$$R_2(x, y, z) = \frac{1}{a + b_1 x^2 + b_2 y^2 + b_3 z^2 + c_1 y^2 z^2 + c_2 x^2 z^2 + c_3 x^2 y^2 + d y^4 z^2 + e z^2 x^4}. \quad (22)$$

The diagonal of this new rational function (22) will be the pullbacked  ${}_2F_1$  exact expression (2) where we change  $x \rightarrow x^2$ . The intersection of the algebraic surface corresponding to the vanishing condition of the denominator of the new rational function (22), with the hyperbola  $p = xyz$  (i.e.  $z = p/x/y$ ), is nothing but the equation (10) where we have changed  $(x, y; p)$  into  $(x^2, y^2; p^2)$

$$a x^2 y^2 + b_1 x^4 y^2 + b_2 x^2 y^4 + b_3 p^2 + c_1 p^2 y^2 + c_2 p^2 x^2 + c_3 x^4 y^4 + d p^2 y^4 + e p^2 x^4 = 0, \quad (23)$$

which is *no longer*† an elliptic curve but a curve of genus 9.

With that example we see that classical modular form results, or pullbacked  ${}_2F_1$  exact expressions like (2), can actually emerge from *higher genus curves* like (23). As far as these diagonals, or telescopers, of rational function calculations are concerned, higher genus curves like (23) must in fact be seen as “almost” elliptic curves up to a  $x \rightarrow x^n$  covering.

Such results for monomial transformations like  $(x, y, z) \rightarrow (x^n, y^n, z^n)$  can, in fact, be generalised to more general (non birational†) monomial transformations. This is sketched in Appendix D.

### 3.4. Changing the parameters into functions of the product $p = xyz$ .

All these results for many parameters families of rational functions can be *drastically generalised* when one remarks that allowing any of these parameters to be a *function* of the product  $p = xyz$  also yields to the previous pullbacked  ${}_2F_1$  exact expression, like (2), *where the parameter is changed into that function of  $x$*  (see [1]). Let us consider a simple (two-parameters) illustration of this general result. Let us consider a subcase of the previous nine or ten parameters families, introducing the two parameters rational function:

$$\frac{1}{1 + 2x + b_2 \cdot y + 5yz + xz + c_3 \cdot xy}. \quad (24)$$

The diagonal of this rational function (24) is the pullbacked hypergeometric function:

$$\frac{1}{P_2(x)^{1/4}} \cdot {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], 43200 \cdot x^4 \cdot \frac{P_4(x)}{P_2(x)^3}\right), \quad (25)$$

‡ If we perform the same calculations with the ten-parameters rational function (4) we get an algebraic curve of genus 10 instead of 9.

† In contrast with transformations like (19).



where

$$P_2(x) = 1 - 8 \cdot (b_2 + 10) \cdot x + 8 \cdot (2b_2^2 - 20b_2 + 15c_3 + 200) \cdot x^2, \quad (26)$$

and

$$\begin{aligned} P_4(x) = & -675c_3^4 \cdot x^4 + 4c_3^2 \cdot (b_2 + 10) \cdot (4b_2^2 - 100b_2 + 45c_3 + 400) \cdot x^3 \\ & + (64b_2^4 - 32b_2^3c_3 - 8b_2^2c_3^2 - 1280b_2^3 + 1280b_2^2c_3 \\ & - 460b_2c_3^2 - 5c_3^3 + 6400b_2^2 - 3200b_2c_3 - 800c_3^2) \cdot x^2 \\ & - (b_2 + 10) \cdot (32b_2^2 - 16b_2c_3 - c_3^2) \cdot x + 2b_2 \cdot (2b_2 - c_3), \end{aligned} \quad (27)$$

Let us now consider the previous rational function (24) where the two parameters  $b_2$  and  $c_3$  become some rational functions of the product  $p = xyz$ , for instance:

$$b_2(p) = \frac{1+3p}{1+7p^2}, \quad c_3(p) = \frac{1+p^2}{1+2p} \quad \text{where:} \quad p = xyz. \quad (28)$$

The new corresponding rational function becomes more involved but one can easily calculate the telescoper of this new rational function of three variables  $x$ ,  $y$  and  $z$ , and find that it is, *again*, an order-two linear differential operator having the pullbacked hypergeometric solution (25) where  $b_2$  and  $c_3$  are, now, replaced by ( $p$  is now  $x$ ) the functions:

$$b_2(x) = \frac{1+3x}{1+7x^2}, \quad c_3(x) = \frac{1+x^2}{1+2x}. \quad (29)$$

In that case (24) with (28), one gets a diagonal which is the pullbacked hypergeometric solution

$$\begin{aligned} & (1+2x)^{1/4} \cdot (1+7x^2)^{1/4} \cdot q_8^{-1/4} \\ & \times {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], \frac{43200 \cdot x^4 \cdot (1+7x^2)^2 \cdot q_{20}}{(1+2x) \cdot q_8^3}\right), \end{aligned} \quad (30)$$

where:

$$\begin{aligned} q_8 = & 5880x^8 + 156800x^7 + 71400x^6 + 35330x^5 + 19985x^4 \\ & + 1332x^3 + 1390x^2 - 86x + 1, \\ q_{20} = & -1620675x^{20} + 1234800x^{18} + 158332230x^{17} + 153642195x^{16} \\ & + 427157990x^{15} + 344201585x^{14} + 367632300x^{13} + 293263834x^{12} \\ & + 229496405x^{11} + 188180096x^{10} + 107454499x^9 + 51936025x^8 \\ & + 21019296x^7 + 6259829x^6 + 1645018x^5 + 266619x^4 \\ & + 40629x^3 - 1110x^2 - 127x + 2, \end{aligned} \quad (31)$$

which is nothing but (25) (with (26) and (27)) where  $b_2$  and  $c_3$  have been replaced by the functions (29). Similar calculations can be performed for more general rational functions (1) or (4), when all the (nine or ten) parameters are more involved rational functions.

When performing our creative telescoping symbolic calculations using the HolonomicFunctions package [3], such results may look quite impressive. From the algebraic geometry viewpoint, it is almost tautological $\ddagger$ , if one takes for granted the result of our previous subsections 3.1 and 3.2, namely that the pullbacked

$\ddagger$  An algebraic geometer will probably see this as a trivial remark: diagonalization is an algebraic procedure and nothing really happens to the coefficients. Therefore if one replaces the coefficients by anything else, one will find those replaced coefficients in the end result.

hypergeometric solution of the telescoper corresponds to the Hauptmodul 1728/ $j$ , where  $j$  is the  $j$ -invariant of the elliptic curve corresponding to the intersection of the algebraic surface corresponding to the vanishing condition of the denominator, with the hyperbola  $p = xyz$ : this calculation of the  $j$ -invariant is performed for  $p$  fixed, and arbitrary (nine or ten) parameters  $a, b_1, \dots$ . It is clearly possible to force the parameters to be functions<sup>†</sup> of  $p$ , the  $j$ -invariant being changed accordingly. Of course, in that case, the parameters in the rational function are the same functions but of the product  $p = xyz$ .

One thus gets pullbacked hypergeometric solutions (classical modular forms) for an (unreasonably ...) large set of rational functions in three variables, namely the families of rational functions (1) or (4), but where, now, the nine or ten parameters are nine, or ten, totally arbitrary rational functions of the product  $p = xyz$ .

**Remark:** When the rational function depends on parameters, one can straightforwardly deduce the solutions of the *telescoper* of the rational function where the parameters are changed into functions. In this example (see (24), (25), (28), (30)), the solution (25) or (30) of the *telescoper* of the rational function is actually the diagonal of the rational function.

We see experimentally that changing the parameters of the rational function into functions, actually works for *diagonals* of rational functions. Let us sketch the demonstration.

*3.4.1. Sketching the demonstration.* Let us introduce the multi-Taylor expansion of the rational function (24) where  $b_2$  and  $c_3$  are parameters (not functions of the product  $p = xyz$ ):

$$R(x, y, z) = \sum_m \sum_n \sum_l a_{m,n,l}(b_2, c_3) \cdot x^n y^m z^l. \quad (32)$$

The diagonal of this rational function (24) reads:

$$\text{Diag}(R(x, y, z)) = \sum_m a_{m,m,m}(b_2, c_3) \cdot x^m. \quad (33)$$

Let us assume that we have an exact closed expression  $\mathcal{E}(b_2, c_3; x)$  for this diagonal (33), like the previous pullbacked hypergeometric functions (2) or (5) (or possibly some Heun functions [17], or more involved exact expressions, like Appell functions, Lauricella functions, ...).

Let us assume that the coefficients  $a_{m,m,m}$ , seen as functions of  $b_2$  and  $c_3$ , have a multi-Taylor expansion in  $b_2$  and  $c_3$ :

$$a_{m,m,m}(b_2, c_3) = \sum_{M,N} A_{M,n} \cdot b_2^M c_3^N. \quad (34)$$

Let us now assume that  $b_2$  and  $c_3$  are *functions of the product*  $p = xyz$  (or more generally functions with Taylor series expansions at  $p = 0$ ). The rational function (24), where  $b_2$  and  $c_3$  are now functions of the product  $p = xyz$ , has the multi-series expansion:

$$R(x, y, z) = \sum_m \sum_n \sum_l a_{m,n,l}(b_2(p), c_3(p)) \cdot x^n y^m z^l. \quad (35)$$

<sup>†</sup> The functions should be rational functions if one wants to stick with diagonals and telescopers of *rational* functions, but the result remains valid for *algebraic functions*, or *even transcendental functions* with reasonable series expansions at  $x = 0$ .

Let us assume that these two functions  $b_2(p)$  and  $c_3(p)$  both have a Taylor series expansion near  $p = 0$ .

Consequently the coefficients  $a_{m,m,m}$  in the multi-Taylor expansion (35) have a Taylor series expansion near  $p = 0$ :

$$a_{m,m,m}(b_2(p), c_3(p)) = \sum_q \alpha_{M,n}^{(q)} \cdot p^q. \quad (36)$$

The diagonal of the rational function (35) is actually (33) where the two parameters  $b_2$  and  $c_3$  are changed into two functions  $b_2(x)$  and  $c_3(x)$  (like (29)):

$$Diag(\mathcal{R}(x, y, z)) = \sum_m a_{m,m,m}(b_2(x), c_3(x)) \cdot x^m. \quad (37)$$

This multi-series (37) has a Taylor series expansion which can be seen to be the Taylor series expansion of the exact closed expression  $\mathcal{E}(b_2(x), c_3(x); x)$ .

Of course this demonstration can be generalised to an arbitrary number of parameters and for an arbitrary numbers of variables.

#### 4. Creative telescoping on rational functions of more than three variables associated with products or foliations of elliptic curves

Let us show that such an algebraic geometry approach of the creative telescoping can be generalised to rational functions of *more than three* variables, when the vanishing condition of the denominator can be associated with *products of elliptic curves*, or more generally, algebraic varieties with *foliations in elliptic curves*.

- The telescoper of the rational function in the *four variables*  $x, y, z$  and  $w$

$$\frac{xyz}{(1+z)^2 - x \cdot (1-x) \cdot (x - xyzw) \cdot y \cdot (1-y) \cdot (y - xyzw)}, \quad (38)$$

gives an order-three *self-adjoint* linear differential operator which is, thus, the *symmetric square* of an order-two linear differential operator. This order-two linear differential operator has the pullbacked hypergeometric solution:

$$\mathcal{S}_1 = (1 - x + x^2)^{-1/4} \cdot {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], \frac{27}{4} \cdot \frac{x^2 \cdot (1-x)^2}{(x^2 - x + 1)^3}\right). \quad (39)$$

This pullbacked hypergeometric solution (39) can also be simply written:

$${}_2F_1\left(\left[\frac{1}{2}, \frac{1}{2}\right], [1], x\right). \quad (40)$$

In [17] we underlined the difference between the *diagonal* of a rational function and *solutions of the telescoper* of the same rational function. In this case, the diagonal of the rational function (38), is zero† and is thus different from the pullbacked hypergeometric solution (39), which is a “*Period*” [36] of the algebraic variety corresponding to the denominator over some (*non-vanishing*‡) *cycle*. From now, we will have a similar situation in most of the following examples of this paper.

† The reason is that the integration takes place over a cycle homologically equivalent to the trivial cycle. The cycle becomes trivial after taking the limit  $p \rightarrow 0$ . Integrals over non vanishing cycles usually give logarithms of  $p$ , like the second solution to the hypergeometric function  ${}_2F_1([1/2, 1/2], [1], x)$ .

‡ Diagonals of the rational functions correspond to periods over *vanishing cycles* [37, 39].

This example is a simple illustration of what we expect for *products of elliptic curves*, or algebraic varieties with *foliations in elliptic curves*. Introducing the product  $p = xyzw$ , the vanishing condition of the denominator of the rational function (38) reads the surface  $S(x, y, z) = 0$ :

$$(1+z)^2 - x \cdot (1-x) \cdot (x-p) \cdot y \cdot (1-y) \cdot (y-p) = 0. \quad (41)$$

For fixed  $p$  and fixed  $y$ , equation (41) can be seen as an algebraic curve

$$(1+z)^2 - \lambda \cdot x \cdot (1-x) \cdot (x-p) = 0 \quad (42)$$

for:  $\lambda = y \cdot (1-y) \cdot (y-p).$

For fixed  $p$  and fixed  $y$ ,  $\lambda$  can be seen as a constant, the algebraic curve (42) being an *elliptic curve* with an obvious Weierstrass form:

$$Z^2 - x \cdot (1-x) \cdot (x-p) = 0. \quad (43)$$

The  $j$ -invariant of (42), or‡ (43), is well-known and yields the Hauptmodul  $\mathcal{H}$ :

$$\mathcal{H} = \frac{1728}{j} = \frac{27}{4} \cdot \frac{p^2 \cdot (1-p)^2}{(p^2 - p + 1)^3}. \quad (44)$$

For fixed  $p$  and fixed  $x$ , equation (41) can be seen as an algebraic curve

$$(1+z)^2 - \mu \cdot y \cdot (1-y) \cdot (y-p) = 0 \quad (45)$$

for:  $\mu = x \cdot (1-x) \cdot (x-p),$

which is also an elliptic curve with an obvious Weierstrass form and the *same* Hauptmodul (44).

More generally, the rational function of the *four variables*  $x$ ,  $y$ ,  $z$  and  $w$

$$\frac{xyz}{(1+z)^2 - x \cdot (1-x) \cdot (x - R_1(p)) \cdot y \cdot (1-y) \cdot (y - R_2(p))}, \quad (46)$$

where  $p = xyzw$ , and where  $R_1(p)$  and  $R_2(p)$  are two arbitrary rational functions of the product  $p = xyzw$ , yields a telescoper which has an *order-four* linear differential operator which is the *symmetric product*¶ of two order-two linear differential operators having respectively the pullbacked hypergeometric solutions (39) where  $x$  is replaced by  $R_1(x)$  and  $R_2(x)$ . These two hypergeometric solutions thus have the two Hauptmodul pullbacks:

$$\mathcal{H}_1 = \frac{1728}{j_1} = \frac{27}{4} \cdot \frac{R_1(p)^2 \cdot (1 - R_1(p))^2}{(R_1(p)^2 - R_1(p) + 1)^3}, \quad (47)$$

$$\mathcal{H}_2 = \frac{1728}{j_2} = \frac{27}{4} \cdot \frac{R_2(p)^2 \cdot (1 - R_2(p))^2}{(R_2(p)^2 - R_2(p) + 1)^3}. \quad (48)$$

A solution of the telescoper of (46) is thus the *product* of these two pullbacked hypergeometric functions. Let us give two simple illustrations of this general result, with the two next examples.

‡ A shift  $z \rightarrow z+1$  or a rescaling  $z^2 \rightarrow z^2/\lambda$  does not change the  $j$ -invariant of the Weierstrass elliptic form.

¶ This paper belonging to the symbolic computation literature and not pure mathematics for algebraic geometers, we use the standard Maple (DEtools) terminology of symmetric powers and symmetric products of linear differential operators [47]. Note that "symmetric product" is not a proper mathematical name for this construction on the solution space; it is a homomorphic image of the tensor product. The (Maple/DEtools) reason for choosing the name `symmetric_product` is the resemblance with the function `symmetric_power`.

- The telescoper of the rational function in the *four variables*  $x, y, z$  and  $w$

$$\frac{xyz}{(1+z)^2 - x \cdot (1-x) \cdot (x - xyzw) \cdot y \cdot (1-y) \cdot (y - x^2 y^2 z^2 w^2)}, \quad (49)$$

gives an *order-four* linear differential operator which is the *symmetric product* of *two order-two* linear differential operators having respectively the pullbacked hypergeometric solution (39) and the solution (39) where  $x$  has been changed into  $x^2$ :

$$(1 - x^2 + x^4)^{-1/4} \cdot {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], \frac{27}{4} \cdot \frac{x^4 \cdot (1 - x^2)^2}{(x^4 - x^2 + 1)^3}\right). \quad (50)$$

- The telescoper of the rational function in the *four variables*  $x, y, z$  and  $w$

$$\frac{xyz}{(1+z)^2 - x \cdot (1-x) \cdot (x - xyzw) \cdot y \cdot (1-y) \cdot (y - 3xyzw)}, \quad (51)$$

gives an *order-four* linear differential operator which is the *symmetric product* of two order-two operators having respectively the pullbacked hypergeometric solution (39) and the solution (39) where the variable  $x$  has been changed into  $3x$ :

$$\mathcal{S}_2 = (1 - 3x + 9x^2)^{-1/4} \cdot {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], \frac{243}{4} \cdot \frac{x^2 \cdot (1 - 3x)^2}{(1 - 3x + 9x^2)^3}\right). \quad (52)$$

#### 4.1. Creative telescoping on rational functions of five variables associated with products or foliations of three elliptic curves

Let us, now, introduce the rational function in *five variables*  $x, y, z, v$  and  $w$

$$\frac{xyzv}{D(x, y, z, v, w)}, \quad (53)$$

where the denominator  $D(x, y, z, v, w)$  reads:

$$D_p = (1+v)^2 - x \cdot (1-x) \cdot (x-p) \cdot y \cdot (1-y) \cdot (y-3p) \cdot z \cdot (1-z) \cdot (z-5p), \quad (54)$$

where:  $p = xyzvw$ .

The telescoper of the rational function (53) of *five variables* gives<sup>‡</sup> an *order-eight* linear differential operator which is the *symmetric product* of *three order-two* linear differential operators having respectively the pullbacked hypergeometric solution (39), the solution (39) where  $x$  has been changed into  $3x$ , namely (52), and the solution (39), where  $x$  has been changed into  $5x$ :

$$\mathcal{S}_3 = (1 - 5x + 25x^2)^{-1/4} \cdot {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], \frac{675}{4} \cdot \frac{x^2 \cdot (1 - 5x)^2}{(1 - 5x + 25x^2)^3}\right). \quad (55)$$

In other words, the order-eight linear differential telescoper of the rational function (53) has the *product*  $\mathcal{S} = \mathcal{S}_1 \cdot \mathcal{S}_2 \cdot \mathcal{S}_3$ , of (39), (52) and (55) as a solution. From an algebraic geometry viewpoint this is a consequence of the fact that, for fixed  $p$ , the algebraic variety  $D_p = 0$ , where  $D_p$  is given by (54), can be seen, for fixed  $y$  and  $z$ , as an *elliptic curve*  $\mathcal{E}_1$  of equation  $D_{y,z,p}(v, x) = 0$ , for fixed  $x$  and  $z$  as an elliptic curve  $\mathcal{E}_2$  of equation  $D_{x,z,p}(v, y) = 0$ , and for fixed  $x$  and  $y$  also as an elliptic curve  $\mathcal{E}_3$  of equation  $D_{x,y,p}(v, z) = 0$ , the  $j$ -invariants  $j_k$ ,  $k = 1, 2, 3$  of these

<sup>‡</sup> Such a creative telescoping calculation requires “some” computing time to achieve the result ...

three elliptic curves  $\mathcal{E}_k$  yielding (in terms of  $p$ ), precisely, the three Hauptmoduls  $\mathcal{H}_k = 1728/j_k$

$$\frac{27}{4} \cdot \frac{x^2 \cdot (1-x)^2}{(x^2-x+1)^3}, \quad \frac{243}{4} \cdot \frac{x^2 \cdot (1-3x)^2}{(1-3x+9x^2)^3}, \quad \frac{675}{4} \cdot \frac{x^2 \cdot (1-5x)^2}{(1-5x+25x^2)^3}, \quad (56)$$

occurring as pullbacks in the three  $\mathcal{S}_k$ 's of the solution  $\mathcal{S} = \mathcal{S}_1 \cdot \mathcal{S}_2 \cdot \mathcal{S}_3$ , of the telescoper of (53).

