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Factorization of Ising correlations $C(M, N)$ for $\nu = -k$ and $M + N$ odd, $M \leq N$, $T < T_c$ and their lambda extensions

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Abstract.
We study the factorizations of Ising low-temperature correlations $C(M, N)$ for $\nu = -k$ and $M + N$ odd, $M \leq N$, for both the cases $M \neq 0$ where there are two factors, and $M = 0$ where there are four factors. We find that the two factors for $M \neq 0$ satisfy the same non-linear differential equation and, similarly, for $M = 0$ the four factors each satisfy Okamoto sigma-form of Painlevé VI equations with the same Okamoto parameters. Using a Landen transformation we show, for $M \neq 0$, that the previous non-linear differential equation can actually be reduced to an Okamoto sigma-form of Painlevé VI equation. For both the two and four factor case, we find that there is a one parameter family of boundary conditions on the Okamoto sigma-form of Painlevé VI equations which generalizes the factorization of the correlations $C(M, N)$ to an additive decomposition of the corresponding sigma's solutions of the Okamoto sigma-form of Painlevé VI equation which we call lambda extensions. At a special value of the parameter, the lambda-extensions of the factors of $C(M, N)$ reduce to homogeneous polynomials in the complete elliptic functions of the first and second kind. We also generalize some Tracy-Widom (Painlevé V) relations between the sum and difference of sigma's to this Painlevé VI framework.

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1. Introduction

In a previous paper [1] we considered the two-point correlation $C(M,N)$ of spins at sites $(0,0)$ and (M,N) of the anisotropic Ising model defined by the interaction energy

$$\mathcal{E} \;=\; -\sum_{j,k}\{E_v\sigma_{j,k}\sigma_{j+1,k} \,+E_h\sigma_{j,k}\sigma_{j,k+1}\}, \tag{1}$$

where $\sigma_{j,k} \,=\, \pm 1$ is the spin at row j and column k, and where the sum is over all lattice sites. Defining

$$k \;=\; (\sinh 2E_v/k_BT\, \sinh 2E_h/k_BT)^{-1} \qquad \text{and} \qquad \nu \;=\; \frac{\sinh 2E_h/k_BT}{\sinh 2E_v/k_BT}, \tag{2}$$

we found [1] that in the special case¶

$$\nu \;=\; -k, \tag{3}$$

the correlation† $C(M,N)$ satisfies an Okamoto sigma-form of the Painlevé VI equation.

For $T < T_c$, $M \le N$ and $\nu = -k$ with $t = k^2$ and

$$\sigma \;=\; t\cdot(t-1)\cdot \frac{d\ln C(M,N)}{dt} \;-\frac{t}{4}, \tag{4}$$

we have [1]:

$$\begin{aligned} & t^2\cdot(t-1)^2\cdot\sigma''^2 \;\; +4\cdot\sigma'\cdot(t\,\sigma' -\sigma)\cdot\Big((t-1)\cdot\sigma' -\sigma\Big) \\ & \quad -M^2\cdot(t\,\sigma'-\sigma)^2 \;\; -N^2\cdot\sigma'^2 \\ & \quad +\Big(M^2+N^2-\frac{1}{2}\cdot(1+(-1)^{M+N})\Big)\cdot\sigma'\cdot(t\,\sigma'-\sigma) \;=\; 0. \end{aligned} \tag{5}$$

When $M+N$ is odd, $M \le N$, the previous Okamoto sigma-form of the Painlevé VI equation (5) becomes:

$$\begin{aligned} & t^2\cdot(t-1)^2\cdot\sigma''^2 \;\; +4\cdot\sigma'\cdot(t\,\sigma' -\sigma)\cdot\Big((t-1)\cdot\sigma' -\sigma\Big) \\ & \quad -M^2\cdot(t\,\sigma'-\sigma)^2 \;\; -N^2\cdot\sigma'^2 \;+(M^2+N^2)\cdot\sigma'\cdot(t\,\sigma'-\sigma) \;=\; 0. \end{aligned} \tag{6}$$

We noted [1], when $M+N$ is odd, that the *low-temperature* correlation $C(M,N)$ factors into two terms. In the even more special case of $M = 0$ and N odd, the previous sigma-form of Painlevé VI non-linear ODE (6) reads

$$\begin{aligned} & t^2\cdot(t-1)^2\cdot\sigma''^2 \;\; +4\cdot\sigma'\cdot(t\,\sigma' -\sigma)\cdot\Big((t-1)\cdot\sigma' -\sigma\Big) \\ & \quad -N^2\cdot\sigma'^2 \;+N^2\cdot\sigma'\cdot(t\,\sigma'-\sigma) \;=\; 0, \end{aligned} \tag{7}$$

and the *low-temperature* correlation $C(M,N)$ factors into *four* terms‡, each of which were shown to satisfy sigma-form Painlevé VI non-linear differential equations with the same Okamoto parameters♯.

¶ The condition $\nu \;=\; -k$ (as well as the isotropic case $\nu \;=\; 1$) is special because it is such that the complete elliptic integrals of the third kind reduce to complete elliptic integrals of the second kind (see equation (30) in [1]).

† Which is the same as the Toeplitz determinants [3] of Forrester-Witte [2] as given in [4].

‡ Homogeneous polynomial in the complete elliptic integrals of the first and second kind.

♯ But with different boundary conditions.

In [1] we gave the four Okamoto parameters for Okamoto sigma-form of Painlevé VI equations which are satisfied by each of the four factors. In this paper we continue the study of [1] and find the second-order non-linear differential equation for the sigma's of the two factors of $C(M,N)$ with $M+N$ odd and $M \neq 0$. We study the necessary boundary conditions for both the two and four factor decompositions and show that the factors of $C(M,N)$ can be generalized to a *one-parameter* family of solutions of the Okamoto sigma-form of Painlevé VI equation analytic at $t = k^2 = 0$. In the remainder of this introduction we outline the methods and results of this study.

1.1. *Outline of the methods and the results*

We begin by recalling that in [1] we showed, for $M+N$ odd, $M \le N$, that the representation of $C(M,N)$ for $k = -\nu$ as a Toeplitz determinant is symmetric. In [5] it is shown, by elementary row column operations, that any $N \times N$ symmetric Toeplitz determinant $\det(a_{i-j})$ with $a_j = a_{-j}$ has a factorization for N even ($N = 2m$) into two $m \times m$ determinants:

$$\det(a_{i-j})_{i,j=1,\cdots,2m} \\ = \det(a_{i-j} - a_{i+j-1})_{i,j=1,\cdots,m} \cdot \det(a_{i-j} + a_{i+j-1})_{i,j=1,\cdots,m}. \qquad (8)$$

This can be extended to N odd ($N = 2m+1$) as a factorization into an $m \times m$ determinant and a $(m+1) \times (m+1)$ determinant:

$$\det(a_{i-j})_{i,j=1,\cdots,2m+1} \\ = \frac{1}{2} \cdot \det(a_{i-j} - a_{i+j})_{i,j=1,\cdots,m} \cdot \det(a_{i-j} + a_{i+j-2})_{i,j=1,\cdots,m+1}. \qquad (9)$$

Thus the existence of factorizations of the $C(M,N)$'s into two factors is not surprising.

To obtain explicit expressions for the factors we use the method discussed in [1] expressing $C(M,N)$ as *homogeneous* polynomials in terms of the complete elliptic integrals of the first and second kind

$$\tilde{K}(k) = \frac{2}{\pi} \cdot K(k) = {}_2F_1\Big([\frac{1}{2}, \frac{1}{2}], [1], k^2\Big),$$
$$\tilde{E}(k) = \frac{2}{\pi} \cdot E(k) = {}_2F_1\Big(\frac{1}{2}, -\frac{1}{2}], [1], k^2\Big), \qquad (10)$$

by first solving the *quadratic difference*† *equations* [7, 8, 9, 10], the $C(M,\, N)$'s being then factored. We will write the factorizations of $C(M,N)$ as

$$(1-t)^{-1/4} \cdot C(M,N;t) = g_+(M,N;t) \cdot g_-(M,N;t), \qquad (11)$$

with

$$t = k^2, \qquad (12)$$

where the two factors $g_\pm$ are homogeneous polynomials of the complete elliptic integrals of the first and second kind and have the expansion about $t = 0$

$$g_\pm(M,N;t) = 1 \pm t^{(N+1)/2} \cdot f_1(t) + t^{N+2} \cdot f_2(t), \qquad (13)$$

where both $f_1(t)$ and $f_2(t)$ are analytic at $t = 0$. Thus from (11) we have:

$$C(M,N;t) = (1-t)^{1/4} \cdot \Big(-t^{N+1} \cdot f_1^2 + \Big(1 + t^{N+2} \cdot f_2\Big)^2\Big). \qquad (14)$$

† See for instance equations (41) and (42) in [6]. Note that these quadratic difference equations [7, 8, 9, 10] are actually valid for the *anisotropic* Ising model. Do note that the *lambda-extensions* [11] of the $C(M,\, N)$ correlation functions *also verify these quadratic difference equations*.

Examples of the factorizations (11), and of the expansions (13), are given in Appendix A.

We consider the following logarithmic derivatives of the previous two factors:

$$\sigma_\pm(M,N;t) \;=\; t\cdot(t-1)\cdot\frac{d\ln g_\pm(M,N;t)}{dt}. \qquad (15)$$

The sigma functions have *additive* decompositions which follow from the mulplicative decompositions (11)

$$\sigma(M,N;t) \;=\; \sigma_+(M,N;t)\; + \sigma_-(M,N;t), \qquad (16)$$

where $\sigma(M,N;t)$ is defined by (4) and

$$\sigma_\pm(M,N;t) \;=\; \pm\, t^{(N+1)/2}\cdot\rho_1(M,N;t)\; + t^{N+1}\cdot\rho_2(M,N;t), \qquad (17)$$

where ρ_1 and ρ_2 (related to f_1 and f_2 by equation (14)) are power series‡ of t, analytic at $t=0$. Examples are given in Appendix B.

In [1] we found in Appendix D 2.1, in the special case $k=-\nu$, that the sigma-form of Painlevé VI for the sigma function of $C(M,N)$ *admits extensions*¶ *to a one parameter family of solutions* which are analytic at $t=0$. This *one-parameter* family of solutions analytic at $t=0$ extends to the additive decomposition (16) as

$$\sigma(M,N;t;\lambda) \;=\; \sigma_+(M,N;t;\lambda_+)\; + \sigma_-(M,N;t;\lambda_-), \qquad (18)$$

where

$$\sigma_\pm(M,N;t;\,\lambda_\pm) \;=\; \sum_{n=1}^{\infty}\Bigl(\lambda_\pm\cdot t^{(N+1)/2}\Bigr)^n\cdot B_n(M,N;t), \qquad (19)$$

where the $B_n(M,N;t)$'s are power series† analytic at $t=0$, and where we must choose (see (52) below) $\lambda_+ = -\lambda_- = \lambda$ in order to match with the *lambda extension* [11] solutions of (6).

In [1] the second order non-linear differential equations (6), (7) were found to be of the "master Painlevé equation" form (see the so-called SD-I equation (4.9) with $c_1=0,\ c_4=0,\ c_3=-c_2$, in Cosgrove and Scoufis [12])

$$\begin{aligned} x^2\cdot(x-1)^2\cdot y''^2\; &+4\cdot y'\cdot(x\,y'-y)\cdot((x-1)\,y'-y)\\ &+c_5\cdot(x\,y'-y)^2\;+c_6\cdot y'\cdot(xy'-y)\;+c_7\cdot(y')^2\\ &+c_8\cdot(x\,y'-y)\;+c_9\cdot y'\;+c_{10}\;=\;0, \end{aligned} \qquad (20)$$

which has the *Painlevé property* of *fixed critical points* [13, 14]. The non-linear differential equation (20) preserves its form under the linear shift:

$$y \quad\longrightarrow\quad y\;+A\;+B\cdot x. \qquad (21)$$

This shift may be used to eliminate c_5 and c_6 which reduces (20) to the canonical form of the sigma-form of Painlevé VI equation obtained by Okamoto [15] with $c_5=c_6=0$ which is birationally equivalent to the original Gambier form of Painlevé VI:

$$\begin{aligned} \frac{d^2y}{dt^2} \;=\; &\frac{1}{2}\cdot\Bigl(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-t}\Bigr)\cdot\Bigl(\frac{dy}{dt}\Bigr)^2 - \Bigl(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{y-t}\Bigr)\cdot\frac{dy}{dt}\\ &+\frac{y\cdot(y-1)\cdot(y-t)}{t^2\cdot(t-1)^2}\cdot\Bigl(\alpha\;+\beta\cdot\frac{t}{y^2}\;+\gamma\cdot\frac{t-1}{(y-1)^2}\;+\delta\cdot\frac{t\cdot(t-1)}{(y-t)^2}\Bigr). \end{aligned} \qquad (22)$$

‡ From (16) it is clear that $2\cdot t^{N+1}\cdot\rho_2(M,N;t)$ is the sigma function for $C(M,N;t)$.

