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DIAGONALS OF RATIONAL FUNCTIONS: FROM DIFFERENTIAL ALGEBRA TO EFFECTIVE ALGEBRAIC GEOMETRY

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Abstract: We show that the results we had obtained on diagonals of nine and ten parameters families of rational functions using creative telescoping, yielding modular forms expressed as pullbacked ${}_2F_1$ hypergeometric functions, can be obtained, much more efficiently, by calculating the j-invariant of an elliptic curve canonically associated with the denominator of the rational functions. This result can be drastically generalised changing the parameters into arbitrary rational functions. In the case where creative telescoping yields pullbacked ${}_2F_1$ hypergeometric functions, we generalise this algebraic geometry approach to other families of rational functions in three, and even more than three, variables. In particular, we generalise this approach to rational functions in more than three variables when the denominator can be associated to an algebraic variety corresponding to products of elliptic curves, or foliation in elliptic curves. We also extend this approach to rational functions in three variables when the denominator is associated with a *genus-two curve such that its Jacobian is a split Jacobian* corresponding to the product of two elliptic curves. We sketch the situation where the denominator of the rational function is associated with algebraic varieties that are not of the general type, having an infinite set of birational automorphisms. We finally provide some examples of rational functions in more than three variables, where the telescopers have pullbacked ${}_2F_1$ hypergeometric solutions, the denominator corresponding to an algebraic variety having a selected elliptic curve in the variety explaining the pullbacked ${}_2F_1$ hypergeometric solution.

Keywords: Diagonals of rational functions, pullbacked hypergeometric functions, modular forms, Hauptmoduls, creative telescoping, telescopers, elliptic curves, j-invariant, Hauptmodul, K3 surfaces, split Jacobian, extremal rational surfaces, birational automorphisms, algebraic varieties of the general type.

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1. Introduction

In a previous paper [1,2], using creative telescoping [3], we have obtained *diagonals*[1] of nine and ten parameters families of rational functions, given by (classical) *modular*

[1] For the introduction of the concept of *diagonals of rational functions*, see [4–11].

forms expressed as pullbacked ${}_2F_1$ hypergeometric functions [12]. The natural emergence of *diagonals of rational functions*[2] in lattice statistical mechanics is explained in [19,20]. This can be seen as the reason of the frequent occurrence of *modular forms*, *Calabi-Yau operators* in lattice statistical mechanics [21–27]. In another previous paper [17,18], dedicated to Heun functions that are diagonals of simple rational functions, or only solutions of *telescopers* [28,29] of simple rational functions of three variables, but most of the time four variables, we have obtained many solutions of order-three telescopers having squares of Heun functions as solutions that turn out to be squares of pullbacked ${}_2F_1$ hypergeometric solutions corresponding to *classical modular forms* and even *Shimura automorphic forms* [30,31], strongly reminiscent of periods of *extremal rational surfaces* [32,33], and other foliation of K3 surfaces in elliptic curves. In other words one finds experimentally that the ${}_2F_1$ hypergeometric functions emerging in the calculation of diagonal of rational functions, or of solutions of the telescopers of rational functions, seem to be only special ${}_2F_1([a,b],[c],x)$ hypergeometric functions with a selected set of parameters $[a,b],[c]$ (see the list (B.1) in Appendix B of [17], corresponding to classical modular forms[3], together with a finite set of parameters, like $[7/24,11/24],[5/4]$, corresponding to Shimura automorphic forms [30,31]), pullbacked by selected pullbacks. This last paper [17] also underlined the difference between the diagonal of a rational function $Diag(R)$, and the solutions of the telescoper of the same rational function.

These results strongly suggested to find an *algebraic geometry interpretation* for all these exact results, and, more generally, suggested to provide an *alternative algebraic geometry approach of the results emerging from creative telescoping*[4].

This is the purpose of the present paper. In particular, we are going to show that most of these pullbacked ${}_2F_1$ hypergeometric functions can be obtained efficiently through algebraic geometry calculations, thus providing a more intrinsic *algebraic geometry interpretation* of the creative telescoping calculations which are typically *differential algebra calculations* [28,29,34,35].

Creative telescoping [28,29,34,36] is a methodology to deal with parametrized symbolic sums and integrals that yields differential/recurrence equations for such expressions. This methodology became popular in computer algebra in the past twenty five years. By "telescoper" of a rational function, say $R(x,y,z)$, we here refer to the output of the creative telescoping program [3], applied to the *transformed* rational function $\tilde{R} = R(x/y, y/z, z)/(yz)$. Such a telescoper is a linear differential operator T in x and $\frac{\partial}{\partial x}$, such that $T + \frac{\partial U}{\partial y} + \frac{\partial V}{\partial z}$ annihilates $\tilde{R}$, where the so-called "certificates" U, V are rational functions in x,y,z. In other words, the telescoper T represents a linear ODE that is satisfied by $Diag(R)$.

The paper is essentially dedicated to *solutions of telescopers* of rational functions which are *not necessarily diagonals* of rational functions. These solutions correspond to *periods* [37] of algebraic varieties over some cycles which are not necessarily vanishing [38] cycles[5] like in the case of diagonals of rational functions. The reader interested in the connection between the process of taking diagonals, calculating telescopers, and the notion of Periods, deRham cohomology (i.e. differential forms) and other Picard-Fuchs equations can read in detail the thesis of Pierre Lairez [35] (see also [41]). We just sketch some of these ideas in A.

The purpose of this paper is not to give an introduction on creative telescoping [28,29], but to provide many pedagogical (non-trivial) examples of telescopers using[6] extensively the "*HolonomicFunctions*" Mathematica package [3].

The paper is organised as follows. We first recall in section 2 the exact results of [1,2] for nine and ten parameters families of rational functions using creative telescoping,

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2 The lattice Green functions are the simplest examples of such diagonals of rational functions [13–18].

3 See Felix Klein's connection of the ${}_2F_1([1/12,5/12],[1],x)$ Gauss hypergeometric function with modular forms, for instance in the very pedagogical and heuristic paper [12].

4 The reader may refer to [34] for an extensive survey of "creative telescoping" approaches.

5 In french "cycles évanescents" [39,40].

6 One can obtain these telescopers using Chyzak's algorithm [42] or Koutschan's semi-algorithm [3,43] (the termination is not proven). Fo the examples displayed in this paper, Koutschan's package [3] is more efficient.

yielding modular forms expressed as pullbacked ${}_2F_1$ hypergeometric functions. We show in section 3 that these exact results can be obtained, much more efficiently, by calculating the j-invariant of an elliptic curve canonically associated with the denominator of the rational functions, and we underline the fact that one can drastically generalise these results, the *parameters becoming quite arbitrary rational functions*. Section 4 generalises the previous calculations to denominators of the rational functions of more than three variables, corresponding to products (or foliations) of elliptic curves. In section 5 we show how *modular forms* expressed as pullbacked ${}_2F_1$ hypergeometric functions occur for rational functions in three variables when the denominator is associated with a *genus-two curve such that its Jacobian is a split Jacobian* corresponding to the product of two elliptic curves. In section (6) we sketch the situation where the denominator of the rational function is associated with algebraic varieties of *low Kodaira dimension*, having an infinite set of birational automorphisms. We finally provide some examples of rational functions in more than three variables, where the telescopers have pullbacked ${}_2F_1$ hypergeometric solutions, the denominator corresponding to an algebraic variety having a selected elliptic curve in the variety explaining these pullbacked ${}_2F_1$ solutions.

2. Classical modular forms and diagonals of nine and ten parameters family of rational functions

In a previous paper [1,2], using creative telescoping [3], we have obtained diagonals of nine and ten parameters families of rational functions, given by (classical) modular forms expressed as pullbacked ${}_2F_1$ hypergeometric functions. Let us recall these results.

2.1. Nine-parameters rational functions giving pullbacked ${}_2F_1$ hypergeometric functions for their diagonals

Let us recall the *nine-parameters* rational function in three variables x, y and z:

$$\frac{1}{a \,\, + b_1\, x + b_2\, y + b_3\, z \,\, + c_1\, y\, z + c_2\, x\, z + c_3\, x\, y \,\, + d\, y^2\, z \,\, + e\, z\, x^2}. \tag{1}$$

Calculating[7] the telescoper of this rational function (1), one gets an *order-two* linear differential operator annihilating the diagonal of the rational function (1). The diagonal of the rational function (1) can be written [1,2] as a pullbacked hypergeometric function

$$\frac{1}{P_4(x)^{1/4}} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], 1 - \frac{P_6(x)^2}{P_4(x)^3}\Big), \tag{2}$$

where $P_4(x)$ and $P_6(x)$ are two polynomials of degree four and six in x, respectively. The Hauptmodul pullback in (2) has the form

$$\mathcal{H} \,=\, \frac{1728}{j} \,=\, 1 - \frac{P_6(x)^2}{P_4(x)^3} \,=\, \frac{1728 \cdot x^3 \cdot P_8(x)}{P_4(x)^3}, \tag{3}$$

where $P_8(x)$ is a polynomial of degree eight in x. Such a pullbacked ${}_2F_1$ hypergeometric function (2) corresponds to a *classical modular form* [1,2].

2.2. Ten-parameters rational functions giving pullbacked ${}_2F_1$ hypergeometric functions for their diagonals.

Let us recall the *ten-parameters* rational function in three variables x, y and z:

$$R(x, y, z) \,=\, \frac{1}{a \,\, + b_1\, x + b_2\, y + b_3\, z \,\, + c_1\, y\, z + c_2\, x\, z + c_3\, x\, y \,\, + d_1\, x^2\, y \, + d_2\, y^2\, z + d_3\, z^2\, x}. \tag{4}$$

7 Using the "*HolonomicFunctions*" Mathematica package [3].

Calculating the telescoper of this rational function (4), one gets an order-two linear differential operator annihilating the diagonal of the rational function (4). The diagonal of the rational function (4) can be written [1,2] as a pullbacked hypergeometric function

$$\frac{1}{P_3(x)^{1/4}} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], 1 - \frac{P_6(x)^2}{P_3(x)^3}\Big), \tag{5}$$

where $P_3(x)$ and $P_6(x)$ are two polynomials of degree three and six in x, respectively. Furthermore, the Hauptmodul pullback in (5) is seen to be of the form:

$$\mathcal{H} \, = \, \frac{1728}{j} \, = \, 1 \, - \, \frac{P_6(x)^2}{P_3(x)^3} \, = \, \frac{1728 \cdot x^3 \cdot P_9(x)}{P_3(x)^3}. \tag{6}$$

where $P_9(x)$ is a polynomial of degree nine in x. Again, (5) corresponds to a *classical modular form* [1,2].

3. Deducing creative telescoping results from effective algebraic geometry

Obtaining the previous pullbacked hypergeometric results (2) and (5) required [1,2] an accumulation of creative telescoping calculations, and a lot of "guessing" using all the symmetries of the diagonals of these rational functions (1) and (4). We are looking for a more efficient and intrinsic way of obtaining these exact results. These two pullbacked hypergeometric results (2) and (5), are essentially "encoded" by their *Hauptmodul* pullbacks (3) and (6), or, equivalently, their corresponding *j-invariants*. The interesting question, which will be addressed in this paper, is whether it is possible to canonically associate elliptic curves whose j-invariants correspond precisely to these Hauptmoduls $\mathcal{H} \, = \, \frac{1728}{j}$.

3.1. Revisiting the pullbacked hypergeometric results in an algebraic geometry perspective.

One expects such an elliptic curve to correspond to the singular part of the rational function, namely the *denominator* of the rational function. Let us recall that the diagonal of a rational function in (for example) three variables is obtained through its multi-Taylor expansion [19,20]

$$R(x, y, z) \, = \, \sum_m \sum_n \sum_l \, a_{m,n,l} \cdot x^m \, y^n \, z^l, \tag{7}$$

by extracting the "diagonal" terms, i.e. the powers of the product $p \, = \, xyz$:

$$Diag\Big(R(x, y, z)\Big) \, = \, \sum_m \, a_{m,m,m} \cdot x^m. \tag{8}$$

Consequently, it is natural to consider the algebraic curve corresponding to the intersection of the surface defined by the vanishing condition $D(x, y, z) \, = \, 0$ of the denominator $D(x, y, z)$ of these rational functions (1) and (4), with the hyperbola $p \, = \, x\,y\,z$ (where p is seen, here, as a constant). This amounts, for instance, to eliminating the variable z, substituting $z \, = \, \frac{p}{x\,y}$ in $D(x, y, z) \, = \, 0$.

3.1.1. Nine-parameters case:

In the case of the rational functions (1) this corresponds to the (planar) algebraic curve

$$a \, + b_1\, x \, + b_2\, y \, + b_3\, \frac{p}{x\,y} \, + c_1\, y\, \frac{p}{x\,y} \, + c_2\, x\, \frac{p}{x\,y} \, + c_3\, x\,y$$
$$+ d\, y^2\, \frac{p}{x\,y} \, + e\, \frac{p}{x\,y}\, x^2 \, = \, 0, \tag{9}$$

which can be rewritten as a (general, nine-parameters) *biquadratic*:

$$a\,x\,y \;+ b_1\,x^2\,y + b_2\,x\,y^2 + b_3\,p \;+ c_1\,p\,y \;+ c_2\,p\,x \;+ c_3\,x^2\,y^2 \\ +d\,p\,y^2 \;+e\,p\,x^2 \;=\; 0. \tag{10}$$

Using formal calculations[8] one can easily calculate the genus of the planar algebraic curve (10), and find that it *is actually an elliptic curve* (genus-one). Furthermore, one can (almost instantaneously) find the exact expression of the j-invariant of this elliptic curve as a rational function of the nine parameters $a,\, b_1,\, b_2,\, \cdots,\, e$ in (1). One actually finds that this j-invariant *is precisely the* j such that the Hauptmodul $\mathcal{H} \,=\, \frac{1728}{j}$ *is the exact expression* (3). In other words, the classical modular form result (2) could have been obtained, almost instantaneously, by calculating the j-invariant of an elliptic curve canonically associated with the denominator of the rational function (1). The algebraic planar curve (10) *corresponds to the most general biquadratic of two variables*, which depends on nine homogeneous parameters. Such general biquadratic is well-known to be an elliptic curve for *generic values* of the *nine parameters*[9].