#### 4.2. Weierstrass and Legendre forms

The telescoper of the rational function in three variables

$$\frac{xy}{(1+y)^2 - x \cdot (1-x) \cdot (x-xyz)}, \quad (57)$$

associated§ with the *elliptic curve* in a *Weierstrass form*:

$$(1+y)^2 - x \cdot (1-x) \cdot (x-p) = 0, \quad (58)$$

is the order-two linear differential operator

$$L_2 = -1 + 4 \cdot (1-2x) \cdot D_x + 4 \cdot x \cdot (1-x) \cdot D_x^2, \quad (59)$$

which has the hypergeometric solution:

$$\begin{aligned} {}_2F_1\left(\left[\frac{1}{2}, \frac{1}{2}\right], [1], x\right) \\ = (1-x+x^2)^{-1/4} \cdot {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], \frac{27}{4} \cdot \frac{x^2 \cdot (1-x)^2}{(1-x+x^2)^3}\right). \end{aligned} \quad (60)$$

The elliptic curve (58) has the Hauptmodul

$$\mathcal{H} = \frac{27}{4} \cdot \frac{p^2 \cdot (1-p)^2}{(1-p+p^2)^3}. \quad (61)$$

in agreement with the pullback in (60).

*4.2.1. K3 surfaces as products or foliations of two elliptic curves.* All the previous examples of this section correspond to denominators which are algebraic varieties that can be seen as *Weierstrass elliptic curves* for fixed values of all the variables except two. Let us show that one also gets simple telescopers for rational functions with denominators which are *algebraic varieties with some foliation in elliptic curves*‡.

- The telescoper of the rational function in *four* variables

$$\frac{xyz}{(1+z)^2 - x \cdot (1-x) \cdot y \cdot (x-y) \cdot (y-xyzw)}, \quad (62)$$

associated with the  $K_3$  surface written in a *Legendre form*||

$$(1+z)^2 - x \cdot (1-x) \cdot y \cdot (x-y) \cdot (y-p) = 0, \quad (63)$$

is an order-three *self-adjoint*†† linear differential operator  $L_3$

$$L_3 = x \cdot (2\theta + 1)^3 - 8 \cdot \theta^3, \quad (64)$$

§ The diagonal extracts the terms function of the product  $p = xyz$  in the multi-Taylor series.

‡ Like K3 surfaces, or three-fold Calabi-Yau manifolds.

|| Along this line see the first equation page 19 of [48].

†† The order-three linear differential operator is thus the symmetric square of an order-two linear differential operator.

which has the following  ${}_3F_2$  solution (which is also, because of a Clausen formula the square of a  ${}_2F_1$  function):

$${}_3F_2\left(\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right], [1, 1], x\right) = {}_2F_1\left(\left[\frac{1}{4}, \frac{1}{4}\right], [1], x\right)^2. \quad (65)$$

The  $K_3$  surface (63) can be seen as *associated with the product of two Weierstrass elliptic curves*† of Hauptmoduls respectively:

$$\mathcal{H}_x = \frac{27}{4} \cdot \frac{p^2 \cdot (1-p)^2}{(1-p+p^2)^3}, \quad \mathcal{H}_y = \frac{27}{4} \cdot \frac{y^2 \cdot (1-y)^2}{(1-y+y^2)^3}. \quad (66)$$

This order-three linear differential operator  $L_3$  is the *symmetric square* of the order-two linear differential operator

$$M_2 = -1 + 8 \cdot (2-3x) \cdot D_x + 16 \cdot x \cdot (1-x) \cdot D_x^2, \quad (67)$$

which has the hypergeometric solutions:

$${}_2F_1\left(\left[\frac{1}{4}, \frac{1}{4}\right], [1], x\right) = \left(1 - \frac{x}{4}\right)^{-1/4} \cdot {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], -\frac{27 \cdot x^2}{(x-4)^3}\right). \quad (68)$$

One thus finds that the telescoping procedure associates to the  $K_3$  surface, “encoded” by  $(\mathcal{H}_x, \mathcal{H}_y)$ , the Hauptmodul given in (68):

$$\left(\frac{27}{4} \cdot \frac{p^2 \cdot (1-p)^2}{(1-p+p^2)^3}, \frac{27}{4} \cdot \frac{y^2 \cdot (1-y)^2}{(1-y+y^2)^3}\right) \rightarrow \left(-\frac{27 \cdot p^2}{(p-4)^3}, -\frac{27 \cdot p^2}{(p-4)^3}\right). \quad (69)$$

**Remark:** The telescoper of

$$\frac{xy}{(1+z)^2 - x \cdot (1-x) \cdot y \cdot (x-y) \cdot (y-xyzw)}, \quad (70)$$

is a huge (48990 characters ...) order-eleven linear differential operator. The telescoper of

$$\frac{1}{(1+z)^2 - x \cdot (1-x) \cdot y \cdot (x-y) \cdot (y-xyzw)}, \quad (71)$$

is a huge (58702 characters ...) order-twelve linear differential operator. The telescoper of

$$\frac{xzw}{(1+z)^2 - x \cdot (1-x) \cdot y \cdot (x-y) \cdot (y-xyzw)}, \quad (72)$$

is a huge (59754 characters ...) order-eleven linear differential operator. This raises the question of *how telescopers of rational functions are changed when one modifies the numerator of the rational function, keeping the same denominator*. This is a quite involved question that we will address in forthcoming papers.

• Let us now introduce the telescoper of the rational function in *four* variables  $x, y, z$  and  $w$

$$\frac{xyz}{(1+z)^2 - x \cdot (1-x) \cdot y \cdot (x-y) \cdot (y-4 \cdot p \cdot (1-p))}, \quad (73)$$

where  $p$  denotes the product  $p = xyzw$ . This rational function is nothing but (62) where  $p$  has been changed into  $4 \cdot p \cdot (1-p)$ . The telescoper of the rational function of *four* variables (73) is a *self-adjoint* order-three linear differential operator which is,

†  $K_3$  surfaces are not abelian varieties, but they are “close” to abelian varieties: they can be seen as essentially products of two elliptic curves.

thus, the *symmetric square* of an order-two linear differential operator. This order-two linear differential operator has the solution:

$$\begin{aligned} & {}_2F_1\left(\left[\frac{1}{2}, \frac{1}{2}\right], [1], x\right) \\ &= \left(1 - x + x^2\right)^{-1/4} \cdot {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], \frac{27}{4} \cdot \frac{x^2 \cdot (1-x)^2}{(1-x+x^2)^3}\right). \end{aligned} \quad (74)$$

The relation of this result (74) with the previous result (68) corresponds to the following identity for  $X = 4 \cdot x \cdot (1-x)$ :

$$\begin{aligned} & {}_2F_1\left(\left[\frac{1}{4}, \frac{1}{4}\right], [1], X\right) = {}_2F_1\left(\left[\frac{1}{4}, \frac{1}{4}\right], [1], 4x(1-x)\right) = {}_2F_1\left(\left[\frac{1}{2}, \frac{1}{2}\right], [1], x\right) \\ &= \left(1 - \frac{X}{4}\right)^{-1/4} \cdot {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], -\frac{27 \cdot X^2}{(X-4)^3}\right) \\ &= \left(1 - x + x^2\right)^{-1/4} \cdot {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], \frac{27}{4} \cdot \frac{x^2 \cdot (1-x)^2}{(1-x+x^2)^3}\right). \end{aligned} \quad (75)$$

We thus get exactly the same solution (39) or (40) than the one for the telescope of the rational function (38), where the algebraic surface, corresponding to the vanishing condition of the denominator, was clearly the product of two identical elliptic curves with the *same* Hauptmodul (44).

**Question:** Could it be possible that the two algebraic *surfaces*

$$(1+z)^2 - x \cdot (1-x) \cdot y \cdot (x-y) \cdot (y-4 \cdot p \cdot (1-p)) = 0, \quad (76)$$

and

$$(1+z)^2 - x \cdot (1-x) \cdot (x-p) \cdot y \cdot (1-y) \cdot (y-p) = 0. \quad (77)$$

be† birationally equivalent?

4.2.2. *Calabi-Yau three-fold manifolds as foliation in three elliptic curves.* The telescope of the rational function in *five* variables  $x, y, z, v$  and  $w$

$$\frac{xyzv}{(1+w)^2 - x \cdot (1-x) \cdot y \cdot (x-y) \cdot z \cdot (y-z) \cdot (z-xyzvw)}, \quad (78)$$

associated†† with the *Calabi-Yau three-fold* written in a *Legendre form*

$$(1+w)^2 - x \cdot (1-x) \cdot y \cdot (x-y) \cdot z \cdot (y-z) \cdot (z-p) = 0, \quad (79)$$

is an order-four (self-adjoint) linear differential operator  $L_4$

$$L_4 = 16 \cdot \theta^4 - x \cdot (2\theta + 1)^4, \quad (80)$$

which is a *Calabi-Yau operator*¶ with the  ${}_4F_3$  solution:

$${}_4F_3\left(\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right], [1, 1, 1], x\right). \quad (81)$$

For  $y$  and  $z$  fixed, the *Calabi-Yau three-fold* (79) is foliated in *genus-one* curves

$$(1+w)^2 - \lambda \cdot x \cdot (1-x) \cdot (x-y) = 0, \quad (82)$$

† For algebraic curves, the situation is simpler since two elliptic curves are *birationally equivalent* if and only if they have the *same*  $j$ -invariant.

†† The diagonal extracts the terms function of the product  $p = xyzvw$  in the multi-Taylor series.

¶ This linear differential operator is self-adjoint, its exterior square is of order five, it is MUM (maximum unipotent monodromy [49, 50, 51]), ...



where  $\lambda$  is the constant expression ( $p$  is fixed):

$$\lambda = y \cdot z \cdot (y - z) \cdot (z - p). \quad (83)$$

The Hauptmodul of these *genus-one* curves is *independent of*  $p$  and  $z$ , reading:

$$\mathcal{H}_{y,z} = \frac{27}{4} \cdot \frac{y^2 \cdot (1 - y)^2}{(1 - y + y^2)^3}. \quad (84)$$

Similarly for  $x$  and  $z$  fixed, the Calabi-Yau three-fold (79) is foliated in *genus-one* curves

$$(1 + w)^2 - \mu \cdot y \cdot (x - y) \cdot (y - z) = 0, \quad (85)$$

where  $\mu$  is the constant expression ( $p$  is fixed):

$$\mu = x \cdot z \cdot (1 - x) \cdot (z - p). \quad (86)$$

The *genus-one* curves (85) can be written in a simpler Weierstrass form:

$$(1 + w)^2 - \rho \cdot Y \cdot \left(1 - Y\right) \cdot \left(Y - \frac{z}{x}\right) = 0, \quad (87)$$

where the constant  $\rho$  reads  $\rho = \mu \cdot x^3$ , and the variable  $y$  has been rescaled into  $Y = y/x$ . The Hauptmodul of these *genus-one* curves (85) is the same as the Hauptmodul of the *genus-one* curves (82), and corresponds to expression (84) where  $y$  has been changed into  $z/x$  (see the canonical form (87)), namely:

$$\mathcal{H}_{x,z} = \frac{27}{4} \cdot \frac{x^2 \cdot z^2 \cdot (x - z)^2}{(x^2 - xz + z^2)^3}. \quad (88)$$

Similarly for  $x$  and  $y$  fixed, the Calabi-Yau three-fold (79) is foliated in *genus-one* curves,

$$(1 + w)^2 - \nu \cdot z \cdot (y - z) \cdot (z - p) = 0, \quad (89)$$

where  $\nu$  reads:

$$\nu = x \cdot (1 - x) \cdot y \cdot (x - y). \quad (90)$$

A reduction to a canonical Weierstrass form similar to (87) gives immediately the Hauptmodul of the *genus-one* curve (89) which reads:

$$\mathcal{H}_{x,y} = \frac{27}{4} \cdot \frac{y^2 \cdot p^2 \cdot (y - p)^2}{(y^2 - yp + p^2)^3}. \quad (91)$$

The *Calabi-Yau three-fold* (79) thus has a foliation in a triple of elliptic curves  $\mathcal{E}_1$ ,  $\mathcal{E}_2$  and  $\mathcal{E}_3$ .

## 5. Creative telescoping of rational functions in three variables associated with genus-two curves with split Jacobians

In a paper [16, 17], dedicated to Heun functions that are solutions of telescopers of simple rational functions of three and four variables, we have obtained† an order-four telescoper of a rational function of *three* variables, which is the *direct sum of two order-two linear differential operators*, each having *classical modular forms* solutions which can be written as pullbacked  ${}_2F_1$  hypergeometric solutions. Unfortunately, the intersection of the algebraic surface corresponding to the denominator of the rational

† See equation (83) in section 2.2 of [17].

function with the  $p = xyz$  hyperbola, yields a *genus-two* algebraic curve. Note that this is a “true” genus-two curve: it does not correspond to the “almost genus-one curves” situation mentioned in subsection 3.3.

Let us try to understand, in this section, *how a genus-two curve can yield two classical modular forms*. Let us first recall the results in section 2.2 of [17].

### 5.1. Periods of extremal rational surfaces

Let us recall the rational function in just *three* variables [17]:

$$R(x, y, z) = \frac{1}{1 + x + y + z + xy + yz - x^3 yz}. \quad (92)$$

Its telescoper is actually an *order-four* linear differential operator  $L_4$  which, not only factorizes into *two order-two* linear differential operators, but is actually the *direct sum* (LCLM) of *two* order-two linear differential operators  $L_4 = L_2 \oplus M_2$ . These two (non homomorphic) order-two linear differential operators have, respectively, the two pullbacked hypergeometric solutions:

$$\begin{aligned} \mathcal{S}_1 &= Heun\left(\frac{1}{2} - \frac{i\sqrt{3}}{2}, \frac{1}{2} - \frac{i\sqrt{3}}{6}, 1, 1, 1, 1, \frac{3}{2} \cdot (-3 + i\sqrt{3}) \cdot x\right) \\ &= (1 + 9x)^{-1/4} \cdot (1 + 3x)^{-1/4} \cdot (1 + 27x^2)^{-1/4} \\ &\quad \times {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], \frac{1728 \cdot x^3 \cdot (1 + 9x + 27x^2)^3}{(1 + 3x)^3 \cdot (1 + 9x)^3 \cdot (1 + 27x^2)^3}\right), \end{aligned} \quad (93)$$

and:

$$\begin{aligned} \mathcal{S}_2 &= \frac{1}{(1 + 4x - 2x^2 - 36x^3 + 81x^4)^{1/4}} \\ &\quad \times {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], \frac{1728 \cdot x^5 \cdot (1 + 9x + 27x^2) \cdot (1 - 2x)^2}{(1 + 4x - 2x^2 - 36x^3 + 81x^4)^3}\right). \end{aligned} \quad (94)$$

The diagonal of (92) *is actually the half-sum of the two series* (93) and (94):

$$Diag(R(x, y, z)) = \frac{\mathcal{S}_1 + \mathcal{S}_2}{2}. \quad (95)$$

As far as our algebraic geometry approach is concerned, the intersection of the algebraic surface corresponding to the denominator of the rational function (92) with the hyperbola  $p = xyz$  gives the planar algebraic curve (corresponding to the elimination of the  $z$  variable by the substitution  $z = p/x/y$ ):

$$1 + x + y + \frac{p}{xy} + xy + y\frac{p}{xy} - x^3 y\frac{p}{xy} = 0. \quad (96)$$

One easily finds that this algebraic curve is (for  $p$  fixed) a *genus-two* curve, and that this higher genus situation does not correspond to the “almost elliptic curves” described in subsection 3.2 namely an elliptic curve transformed by a monomial transformation. How a “true” *genus-two* curve can give two  $j$ -invariants, namely a telescoper with two Hauptmodul pullbacked  ${}_2F_1$  solutions? We are going to see that the answer is that the Jacobian of this *genus-two* curve<sup>†</sup> is in fact isogenous to a product  $\mathcal{E} \times \mathcal{E}'$  of *two elliptic curves* (split Jacobian).

<sup>‡</sup> These two order-two linear differential operators  $L_2$  and  $M_2$  are *not* homomorphic.

<sup>†</sup> An algebraic geometer will probably recall that it is very well-known that a genus two curve *may* have Jacobian isogeneous to a product of elliptic curves. This is not the case in general. The genus two curves that have a (nonconstant) map to an elliptic curve have this property. Our purpose in section (5.3) is to perform a creative telescoping calculation in such a selected situation.

## 5.2. Split Jacobians

Let us first recall the concept of *split Jacobian* with a very simple example. In [52], one has a crystal-clear example of a genus-two curve  $C$

$$y^2 - (x^3 + 420x - 5600) \cdot (x^3 + 42x^2 + 1120) = 0, \quad (97)$$

such that its *Jacobian*  $J(C)$  is *isogenous to a product of elliptic curves* with  $j$  invariants  $j_1 = -2^7 \cdot 7^2 = -6272$  and  $j_2 = -2^5 \cdot 7 \cdot 17^3 = -1100512$ . These two values correspond to the following two values of the Hauptmodul  $H = 1728/j$ :  $H_1 = -27/98$  and  $H_2 = -54/34391$ . Let us consider the *genus-one* elliptic curve

$$v^2 = u^3 + 4900u^2 + 7031500u + 2401000000, \quad (98)$$

of  $j$ -invariant  $j = j_2 = -2^5 \cdot 7 \cdot 17^3 = -1100512$ . We consider the following transformation§:

$$u = -\frac{882000 \cdot (x - 14)}{x^3 + 420x - 5600}, \quad v = \frac{49000 \cdot (x^3 - 21x^2 - 140)}{(x^3 + 420x - 5600)^2} \cdot y. \quad (99)$$

This change of variable (99) actually transforms the *elliptic curve* (98) into the *genus-two curve* (97). *This provides a simple example of genus-two curve with split Jacobian through K3 surfaces.*

More generally, let us consider the Jacobian of a *genus-two curve*  $C$ . The Jacobian is simple if it does not contain a proper abelian subvariety, otherwise the Jacobian is reducible, or decomposable or “split”. For this latter case, the only possibility for a genus-two curve is that its Jacobian is *isogenous to a product*  $\mathcal{E} \times \mathcal{E}'$  of *two elliptic curves*‡. Equivalently, there is a degree  $n$  map  $C \rightarrow \mathcal{E}$  to some elliptic curves. Classically such pairs‡  $C, \mathcal{E}$  arose in the *reduction of hyperelliptic integrals to elliptic ones* [52]. The  $j$ -invariants correspond, here, to the *two elliptic subfields*: see [52].

## 5.3. Creative telescoping on rational functions in three variables associated with genus-two curves with split Jacobians: a two-parameters example.