¶ For an introduction of the concept of *lambda extension* of correlation functions see for instance equations (9), (10) in [11].

† The $B_n(M,N;t)$'s are D-finite series and, in fact, polynomials in the complete elliptic integrals of the first and second kind $\tilde{K}$ and $\tilde{E}$ (see (54), (55) below).

In section 2 we will obtain non-linear differential equations for $\sigma_\pm(M, N; t)$ by using the method of [1] of expanding the factors as power series in k (or $t = k^2$), and then using Jay Pantone's program *guessfunc* [16] to produce a *non-linear differential equation quadratic in the second derivative.* These non-linear differential equations for $\sigma_\pm(M, N; t)$ are *not of the "master Painlevé equation" form* (20). We will show that they can be (non-trivially) reduced to the form (20), or to the Okamoto sigma-form of Painlevé VI, by introducing a *Landen transformation* [17]. In section 2.3 we find the selected values of λ for which $g_\pm(M, N; t; \lambda)$ reduce to homogeneous polynomials in the complete elliptic integrals of the first and second kind $\tilde{K}(k)$ and $\tilde{E}(k)$. In section 3, recalling the Tracy and Widom paper [18] we introduce, besides the sum (16), the *difference* of the two $\sigma_\pm$, and find the second order non-linear ODE satisfied by this difference.

In section 4 we recall *the Okamoto sigma form of Painlevé VI equations* [1] (having the *same* Okamoto parameters) satisfied by the *four* sigma's corresponding to the four factors of $C(0, N; t)$ with N odd. This allows us to write $\sigma(M, N; t)$ *as the sum of four sigma's.* We find the boundary conditions needed to generalize this additivity relation to one-parameter lambda-extensions of these sigma's. We also find the selected values of the lambda parameters such that the four factors of $C(0, N; t)$ are (homogeneous) polynomial expressions of the complete elliptic integrals. In section 5 we also introduce the *difference* of two sigma's among the four. We find that the second order non-linear ODE, satisfied by this difference, is compatible with the second order non-linear ODE satisfied by the sum of these two sigma's. We also show that the situation, where the four factors of $C(0, N; t)$ are actually homogeneous polynomial expressions of the complete elliptic integrals $\tilde{K}(k)$ and $\tilde{E}(k)$, associated with the previous selected values of the lambda parameters, corresponds, in fact, to the existence of a polynomial relation, $\mathcal{P}_N(\sigma, \sigma', t) = 0$, compatible with the second order non-linear ODE. We finally present, in section 6, a discussion of the Forrester-Witte determinants of [2] and [4] and of the boundary conditions on the Okamoto non-linear differential equation needed to specify these determinants for the factors of $C(M, N)$ when $(M + N)$ is odd.

2. Non-linear equation for the two factors of $C(M, N)$ with $M + N$ odd, $M \le N$

The earliest study of factorizations of Painlevé τ-functions [19] was made by Tracy and Widom [18] in the context of random matrix theory and Painlevé V representation of *Fredholm determinants.*

Here we begin with the factorizations (11) of $C(M, N)$'s with $M + N$ odd, $M \le N$, for miscellaneous values of M and N, and, by use of the methods described in [1] and of the program *guessfunc* of Jay Pantone [16], we find that both $\sigma_+(M, N; t)$ and $\sigma_-(M, N; t)$ in (16) satisfy the *same second-order non-linear differential equation*

$$32\, t^3 \cdot (t-1)^2 \cdot \sigma''^2 \, + 4\, t^2 \cdot (t-1) \cdot \Big(8 \cdot \sigma \, - 8 \cdot (t+1) \cdot \sigma' \, + M^2 - N^2\Big) \cdot \sigma''$$
$$- \Big(8\,\sigma \, - 16 \cdot t\,\sigma' \, + M^2\, t - N^2 \, + 1 \, - t\Big) \cdot \Big(8 \cdot t \cdot (t-1) \cdot \sigma'^2 \, - 16\, t \cdot \sigma \cdot \sigma'$$
$$+ 8 \cdot \sigma^2 \, + (M^2 - N^2) \cdot \sigma\Big) \; = \; 0, \qquad (23)$$

where the prime indicates a derivative with respect to t, and where σ reads:

$$\sigma \;=\; t \cdot (t-1) \cdot \frac{d \ln g}{dt}. \tag{24}$$

The two solutions of (23), $\sigma_+(M,N;t)$ and $\sigma_-(M,N;t)$, have different boundary conditions. Note that $\sigma_\pm \,=\, 0$ *is a selected solution* of (23).

Similar to [1], these non-linear differential equations are obtained for particular values of M and N, when restricted to order three derivatives and, then, finding a first integral to obtain a non-linear differential equation quadratic in the second derivative. For small values of M and N one may get several (compatible) non-linear differential equations, however with larger values of M and N one gets a cleaner situation with a unique and *stable* form corresponding to the previous pattern (23). Note that this form (23) is actually valid for the very small values of M and N when other compatible non-linear differential equations also occur.

The second order non-linear differential equation (23) is *not* of the SD-I "master Painlevé equation" form (20) given in [12]. On the contrary (23) is, at first sight, of the general form studied by Bureau [20, 21]

$$y''^2 \;=\; E(x,\, y,\, y') \cdot y'' \;+ F(x,\, y,\, y'), \tag{25}$$

having the interesting feature that *movable essential singularities* and *movable natural boundaries* are known to be possible. Consequently, it is not garanteed that the non-linear differential equation (23) can be simply reduced to a sigma-form of Painlevé VI or the SD-I "master Painlevé equation" form†.

In the present case (23) this reduction can actually be carried out by making the (Landen [17]) substitution

$$k^2 \;=\; t \;=\; \Big(\frac{1 - \sqrt{1-x}}{1 + \sqrt{1-x}} \Big)^2, \tag{26}$$

which is the (compositional) inverse of

$$x \;=\; \frac{4k}{(1+k)^2}, \qquad \text{where:} \quad x \;=\; k_L^2 \qquad \text{with:} \qquad k_L \;=\; \frac{2\sqrt{k}}{1+k}, \tag{27}$$

together with $\sigma(t) \,=\, \tilde{\sigma}(x)$ given by:

$$\sigma(t) \;=\; \tilde{\sigma}(x)$$
$$= \; \frac{2}{x} \cdot \frac{1 \,-\sqrt{1-x}}{1 \,+\sqrt{1-x}} \cdot \Big(h(x) \;\; - \frac{M^2 - 3N^2 + 1}{16} \;\; + \frac{M^2 - N^2 + 1}{16} \cdot x$$
$$+ \; \frac{M^2 - N^2}{16} \cdot x \cdot \Big(\frac{1 - \sqrt{1-x}}{1 + \sqrt{1-x}} \Big) \Big). \tag{28}$$

With the previous change of variable (27) and function transformation (28), $h(x)$ *satisfies the Okamoto non-linear differential equation*

$$x^2 \cdot (x-1)^2 \cdot h''^2 \;\; + 4\, h' \cdot (x\, h' \,- h) \cdot ((x-1) \cdot h' \,- h)$$
$$+ \, c_7 \cdot h'^2 \;\; + c_8 \cdot (x\, h' \,- h) \;\; + c_9 \cdot h' \;\; + c_{10} \;\;=\;\; 0. \tag{29}$$

where the prime now indicates a derivative with respect to x and where the c_n's read

$$c_7 \;=\; -(n_1^2 \,+ n_2^2 \,+ n_3^2 \,+ n_4^2), \qquad\qquad c_8 \;=\; -4\, n_1 n_2 n_3 n_4,$$
$$c_9 \;=\; -(n_1^2 n_2^2 \,+ n_1^2 n_3^2 \,+ n_1^2 n_4^2 \,+ n_2^2 n_3^2 \,+ n_2^2 n_4^2 \,+ n_3^2 n_4^2 \,- 2\, n_1 n_2 n_3 n_4),$$
$$c_{10} \;=\; -(n_1^2 n_2^2 n_3^2 \,+ n_1^2 n_2^2 n_4^2 \,+ n_1^2 n_3^3 n_4^2 +\, n_2^2 n_3^2 n_4^2), \tag{30}$$

† And if this is the case, one expects quite unpleasant Bäcklund correspondences, like (5.19) in [12], to take place.

the four Okamoto parameters being (unique up to permutations and sign changes of any pair of n_k):

$$n_1 = \frac{M+N+1}{4}, \quad n_2 = \frac{M+N-1}{4}, \quad n_3 = \frac{N-M+1}{4}, \quad n_4 = \frac{N-M-1}{4}. \qquad (31)$$

The previous Okamoto non-linear differential equation (29) can be rewritten:

$$\begin{aligned} x^2 \cdot (x-1)^2 \cdot h''^2 \ \ & + 4\, h' \cdot (x\, h' \, - h) \cdot ((x-1) \cdot h' \, - h) \\ & - \frac{(M^2+N^2+1)}{4} \cdot h'^2 \ \ - \frac{(M^2+N^2+1)^2}{64} \cdot h' \\ & - \frac{(M+N+1) \cdot (M+N-1) \cdot (M-N+1) \cdot (M-N-1)}{64} \cdot (x\, h' \, - h) \\ - \frac{(M^6 - M^4\, N^2 - M^2\, N^4 + N^6 \, - M^4 + 10\, M^2\, N^2 - N^4 \, - M^2 - N^2 \, + 1)}{1024} & \ = \ 0. \end{aligned} \qquad (32)$$

Such a reduction of the non-linear differential equation (23) to the Okamoto sigma-form of Painlevé VI (32), is illustrated in Appendix C on their respective solutions in t and x, associated with the two factors of the low-temperature correlation function $C(2,\, 3)$.

2.1. A few remarks on the previous substitutions (26) and (28)

Even if the sum (16) (see also (11), (15), (33))

$$\sigma(M, N; t) \ = \ \sigma_+(M, N; t) \ + \sigma_-(M, N; t), \qquad (33)$$

satisfies the Okamoto sigma form of Painlevé VI equation (6), and the two $\sigma_\pm(M, N; t)$, in the right-hand-side of (33), verify another non-linear differential equation (23), this is *far from sufficient* to show that (23) has the *Painlevé property*, namely having *fixed critical points* [13, 14]. To prove that a non-linear differential equation like (23) actually has fixed critical points remains a quite technical proof. We have actually achieved such a demonstration, but it is too cumbersome to be given here. Clearly the simplest way to show that (23) has the Painlevé property amounts to reducing (23) to a sigma form of Painlevé VI equation, finding the change of variables (26) and (28) to perform this reduction. The non-linear differential equation (23) is clearly different of an Okamoto form (29) *because of the presence of a term in* σ'' next to the term in σ''^2: in contrast the Okamoto forms (29) have *no term* in h'' next to the term in h''^2.

Finding the well-suited transformations (26) and (28) is, however, far from being obvious. Recalling transformation (28) let us first note that transformations of the general form

$$\sigma(t) \ = \ \ \alpha(t) \cdot h(t) \ \ + \beta(t), \qquad (34)$$

where $\alpha(t)$ and $\beta(t)$ are some functions to be found, *are not sufficient* to reduce (23) into an Okamoto form (29), and *not even sufficient to get rid of the term in* h'' next to the term in h''^2.