Thus, the nine-parameters exact result (2) *can be seen as a simple consequence of the fact that the most general nine-parameters biquadratic is an elliptic curve*.

3.1.2. Ten-parameters case:

In the case of the rational function (4), substituting $z \,=\, \frac{p}{x\,y}$ in $D(x,\, y,\, z) \,=\, 0$, one obtains the *ten-parameters bicubic*:

$$a\,x\,y^2 \;+ b_1\,x^2\,y^2 + b_2\,x\,y^3 + b_3\,p\,y \;+ c_1\,p\,y^2 + c_2\;p x\,y + c_3\,x^2\,y^3 \\ +\, d_1\,x^3\,y^3 \,+\, d_2\,y^3 \,+\, d_3\,p^2 \;=\; 0. \tag{11}$$

As before, we find that this planar algebraic curve is actually an elliptic curve[10] and that the exact expression of its j-invariant is precisely the j of the Hauptmodul $H \,=\, 1728/j$ in (6).

Thus, this ten-parameters result (5) can again be seen as a simple consequence of the fact that *there exists a family of ten-parameters bicubics* (see (11)) *which are elliptic curves for generic values of the ten parameters*.

These preliminary calculations are a strong incentive to try to replace the differential algebra calculations of the *creative telescoping*, by more intrinsic algebraic geometry calculations, or, at least, perform effective algebraic geometry calculations to provide an algebraic geometry interpretation of the exact results obtained from creative telescoping.

3.2. Finding creative telescoping results from j-invariant calculations.

One might think that these results are a consequence of the simplicity of the denominators of the rational functions (1) or (4), being associated with biquadratics or selected bicubics. In fact, these results are very general. Let us, for instance, consider a nine-parameters family of planar algebraic curves that are not biquadratics or (selected) bicubics:

$$a_1\,x^4 \;+ a_2\,x^3 \;+ a_3\,x^2 \;+ a_4\,x \;+ a_5 + a_6\,x^2\,y + a_7\,y^2 \;+ a_8\,y \;+ a_9\,x\,y \;=\; 0. \tag{12}$$

One can easily calculate the genus of this planar curve and see that this genus is actually one for *arbitrary values* of the a_n's. Thus the planar curve (12) *is an elliptic curve for generic*

8 Namely using with(algcurves) in Maple, and, in particular, the command j_invariant.

9 So many results in integrable models correspond to this most general biquadratic: the Bethe ansatz of the Baxter model [44,45], the elliptic curve foliating the sixteen-vertex model [45], so many QRT birational maps [46], ...

10 Generically, the most general planar bicubic is *not* a genus-one algebraic curve. It is a genus-four curve.

values of the nine parameters $a_1, \cdots, a_9$. It is straightforward to see that the algebraic surface $S(x, y, z) = 0$, corresponding to

$$z \cdot (a_1 x^4 + a_2 x^3 + a_3 x^2 + a_4 x + a_5 + a_6 x^2 y + a_7 y^2 + a_8 y) + a_9 p = 0, \tag{13}$$

will automatically be such that its intersection with the hyperbola $p = x y z$ gives back the elliptic curve (12).

Using this kind of "reverse engineering" yields to consider the rational function in three variables x, y and z

$$R(x, y, z) = \frac{1}{1 + z \cdot (a_1 x^4 + a_2 x^3 + a_3 x^2 + a_4 x + a_5 + a_6 x^2 y + a_7 y^2 + a_8 y)}, \tag{14}$$

which will be such that *its denominator is canonically associated with an elliptic curve*. Again we can immediately calculate the j-invariant of that elliptic curve. If one calculates the telescoper of this eight-parameters family of rational functions (14), one finds that this telescoper is an order-two linear differential operator with pullbacked hypergeometric solutions of the form

$$\mathcal{A}(x) \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], \mathcal{H}\Big), \tag{15}$$

where $\mathcal{A}(x)$ is an algebraic function and, where again, the pullback-Hauptmodul $\mathcal{H} = 1728/j$, *precisely corresponds to the* j*-invariant of the elliptic curve*.

More generally, seeking for planar elliptic curves, one can, for given values of two integers M and N, look for planar algebraic curves

$$\sum_{n=0}^{n=N} \sum_{m=0}^{m=M} a_{m,n} \cdot x^n y^m = 0, \tag{16}$$

defined by the set of $a_{m,n}$'s which are equal to zero, apart of $\mathcal{N}$ homogeneous parameters $a_{m,n}$ being, as in (10) or (11) or (13), *independent parameters*. Finding such an $\mathcal{N}$-parameters family of (planar) elliptic curves automatically provides an $\mathcal{N}$-parameters family of rational functions such that their telescopers have a pullbacked ${}_2F_1$ hypergeometric solution we can simply deduce from the j-invariant of that elliptic curve.

Recalling the results of section 2.2, the quite natural question to ask now is whether it is possible to find families of such (planar) elliptic curves *which depend on more than ten independent parameters*?

Before addressing this question, let us recall the concept of *birationally equivalent elliptic curves*. Let us consider the monomial transformation:

$$(x, y) \quad \longrightarrow \quad (x^M y^N, \; x^P y^Q), \tag{17}$$

where M, N, P, Q are integers such that $M \cdot Q - P \cdot N = 1$, then its compositional inverse is the monomial transformation:

$$(x, y) \quad \longrightarrow \quad \Big(\frac{x^Q}{y^N}, \frac{y^M}{x^P}\Big). \tag{18}$$

This monomial transformation (17) is thus a *birational*[11] transformation. A birational transformation transforms an elliptic curve, like (12), into another elliptic curve *with the*

[11] This transformation is rational and its compositional inverse is also rational (here monomial).

same j-invariant: these two elliptic curves are called *birationally equivalent*. In the case of the birational and monomial transformation (17), the elliptic curve (12) is changed into[12]:

$$a_1 \cdot x^{4M} y^{4N} + a_2 \cdot x^{3M} y^{3N} + a_3 \cdot x^{2M} y^{2N} + a_4 \cdot x^M y^N + a_5 \\ + a_6 \cdot x^{2M+P} y^{2N+Q} + a_7 \cdot x^{2P} y^{2Q} + a_8 \cdot x^P y^Q + a_9 \cdot x^{M+P} y^{N+Q} = 0. \tag{19}$$

With this kind of birational monomial transformation (17), we see that one can obtain *families of elliptic curves* (19) *of arbitrary large degrees* in x and y. Consequently one can find nine or ten parameters families of rational functions of *arbitrary large degrees* yielding pullbacked ${}_2F_1$ hypergeometric functions. There is no constraint on the degree of the planar algebraic curves (19): *the only relevant question is the question of the maximum number of (linearly) independent parameters of families of planar elliptic curves* which is shown *to be ten*. The demonstration[13] is sketched in B.

3.3. *Pullbacked ${}_2F_1$ functions for higher genus curves: monomial transformations.*

Let us recall another important point. We have already remarked in [1,2] that once we have an exact result for a diagonal of a rational function of three variables $R(x, y, z)$, we immediately get another exact result for the diagonal of the rational function $R(x^n, y^n, z^n)$ for any positive integer n. As a result we obtain a new expression for the diagonal changing x into x^n. In fact, this is also a result on the telescoper of the rational function $R(x, y, z)$: the telescoper of the rational function $R(x^n, y^n, z^n)$ is the $x \to x^n$ pullback of the telescoper of the rational function $R(x, y, z)$. Having a pullbacked ${}_2F_1$ solution for the telescoper of the rational function $R(x, y, z)$ (resp. the diagonal of the rational function $R(x, y, z)$), we will immediately deduce a pullbacked ${}_2F_1$ solution for the telescoper of the rational function $R(x^n, y^n, z^n)$ (resp. the diagonal of the rational function $R(x^n, y^n, z^n)$).

Along this line, let us change in the rational function (1), (x, y, z) into (x^2, y^2, z^2):

$$R_2(x, y, z) = \\ \frac{1}{a + b_1 x^2 + b_2 y^2 + b_3 z^2 + c_1 y^2 z^2 + c_2 x^2 z^2 + c_3 x^2 y^2 + d y^4 z^2 + e z^2 x^4}. \tag{20}$$

The diagonal of this new rational function (20) will be the pullbacked ${}_2F_1$ exact expression (2) where we change $x \to x^2$. The intersection of the algebraic surface corresponding to the vanishing condition of the denominator of the new rational function (20), with the hyperbola $p = xyz$ (i.e. $z = \frac{p}{xy}$), is nothing but the equation (10) where we have changed $(x, y; p)$ into $(x^2, y^2; p^2)$

$$a\, x^2 y^2 + b_1 x^4 y^2 + b_2 x^2 y^4 + b_3 p^2 + c_1 p^2 y^2 + c_2 p^2 x^2 + c_3 x^4 y^4 \\ + d\, p^2 y^4 + e\, p^2 x^4 = 0, \tag{21}$$

which is *no longer*[14] *an elliptic curve but a curve of genus* 9.

With that example we see that classical modular form results, or pullbacked ${}_2F_1$ exact expressions like (2), can actually emerge from *higher genus curves* like (21). As far as these diagonals, or telescopers, of rational function calculations are concerned, higher genus curves like (21) must in fact be seen as "almost" elliptic curves up to an $x \to x^n$ covering.

Such results for monomial transformations like $(x, y, z) \to (x^n, y^n, z^n)$ can, in fact, be generalised to more general (non birational[15]) monomial transformations. This is sketched in C.

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[12] One can easily verify for particular values of the M, N, P, Q and a_k's, using with(algcurves) in Maple, that the j-invariants of (12) and (19) are actually equal.

[13] We thank Josef Schicho for providing this demonstration.

[14] If we perform the same calculations with the ten-parameters rational function (4) we get an algebraic curve of genus 10 instead of 9.

[15] In contrast with transformations like (17).

3.4. *Changing the parameters into functions of the product* $p = xyz$.

All these results for many parameters families of rational functions can be *drastically generalised* when one remarks that allowing any of these parameters to be a *rational function* of the product $p = xyz$ also yields to the previous pullbacked ${}_2F_1$ exact expression, like (2), *where the parameter is changed into that rational function of* x (see [1]). Let us consider a simple (two-parameters) illustration of this general result. Let us consider a subcase of the previous nine or ten parameters families, introducing, for example, the two parameters rational function:

$$\frac{1}{1 \ + 2x + b_2 \cdot y \ + 5yz \ + xz \ + c_3 \cdot xy}. \tag{22}$$

The diagonal of this rational function (22) is the pullbacked hypergeometric function:

$$\frac{1}{P_2(x)^{1/4}} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], \ 43200 \cdot x^4 \cdot \frac{P_4(x)}{P_2(x)^3}\Big), \tag{23}$$

where

$$P_2(x) \ = \ 1 \ \ - 8 \cdot (b_2 + 10) \cdot x \ \ + 8 \cdot (2b_2^2 - 20b_2 + 15c_3 + 200) \cdot x^2, \tag{24}$$

and

$$\begin{aligned} P_4(x) \ = \ & -675 c_3^4 \cdot x^4 \ \ + 4c_3^2 \cdot (b_2 + 10) \cdot (4b_2^2 - 100b_2 + 45c_3 + 400) \cdot x^3 \\ & +(64b_2^4 - 32b_2^3 c_3 - 8b_2^2 c_3^2 - 1280 b_2^3 + 1280 b_2^2 c_3 \\ & \quad -460 b_2 c_3^2 - 5c_3^3 + 6400 b_2^2 - 3200 b_2 c_3 - 800 c_3^2) \cdot x^2 \\ & \quad -(b_2 + 10) \cdot (32b_2^2 - 16b_2 c_3 - c_3^2) \cdot x \ \ + 2b_2 \cdot (2b_2 - c_3), \end{aligned} \tag{25}$$

Let us now consider the previous rational function (22) where the two parameters b_2 and c_3 *become some rational functions of the product* $p = xyz$, for instance:

$$b_2(p) \ = \ \frac{1 + 3p}{1 + 7p^2}, \qquad c_3(p) \ = \ \frac{1 + p^2}{1 + 2p} \qquad \text{where:} \qquad p \ = \ xyz. \tag{26}$$

The new corresponding rational function becomes more involved but one can easily calculate the telescoper of this new rational function of three variables x, y and z, and find that it is, *again*, an order-two linear differential operator having the pullbacked hypergeometric solution (23) *where* b_2 *and* c_3 *are, now, replaced by* (p is now x) the functions:

$$b_2(x) \ = \ \frac{1 + 3x}{1 + 7x^2}, \qquad c_3(x) \ = \ \frac{1 + x^2}{1 + 2x}. \tag{27}$$

In that case (22) with (26), one gets a diagonal which is the pullbacked hypergeometric solution

$$\begin{aligned} & (1 + 2x)^{1/4} \cdot (1 + 7x^2)^{1/4} \cdot q_8^{-1/4} \\ & \quad \times {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], \ \frac{43200 \cdot x^4 \cdot (1 + 7x^2)^2 \cdot q_{20}}{(1 + 2x) \cdot q_8^3}\Big), \end{aligned} \tag{28}$$

where q_8 and q_{20} are two polynomials with integer coefficients of degree eight and twenty in x. The exact expression (28) is nothing but (23) (with (24) and (25)) where b_2 and c_3 have been replaced by the rational functions (27). Similar calculations can be performed for more general rational functions (1) or (4), *when all the (nine or ten) parameters are more involved rational functions.*

When performing our creative telescoping symbolic calculations using the HolonomicFunctions package [3], such results may look quite impressive. From the algebraic geometry viewpoint, it is almost tautological[16], if one takes for granted the result of our previous subsections 3.1 and 3.2, namely that the pullbacked hypergeometric solution of the telescoper corresponds to the Hauptmodul $1728/j$, where j is the j-invariant of the elliptic curve corresponding to the intersection of the algebraic surface corresponding to the vanishing condition of the denominator, with the hyperbola $p = x\,y\,z$: this calculation of the j-invariant is performed for p fixed, and arbitrary (nine or ten) parameters $a, b_1, \cdots$. It is clearly possible to force the parameters to be functions[17] of p, the j-invariant being changed accordingly. Of course, in that case, the parameters in the rational function are the same functions but of the product $p = x\,y\,z$.