Let us now consider the example with *two-parameters*,  $a$  and  $b$ , given in section 4.5 page 12 of [52]. Let us substitute the rational parametrisation¶

$$u = \frac{x^2}{x^3 + ax^2 + bx + 1}, \quad v = \frac{y \cdot (x^3 - bx - 2)}{(x^3 + ax^2 + bx + 1)^2}, \quad (100)$$

in the *elliptic curve*

$$R \cdot v^2 = R \cdot u^3 + 2 \cdot (ab^2 - 6a^2 + 9b) \cdot u^2 + (12a - b^2) \cdot u - 4, \quad (101)$$

where

$$R = 4 \cdot (a^3 + b^3) - a^2b^2 - 18ab + 27. \quad (102)$$

This gives the *genus-two curve*  $C_{a,b}(x, y) = 0$  with:

$$C_{a,b}(x, y) = R \cdot y^2 + (4x^3 + b^2x^2 + 2bx + 1) \cdot (x^3 + ax^2 + bx + 1). \quad (103)$$

§ This transformation is rational but *not birational*. If it were birational, then it would preserve the genus. Here, one goes from genus one to genus two.

‡ Along these lines, see also the concepts of Igusa-Clebsch invariants and Hilbert modular surfaces [52, 53, 54, 55].

† One also has an anti-isometry Galois invariant  $\mathcal{E}' \simeq \mathcal{E}$  under Weil pairing. The decomposition corresponds to real multiplication by quadratic ring of discriminant  $n^2$ .

¶ See also [56] section 6 page 48.

The  $j$ -invariant of the elliptic curve (101) reads:

$$j = \frac{16 \cdot (a^2b^4 + 12b^5 - 126ab^3 + 216ba^2 + 405b^2 - 972a)^3}{(4a^3 + 4b^3 - a^2b^2 - 18ab + 27)^2 \cdot (b-3)^3 \cdot (b^2 + 3b + 9)^3}. \quad (104)$$

The Hauptmodul  $\mathcal{H} = 1728/j$  thus reads

$$\mathcal{H} = \frac{108 \cdot (b-3)^3 \cdot (4a^3 + 4b^3 - a^2b^2 - 18ab + 27)^2 \cdot (b^2 + 3b + 9)^3}{(a^2b^4 + 12b^5 - 126ab^3 + 216ba^2 + 405b^2 - 972a)^3}. \quad (105)$$

For  $b = 3 + x$  this Hauptmodul (105) reads

$$\mathcal{H}_x = \frac{108 \cdot x^3 \cdot (x^2 + 9x + 27)^3 \cdot P_2^2}{P_4^3}, \quad (106)$$

where:

$$\begin{aligned} P_2 &= 4x^3 - (a-6) \cdot (a+6) \cdot x^2 - 6 \cdot (a+6) \cdot (a-3) \cdot x + (4a+15) \cdot (a-3)^2, \\ P_4 &= 12x^5 + (a^2+180) \cdot x^4 + 6 \cdot (2a^2 - 21a + 180) \cdot x^3 \\ &\quad + 27 \cdot (2a^2 - 42a + 135) \cdot x^2 \\ &\quad + 162 \cdot (a-3) \cdot (2a-15) \cdot x + 729 \cdot (a-3)^2. \end{aligned} \quad (107)$$

Let us consider the telescoper of the rational function of three variables  $xy/D_a(x, y, z)$  where the denominator  $D_a(x, y, z)$  is  $C_{a,b}(x, y)$  given by (103), but for  $b = 3 + xyz$ :

$$\begin{aligned} D_a(x, y, z) &= C_{a,3+xyz}(x, y) \\ &= x^6y^3z^3 + x^7y^2z^2 + 4x^3y^5z^3 + 9x^5y^2z^2 + 6x^6yz + 3x^4y^2z^2 + 36y^4x^2z^2 \\ &\quad + 6x^5yz + 4x^6 + 27x^4yz + 9x^5 + 18x^3yz + 108xy^3z + 18x^4 + 3x^2yz \\ &\quad + 32x^3 + 27x^2 + 135y^2 + 9x + 1 \\ &\quad + (x^6y^2z^2 + 6x^5yz + 2x^4yz + 4x^5 - 18xy^3z + 9x^4 + 6x^3 + x^2 - 54y^2) \cdot a \\ &\quad - y^2 \cdot (xyz + 3)^2 \cdot a^2 + 4y^2 \cdot a^3. \end{aligned} \quad (108)$$

This telescoper of the rational function

$$R_a(x, y, z) = \frac{xy}{D_a(x, y, z)}, \quad (109)$$

is an *order-four* linear differential operator  $L_4$  which is actually the direct-sum,  $L_4 = LCLM(L_2, M_2) = L_2 \oplus M_2$ , of two *order-two* linear differential operators, having two pullbacked hypergeometric solutions (see Appendix E). One finds out that one of the two pullbacks *precisely corresponds to the Hauptmodul*  $\mathcal{H}_x$  given by (106). This general case is detailed in Appendix E.

Let us consider the  $a = 3$  subcase<sup>†</sup>. For  $a = 3$ , the Hauptmodul  $\mathcal{H} = 1728/j$ , corresponding to the  $j$ -invariant (104), reads:

$$\mathcal{H} = \frac{4 \cdot (b-3) \cdot (4b+15)^2 \cdot (b^2 + 3b + 9)^3}{(b+6)^3 \cdot (4b^2 + 3b - 18)^3}. \quad (110)$$

This Hauptmodul becomes for  $b = 3 + x$

$$\mathcal{H} = \frac{4 \cdot x \cdot (27 + 4x)^2 \cdot (x^2 + 9x + 27)^3}{(9+x)^3 \cdot (4x^2 + 27x + 27)^3}. \quad (111)$$

<sup>†</sup> The discriminant in  $b$  of  $4a^3 + 4b^3 - a^2b^2 - 18ab + 27$  reads:  $(a-3)^3 \cdot (a^2 + 3a + 9)^3$ , consequently the exact expressions are simpler at  $a = 3$ .

The telescoper of the rational function (109) with  $D_a(x, y, z)$  given by (108) for  $a = 3$ , is an *order-four* linear differential operator which is the direct-sum of two order-two linear differential operators  $L_4 = LCLM(L_2, M_2) = L_2 \oplus M_2$ , these *two order-two* linear differential operators having the pullbacked hypergeometric solutions

$$(27 + 4x)^{-1/2} \cdot x^{-5/4} \cdot {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], 1 + \frac{27}{4x}\right), \quad (112)$$

for  $L_2$ , and

$$\frac{3+x}{(9+x)^{1/4} \cdot (4x^2+27x+27)^{1/4} \cdot x^{3/2} \cdot (27+4x)^{1/2}} \times {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], \frac{4 \cdot x \cdot (27+4x)^2 \cdot (x^2+9x+27)^3}{(9+x)^3 \cdot (4x^2+27x+27)^3}\right), \quad (113)$$

for  $M_2$ , where we see clearly that the Hauptmodul in (113) is precisely the Hauptmodul (111). The Jacobian of the genus-two curve is a split Jacobian corresponding to the product  $\mathcal{E}_1 \times \mathcal{E}_2$  of two elliptic curves, the  $j$ -invariant of the second elliptic curve being (104), when the  $j$ -invariant of the first elliptic curve reads

$$j_1 = \frac{6912x}{27+4x}, \quad (114)$$

corresponding to the Hauptmodul  $1728/j_1 = 1 + \frac{27}{4x}$  in (112). This second invariant is, as it should, *exactly the  $j$ -invariant of the second elliptic curve  $\mathcal{E}'$* , given page 48 in [56]:

$$j(\mathcal{E}') = \frac{256 \cdot (3b - a^2)^3}{4a^3c - a^2b^2 - 18abc + 4b^3 + 27c^2}, \quad (115)$$

for  $c = 1$ ,  $a = 3$  and  $b = 3 + x$ .

This subcase  $a = 3$  is very special. The general case of arbitrary value  $a$  is sketched in Appendix E.

For some general facts‡ about algebraic curves, their Jacobians and algebraic correspondences, see [57, 58, 59] and page 301 in [60]: the multiplier equation is seen to contain the modular equation as a particular case. In [61] Felix Klein *also defined the associated idea of modular correspondence*¶.

#### 5.4. Creative telescoping on rational functions of three variables associated with genus-two curves with split Jacobians: a simple example

Let us consider another simpler example of *genus-two* curve with pullbacked  ${}_2F_1$  solution (not product of pullbacked  ${}_2F_1$ ) of the telescoper.

Let us consider the *genus-two algebraic curve*  $C_p(x, y) = 0$  given in lemma 7 of [63] (see also [64, 65])

$$C_p(x, y) = x^5 + x^3 + p \cdot x - y^2, \quad (116)$$

where  $C_p(x, y)$  is given in lemma 7 of [63]. Let us introduce the rational function  $xy/D(x, y, z)$  where the denominator  $D(x, y, z)$  is given by:

$$D(x, y, z) = C_{p=xyz}(x, y) = x^5 + x^3 + x^2yz - y^2. \quad (117)$$

‡ Explicit calculations require to use various tools in Magma: AnalyticJacobian, EndomorphismRing, ToAnalyticJacobian, FromAnalyticJacobian, ...

¶ See also F. Klein and R. Fricke in [62]

The telescoper of this rational function is an order-two linear differential operator which has the two hypergeometric solutions

$$x^{-1/4} \cdot {}_2F_1\left(\left[\frac{1}{8}, \frac{5}{8}\right], \left[\frac{3}{4}\right], 4x\right) \quad (118)$$

which is a Puiseux series at  $x = 0$  and:

$$x^{-1/4} \cdot {}_2F_1\left(\left[\frac{1}{8}, \frac{5}{8}\right], [1], 1 - 4x\right). \quad (119)$$

These two hypergeometric solutions can be rewritten as†

$$\mathcal{A}(x) \cdot {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], \frac{1728}{J}\right), \quad (120)$$

where the  $j$ -invariant  $J$ , in the Hauptmodul  $1728/J$  in (120), *corresponds exactly to the degree-two elliptic subfields*

$$J^2 - 128 \cdot \frac{(2000x^2 + 1440x + 27)}{(1 - 4x)^2} \cdot J - 4096 \cdot \frac{(100x - 9)^3}{(1 - 4x)^3} = 0, \quad (121)$$

given in the first equation of page 6 of [63].

Of course, if we change  $p$  into  $p \rightarrow (1 - p)/4$  in subsection 5.3, the telescoper of the rational function  $xy/D(x, y, z)$  where the denominator  $D(x, y, z)$  is given by:

$$D(x, y, z) = C_{p=xyz}(x, y) = x^5 + x^3 + \left(\frac{1 - xyz}{4}\right) \cdot x - y^2, \quad (122)$$

is the order-two linear differential operator corresponding to the  $x \rightarrow (1 - x)/4$  pullback of the previous one. It has the two hypergeometric solutions

$$(1 - x)^{-1/4} \cdot {}_2F_1\left(\left[\frac{1}{8}, \frac{5}{8}\right], [1], x\right), \quad (123)$$

and:

$$(1 - x)^{-1/4} \cdot {}_2F_1\left(\left[\frac{1}{8}, \frac{5}{8}\right], \left[\frac{3}{4}\right], 1 - x\right). \quad (124)$$

**Remark:** In contrast with the previous example of subsection 5.3 where we had two  $j$ -invariants corresponding to the *two order-two* linear differential operators  $L_2$  and  $M_2$  of the direct-sum decomposition of the order-four telescoper, we have, here, *just one order-two* telescoper, which is enough to “encapsulate” two  $j$ -invariants (121). One order-two linear differential operator is enough because the two  $j$ -invariants are Galois-conjugate (see (121)).

### 5.5. Creative telescoping on rational functions of three variables associated with genus-two curves with split Jacobians: another simple example

Another similar example of *genus-two algebraic curve*  $C_p(x, y) = 0$  given in equation (5) of lemma 4 of [63]

$$C_p(x, y) = x^6 + x^3 + p - y^2, \quad (125)$$

with a split Jacobian, yields an order-two telescoper for the corresponding rational function, with pullbacked hypergeometric solutions, where, again, the  $j$ -invariant  $J$ , in the Hauptmodul  $1728/J$  *corresponds exactly to the degree-two elliptic subfields* of the split Jacobian of the *genus-two* curve. More details are sketched in Appendix F.

† The fact that  ${}_2F_1\left(\left[\frac{1}{8}, \frac{5}{8}\right], [1], z\right)$  can be rewritten as  ${}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], H(z)\right)$  where the Hauptmodul  $H(z)$  is solution of a quadratic equation is given in equation (H.14) of Appendix H of [17].

5.6. Creative telescoping on rational functions of three variables associated with genus-two curves with split Jacobians: another example

Let us now recall the paper [66] by K. Diarra. Similarly to subsection 5.3, we consider the *one-parameter*  $a$ -example given in section 6 page 52 of [66]. Let us substitute the rational parametrisation

$$u = -\frac{x^2 + a}{x^2 - 1}, \quad v = \frac{y}{(x^2 - 1)^2}, \quad (126)$$

in the *elliptic curve*  $P(u, v) = 0$

$$u^3 + 1 - v^2 = 0, \quad (127)$$

which  $j$ -invariant is 0 (and thus the Hauptmodul is  $\infty$ ). This gives the *genus-two curve*  $C_a(x, y) = 0$  with:

$$C_a(x, y) = y^2 + (a + 1) \cdot (x^2 - 1) \cdot \left( 3x^4 + 3 \cdot (a - 1) \cdot x^2 + a^2 - a + 1 \right). \quad (128)$$

Let us denote  $\mathcal{C}(x, y, z)$  the previous polynomial (128) where the parameter  $a$  becomes the product  $a = xyz$ . The telescoper of the rational function

$$R(x, y, z) = \frac{xy}{\mathcal{C}(x, y, z)}, \quad (129)$$

is an *order-two* linear differential operator

$$L_2 = 3 \cdot (3x - 1) + 4 \cdot (4x^2 - x + 1) \cdot D_x + 4 \cdot (1 + x^3) \cdot D_x^2, \quad (130)$$

which has the following  ${}_2F_1$  hypergeometric solutions

$$(1 + x)^{-1} \cdot (1 - 2x)^{-1/2} \cdot {}_2F_1\left(\left[\frac{1}{4}, \frac{3}{4}\right], [1], -\frac{3}{(1 - 2x)^2}\right), \quad (131)$$

or:

$$(1 + x)^{-1} \cdot (1 - 2x)^{-1/2} \cdot {}_2F_1\left(\left[\frac{1}{4}, \frac{3}{4}\right], [1], \frac{4 \cdot (x^2 - x + 1)}{(1 - 2x)^2}\right), \quad (132)$$

and:

$$(1 + x)^{-5/4} \cdot (x - 2)^{-1/4} \times {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], -\frac{81 \cdot (x^2 - x + 1)^2}{4 \cdot (x - 2)^3 \cdot (x + 1)^3}\right). \quad (133)$$

Again the telescoper is *not an order-four* operator but an *order-two operator*. This is a consequence of the fact that, among the two  $j$ -invariants of the split Jacobian, one is trivial ( $j = 0$ ). Note that the Hauptmodul in (133) is simply related to the Hauptmodul  $1728z/(z + 16)^3$  in [32] (see  $N = 2$  in Table 4 page 11 in [32]):

$$-\frac{81 \cdot (x^2 - x + 1)^2}{4 \cdot (x - 2)^3 \cdot (x + 1)^3} = \frac{1728z}{(z + 16)^3} \quad \text{when:} \quad z = \frac{-48}{x^2 - x + 1}. \quad (134)$$

Consequently we can also write (see  $N = 2$  in Table 5 page 12 in [32]) the solution of the order-two telescoper (130) in terms of the alternative Hauptmodul:

$$\frac{972 \cdot (x^2 - x + 1)}{(16x^2 - 16x + 13)^3} = \frac{1728z^2}{(z + 256)^3} \quad \text{with:} \quad z = \frac{-48}{x^2 - x + 1}. \quad (135)$$

This alternative writing of the solution reads:

$$\frac{(16x^2 - 16x + 13)^{-1/4}}{1 + x} \cdot {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], \frac{972 \cdot (x^2 - x + 1)}{(16x^2 - 16x + 13)^3}\right). \quad (136)$$

## 6. Rational functions with tri-quadratic denominator and $N$ -quadratic denominator.

We try to find telescopers of rational functions corresponding to (factors of) linear differential operators of small orders, for instance order-two linear differential operators with pullbacked  ${}_2F_1$  hypergeometric functions, classical modular forms, or their modular generalisations (order-four Calabi-Yau linear differential operators [50], etc ...). As we saw in the previous sections, this corresponds to the fact that the denominator of these rational functions is associated with an elliptic curve, or products of elliptic curves, with K3 surfaces or with threefold Calabi-Yau manifolds corresponding to algebraic varieties with foliations in elliptic curves $\parallel$ . Since this paper tries to reduce the *differential algebra* creative telescoping calculations to *effective algebraic geometry* calculations $\S$  and structures, we want to focus on rational functions with denominators that correspond to *selected* algebraic varieties [44, 69], beyond algebraic varieties corresponding to products of elliptic curves or foliations in elliptic curves $\#$ , namely algebraic varieties *with an infinite number of birational automorphisms* $\dagger$ . This *infinite number of birational symmetries*, excludes algebraic varieties of the “general type” (with *finite* numbers $\ddagger$  of birational symmetries). For algebraic surfaces, this amounts to discarding the surfaces of the “general type” which have Kodaira dimension 2, focusing on Kodaira dimension one (elliptic surfaces), or Kodaira dimension zero (abelian surfaces, hyperelliptic surfaces, K3 surfaces, Enriques surfaces), or even Kodaira dimension  $-\infty$  (ruled surfaces, rational surfaces).

In contrast with algebraic *curves* where one can easily, and very efficiently, calculate the genus of the curves to discard the algebraic curves of higher genus and, in the case of genus-one, obtain the  $j$ -invariant using formal calculations $\P$ , it is, in practice, quite difficult to see for higher dimensional algebraic varieties, that the algebraic variety is not of the “general type”, because it has an *infinite number of birational symmetries*. For these “selected cases” we are interested in, calculating the generalisation of the  $j$ -invariant (Igusa-Shiode invariants, etc ...) is quite hard.

Along this line we want to underline that there exists a remarkable set of algebraic surfaces, namely the algebraic surfaces corresponding to tri-quadratic equations:

$$\sum_{m=0,1,2} \sum_{n=0,1,2} \sum_{l=0,1,2} a_{m,n,l} \cdot x^m y^n z^l = 0, \quad (137)$$

depending on  $27 = 3^3$  parameters  $a_{m,n,l}$ . More generally, one can introduce algebraic

$\parallel$  Even if K3 surfaces, or threefold Calabi-Yau manifolds, are *not* abelian varieties, the Weierstrass-Legendre forms introduced in the previous section, amounts to saying that K3 surfaces can be “essentially viewed” (as far as creative telescoping is concerned) as foliation in two elliptic curves, and threefold Calabi-Yau manifolds as foliation in three elliptic curves.

$\S$  One has birational automorphisms in projective spaces [67, 68], but since this paper is dedicated to (efficient) formal calculations we work exclusively in affine coordinates (see for instance (160), (176), (177) below). For algebraic geometers an elliptic curve is a smooth complete genus 1 curve with a choice of a base point. Here our elliptic curves are, in fact, an affine piece of a genus 1 curve with no base point, but this does not really matter because the  $j$ -invariant which is all we care about in this kind of creative telescoping calculations, is determined by that much information.

$\#$  K3 surfaces, threefold Calabi-Yau manifolds, higher curves with split Jacobian corresponding to products of elliptic curves, ...

$\dagger$  The best explicit illustration of this situation emerges in integrable models [44, 69, 70, 71]

$\ddagger$  There are even precise bounds for the number of automorphisms. The upper bound is  $84(g-1)$  for curves of genus  $g$  and these bounds have been extensively studied in higher dimensions [72, 73, 74].

$\P$  Use with(algcurves) in Maple and the command “genus” and “j\_invariant”.



varieties corresponding to  $N$ -quadratic equations:

$$\sum_{m_1=0,1,2} \sum_{m_2=0,1,2} \cdots \sum_{m_N=0,1,2} a_{m_1, m_2, \dots, m_N} \cdot x_1^{m_1} x_2^{m_2} \cdots x_N^{m_N} = 0. \quad (138)$$

With these tri-quadratic (137), or  $N$ -quadratic (138) equations, we will see, in subsection 6.3, that we have *automatically* (selected) algebraic varieties that are not of the “general type” having an *infinite number of birational symmetries*, which is precisely our requirement for the denominator of rational functions with remarkable telescopers<sup>†</sup>.