The (Landen [17]) change of variable (26) *is in fact crucial to achieve that goal.* Once one has discovered this key change of variable (26) one can, for instance, seek for transformations of the form

$$\sigma(t) \ = \ \alpha(x) \cdot \Big(h(x) \ + \beta(x) \Big), \qquad (35)$$

where $\alpha(x)$ and $\beta(x)$ are arbitrary functions, such that one gets *no term* in h'' next to the term in h''^2, which is a first *necessary condition* to be an Okamoto sigma form of Painlevé VI. One first finds that $\alpha(x)$ must necessarily be a solution of the following linear differential equation

$$\Big(\sqrt{1-x}\, + 1 - x\Big) \cdot \alpha(x) \, + (x\, - 1) \cdot \Big(2 \cdot \sqrt{1-x}\, + 2\, - x\Big) \cdot {{d\alpha(x)} \over {dx}} \,\, = \,\, 0, \quad (36)$$

which has the following solution

$$\alpha(x) \,\, = \,\, {{\rho} \over {(1+ \sqrt{1-x})^2}} \,\, = \,\, {{\rho} \over {x}} \cdot {{1 - \sqrt{1-x}} \over {1 + \sqrt{1-x}}} \,\, = \,\, {{\rho} \over {x^2}} \cdot \Big(2\, - x\, - 2 \cdot \sqrt{1-x}\Big), \quad (37)$$

where ρ is an arbitrary constant. One finds another second order linear ODE which also has (37) as a solution, and another linear ODE, in $\alpha(x)$ and $\beta(x)$, of the form (the a_n's are simple algebraic expressions of x with $\sqrt{1\, - x}$):

$$a_0(x) \, + \Big(a_1(x) \cdot \alpha(x) \, + a_2(x) \cdot {{d\alpha(x)} \over {dx}} \, + a_3(x) \cdot {{d^2\alpha(x)} \over {dx^2}}\Big) \cdot \beta(x)$$
$$+ \Big(a_4(x) \cdot \alpha(x) \, + a_5(x) \cdot {{d\alpha(x)} \over {dx}}\Big) \cdot {{d\beta(x)} \over {dx}} \, + a_6(x) \cdot \alpha(x) \cdot {{d^2\beta(x)} \over {dx^2}} \,\, = \,\, 0. \quad (38)$$

Taking into account (37), equation (38) reduces to

$$16 \cdot \rho \cdot (1\, - x)^{3/2} \cdot {{d^2\beta(x)} \over {dx^2}} \, + (N^2 - M^2) \,\, = \,\, 0, \quad (39)$$

yielding the following expression for $\beta(x)$

$$\beta(x) \,\, = \,\, - {{M^2 - N^2} \over {4 \cdot \rho}} \cdot \sqrt{1-x} \,\,\, + \alpha_0 \cdot x \, + \beta_0 \quad (40)$$
$$= \,\, {{M^2 - N^2} \over {8 \cdot \rho}} \cdot x \cdot {{1 - \sqrt{1-x}} \over {1 + \sqrt{1-x}}} \,\,\, + \Big(\alpha_0 + {{1} \over {4 \cdot \rho}}\Big) \cdot x \,\, + \Big(\beta_0 - {{1} \over {4 \cdot \rho}}\Big),$$

where α_0 and β_0 are arbitrary constants. This yields to the following form†

$$\sigma(t) = {{2} \over {x}} \cdot {{1 - \sqrt{1-x}} \over {1 + \sqrt{1-x}}} \cdot \Big(\gamma \cdot h(x) \, + \alpha \, + \beta \cdot x \, + {{M^2 - N^2} \over {16}} \cdot x \cdot {{1 - \sqrt{1-x}} \over {1 + \sqrt{1-x}}}\Big), \quad (41)$$

where $\gamma \, = \, \rho/2$ and where α and β are arbitrary constants. This form is such that one reduces to an Okamoto form (29) up to the usual $h(x) \, \rightarrow \, \gamma \cdot h(x) \, + \alpha \, + \, \beta \cdot x$ transformations. To sum-up (28) can be deduced from (26).

The main question is how to discover the key (Landen) change of variable (26)? Assuming that the non-linear differential equation (23) has the Painlevé property, one can probably assume, because of the explicit form of (23), that its critical points are the three points $0,\, 1,\, \infty$. Consequently, a change of variable to reduce (23) to an Okamoto form (29), must map the three critical points $0,\, 1,\, \infty$ of (23) onto the three critical points $0,\, 1,\, \infty$ of an Okamoto form (29). Unfortunately this condition, reminiscent of *Belyi maps* [22], is *not* sufficient enough to actually discover the well-suited change of variable (26). At this step it is worth recalling that Painlevé VI functions *can be seen as deformations of elliptic functions* [23] (see also Appendix D). A *lattice of periods* is canonically attached to elliptic functions. If a change of variable maps a non-linear differential equation of the Painlevé type onto another non-linear differential equation of the Painlevé type, namely (23) onto an Okamoto form (29), *it must map the two lattice of periods* of the underlying elliptic functions. In other words the change of variable *must correspond to a quite selected and rigid set of transformations*: it has to be a *modular correspondence* [24]. These are *algebraic transformations* given by the

† To be compared with (28).

so-called *modular equations*† [24, 26]. The Landen (or inverse Landen) transformation is the simplest example of *modular correspondence*. In Appendix D we recall Manin's viewpoint showing explicitely that Painlevé VI functions can be seen as *deformations of elliptic functions*, and also underlying the *Landen transformation as a symmetry in the family of Painlevé VI equations.* The crucial role of other modular correspondences for Painlevé VI equations is also underlined in Appendix D.2.

2.2. Completing the characterization of the factors $g_\pm$.

Both of the sigma functions $\sigma_\pm$ satisfy the same Painlevé-type non-linear differential equation (23). To complete the characterization of the factors $g_\pm$ we need to obtain the boundary conditions on the equations for $\sigma_\pm$ which allow (homogeneous) polynomial in $\tilde{K}(k)$ and $\tilde{E}(k)$ factors to occur. By direct substitution in (28) we see that

$$h_0(x) \; = \; {{M^2 -3N^2+1} \over {16}} \quad - {{M^2 -N^2+1} \over {16}} \cdot x \\ - {{M^2 -N^2} \over {16}} \cdot x \cdot \Bigl({{1 \, - \sqrt{1-x}} \over {1 \, + \sqrt{1-x}}} \Bigr), \qquad (42)$$

is *an exact solution of* (29) with Okamoto parameters (31). This exact *algebraic* solution of (29) precisely corresponds to the exact solution $\sigma \, = \, 0$ of (23). This algebraic function is in fact of the form

$$h_0(x) \; = \; x \cdot (x \, -1) \cdot {{d \ln(\mathcal{H}_0(x))} \over {dx}}, \qquad (43)$$

where $\mathcal{H}_0(x)$ is *an algebraic function*:

$$\mathcal{H}_0(x) \; = \qquad (44) \\ \Bigl(1 - \sqrt{1-x}\Bigr)^{-(M^2-3\,N^2+1)/16} \cdot \Bigl(1 + \sqrt{1-x}\Bigr)^{(3\,M^2-N^2-1)/16} \cdot (1-x)^{-(M^2+N^2)/16}.$$

Thus we may write:

$$h(x) \; = \; H(x) \quad + h_0(x). \qquad (45)$$

We need only the power series solutions of (29) which are analytic at $x \, = \, 0$:

$$h(x) \; = \; \sum_{n=0} a_n \, x^n. \qquad (46)$$

We found, in Appendix D of [1], that there are, in general, four classes of these analytic solutions which are related by changing the signs of any pair of n_k. For the present purpose, we need the class 4 solutions given by (D.7), (D.11) and (D.15)

$$a_0^{(4)} \; = \; {{-n_1n_2 - n_3n_4 \, - (n_1+n_2) \cdot \, (n_3+n_4)} \over {2}}, \\ a_1^{(4)} \; = \; {{(n_1+n_2) \cdot \, n_3n_4 \;\, + (n_3+n_4) \cdot \, n_1n_2} \over {n_1+n_2+n_3+n_4}}, \qquad (47) \\ a_2^{(4)} = {{(n_1+n_2) \cdot \, (n_1+n_3) \cdot \, (n_1+n_4) \cdot \, (n_2+n_3) \cdot \, (n_2+n_4) \cdot \, (n_3+n_4)} \over {(n_1+n_2+n_3+n_4)^2 \cdot \, (n_1+n_2+n_3+n_4+1)(n_1+n_2+n_3+n_4-1)}},$$

† We must consider *modular curves* associated with modular forms. One excludes Shimura curves associated with automorphic forms [25].

which, with the Okamoto parameters (31), read:

$$a_0^{(4)} \; = \; \frac{1}{16} \cdot (M^2 - 3N^2 + 1), \qquad a_1^{(4)} \; = \; -\frac{1}{16} \cdot (M^2 - N^2 + 1),$$
$$a_2^{(4)} \; = \; \frac{1}{64} \cdot (M^2 - N^2). \tag{48}$$

These agree with the expansion at $x = 0$ of (42). Because

$$n_1 + n_2 + n_3 + n_4 \; = \; N, \tag{49}$$

we see, from (D.23) of [1], that $a_{N+1}^{(4)}$, the coefficient of x^{N+1}, is an arbitrary constant. To proceed further we extend the recursive analysis of [1] (see appendix D in [1]) beyond the term x^{N+1}. We find that the coefficients of x^n for $(N+1) \le n \le (2N+1)$ depend only on c_{N+1}, the coefficient of x^{N+1} but that starting with $c_{2(N+1)}$ the coefficients depend on c_{N+1}^2 as well as c_{N+1}. Continuing in this fashion we obtain the form (19)

$$\sigma_\pm(M,N;t;\,\lambda_\pm) \; = \; \sum_{n=1}^{\infty} \Bigl(\lambda_\pm \cdot t^{(N+1)/2}\Bigr)^n \cdot B_n(M,N;t), \tag{50}$$

where the $B_n(M,N;t)$'s are power series of t, analytic at $t = 0$, such that:

$$\Bigl(\frac{N+1}{2}\Bigr)^{n-1} \cdot B_n(M,N;t) \; = \; 1 \; + o(t). \tag{51}$$

We must choose

$$\lambda_+ \; = \; -\lambda_- \; = \; \lambda, \tag{52}$$

in order to match with the *lambda extension* solutions of (6):

$$\sigma(M,N;t;\,\lambda) \; = \; 2 \cdot \sum_{n=1}^{\infty} \Bigl(\lambda^2 \cdot t^{N+1}\Bigr)^n \cdot B_{2n}(M,N;t). \tag{53}$$

However, these lambda extensions of $\sigma_\pm(M,N;t;\lambda_\pm)$ *do not in general have a representation as homogenerous polynomials* in $\tilde{K}$ and $\tilde{E}$ for the corresponding $g_\pm(M,N;t;\lambda_\pm)$. We note in particular that B_1 reads

$$B_1 \; = \; {}_2F_1\Bigl([\frac{N+M}{2}, \frac{N-M}{2}], \, [N+1], \, t\Bigr), \tag{54}$$

which may be conjectured from the expansions in Appendix B, and proven by the recursive procedure outlined in section 3. A step further one can find that:

$$\begin{aligned}
&\frac{N+1}{2} \cdot B_2 \\
&\quad = \; \frac{N^2 - M^2}{4 \cdot (N+1)} \cdot t \cdot (t-1) \cdot {}_2F_1\Bigl([\frac{N+M+2}{2}, \frac{N-M+2}{2}], \, [N+2], \, t\Bigr)^2 \\
&\quad\quad + (N+1) \cdot {}_2F_1\Bigl([\frac{N+M}{2}, \frac{N-M}{2}], \, [N+1], \, t\Bigr)^2 \\
&\quad\quad\quad + N \cdot (t-1) \cdot {}_2F_1\Bigl([\frac{N+M}{2}, \frac{N-M}{2}], \, [N+1], \, t\Bigr) \\
&\quad\quad\quad\quad \times \, {}_2F_1\Bigl([\frac{N+M+2}{2}, \frac{N-M+2}{2}], \, [N+2], \, t\Bigr).
\end{aligned} \tag{55}$$

2.3. Selected values of λ for which $g_\pm(M,N;t;\lambda)$ reduce to polynomials in $\tilde{K}(k)$ and $\tilde{E}(k)$.

To complete the illustration of the factorization of the Toeplitz determinant for $C(M,N;\,t)$, we need to determine the value of λ for which the (differentially algebraic) lambda extension $g_\pm(M,N;t;\lambda_\pm)$ reduces to a determinant of *finite-dimensional matrices*†. In table 1, we list the coefficients of the terms $\pm\, k^{N+1}$ and $k^{2\,(N+1)}$ in $\sigma_\pm(M,N)$ for some low values of M and N. From this table we see that the