One thus gets pullbacked hypergeometric solutions (classical modular forms) for an (unreasonably ...) large set of rational functions in three variables, namely the families of rational functions (1) or (4), *but where, now, the nine or ten parameters are nine, or ten, totally arbitrary rational functions* (with Taylor series expansions) *of the product* $p = x\,y\,z$.

We see experimentally that changing the parameters of the rational function into functions, actually works for *diagonals* of rational functions, as well as for solutions of telescopers of rational functions depending on parameters.

4. Creative telescoping on rational functions of more than three variables associated with products or foliations of elliptic curves

Let us show that such an algebraic geometry approach to creative telescoping can be generalised to rational functions of *more than three* variables, when the vanishing condition of the denominator can be associated with *products of elliptic curves*, or more generally, algebraic varieties with *foliations in elliptic curves*.

- The telescoper of the rational function in the *four variables* x, y, z and w

$$\frac{x\,y\,z}{(1+z)^2 \quad - x \cdot (1-x) \cdot (x - x\,y\,z\,w) \cdot y \cdot (1-y) \cdot (y - x\,y\,z\,w)}, \tag{29}$$

gives an order-three *self-adjoint* linear differential operator which is, thus, the *symmetric square* of an order-two linear differential operator. The latter has the pullbacked hypergeometric solution:

$$\begin{aligned} \mathcal{S}_1(x) \;\; &= \;\; (1 - x + x^2)^{-1/4} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], \frac{27}{4} \cdot \frac{x^2 \cdot (1-x)^2}{(x^2 - x + 1)^3}\Big) \\ &= \;\; {}_2F_1\Big([\frac{1}{2}, \frac{1}{2}], [1], \; x\Big). \end{aligned} \tag{30}$$

In [18] we underlined the difference between the *diagonal* of a rational function and *solutions of the telescoper* of the same rational function. In this case, the diagonal of the rational function (29) is zero[18] and is thus different from the pullbacked hypergeometric solution (30), which is a *"Period"* [37] of the algebraic variety corresponding to the denominator over some (*non-vanishing*[19]) *cycle*. From now, we will have a similar situation in most of the following examples of this paper.

[16] An algebraic geometer will probably see this as a trivial remark: diagonalization is an algebraic procedure and nothing really happens to the coefficients. Therefore if one replaces the coefficients by anything else, one will find those replaced coefficients in the end result.

[17] The functions should be rational functions if one wants to stick with diagonals and telescopers of *rational* functions, but the result remains valid for *algebraic functions*, or *even transcendental functions* with reasonable Taylor series expansions at $x = 0$: for instance, for ${}_2F_1$ hypergeometric functions, one gets a *differentially algebraic function* corresponding to the composition of ${}_2F_1$ hypergeometric functions.

[18] The reason is that the integration takes place over a cycle homologically equivalent to the trivial cycle. The cycle becomes trivial after taking the limit $p \to 0$. Integrals over non vanishing cycles usually give logarithms of p, like the second solution to the hypergeometric function ${}_2F_1([1/2, 1/2], [1], , x)$.

[19] Diagonals of the rational functions correspond to periods over *vanishing cycles* [38,40].

This example is a simple illustration of what we expect for *products of elliptic curves*, or algebraic varieties with *foliations in elliptic curves*. Introducing the product $p = xyzw$, the vanishing condition of the denominator of the rational function (29) reads the surface $S(x, y, z) = 0$:

$$(1+z)^2 \quad - x \cdot (1-x) \cdot (x-p) \cdot y \cdot (1-y) \cdot (y-p) \quad = \quad 0. \tag{31}$$

For fixed p and fixed y, equation (31) can be seen as an algebraic curve

$$(1+z)^2 \quad - \lambda \cdot x \cdot (1-x) \cdot (x-p) \quad = \quad 0 \qquad \text{with:} \qquad \lambda \, = \, y \cdot (1-y) \cdot (y-p). \tag{32}$$

For fixed p and fixed y, λ can be considered as a constant, the algebraic curve (32) being an *elliptic curve* with an obvious Weierstrass form:

$$Z^2 \quad - \quad x \cdot (1-x) \cdot (x-p) \quad = \quad 0 \qquad \text{where:} \qquad Z \, = \, \frac{1+z}{\sqrt{\lambda}}. \tag{33}$$

The j-invariant of (32), or[20] (33), is well-known and yields the Hauptmodul $\mathcal{H}$:

$$\mathcal{H} \, = \, \frac{1728}{j} \, = \, \frac{27}{4} \cdot \frac{p^2 \cdot (1-p)^2}{(p^2-p+1)^3} \tag{34}$$

For fixed p and fixed x, equation (31) can be seen as an algebraic curve

$$(1+z)^2 \quad - \mu \cdot y \cdot (1-y) \cdot (y-p) \quad = \quad 0 \qquad \text{for:} \qquad \mu \, = \, x \cdot (1-x) \cdot (x-p), \tag{35}$$

which is also an elliptic curve with an obvious Weierstrass form and the *same* Hauptmodul (34). This Hauptmodul is precisely the one occurring in the pullbacked hypergeometric solution (30).

More generally, the rational function of the *four variables* x, y, z and w

$$\frac{x\,y\,z}{(1+z)^2 \quad - x \cdot (1-x) \cdot (x - R_1(p)) \cdot y \cdot (1-y) \cdot (y - R_2(p))}, \tag{36}$$

where $p = x\,y\,z\,w$, and where $R_1(p)$ and $R_2(p)$ are two arbitrary rational functions (with Taylor series expansions) of the product $p = x\,y\,z\,w$, yields a telescoper which has an *order-four* linear differential operator which is the *symmetric product*[21] of two order-two linear differential operators having respectively the pullbacked hypergeometric solutions (30) where x is replaced by $R_1(x)$ and $R_2(x)$. These two hypergeometric solutions thus have the two Hauptmodul pullbacks

$$\mathcal{H}_1 \, = \, \frac{1728}{j_1} \, = \, \frac{27}{4} \cdot \frac{R_1(p)^2 \cdot (1-R_1(p))^2}{(R_1(p)^2 - R_1(p) + 1)^3}, \tag{37}$$

$$\mathcal{H}_2 \, = \, \frac{1728}{j_2} \, = \, \frac{27}{4} \cdot \frac{R_2(p)^2 \cdot (1-R_2(p))^2}{(R_2(p)^2 - R_2(p) + 1)^3}, \tag{38}$$

obtained by calculations similar to the ones previously performed on (31) but, now, for the Weierstrass form corresponding to the denominator (36).

[20] A shift $z \to z+1$ or a rescaling $z^2 \to \frac{z^2}{\lambda}$ does not change the j-invariant of the Weierstrass elliptic form.

[21] This paper belonging to the symbolic computation literature and not pure mathematics for algebraic geometers, we use the standard Maple (DEtools) terminology of symmetric powers and symmetric products of linear differential operators [47]. Note that "symmetric product" is not a proper mathematical name for this construction on the solution space; it is a homomorphic image of the tensor product. The (Maple/DEtools) reason for choosing the name symmetric_product is the resemblance with the function symmetric_power.

A solution of the telescoper of (36) is thus the *product* of these two pullbacked hypergeometric functions. Let us give a simple illustration of this general result, with the next example.

• The telescoper of the rational function in the *four variables* x, y, z and w

$$\frac{x\,y\,z}{(1+z)^2 \quad - x \cdot (1-x) \cdot (x - x\,y\,z\,w) \cdot y \cdot (1-y) \cdot (y - 3\,x\,y\,z\,w)}, \tag{39}$$

corresponding to (36) with $R_1(p) = p$ and $R_1(p) = 3\,p$, gives an *order-four* linear differential operator which is the *symmetric product* of two order-two operators having respectively the pullbacked hypergeometric solution (30) and the solution (30) where the variable x has been changed into $3\,x$:

$$\begin{aligned} \mathcal{S}_2(x) \; &= \; \mathcal{S}_1(3\,x) \\ &= \; (1 - 3\,x + 9\,x^2)^{-1/4} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], \frac{243}{4} \cdot \frac{x^2 \cdot (1-3\,x)^2}{(1-3\,x+9\,x^2)^3}\Big). \end{aligned} \tag{40}$$

4.1. *Creative telescoping on rational functions of five variables associated with products or foliations of three elliptic curves*

Let us, now, introduce the rational function in *five* variables x, y, z, v and w

$$\frac{x\,y\,z\,v}{D(x, y, z, v, w)}, \tag{41}$$

where the denominator $D(x, y, z, v, w)$ reads:

$$\begin{aligned} D_p \; &= \\ &(1+v)^2 \quad - x \cdot (1-x) \cdot (x-p) \cdot y \cdot (1-y) \cdot (y - 3\,p) \cdot z \cdot (1-z) \cdot (z - 5\,p), \\ &\qquad \text{where:} \qquad\qquad p \; = \; x\,y\,z\,v\,w. \end{aligned} \tag{42}$$

The telescoper of the rational function (41) of *five* variables gives[22] an *order-eight* linear differential operator which is the *symmetric product* of *three order-two* linear differential operators having respectively the pullbacked hypergeometric solution (30), the solution (30) where x has been changed into $3\,x$, namely (40), and the solution (30), where x has been changed into $5\,x$:

$$\begin{aligned} \mathcal{S}_3(x) \; &= \; \mathcal{S}_1(5\,x) \\ &= \; (1 - 5\,x + 25\,x^2)^{-1/4} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], \frac{675}{4} \cdot \frac{x^2 \cdot (1-5\,x)^2}{(1-5\,x+25\,x^2)^3}\Big). \end{aligned} \tag{43}$$

In other words, the order-eight telescoper of the rational function (41) has the *product* $\mathcal{S} = \mathcal{S}_1 \cdot \mathcal{S}_2 \cdot \mathcal{S}_3$, of (30), (40) and (43) as a solution. From an algebraic geometry viewpoint, this is a consequence of the fact that, for fixed p, the algebraic variety $D_p = 0$, where D_p is given by (42), can be seen, for fixed y and z, *as an elliptic curve* $\mathcal{E}_1$ of equation $D_{y,z,p}(v, x) = 0$, for fixed x and z as an elliptic curve $\mathcal{E}_2$ of equation $D_{x,z,p}(v, y) = 0$, and for fixed x and y also as an elliptic curve $\mathcal{E}_3$ of equation $D_{x,y,p}(v, z) = 0$, the j-invariants j_k, $k = 1, 2, 3$ of these three elliptic curves $\mathcal{E}_k$ yielding (in terms of p), precisely, the three Hauptmoduls $\mathcal{H}_k = \frac{1728}{j_k}$

$$\frac{27}{4} \cdot \frac{x^2 \cdot (1-x)^2}{(x^2 - x + 1)^3}, \qquad \frac{243}{4} \cdot \frac{x^2 \cdot (1-3\,x)^2}{(1-3\,x+9\,x^2)^3}, \qquad \frac{675}{4} \cdot \frac{x^2 \cdot (1-5\,x)^2}{(1-5\,x+25\,x^2)^3}, \tag{44}$$

[22] Such a creative telescoping calculation requires "some" computing time to achieve the result.

occurring as pullbacks in the three $\mathcal{S}_k$'s of the solution $\mathcal{S} = \mathcal{S}_1 \cdot \mathcal{S}_2 \cdot \mathcal{S}_3$, of the telescoper of (41).

Remark: Other examples of rational functions of three, four, five variables where the denominators also correspond to Weierstrass (resp. Legendre) forms, are displayed in D. They provide simple illustrations of rational functions where the denominator is associated with *K3 surfaces*[23], or *Calabi-Yau three-folds*. In these cases the algebraic varieties have simple foliations in terms of two or three families of elliptic curves, and the solutions of the corresponding telescopers can be selected ${}_3F_2$ and ${}_4F_3$ hypergeometric functions (see (A28) in D), naturally associated with K3 surfaces and Calabi-Yau operators [27].

5. Creative telescoping of rational functions in three variables associated with genus-two curves with split Jacobians

In a paper [17,18], dedicated to Heun functions that are solutions of telescopers of simple rational functions of three and four variables, we have obtained[24] an order-four telescoper of a rational function of *three* variables, which is the *direct sum of two order-two linear differential operators*, each having *classical modular forms* solutions which can be written as pullbacked ${}_2F_1$ hypergeometric solutions. Unfortunately, the intersection of the algebraic surface corresponding to the denominator of the rational function with the $p = xyz$ hyperbola, yields a *genus-two* algebraic curve.

Let us try to understand, in this section, *how a genus-two curve can yield two classical modular forms*. Let us first recall the results in section 2.2 of [18].