Let us first, as a warm-up, consider, in the next subsection, a remarkable example of tri-quadratic (137), where the underlying foliation in elliptic curves is crystal clear.

### 6.1. Rational functions with tri-Quadratic denominator simply corresponding to elliptic curves.

Let us first recall the tri-quadratic equation in three variables  $x$ ,  $y$  and  $z$ :

$$\begin{aligned} x^2 y^2 z^2 - 2 \cdot M \cdot xyz \cdot (x + y + z) + 4 \cdot M \cdot (M + 1) \cdot xyz \\ + M^2 \cdot (x^2 + y^2 + z^2) - 2 M^2 \cdot (xy + xz + yz) = 0, \end{aligned} \quad (139)$$

already introduced in Appendix C of [75]. This algebraic surface, symmetric in  $x$ ,  $y$  and  $z$ , can be seen for  $z$  (resp.  $x$  or  $y$ ) fixed, as an *elliptic curve* which  $j$ -invariant is *independent of  $z$*  and reads

$$j = 256 \cdot \frac{(M^2 - M + 1)^3}{M^2 \cdot (M - 1)^2}, \quad (140)$$

the corresponding Hauptmodul reading:

$$\mathcal{H} = \frac{27 \cdot M^2 \cdot (M - 1)^2}{4 \cdot (M^2 - M + 1)^3}. \quad (141)$$

This corresponds to the fact that this algebraic surface (139) can be seen as a product of two times the same elliptic curve with the  $j$ -invariant (140) or the Hauptmodul (141). This is a consequence of the fact that, introducing  $x = sn(u)^2$ ,  $y = sn(v)^2$  and  $z = sn(u + v)^2$ , and  $M = 1/k^2$ , this algebraic surface (139) corresponds to the well-known formula for the *addition on elliptic sine*<sup>¶</sup>:

$$sn(u + v) = \frac{sn(u) cn(v) dn(v) + sn(v) cn(u) dn(u)}{1 - k^2 sn(u)^2 sn(v)^2}. \quad (142)$$

For  $M = xyzw$ , the LHS of the tri-quadratic equation (139) yields a polynomial of *four variables*  $x$ ,  $y$ ,  $z$  and  $w$ , that we denote  $T(x, y, z, w)$ :

$$\begin{aligned} T(x, y, z, w) = \\ x^2 y^2 z^2 - 2 \cdot x^2 y^2 z^2 w \cdot (x + y + z) + 4 \cdot (xyzw + 1) \cdot x^2 y^2 z^2 w \\ + x^2 y^2 z^2 w^2 \cdot (x^2 + y^2 + z^2) - 2 x^2 y^2 z^2 w^2 \cdot (xy + xz + yz). \end{aligned} \quad (143)$$

The telescoper of the rational function in *four variables*  $x$ ,  $y$ ,  $z$  and  $w$ ,

$$\frac{xyz}{T(x, y, z, w)}, \quad (144)$$

<sup>†</sup> Telescopers with factors of small enough order, possibly yielding classical modular forms, Calabi-Yau operators, ... Rational functions with denominators of the “general type” will yield telescopers of very large orders.

<sup>¶</sup> See equation (C.3) in Appendix C of [75].

is an order-three (self-adjoint) linear differential operator which is the *symmetric square* of the order-two linear differential operator having the following pullbacked  ${}_2F_1$  hypergeometric solution:

$$x^{-1/2} \cdot (x^2 - x + 1)^{-1/4} \cdot {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], \frac{27 \cdot x^2 \cdot (x-1)^2}{4 \cdot (x^2 - x + 1)^3}\right). \quad (145)$$

As it should the Hauptmodul in (145) is the same as the Hauptmodul (141). The algebraic surface (139) can be seen as the product of *two times the same elliptic curve* with the Hauptmodul (141). As expected the solution of the order-three telescoper is the *square* of the pullbacked  ${}_2F_1$  hypergeometric function (145) with that Hauptmodul.

More generally, we can also introduce the tri-quadratic equation of three variables  $x$ ,  $y$  and  $z$  and two parameters  $M$  and  $N$ :

$$x^2 y^2 z^2 - 2M \cdot xyz \cdot (x + y + z) + N \cdot xyz + M^2 \cdot (x^2 + y^2 + z^2) - 2M^2 \cdot (xy + xz + yz) = 0. \quad (146)$$

This surface, symmetric in  $x$ ,  $y$  and  $z$ , can be seen for  $z$  (resp.  $x$  or  $y$ ) fixed as an elliptic curve which  $j$ -invariant is, again, *independent of  $z$*  and reads

$$j = \frac{(48M^3 - N^2)^3}{M^6 \cdot (64M^3 - N^2)}. \quad (147)$$

the corresponding Hauptmodul reading:

$$\mathcal{H} = \frac{1728 \cdot M^6 \cdot (64M^3 - N^2)}{(48M^3 - N^2)^3}. \quad (148)$$

Let us consider the following change of variables  $M = m^2$  and  $N = 8 \cdot m^3 + p$  in (146). For  $p = xyzw$ , the LHS of the tri-quadratic equation (146) yields a polynomial in *four variables*  $x$ ,  $y$ ,  $z$  and  $w$ , that we denote  $\mathcal{T}_m(x, y, z, w)$ :

$$\begin{aligned} \mathcal{T}_m(x, y, z, w) = & x^2 y^2 z^2 - 2m^2 \cdot xyz \cdot (x + y + z) + (8 \cdot m^3 + xyzw) \cdot xyz \\ & + m^4 \cdot (x^2 + y^2 + z^2) - 2m^4 \cdot (xy + xz + yz). \end{aligned} \quad (149)$$

For  $z$  (resp.  $x$  or  $y$ ) fixed the corresponding Hauptmodul (148) reads:

$$\mathcal{H} = \frac{1728 \cdot m^{12} \cdot p \cdot (16m^3 + p)}{(16m^6 + 16m^3 \cdot p + p^2)^3}. \quad (150)$$

The telescoper of the rational function in *four variables*  $x$ ,  $y$ ,  $z$  and  $w$ ,

$$\frac{xyz}{\mathcal{T}_m(x, y, z, w)}, \quad (151)$$

is an order-three (self-adjoint) linear differential operator which is the *symmetric square* of an order-two linear differential operator having the following pullbacked  ${}_2F_1$  hypergeometric solution:

$$(16m^6 + 16m^3 \cdot x + x^2)^{-1/4} \cdot {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], \frac{1728 \cdot m^{12} \cdot x \cdot (16m^3 + x)}{(16m^6 + 16m^3 \cdot x + x^2)^3}\right). \quad (152)$$

As it should the Hauptmodul in (152) is the same as the Hauptmodul (150). The algebraic surface (146) can be seen as the product of *two times the same elliptic curve*

with the Hauptmodul (148) (or (150)). As expected the solution of the order-three telescoper is the *square* of the pullbacked  ${}_2F_1$  hypergeometric function (152) with the Hauptmodul (150).

**Remark:** Let us perform some deformation of the rational function (144), changing the first  $-2$  coefficient in (143) into a  $-3$  coefficient. The polynomial  $T(x, y, z, w)$ :

$$\begin{aligned} T(x, y, z, w) = & \quad (153) \\ & x^2 y^2 z^2 - 3 \cdot x^2 y^2 z^2 w \cdot (x + y + z) + 4 \cdot (xyzw + 1) \cdot x^2 y^2 z^2 w \\ & + x^2 y^2 z^2 w^2 \cdot (x^2 + y^2 + z^2) - 2 \cdot x^2 y^2 z^2 w^2 \cdot (xy + xz + yz). \end{aligned}$$

The telescoper of the rational function in *four variables*,

$$\frac{xyz}{T(x, y, z, w)}, \quad (154)$$

is an (irreducible) of (only) *order-four* linear differential operator  $L_4$  which is non-trivially homomorphic to its adjoint $\ddagger$ . A priori, we cannot exclude the fact that  $L_4$  could be homomorphic to the symmetric cube of a second-order linear differential operator, or to a symmetric product of two second-order operators. Furthermore, it could also be, in principle, that these second-order operators admit classical modular forms as solutions (pullbacks of special  ${}_2F_1$  hypergeometric functions). However, these options can both be excluded by using some results from differential Galois theory [76], specifically from [77, Prop. 7, p. 50] for the symmetric cube case, and from [77, Prop. 10, p. 69] for the symmetric product case, see also [79, §3]. Indeed, if  $L_4$  were either a symmetric cube or a symmetric product of order-two operators, then its symmetric square would contain a (direct) factor of order 3 or 1. This is ruled out by a factorization procedure which shows that the symmetric square of  $L_4$  is (LCLM-)irreducible.

This example does not correspond to an addition formula like (142), but the polynomial  $T(x, y, z, w)$  still corresponds to a tri-quadratic (and thus an algebraic variety with an infinite number of birational automorphisms).

## 6.2. Rational functions with tri-quadratic denominator: another example.

The telescoper of the rational function in four variables  $x, y, z$  and  $w$ ,

$$\frac{xyz}{T(x, y, z, w)}, \quad (155)$$

where the polynomial  $T(x, y, z, w)$  almost corresponds to the tri-quadratic (146):

$$\begin{aligned} T(x, y, z, w) = & x^2 y^2 z^2 - 2 \cdot xyz \cdot (x + y + z) + 8 \cdot xyz \\ & + (x^2 + y^2 + z^2) - 2 \cdot (xy + xz + yz) + xyzw. \end{aligned} \quad (156)$$

is an *order-four* linear differential operator non-trivially $\blacklozenge$  homomorphic to its adjoint.

**Remark:** If one (slightly) changes the first coefficient  $-2$  into  $-3$  in (156)

$$\begin{aligned} T(x, y, z, w) = & x^2 y^2 z^2 - 3 \cdot xyz \cdot (x + y + z) + 8 \cdot xyz \\ & + (x^2 + y^2 + z^2) - 2 \cdot (xy + xz + yz) + xyzw. \end{aligned} \quad (157)$$

$\ddagger$  Its exterior square has a rational solution. However this order-four linear differential operator is not MUM (maximum unipotent monodromy [49, 50, 51])

$\blacklozenge$  The intertwiners are of order one and order three. This order-four linear differential operator is not MUM (maximum unipotent monodromy [49, 50, 51]).

one obtains an *order-six* telescoper for rational function of four variables (155). This order-six linear differential operator is non trivially<sup>‡</sup> homomorphic to its adjoint.

6.3. Rational functions with tri-quadratic denominator.

Let us consider the most general *tri-quadratic surface*

$$\sum_{m=0,1,2} \sum_{n=0,1,2} \sum_{l=0,1,2} a_{m,n,l} \cdot x^m y^n z^l = 0, \tag{158}$$

depending on  $27 = 3^3$  parameters  $a_{m,n,l}$ . It can be rewritten as:

$$A(x, y) \cdot z^2 + B(x, y) \cdot z + C(x, y) = 0. \tag{159}$$

It is straightforward to see that condition (159) is preserved by the *birational involution*  $I_z$

$$I_z : \left( x, y, z \right) \longrightarrow \left( x, y, \frac{C(x, y)}{A(x, y)} \cdot \frac{1}{z} \right), \tag{160}$$

and we have of course two other similar *birational involutions*  $I_x$  and  $I_y$  that single out  $x$  and  $y$  respectively. The (generically) *infinite-order* birational transformations  $K_x = I_y \cdot I_z$ ,  $K_y = I_z \cdot I_x$  and  $K_z = I_x \cdot I_y$  are birational symmetries of the surface (158) or (159). They are related by  $K_x \cdot K_y \cdot K_z = \textit{identity}$ . Note that the *birational transformation*  $K_x$  preserves  $x$ . The iteration of the (generically) *infinite-order* birational transformation  $K_x$  gives *elliptic curves*. Since equation (158) or (159) is preserved by  $K_x$ , which also preserves  $x$ , the equation of the *elliptic curves* corresponding to the iteration<sup>†</sup> of  $K_x$  is (158) for fixed values of  $x$ . Equation (158), for fixed values of  $x$ , is a (general) biquadratic curve in  $y$  and  $z$  and is thus *an elliptic curve depending on  $x$* . Therefore one has a canonical foliation of the algebraic surface (158) in elliptic curves. Of course the iteration of  $K_y$  (resp.  $K_z$ ) also yields elliptic curves, and similarly yields two other foliations in elliptic curves.

We have a foliation in two families of elliptic curves  $\mathcal{E}$  and  $\mathcal{E}'$  of the surface. Consequently, this tri-quadratic surface (158), having an *infinite set* of *birational automorphisms*, an *infinite set* of *birational symmetries*, cannot be of the “general type” (it has Kodaira dimension less than 2).

6.4. Rational functions with tri-Quadratic denominator: Fricke cubics examples associated with Painlevé VI equations

Let us consider more very simple examples of tri-quadratic surfaces that occur in different domains of mathematics and physics.

Among the *Fricke families* of cubic surfaces, the family [81, 82, 83]

$$x y z + x^2 + y^2 + z^2 + b_1 x + b_2 y + b_3 z + c = 0, \tag{161}$$

of affine cubic surfaces parametrised by the four constants  $(b_1, b_2, b_3, c)$  is known [82] to be a deformation of a  $D_4$  singularity which occurs at the symmetric (Manin’s) case  $b_1 = b_2 = b_3 = -8, c = 28$ .

<sup>‡</sup> The intertwiners are of order three and order five. This order-six linear differential operator is not MUM [49, 50, 51].

<sup>†</sup> The birational transformation  $K_x$  maps the elliptic curve onto itself (self-map). One can use the iteration of the birational transformation  $K_x$  to actually visualise the elliptic curve [44, 80].

Among the symmetric  $b_1 = b_2 = b_3$  cases some selected sets of the four constants  $(b_1, b_2, b_3, c)$  emerge: the Markov cubic  $b_1 = b_2 = b_3 = c = 0$ , Cayley's nodal cubic  $b_1 = b_2 = b_3 = 0, c = -4$ , Clebsch diagonal cubic  $b_1 = b_2 = b_3 = 0, c = -20$ , and Klein's cubic  $b_1 = b_2 = b_3 = -1, c = 0$ .

Some of these symmetric cubics play can be seen as the *monodromy manifold* of the *Painlevé VI equation* (see equation (1.7) in [84], see also equations (1.2) and (1.4) in [83]): the Picard-Hitchin cases  $(0, 0, 0, 4)$ ,  $(0, 0, 0, -4)$ ,  $(0, 0, 0, -32)$ , the Kitaev's cases  $(0, 0, 0, 0)$ ,  $(-8, -8, -8, -64)$ , and especially the Manin's case  $(-8, -8, -8, 28)$ .

Let us consider the Picard-Hitchin example  $(0, 0, 0, -4)$  as a denominator of a rational function [82]. Let us consider the rational function in three variables  $x, y$  and  $z$  [82]:

$$R(x, y, z) = \frac{1}{x^2 + y^2 + z^2 + xyz - 4}. \quad (162)$$

The telescoper of the rational function (162) is actually an *order-two* linear differential operator  $L_2$

$$L_2 = 2 + x + (3x^2 + 14x - 8) \cdot D_x + x \cdot (x + 8) \cdot (x - 1) \cdot D_x^2, \quad (163)$$

which has the pullbacked hypergeometric solution†:

$$\begin{aligned} \frac{1}{x+2} \cdot {}_2F_1\left(\left[\frac{1}{3}, \frac{2}{3}\right], [1], \frac{27x}{(x+2)^3}\right) &= -\frac{2}{x-4} \cdot {}_2F_1\left(\left[\frac{1}{3}, \frac{2}{3}\right], [1], -\frac{27x^2}{(x-4)^3}\right) \\ &= \left((x+2) \cdot (x^3 + 6x^2 - 12x + 8)\right)^{-1/4} \\ &\quad \times {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], \frac{1728 \cdot x^3 \cdot (x+8) \cdot (x-1)^2}{(x+2)^3 \cdot (x^3 + 6x^2 - 12x + 8)^3}\right) \\ &= 2 \cdot \left((x-4) \cdot (x^3 + 12x^2 + 48x - 64)\right)^{-1/4} \\ &\quad \times {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], -\frac{1728 \cdot x^6 \cdot (x-1) \cdot (x+8)^2}{(x-4)^3 \cdot (x^3 + 12x^2 + 48x - 64)^3}\right). \end{aligned} \quad (164)$$

**Remark:** Note that the two Hauptmoduls in (164) and (165) are related by the involution  $x \longleftrightarrow -8/x$ . This symmetry of the problem corresponds to the fact that the order-two telescoper  $L_2$  is simply conjugated to its pullback by  $x \rightarrow -8/x$ .

Eliminating  $z = p/x/y$  in the denominator of (162) gives the *genus-four* algebraic curve:

$$x^4 y^2 + x^2 y^4 + (p-4) \cdot x^2 y^2 + p^2 = 0. \quad (166)$$

The question is to see whether the Jacobian of this *genus-four* algebraic curve (166) could also correspond to a split Jacobian, with a  $j$ -invariant corresponding to the Hauptmoduls in (164) or (165).

More generally the symmetric rational function in three variables  $x, y$  and  $z$  [82]:

$$R(x, y, z) = \frac{1}{x^2 + y^2 + z^2 + xyz + c}, \quad (167)$$

† Note the emergence of the pullback  $-27x^2/(x-4)^3$  that we already saw in (68) and in (75).

which takes into account the other Picard-Hitchin cases<sup>‡</sup>  $(0, 0, 0, 4)$ ,  $(0, 0, 0, -4)$ ,  $(0, 0, 0, 32)$ , also has an *order-two* telescoper which has a simple pullbacked hypergeometric solution:

$$\begin{aligned} & \frac{1}{x+c} \cdot {}_2F_1\left(\left[\frac{1}{3}, \frac{2}{3}\right], [1], -\frac{27x^2}{(x+c)^3}\right) \\ &= p_6(x)^{-1/6} \cdot {}_2F_1\left(\left[\frac{1}{12}, \frac{7}{12}\right], [1], \frac{1728 \cdot x^6 \cdot p_3(x)}{p_6(x)^2}\right) \\ &= (x+c)^{-1/4} \cdot q_3(x)^{-1/4} \cdot {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], -\frac{1728 \cdot x^6 \cdot p_3(x)}{(x+c)^3 \cdot q_3(x)^3}\right), \end{aligned} \quad (168)$$

where<sup>†</sup>:

$$\begin{aligned} p_3(x) &= x^3 + 3 \cdot (c+8) \cdot x^2 + 3 \cdot c^2 \cdot x + c^3, \\ q_3(x) &= x^3 + 3 \cdot (c+9) \cdot x^2 + 3 \cdot c^2 \cdot x + c^3, \\ p_6(x) &= x^6 + 6 \cdot (c+6) \cdot x^5 + (216 + 108c + 15c^2) \cdot x^4 \\ &\quad + (20c + 108) \cdot c^2 \cdot x^3 + (15c + 36) \cdot c^3 \cdot x^2 + 6 \cdot c^5 \cdot x + c^6. \end{aligned}$$

Eliminating  $z = p/x/y$  in the denominator of (162) gives the *genus-four* algebraic curve:

$$x^2 y^2 \cdot (x^2 + y^2) + (p+c) \cdot x^2 y^2 + p^2 = 0. \quad (169)$$

Again, the question is to see whether the Jacobian of this *genus-four* algebraic curve (169) could also correspond to a split Jacobian, with a  $j$ -invariant corresponding to the Hauptmodul in (168).