Table 1. Coefficients of $\pm k^{N+1}$ and $k^{2(N+1)}$ in $\sigma_\pm(M,N)$

M, N	$\pm k^{N+1}$ coefficient	$k^{2(N+1)}$ coefficient
1, 2	$\frac{3}{2^5} = \frac{3}{2}\left(\frac{1}{2^4}\right)$	$\frac{3}{2^9} = \frac{3}{2}\left(\frac{1}{2^4}\right)^2$
1, 4	$\frac{3\cdot 5}{2^9} = \frac{5}{2}\left(\frac{3}{2^8}\right)$	$\frac{3^2\cdot 5}{2^{17}} = \frac{5}{2}\left(\frac{3}{2^8}\right)^2$
3, 4	$\frac{5\cdot 7}{2^9} = \frac{5}{2}\left(\frac{7}{2^8}\right)$	$\frac{5\cdot 7^2}{2^{17}} = \frac{5}{2}\left(\frac{7}{2^8}\right)^2$
1, 6	$\frac{5\cdot 7}{2^{12}} = \frac{7}{2}\left(\frac{5}{2^{11}}\right)$	$\frac{5^2\cdot 7}{2^{23}} = \frac{7}{2}\left(\frac{5}{2^{11}}\right)^2$
3, 6	$\frac{7\cdot 9}{2^{12}} = \frac{7}{2}\left(\frac{9}{2^{11}}\right)$	$\frac{7\cdot 9^2}{2^{23}} = \frac{7}{2}\left(\frac{9}{2^{11}}\right)^2$
5, 6	$\frac{3\cdot 7\cdot 11}{2^{12}} = \frac{7}{2}\left(\frac{3\cdot 11}{2^{11}}\right)$	$\frac{3^2\cdot 7\cdot 11^2}{2^{23}} = \frac{7}{2}\left(\frac{3\cdot 11}{2^{11}}\right)^2$
2, 3	$\frac{5}{2^6} = 2\left(\frac{5}{2^7}\right)$	$\frac{5^2}{2^{13}} = 2\left(\frac{5}{2^7}\right)^2$
2, 5	$\frac{3\cdot 7}{2^{10}} = 3\left(\frac{7}{2^{10}}\right)$	$\frac{3\cdot 7^2}{2^{20}} = 3\left(\frac{7}{2^{10}}\right)^2$
4, 5	$\frac{3^2\cdot 7}{2^{10}} = 3\left(\frac{3\cdot 7}{2^{10}}\right)$	$\frac{3^3\cdot 7^2}{2^{20}} = 3\left(\frac{3\cdot 7}{2^{10}}\right)^2$
2, 7	$\frac{5\cdot 9}{2^{13}} = 4\left(\frac{5\cdot 9}{2^{15}}\right)$	$\frac{3^4\dot{5}^2}{2^{28}} = 4\left(\frac{5\cdot 9}{2^{15}}\right)^2$
4, 7	$\frac{9\cdot 11}{2^{13}} = 4\left(\frac{11\cdot 9}{2^{15}}\right)$	$\frac{11^2\cdot 9^2}{2^{28}} = 4\left(\frac{11\cdot 9}{2^{15}}\right)^2$

coefficients of $\pm\, k^{N+1}$, and the coefficients of $k^{2(N+1)}$ (or $\pm\, t^{(N+1)/2}$ and t^{N+1} in (19)), have respectively the form

$$\frac{N+1}{2}\cdot\,\alpha_{M,N}, \qquad\qquad \frac{N+1}{2}\cdot\,\alpha_{M,N}^2. \tag{56}$$

where¶

$$\alpha_{M,N} \;=\; \frac{(N+M)!\cdot\,(N-M)!}{2^{2N}\cdot\,(N+1)!\cdot\,((N+M-1)/2)!\cdot\,((N-M-1)/2)!}. \tag{57}$$

The selected values of $\lambda \;=\; \lambda_+$ read:

$$\lambda_+ \;=\; \lambda \;=\; \frac{N+1}{2}\cdot\,\alpha_{M,N}. \tag{58}$$

3. Tracy Widom viewpoint

Recalling the Tracy-Widom paper [18] we introduce, besides the sum (16), the *difference*:

$$\delta(M,N;t) \;=\; \sigma_+(M,N;t) \;\;-\sigma_-(M,N;t). \tag{59}$$

† In contrast with Fredholm determinants. For generic values of λ the lambda extensions of $C(M,N;\,t;\,\lambda)$ are Fredholm determinants.
¶ A demonstration of this result wich requires the introduction of Schlesinger's transformations will not be given here.

In this section we simply denote the difference $\delta(M,N;t)$ by δ, and the sum $\sigma(M,N;t) \;=\; \sigma_+(M,N;t) \;+\sigma_-(M,N;t)$ by σ. One has the following non-trivial relation‡ between the sum (16) and this difference (59)

$$\delta^2 \;+\, t \cdot (t-1) \cdot \sigma' \;\; - t \cdot \sigma \;\;=\;\; 0. \qquad (60)$$

Relation (60) also yields

$$\sigma'' \;\;=\;\; {{\delta^2 } \over {t^2 \cdot \, (t \, -1)}} \;\; -2 \cdot \, {{\delta \cdot \, \delta' } \over {t \cdot \, (t \, -1)}}, \qquad (61)$$

or:

$$\sigma \;\;=\;\; -(t-1) \cdot \, \int^t \, {{\delta^2} \over {(t-1)^2 \cdot \, t}} \cdot \, dt. \qquad (62)$$

Relation (60) is the generalization to Painlevé VI of the Tracy and Widom relation (82) in [18] associated with Painlevé V, which reads:

$$\delta^2 \;\; + t \cdot \sigma' \;\; - \sigma \;\;=\;\; 0. \qquad (63)$$

Using (60) and (61) one can eliminate σ' and σ'' in the Okamoto relation (6), and deduce†:

$$\sigma \;=\; -{{t^2 \cdot \, (t\, -1)^2 } \over {t\, +1}} \cdot \, {{\delta''} \over {\delta}} \, +2 \cdot \, {{\delta^2} \over {t \, +1}} \, + \, {{(t\,-1) \cdot \, \Big((M^2\,-1) \cdot \, t \, -(N^2\,-1)\Big) } \over {4 \cdot \, (t\,+1) }}. \qquad (64)$$

Furthermore, using Pantone's program one can first find that the *difference* (59) actually satisfies an *order-three* non-linear differential equation:

$$\begin{aligned} & 4 \cdot t^3 \cdot (t-1)^2 \cdot (t+1) \cdot \delta \cdot \delta''' \\ & \quad +4 \cdot t^2 \cdot (t-1) \cdot \Big(2 \cdot (t^2+t-1) \cdot \delta \; - t \cdot (t^2-1) \cdot \delta'\Big) \cdot \delta'' \\ & \qquad - 16 \cdot t \cdot (t+1) \cdot \delta^3 \cdot \delta' \\ & \qquad \quad +4 \cdot (3\,t+1) \cdot \delta^4 \; - (M^2+N^2-2) \cdot t \cdot (t-1) \cdot \delta^2 \;=\; 0. \end{aligned} \qquad (65)$$

Injecting the expression (64) of σ in terms of δ and δ'' in the Okamoto relation (6), one find a non-linear ODE on δ of *order four*. One can use the order three non-linear ODE (65) to express δ''' in terms of δ, δ', δ'', but also the fourth derivative $\delta^{(4)}$ in terms of δ, δ', σ''. Injecting these expressions of $\delta^{(4)}$ and δ''' in the previous *order four* non-linear ODE, one finally finds¶ the *order-two* non-linear ODE (quite similar to (23))

$$\begin{aligned} & 16 \cdot t^5 \cdot (t\,-1)^2 \cdot \delta''^{\,2} \;+4 \cdot t^2 \cdot (t\,-1)^2 \cdot \Big(4 \cdot \delta^2 \;-(M^2+N^2-2) \cdot t\Big) \cdot \delta \cdot \delta'' \\ & \quad - 16 \cdot t \cdot (t\,+1)^2 \cdot \Big(t \cdot \delta' - \delta\Big) \cdot \delta^2 \cdot \delta' \;\; - 16 \cdot \delta^6 \;\; +8 \cdot t \cdot (M^2+N^2-2) \cdot \delta^4 \\ & \quad +t \cdot \Big((M^2-1) \cdot t \; -(N^2\,-1)\Big) \cdot \Big((N^2\,-1) \cdot t \; -(M^2-1)\Big) \cdot \delta^2 \;=\; 0, \end{aligned} \qquad (66)$$

which is *not* of the SD-I "master of Painlevé form" (20).

‡ One can easily verify this relation for the two factors of $C(2,\,3)$ (see (C.1) below).
† Again one can easily verify this relation for the two factors of $C(2,\,3)$ (see (C.1) below).
¶ Note that this order-two non-linear ODE (66) could have been obtained directly using Pantone's program, but this requires many more coefficients of the power series of δ to be found (1600 coefficients in k).

Let us denote the LHS of the order-three non-linear ODE (65) by $\mathcal{R}_3$, and the LHS of the order-two non-linear ODE (66) by $\mathcal{R}_2$. We have the following relation:

$$\Bigl(8 \cdot t^3 \cdot \delta'' \ + \Bigl(4 \cdot \delta^2 - (M^2+N^2-2) \cdot t\Bigr) \cdot \delta \cdot \Bigr) \cdot \mathcal{R}_3$$
$$= \ t \cdot (t+1) \cdot \delta \cdot \frac{d\mathcal{R}_2}{dt} \ - \Bigl(2 \cdot t \cdot (t+1) \cdot \delta' \ + (3\,t+1) \cdot \delta\Bigr) \cdot \mathcal{R}_2. \qquad (67)$$

Similar to what has been performed in section 2, one would like to find the change of variable, and function transformation, enabling the reduction of the order-two non-linear ODE (66) to an Okamoto sigma-form of Painlevé VI. Again one notes (see (35)) that a transformation of the form $\delta(t) = \alpha(t) \cdot h(t) + \beta(t)$ is *not* sufficient to get rid of the h'' term next to the h''^2 term. One does need to find a change of variable like the Landen transformation (26).

Another (simpler) route amounts to saying that the Tracy-Widom-like transformation (60) will change the second order non-linear ODE (66) into a third-order non-linear ODE in σ, $\mathcal{S}_3 = 0$, that will eventually reduce to (6) because of the compatibility of all these equations. Let us write (6) as $\mathcal{S}_2 = 0$, we actually have the following compatibility relation:

$$\sigma''^2 \cdot \mathcal{S}_3 \ = \ t^5 \cdot \Bigl((t-1) \cdot \sigma' - \sigma\Bigr)^2 \cdot \Bigl(\frac{d\mathcal{S}_2}{dt}\Bigr)^2 \ + \sigma'' \cdot t^5 \cdot (t-1)^2 \cdot \mathcal{S}_2^2$$
$$+ \sigma'' \cdot t^5 \cdot (t+1) \cdot \Bigl((t-1) \cdot \sigma' - \sigma\Bigr)^3 \cdot \Bigl(M^2 - N^2 \ - 4 \cdot \Bigl((t+1) \cdot \sigma' - \sigma\Bigr)\Bigr) \cdot \frac{d\mathcal{S}_2}{dt}$$
$$- \sigma'' \cdot t^5 \cdot \Bigl((t-1) \cdot \sigma' - \sigma\Bigr) \cdot \Bigl(2 \cdot (t-1) \cdot \frac{d\mathcal{S}_2}{dt} \ + \sigma'' \cdot (t-1) \cdot \Bigl((t-1) \cdot \sigma' - \sigma\Bigr)$$
$$\times \Bigl(8 \cdot t \cdot \sigma \ - 8 \cdot (t^2-1) \cdot \sigma' \ + (t-1) \cdot (M^2 - N^2)\Bigr)\Bigr) \cdot \mathcal{S}_2. \qquad (68)$$

4. Boundary conditions for the four factors of $C(0,N)$ with N odd

In [1], we discovered that $C(0,N)$ with N odd and $k = -\nu$, in the low-temperature regime, *factors into four terms* instead of two. The four factors for $C(0,N)$ were presented as

$$C(0,N) \ = \ \text{constant} \cdot (1-t)^{1/2} \cdot t^{(1-N^2)/4} \cdot f_1 f_2 f_3 f_4, \qquad (69)$$

where the factors f_j all vanish at $t = 0$ in such a way to cancel the factor $t^{(1-N^2)/4}$. Here again we change the factors f_i in (69) in such a way to extract a factor of $(1-t)^{1/4}$ which is the limiting behavior of $C(0,N)$ as $N \to \infty$, and we impose the condition that *the four new factors satisfy the same non-linear differential equation.* The previous factorization (69) in *four* factors† now reads:

$$(1-t)^{-1/4} \cdot C(0,N) \ = \ g_1(0,N) \cdot g_2(0,N) \cdot g_3(0,N) \cdot g_4(0,N). \qquad (70)$$

If one defines

$$\sigma_j \ = \ t \cdot (t-1) \cdot \frac{d \ln g_j(t)}{dt}, \qquad (71)$$

the previous factorization (70) in four factors becomes an *additivity property* of the corresponding σ_i's:

$$\sigma(0,\,N) \ = \ \sigma_1(0,\,N) \ + \sigma_2(0,\,N) \ + \sigma_3(0,\,N) \ + \sigma_4(0,\,N). \qquad (72)$$

† Examples of $g_1(0,N)$'s for $C(0,5)$ and $C(0,7)$ are given in Appendix E.