5.1. Periods of extremal rational surfaces

Let us recall the rational function in just *three* variables [18]:

$$R(x, y, z) \;=\; \frac{1}{1 \;+x\;+y\;+z\;\;+xy\;+yz\;\;-x^3yz}. \tag{45}$$

Its telescoper is actually an *order-four* linear differential operator L_4 which, not only factorizes into *two order-two* linear differential operators, but is actually the *direct sum* (LCLM) of *two*[25] *order-two* linear differential operators $L_4 = L_2 \oplus M_2$. These two (non homomorphic) order-two linear differential operators have, respectively, the two pullbacked hypergeometric solutions:

$$\begin{aligned}\mathcal{S}_1 \;=\;& (1+9x)^{-1/4}\cdot(1+3x)^{-1/4}\cdot(1+27x^2)^{-1/4} \\ &\times {}_2F_1\Big([\frac{1}{12},\frac{5}{12}],[1],\;\frac{1728\cdot x^3\cdot(1+9x+27x^2)^3}{(1+3x)^3\cdot(1+9x)^3\cdot(1+27x^2)^3}\Big),\end{aligned} \tag{46}$$

and:

$$\begin{aligned}\mathcal{S}_2 \;=\;& \frac{1}{(1+4x-2x^2-36x^3+81x^4)^{1/4}} \\ &\times {}_2F_1\Big([\frac{1}{12},\frac{5}{12}],[1],\;\frac{1728\cdot x^5\cdot(1+9x+27x^2)\cdot(1-2x)^2}{(1+4x-2x^2-36x^3+81x^4)^3}\Big).\end{aligned} \tag{47}$$

The diagonal of (45) *is actually the half-sum of the two series* (46) and (47):

$$Diag\Big(R(x, y, z)\Big) \;=\; \frac{\mathcal{S}_1+\mathcal{S}_2}{2}. \tag{48}$$

[23] See the emergence of *product of elliptic curves* from *Shioda-Inose structure* on surfaces with *Picard number 19* in [48]. In [48], Ling Long considers one-parameter families of K3 surfaces with generic Picard number 19. The existence of a Shioda-Inose structure implies that *there is a one-parameter family of elliptic curves*.

[24] See equation (83) in section 2.2 of [18].

[25] These two order-two linear differential operators L_2 and M_2 are *not* homomorphic.

As far as our algebraic geometry approach is concerned, the intersection of the algebraic surface corresponding to the denominator of the rational function (45) with the hyperbola $p = xyz$ gives the planar algebraic curve (corresponding to the elimination of the z variable by the substitution $z = \frac{p}{xy}$):

$$1 + x + y + \frac{p}{xy} + xy + y\frac{p}{xy} - x^3 y \frac{p}{xy} = 0. \tag{49}$$

One easily finds that this algebraic curve is (for p fixed) a *genus-two* curve, and that this higher genus situation *does not* correspond to the "almost elliptic curves" described in subsection 3.2 namely an elliptic curve transformed by a monomial transformation. How can a "true" *genus-two* curve give two j-invariants, namely a telescoper with two Hauptmodul pullbacked ${}_2F_1$ solutions? We are going to see that the answer is that the Jacobian of this *genus-two* curve[26] is in fact isogenous to a product $\mathcal{E} \times \mathcal{E}'$ of *two elliptic curves* (split Jacobian).

5.2. Split Jacobians

Let us first recall the concept of *split Jacobian* with a simple example. In [49], one has a crystal-clear example of a genus-two curve C

$$y^2 - (x^3 + 420x - 5600) \cdot (x^3 + 42x^2 + 1120) = 0, \tag{50}$$

such that its *Jacobian* $J(C)$ *is isogenous to a product of elliptic curves* with j-invariants $j_1 = -2^7 \cdot 7^2 = -6272$ and $j_2 = -2^5 \cdot 7 \cdot 17^3 = -1100512$, corresponding to the following two values of the Hauptmodul $H = \frac{1728}{j}$: $H_1 = -27/98$ and $H_2 = -54/34391$. Let us consider the *genus-one* elliptic curve

$$v^2 = u^3 + 4900u^2 + 7031500u + 2401000000, \tag{51}$$

of j-invariant $j = j_2 = -2^5 \cdot 7 \cdot 17^3$. We consider the following transformation[27]:

$$u = -\frac{882000 \cdot (x-14)}{x^3 + 420x - 5600}, \qquad v = \frac{49000 \cdot (x^3 - 21x^2 - 140)}{(x^3 + 420x - 5600)^2} \cdot y. \tag{52}$$

This change of variable (52) actually transforms the *elliptic curve* (51) in u and v into the *genus-two* curve (50) in x and y. *This provides a simple example of a genus-two curve with split Jacobian through K3 surfaces*.

More generally, let us consider the Jacobian of a *genus-two* curve C. The Jacobian is simple if it does not contain a proper abelian subvariety, otherwise the Jacobian is reducible, or decomposable or "split". For this latter case, the only possibility for a *genus-two curve* is that its Jacobian is *isogenous to a product* $\mathcal{E} \times \mathcal{E}'$ of *two elliptic curves*[28]. Equivalently, there is a degree n map $C \to \mathcal{E}$ to some elliptic curves. Classically such pairs[29] $C, \mathcal{E}$ arose in the *reduction of hyperelliptic integrals to elliptic ones* [49]. The j-invariants correspond, here, to the *two elliptic subfields*: see [49].

26 An algebraic geometer will probably recall that it is very well-known that a genus two curve *may* have Jacobian isogeneous to a product of elliptic curves. This is not the case in general. The genus two curves that have a (nonconstant) map to an elliptic curve have this property. Our purpose in section (5.3) is to perform a creative telescoping calculation in such a selected situation.

27 This transformation is rational but *not birational*. If it were birational, then it would preserve the genus. Here, one goes from genus one to genus two.

28 Along these lines, see also the concepts of Igusa-Clebsch invariants and Hilbert modular surfaces [49–52].

29 One also has an anti-isometry Galois invariant $\mathcal{E}' \simeq \mathcal{E}$ under Weil pairing. The decomposition corresponds to real multiplication by quadratic ring of discriminant n^2.

5.3. Creative telescoping on rational functions in three variables associated with genus-two curves with split Jacobians: a two-parameters example.

Let us now consider the example with *two parameters*, a and b, given in section 4.5 page 12 of [49]. Let us substitute the rational parametrisation[30]

$$u \;=\; \frac{x^2}{x^3 \, + a\, x^2 \, + b\, x \, +1}, \qquad v \;=\; \frac{y \cdot \, (x^3 \, - b\, x \, -2)}{(x^3 + a\, x^2 \, + b\, x \, +1)^2}, \tag{53}$$

in the *elliptic curve*

$$R \cdot \, v^2 \;=\; R \cdot \, u^3 \;\; +2 \cdot \, (ab^2 \, -6\, a^2 \, +9\, b) \cdot \, u^2 \; + (12\, a - b^2) \cdot \, u \;\; -4, \tag{54}$$

where

$$R \;=\; 4 \cdot \, (a^3 \, + b^3) \;\; - a^2 b^2 \; -18\, ab \;\; +27. \tag{55}$$

This gives the *genus-two curve* $C_{a,b}(x, y) \;=\; 0$ with:

$$C_{a,b}(x, y) \;=\; R \cdot \, y^2 \;\; + (4\, x^3 \, + b^2\, x^2 \, + 2\, b\, x \, +1) \cdot \, (x^3 \, + a\, x^2 \, + b\, x \, +1). \tag{56}$$

The j-invariant of the elliptic curve (54) gives the following exact expression for the Hauptmodul $\mathcal{H} \;=\; \frac{1728}{j}$:

$$\mathcal{H} \;=\; \frac{108 \cdot \, (b-3)^3 \cdot \, (4\, a^3 + 4\, b^3 - a^2 b^2 - 18\, ab + 27)^2 \cdot \, (b^2 + 3\, b + 9)^3}{(a^2 b^4 + 12\, b^5 - 126\, ab^3 + 216\, ba^2 + 405\, b^2 - 972\, a)^3}. \tag{57}$$

Let us consider the telescoper of the rational function of three variables $x\, y / D_a(x, y, z)$ where the denominator $D_a(x, y, z)$ is $C_{a,b}(x, y)$ given by (56), but for $b \;=\; 3 + \; x y z$:

$$\begin{aligned} D_a(x, y, z) \;&=\; C_{a, 3+xyz}(x, y) \\ &=\; x^6 y^3 z^3 \; + x^7 y^2 z^2 \; + 4\, x^3 y^5 z^3 + 9\, x^5 y^2 z^2 + 6\, x^6 yz + 3\, x^4 y^2 z^2 + 36\, y^4 x^2 z^2 \\ &\quad +6\, x^5 yz + 4\, x^6 + 27\, x^4 yz + 9\, x^5 + 18\, x^3 yz + 108\, xy^3 z + 18\, x^4 + 3\, x^2 yz \\ &\quad +32\, x^3 + 27\, x^2 + 135\, y^2 \; +9\, x \; +1 \\ &\quad + (x^6 y^2 z^2 + 6\, x^5 yz + 2\, x^4 yz + 4\, x^5 - 18\, xy^3 z + 9\, x^4 + 6\, x^3 + x^2 - 54\, y^2) \cdot \, a \\ &\quad -y^2 \cdot \, (xyz \, +3)^2 \cdot \, a^2 \;\; +4\, y^2 \cdot \, a^3. \end{aligned} \tag{58}$$

This telescoper of the rational function

$$R_a(x, y, z) \;=\; \frac{x\, y}{D_a(x, y, z)}, \tag{59}$$

is an *order-four* linear differential operator L_4 which *is actually the direct-sum*, $L_4 \;=\; LCLM(L_2, M_2) \;=\; L_2 \oplus M_2$, of *two order-two* linear differential operators, having two pullbacked hypergeometric solutions. One finds out that one of the two pullbacks *precisely corresponds to the Hauptmodul* $\mathcal{H}$ given by (57) for $b \;=\; 3 + x$.

Let us consider the $a \;=\; 3$ subcase[31]. For $a \;=\; 3$, the Hauptmodul $\mathcal{H} \;=\; \frac{1728}{j}$, given by (57) becomes for $b \;=\; 3 \; + x$:

$$\mathcal{H} \;=\; \frac{4 \cdot \, x \cdot \, (27 + 4\, x)^2 \cdot \, (x^2 + 9\, x + 27)^3}{(9+x)^3 \cdot \, (4\, x^2 + 27\, x + 27)^3}. \tag{60}$$

[30] See also [53] section 6 page 48.

[31] The discriminant in b of $4\, a^3 + 4\, b^3 - a^2 b^2 \, - 18\, ab \, +27$ reads: $(a-3)^3 \cdot \, (a^2 \, +3\, a \, +9)^3$, consequently the exact expressions are simpler at $a = 3$.

The telescoper of the rational function (59) with $D_a(x, y, z)$ given by (58) for $a = 3$, is an *order-four* linear differential operator which is the direct-sum of two order-two linear differential operators $L_4 = LCLM(L_2, M_2) = L_2 \oplus M_2$, these *two order-two* linear differential operators having the pullbacked hypergeometric solutions

$$(27 + 4x)^{-1/2} \cdot x^{-5/4} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], 1 + \frac{27}{4x}\Big), \tag{61}$$

for L_2, and

$$\frac{3+x}{(9+x)^{1/4} \cdot (4x^2+27x+27)^{1/4} \cdot x^{3/2} \cdot (27+4x)^{1/2}} \\ \times {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], \frac{4 \cdot x \cdot (27+4x)^2 \cdot (x^2+9x+27)^3}{(9+x)^3 \cdot (4x^2+27x+27)^3}\Big), \tag{62}$$

for M_2, *where we see clearly that the Hauptmodul in* (62) *is precisely the Hauptmodul* (60). The *Jacobian of the genus-two curve is a split Jacobian corresponding to the product* $\mathcal{E}_1 \times \mathcal{E}_2$ *of two elliptic curves*, the j-invariant of the second elliptic curve corresponds to the Hauptmodul $\mathcal{H} = \frac{1728}{j}$ given by (57) when the j-invariant of the first elliptic curve reads

$$j_1 = \frac{6912x}{27 + 4x}, \tag{63}$$

corresponding to the Hauptmodul $\frac{1728}{j_1} = 1 + \frac{27}{4x}$ in (61). This second invariant is, as it should, *exactly the* j*-invariant of the second elliptic curve* $\mathcal{E}'$, given page 48 in [53]:

$$j(\mathcal{E}') = \frac{256 \cdot (3b - a^2)^3}{4a^3c - a^2b^2 - 18abc + 4b^3 + 27c^2}, \tag{64}$$

for $c = 1$, $a = 3$ and $b = 3 + x$.

5.4. Creative telescoping on rational functions of three variables associated with genus-two curves with split Jacobians: a simple example

Another simpler example of a *genus-two* curve with pullbacked ${}_2F_1$ solution (not product of pullbacked ${}_2F_1$) of the telescoper can be given if one considers the *genus-two algebraic curve* $C_p(x, y) = 0$ given in Lemma 7 of [54] (see also [55,56])

$$C_p(x, y) = x^5 + x^3 + p \cdot x - y^2. \tag{65}$$

Let us introduce the rational function $xy/D(x, y, z)$ where the denominator $D(x, y, z)$ is given by:

$$D(x, y, z) = C_{(p=xyz)}(x, y) = x^5 + x^3 + x^2yz - y^2. \tag{66}$$

The telescoper of this rational function is an order-two linear differential operator which has the two hypergeometric solutions

$$x^{-1/4} \cdot {}_2F_1\Big([\frac{1}{8}, \frac{5}{8}], [\frac{3}{4}], 4x\Big) \tag{67}$$

which is a Puiseux series at $x = 0$ and:

$$x^{-1/4} \cdot {}_2F_1\Big([\frac{1}{8}, \frac{5}{8}], [1], 1 - 4x\Big). \tag{68}$$

These two hypergeometric solutions can be rewritten as[32]

$$\mathcal{A}(x) \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], \frac{1728}{J}\Big), \tag{69}$$

where the j-invariant J, in the Hauptmodul $\frac{1728}{J}$ in (69), *corresponds exactly to the degree-two elliptic subfields*

$$J^2 \quad - 128 \cdot \frac{(2000\, x^2 \, + 1440\, x \, + 27)}{(1 \, - 4\, x)^2} \cdot J \quad - 4096 \cdot \frac{(100\, x - 9)^3}{(1 \, - 4\, x)^3} \quad = \quad 0, \tag{70}$$

given in the first equation of page 6 of [54].