**Remark:** Note after [82] that the value  $c = -4$  is particular. It is such that the denominator

$$f(x, y, z; c) = x^2 + y^2 + z^2 + x y z + c, \quad (170)$$

when transformed by the simple quadratic transformation

$$(x, y, z) \longrightarrow (2 - x^2, 2 - y^2, 2 - z^2), \quad (171)$$

factorises nicely:

$$f(2 - x^2, 2 - y^2, 2 - z^2; -4) = f(x, y, z; -4) \cdot f(-x, y, z; -4). \quad (172)$$

In other words we have an *endomorphism of the Cayley cubic surface*.

**6.4.1. Singular symmetric Fricke surface.** Let us consider the rational function

$$R(x, y, z) = \frac{1}{xyz + x^2 + y^2 + z^2 + b \cdot (x + y + z) + c}. \quad (173)$$

The vanishing condition of the denominator of (173) is a symmetric Fricke surface which, according to Lemma 9 in [82], is singular for

$$b^2 - 8b - 16 - 4c = 0, \quad (174)$$

<sup>‡</sup> As well as the Markov cubic  $b_1 = b_2 = b_3 = c = 0$ , Cayley's nodal cubic  $b_1 = b_2 = b_3 = 0$ ,  $c = -4$ , and Clebsch diagonal cubic  $b_1 = b_2 = b_3 = 0$ ,  $c = -20$  cases.

<sup>†</sup> The values  $c = 0$  and  $c = -4$  are the only values such that the discriminant in  $x$  of  $p_3(x)$  can be zero.

and:

$$4b^3 - 3b^2 - 6bc + c^2 + 4c = 0. \quad (175)$$

For instance, for  $b = -8$ , the first condition (174) gives  $c = 28$  (i.e. Manin's case) and the second condition (175) gives  $c = 28$  and  $c = -80$ .

The calculation of the telescoper of (173) in the singular  $(b, c) = (-8, 28)$  case gives an (irreducible) *order-four* linear differential operator which is (non-trivially) *homomorphic to its adjoint*‡.

### 6.5. Rational functions with $N$ -Quadratic denominator.

The calculations of subsection 6.3 can straightforwardly be generalised to  $N$ -quadratic equations, writing the  $N$ -quadratic (138) as

$$\begin{aligned} A(x_1, x_2, \dots, x_{N-1}) \cdot x_N^2 + B(x_1, x_2, \dots, x_{N-1}) \cdot x_N \\ + C(x_1, x_2, \dots, x_{N-1}) = 0, \end{aligned} \quad (176)$$

and introducing the *birational involution*  $I_N$

$$\begin{aligned} I_N : \quad & (x_1, x_2, \dots, x_N) \\ & \longrightarrow \left( x_1, x_2, \dots, x_{N-1}, \frac{C(x_1, x_2, \dots, x_{N-1})}{A(x_1, x_2, \dots, x_{N-1})} \cdot \frac{1}{x_N} \right). \end{aligned} \quad (177)$$

Similarly to subsection 6.3, we can introduce  $N$  involutive birational transformations  $I_m$  and consider the products of two such involutive birational transformations  $K_{m,n} = I_m \cdot I_n$ . These  $K_{m,n}$ 's are (generically) infinite order birational transformations preserving the  $N - 2$  variables that are not  $x_m$  and  $x_n$ .

Using such remarkable  $N$  variables algebraic varieties, with an *infinite set of birational automorphisms*, one can build rational functions of  $N + 1$  variables, any of the parameter of the algebraic variety, becoming an arbitrary rational† function of the product  $p = x_1 x_2 \cdots x_N$  in order to build the denominator of the rational function. The telescopers of such rational functions is seen (experimentally using creative telescoping) to be of substantially smaller order than the one for rational functions where their denominators are, after reduction by  $p = x_1 x_2 \cdots x_N$ , associated with algebraic varieties of the “general type”.

## 7. Telescopers of rational functions of several variables

In our previous paper [16, 17], dedicated to Heun functions that are solutions of telescopers of simple rational functions of (most of the time) four variables, we have obtained many order-three telescopers having square of pullbacked  ${}_2F_1$  hypergeometric solutions. Recalling sections 4, 4.2, or even 5.3 in [17], it is natural to imagine, for these examples in [16, 17] yielding square of pullbacked  ${}_2F_1$  hypergeometric functions, a scenario where, after elimination of the fourth variable

‡ The intertwiners are of order-two. The exterior square of that operator has a simple rational solution  $(x^2 + 39x - 168)/(x + 343)/x/(x - 8)^2/(x - 9)$ . We have a similar result for  $(b, c) = (-8, -80)$ , the exterior square of that operator having the rational solution  $p_3(x)/x/(x + 64)/(x - 125)/q_3(x)$ , where  $p_3(x) = x^3 - 149x^2 + 34080x - 3010560$  and  $q_3(x) = x^3 - 349x^2 + 38656x - 1032192$ .

† Or even an arbitrary algebraic function of the product  $p = x_1 x_2 \cdots x_N$ , or a transcendent series analytic at  $p = 0$ .

( $w = p/x/y/z$ ) in the denominator of the rational function of four variables, the corresponding algebraic surface  $S(x, y, z) = 0$ , in the remaining three variables, could be seen as  $K_3$  surface (63) which can be seen as *associated with the product of two times the same elliptic curve*, or other “Periods [36] of extremal rational surfaces” scenario. Some other cases of similar rather simple rational functions of four variables, yield order-two telescopers with pullbacked  ${}_2F_1$  hypergeometric functions (but not square or products of pullbacked  ${}_2F_1$  hypergeometric functions).

- Let us consider the rational function in *four* variables  $x, y, z, u$ :

$$R(x, y, z, u) = \frac{1}{1 + 3y + z + 9yz + 11z^2y + 3ux}. \quad (178)$$

The telescoper of this rational function of four variables is an order-two linear differential operator  $L_2$  which has the pullbacked hypergeometric solution:

$$(1 - 2592x^2)^{-1/4} \times {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], -\frac{419904 \cdot x^3 \cdot (5 - 12x - 19440x^2 + 2665872x^3)}{(1 - 2592x^2)^3}\right). \quad (179)$$

The diagonal of (178) is the expansion of this pullbacked hypergeometric function (179):

$$1 + 648x^2 - 72900x^3 + 1224720x^4 - 330674400x^5 + 23370413220x^6 - 1276733858400x^7 + 180019474034400x^8 - 12013427240614800x^9 + \dots \quad (180)$$

If one considers the intersection of the vanishing condition of the denominator of (178) with the hyperbola  $p = xyz u$ , eliminating for instance  $u = p/x/y/z$  in the vanishing condition of the denominator of (178), one gets a condition, *independent of*  $x$ , which corresponds to a *genus-one* curve

$$11y^2z^3 + 9y^2z^2 + 3y^2z + yz^2 + yz + 3p = 0. \quad (181)$$

The Hauptmodul of this elliptic curve (181) reads:

$$\mathcal{H} = -\frac{419904 \cdot p^3 \cdot (5 - 12p - 19440p^2 + 2665872p^3)}{(1 - 2592p^2)^3}, \quad (182)$$

which corresponds precisely to the Hauptmodul pullback in (179).

- Let us, now, generalize the rational function (178) of *four* variables  $x, y, z, u$ , introducing the rational function of  $N + 3$  variables  $x, y, z, u_1, u_2, \dots, u_N$ :

$$R(x, y, z, u_1, u_2, \dots, u_N) = \frac{1}{1 + 3y + z + 9yz + 11z^2y + 3x \cdot u_1 u_2 \dots u_N}. \quad (183)$$

The telescoper of this rational function of  $N + 3$  variables is the same order-two telescoper as for (178), which has the pullbacked hypergeometric solution (179). Again one can verify that the diagonal of (183) is the expansion (180) of the pullbacked hypergeometric function<sup>‡</sup> (179). If one considers the intersection of the vanishing condition of the denominator of (183) with the hyperbola  $p = xyz u_1 u_2 \dots u_N$ , eliminating for instance  $u_N = p/x/y/z/u_1/\dots/u_{N-1}$  in the vanishing condition

<sup>‡</sup> A pure algebraic geometer will probably consider this result as trivial from the computational point of view, saying that the variety is a fiber bundle over a family of elliptic curves with constant fiber (see also below).



of the denominator of (183), one gets again a condition, *independent of  $x$* , which corresponds to a *genus-one* curve (181):

$$11y^2z^3 + 9y^2z^2 + 3y^2z + yz^2 + yz + 3p = 0. \quad (184)$$

The Hauptmodul of this elliptic curve (184), or (181) reads again the Hauptmodul (182) which corresponds precisely to the Hauptmodul pullback in (179).

**Remark:** Recalling subsections 2.1 and 2.2, one can consider a nine-parameters biquadratic in two variables, or a selected ten-parameters bicubic like (4), where the parameters are now *functions* of the product of  $N$ -variables  $p = x_1 x_2 \cdots x_N$ . This will yield an algebraic variety of  $N$  variables (that are not on the same footing) that will automatically be foliated in elliptic curves.

Simple other examples are displayed in Appendix G.3, and one sees (experimentally) that the Hauptmodul of the pullbacked  ${}_2F_1$  hypergeometric functions can be seen as corresponding to some  $x \rightarrow 0$  limit of Hauptmoduls of the elliptic curves foliating the previous algebraic surface. In contrast with the other examples and results of this paper, we have no algebraic geometry interpretation of this experimental result yet.

## 8. Conclusion

We have shown that the results we had obtained on diagonals of nine and ten parameters families of rational functions, using creative telescoping yielding classical modular forms expressed as pullbacked  ${}_2F_1$  hypergeometric functions [1, 2], can be obtained much more efficiently calculating the  $j$ -invariant of an *elliptic curve canonically associated with the denominator of the rational functions*. In the case where creative telescoping yields pullbacked  ${}_2F_1$  hypergeometric functions, we generalize this result to other families of rational functions of three, and even more than three, variables, when the denominator can be associated with products of elliptic curves or foliation in terms of elliptic curves, or when the denominator is associated with a *genus-two curve with a split Jacobian* corresponding to *products of elliptic curves*.

We have seen different scenarii. In the first cases, we have considered denominators corresponding to *products* of elliptic curves: in these cases the solutions of the telescoper were *products* of pullbacked  ${}_2F_1$  hypergeometric functions. We have also considered denominators corresponding to *genus-two* curves with *split Jacobians isogenous to products of two elliptic curves*, and in these cases the solutions of the telescoper were *sums* of two pullbacked  ${}_2F_1$  hypergeometric functions, sometimes one pullbacked  ${}_2F_1$  hypergeometric function being enough to describe the two Galois-conjugate  $j$ -invariants (see 5.4). We also considered denominators corresponding to algebraic varieties with elliptic foliations, the Hauptmodul pullback in the pullbacked  ${}_2F_1$  hypergeometric functions emerging from a selected elliptic curve of the foliation ( $x = 0$ , see Appendix G.1, Appendix G.2). We also encountered denominators corresponding to algebraic manifolds with an infinite set of birational automorphisms and elliptic curves foliation yielding, no longer classical modular forms represented as pullbacked  ${}_2F_1$  hypergeometric functions, but more general modular structures associated with selected linear differential operators like Calabi-Yau linear differential operators [49, 50] and their generalisations.

The creative telescoping method on a rational function is a way to find the periods of an algebraic variety over *all possible cycles*‡. The fact that the solution of the telescoper corresponds to “Periods” [36] *over all possible cycles* is a simple consequence of the fact that creative telescoping corresponds to *purely differential algebraic manipulations* on the integrand *independently of the cycles*, thus *being blind to analytical details*. In this paper, we show that the final result emerging from differential algebra procedure (which can be cumbersome when the result depends on nine or ten parameters), can be obtained almost instantaneously from a more fundamental intrinsic pure algebraic geometry approach, calculating the  $j$ -invariant of some canonical elliptic curve. This corresponds to a shift Analysis  $\rightarrow$  Differential Algebra  $\rightarrow$  Algebraic Geometry. Ironically, algebraic geometry studies of more involved algebraic varieties than product of elliptic curves, foliation in elliptic curves (Calabi-Yau manifolds, ...) is often a tedious and/or difficult task (finding Igusa-Shiode invariants, ...), and formal calculations tools are not always available or user-friendly. For such involved algebraic varieties the creative telescoping may then become a simple and efficient tool to perform effective algebraic geometry studies.

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## Appendix A. Diagonals of rational functions and Picard-Fuchs equations

For simplicity let us consider a rational function of three variables. The diagonal of a rational function of three variables is obtained through its multi-Taylor expansion [18, 19]

$$R(x, y, z) = \sum_m \sum_n \sum_l a_{m,n,l} \cdot x^m y^n z^l, \quad (\text{A.1})$$

by extracting the “diagonal” terms, i.e. the powers of the product  $p = xyz$ :

$$\text{Diag}\left(R(x, y, z)\right) = \sum_m a_{m,m,m} \cdot p^m. \quad (\text{A.2})$$

Such diagonals are closely related to the integrals of rational functions. For example  $\text{Diag}\left(R(x, y, z)\right)$  is the constant term (in  $y, z$ ) in the infinite expansion

$$R\left(\frac{p}{yz}, y, z\right) = \sum_{m,n,l \geq 0} a_{m,n,l} \cdot p^m y^{n-m} z^{l-m}, \quad (\text{A.3})$$

‡ Not only the *vanishing cycles* corresponding to *diagonals* of rational functions.

which can be represented by the integral [34]

$$\frac{1}{(2\pi i)^2} \oint \oint R\left(\frac{p}{yz}, y, z\right) \frac{dy}{y} \wedge \frac{dz}{z}. \quad (\text{A.4})$$

The diagonal (A.2) is also the constant term (in  $y, z$ ) of

$$R\left(\frac{p}{y}, \frac{y}{z}, z\right) = \sum_{m, n, l \geq 0} a_{m, n, l} \cdot p^m y^{n-m} z^{l-n}, \quad (\text{A.5})$$

wich is of the form

$$\frac{1}{(2\pi i)^2} \oint \oint \frac{N_p(y, z)}{D_p(y, z)} \frac{dy}{y} \wedge \frac{dz}{z}, \quad (\text{A.6})$$

where the numerator  $N_p(y, z)$  and the denominator  $D_p(y, z)$  are polynomials. it is well-known that such integrals satisfy a linear differential equation with respect to  $p$  having rational functions in  $p$  as coefficients, called the Picard-Fuchs equation<sup>‡</sup>. the problem of determining such linear differential equations has been started by Griffiths [78] with the assumption that the variety  $D_p(y, z) = 0$  is smooth, but later techniques were developed to include examples with singular points [34, 40]. The linear differential equations (Gauss-Manin systems, telescopers) occurring in integrable models [15, 22, 23] are of order much larger than order two<sup>¶</sup> and almost never correspond to smooth varieties. Creative telescoping<sup>†</sup> and more specifically the programs [3] corresponding to a fast approach to creative telescoping [42], are a powerfull way to find these linear differential operators annihilating these diagonal of rational functions in the cases emerging naturally in theoretical physics, integrable models, enumerative combinatorics, for which the order of the linear differential operators is quite large [15, 22, 23] and the variety  $D_p(y, z) = 0$  is (most of the time) not a smooth one. All the pedagogical (but non-trivial) examples of telescopers displayed in this paper can be viewed by an algebraic geometer as a presentation of examples of families of varieties and their Picard-Fuchs equations.

## Appendix B. A simple example corresponding to planar elliptic curves obtained as intersection of quadrics

Let us consider the rational function in three variables  $x, y$  and  $z$

$$R(x, y, z) = \frac{xy^2}{D(x, y, z)}, \quad (\text{B.1})$$

where:

$$D(x, y, z) = 4x^4 \cdot xyz + 16y^2x^2 + 16xy^3 + 16y^4 + 32yx^2 + 40xy^2 + 40y^3 + 15x^2 + 25yx + 41y^2 + 40y + 25, \quad (\text{B.2})$$

which corresponds (with  $p = xyz$ ) to the *elliptic curve*

$$C_p(x, y) = 4x^4 \cdot p + 16y^2x^2 + 16xy^3 + 16y^4 + 32yx^2 + 40xy^2 + 40y^3 + 15x^2 + 25yx + 41y^2 + 40y + 25 = 0, \quad (\text{B.3})$$

<sup>‡</sup> The order of this linear differential equation is generally equal to the rank of the algebraic deRham cohomology of  $D_p(y, z) = 0$ . For curves of genus  $g$  this rank is  $2g$ .

<sup>¶</sup> Since Felix Klein it is well-known that the Picard-Fuchs equation corresponding to the (Weierstrass) elliptic curve corresponds to the hypergeometric function  ${}_2F_1([1/12, 5/12], [1], 1/J)$ .

<sup>†</sup> For a detailed introduction to creative telescoping [35] see for instance [33].

corresponding to the intersection (elimination of  $u$  at  $z = 1$ ) of the two *quadrics*

$$p \cdot u^2 + uz + x^2 + yx + y^2 + z^2 = 0, \quad 4uy + 5uz + 2x^2 = 0. \quad (\text{B.4})$$

The  $j$ -invariant of elliptic curve (B.3) reads:

$$J = \frac{27 \cdot (3523 + 10496 p)^3}{6724 \cdot (2686976 p^3 - 1614336 p^2 + 4051257 p - 470096)}. \quad (\text{B.5})$$

The telescoper of the rational function (B.1) is an order-three linear differential operator which can be factorized as

$$L_3 = L_2 \cdot \left( D_x + \frac{41x + 8}{2x \cdot (41x - 4)} \right), \quad (\text{B.6})$$

where the order-two linear differential operator  $L_2$  is homomorphic to an order-two linear differential operator  $Z_2$  such that

$$L_2 \cdot \rho(x) \cdot X_1 = Y_1 \cdot Z_2 \quad \text{where:} \quad (\text{B.7})$$

$$\begin{aligned} \rho(x) &= \frac{5}{16x \cdot (41x - 4)(6400x - 11281)}, \\ X_1 &= (839680x^3 - 16606384x^2 - 6835099x + 2350480) \cdot D_x \\ &\quad + 656 \cdot (2720x^2 - 9447x - 2096), \\ Z_2 &= (2686976x^3 - 1614336x^2 + 4051257x - 470096)(6400x - 11281) \cdot D_x^2 \\ &\quad + (34393292800x^3 - 101267079168x^2 + 36422648832x \\ &\quad - 42693615817) \cdot D_x + 1968 \cdot (1638400x^2 - 6531584x + 79633), \end{aligned} \quad (\text{B.8})$$

where the order-two operator  $Z_2$  has the pullbacked  ${}_2F_1$  hypergeometric solution

$$\left(3523 + 10496x\right)^{-1/4} \cdot {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], \mathcal{H}\right), \quad (\text{B.9})$$

where the Hauptmodul  $\mathcal{H}$  reads:

$$\mathcal{H} = 1 - \frac{27 \cdot (95457 - 262400x)^2}{(3523 + 10496x)^3}, \quad (\text{B.10})$$

which is nothing but the Hauptmodul associated to (B.5) (with, of course,  $p$  changed into  $x$ ).

### Appendix C. Maximum number of parameters for families of planar elliptic curves.

We have seen, in section 3, that the previous results on diagonals of nine or ten parameters families of rational functions of three variables being pullbacked  ${}_2F_1$  hypergeometric functions (and in fact classical modular forms) can actually be seen as corresponding to the (well-known in integrable models and integrable mappings) fact that the most general biquadratic corresponding to *elliptic curves* is a *nine-parameters* family and that the most general ternary cubic corresponding to elliptic curves is a *ten-parameters* family. One can, for instance recall page 238 of [85], which amounts to considering the collection of all cubic curves in  $\mathbb{C}P_2$  with the homogeneous equation

$$\begin{aligned} ax^3 + bx^2y + cxy^2 + dy^3 + ex^2z + fxz^2 + gy^2z \\ + hyz^2 + iz^3 + jxyz = 0, \end{aligned} \quad (\text{C.1})$$

and the associated problems of passing through nine given points. One can also recall the ternary cubics in [86, 87] and other problems of elliptic curves of high rank [88] (see the concept of Neron-Severy rank).