In [1], we showed that the sigma's, associated with the four factors f_j in (69), satisfy Okamoto sigma-form of Painlevé VI equations (29) with the *same* Okamoto parameters n_i (unique up to permutations and sign changes of any pair)

$$n_1 = \frac{N+1}{4}, \qquad n_2 = \frac{N-1}{4}, \qquad n_3 = -\frac{1}{2}, \qquad n_4 = 0, \tag{73}$$

which specializes to

$$t^2 \cdot (t-1)^2 \cdot h''^2 \quad + 4\,h' \cdot (t \cdot h' - h) \cdot \Big((t-1) \cdot h' - h\Big)$$
$$- \frac{1}{8} \cdot (N^2+3) \cdot h'^2 \;\; - \frac{1}{2^8} \cdot (N^2+3)^2 \cdot h' \;\; - \frac{1}{2^{10}} \cdot (N^2-1)^2 \;\; = \;\; 0, \tag{74}$$

where four functions h_j are solutions of (74), and are related to $t\,(t-1)\,df_j/dt$ by (153)-(156) of [1]:

$$h_1 \;=\; t \cdot (t-1) \cdot \frac{d \ln f_1}{dt} \;\; - \frac{N^2+3}{16} \cdot t \;\; + \frac{N^2+3}{32}, \tag{75}$$

$$h_2 \;=\; t \cdot (t-1) \cdot \frac{d \ln f_2}{dt} \;\; - \frac{N^2-1}{16} \cdot t \;\; + \frac{N^2+3}{32}, \tag{76}$$

$$h_3 \;=\; t \cdot (t-1) \cdot \frac{d \ln f_3}{dt} \;\; - \frac{N^2-1}{16} \cdot t \;\; + \frac{N^2-5}{32}, \tag{77}$$

$$h_4 \;=\; t \cdot (t-1) \cdot \frac{d \ln f_4}{dt} \;\; - \frac{N^2-5}{16} \cdot t \;\; + \frac{N^2-5}{32}. \tag{78}$$

From (69), (75), (76), (77), (78), one gets:

$$\sigma(0,\, N) \;\;=\;\; h_1 + h_2 + h_3 + h_4 \;\; + 4 \cdot \Big(\frac{t}{16} \; + \frac{(N^2-1)}{32}\Big). \tag{79}$$

The σ_i's in the additive relation (72) such that they satisfy the *same non-linear differential equation*, are, thus, simply related to the previous h_i's:

$$\sigma_i(0, N; t) \;\;=\;\; h_i \;\; + \frac{t}{16} \;\; + \frac{(N^2-1)}{32}. \tag{80}$$

These σ_i's are solutions of the *same* non-linear differential equation obtained from (74) by (80), which reads:

$$t^2 \cdot (t-1)^2 \cdot \sigma''^2 \quad + 4\,\sigma' \cdot (t \cdot \sigma' - \sigma) \cdot \Big((t-1) \cdot \sigma' - \sigma\Big)$$
$$+ \frac{1}{4} \cdot \Big((N^2+1) \cdot (t-1) \; - t^2\Big) \cdot \sigma'^2 \;\; - \frac{1}{2^6} \cdot \Big(16 \cdot (N^2 + 1 - 2\,t) \cdot \sigma \; + N^2 \cdot t\Big) \cdot \sigma'$$
$$- \frac{1}{4} \cdot \sigma^2 \;\; + \frac{N^2}{2^6} \cdot \sigma \;\; - \frac{N^2 \cdot (N^2-3)}{2^{10}} \;\; = \;\; 0. \tag{81}$$

This non-linear differential equation of the Painlevé type (81) is of course of the Cosgrove-Scoufis form (20), being reducible to an Okamoto sigma-form of Painlevé VI equation (74) up to a simple shift (see (80)).

Do note that the four h_i's are solutions of the *same* Okamoto sigma-form of Painlevé VI (74). The boundary conditions for each h_j were not discussed in [1], and must be properly chosen for additivity (72) to occur.

To do this we recall that in [1], we found that there are in general four different possible boundary conditions for the expansion of solutions analytic at $t = 0$ of any Okamoto sigma form of Painlevé VI equation

$$h^{(i)}(t) \;=\; \sum_{n=0} a_n^{(i)}\, t^n, \tag{82}$$

where we denote by $h^{(i)}$ the solution of (74) with boundary conditions of class (i) of appendix D of [1].

The first few $a_n^{(i)}$ were determined analytically in [1]. For the present case with parameters (73) we only need cases 1 and 4 of appendix D of [1] where we find for case 1 that

$$a_0^{(1)} \;=\; -\frac{N^2+4N-1}{32}, \quad a_1^{(1)} \;=\; \frac{N-1}{16}, \quad a_2^{(1)} \;=\; \frac{N}{2^6}, \quad a_3^{(1)} \;=\; \frac{N}{2^7}, \qquad (83)$$

with $a_{(N+3)/2}^{(1)}$ arbitrary and for case 4 that

$$a_0^{(4)} \;=\; -\frac{N^2-4N-1}{32}, \quad a_1^{(4)} \;=\; -\frac{N+1}{16}, \quad a_2^{(4)} \;=\; -\frac{N}{2^6}, \quad a_3^{(4)} \;=\; \frac{N}{2^7}, \qquad (84)$$

with $a_{(N+1)/2}^{(4)}$ arbitrary.

The four h_i's solutions of (74) can be written

$$h_j \;=\; t \cdot (t-1) \cdot \frac{d \ln \mathcal{H}_j(0,N;\,t)}{dt}, \qquad (85)$$

where the $\mathcal{H}_j(0,N;\,t)$'s read¶ for instance for $N\,=\,5$:

$$\mathcal{H}_1(0,5;\,t) \;=\; \frac{2}{3} \cdot (1-t)^{-7/8} \cdot t^{-7/8} \cdot \Big((2t-1) \cdot \tilde{E} \;-(t-1) \cdot \tilde{K}\Big), \qquad (86)$$

$$\mathcal{H}_2(0,5;\,t) \;=\; \frac{2}{3} \cdot (1-t)^{-5/8} \cdot t^{-7/8} \Big((t+1) \cdot \tilde{E} \;+(t-1) \cdot \tilde{K}\Big), \qquad (87)$$

$$\mathcal{H}_3(0,5;\,t) \;=\; -\frac{8}{3} \cdot (1-t)^{-7/8} \cdot t^{-5/8} \cdot \Big((t-2) \cdot \tilde{E} \;-2 \cdot (t-1) \cdot \tilde{K}\Big), \qquad (88)$$

$$\mathcal{H}_4(0,5;\,t) \;=\; -\frac{8}{3} \cdot (1-t)^{-5/8} \cdot t^{-5/8} \cdot \Big(3\,\tilde{E}^2 \;+2 \cdot (t-2) \cdot \tilde{E}\tilde{K} \;-(t-1) \cdot \tilde{K}^2\Big). \qquad (89)$$

Let us display some expansions of σ_i's for example for $C(0,5)$:

$$\sigma_1(0,5) \;=\; \frac{5}{8} - \frac{5}{16}\,t - \frac{5}{2^6}\,t^2 - \frac{5 \cdot 11}{2^{10}}\,t^3 - \frac{5}{2^7}\,t^4 - \frac{3^2 \cdot 5 \cdot 43}{2^{16}}\,t^5 - \frac{5 \cdot 4817}{2^{20}}\,t^6 - \frac{5 \cdot 241 \cdot 509}{2^{25}}\,t^7 - \frac{5 \cdot 397811}{2^{27}}\,t^8 - \frac{3 \cdot 5 \cdot 13 \cdot 134401}{2^{31}}\,t^9 + \cdots \qquad (90)$$

$$\sigma_2(0,5) \;=\; \frac{5}{8} - \frac{5}{16}\,t - \frac{5}{2^6}\,t^2 - \frac{5^2}{2^{10}}\,t^3 - \frac{5}{2^9}\,t^4 - \frac{5 \cdot 61}{2^{16}}\,t^5 - \frac{5 \cdot 23^2}{2^{20}}\,t^6 - \frac{5 \cdot 10099}{2^{25}}\,t^7 - \frac{5^2 \cdot 71 \cdot 73}{2^{27}}\,t^8 - \frac{5 \cdot 281321}{2^{31}}\,t^9 + \cdots \qquad (91)$$

$$\sigma_3(0,5) \;=\; -\frac{5}{8} + \frac{5}{16}\,t + \frac{5}{2^6}\,t^2 + \frac{5}{2^7}\,t^3 + \frac{3 \cdot 5 \cdot 13}{2^{13}}\,t^4 + \frac{5 \cdot 53}{2^{14}}\,t^5 + \frac{5 \cdot 11 \cdot 449}{2^{21}}\,t^6 + \frac{5 \cdot 19 \cdot 397}{2^{22}}\,t^7 + \frac{3 \cdot 5 \cdot 15907}{2^{25}}\,t^8 + \frac{5 \cdot 77527}{2^{26}}\,t^9 + \cdots \qquad (92)$$

$$\sigma_4(0,5) \;=\; -\frac{5}{8} + \frac{5}{16}\,t + \frac{5}{2^6}\,t^2 + \frac{5}{2^7}\,t^3 + \frac{5 \cdot 41}{2^{13}}\,t^4 + \frac{5 \cdot 59}{2^{14}}\,t^5 + \frac{5 \cdot 5813}{2^{21}}\,t^6 + \frac{5 \cdot 47 \cdot 199}{2^{22}}\,t^7 + \frac{5 \cdot 13 \cdot 97 \cdot 197}{2^{27}}\,t^8 + \frac{5 \cdot 13 \cdot 97 \cdot 197}{2^{28}}\,t^9 + \cdots \qquad (93)$$

¶ Note that the $\mathcal{H}_j(0,N;\,t)$'s are Puiseux series : $\mathcal{H}_1(0,5;\,t) \;=\; t^{1/8} + \cdots$, $\mathcal{H}_2(0,5;\,t) \;=\; t^{1/8} + \cdots$, $\mathcal{H}_3(0,5;\,t) \;=\; t^{11/8} + \cdots$, $\mathcal{H}_4(0,5;\,t) \;=\; t^{11/8} + \cdots$.

to be compared‡ with the expansion of $\sigma_+(0,5)$ solution of (23):

$$\sigma_+(0,5) \; = \; -\frac{3 \cdot 5}{2^{10}} \cdot t^3 \; - \frac{5^3}{2^{13}} \cdot t^4 \; - \frac{5^3 \cdot 7}{2^{16}} \cdot t^5 \; - \frac{3 \cdot 5^2 \cdot 313}{2^{21}} \cdot t^6 \; + \; \cdots \qquad (94)$$

The corresponding expansions for $C(0,7)$ are given in Appendix E.

4.1. Algebraic solutions of (74) and (81)

Let us now define

$$A(t;N) \; = \; \frac{N}{2^3} \cdot \Bigl(1 - \frac{t}{2} - \sqrt{1-t}\Bigr) \; = \; -\frac{N}{2^3} \cdot \sum_{n=2} \Bigl(-\frac{1}{2}\Bigr)_n \cdot \frac{t^n}{n!}, \qquad (95)$$

where $(a)_n \, = \, a\,(a+1) \cdots (a+n-1)$ is the Pochhammer symbol. We have the remarkable (algebraic) result that (for case 1)

$$\begin{aligned} h_A^{(1)} \; &= \; A(t;N) \quad + \Bigl(\frac{N-1}{16}\Bigr) \cdot t \quad - \Bigl(\frac{N^2+4N-1}{32}\Bigr) \\ &= \; -\frac{N}{8} \cdot \sqrt{1-t} \quad - \frac{t}{16} \; - \frac{(N^2-1)}{32}, \end{aligned} \qquad (96)$$

and (for case 4)

$$\begin{aligned} h_A^{(4)} \; &= \; -A(t;N) \quad - \Bigl(\frac{N+1}{16}\Bigr) \cdot t \quad - \Bigl(\frac{N^2-4N-1}{32}\Bigr) \\ &= \; \frac{N}{8} \cdot \sqrt{1-t} \quad - \frac{t}{16} \; - \frac{(N^2-1)}{32}, \end{aligned} \qquad (97)$$

are actually exact (algebraic) solutions of (74). Using (80) these algebraic results correspond, in fact, and more simply, to the fact that the corresponding algebraic sigma's

$$\sigma_A^{(1)}(t;\, N) \; = \; -\frac{N}{8} \cdot \sqrt{1-t}, \qquad \sigma_A^{(4)}(t;\, N) \; = \; \frac{N}{8} \cdot \sqrt{1-t}, \qquad (98)$$

are, actually, (algebraic) solutions of (81).

The introduction of $A(t;N)$ corresponds to the *remarkable existence of algebraic solutions* for the g_j's. Actually $A(t;N)$ reads

$$A(t;N) \; = \; \frac{N}{2^3} \cdot \Bigl(1 - \frac{t}{2} - \sqrt{1-t}\Bigr) \; = \; t \cdot (t-1) \cdot \frac{d\ln(\mathcal{A}(t))}{dt}, \qquad (99)$$

where $\mathcal{A}(t)$ is the *algebraic function*†:

$$\mathcal{A}(t) \; = \; (1-t)^{N/16} \cdot \Bigl(\frac{1+(1-t)^{1/2}}{2}\Bigr)^{-N/4}. \qquad (100)$$

4.2. Lambda extension of the four factors of $C(0,N)$ with N odd

Using the *exact algebraic solutions* (96) and (97), as well as the relations (80) between the σ_i's and the h_i's, the recursive expansions¶ of [1] can be extended to an arbitrary

‡ One verifies easily that $\sigma_+(0,5) \, = \, \sigma_1(0,5) + \sigma_3(0,5)$.
† A function $\mathcal{A}(t)$ which is the exponential of the integral of an algebraic function (here $A(t;N)/t/(t-1)$) is called a Liouvillian function. Here we see that $\mathcal{A}(t)$ is not only Liouvillian, it is algebraic.
¶ By recursive we mean using the non-linear differential equation to get order by order a power series analytic at $t=0$ solution of that equation.

order and generalized with the arbitrary boundary condition constant, to find the pattern of the lambda extensions of the σ_i's.