Remark: In contrast with the previous example of subsection 5.3 where we had two j-invariants corresponding to the *two order-two* linear differential operators L_2 and M_2 of the direct-sum decomposition of the order-four telescoper, we have, here, *just one order-two* telescoper, which is enough to "encapsulate" the two j-invariants (70), since they are Galois-conjugate.

6. Rational functions with tri-quadratic denominator and N-quadratic denominator.

We try to find telescopers of rational functions corresponding to (factors of) linear differential operators of "small" orders, for instance order-two linear differential operators with pullbacked ${}_2F_1$ hypergeometric functions, classical modular forms, or their modular generalisations (order-four Calabi-Yau linear differential operators [27], etc ...). As we saw in the previous sections, this corresponds to the fact that the denominator of these rational functions is associated with an elliptic curve, or products of elliptic curves, with K3 surfaces or with threefold Calabi-Yau manifolds corresponding to algebraic varieties with foliations in elliptic curves[33]. Since this paper tries to reduce the *differential algebra* creative telescoping calculations to *effective algebraic geometry* calculations[34] and structures, we want to focus on rational functions with denominators that correspond to *selected* algebraic varieties [45,59], beyond algebraic varieties corresponding to products of elliptic curves or foliations in elliptic curves[35], namely algebraic varieties *with an infinite number of birational automorphisms*[36]. This *infinite number of birational symmetries*, excludes algebraic varieties of the "general type" (with *finite* numbers[37] of birational symmetries). For algebraic surfaces, this amounts to discarding the surfaces of the "general type" which have Kodaira dimension 2, focusing on *Kodaira dimension one* (elliptic surfaces), or *Kodaira dimension zero* (abelian surfaces, hyperelliptic surfaces, K3 surfaces, Enriques surfaces), or even *Kodaira dimension* $-\infty$ (ruled surfaces, rational surfaces).

In contrast with algebraic *curves* where one can easily, and very efficiently, calculate the genus of the curves to discard the algebraic curves of higher genus and, in the case

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32 The fact that ${}_2F_1\Big([\frac{1}{8}, \frac{5}{8}], [1], z\Big)$ can be rewritten as ${}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], H(z)\Big)$ where the Hauptmodul $H(z)$ is solution of a quadratic equation is given in equation (H.14) of Appendix H of [18].

33 Even if K3 surfaces, or threefold Calabi-Yau manifolds, are *not* abelian varieties, the Weierstrass-Legendre forms introduced in D, amounts to saying that K3 surfaces can be "essentially viewed" (as far as creative telescoping is concerned) as foliations in two elliptic curves, and threefold Calabi-Yau manifolds as foliations in three elliptic curves.

34 One has birational automorphisms in projective spaces [57,58], but since this paper is dedicated to (efficient) formal calculations we work exclusively in affine coordinates (see for instance (A41), (A42), (A43) below). For algebraic geometers an ellitic curve is a smooth complete genus 1 curve with a choice of a base point. Here our elliptic curves are, in fact, an affine piece of a genus 1 curve with no base point, but this does not really matter because the j-invariant which is all we care about in this kind of creative telescoping calculations, is determined by that much information.

35 K3 surfaces, threefold Calabi-Yau manifolds, higher curves with split Jacobian corresponding to products of elliptic curves, ...

36 The best explicit illustration of this situation emerges in integrable models [45,59–61]

37 There are even precise bounds for the number of automorphisms. The upper bound is $84\,(g-1)$ for curves of genus g and these bounds have been extensively studied in higher dimensions [62–64].

of genus-one, obtain the j-invariant using formal calculations[38], it is, in practice, quite difficult to see for higher dimensional algebraic varieties, that the algebraic variety is not of the "general type", because it has an *infinite number of birational symmetries*. For these (low Kodaira dimension) "selected cases" we are interested in, calculating the generalisation of the j-invariant (Igusa-Shiode invariants, etc ...) is quite hard.

Along this line we want to underline that there exists a remarkable set of algebraic surfaces, namely the algebraic surfaces corresponding to *tri-quadratic* equations:

$$\sum_{m=0,1,2} \sum_{n=0,1,2} \sum_{l=0,1,2} a_{m,n,l} \cdot x^m y^n z^l \quad = \quad 0, \tag{71}$$

depending on $27 = 3^3$ parameters $a_{m,n,l}$. More generally, one can introduce algebraic varieties corresponding to N-quadratic equations:

$$\sum_{m_1=0,1,2} \sum_{m_2=0,1,2} \cdots \sum_{m_N=0,1,2} a_{m_1,m_2,\cdots,m_N} \cdot x_1^{m_1} x_2^{m_2} \cdots x_N^{m_N} \quad = \quad 0. \tag{72}$$

With these tri-quadratic (71), or N-quadratic (72) equations, we will see, in E.1 and E.2, that we have *automatically* (selected) algebraic varieties that are not of the "general type" having an *infinite number of birational symmetries*, which is precisely our requirement for the denominator of rational functions with remarkable telescopers[39].

Let us first, as a warm-up, consider, in the next subsection, a remarkable example of tri-quadratic (71), where the underlying foliation in elliptic curves is crystal clear.

6.1. *Rational functions with tri-quadratic denominator simply corresponding to elliptic curves.*

Let us first recall the tri-quadratic equation in three variables x, y and z

$$\begin{aligned} & x^2y^2z^2 \quad - 2 \cdot M \cdot xyz \cdot (x + y + z) \quad + 4 \cdot M \cdot (M + 1) \cdot xyz \\ & \quad + M^2 \cdot (x^2 + y^2 + z^2) \quad - 2\, M^2 \cdot (xy + xz + yz) \quad = \quad 0, \end{aligned} \tag{73}$$

already introduced in Appendix C of [65]. This algebraic surface, symmetric in x, y and z, can be seen for z (resp. x or y) fixed, as an *elliptic curve* which j-invariant is *independent of* z yielding the corresponding Hauptmodul:

$$\mathcal{H} \;=\; \frac{1728}{j} \;=\; \frac{27 \cdot M^2 \cdot (M-1)^2}{4 \cdot (M^2 - M + 1)^3}. \tag{74}$$

This corresponds to the fact that this algebraic surface (73) can be seen as a product of two times the same elliptic curve with the Hauptmodul (74). This is a consequence of the fact that, introducing $x = sn(u)^2$, $y = sn(v)^2$ and $z = sn(u + v)^2$, and $M = 1/k^2$, this algebraic surface (73) corresponds to the well-known formula for the *addition on elliptic sine*[40]:

$$sn(u \,+\, v) \quad = \quad \frac{sn(u)\, cn(v)\, dn(v) \;+\; sn(v)\, cn(u)\, dn(u)}{1 \;- k^2\, sn(u)^2\, sn(v)^2}. \tag{75}$$

38 Use with(algcurves) in Maple and the command "genus" and "j_invariant".

39 Telescopers with factors of "small enough" order, possibly yielding classical modular forms, Calabi-Yau operators, ... Rational functions with denominators of the "general type" will yield telescopers of very large orders.

40 See equation (C.3) in Appendix C of [65].

For $M = x\,y\,z\,w$, the LHS of the tri-quadratic equation (73) yields a polynomial of *four variables* $x,\, y,\, z$ and w, that we denote $T(x,\, y,\, z,\, w)$:

$$T(x,\, y,\, z,\, w) \,\, = \tag{76}$$
$$x^2y^2z^2 \,\,\, -2\, \cdot\, x^2y^2z^2\, w \cdot\, (x\, +y\, +z) \,\,\, +4 \cdot\, (xyzw\, +1) \cdot\, x^2y^2z^2\, w$$
$$+x^2y^2z^2w^2 \cdot\, (x^2+y^2+z^2) \,\,\, -2\, x^2y^2z^2w^2 \cdot\, (xy\, +xz\, +yz).$$

The telescoper of the rational function in *four variables* $x,\, y,\, z$ and w,

$$\frac{x\,y\,z}{T(x,\, y,\, z,\, w)}, \tag{77}$$

is an order-three (self-adjoint) linear differential operator which is the *symmetric square* of the order-two linear differential operator having the following pullbacked ${}_2F_1$ hypergeometric solution:

$$x^{-1/2} \cdot\, (x^2-x+1)^{-1/4}$$
$$\times\, {}_2F_1\Big([\frac{1}{12},\, \frac{5}{12}],\, [1],\, \frac{27 \cdot\, x^2 \cdot\, (x-1)^2}{4\, \cdot\, (x^2-x+1)^3}\Big). \tag{78}$$

As it should the Hauptmodul in (78) is the same as the Hauptmodul (74). The algebraic surface (73) can be seen as the product of *two times the same elliptic curve* with the Hauptmodul (74): as expected the solution of the order-three telescoper is the *square* of the pullbacked ${}_2F_1$ hypergeometric function (78) with that Hauptmodul.

More generally, we can also consider another tri-quadratic equation of three variables $x,\, y$ and z and *two* parameters M and N:

$$x^2y^2z^2 \,\,\, -2\, M \cdot\, xyz \cdot\, (x\, +y\, +z) \,\,\, +N \cdot\, xyz \tag{79}$$
$$+M^2 \cdot\, (x^2+y^2+z^2) \,\,\, -2\, M^2 \cdot\, (xy\, +xz\, +yz) \,\, = \,\, 0.$$

This surface, symmetric in $x,\, y$ and z, can be seen for z (resp. x or y) fixed as an elliptic curve which j-invariant is, again, *independent of* z yielding the corresponding Hauptmodul:

$$\mathcal{H} \,\, = \,\, \frac{1728}{j} \,\, = \,\, \frac{1728 \cdot\, M^6 \cdot\, (64\, M^3-N^2)}{(48\, M^3-N^2)^3}. \tag{80}$$

Let us consider the following change of variables $M = m^2$ and $N = 8 \cdot m^3 + p$ in (79). For $p = x\,y\,z\,w$, the LHS of the tri-quadratic equation (79) yields a polynomial in *four variables* $x,\, y,\, z$ and w, that we denote $\mathcal{T}_m(x,\, y,\, z,\, w)$:

$$\mathcal{T}_m(x,\, y,\, z,\, w) \,\, =$$
$$x^2y^2z^2 \,\,\, -2\, m^2 \cdot\, xyz \cdot\, (x\, +y\, +z) \,\,\, +(8 \cdot\, m^3\, +\, x\,y\,z\,w) \cdot\, xyz$$
$$+m^4 \cdot\, (x^2+y^2+z^2) \,\,\, -2\, m^4 \cdot\, (xy\, +xz\, +yz). \tag{81}$$

For z (resp. x or y) fixed the corresponding Hauptmodul (80) reads:

$$\mathcal{H} \,\, = \,\, \frac{1728 \cdot\, m^{12} \cdot\, p \cdot\, (16\, m^3\, +p)}{(16\, m^6\, +16\, m^3 \cdot\, p\, +p^2)^3}. \tag{82}$$

The telescoper of the rational function in *four variables* $x,\, y,\, z$ and w,

$$\frac{x\,y\,z}{\mathcal{T}_m(x,\, y,\, z,\, w)}, \tag{83}$$

is an order-three (self-adjoint) linear differential operator which is the *symmetric square* of an order-two linear differential operator having the following pullbacked ${}_2F_1$ hypergeometric solution:

$$(16\,m^6 \,+16\,m^3 \cdot\, x \,+x^2)^{-1/4} \cdot \\ \times\, {}_2F_1\Big([\frac{1}{12}, \, \frac{5}{12}], \, [1], \, \frac{1728 \cdot \, m^{12} \cdot \, x \cdot \, (16\,m^3 \, + x)}{(16\,m^6 \, + 16\,m^3 \cdot \, x \, + x^2)^3}\Big). \tag{84}$$

As it should the Hauptmodul in (84) is the same as the Hauptmodul (82). The algebraic surface (79) can be seen as the product of *two times the same elliptic curve* with the Hauptmodul (80) (or (82)). As expected the solution of the order-three telescoper is the *square* of the pullbacked ${}_2F_1$ hypergeometric function (84) with the Hauptmodul (82).

Remark: Let us perform some (slight) deformation of the rational function (77), changing the first -2 coefficient in (76) into a -3 coefficient. One thus considers the polynomial $T(x, y, z, w)$:

$$\begin{aligned} T(x, \, y, \, z, \, w) \,\, &= \\ &x^2y^2z^2 \,\,\, - 3\, \cdot \, x^2y^2z^2\, w \cdot \, (x \, + y \, + z) \,\,\, + 4 \cdot \, (xyzw \, + 1) \cdot \, x^2y^2z^2\, w \\ &\quad + x^2y^2z^2w^2 \cdot \, (x^2 + y^2 + z^2) \,\,\, - 2 \cdot \, x^2y^2z^2w^2 \cdot \, (xy \, + xz \, + yz). \end{aligned} \tag{85}$$

The telescoper of the rational function in *four variables*,

$$\frac{x\,y\,z}{T(x, \, y, \, z, \, w)}, \tag{86}$$

is an (irreducible) linear differential operator of (only) *order-four* L_4 which is non-trivially homomorphic to its adjoint[41]. A priori, we cannot exclude the fact that L_4 could be homomorphic to the symmetric cube of a second-order linear differential operator, or to a symmetric product of two second-order operators. Furthermore, it could also be, in principle, that these second-order operators admit classical modular forms as solutions (pullbacks of special ${}_2F_1$ hypergeometric functions). However, these options can both be excluded by using some results from differential Galois theory [68], specifically from [69, Prop. 7, p. 50] for the symmetric cube case, and from [69, Prop. 10, p. 69] for the symmetric product case, see also [70, §3]. Indeed, if L_4 were either a symmetric cube or a symmetric product of order-two operators, then its symmetric square would contain a (direct) factor of order 3 or 1. This is ruled out by a factorization procedure which shows that the symmetric square of L_4 is (LCLM-)irreducible.