Since the rational functions of three variables we consider are essentially encoded by the denominator of these rational functions, and in the cases we have considered, the emergence of pullbacked  ${}_2F_1$  hypergeometric functions (and in fact classical modular forms) corresponds to the fact that the intersection of these denominators with the hyperbola  $p = xyz$  corresponds to elliptic curves, one sees that these rational functions are essentially classified by the possible  $n$ -parameters families  $P(x, y) = 0$  of elliptic curves.

If one considers a polynomial

$$P(x, y) = \sum_m \sum_n a_{m,n} \cdot x^m y^n, \quad (\text{C.2})$$

with generic coefficients  $a_{m,n} \in \mathbb{C}$ , then the genus of the algebraic curve defined by  $P$  is determined by the support  $\text{supp}(P) = \{(m, n) \in \mathbb{N}^2 : a_{m,n} \neq 0\}$ . More precisely, the genus equals the number of interior integer lattice points inside the convex hull of  $\text{supp}(P)$  [89] (see also the discussion in [90]). For example, the support of the ten-parameters family (12) consists of the following 10 points in  $\mathbb{N}^2$ :

$$(0, 0), (0, 1), (0, 2), (0, 3), (1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)$$

which form a right triangle of side length 3. Only one of these points is an interior point, namely  $(1, 2)$ , hence the genus is 1.

Therefore we may ask: which integer lattice polytopes exist which have exactly one interior point and what is the largest such polytope? Not surprisingly, the answer is known: there are (up to transformations like translation, rotation, shearing) exactly 16 different polytopes with a single interior point [91] (see also Figure 5, page 548 in [92]), the above-mentioned right triangle being the one with the highest total number of lattice points.

This shows that *there cannot be a family of elliptic curves with more than ten parameters.*

#### Appendix D. Monomial transformations preserving pullbacked hypergeometric results

More generally, recalling subsection 4.2 in [2] and subsection 4.2 page 17 in [1], let us consider the monomial transformation

$$\begin{aligned} (x, y, z) &\longrightarrow M(x, y, z) = (x_M, y_M, z_M) \\ &= \left( x^{A_1} \cdot y^{A_2} \cdot z^{A_3}, x^{B_1} \cdot y^{B_2} \cdot z^{B_3}, x^{C_1} \cdot y^{C_2} \cdot z^{C_3} \right), \end{aligned} \quad (\text{D.1})$$

where the  $A_i$ 's,  $B_i$ 's and  $C_i$ 's are positive integers such that  $A_1 = A_2 = A_3$  is excluded (as well as  $B_1 = B_2 = B_3$  as well as  $C_1 = C_2 = C_3$ ), and that the determinant of the  $3 \times 3$  matrix [1, 2]

$$\begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix}, \quad (\text{D.2})$$

is not equal to zero<sup>††</sup>, and that:

$$A_1 + B_1 + C_1 = A_2 + B_2 + C_2 = A_3 + B_3 + C_3. \quad (\text{D.3})$$

We will denote by  $n = A_i + B_i + C_i$  the integer<sup>†</sup> in these three equal sums (D.3). The condition (D.3) is introduced in order to impose that the product<sup>¶</sup> of  $x_M y_M z_M$  is an integer power of the product of  $x y z$ :  $x_M y_M z_M = (x y z)^n$ .

If we take a rational function  $\mathcal{R}(x, y, z)$  in three variables and perform such a monomial transformation (D.1)  $(x, y, z) \rightarrow M(x, y, z)$ , on this rational function  $\mathcal{R}(x, y, z)$ , we get another rational function that we denote by  $\tilde{\mathcal{R}} = \mathcal{R}(M(x, y, z))$ . Now the diagonal of  $\tilde{\mathcal{R}}$  is the diagonal of  $\mathcal{R}(x, y, z)$  where we have changed  $x$  into  $x^n$ :

$$\Phi(x) = \text{Diag}\left(\mathcal{R}(x, y, z)\right), \quad \text{Diag}\left(\tilde{\mathcal{R}}(x, y, z)\right) = \Phi(x^n). \quad (\text{D.4})$$

### Appendix E. Telescopers of rational functions associated with a split Jacobian: the general case.

Following calculations in subsection 5.3, we have seen that for arbitrary values of the parameter  $a$  and  $b = 3 + x$ , the Hauptmodul of one of the two elliptic curves of the split Jacobian reads (see eq. (106)):

$$\mathcal{H}_x = \frac{108 \cdot x^3 \cdot (x^2 + 9x + 27)^3 \cdot P_2^2}{P_4^3}, \quad (\text{E.1})$$

where:

$$\begin{aligned} P_2 &= 4x^3 - (a-6) \cdot (a+6) \cdot x^2 - 6 \cdot (a+6) \cdot (a-3) \cdot x + (4a+15) \cdot (a-3)^2, \\ P_4 &= 12x^5 + (a^2+180) \cdot x^4 + 6 \cdot (2a^2-21a+180) \cdot x^3 \\ &\quad + 27 \cdot (2a^2-42a+135) \cdot x^2 \\ &\quad + 162 \cdot (a-3) \cdot (2a-15) \cdot x + 729 \cdot (a-3)^2. \end{aligned} \quad (\text{E.2})$$

Let us consider the telescoper of the rational function of three variables  $xy/D_a(x, y, z)$  where the denominator  $D_a(x, y, z)$  is  $C_{a,b}(x, y)$  given by (103) for  $b = 3 + xyz$ , namely (108). We have calculated this telescoper for an arbitrary value of the parameter  $a$ .

This telescoper is an order-four linear differential operator  $L_4$  which is actually the *direct-sum* of two *order-two* linear differential operators  $L_4 = LCLM(L_2, M_2) = L_2 \oplus M_2$ , these two order-two linear differential operators having respectively the head polynomials  $H(L_2)$  and  $H(M_2)$ :

$$\begin{aligned} H(L_2) &= 4 \cdot x \cdot (x+3)^2 \cdot \left( a \cdot x^2 + 3 \cdot (2a-15) \cdot x + 3 \cdot (4a+15) \cdot (a-3) \right) \\ &\quad \times (x^2 + 9x + 27) \cdot P_2^2, \end{aligned} \quad (\text{E.3})$$

$$H(M_2) = 4 \cdot \left( a \cdot x + 3 \cdot (a-3) \right) \cdot P_2^2. \quad (\text{E.4})$$

<sup>††</sup>We want the rational function  $\tilde{\mathcal{R}} = \mathcal{R}(M(x, y, z))$  deduced from the monomial transformation (D.1) to remain a rational function of *three* variables and not of two, or one, variables.

<sup>†</sup>Note a typo in the footnote 28 page 17 of [1] as well as in the second footnote page 18 in [2]. The sentence has been truncated. One should read: For  $n = 1$ , the  $3 \times 3$  matrix (D.2) is stochastic and transformation (D.1) is a *birational transformation* if the determinant of the matrix (D.2) is  $\pm 1$ .

<sup>¶</sup>Recall that taking the diagonal of a rational function of three variables extracts, in the multi-Taylor expansion, only the terms that are  $n$ -th power of the *product*  $x y z$ .

These two order-two linear differential operators *cannot be homomorphic* since they do not have exactly<sup>‡</sup> the same singularities. In view of the  $x^2 + 9x + 27$  term, we expect the order-two linear differential operator  $L_2$  to have a pullbacked  ${}_2F_1$  hypergeometric solution with a Hauptmodul (E.1), or, at least, related to (E.1) by some isogeny (modular correspondence) [61, 62].

Unfortunately the Maple command "hypergeometricsols" of Mark van Hoeij [93] has not been able to find the pullbacked  ${}_2F_1$  hypergeometric solution of  $L_2$  for values different from  $a = 3$  and  $a = \infty$ . For  $a = \infty$  we find the following pullbacked  ${}_2F_1$  hypergeometric solution for  $L_2$ :

$$\mathcal{S} = \left( \frac{(x+3)^3}{x^3 + 9x^2 + 27x + 3} \right)^{1/4} \times {}_2F_1 \left( \left[ \frac{1}{12}, \frac{5}{12} \right], [1], \frac{1728 \cdot x \cdot (x^2 + 9x + 27)}{(x+3)^3 \cdot (x^3 + 9x^2 + 27x + 3)^3} \right). \quad (\text{E.5})$$

In the  $a = \infty$  limit the Hauptmodul (E.1) reads:

$$\mathcal{H} = \frac{1728 \cdot x^3 \cdot (x^2 + 9x + 27)^3}{(x+9)^3 \cdot (x+3)^3 \cdot (x^2 + 27)^3}, \quad (\text{E.6})$$

to be compared with the pullback in the pullbacked  ${}_2F_1$  hypergeometric solution (E.5). This does not seem to match at first sight. In fact, we have a remarkable identity: the pullbacked  ${}_2F_1$  hypergeometric solution (E.5) *can also be written*:

$$\mathcal{S} = \left( \frac{(x+3)^3}{x^2 + 27} \cdot (x+9) \right)^{1/4} \times {}_2F_1 \left( \left[ \frac{1}{12}, \frac{5}{12} \right], [1], \frac{1728 x^3 \cdot (x^2 + 9x + 27)^3}{(9+x)^3 \cdot (3+x)^3 \cdot (x^2 + 27)^3} \right). \quad (\text{E.7})$$

We have a *modular correspondence* between these two Hauptmoduls appearing in (E.5) and (E.7). The algebraic relation between these two Hauptmoduls corresponds to the *mmodular equation*:

$$\begin{aligned} & 26214400000000 A^3 B^3 \cdot (A+B) + 4096000000 A^2 B^2 \cdot (27 A^2 - 45946 AB + 27 B^2) \\ & + 15552000 \cdot AB \cdot (A+B) \cdot (A^2 + 241433 AB + B^2) \\ & + 729 \cdot (A^4 + B^4) - 779997924 \cdot AB \cdot (A^2 + B^2) + 1886592284694 \cdot A^2 B^2 \\ & + 2811677184 \cdot AB \cdot (A+B) - 2176782336 \cdot AB = 0, \end{aligned} \quad (\text{E.8})$$

which is a representation of  $\tau \rightarrow 3\tau$  where  $\tau$  is the ratio of the periods.

In fact, we have been able to find the two pullbacked hypergeometric solutions of  $L_2$  and  $M_2$ . One can actually discover that the pullbacked hypergeometric solutions of  $L_2$  have the form

$$\frac{(x+3)}{P_2^{1/2} \cdot P_4^{1/4}} \cdot {}_2F_1 \left( \left[ \frac{1}{12}, \frac{5}{12} \right], [1], \frac{108 \cdot x^3 \cdot (x^2 + 9x + 27)^3 \cdot P_2^2}{P_4^3} \right), \quad (\text{E.9})$$

where the pullback in (E.9) is *exactly the same Hauptmodul*  $\mathcal{H}_x$  as (E.1) corresponding to the  $j$ -invariant of the elliptic curve of the split Jacobian of the genus-two curve !! The pullbacked hypergeometric solutions of  $M_2$  reads:

$$\begin{aligned} & \frac{1}{P_2^{1/2} \cdot (a^2 - 9 - 3x)^{1/4}} \\ & \times {}_2F_1 \left( \left[ \frac{1}{12}, \frac{5}{12} \right], [1], 1 - \frac{\left( (a-3) \cdot (2a^2 + 6a - 9) - 9ax \right)^2}{4 \cdot (a^2 - 9 - 3x)^3} \right). \end{aligned} \quad (\text{E.10})$$

<sup>‡</sup> They share, however, the singularities  $P_2 = 0$ .

Note that this Hauptmodul in (E.10)

$$\mathcal{H}_2 = 1 - \frac{\left((a-3) \cdot (2a^2 + 6a - 9) - 9ax\right)^2}{4 \cdot (a^2 - 9 - 3x)^3}, \quad (\text{E.11})$$

is not expandable at  $x = 0$ . That way we finally find the second Hauptmodul corresponding to the second elliptic curve in the split Jacobian  $\mathcal{E}_1 \times \mathcal{E}_2$ . Note that, in terms of  $a$  and  $b$  this second  $j$ -invariant has a quite simple form:

$$j_2 = \frac{256 \cdot (a^2 - 3b)^3}{a^2b^2 - 4a^3 - 4b^3 + 18ab - 27}. \quad (\text{E.12})$$

Let us denote  $A$  the first Hauptmodul (E.1) and  $B$  this last Hauptmodul in (E.10). For arbitrary values of the parameter  $a$  they are not related by a modular correspondence. The corresponding  $j$ -invariants must be seen as two independent†  $j$ -invariants. Of course, eliminating  $x$  one can find, for arbitrary values of the parameter  $a$ , some quite involved (non-symmetric) polynomial relation  $P(A, B) = 0$  between these two Hauptmoduls. We have, however, rather simple relation for selected values of the parameter  $a$ , namely:

$$A = \frac{27 \cdot B^2 \cdot (16B^2 + 4B + 7)^3}{(4B - 1)^3 \cdot (4B^2 + 19B + 4)^3}, \quad \text{for } a = 3, \quad (\text{E.13})$$

$$A = -\frac{B^2 \cdot (4B - 5)^3}{(5B - 4)^3}, \quad \text{for } a = 0. \quad (\text{E.14})$$

Note that, in the  $a = \infty$  limit, the second Hauptmodul (E.11) trivialises and becomes  $\mathcal{H}_2 = 0$ . The  $a = \infty$  limit is, in fact, a bit tricky. The genus-two curve (103) becomes

$$4 \cdot y^2 + \alpha^2 \cdot (b^2 x^2 + 4x^3 + 2bx + 1) \cdot x^2 = 0, \quad (\text{E.15})$$

where  $\alpha = 1/a$ ,  $a \rightarrow \infty$  (i.e.  $\alpha \rightarrow 0$ ). Let us introduce  $Y = y/\alpha = a \cdot y$  (with  $a \rightarrow \infty$ ). The previous curve (E.15) becomes the elliptic curve

$$4 \cdot Y^2 + (b^2 x^2 + 4x^3 + 2bx + 1) \cdot x^2 = 0, \quad (\text{E.16})$$

with a  $j$ -invariant giving the Hauptmodul  $\mathcal{H} = 1728/j$ :

$$\mathcal{H} = \frac{1728 \cdot (b-3) \cdot (b^2 + 3b + 9)}{b^3 \cdot (b^3 - 24)^3}, \quad (\text{E.17})$$

which, for  $b = 3 + x$ , gives exactly the Hauptmodul in (E.7), namely:

$$\mathcal{H}_x = \frac{1728 \cdot x \cdot (x^2 + 9x + 27)}{(x+3)^3 \cdot (x^3 + 9x^2 + 27x + 3)^3}. \quad (\text{E.18})$$

In that  $a = \infty$  limit, the genus-two curve (103) degenerates into a *genus-one* curve. As far as creative telescoping is concerned, this amounts to calculating the telescoper of the rational function

$$\frac{xy}{4 \cdot y^2 + \left((3 + xyz)^2 \cdot x^2 + 4x^3 + 2 \cdot (3 + xyz) \cdot x + 1\right) \cdot x^2}. \quad (\text{E.19})$$

This telescoper is an order-three linear differential  $L_3$  which is the *direct-sum*  $L_3 = D_x \oplus \mathcal{L}_2$ , where the order-two linear differential  $\mathcal{L}_2$  is exactly the  $a = \infty$  limit of the order-two linear differential operator  $L_2$  in the order-four linear differential operator

† Emerging from the Igusa-Shiode invariants of the Jacobian.



$L_4 = LCLM(L_2, M_2) = L_2 \oplus M_2$ , with head polynomial (E.3), and thus has (E.7) as a solution. With these calculations we see, quite clearly, how the split Jacobian, which is *isogenous to the product of two elliptic curves*, degenerates, in the  $a = \infty$  limit, into an elliptic curve.

Curiously, as a by-product of the calculation of the (non-symmetric) polynomial relation  $P(A, B) = 0$  between these two Hauptmoduls, we find in the  $a = \infty$  case, a (spurious) genus-zero algebraic symmetric artefact relation

$$1953125 \cdot A^3 B^3 - 187500 \cdot A^2 B^2 \cdot (A + B) + 375 \cdot AB \cdot (16 A^2 - 4027 AB + 16 B^2) - 64 \cdot (A + B) \cdot (A^2 + 1487 AB + B^2) + 110592 \cdot AB = 0, \quad (\text{E.20})$$

which turns out to be the fundamental modular equation, parametrised by:

$$A = \frac{1728 z}{(z + 16)^3}, \quad B = \frac{1728 z^2}{(z + 256)^3}. \quad (\text{E.21})$$

### Appendix F. Creative telescoping on rational functions of three variables associated with genus-two curves with split Jacobians: another example

Let us consider the *genus-two curve*  $C_p(x, y) = 0$  given in equation (5) of lemma 4 of [63]:

$$C_p(x, y) = x^6 + x^3 + p - y^2. \quad (\text{F.1})$$

Let us introduce the rational function  $xy/D(x, y, z)$  where the denominator  $D(x, y, z)$  is given by:

$$D(x, y, z) = C_{p=xyz}(x, y) = x^6 + x^3 + xyz - y^2. \quad (\text{F.2})$$

The telescoper of this rational function is an order-two linear differential operator which has the two hypergeometric solutions

$$x^{-1/6} \cdot {}_2F_1\left(\left[\frac{1}{6}, \frac{2}{3}\right], \left[\frac{5}{6}\right], 4x\right), \quad (\text{F.3})$$

which is a Puiseux series at  $x = 0$  and:

$$x^{-1/6} \cdot {}_2F_1\left(\left[\frac{1}{6}, \frac{2}{3}\right], [1], 1 - 4x\right). \quad (\text{F.4})$$

These two hypergeometric solutions can be rewritten as<sup>‡</sup>

$$\mathcal{A}(x) \cdot {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], \frac{1728}{J}\right). \quad (\text{F.5})$$

where the  $j$ -invariant  $J$ , in the Hauptmodul  $1728/J$  corresponds exactly to the degree-two elliptic subfields of the the split Jacobian of the genus-two curve.

Of course, if we change  $p$  into  $p \rightarrow (1 - p)/4$  in (F.1), the telescoper of the rational function  $xy/D(x, y, z)$  where the denominator  $D(x, y, z)$  is given by

$$D(x, y, z) = C_{p=xyz}(x, y) = x^6 + x^3 + \left(\frac{1 - xyz}{4}\right) - y^2, \quad (\text{F.6})$$

is the order-two linear differential operator corresponding to the  $x \rightarrow (1 - x)/4$  pullback of the previous one. It has the two hypergeometric solutions

$$(1 - x)^{-1/6} \cdot {}_2F_1\left(\left[\frac{1}{6}, \frac{2}{3}\right], [1], x\right), \quad (\text{F.7})$$

<sup>‡</sup> The fact that  ${}_2F_1\left(\left[\frac{1}{6}, \frac{2}{3}\right], [1], z\right)$  can be rewritten as  ${}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], H(z)\right)$  where the Hauptmodul  $H(z)$  is solution of a quadratic equation is given in equation (H.16) of Appendix H of [17].

and:

$$(1-x)^{-1/6} \cdot {}_2F_1\left(\left[\frac{1}{6}, \frac{2}{3}\right], \left[\frac{5}{6}\right], 1-x\right). \quad (\text{F.8})$$

The pullbacked hypergeometric solution (F.7) can also be written

$$(1-x)^{-1/6} \cdot {}_2F_1\left(\left[\frac{1}{6}, \frac{2}{3}\right], [1], x\right) = \mathcal{A}(x) \cdot {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], \mathcal{H}(x)\right), \quad (\text{F.9})$$

where  $\mathcal{H}(x)$  reads

$$\mathcal{H}(x) = 4 \cdot x \cdot \frac{1458 - 1215x + 125x^2}{(25x - 9)^3} + 8 \cdot x \cdot \frac{(27 - 11x) \cdot (27 - 25x)}{\sqrt{1-x} \cdot (9 - 25x)^3}, \quad (\text{F.10})$$

and where  $\mathcal{A}(x)$  reads:

$$\mathcal{A}(x) = (1-x)^{-5/24} \cdot \left(\frac{81}{9-25x}\right)^{1/8} \cdot \left(\frac{5 \cdot (1-x)^{1/2} - 4}{5 \cdot (1-x)^{1/2} + 4}\right)^{1/8}. \quad (\text{F.11})$$

## Appendix G. Telescopers of rational functions of several variables: some examples

*Appendix G.1. Telescopers of rational functions of several variables: a second example with four variables*

Let us now consider the rational function in *four* variables  $x, y, z, u$ :

$$R(x, y, z, u) = \frac{1}{1 + 9x + 3y + z + 9yz + 3ux + 2xy + 5xz + 7x^2y + 11z^2y}. \quad (\text{G.1})$$

The telescoper of this rational function of four variables is the *same order-two linear differential operator*  $L_2$  as for the telescoper of (178). It has the same pullbacked hypergeometric solution (179).