For $j = 1, 2$ (case 4) we found experimentally for $N = 5$

$$\sigma_j(0,5;\lambda_j) \; = \; \frac{5}{8} \cdot \sqrt{1-t} \; + \lambda_j\, t^3 \; + \lambda_j\, t^4 \; + \frac{163}{3 \cdot 2^6}\, \lambda_j\, t^5 \; + \frac{1}{3} \cdot \Bigl(\lambda_j^2 + \frac{67}{2^5}\, \lambda_j\Bigr) \cdot t^6 + \frac{1}{3} \cdot \Bigl(\frac{5}{2}\, \lambda_j^2 + \frac{5 \cdot 11257}{2^{15}}\, \lambda_j\Bigr) \cdot t^7 \; + \Bigl(\frac{173}{2^7}\, \lambda_j^2 \; + \frac{7 \cdot 29 \cdot 229}{3 \cdot 2^{15}}\, \lambda_j\Bigr) \cdot t^8 + \Bigl(\frac{1}{3^2}\, \lambda_j^3 \; + \frac{7 \cdot 199}{3 \cdot 2^8}\, \lambda_j^2 + \frac{7 \cdot 347 \cdot 1021}{3 \cdot 2^{21}}\, \lambda_j\Bigr) \cdot t^9 \quad + \cdots, \qquad (101)$$

where

$$\lambda_2 \; = \; -\lambda_1, \qquad (102)$$

with λ_j for $j = 1, 2$, the arbitrary constant $a_{(N+1)/2}$ of class 4 solutions.

For $j = 3, 4$ (case 1) we found experimentally for $N = 5$

$$\sigma_j(0,5;\lambda_j) \; = \; -\frac{5}{8} \cdot \sqrt{1-t} \; + \lambda_j\, t^4 \; + \frac{3}{2}\, \lambda_j\, t^5 \; + \frac{19 \cdot 23}{2^8}\, \lambda_j\, t^6 \; + \frac{5 \cdot 181}{2^9}\, \lambda_j\, t^7 + \Bigl(\frac{1}{4}\lambda_2^2 \; + \frac{7 \cdot 8219}{2^{15}}\, \lambda_j\Bigr) \cdot t^8 \; + \Bigl(\frac{7}{8}\lambda_j^2 \; + \frac{7 \cdot 17 \cdot 941}{2^{16}}\, \lambda_j\Bigr) \cdot t^9 \quad + \ldots \qquad (103)$$

where

$$\lambda_4 \; = \; -\lambda_3, \qquad (104)$$

with λ_j for $j = 3, 4$ the arbitrary constant $a_{(N+3)/2}$ for class 1 solutions.

When

$$\lambda_2 \; = \; -\lambda_1 \; = \; \frac{3 \cdot 5}{2^{10}}, \qquad (105)$$

the lambda expansions (101) *actually reduce to* (90) and (91), and for

$$\lambda_4 \; = \; -\lambda_3 \; = \; \frac{5}{2^{13}}, \qquad (106)$$

the lambda expansions (103) *actually reduce to* (92) and (93).

More generally, for arbitrary N for $j = 1, 2$ (case 4), one has the following general form for the lambda extension of the $\sigma_j(0, N)$'s

$$\sigma_j(0,N;t;\lambda_j) \; = \; \frac{N}{8} \cdot \sqrt{1-t} \; + \sum_{n=1}^{\infty} \Bigl(\lambda_j \cdot t^{(N+1)/2}\Bigr)^n \cdot B_n^{(4)}(0,N;t), \qquad (107)$$

and for $j = 3, 4$ (case 1)

$$\sigma_j(0,N;t;\lambda_j) \; = \; -\frac{N}{8} \cdot \sqrt{1-t} \; + \sum_{n=1}^{\infty} \Bigl(\lambda_j \cdot t^{(N+3)/2}\Bigr)^n \cdot B_n^{(1)}(0,N;t), \qquad (108)$$

where $B_n^{(4)}(0,N;t)$ and $B_n^{(1)}(0,N;t)$ are power series‡, in t, and where we have the following normalization for both $i = 1$ and 4 and all N:

$$B_1^{(i)}(0,N;0) \; = \; 1, \qquad\qquad i \; = \; 1,\, 4. \qquad (109)$$

In other words the two one-parameter solutions $\sigma_1(0,N;t;\lambda_1)$ and $\sigma_2(0,N;t;\lambda_2)$ can be seen as a deformation of the same algebraic function $N \cdot \sqrt{1-t}/8$, when the

‡ They are in fact D-finite series (see Appendix F).

other two one-parameter solutions $\sigma_3(0,N;t;\lambda_3)$ and $\sigma_4(0,N;t;\lambda_4)$ can be seen as a deformation of the same algebraic function $-N \cdot \sqrt{1-t}/8$. We illustrate this for the series expansions of the lambda-extensions of $\sigma_j(0,7)$ in Appendix E.1 (see also (101) and (103) for $N = 5$).

When $M = 0$ with N odd, the additivity property (16) of the σ's becomes the additivity property (72) as a consequence of the additional factorization of the $g_\pm(0,N)$ in (11), yielding the factorization in four factors (70). Similarly the lambda extension (18) of the additivity property (16) becomes a lambda extension of the additivity property (72). One thus has

$$\sigma(0,\, N;\, \lambda) \; = \; \sigma_+(0,N;t;\lambda_+) \; + \sigma_-(0,N;t;\lambda_-), \tag{110}$$

and also

$$\sigma(0,\, N;\, \lambda) \; = \tag{111}$$
$$\sigma_1(0,\, N;\, t;\, \lambda_1) \, + \sigma_2(0,\, N;\, t;\, \lambda_2) \; + \sigma_3(0,\, N;\, t;\, \lambda_3) \; + \sigma_4(0,\, N;\, t;\, \lambda_4),$$

where $\lambda_+ \, = \, -\lambda_- \, = \, \lambda$ (see (52)) and where the well-suited λ_i's and $\lambda_\pm$'s remain to be found.

4.3. Constraints on the λ_i's

One has the two following relations:

$$\sigma_+(0,N;t;\lambda_+) \; = \; \sigma_1(0,N;t;\lambda_1) \; + \sigma_3(0,N;t;\lambda_3), \tag{112}$$

and

$$\sigma_-(0,N;t;\lambda_-) \; = \; \sigma_2(0,N;t;\lambda_2) \; + \sigma_4(0,N;t;\lambda_4), \tag{113}$$

where $\sigma_\pm(0,N;t;\lambda_\pm)$ are the sigma functions (19) for the factors $g_\pm(0,N;t)$ and where $\lambda_+ \, = \, -\lambda_- \, = \, \lambda$ (see (52)). These two relations (112) and (113) will only hold if there is a relation between λ_1 and λ_3, as well as a similar relation between λ_2 and λ_4.

For different values of N let us recall the form for the σ_i's (see (107) and (108)) such that the σ_i's satisfy their respective non-linear differential ODE's. Imposing that the RHS of (111) is solution the non-linear differential ODE (7) for $\sigma(0,\, N;\, \lambda)$, one finds experimentally, for different values of N, that

$$\lambda_2 \; = \; -\lambda_1, \qquad \text{and:} \qquad \lambda_4 \; = \; -\lambda_3. \tag{114}$$

Similarly imposing that $\sigma_+(0,N;t;\lambda_+)$ and $\sigma_-(0,N;t;\lambda_-)$, given respectively by (112) and (113), both verify the *same* non-linear ODE (23), one finds that:

$$\lambda_3 \; = \; \frac{\lambda_1}{4 \cdot (N+1)}, \tag{115}$$

To determine this relation (115) it is, for instance, sufficient to consider the term $n = 1$ in (107) and (108). From (112) and (19) we obtain the condition

$$\lambda_1 \cdot B_1^{(4)}(t;N) \; + \lambda_3 \cdot t \cdot B_1^{(1)}(t;N) \; = \; \lambda_+ \cdot B_1(0,N;t), \tag{116}$$

where we recall from (54) that B_1 reads:

$$B_1 \; = \; {}_2F_1\Big([\frac{N}{2}, \frac{N}{2}],\, [N+1],\, t\Big). \tag{117}$$

To proceed further, we require explicit forms for $B_1^{(4)}(t;N)$ and $B_1^{(1)}(t;N)$ which are computed in Appendix F as solutions (normalized to unity at $t\,=\,0$) of linear differential equations:

$$B_1^{(4)}(t;N) \;=\; \frac{1}{2N}\cdot\Bigl(2\cdot t\cdot\sqrt{1-t}\cdot\frac{dB_1(t)}{dt}\;+N\cdot(1+\sqrt{1-t})\cdot B_1(t)\Bigr), \tag{118}$$

$$\begin{aligned} B_1^{(1)}(t,N) \;&=\; \\ &=\; -\frac{2\,(N+1)}{N\,t}\cdot\Bigl(2\cdot t\cdot\sqrt{1-t}\cdot\frac{dB_1(t)}{dt}\;-N\cdot(1-\sqrt{1-t})\cdot B_1(t)\Bigr) \\ &=\; 1\;+\Bigl(\frac{N+1}{4}\Bigr)\cdot t\;+\Bigl(\frac{N^3\,+8\,N^2\,+20\,N\,+12}{32\cdot(N\,+3)}\Bigr)\cdot t^2\;+\;\cdots \end{aligned} \tag{119}$$

Thus we find:

$$\begin{aligned} &\lambda_1\cdot B_1^{(4)}(t;N)\;+\lambda_3\cdot t\cdot B_1^{(1)}(t;N) \\ &\quad=\;\frac{t}{N}\cdot\sqrt{1-t}\cdot\Bigl(\lambda_1\;-4\cdot(N+1)\cdot\lambda_4\Bigr)\cdot\frac{dB_1(t)}{dt} \\ &\qquad+\frac{1}{2}\cdot\Bigl(\lambda_1\cdot(1+\sqrt{1-t})\;+\lambda_3\cdot 4\cdot(N+1)\cdot(1-\sqrt{1-t})\Bigr)\cdot B_1(t). \end{aligned} \tag{120}$$

By setting

$$\lambda_3 \;=\; \frac{\lambda_1}{4\cdot(N+1)}, \tag{121}$$

the coefficient of $dB_1(t)/dt$ vanishes. We find

$$\lambda_1\cdot B_1^{(4)}(t;N)\;+\lambda_3\cdot t\cdot B_1^{(1)}(t,N)\;=\;\lambda_1\cdot B_1(t), \tag{122}$$

which setting

$$\lambda\;=\;\lambda_+\;=\;\lambda_1, \tag{123}$$

verifies (116) as desired.

To sum-up: There is a *one-parameter* family of $\sigma_j(0,N;t;\lambda)$'s solutions of Okamoto sigma form of Painlevé VI for which the lambda extension of the additive decomposition (111) holds, namely, taken into account (121) and (123)

$$\begin{aligned} &\sigma(0,N;\lambda)\;= \\ &\sigma_1(0,N;\lambda)\;+\sigma_2(0,N;-\lambda)\;+\sigma_3\Bigl(0,N;\frac{\lambda}{4\cdot(N+1)}\Bigr)\;+\sigma_4\Bigl(0,N;-\frac{\lambda}{4\cdot(N+1)}\Bigr). \end{aligned} \tag{124}$$

Selected values of the λ_i's: The selected values of the λ_i's, such that the g_i's are homogeneous polynomial expressions of the elliptic integrals of the first and second kind $\tilde{K}$ and $\tilde{E}$, read

$$\lambda_2\;=\;-\lambda_1\;=\;\frac{N+1}{2}\cdot\alpha_{0,N}\;=\;\frac{N!}{2^{2\,N\,+1}\cdot\Bigl((\frac{N-1}{2})!\Bigr)^2}, \tag{125}$$

and

$$\lambda_4\;=\;-\lambda_3\;=\;\frac{\alpha_{0,N}}{8}\;=\;\frac{N!}{2^{2\,N\,+3}\cdot(N+1)\cdot\Bigl((\frac{N-1}{2})!\Bigr)^2}, \tag{126}$$

where $\alpha_{0,N}$ is given by (57) for $M\,=\,0$.