This example does not correspond to an addition formula like (75), but the polynomial $T(x, y, z, w)$ still corresponds to a tri-quadratic. Consequently it is an algebraic variety with an *infinite number of birational automorphisms*, as shown in E.1.

6.2. Rational functions with tri-quadratic denominator: Fricke cubics examples associated with Painlevé VI equations

Let us consider other simple examples of tri-quadratic surfaces that occur in different domains of mathematics and physics.

Among the *Fricke families* of cubic surfaces, the family [71–73]

$$x\,y\,z \,\,\, + x^2 \, + y^2 \, + z^2 \,\,\, + b_1\, x \, + b_2\, y \, + b_3\, z \,\,\, + c \,\, = \,\, 0, \tag{87}$$

of affine cubic surfaces parametrised by the four constants $(b_1, \, b_2, \, b_3, \, c)$ is known [72] to be a deformation of a D_4 singularity which occurs at the symmetric (Manin's) case $b_1 \, = \, b_2 \, = \, b_3 \, = \, -8, \, c \, = \, 28$.

[41] Its exterior square has a rational solution. However this order-four linear differential operator is not MUM (maximum unipotent monodromy [27,66,67])

Among the symmetric $b_1 = b_2 = b_3$ cases some selected sets of the four constants (b_1, b_2, b_3, c) emerge: the Markov cubic $b_1 = b_2 = b_3 = c = 0$, Cayley's nodal cubic $b_1 = b_2 = b_3 = 0,\ c = -4$, Clebsch diagonal cubic $b_1 = b_2 = b_3 = 0,\ c = -20$, and Klein's cubic $b_1 = b_2 = b_3 = -1,\ c = 0$.

Some of these symmetric cubics can be seen as the monodromy manifold of the *Painlevé VI equation* (see equation (1.7) in [74], see also equations (1.2) and (1.4) in [73]): the Picard-Hitchin cases $(0,0,0,4)$, $(0,0,0,-4)$, $(0,0,0,-32)$, Kitaev's cases $(0,0,0,0)$, $(-8,-8,-8,-64)$, and especially Manin's case $(-8,-8,-8,28)$.

Let us consider the (symmetric) rational function in three variables x, y and z [72]:

$$R(x, y, z) \quad = \quad \frac{1}{x^2 + y^2 + z^2 \ + x\,y\,z \ + c}, \tag{88}$$

which takes into account the other Picard-Hitchin cases[42] $(0,0,0,4)$, $(0,0,0,-4)$, $(0,0,0,32)$. The rational function (88) has an *order-two* telescoper which has a simple pullbacked hypergeometric solution:

$$\begin{aligned} &\frac{1}{x+c} \cdot {}_2F_1\Big([\frac{1}{3}, \frac{2}{3}], [1], \ -\frac{27\,x^2}{(x+c)^3}\Big) \\ &\quad = \quad (x+c)^{-1/4} \cdot q_3(x)^{-1/4} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], \ -\frac{1728 \cdot x^6 \cdot p_3(x)}{(x+c)^3 \cdot q_3(x)^3}\Big), \end{aligned} \tag{89}$$

where[43]:

$$p_3(x) \ = \ x^3 \ + 3 \cdot (c+9) \cdot x^2 \ + 3 \cdot c^2 \cdot x \ + c^3,$$
$$q_3(x) \ = \ x^3 \ + 3 \cdot (c+8) \cdot x^2 \ + 3 \cdot c^2 \cdot x \ + c^3,$$

Eliminating $z = \frac{p}{x\,y}$ in the denominator of (88) gives the *genus-four* algebraic curve:

$$x^2y^2 \cdot (x^2 + y^2) \ + (p+c) \cdot x^2y^2 \ + p^2 \ = \ 0. \tag{90}$$

Again, the question is to see whether the Jacobian of this *genus-four* algebraic curve (88) could also correspond to a split Jacobian, with a j-invariant corresponding to the Hauptmodul in (89).

7. Telescopers of rational functions of several variables

Let us consider the rational function in *four* variables x, y, z, u:

$$R(x, y, z, u) \ = \ \frac{1}{1 \ + 3\,y \ + z \ + 9\,y\,z \ + 11\,z^2\,y \ + 3\,u\,x}. \tag{91}$$

The telescoper of this rational function of four variables is an order-two linear differential operator L_2 which has the pullbacked hypergeometric solution:

$$\begin{aligned} &(1 - 2592\,x^2)^{-1/4} \\ &\times {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], \ -\frac{419904 \cdot x^3 \cdot (5 - 12\,x - 19440\,x^2 + 2665872\,x^3)}{(1 - 2592\,x^2)^3}\Big). \end{aligned} \tag{92}$$

[42] As well as the Markov cubic $b_1 = b_2 = b_3 = c = 0$, Cayley's nodal cubic $b_1 = b_2 = b_3 = 0,\ c = -4$, and Clebsch diagonal cubic $b_1 = b_2 = b_3 = 0,\ c = -20$ cases.

[43] The values $c = 0$ and $c = -4$ are the only values such that the discriminant in x of $p_3(x)$ can be zero.

The diagonal of (91) is the expansion of this pullbacked hypergeometric function (92):

$$1 \ + 648\,x^2 \ - 72900\,x^3 \ + 1224720\,x^4 - 330674400\,x^5 \ + 23370413220\,x^6 \\ -1276733858400\,x^7 \ + 180019474034400\,x^8 \ - 12013427240614800\,x^9 \ + \ \cdots \tag{93}$$

If one considers the intersection of the vanishing condition of the denominator of (91) with the hyperbola $p \, = \, x\,y\,z\,u$, eliminating for instance $u \, = \, \frac{p}{x\,y\,z}$ in the vanishing condition of the denominator of (91), one gets a condition, *independent of* x, which corresponds to a *genus-one* curve

$$11\,y^2z^3 \ + 9\,y^2z^2 \ + 3\,y^2z \ + yz^2 \ + yz \ + 3\,p \ = \ 0. \tag{94}$$

The Hauptmodul of this elliptic curve (94) reads:

$$\mathcal{H} \ = \ - \frac{419904 \cdot \, p^3 \cdot \, (5 \, - 12\,p \, - 19440\,p^2 \, + 2665872\,p^3)}{(1 \, - \, 2592\,p^2)^3}, \tag{95}$$

which corresponds precisely to the Hauptmodul pullback in (92).

Remark : The expansion (93) of (92) is not only the diagonal of the rational function $R(x, \, y, \, z, \, u)$ in *four* variables (91), it is also the diagonal of the rational function of *three* variables $R(x, \, y, \, z, \, 1)$. Actually, using section (3), one sees easily that eliminating $x \, = \, \frac{p}{y\,z}$ in the the vanishing condition of the denominator of $R(x, \, y, \, z, \, 1)$ gives exactly the same elliptic curve (94).

Let us, now, generalize the rational function (91) of *four* variables $x, \, y, \, z, \, u$, introducing the rational function of $N+3$ variables $x, \, y, \, z, \, u_1, \, u_2, \, \cdots, \, u_N$:

$$R(x, \, y, \, z, \, u_1, \, u_2, \, \cdots, \, u_N) \\ = \ \frac{1}{1 \ + 3\,y \ + z \ + 9\,y\,z \ + 11\,z^2\,y \ + 3\,x \cdot \, u_1\,u_2 \ \cdots\, u_N}. \tag{96}$$

The telescoper of this rational function of $N+3$ variables is the same order-two telescoper as for (91), which has the pullbacked hypergeometric solution (92). Again one can verify that the diagonal of (96) is the expansion (93) of the pullbacked hypergeometric function[44] (92). If one considers the intersection of the vanishing condition of the denominator of (96) with the hyperbola $p \, = \, x\,y\,z\,u_1\,u_2 \ \cdots\, u_N$, eliminating for instance $u_N \, = \, \frac{p}{x\,y\,z\,u_1\,\cdots\,u_{N-1}}$ in the vanishing condition of the denominator of (96), one gets again a condition, *independent of* x but also of $u_1, \, \cdots, \, u_N$, which corresponds to a *genus-one* curve (94):

$$11\,y^2z^3 \ + 9\,y^2z^2 \ + 3\,y^2z \ + yz^2 \ + yz \ + 3\,p \ = \ 0. \tag{97}$$

The Hauptmodul of this elliptic curve (97), or (94) reads again the Hauptmodul (95) which corresponds precisely to the Hauptmodul pullback in (92).

Other examples, corresponding to simple polynomial deformations of (91), such that their diagonal is the pullbacked $_2F_1$ hypergeometric function (92) are displayed in F. This (infinite) family of rational functions correspond to a different algebraic geometry scenario: the "canonical" algebraic surface corresponding to the intersection of the vanishing condition of the denominator of the rational function with the hyperbola $p \, = \, xyz$, is foliated in (generically high genus) algebraic curves depending on the variable x. One sees (experimentally) that the Hauptmodul of the pullbacked $_2F_1$ hypergeometric functions corresponds to the Hauptmodul of the $x \, = \, 0$ algebraic curve, which is an elliptic curve[45].

[44] A pure algebraic geometer will probably consider this result as trivial from the computational point of view, saying that the variety is a fiber bundle over a family of elliptic curves with constant fiber (see also below).

[45] The algebraic curves for other values of x are not necessarily elliptic curves, they can be algebraic curves of quite large genus.

In contrast with the other examples and results of this paper, we have no algebraic geometry interpretation of this experimental result yet.

8. Conclusion

Diagonals of rational functions emerge quite naturally in lattice statistical mechanics [19,20]. This explains the frequent occurrence of *modular forms*, represented as pullbacked ${}_2F_1$ hypergeometric functions [1,2] in lattice statistical mechanics [21–27].

We have shown that the results we had obtained on diagonals of nine and ten parameters families of rational functions in three variables, using creative telescoping yielding *classical modular forms* expressed as pullbacked ${}_2F_1$ hypergeometric functions [1,2], can be obtained much more efficiently calculating the j-invariant of an *elliptic curve canonically associated with the denominator of the rational functions*. In the case where creative telescoping yields pullbacked ${}_2F_1$ hypergeometric functions, we generalize this result to other families of rational functions of three, and even more than three, variables, when the denominator can be associated with products of elliptic curves or foliation in terms of elliptic curves, or when the denominator is associated with a *genus-two curve* with *a split Jacobian* corresponding to *products of elliptic curves*.

We have seen different scenarii. In the first cases, we have considered denominators corresponding to *products* of elliptic curves: in these cases the solutions of the telescoper were *products* of pullbacked ${}_2F_1$ hypergeometric functions. We have also considered denominators corresponding to *genus-two* curves with *split Jacobians isogenous to products of two elliptic curves*, and in these cases the solutions of the telescoper were *sums* of two pullbacked ${}_2F_1$ hypergeometric functions, sometimes one pullbacked ${}_2F_1$ hypergeometric function being enough to describe the two Galois-conjugate j-invariants (see 5.4). We also considered denominators corresponding to algebraic varieties where the Hauptmodul pullback in the pullbacked ${}_2F_1$ hypergeometric functions emerges from a selected ($x = 0$, see F.1, F.2) elliptic curve of the algebraic variety. We also encountered denominators corresponding to algebraic manifolds with an infinite set of birational automorphisms and elliptic curves foliation yielding, no longer classical modular forms represented as pullbacked ${}_2F_1$ hypergeometric functions, but more general modular structures associated with selected linear differential operators like Calabi-Yau linear differential operators [27,66] and their generalisations.

The creative telescoping method on a rational function is an efficient way to find the periods of an algebraic variety over *all possible cycles*[46]. The fact that the solution of the telescoper corresponds to “periods" [37] *over all possible cycles* is a simple consequence of the fact that creative telescoping corresponds to *purely differential algebraic manipulations* on the integrand *independently of the cycles*, thus *being blind to analytical details*. In this paper, we show that the final result emerging from such differential algebra procedures (which can be cumbersome when the result depends on nine or ten parameters), can be obtained almost instantaneously from a more fundamental intrinsic pure algebraic geometry approach, calculating, for instance, the j-invariant of some canonical elliptic curve. This corresponds to a shift Analysis $\rightarrow$ Differential Algebra $\rightarrow$ Algebraic Geometry. Algebraic geometry studies of more involved algebraic varieties than product of elliptic curves, foliation in elliptic curves (Calabi-Yau manifolds, ...) is often a tedious and/or difficult task (finding Igusa-Shiode invariants, ...), and formal calculations tools are not always available or user-friendly. Ironically, for such involved algebraic varieties the creative telescoping may then become a simple and efficient tool to perform effective algebraic geometry studies.

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[46] Not only the *vanishing cycles* [38,40] corresponding to *diagonals* of rational functions.

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Appendix A Diagonals of rational functions and Picard-Fuchs equations

For simplicity let us consider rational functions of three variables, and double integrals [86]. The diagonal of a rational function of three variables is obtained through its multi-Taylor expansion [19,20]

$$R(x, y, z) \; = \; \sum_m \sum_n \sum_l \, a_{m,n,l} \cdot x^m \, y^n \, z^l, \tag{A1}$$

by extracting the "diagonal" terms, i.e. the powers of the product $p \, = \, xyz$:

$$Diag\Big(R(x, y, z)\Big) \; = \; \sum_m \, a_{m,m,m} \cdot p^m. \tag{A2}$$

Such diagonals are closely related to the integrals of rational functions. For example $Diag\Big(R(x, y, z)\Big)$ is the constant term (in y, z) in the infinite expansion

$$R\Big(\frac{p}{y\,z}, y, z\Big) \; = \; \sum_{m,n,l \ge 0} a_{m,n,l} \cdot p^m \, y^{n-m} \, z^{l-m}, \tag{A3}$$

which can be represented by the integral [35]

$$\frac{1}{(2\,\pi\,i)^2} \oint\oint R\Big(\frac{p}{y\,z}, y, z\Big) \, \frac{dy}{y} \wedge \frac{dz}{z}. \tag{A4}$$

The diagonal (A2) is also the constant term (in y, z) of

$$R\Big(\frac{p}{y}, \frac{y}{z}, z\Big) \; = \; \sum_{m,n,l \ge 0} a_{m,n,l} \cdot p^m \, y^{n-m} \, z^{l-n}, \tag{A5}$$

wich is of the form

$$\frac{1}{(2\,\pi\,i)^2} \oint\oint \frac{N_p(y, z)}{D_p(y, z)} \, \frac{dy}{y} \wedge \frac{dz}{z}, \tag{A6}$$

where the numerator $N_p(y, z)$ and the denominator $D_p(y, z)$ are polynomials. it is well-known that such integrals satisfy a linear differential equation with respect to p having rational functions in p as coefficients, called the Picard-Fuchs equation[47]. the problem of determining such linear differential equations has been started by Griffiths [75] with the assumption that the variety $D_p(y, z) \, = \, 0$ is smooth, but later techniques were developed to include examples with singular points [35,41]. The linear differential equations (Gauss-Manin systems, telescopers) occuring in integrable models [16,23,24] are of order much larger than order two[48] and almost never correspond to smooth varieties. Creative telescop-

47 The order of this linear differential equation is generally equal to the rank of the algebraic deRham cohomology of $D_p(y, z) \, = \, 0$. For curves of genus g this rank is $2\,g$.