Performing the intersection of the codimension-one algebraic variety

$$1 + 9x + 3y + z + 3ux + 9yz + 2xy + 5xz + 7x^2y + 11z^2y = 0,$$

corresponding to the denominator of (G.1), with the hyperbolae  $p = xyz u$  amounts to eliminating, for instance  $u$  (writing  $u = p/x/y/z$ ). This gives  $P_u = 0$  where  $P_u$  reads:

$$P_u = 7x^2y^2z + 11y^2z^3 + 2xy^2z + 5xyz^2 + 9y^2z^2 + 9xyz + 3y^2z + yz^2 + yz + 3p. \quad (\text{G.2})$$

Assuming  $x$  to be constant<sup>†</sup> the previous condition  $P_u(y, z) = 0$  is an algebraic curve. Calculating its genus, one finds immediately that it is *genus-one*. Calculating its  $j$ -invariant, one finds

$$J = -\frac{N^3}{27 \cdot p^3 \cdot (7x^2 + 2x + 3)^2 \cdot D} \quad \text{where:} \quad (\text{G.3})$$

$$N = 81 \cdot (280p + 81) \cdot x^4 + 36(376p + 81) \cdot x^3 - 18 \cdot (1848p^2 + 292p - 27) \cdot x^2 - 36 \cdot (264p^2 + 20p - 1) \cdot x - 2592p^2 + 1,$$

<sup>†</sup> If one assumes  $z$  to be constant, the previous condition  $P_u(y, z) = 0$  becomes a genus-zero curve. If one assumes  $y$  to be constant, the previous condition  $P_u(y, z) = 0$  is again a genus-one curve, but the corresponding Hauptmodul, which depends on  $y$  is not simply related to (G.5) for any selected value of  $y$ .

and where:

$$\begin{aligned}
D = & 70875 \cdot (35p + 9) \cdot x^8 + 97200 \cdot (35p + 8) \cdot x^7 \\
& + 270 \cdot (38655p + 9047) \cdot x^6 + 4 \cdot (2000295p + 393278) \cdot x^5 \\
& - (21704760p^2 + 1329219p - 446680) \cdot x^4 \\
& - 36 \cdot (332112p^2 + 28965p - 1888) \cdot x^3 \\
& + 2 \cdot (8049888p^3 - 864324p^2 - 70038p + 2903) \cdot x^2 \\
& + 24 \cdot (191664p^3 - 16218p^2 - 246p + 11) \cdot x \\
& + 2665872p^3 - 19440p^2 - 12p + 5.
\end{aligned} \tag{G.4}$$

In the  $x \rightarrow 0$  limit of the Hauptmodul  $H_{p,x} = 1728/J$ , one finds:

$$H_p = -\frac{419904 \cdot p^3 \cdot (5 - 12p - 19440p^2 + 2665872p^3)}{(1 - 2592p^2)^3}. \tag{G.5}$$

which actually corresponds to the Hauptmodul in (179).

### Appendix G.2. Telescopers of rational functions of several variables: a third example with four variables

Let us consider the rational function in *four* variables  $x, y, z, u$ :

$$R(x, y, z, u) = \frac{1}{1 + 3y + z + 9yz + 11z^2y + 3ux + x \cdot P_1(y, z) + x^2 \cdot P_2(y, z)}, \tag{G.6}$$

where  $P_1(y, z)$  and  $P_2(y, z)$  are the two simple polynomials  $P_1(y, z) = y^2 z^2$  and  $P_2(y, z) = y^3$ . The telescoper of this rational function of *four variables* is the *same order-two linear differential operator*  $L_2$  as for the telescoper of (178). It has the same pullbacked hypergeometric solution (179). Actually the diagonal of the rational function (178) is the expansion (180) of the pullbacked hypergeometric function (179). For  $P_1(y, z) = y^2 z^2$  and  $P_2(y, z) = y^3$  the elimination of  $u = p/x/y/z$  in the vanishing condition of the denominator (G.6) gives the algebraic curve:

$$x^2 y^4 z + x y^3 z^3 + 11 y^2 z^3 + 9 y^2 z^2 + 3 y^2 z + y z^2 + y z + 3p = 0. \tag{G.7}$$

For  $x$  fixed (and of course  $p$  fixed) this algebraic curve (G.7) is a *genus-five* curve, but in the  $x \rightarrow 0$  limit it reduces to the *same genus-one* curve as for the first example (178), namely:

$$11 y^2 z^3 + 9 y^2 z^2 + 3 y^2 z + y z^2 + y z + 3p = 0. \tag{G.8}$$

which corresponds to the Hauptmodul (G.5).

The generalisation of this result is straightforward. Let us consider the rational function in *four variables*  $x, y, z$  and  $u$

$$R(x, y, z, u) = \frac{1}{1 + 3y + z + 9yz + 11z^2y + 3ux + x \cdot P(x, y, z)}, \tag{G.9}$$

where  $P(x, y, z)$  is an *arbitrary polynomial* of the three variables  $x, y$  and  $z$ . On a large set of examples one verifies that the *diagonal* of (G.9) is actually the expansion (180) of the pullbacked hypergeometric function (179):

$$\begin{aligned}
& 1 + 648x^2 - 72900x^3 + 1224720x^4 - 330674400x^5 + 23370413220x^6 \\
& - 1276733858400x^7 + 180019474034400x^8 - 12013427240614800x^9 + \dots
\end{aligned} \tag{G.10}$$

However, as far as creative telescoping calculations are concerned the telescoper corresponding to different polynomials  $P(x, y, z)$  becomes quickly a quite large non-minimal linear differential operator. For instance, even for the simple polynomial  $P(x, y, z) = x + y$ , one obtains a quite large order-ten telescoper. Of course, since this telescoper has the pullbacked hypergeometric function (179) as a solution, it is not minimal, it is righdivisible by the order-two linear differential operator having (179) as a solution. It is straightforward to see that the previous elimination of  $u = p/x/y/z$  in the vanishing condition of the denominator (G.9) gives an algebraic curve†

$$11y^2z^3 + 9y^2z^2 + 3y^2z + yz^2 + yz + 3p + yz \cdot P(x, y, z) = 0. \quad (\text{G.11})$$

which reduces again, in the  $x \rightarrow 0$  limit, to the *same genus-one* curve (G.8).

With that general example (G.9) we see that there is *an infinite set of rational functions depending on an arbitrary polynomials  $P(x, y, z)$  of three variables* which diagonals are a pullbacked  ${}_2F_1$  hypergeometric solution.

### Appendix G.3. Telescopers of rational functions of several variables: some examples

More generally we find that the diagonal of the rational function in  $x, y, z, u$

$$R(x, y, z, u) = \frac{1}{a + b_1y + c_1/y + b_2z + c_2/z + d_1yz + e_1y/z + f_1z/y + g_1ux + x^N \cdot P(y, z)}, \quad (\text{G.12})$$

gives, for every integer  $N \geq 1$ , a telescoper *independent of the arbitrary polynomial  $P(y, z)$* , namely the *same* telescoper that the rational function of  $x, y, z, u$

$$R(x, y, z, u) = \frac{1}{a + b_1y + c_1/y + b_2z + c_2/z + d_1yz + e_1y/z + f_1z/y + g_1ux}. \quad (\text{G.13})$$

The telescoper annihilates the pullbacked hypergeometric function:

$$D_H^{-1/12} \cdot {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], \frac{1728}{J}\right), \quad (\text{G.14})$$

where  $D_H$  denotes the denominator of the Hauptmodul  $H = 1728/J$  and where the  $j$ -invariant  $J$  is the  $j$ -invariant of the elliptic curve corresponding to the  $x = 0$  limit of

$$D\left(x, y, z, \frac{p}{xyz}\right) = 0, \quad (\text{G.15})$$

namely the most general *nine-parameters* biquadratic  $B(y, z) = 0$ :

$$d_1y^2z^2 + b_1y^2z + b_2yz^2 + ayz + e_1y^2 + f_1z^2 + c_1z + c_2y + g_1p = 0. \quad (\text{G.16})$$

### Appendix G.4. A simple $u$ -extension of the bicubic case.

Let us perform a similar “reversed engineering” with the (selected) ten-parameters bicubics like (12) that are elliptic curves. Let us consider one of these (selected) bicubic  $B(x, y)$ :

$$B(x, y) = 2x^3y^3 + 5x^2y^3 + 3x^2y^2 + xy^3 + xy^2 + 3y^3 + xy + 3y^2 + 2y + 5 \quad (\text{G.17})$$

† Of arbitrary large genus for increasing degrees of the polynomial  $P(x, y, z)$ .

The bicubic equation  $B(x, y) + p = 0$  is an elliptic curve. One can calculate its  $j$ -invariant and the corresponding Hauptmodul  $1728/j$ :

$$\mathcal{H} = -1728 \frac{p_4(p)}{(2424p + 11305)^3}, \quad \text{where:} \quad (\text{G.18})$$

$$p_4(p) = 99015075p^4 + 1743092117p^3 + 11512110810p^2 + 33804556190p + 37237506697. \quad (\text{G.19})$$

Let us now consider the rational function of four variables

$$R(x, y, z, u) = \frac{xy^2}{B(x, y) + uxyz}. \quad (\text{G.20})$$

Its telescoper is an order-two linear differential operator  $L_2$  with pullbacked  ${}_2F_1$  hypergeometric solutions

$$(2424x + 11305)^{-1/4} \cdot {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], -1728 \frac{p_4(x)}{(2424x + 11305)^3}\right),$$

where:  $p_4(x) = 99015075x^4 + 1743092117x^3 + 11512110810x^2 + 33804556190x + 37237506697,$  (G.21)

which actually corresponds to the Hauptmodul (G.18) of the bicubic equation  $B(x, y) + p = 0$ .

**Comment:** if one considers, instead of (G.20), the rational function

$$R(x, y, z, u) = \frac{xy}{B(x, y) + uxyz}, \quad (\text{G.22})$$

one finds an *order-four* telescoper which factorises into *two order-two* linear differential operators  $M_4 = M_2 \cdot N_2$ , where  $N_2$  has algebraic functions solutions, and where  $M_2$  is homomorphic to the previous order-two linear differential operator  $L_2$ .

#### Appendix G.5. Another simple $u$ -extension of the bicubic case.

Let us consider the rational function in three variables

$$R(x, y, z) = \frac{1}{1 + x + 2y + 3z + 2yz + 5xz + 7xz + x^2y + y^2z + 2z^2x}. \quad (\text{G.23})$$

If one substitute  $z = p/x/y$  in the rational function (G.23), one gets

$$\mathcal{R}_p(x, y) = \frac{xy^2}{\mathcal{B}_p(x, y)} \quad \text{where:} \quad (\text{G.24})$$

$$\mathcal{B}_p(x, y) = 2p^2 + y \cdot (y^2 + 12x + 2y + 3) \cdot p + x^3y^3 + x^2y^2 + 2xy^3 + xy^2.$$

Let us now consider the rational function in *four* variables  $x, y, z$  and  $u$  which is  $\mathcal{R}_p(x, y)$  given by (G.24) where  $p = xyz u$ :

$$\frac{xy^2}{2(xyz u)^2 + y \cdot (y^2 + 12x + 2y + 3) \cdot xyz u + x^3y^3 + x^2y^2 + 2xy^3 + xy^2}. \quad (\text{G.25})$$

The telescoper of the rational function in three variables (G.23), and the telescoper of the rational function of four variables (G.25), *are actually equal*, having the pullbacked

${}_2F_1$  hypergeometric solution given by (5) in subsection 2.2. In this particular case the pullbacked  ${}_2F_1$  hypergeometric solution reads

$$(1 - 64x + 7552x^2 + 3600x^3)^{-1/4} \times {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], \frac{1728 \cdot x^3 \cdot p_9(x)}{(1 - 64x + 7552x^2 + 3600x^3)^3}\right), \quad (\text{G.26})$$

where:

$$p_9(x) = 675 - 46908x + 7579422x^2 - 256103188x^3 + 748623104x^4 - 1361870768x^5 + 554260968x^6 - 1071752256x^7 - 36904896x^8 - 314928x^9. \quad (\text{G.27})$$

The Hauptmodul of the elliptic curve  $\mathcal{B}_p(x, y) = 0$  corresponds to the pullback in the pullbacked  ${}_2F_1$  hypergeometric solution (G.26):

$$\mathcal{H} = \frac{1728 \cdot p^3 \cdot p_9(p)}{(1 - 64p + 7552p^2 + 3600p^3)^3}. \quad (\text{G.28})$$

*Appendix G.6. One more simple u-extension of the bicubic case.*

Let us perform a similar “reversed engineering” with the (selected) ten-parameters bicubics like (12) that are elliptic curves. Let us recall the fact that the (selected) bicubic  $B(x, y) = 0$  where  $B(x, y)$  reads:

$$B(x, y) = ax^2y^2 + b_1x^2y^2 + b_2xy^3 + b_3y + c_1y^2 + c_2pxy + c_3x^2y^3 + d_1x^3y^3 + d_2y^3 + d_3. \quad (\text{G.29})$$

is an elliptic curve. Let us consider the denominator of four variables  $x, y, z$  and  $u$ :

$$D(x, y, z, u) = B(x, y) + u \cdot xyz. \quad (\text{G.30})$$

It is straightforward to see that, imposing that the product of the four variables  $xyz u = p$  is fixed, the elimination of the fourth variable  $u$ , by the substitution  $u = p/x/y/z$ , yields a bicubic  $B(x, y) + p = 0$ , which is an elliptic curve. Introducing the rational function of four variables

$$R(x, y, z, u) = \frac{xy^2}{D(x, y, z, u)}, \quad (\text{G.31})$$

one finds that the telescoper of this rational function of four variables (G.31) is an order-two linear differential operator

$$\mathcal{A}(x) \cdot {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], \mathcal{H}\right) \quad (\text{G.32})$$

where the Hauptmodul  $\mathcal{H} = 1728/j$  corresponds to the  $j$ -invariant of the elliptic curve  $B(x, y) + p = 0$ . Let us just give here a simple example.

The telescoper of (G.31) with

$$B(x, y) = 17xy^2 + x^2y^2 + 5xy^3 + 13y + y^2 + 7yx + x^2y^3 + 2x^3y^3 + y^3 + 3,$$

is an order-two linear differential operator having as solution:

$$\frac{1}{(15373 + 1656x)^{1/4}} \cdot {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], \frac{64q_4(x)}{(15373 + 1656x)^3}\right), \quad \text{where:} \quad (\text{G.33})$$

$$q_4(x) = 1143420561541 - 475427554218x + 67894132770x^2 - 2161434807x^3 + 22211523x^4.$$

The Hauptmodul appearing in this solution is, as it should, of the form  $1728/j$  where  $j$  is the  $j$ -invariant of the bicubic  $B(x, y) + p = 0$ .

Appendix G.7. A simpler  $u$ -extension.

The telescoper of the rational function

$$R(x, y, z, u) = \frac{1}{1 + 2y + 3z + 5yz + 4y^2z + 3xu}, \quad (\text{G.34})$$

or the telescoper of the rational function

$$R(x, y, z, u) = \frac{1}{1 + 2y + 3z + 5yz + 4y^2z + 3xu + (\alpha + \beta z + \gamma y) \cdot x^n}, \quad (\text{G.35})$$

for  $n = 1, 2, \dots, 5, \dots$ , are identical (whatever the values of  $\alpha, \beta, \gamma$ ). This telescoper is an order-two linear differential operator with the pullbacked hypergeometric solution:

$$(1 + 312x - 1584x^2)^{-1/4} \times {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], \frac{5038848x^3 \cdot (13248x^3 + 2928x^2 + 368x + 1)}{(1584x^2 - 312x - 1)^3}\right). \quad (\text{G.36})$$

One can verify that the diagonals of the rational functions (G.34) and (G.35) are actually equal and correspond to the expansion of the pullbacked hypergeometric solution (G.36):

$$1 - 78x + 15606x^2 - 3888540x^3 + 1069866630x^4 - 311621002308x^5 + 94190901642684x^6 - 29220290149904568x^7 + \dots \quad (\text{G.37})$$

The elimination of  $u$ , with  $u = p/x/y/z$  in the vanishing condition of the denominator of (G.34) gives the elliptic curve

$$4y^3z^2 + 5y^2z^2 + 2y^2z + 3yz^2 + yz + 3p = 0, \quad (\text{G.38})$$

which has a  $j$ -invariant yielding the Hauptmodul

$$\mathcal{H} = \frac{5038848p^3 \cdot (13248p^3 + 2928p^2 + 368p + 1)}{1584p^2 - 312p - 1)^3}, \quad (\text{G.39})$$

which is precisely the pullback in (G.36).

**Remark:** note that the telescoper of

$$R(x, y, z, u) = \frac{1}{1 + 2y + 3z + 5yz + 4y^2z + 3xu + 11zx^2 + 7yx}, \quad (\text{G.40})$$

is a pretty large order-seven linear differential operator, however, this operator is not the minimal order operator. The *minimal order* linear differential operator for the diagonal of (G.40) is actually the previous order-two linear differential operator having the pullbacked hypergeometric solution (G.36). One can verify directly that the *diagonal* of (G.40) is actually the expansion (G.37) for the pullbacked hypergeometric solution (G.36).

## Appendix G.8. Examples with five, six, ... variables.

Let us generalise the four variables rational function (G.13) introducing the *five* and *six* variables rational functions

$$R(x, y, z, u, v) = \frac{1}{a + b_1y + c_1/y + b_2z + c_2/z + d_1yz + e_1y/z + f_1z/y + g_1uvx}, \quad (\text{G.41})$$

and

$$R(x, y, z, u, v, w) = \frac{1}{a + b_1 y + c_1/y + b_2 z + c_2/z + d_1 y z + e_1 y/z + f_1 z/y + g_1 u v w x}. \quad (\text{G.42})$$

Their telescopers are the same as the telescoper of (G.13) which annihilates the pullbacked hypergeometric function (G.15) which Hauptmodul is associated with the elliptic (biquadratic) curve (G.16).