5. Tracy Widom viewpoint on the four σ_i's

Recalling section 3 one can try to see if, instead of the sums (112) and (113), the difference† $\delta = \sigma_1(0, N; t) - \sigma_3(0, N; t)$ verifies a simple enough non-linear ODE. In that case the equivalent of the Tracy-Widom-like relation (60) becomes

$$\delta^2 + t \cdot (t-1) \cdot \sigma' - \frac{t-1}{2} \cdot \sigma + \frac{N^2}{16} \cdot (t-1) = 0, \qquad (127)$$

where σ denotes, here, the sum (see (112)) $\sigma_+ = \sigma_1(0, N; t) + \sigma_3(0, N; t)$. Again, using Pantone's program one first finds that this difference δ verifies an order-three non-linear ODE

$$\begin{aligned} & 8 \cdot (t-1)^3 \cdot t^2 \cdot (t+1) \cdot \delta \cdot \delta''' \\ & + 8 \cdot t \cdot (t-1)^2 \cdot \Big(2 \cdot (t^2+t-1) \cdot \delta - t \cdot (t^2-1) \cdot \delta'\Big) \cdot \delta'' \\ & - 32 \cdot (t^2-1) \cdot \delta^3 \cdot \delta' - 2 \cdot (t-1) \cdot (t+1)^2 \cdot (3t-1) \cdot \delta \cdot \delta' \\ & + 4 \cdot t \cdot (t^2-1)^2 \cdot \delta'^2 + 32 \cdot t \cdot \delta^4 \\ & + \Big(2t^2 + (N^2+5) \cdot t - (N^2-1)\Big) \cdot (t-1) \cdot \delta^2 = 0, \end{aligned} \qquad (128)$$

which is similar to (65). In fact, with more coefficients, one can find a second-order non-linear ODE like (66):

$$\begin{aligned} & 16 \cdot t^3 \cdot (t-1)^4 \cdot \delta \cdot \delta''^2 - 8 \cdot (t-1)^2 \cdot t^2 \cdot \Big(2 \cdot (t-1)^3 \cdot (t+1) \cdot \delta' \\ & \qquad + (t-1)^2 \cdot (N^2 - 2t - 1) \cdot \delta - 32 \cdot \delta^3\Big) \cdot \delta'' \\ & - 4 \cdot t \cdot (t^2-1) \cdot \Big(16 \cdot \delta^2 - (1-t)^2\Big) \cdot \delta'^2 \\ & + 4 \cdot t \cdot (t^2-1) \cdot \Big(32 \cdot t \cdot \delta^2 + (1-t)^2 \cdot \Big(N^2 - (2t+1)\Big)\Big) \cdot \delta \cdot \delta' \\ & + \Big((t-1) \cdot (N^2 - t) + 16 \cdot \delta^2\Big) \cdot \Big((t-1) \cdot (N^2 t - (2t+1)^2) - 16 \cdot \delta^2\Big) \cdot \delta^2 = 0. \end{aligned} \qquad (129)$$

Again, similar to what has been achieved in section 3, we can say that the Tracy-Widom-like transformation (127) will change the second order non-linear ODE (129) into a third-order non-linear ODE in σ. Using some differential algebra elimination, one can check that this last order-three non-linear ODE is *actually compatible* with¶ (23).

Let us denote, again, the LHS of the order-three non-linear ODE (128) by $\mathcal{R}_3$, and the LHS of the order-two non-linear ODE (129) by $\mathcal{R}_2$, we have the following relation similar to (67):

$$\begin{aligned} & \Big(4 \cdot t \cdot (t-1)^4 \cdot \delta'' - 2 \cdot (t+1) \cdot (t-1)^3 \cdot \delta' \\ & \qquad - (t-1)^2 \cdot (N^2 - 2t - 1) \cdot \delta + 32 \cdot \delta^3\Big) \cdot \mathcal{R}_3 \\ & = (t^2-1) \cdot \delta \cdot \frac{d\mathcal{R}_2}{dt} - 2 \cdot \Big((t^2-1) \cdot \delta' + 2 \cdot \delta'\Big) \cdot \mathcal{R}_2. \end{aligned} \qquad (130)$$

Let us remark that for small values of N, for instance $N = 9$, we get also another second-order non-linear ODE in δ. Combined with the previous second order

† Note that the difference $\delta = \sigma_2(0, N; t) - \sigma_4(0, N; t)$ yields the same results (128), (129), (130).
¶ And we have seen that (23) actually reduces to an Okamoto sigma-form of Painlevé VI using a Landen change of variable (26) together with transformation (28).

non-linear ODE (129) one eventually finds, eliminating δ'', a polynomial relation† $P_N(\delta, \delta', t) = 0$ which, for $N = 9$, is of the form

$$(16 \cdot \delta' + 35) \cdot (16 \cdot \delta' - 15) \cdot (16 \cdot \delta' + 3) \cdot (16 \cdot \delta' - 63) \cdot t^8 \quad + \cdots$$
$$+ (4 \cdot \delta - 1) \cdot (4 \cdot \delta - 9) \cdot (4 \cdot \delta - 5)^2 \cdot (4 \cdot \delta + 7)^2 \cdot (4 \cdot \delta + 3)^2 \;=\; 0, \qquad (131)$$

this relation being *compatible* with the two previous second-order non-linear ODEs in δ. In fact we have the following situation. Recalling the definition of the σ_i's in terms of log-derivatives of factors of the $C(0, N)$'s, expressed in terms of the complete elliptic integrals $\tilde{K}(k)$ and $\tilde{E}(k)$, one verifies easily that δ is a solution of (129) *as well as* (131). The power-series solutions of second-order non-linear ODE (129) are actually *one-parameter families of solutions* of (129), which correspond to *lambda extension of the previous* δ expressed in terms of $\tilde{K}$ and $\tilde{E}$. By contrast, the power-series solutions of relation (131), valid for $N = 9$, correspond to power series solutions of the form

$$\delta \;=\; \sum_{n=0}^{\infty} \delta_n \cdot t^n, \qquad (132)$$

where the first coefficient δ_0 can only take the following values $1/4, -3/4, 5/4, -7/4, 9/4$. For $\delta_0 = 9/4$ one finds easily, that the power series solution of (131) is unique, and can be obtained order by order:

$$\delta \;=\; {{9} \over {4}} \, - {{9} \over {8}} \cdot t \, - {{9} \over {32}} \cdot t^2 \, - {{9} \over {64}} \cdot t^3 \, - {{45} \over {512}} \cdot t^4 \, - {{16443} \over {262144}} \cdot t^5 \quad + \cdots \qquad (133)$$

This series (133) is nothing else but the expansion of the δ expressed in terms of the complete elliptic integrals $\tilde{K}(k)$ and $\tilde{E}(k)$ corresponding to the factors of $C(0, 9)$. In other words the polynomial relation (131), which is compatible with the order-two non-linear ODE (129), *actually selects in the one-parameter* (lambda-extension) *family of solutions of* (129), *the one corresponding to the "physical"* $C(0, 9)$ (i.e. Toeplitz determinants and no longer Fredholm determinants).

Remark 1: Let us also note that the order-two non-linear ODE (129) can, also be obtained performing some differential algebra eliminations using relations (127) and (23) (for $M = 0$) for $\sigma_+ = \sigma_1 + \sigma_3$.

Remark 2: Let us also note that performing some differential algebra eliminations using relations (127), (131), and (23) for $M = 0$, *one also finds a polynomial relation* $\mathcal{P}_N(\sigma, \sigma', t) = 0$, for $\sigma = \sigma_\pm$, which, for $N = 9$, is of the form:

$$(64 \cdot \sigma' + 225) \cdot (64 \cdot \sigma' + 3969) \cdot (64 \cdot \sigma' + 1225) \cdot (64 \cdot \sigma' + 9) \cdot t^8 \quad + \cdots$$
$$+ 2^{24} \cdot \sigma \cdot (\sigma - 4)^2 \cdot (\sigma - 7)^2 \cdot (\sigma - 9)^2 \cdot (\sigma - 10) \;=\; 0. \qquad (134)$$

The power-series solutions of relation (134), valid for $N = 9$, correspond to power series solutions of the form

$$\sigma \;=\; \sum_{n=0}^{\infty} \sigma_n \cdot t^n, \qquad (135)$$

where the first coefficient σ_0 can only take the following values $0, 4, 7, 9, 10$. For $\sigma_0 = 0$ one finds easily, that the power series solution of (134) is unique, and can be obtained order by order:

$$\sigma_+ \;=\; {{-315} \over {262144}} \cdot t^5 \, - {{5103} \over {2097152}} \cdot t^6 \, - {{56133} \over {16777216}} \cdot t^7 \, - {{1054053} \over {268435456}} \cdot t^8 \, + \cdots \qquad (136)$$

† For fixed value of t and for $N = 9$, the genus of the curve $P_N(x, y; t) = 0$ is zero (rational curve).

This series (136) is nothing else but the expansion of the σ_+ corresponding to the factors of $C(0, 9)$, expressed in terms of the complete elliptic integrals $\tilde{K}$ and $\tilde{E}$.

Remark 3: Let us recall that

$$\sigma_+ \; = \; \sigma_1 \, + \sigma_3 \; = \; t \cdot (t-1) \cdot \frac{d \ln(g_1 \cdot g_3)}{dt}, \qquad (137)$$

where the product $g_{13} \, = \, g_1 \cdot g_3$ is D-finite: it is solution of a linear differential equation of order *five*. Denoting Σ the log-derivative of the product g_{13}, one gets:

$$\frac{g_{13}'}{g_{13}} \, = \, \frac{\sigma_+}{t \cdot (t-1)} \, = \, \Sigma, \quad \frac{g_{13}''}{g_{13}} \, = \, \Sigma' \, + \Sigma^2, \quad \frac{g_{13}'''}{g_{13}} \, = \, \Sigma'' \, + 3 \, \Sigma \, \Sigma' \, + \Sigma^3, \;\; \cdots \qquad (138)$$

This order-*five* linear differential equation can thus be rewritten in an order-four Ricatti polynomial form:

$$\mathcal{R}(\sigma, \, \sigma', \, \sigma'', \, \sigma^{(3)}, \, \sigma^{(4)}; \; t) \quad = \quad 0. \qquad (139)$$

We underline that the polynomial relation (134) can also be obtained performing some differential algebra elimination between (23) and (139). This is a general result: the existence of polynomial relations†, like $\mathcal{P}_N(\sigma, \, \sigma', \, t) \, = \, 0$, which selects the D-finite (homogeneous polynomials of $\tilde{K}$ and $\tilde{E}$) factors scenario in the one-parameter families of solutions of a non-linear second order differential equation (like (23)), is precisely a consequence of this D-finite character (combined with the non-linear second order differential equation).

Remark 4: One should note that the differences of any of two σ_i's give similar results. Similarly, the sum of any of two σ_i's give results similar to (23). However the sums of three among the four σ_i's yield much more involved non-linear ODEs as well as the linear combinations $\sigma_i \, + \mu \cdot \sigma_j$ when μ is no longer equal to ± 1.

6. The Determinants of Forrester-Witte

We began this paper with examples of factorizations of Toeplitz determinants (see (8), (9)) and proceeded to show that this leads to a *one parameter family* of sigma forms of Painlevé VI which have *additive decompositions*. To complete the discussion we need to determine the determinants of the factors from the sigma functions which satisfy the Painlevé equations.

The $\tilde{N} \times \tilde{N}$ Toeplitz determinants of Forrester-Witte [2] as given in [4] are

$$D_{\tilde{N}}^{(p,p',\eta,\xi)}(t) \quad = \quad \det \left[A_{j-k}^{(p,p',\eta,\xi)}(t) \right]_{j,k=0}^{\tilde{N}-1}, \qquad (140)$$

where

$$A_m^{(p,p',\eta,\xi)}(t) \quad = \quad A^{(1)}(t) \;\; + \xi \cdot \, A^{(2)}(t), \qquad (141)$$

where in [1] $A^{(1)}(t)$ and $A^{(2)}(t)$ may be written as

$$A_m^{(1)}(t) \;\; = $$
$$\frac{\Gamma(1+p') \cdot \, t^{(\eta-m)/2}}{\Gamma(1+\eta-m) \, \Gamma(1-\eta+m+p')} \cdot \, {}_2F_1[-p, \, -p'+\eta-m], \, [1+\eta-m], \; t), \qquad (142)$$

$$A_m^{(2)}(t) \;\; = $$
$$\frac{\Gamma(1+p) \cdot \, t^{(m-\eta)/2}}{\Gamma(1-\eta+m) \, \Gamma(1+\eta-m+p)} \cdot \, {}_2F_1([-p', \, -p-\eta+m], \, [1-\eta+m], \; t), \qquad (143)$$

† Reminiscent of the "invariants" one obtains for linear differential operators with selected differential Galois groups (see the relation $Q(f, \, f', \, f'') = Constant$ given in the introduction of [27]).

where each $A_m^{(1)}$ and $A_m^{(2)}$ separately gives a Toeplitz matrix. For the purposes of this paper *it is sufficient to consider* $\xi = 0$ and see that after taking suitable limits (see eqs (128-130) of [1])

$$A_m^{(p,p',\eta)}(t) \, = \, \begin{cases} A_m^{(1)}(t) & \text{for } m \, \leq \, \eta \\ A_m^{(2)}(t) & \text{for } m \, \geq \, \eta \end{cases} \qquad (144)$$

In [4] it was shown that the sigma equations of these determinants satisfy Painlevé VI sigma equations with the Okamoto parameters:

$$\begin{aligned} n_1 \, &= \, (\tilde{N} + \eta + p - p')/2, \qquad & n_2 \, &= \, (\tilde{N} - \eta - p + p')/2, \\ n_3 \, &= \, (\eta - \tilde{N} - p - p')/2, \qquad & n_4 \, &= \, (\eta + \tilde{N} + p + p')/2. \end{aligned} \qquad (145)$$

In [1] we have computed the Okamoto parameters for the sigma non-linear differential equations for the two factors of $C(M,N)$ for $\nu = -k$ with $M+N$ odd, $M \leq N$, and for the four factors of $C(0,N)$ with N odd. From (145) we see that the parameters of the associated Forrester-Witte determinants are:

$$\tilde{N} = n_1 + n_2, \qquad \eta = n_3 + n_4, \qquad p = -n_2 - n_3, \qquad p' = -n_1 + n_4. \qquad (146)$$

The non-linear differential equation is invariant under permutations of the parameters n_i and the change of sign of any pair. However, the parameters of the determinants (146) do not share this symmetry which means that several different determinants have the same sigma equation. We must, of course select those choices for n_i which make $\tilde{N}$, the size of the Toeplitz matrix, an integer.