48 Since Felix Klein it is well-known that the Picard-Fuchs equation corresponding to the (Weierstrass) elliptic curve corresponds to the hypergeometric function ${}_2F_1([1/12, 5/12], [1], 1/J)$.

ing[49] and more specifically the programs [3] corresponding to a fast approach to creative telescoping [43], are a powerfull way to find these linear differential operators annihilating these diagonal of rational functions in the cases emerging naturally in theoretical physics, integrable models, enumerative combinatorics, for which the order of the linear differential operators is quite large [16,23,24] and the variety $D_p(y, z) = 0$ is (most of the time) not a smooth one. All the pedagogical (but non-trivial) examples of telescopers displayed in this paper can be viewed by an algebraic geometer as a presentation of examples of families of varieties and their Picard-Fuchs equations.

Appendix B Maximum number of parameters for families of planar elliptic curves.

We have seen, in section 3, that the previous results on diagonals of nine or ten parameters families of rational functions of three variables being pullbacked ${}_2F_1$ hypergeometric functions (and in fact classical modular forms) can actually be seen as corresponding to the (well-known in integrable models and integrable mappings) fact that the most general biquadratic corresponding to *elliptic curves* is a *nine-parameters* family and that the most general ternary cubic corresponding to elliptic curves is a *ten-parameters* family. One can, for instance recall page 238 of [76], which amounts to considering the collection of all cubic curves in $\mathbb{C}P_2$ with the homogeneous equation

$$\begin{aligned} a\,x^3 \;+b\,x^2\,y \;+c\,x\,y^2 \;+d\,y^3 \;+e\,x^2\,z \;+f\,x\,z^2 \;+g\,y^2\,z \\ +h\,y\,z^2 \;+i\,z^3 \;+j\,x\,y\,z \;=\; 0, \end{aligned} \tag{A7}$$

and the associated problems of passing through nine given points. One can also recall the ternary cubics in [77,78] and other problems of elliptic curves of high rank [79] (see the concept of Neron-Severi rank).

Since the rational functions of three variables we consider are essentially encoded by the denominator of these rational functions, and in the cases we have considered, the emergence of pullbacked ${}_2F_1$ hypergeometric functions (and in fact classical modular forms) corresponds to the fact that the intersection of these denominators with the hyperbola $p = x\,y\,z$ corresponds to elliptic curves, one sees that these rational functions are essentially classified by the possible n-parameters families $P(x, y) = 0$ of elliptic curves.

If one considers a polynomial

$$P(x,y) \;=\; \sum_m \sum_n a_{m,n} \cdot x^m\, y^n, \tag{A8}$$

with generic coefficients $a_{m,n} \in \mathbb{C}$, then the genus of the algebraic curve defined by P is determined by the support $supp(P) = \{(m,n) \in \mathbb{N}^2 : a_{m,n} \neq 0\}$. More precisely, the genus equals the number of interior integer lattice points inside the convex hull of $supp(P)$ [80] (see also the discussion in [81]). For example, the support of the ten-parameters family (11) consists of the following 10 points in $\mathbb{N}^2$:

$$(0,0),\ (0,1),\ (0,2),\ (0,3),\ (1,1),\ (1,2),\ (1,3),\ (2,2),\ (2,3),\ (3,3)$$

which form a right triangle of side length 3. Only one of these points is an interior point, namely $(1,2)$, *hence the genus is* 1.

Therefore we may ask: which integer lattice polytopes exist which have exactly one interior point and what is the largest such polytope? Not surprisingly, the answer is known: there are (up to transformations like translation, rotation, shearing) exactly 16 different polytopes with a single interior point [82] (see also Figure 5, page 548 in [83]), the above-mentioned right triangle being the one with the highest total number of lattice points.

This shows that *there cannot be a family of elliptic curves with more than ten parameters*.

[49] For a detailed introduction to creative telescoping [36] see for instance [34].

Appendix C Monomial transformations preserving pullbacked hypergeometric results

More generally, recalling subsection 4.2 in [2] and subsection 4.2 page 17 in [1], let us consider the monomial transformation

$$\begin{aligned}(x, y, z) \quad &\longrightarrow \quad M(x, y, z) \;=\; (x_M, y_M, z_M)\\ &= \; \Big(x^{A_1} \cdot y^{A_2} \cdot z^{A_3}, \;\; x^{B_1} \cdot y^{B_2} \cdot z^{B_3}, \;\; x^{C_1} \cdot y^{C_2} \cdot z^{C_3}\Big), \end{aligned} \tag{A9}$$

where the A_i's, B_i's and C_i's are positive integers such that $A_1 = A_2 = A_3$ is excluded (as well as $B_1 = B_2 = B_3$ as well as $C_1 = C_2 = C_3$), and that the determinant[50] of the 3×3 matrix [1,2]

$$\left[\begin{array}{ccc} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{array}\right], \tag{A10}$$

is not equal to zero[51], and that:

$$A_1 + B_1 + C_1 \;=\; A_2 + B_2 + C_2 \;=\; A_3 + B_3 + C_3. \tag{A11}$$

We will denote by $n = A_i + B_i + C_i$ the integer in these three equal sums (A11). Condition (A11) is introduced in order to impose that the product[52] of $x_M\, y_M\, z_M$ is *an integer power of the product* of $x\,y\,z$: $x_M\, y_M\, z_M = (x\,y\,z)^n$.

If we take a rational function $\mathcal{R}(x, y, z)$ in three variables and perform such a monomial transformation (A9) $(x, y, z) \rightarrow M(x, y, z)$, on this rational function $\mathcal{R}(x, y, z)$, we get another rational function that we denote by $\tilde{\mathcal{R}} = \mathcal{R}(M(x, y, z))$. Now the diagonal of $\tilde{\mathcal{R}}$ is the diagonal of $\mathcal{R}(x, y, z)$ where we have changed x into x^n:

$$\Phi(x) \;=\; Diag\Big(\mathcal{R}\Big(x, y, z\Big)\Big), \qquad Diag\Big(\tilde{\mathcal{R}}\Big(x, y, z\Big)\Big) \;=\; \Phi(x^n). \tag{A12}$$

Appendix D Weierstrass and Legendre forms

The telescoper of the rational function in three variables

$$\frac{x\,y}{(1+y)^2 \;\; - x \cdot (1-x) \cdot (x - x\,y\,z)}, \tag{A13}$$

associated[53] with the *elliptic curve* in a *Weierstrass form*:

$$(1+y)^2 \;\; - x \cdot (1-x) \cdot (x-p) \;=\; 0, \tag{A14}$$

is the order-two linear differential operator

$$L_2 \;=\; -1 \;\; + 4 \cdot (1-2\,x) \cdot D_x \;\; + 4 \cdot x \cdot (1-x) \cdot D_x^2, \tag{A15}$$

[50] Note a typo in the footnote 28 page 17 of [1] as well as in the second footnote page 18 in [2]. The sentence has been truncated. One should read: For $n = 1$, the 3×3 matrix (A10) is stochastic and transformation (A9) is a *birational transformation* if the determinant of the matrix (A10) is ± 1.

[51] We want the rational function $\tilde{\mathcal{R}} = \mathcal{R}(M(x, y, z))$ deduced from the monomial transformation (A9) to remain a rational function of *three* variables and not of two, or one, variables.

[52] Recall that taking the diagonal of a rational function of three variables extracts, in the multi-Taylor expansion, only the terms that are n-th power of the *product* $x\,y\,z$.

[53] The diagonal extracts the terms function of the product $p = x\,y\,z$ in the multi-Taylor series.

which has the hypergeometric solution:

$$
{}_2F_1\Big([\frac{1}{2}, \frac{1}{2}], [1], x\Big) \\
= (1 - x + x^2)^{-1/4} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], \frac{27}{4} \cdot \frac{x^2 \cdot (1-x)^2}{(1-x+x^2)^3}\Big). \tag{A16}
$$

The elliptic curve (A14) has the Hauptmodul

$$
\mathcal{H} = \frac{27}{4} \cdot \frac{p^2 \cdot (1-p)^2}{(1-p+p^2)^3}. \tag{A17}
$$

in agreement with the pullback in (A16).

Appendix D.1 K3 surfaces as products or foliations of two elliptic curves.

The examples of section 4 correspond to denominators which are algebraic varieties that can be seen as *Weierstrass elliptic curves* for fixed values of all the variables except two. Let us show other simple telescopers for rational functions with denominators which are *algebraic varieties with some foliation in elliptic curves*[54].

The telescoper of the rational function in *four* variables

$$
\frac{x\,y\,z}{(1+z)^2 \;\; - x \cdot (1-x) \cdot y \cdot (x-y) \cdot (y - x\,y\,z\,w)}, \tag{A18}
$$

associated with the K_3 *surface* written in a *Legendre form*[55]

$$
(1+z)^2 \;\; - x \cdot (1-x) \cdot y \cdot (x-y) \cdot (y-p) \;\; = \;\; 0, \tag{A19}
$$

is an order-three *self-adjoint*[56] linear differential operator L_3

$$
L_3 \;=\; x \cdot (2\,\theta + 1)^3 \;\; - 8 \cdot \theta^3, \tag{A20}
$$

which has the following ${}_3F_2$ solution (which is also, because of Clausen's formula, the square of a ${}_2F_1$ function):

$$
{}_3F_2\Big([\frac{1}{2}, \frac{1}{2}, \frac{1}{2}], [1, 1], x\Big) \;=\; {}_2F_1\Big([\frac{1}{4}, \frac{1}{4}], [1], x\Big)^2. \tag{A21}
$$

The K_3 surface (A19) can be seen as *associated with the product of two Weierstrass elliptic curves*[57] of Hauptmoduls respectively:

$$
\mathcal{H}_x \;=\; \frac{27}{4} \cdot \frac{p^2 \cdot (1-p)^2}{(1-p+p^2)^3}, \qquad \mathcal{H}_y \;=\; \frac{27}{4} \cdot \frac{y^2 \cdot (1-y)^2}{(1-y+y^2)^3}. \tag{A22}
$$

This order-three linear differential operator L_3 is the *symmetric square* of the order-two linear differential operator

$$
M_2 \;=\; -1 \;\; + 8 \cdot (2 - 3\,x) \cdot D_x \;\; + 16 \cdot x \cdot (1-x) \cdot D_x^2, \tag{A23}
$$

54 Like K3 surfaces, or three-fold Calabi-Yau manifolds.
55 Along this line see the first equation page 19 of [84].
56 The order-three linear differential operator is thus the symmetric square of an order-two linear differential operator.
57 K_3 surfaces *are not abelian varieties*, but they are "close" to abelian varieties: from a creative telescoping viewpoint they can be seen as essentially products of two elliptic curves.

which has the hypergeometric solutions:

$$ {}_2F_1\Big([\frac{1}{4}, \frac{1}{4}], [1], x\Big) \; = \; \Big(1 - \frac{x}{4}\Big)^{-1/4} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], -\frac{27 \cdot x^2}{(x-4)^3}\Big). \tag{A24}$$

Appendix D.2 Calabi-Yau three-fold manifolds as foliation in three elliptic curves.