## References

- [1] Y. Abdelaziz, S. Boukraa, C. Koutschan, J-M. Maillard, *Diagonals of rational functions, pullbacked  ${}_2F_1$  hypergeometric functions and modular forms*, J. Phys. **A 51**, Number 45 (2018) 455201 (30 pages)
- [2] Y. Abdelaziz, S. Boukraa, C. Koutschan, J-M. Maillard, *Diagonals of rational functions, pullbacked  ${}_2F_1$  hypergeometric functions and modular forms (unabridged version)*, arXiv:1805.04711v1 [math-ph] (2018)
- [3] HolonomicFunctions Package version 1.7.1 (09-Oct-2013) written by Christoph Koutschan, Copyright 2007-2013, Research Institute for Symbolic Computation (RISC), Johannes Kepler University, Linz, Austria
- [4] G. Christol, 1984 Diagonales de fractions rationnelles et équations différentielles *Study group on ultrametric analysis, 10th year: 1982/83, No. 2, Exp. No. 18* (Paris: Inst. Henri Poincaré) pp 1–10 [http://archive.numdam.org/article/GAU\\_1982-1983\\_\\_10\\_2\\_A4\\_0.pdf](http://archive.numdam.org/article/GAU_1982-1983__10_2_A4_0.pdf)
- [5] G. Christol, 1985 Diagonales de fractions rationnelles et équations de Picard-Fuchs *Study group on ultrametric analysis, 12th year, 1984/85, No. 1, Exp. No. 13* (Paris: Secrétariat Math.) pp 1–12 [http://archive.numdam.org/article/GAU\\_1984-1985\\_\\_12\\_1\\_A8\\_0.pdf](http://archive.numdam.org/article/GAU_1984-1985__12_1_A8_0.pdf)
- [6] G. Christol, 1988 Diagonales de fractions rationnelles *Séminaire de Théorie des Nombres, Paris 1986–87 (Progr. Math. vol 75)* (Boston, MA: Birkhäuser Boston) pp. 65-90
- [7] G. Christol. 1990 *Globally bounded solutions of differential equations, Analytic number theory (Tokyo, 1988) (Lecture Notes in Math. 1434)* (Berlin: Springer) pp. 45-64 <http://dx.doi.org/10.1007/BFb0097124>
- [8] L. Lipshitz and A J van der Poorten, 1990 Rational functions, diagonals, automata and arithmetic *Number theory (Banff, AB, 1988)* (Berlin: de Gruyter) pp. 339-358
- [9] L. Lipshitz, 1988 The diagonal of a  $D$ -finite power series is  $D$ -finite *J. Algebra* **113** 373–378 [http://dx.doi.org/10.1016/0021-8693\(88\)90166-4](http://dx.doi.org/10.1016/0021-8693(88)90166-4)
- [10] J. Denef and L. Lipshitz, 1987 Algebraic power series and diagonals *J. Number Theory* **26** 46–67 [http://dx.doi.org/10.1016/0022-314X\(87\)90095-3](http://dx.doi.org/10.1016/0022-314X(87)90095-3)
- [11] A. Bostan, S. Boukraa, J-M. Maillard, J-A. Weil, *Diagonal of rational functions and selected differential Galois groups*, (2015), J. Phys. **A 48**: Math. Theor. 504001 (29 pages) arXiv:1507.03227v2 [math-ph]
- [12] M L. Glasser and A J. Guttman, 1994 Lattice Green function (at 0) for the 4D hypercubic lattice *J. Phys. A* **27** 7011–7014, <http://arxiv.org/abs/cond-mat/9408097>
- [13] A J. Guttman, 2010 Lattice Green's functions in all dimensions *J. Phys. A* **43** 305205, 26, <http://arxiv.org/abs/1004.1435>
- [14] N. Zenine, S. Hassani, J-M. Maillard, *Lattice Green Functions: the seven-dimensional face-centred cubic lattice*, (2015), J. Phys. **A 48**: Math. Theor. 035205 (18 pages) arXiv:1409.8615v1 [math-ph]
- [15] S. Hassani, C. Koutschan, J-M. Maillard and N. Zenine, *Lattice Green Functions: the d-dimensional face-centred cubic lattice,  $d = 8, 9, 10, 11, 12$* , (2016), J. Phys. **A 49**: Math. Theor. 164003 (30 pages) arXiv:1601.05657v2 [math-ph]
- [16] Y. Abdelaziz, S. Boukraa, C. Koutschan and J-M. Maillard, *Heun functions and diagonals of rational functions*, 2020 J. Phys. **A 53**: Math. Theor. 075206 (24 pages).
- [17] Y. Abdelaziz, S. Boukraa, C. Koutschan and J-M. Maillard, *Heun functions and diagonals of rational functions (unabridged version)*, (2019), <https://arxiv.org/abs/1910.10761>
- [18] A. Bostan, S. Boukraa, G. Christol, S. Hassani, J-M. Maillard, *Ising  $n$ -fold integrals as diagonals of rational functions and integrality of series expansions*, (2013), J. Phys. **A 46**: Math. Theor. 185202 (44 pages), <http://arxiv.org/abs/1211.6645v2>
- [19] A. Bostan, S. Boukraa, G. Christol, S. Hassani and J-M. Maillard, *Ising  $n$ -fold integrals*



- as diagonals of rational functions and integrality of series expansions: integrality versus modularity Preprint, <http://arxiv.org/abs/1211.6031>
- [20] M. Assis, S. Boukraa, S. Hassani, M. van Hoeij, J-M. Maillard and B M. McCoy, 2012 Diagonal Ising susceptibility: elliptic integrals, modular forms and Calabi-Yau equations *J. Phys. A* **45** 075205, 32 <http://arxiv.org/abs/1110.1705>
- [21] S. Boukraa, S. Hassani, J-M. Maillard, J-A. Weil, *Differential algebra on lattice Green functions and Calabi-Yau operators*, (2014), *J. Phys. A* **48**: Math. Theor. 095203 (37 pages)
- [22] A. Bostan, S. Boukraa, A-J. Guttman, S. Hassani, I. Jensen, J-M. Maillard and N. Zenine, 2009 High order Fuchsian equations for the square lattice Ising model:  $\chi^{(5)}$  *J. Phys. A* **42** 275209, 32, <http://arxiv.org/abs/0904.1601>
- [23] S. Boukraa, S. Hassani, I. Jensen, J-M. Maillard and N. Zenine, 2010 High-order Fuchsian equations for the square lattice Ising model:  $\chi^{(6)}$  *J. Phys. A* **43** 115201, 22, <http://arxiv.org/abs/0912.4968>
- [24] S. Boukraa, S. Hassani, J-M. Maillard and N. Zenine, 2007 Singularities of  $n$ -fold integrals of the Ising class and the theory of elliptic curves *J. Phys. A* **40** pp. 11713-11748, <http://arxiv.org/abs/0706.3367>
- [25] M. Assis, M. van Hoeij, J-M. Maillard, *The perimeter generating functions of three-choice, imperfect, and one-punctured staircase polygons*, (2016), *J. Phys. A* **49**: Math. Theor. 214002 (29 pages), <https://arxiv.org/abs/1602.00868>
- [26] A. Bostan, P. Lairez and B. Salvy, *Creative telescoping for rational functions using the Griffiths-Dwork method*, Proceedings ISSAC'13, pp. 93-100, ACM Press, 2013. <http://specfun.inria.fr/bostan/publications/BoLaSa13.pdf>
- [27] S. Chen, M. Kauers and M.F. Singer, *Telescopers for Rational and Algebraic Functions via Residues*, 2012, Proceedings of the 37th International Symposium on Symbolic and Algebraic Computation, ed. by Joris van der Hoeven and Mark van Hoeij, 2012 pp. 130-137. <https://arxiv.org/abs/1201.1954>
- [28] K. Takeuchi, *Commensurability classes of arithmetic triangle groups*, *J. Fac. Science Univ. Tokyo Sec. IA. Math.* **24**, (1977) pp. 201-212.
- [29] J. Voight, *Shimura curves of genus at most two*, *Math. Comp.* **78** pp 1155-1172, (2009).
- [30] C. F. Doran and A. Malmendier, *Calabi-Yau Manifolds Realizing Symplectically Rigid Monodromy Tuples*, <https://arxiv.org/pdf/1503.07500.pdf>
- [31] A. Malmendier, T. Shaska, editors *Higher Genus Curves in Mathematical Physics and Arithmetic Geometry*, Contemporary Mathematics, AMS Special Session Higher Genus Curves and Fibrations in Mathematical Physics and Arithmetic Geometry January 2016.
- [32] R. S. Maier, (2009), *On rationally parametrized modular equations*, *J. Ramanujan Math. Soc.* **24** pp. 1-73 <http://arxiv.org/abs/math/0611041>
- [33] F. Chyzak, *The ABC of Creative Telescoping - Algorithms, Bounds, Complexity*, 2014, <https://hal.inria.fr/tel-01069831>
- [34] Pierre Lairez, *Périodes d'intégrales rationnelles: algorithmes et applications*, Thèse de doctorat, <https://pierre.lairez.fr/these.pdf>
- [35] D. Zeilberger, *The Method of Creative Telescoping*, *J. Symbolic Computation* (1991) **11**, pp. 195-204
- [36] M. Kontsevich and D. Zagier, *Periods*, IHES/M/01/22 2001, <https://www.maths.ed.ac.uk/~v1ranick/papers/kontzag1.pdf>
- [37] J. I. Igusa, *Abstract vanishing cycle theory*, *Proc. Japan Acad.*, **34**, (1958) pp. 589-593.
- [38] Deligne, *Intégration sur un cycle évanescant*, (1983), *Inventiones Math.* **76**, 1-29-1-43, Springer-Verlag.
- [39] G. Christol, *Diagonales de fractions rationnelles et équations de Picard-Fuchs*, (1984), Groupe de travail d'analyse ultramétrique, Tome 12 (1984-1985) no. 1, *Expos. Math.* no. 13, 12 p. [http://www.numdam.org/item/GAU\\_1984-1985\\_\\_12\\_1\\_A8\\_0/](http://www.numdam.org/item/GAU_1984-1985__12_1_A8_0/)
- [40] P. Lairez, *Computing periods of rational integrals*, *Mathematics of Computation*, **85**, Number 300, pp. 1719-1752 and arXiv:1404.5069v3 [cs.SC]
- [41] F. Chyzak, *An extension of Zeilberger's fast algorithm to general holonomic functions*, *Discrete Math.* **217** 1-3 (2000) Formal power series and algebraic combinatorics (Vienna 1997) pp. 115-134
- [42] C. Koutschan, *A fast approach to creative telescoping*, *Math. Comput. sci.* **4**, (2-3) pp. 259-266.
- [43] S. Boukraa and J-M. Maillard, *Symmetries of lattice models in statistical mechanics and effective algebraic geometry*, *J. Phys. I France*, (1993) pp. 293-258.
- [44] M.P. Bellon, J-M. Maillard and C-M. Viallet, *Quasi integrability of the sixteen-vertex model*, *Phys. Letters B* **281**, (1992) pp. 315-319.
- [45] S. Boukraa, J-M. Maillard and G. Rollet, *Determinantal identities on integrable mappings*,

- International Journal of Modern Physics **B 8**, (1994) pp. 2157-2201.
- [46] H. Knaff, E. Selder and K. Spindler, *Explicit transformation of an intersection of quadrics to an elliptic curve in Weierstrass form*, <https://arxiv.org/pdf/1906.10230.pdf>
- [47] M. Bronstein, T. Mudders and J-A. Weil, *On Symmetric Powers of Differential Operators*, ISSAC '97: Proceedings of the 1997 international symposium on Symbolic and algebraic computation
- [48] S. Algreen, *The Point of a Certain Fivefold over Finite Fields and the Twelfth Power of the eta Function* Finite Fields and Their applications **8**, 18-33 (2002).
- [49] G. Almkvist, Ch. van Enckevort, D. van Straten and W. Zudilin, *Tables of Calabi-Yau equations*, (2010), [arXiv:math/0507430v2](https://arxiv.org/abs/math/0507430v2) [math.AG]
- [50] D. van Straten, *Calabi-Yau operators*, (2017), [arXiv:1704.00164v1](https://arxiv.org/abs/1704.00164v1) [math.AG]
- [51] A. Bostan, S. Boukraa, S. Hassani, M. van Hoeij, J-M. Maillard, J-A. Weil, N. J. Zenine, *The Ising model: from elliptic curves to modular forms and Calabi-Yau equations*, J. Phys. **A 44**: Math. Theor. (2011) (43 pp) 045204 and [arXiv:1007.69804v1](https://arxiv.org/abs/1007.69804v1) [math-ph] and [hal-00684883](https://arxiv.org/abs/hal-00684883), version 1
- [52] A. Kumar, *Hilbert Modular Surfaces for square discriminants and elliptic subfields of genus 2 function fields*, Mathematical Sciences (2015) 2:24, Reserach in the mathematical Sciences, a SpringerOpen Journal, [arXiv:1412.2849v2](https://arxiv.org/abs/1412.2849v2) [math.AG] (2016).
- [53] A. Kumar, *Elliptic Fibrations on a Generic Jacobian Kummer Surface* (2014), [arXiv:1105.1715v3](https://arxiv.org/abs/1105.1715v3) [math.AG]
- [54] A. Kumar and R. Mukamel, *Algebraic models and arithmetic geometry of Teichmüller curves in genus two*, <https://arxiv.org/abs/1406.7057>.pdf
- [55] N. Elkies and A. Kumar, *K3 surfaces and equations for Hilbert modular surfaces*, Algebra and Number Theory **8** (2014), no. 10, 2297-2411. [arXiv: 1209.3527](https://arxiv.org/abs/1209.3527) [https://arxiv.org/pdf/1209.3527.pdf](https://arxiv.org/abs/1209.3527)
- [56] R. M. Kuhn, *Curves of genus 2 with split Jacobian* (2014), Transactions of the American Mathematical Society Vol. **307**, No. 1 (May, 1988), pp. 41-49 Published by: American Mathematical Society
- [57] A. Kumar and R. E. Mukamel, *Real multiplication through explicit correspondences*, <https://math.rice.edu/~rm51/papers/correspondences.pdf>
- [58] G. Shimura, *Correspondances modulaires et les fonctions  $\zeta$  de courbes algébriques*, (1958) Journal of the mathematical society of Japan, **10**, pp. 1-28.
- [59] M. Eichler, *Lectures on Modular Correspondences*, Tata Institute of Fund. Research, 1957, <http://www.math.tifr.res.in/~publ/ln/tifr09.pdf>
- [60] R. Roy, *Elliptic and Modular Functions from Gauss to Dedekind to Hecke*, Cambridge University Press, 2017
- [61] F. Klein, *Zur [Systematik der] Theorie der ellitischen Modulfunctionen*, Akad. Wiss. München, 1879, WA [348] pp. 169-178.
- [62] F. Klein and R. Fricke, (1890-1892), *Vorlesungen über die Theorie der elliptischen Modulfunctionen*, vol 2, pp. 596-634.
- [63] T. Shaska, *Genus 2 curves with (3,3)-split Jacobian and large automorphism group*, [arXiv:math/0201008v1](https://arxiv.org/abs/math/0201008v1) [math.AG], 1 Jan 2002.
- [64] T. Shaska, *Genus 2 fields with degree 3 elliptic subfields*, Forum Math. **16**, (2004) pp. 263-280, [arXiv:math/0109155](https://arxiv.org/abs/math/0109155) [math.AG]
- [65] T. Shaska and H. Völklein, *Elliptic subfields and automorphisms of genus 2 function fields*, [arXiv:math/0107142v1](https://arxiv.org/abs/math/0107142v1) [math.AG], 19 Jul 2001.
- [66] K. Diarra, *Solutions algébriques des équations isomonodromiques sur les courbes de genre 2*, Annales de la Faculté des Sciences de Toulouse, Mathématiques, Tome XXIV, (2015), p.39-54.
- [67] E. Bedford, K. Kim, T. T. Truong, N. Abarenkova and J-M. Maillard, *Degree Complexity of a Family of Birational Maps*, Math. Phys. Anal. Geom. Springer
- [68] J-Ch Anglès d'Auriac, J-M Maillard and C M Viallet, *On the complexity of some birational transformations*, J. Phys. **A 39**: Math. Gen. (2006) pp. 3641-3654
- [69] J-M. Maillard, *Automorphisms of algebraic varieties and Yang-Baxter equations*, J. Math. Phys. **27**, 2776, (1986), [doi:10.1063/1.527303](https://doi.org/10.1063/1.527303)
- [70] M.P. Bellon, J-M. Maillard and C-M. Viallet, *Infinite discrete symmetry group for the Yang-Baxter equations: spin models*, Phys. Lett. **A 157** (1991) 343-353.
- [71] M.P. Bellon, J-M. Maillard and C-M. Viallet, *Infinite discrete symmetry group for the Yang-Baxter equations: vertex models*, Phys. Lett. **B 260** (1991) 87-100.
- [72] A. Corti, *Polynomial bounds for the number of automorphisms of a surface of general type*, Ann. Sci. Ecole Norm. Sup. **24** (1991) pp. 113-137

- [73] E. Szabó, *Bounding automorphism groups*, Math. Ann. **304** (1996) pp. 801-811.
- [74] C.D. Hacon, J. McKernan and C. Xu, *On the birational automorphisms of varieties of general type*, Annals of Mathematics **177**, (2013) pp. 1077-1111 and arXiv:1011.1464v2 [math.AG]
- [75] Y. Abdelaziz, J.-M. Maillard, *Modular forms, Schwarzian conditions, and symmetries of differential equations in physics*, (2017), J. Phys. **A 50**: Math. Theor. 215203 (44 pages), arXiv:1611.08493v3[math-ph]
- [76] M. F. Singer, *Solving homogeneous linear differential equations in terms of second order linear differential equations*, Amer. J. Math. **107** (1985), no. 3, 663-696.
- [77] A. C. Person, *Solving Homogeneous Linear Differential Equations of Order 4 in Terms of Equations of Smaller Order*, PhD thesis, Raleigh, North Carolina, 2002. <http://www.lib.ncsu.edu/resolver/1840.16/3059>
- [78] P. A. Griffiths, *On the periods of certain rational integrals I, II*. Ann. of Math. (2) **90** (1969), pp. 460-495 and pp. 496-541.
- [79] M. van Hoeij, *Solving third order linear differential equations in terms of second order equations*, Proc. ISSAC'07, 355-360, ACM, 2007.
- [80] S. Boukraa, S. Hassani and J.-M. Maillard, *Noetherian mappings*, Physica **D 185** (2003) pp. 3-44
- [81] R. Fricke and F. Klein, *Vorlesungen über die Theorie der automorphen Funktionen. I*, Druckund Verlag von B. G. Teubner, Leipzig, 1897, p. 366
- [82] P. Boalch and R. Paluba, *Symmetric cubic surfaces and  $G_2$  character varieties*, arXiv:1305.6594v2[math.AG]
- [83] S. Cantat and F. Loray, *Holomorphic dynamics, Painlevé@ VI equation and Character Varieties*, <https://hal.archives-ouvertes.fr/hal-00186558v2>
- [84] M. Mazzocco and R. Vidunas, *Cubic and Quartic Transformations of the Sixth Painlevé Equation in Terms of Riemann-Hilbert Correspondence*, arXiv:1305.6594v2[math.AG]
- [85] J. H. Silverman and J. Tate, *Rational Points on elliptic Curves*, Undergraduate Texts in PMathematics, Springer 1992
- [86] M. Sadek, *Minimal genus one curves*, Functiones et Approximatio, 46.1 (2012) 117-131, arXiv:1002.0451v1[math.NT]
- [87] B. Poonen, *An explicit algebraic family of genus-one curves violating the Hasse principle*, arXiv:math/9910124v1[math.NT]
- [88] N. Elkies, *Three lectures on elliptic surfaces and curves of high rank*, (2007) , arXiv:0709.2908v1[math.NT]
- [89] A. G. Khovanskii, *Newton polyhedra and the genus of complete intersections*, Funct. Anal. i ego pril. English translation: Functional Anal. Appl., 12 (1978), 38-46.
- [90] <https://mathoverflow.net/questions/16615/calculating-the-genus-of-a-curve-using-the-newton-polygon>
- [91] S. Rabinowitz, *A census of convex lattice polygons with at most one interior lattice point*, Ars Combinatoria 28 (1989), pp. 83-96.
- [92] Josef Schicho, *Simplification of surface parametrizations—a lattice polygon approach*, Journal of Symbolic Computation 36 (2003), pp. 535-554.
- [93] To find the implementation of “hypergeometricsols” see:  
<https://www.math.fsu.edu/~hoeij/algorithms/Erdal/> [https://www.math.fsu.edu/~\\$hoeij/maple.html](https://www.math.fsu.edu/~$hoeij/maple.html)  
[https://www.math.fsu.edu/~\\$hoeij/comparison-hypergeomsols](https://www.math.fsu.edu/~$hoeij/comparison-hypergeomsols)