The telescoper of the rational function in *five* variables x, y, z, v and w

$$ \frac{x\,y\,z\,v}{(1+w)^2 \;\; - x \cdot (1-x) \cdot y \cdot (x-y) \cdot z \cdot (y-z) \cdot (z - x\,y\,z\,v\,w)}, \tag{A25}$$

associated[58] with the *Calabi-Yau three-fold* written in a *Legendre form*

$$ (1+w)^2 \;\; - x \cdot (1-x) \cdot y \cdot (x-y) \cdot z \cdot (y-z) \cdot (z-p) \; = \; 0, \tag{A26}$$

is an order-four (self-adjoint) linear differential operator L_4

$$ L_4 \; = \; 16 \cdot \theta^4 \;\; - x \cdot (2\,\theta + 1)^4, \tag{A27}$$

which is a *Calabi-Yau operator*[59] with the ${}_4F_3$ solution:

$$ {}_4F_3\Big([\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}], [1, 1, 1], x\Big). \tag{A28}$$

For y and z fixed, the Calabi-Yau three-fold (A26) is foliated in *genus-one* curves

$$ (1+w)^2 \;\; - \lambda \cdot x \cdot (1-x) \cdot (x-y) \; = \; 0, \tag{A29}$$

where λ is the constant expression (p is fixed):

$$ \lambda \; = \; y \cdot z \cdot (y-z) \cdot (z-p). \tag{A30}$$

The Hauptmodul of these *genus-one* curves is *independent of* p and z, reading:

$$ \mathcal{H}_{y,z} \; = \; \frac{27}{4} \cdot \frac{y^2 \cdot (1-y)^2}{(1-y+y^2)^3}. \tag{A31}$$

Similarly for x and z fixed, the Calabi-Yau three-fold (A26) is foliated in *genus-one* curves

$$ (1+w)^2 \;\; - \mu \cdot y \cdot (x-y) \cdot (y-z) \; = \; 0, \tag{A32}$$

where μ is the constant expression (p is fixed):

$$ \mu \; = \; x \cdot z \cdot (1-x) \cdot (z-p). \tag{A33}$$

The *genus-one* curves (A32) can be written in a simpler Weierstrass form:

$$ (1+w)^2 \;\; - \rho \cdot Y \cdot \Big(1 - Y\Big) \cdot \Big(Y - \frac{z}{x}\Big) \; = \; 0, \tag{A34}$$

where the constant ρ reads $\rho = \mu \cdot x^3$, and the variable y has been rescaled into $Y = y/x$. The Hauptmodul of these *genus-one* curves (A32) is the same as the Hauptmodul of the

58 The diagonal extracts the terms function of the product $p = x\,y\,z\,v\,w$ in the multi-Taylor series.

59 This linear differential operator is self-adjoint, its exterior square is of order five, it is MUM (maximum unipotent monodromy [27,66,67]), ...

genus-one curves (A29), and corresponds to expression (A31) where y has been changed into z/x (see the canonical form (A34)), namely:

$$\mathcal{H}_{x,z} \;=\; \frac{27}{4} \cdot \frac{x^2 \cdot z^2 \cdot (x-z)^2}{(x^2 - xz + z^2)^3}. \tag{A35}$$

Similarly for x and y fixed, the Calabi-Yau three-fold (A26) is foliated in *genus-one* curves,

$$(1+w)^2 \;-\nu \cdot z \cdot (y-z) \cdot (z-p) \;=\; 0, \tag{A36}$$

where ν reads:

$$\nu \;=\; x \cdot (1-x) \cdot y \cdot (x-y). \tag{A37}$$

A reduction to a canonical Weierstrass form similar to (A34) gives immediately the Hauptmodul of the *genus-one* curve (A36) which reads:

$$\mathcal{H}_{x,y} \;=\; \frac{27}{4} \cdot \frac{y^2 \cdot p^2 \cdot (y-p)^2}{(y^2 - yp + p^2)^3}. \tag{A38}$$

The *Calabi-Yau three-fold* (A26) thus has a foliation in a triple of elliptic curves $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{E}_3$.

Appendix E Rational functions with tri-quadratic and N-quadratic denominators.

Appendix E.1 Rational functions with tri-quadratic denominators.

Let us consider the most general *tri-quadratic surface*

$$\sum_{m=0,1,2} \sum_{n=0,1,2} \sum_{l=0,1,2} a_{m,n,l} \cdot x^m y^n z^l \;=\; 0, \tag{A39}$$

depending on $27 = 3^3$ parameters $a_{m,n,l}$. It can be rewritten as:

$$A(x,y) \cdot z^2 \; + B(x,y) \cdot z \; + C(x,y) \;=\; 0. \tag{A40}$$

It is straightforward to see that condition (A40) is preserved by the *birational involution* I_z

$$I_z : \qquad \Big(x,\, y,\, z\Big) \quad \longrightarrow \quad \Big(x,\, y,\, \frac{C(x,y)}{A(x,y)} \cdot \frac{1}{z}\Big), \tag{A41}$$

and we have of course two other similar *birational involutions* I_x and I_y that single out x and y respectively. The (generically) *infinite-order* birational transformations $K_x = I_y \cdot I_z$, $K_y = I_z \cdot I_x$ and $K_z = I_x \cdot I_y$ are birational symmetries of the surface (A39) or (A40). They are related by $K_x \cdot K_y \cdot K_z = \mathit{identity}$. Note that the *birational transformation* K_x *preserves* x. The iteration of the (generically) *infinite-order* birational transformation K_x gives *elliptic curves*. Since equation (A39) or (A40) is preserved by K_x, *which also preserves* x, the equation of the *elliptic curves* corresponding to the iteration[60] of K_x is actually (A39) *for fixed values of* x. Equation (A39), for fixed values of x, is a (general) biquadratic curve in y and z and is thus *an elliptic curve depending on* x. Therefore one has a canonical foliation of the algebraic surface (A39) in elliptic curves. Of course the iteration of K_y (resp. K_z) also yields elliptic curves, and similarly yields two other foliations in elliptic curves.

We have a foliation in two families of elliptic curves $\mathcal{E}$ and $\mathcal{E}'$ of the surface. Consequently, this tri-quadratic surface (A39), having an *infinite set* of *birational automorphisms*, an

[60] The birational transformation K_x maps the elliptic curve onto itself (self-map). One can use the iteration of the birational transformation K_x to actually visualise the elliptic curve [45,85].

infinite set of *birational symmetries*, cannot be of the "general type" (it has Kodaira dimension less than 2).

Appendix E.2 Rational functions with N-quadratic denominators.

The calculations of E.1 can straightforwardly be generalised to N-quadratic equations, writing the N-quadratic (72) as

$$A(x_1, x_2, \cdots, x_{N-1}) \cdot x_N^2 \quad + B(x_1, x_2, \cdots, x_{N-1}) \cdot x_N \\ +C(x_1, x_2, \cdots, x_{N-1}) \quad = \quad 0, \tag{A42}$$

and introducing the *birational involution* I_N

$$I_N : \qquad \Big(x_1, x_2, \cdots, x_N\Big) \\ \longrightarrow \qquad \Big(x_1, x_2, \cdots, x_{N-1}, \; \frac{C(x_1, x_2, \cdots, x_{N-1})}{A(x_1, x_2, \cdots x_{N-1})} \cdot \frac{1}{x_N}\Big). \tag{A43}$$

Similarly to E.1, we can introduce N involutive birational transformations I_m and consider the products of two such involutive birational transformations $K_{m,n} = I_m \cdot I_n$. These $K_{m,n}$'s are (generically) *infinite order birational transformations* preserving the $N-2$ variables that are not x_m and x_n.

Using such remarkable N variables algebraic varieties, with an *infinite set of birational automorphisms*, one can build rational functions of $N+1$ variables, any of the parameter of the algebraic variety, becoming an arbitrary rational[61] function of the product $p = x_1 x_2 \cdots x_N$ in order to build the denominator of the rational function. The telescopers of such rational functions are seen (experimentally using creative telescoping) *to be of substantially smaller order* than the ones for rational functions where their denominators are, after reduction by $p = x_1 x_2 \cdots x_N$, associated with algebraic varieties of the "general type".

Appendix F Telescopers of rational functions of several variables: some examples

Let us consider here the following family of rational functions in four variables

$$R(x, y, z, u) \quad = \\ \frac{1}{1 \;\; +3y \;\; +z \;\; +9yz \;\; +11z^2y \;\; +3ux \;\; +x \cdot P(x, y, z)}, \tag{A44}$$

where $P(x, y, z)$ is an *arbitrary polynomial* of the three variables x, y and z.

Appendix F.1 Telescopers of rational functions of several variables: a second example with four variables

Let us now consider the rational function in *four* variables x, y, z, u:

$$R(x, y, z, u) \quad = \\ \frac{1}{1 \;\; +3y \;\; +z \;\; +9yz \;\; +11z^2y +3ux \;\; +9x +2xy \;\; +5xz \;\; +7x^2y}. \tag{A45}$$

which corresponds to $P(x, y, z) = 9 + 2y + 5z + 7xy$. The telescoper of this rational function of four variables is the *same order-two linear differential operator* L_2 as for the telescoper of (91). It has the same pullbacked hypergeometric solution (92). The diagonal of the rational function (A45) is the expansion of (92), namely (93).

---

[61] Or even an arbitrary algebraic function of the product $p = x_1 x_2 \cdots x_N$, with a Taylor series expansion at $p = 0$, the diagonal of rational functions becoming diagonal of algebraic functions.

Performing the intersection of the codimension-one algebraic variety

$$1 \, + 3\, y \, + z \, \, + 9\, y\, z \, \, + 11\, z^2\, y \, + 3\, u\, x \, \, \, + 9\, x \, \, + 2\, x\, y \, + 5\, x\, z \, + 7\, x^2\, y \, \, = \, \, 0,$$

corresponding to the denominator of (A45), with the hyperbola $p \, = \, x\, y\, z\, u$ amounts to eliminating, for instance u (writing $u \, = \, \frac{p}{x\, y\, z}$). This gives $P_u \, = \, 0$ where P_u reads:

$$P_u \, \, = \, \, 7\, x^2 y^2 z \, + 2\, xy^2 z + 5\, xyz^2 + 9\, xyz \, \, + 11\, y^2 z^3 \, \, \, + 9\, y^2 z^2 \\ +3\, y^2 z \, + yz^2 \, + yz \, \, + 3\, p. \tag{A46}$$

Assuming x to be constant the previous condition $P_u(y, \, z) \, = \, 0$ is an algebraic curve. Calculating its genus, one finds immediately that it is *genus-one*. Calculating its j-invariant, one deduces the expression of the Hauptmodul $H_{p,x} \, = \, \frac{1728}{J}$ as a rational expression of p and x:

$$H_{p,x} \, \, = \, \, \frac{1728}{J} \, \, = \, \, - \, \frac{46656\, p^3 \cdot \, (7x^2 + 2x + 3)^2 \cdot N}{D^3}, \tag{A47}$$

where N is a polynomial expression of degree eight in w and three in p, and D is a polynomial expression of degree four in w and two in p. In the $x \, \rightarrow \, 0$ limit of the Hauptmodul $H_{p,x} \, = \, \frac{1728}{J}$, one finds:

$$H_p \, \, = \, \, - \, \frac{419904 \cdot \, p^3 \cdot \, (5 \, - 12\, p \, - 19440\, p^2 \, + 2665872\, p^3)}{(1 \, - \, 2592\, p^2)^3}, \tag{A48}$$

which is actually the Hauptmodul in (92). In other words, the exact expression of the diagonal of the rational function (A45), which is (92), and is essentially encapsulated in the Hauptmodul in (92), could have been obtained from the $x \, = \, 0$ selection of the Hauptmoduls $H_{p,x}$.

Appendix F.2 Telescopers of rational functions of several variables: a third example with four variables

Let us consider the rational function in *four* variables $x, \, y, \, z, \, u$:

$$R(x, \, y, \, z, \, u) \, \, = \, \, \frac{1}{1 \, \, + 3\, y \, \, + z \, \, \, + 9\, y\, z \, \, \, + 11\, z^2\, y \, \, + 3\, u\, x \, \, \, + x \cdot \, (y^2\, z^2 \, + \, x\, y^3)}, \tag{A49}$$

which corresponds to $P(x,y,z) \, = \, y^2\, z^2 \, + \, x\, y^3$ in the family (A44). Again, the telescoper of this rational function of *four variables* is the *same order-two linear differential operator* L_2 as for the telescoper of (91). It has the same pullbacked hypergeometric solution (92). Actually the diagonal of the rational function (91) is the expansion (93) of the pullbacked hypergeometric function (92). In this case (A49), the elimination of $u \, = \, \frac{p}{x\, y\, z}$ in the vanishing condition of the denominator (A49) gives the algebraic curve:

$$x^2\, y^4\, z \, \, + x\, y^3\, z^3 \, \, \, + 11\, y^2\, z^3 \, \, + 9\, y^2\, z^2 \, + 3\, y^2\, z \, \, + y\, z^2 \, \, \, + y\, z \, \, + 3\, p \, \, = \, \, 0. \tag{A50}$$

For x fixed (and of course p fixed) this algebraic curve (A50) is a *genus-five* curve, but, of course, in the $x \, = \, 0$ case it reduces to the *same genus-one* curve as for the first example (91), namely:

$$11\, y^2\, z^3 \, \, + 9\, y^2\, z^2 \, \, + 3\, y^2\, z \, \, + y\, z^2 \, \, \, + y\, z \, \, + 3\, p \, \, \, = \, \, \, 0. \tag{A51}$$

which corresponds to the Hauptmodul (A48).

The generalisation of this result is straightforward. Let us consider the rational function in *four variables* x, y, z and u

$$R(x, y, z, u) \;=\; \frac{1}{1 \;+3y \;+z \;+9yz \;+11z^2y \;+3ux \;+x\cdot P(x, y, z)}, \tag{A52}$$

where $P(x, y, z)$ is an *arbitrary polynomial* of the three variables x, y and z. On a large set of examples one verifies that the *diagonal* of (A52) is actually the expansion (93) of the pullbacked hypergeometric function (92):

$$1 \;+648x^2 \;-72900x^3 \;+1224720x^4 \;-330674400x^5 \;+23370413220x^6 \\ -1276733858400x^7 \;+180019474034400x^8 \;-12013427240614800x^9 \;+\;\cdots \tag{A53}$$

However, as far as creative telescoping calculations are concerned[62], the telescoper corresponding to different polynomials $P(x, y, z)$ becomes quickly a quite large *non-minimal* linear differential operator. For instance, even for the simple polynomial $P(x, y, z) = x + y$, one obtains a quite large *order-ten* telescoper. Of course, since this telescoper has the pullbacked hypergeometric function (92) as a solution, it is not minimal, it is rightdivisible by the order-two linear differential operator having (92) as a solution. It is straightforward to see that the previous elimination of $u = \frac{p}{xyz}$ in the vanishing condition of the denominator (A52) gives an algebraic curve[63]

$$11y^2z^3 \;+9y^2z^2 \;+3y^2z \;+yz^2 \;+yz \;+3p \;+yz\cdot P(x, y, z) = \;0. \tag{A54}$$

which reduces again, in the $x = 0$ case, to the *same genus-one* curve (A51).

With that general example (A52) we see that there is *an infinite set of rational functions depending on an arbitrary polynomial* $P(x, y, z)$ *of three variables* whose diagonals are actually a pullbacked ${}_2F_1$ hypergeometric solution, namely (92).

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