We must, therefore consider all possible $\tilde{N} \times \tilde{N}$ determinants which can be obtained from a given set of Okamoto parameters. This is done in Appendix G.1 Appendix G.3 and Appendix G.5, where we see that for both the two and four factor cases, the determinants can be grouped in sets of four, and it can be shown, by direct computation that the ratios of these four determinants are up to constants, powers of t and $(1-t)$. These powers of t and $(1-t)$ up to constants do not contribute to the sigma equations and must be studied independenly to obtain the associated factorization of the Forrester-Witte determinants.

7. Conclusion

The factorization of the low temperature correlation functions $C(M,N)$ for $\nu = -k$ with $M+N$ odd, $M \leq N$ considered in this paper, corresponds to a factorization of Toeplitz determinants that has been seen in many papers, in particular miscellaneous contexts (random matrices, see [18, 28, 29]). Here we address a much more rigid and strong property than a simple factorization property. We try to understand how a sigma function, solution of an Okamoto sigma form of Painlevé VI non-linear differential equation, can actually be the *sum* of several sigma functions each being *also solution* of non-linear ODEs with a Painlevé property that can be reduced to Okamoto sigma form of Painlevé VI. This is some kind of *addition formula of Painlevé transcendental functions*, similar to formulae of addition of elliptic functions. The case of the factorization of $C(0,N)$, for $\nu = -k$ with N odd, in *four* factors corresponds to a quite remarkable situation of four sigma functions solutions of Okamoto sigma form of Painlevé VI, their sum being also solution of an Okamoto sigma form of Painlevé VI. In that case we do not have a change of the variable t. The more general case of the factorization of the low-temperature $C(M,N)$, for $\nu = -k$ with $M+N$ odd, $M \leq N$, in *two* factors is more illuminating, since it actually introduces *selected* (Landen) *changes of variables* and functions, enabling to understand what kind of mathematical structures have to be introduced. With this last example, we provide the simplest example of such kind of sum of two sigma Painlevé transcendental functions being sigma Painlevé transcendental. This paper showed that these factorization properties for the correlation functions, or the additivity properties on the corresponding sigma's, *can actually be lambda-extended to one-parameter family of solutions* of the corresponding Okamoto sigma form of Painlevé VI and non-linear ODE's reducible to Okamoto sigma form of Painlevé VI.

The Painlevé transcendentals can be seen as *deformations of elliptic functions* as very well illustrated in [23]. If one assumes that a non-linear ODE with the Painlevé property, namely having fixed critical points, can be reduced to Painlevé VI, more precisely to an Okamoto sigma form of Painlevé VI equations, these *algebraic* changes of variables *cannot be arbitrary*: they have to be "compatible" with the underlying elliptic curve structure. These algebraic change of variables must not only preserve the set† of critical points $0, 1$ and ∞, they must be compatible¶ *with the lattice of periods*: in fact these transformations are highly selected, they must be *isogenies, modular correspondences associated with modular curves* [17, 24, 30]. Other non-linear differential equations with the Painlevé property of having fixed critical points, have been found to reduce to Painlevé transcendentals†† up to change of variables and functions [12, 32, 33]. It would be interesting to see if more involved non-linear ODEs for other Ising correlations $C(M, N)$ $(\nu \neq -k)$, can also be reduced to Painlevé VI transcendentals, possibly up to changes of variables corresponding to selected modular correspondences.

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Appendix A. Examples of factorizations of $C(M, N)$, $M + N$ odd, $M \le N$

We give here examples of factors $g_\pm(M, N)$ of (11) for $C(M, N)$ with $k = -\nu$ and $M + N$ odd. We use the previous notation (10)

$$\tilde{E}(k) = \frac{2}{\pi} \cdot E(k), \qquad \tilde{K}(k) = \frac{2}{\pi} \cdot K(k), \tag{A.1}$$

with $K(k)$ and $E(k)$ the complete elliptic integrals of the first and second kind.

We note that the first $N+4$ terms in the term $\pm\, k^{N+1} = \pm\, t^{(N+1)/2}$ in the form (13) are fitted by a constant times

$$ {}_3F_2\Big([\frac{N+1}{2}, \frac{N+M+2}{2}, \frac{N-M+2}{2}], [N+1, \frac{N+3}{2}], \, k^2\Big). \tag{A.2}$$

This holds until large primes appear in the coefficients.

Appendix A.1. Factors for $C(M, N)$ with $M + N$ odd and N even

In all the examples shown the expansions are carried out to the point that large primes appear in both the terms with even and odd powers of k.

† Each of the critical points $0, 1$ and ∞ do not have to be preserved individually.
¶ The new lattice of periods must be included in the original lattice, or conversely the original lattice of periods must be included in the new lattice of periods.
††In particular it turns out that there are two second-order (but *fourth*-degree) Painlevé-type equations, labelled as BP-IX and BP-X (see (1.10) with $m = 4$ in [31]). BP-IX and BP-X were solved in terms of elliptic functions or the special case of the second Painlevé transcendent [31].

Appendix A.1.1. Factors for $C(1,2)$

The two factors of $C(1,2)$ read:

$$g_\pm(1,2) \;=\; \frac{(1-k^2)^{-1/8}}{k} \cdot (1 \pm k)^{1/2} \cdot \Bigl(\tilde{E} \,-(1 \mp k) \cdot \tilde{K}\Bigr)$$

$$= \; 1 \; \pm k^3 \cdot \Bigl(\frac{1}{2^4} \; + \frac{3}{2^6}\,k^2 \; + \frac{3 \cdot 5^2}{2^{11}}\,k^4 \; + \frac{5 \cdot 7^2}{2^{13}}\,k^6 \; + \frac{3^3 \cdot 5 \cdot 7^2}{2^{18}}\,k^8$$

$$+ \frac{3^3 \cdot 7 \cdot 11^2}{2^{20}}\,k^{10} \; + \frac{661 \cdot 1949}{2^{26}}\,k^{12} \; + \cdots\Bigr) \qquad \text{(A.3)}$$

$$+ k^8 \cdot \Bigl(\frac{3}{2^{14}} \; + \frac{3}{2^{13}}\,k^2 \; + \frac{3^3 \cdot 5}{2^{18}}\,k^4 \; + \frac{3 \cdot 5 \cdot 11}{2^{18}}\,k^6 \; + \frac{3 \cdot 5 \cdot 7 \cdot 7321}{2^{30}}\,k^8 \; + \cdots\Bigr).$$

Appendix A.1.2. Factors for $C(1,4)$

The two factors of $C(1,4)$ read:

$$g_\pm(1,4) \;=\; -\frac{4}{3} \cdot \frac{(1-k^2)^{-1/8}}{k^4} \cdot (1 \pm k)^{1/2} \cdot \Bigl((k^2 \mp 3k \, + 1) \cdot \tilde{E}^2$$

$$+ 2 \cdot (1 \mp k) \cdot (k^2 \pm k \, - 1)) \cdot \tilde{E}\tilde{K} \; + (1 \mp k)\,(1 - k^2) \cdot \tilde{K}^2\Bigr)$$

$$= \; 1 \; \pm k^5 \cdot \{\frac{3}{2^8} \; + \frac{3 \cdot 5}{2^{10}}\,k^2 \; + \frac{5 \cdot 7^2}{2^{14}}\,k^4 \; + \frac{3^3 \cdot 5 \cdot 7}{2^{16}}\,k^6$$

$$+ \frac{3^3 \cdot 5 \cdot 7 \cdot 11^2}{2^{23}}\,k^8 \; + \frac{3 \cdot 7 \cdot 11^2 \cdot 13^2}{2^{25}}\,k^{10} \; + \frac{3^2 \cdot 5 \cdot 7 \cdot 11^2 \cdot 13^2}{2^{29}}\,k^{12}$$

$$+ \frac{3^2 \cdot 5 \cdot 11 \cdot 13^2 \cdot 17^2}{2^{31}}\,k^{14} \; + \frac{5 \cdot 1087 \cdot 267637}{2^{37}}\,k^{16} \; + \cdots\Bigr)$$

$$+ k^{12} \cdot \Bigl(\frac{5}{2^{20}} \; + \frac{3 \cdot 5}{2^{20}}\,k^2 \; + \frac{3 \cdot 5 \cdot 7 \cdot 139}{2^{29}}\,k^4 \; + \frac{5 \cdot 7^2 \cdot 23}{2^{27}}\,k^6$$

$$+ \frac{3^2 \cdot 7 \cdot 19 \cdot 827}{2^{34}}\,k^8 \; + \cdots\Bigr).$$

Appendix A.1.3. Factors for $C(3,4)$

The two factors of $C(3,4)$ read:

$$g_\pm(3,4) \;=\; \frac{4}{45} \cdot \frac{(1-k^2)^{-1/8}}{k^4} \cdot (1 \pm k)^{1/2} \cdot \Bigl((k^4 \, \pm 15k^3 - 16k^2 \, \pm 15k \, + 1) \cdot \tilde{E}^2$$

$$- 2 \cdot (1 \mp k) \cdot (3k^4 \, \pm k^3 \, - 2k^2 \, \pm 13k \, + 1) \cdot \tilde{E}\tilde{K}$$

$$- (1 - k^2) \cdot (1 \mp k) \cdot (3k^2 \mp 10k \, - 1) \cdot \tilde{K}^2\Bigr)$$

$$= \; 1 \; \pm k^5 \cdot \Bigl(\frac{7}{2^8} \; + \frac{3^3}{2^{10}}\,k^2 \; + \frac{5 \cdot 7 \cdot 11}{2^{14}}\,k^4 \; + \frac{3 \cdot 5 \cdot 7 \cdot 13}{2^{16}}\,k^6$$

$$+ \frac{3^4 \cdot 5^2 \cdot 7 \cdot 11}{2^{23}}\,k^8 \; + \frac{3 \cdot 7 \cdot 11^2 \cdot 13 \cdot 17}{2^{25}}\,k^{10} \; + \frac{3 \cdot 7 \cdot 11^2 \cdot 13^2 \cdot 19}{2^{29}}\,k^{12}$$

$$+ \frac{3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13^2 \cdot 17}{2^{31}}\,k^{14} \; + \frac{3^3 \cdot 5 \cdot 7^3 \cdot 11 \cdot 3457}{2^{37}}\,k^{16} \; + \cdots\Bigr)$$

$$+ k^{12} \cdot \Bigl(\frac{3 \cdot 7}{2^{20}} \; + \frac{5 \cdot 11}{2^{20}}\,k^2 \; + \frac{3^4 \cdot 5 \cdot 7 \cdot 17}{2^{29}}\,k^4 \; + \frac{3 \cdot 5 \cdot 7 \cdot 163}{2^{27}}\,k^6$$

$$+ \frac{3^2 \cdot 7 \cdot 11 \cdot 4051}{2^{34}}\,k^8 \; + \cdots\Bigr). \qquad \text{(A.4)}$$

Appendix A.2. Factors of $C(M,N)$ with $M+N$ odd, N odd, $M \neq 0$

Appendix A.2.1. Factors for $C(2,3;t)$

The two factors of $C(2,3)$ read

$$g_+(2,3) \;=\; -\frac{2}{3} \cdot \frac{(1-k^2)^{-1/8}}{k^2} \cdot (1-k^2)^{1/2} \\ \times \Big(3\,\tilde{E}^2 \;+ (k^2-5) \cdot \tilde{E}\tilde{K} \;\; -2 \cdot (k^2-1) \cdot \tilde{K}^2\Big),$$

$$g_-(2,3) \;=\; \frac{2}{3} \cdot \frac{(1-k^2)^{-1/8}}{k^2} \cdot \Big((k^2+1) \cdot \tilde{E} \;+ (k^2-1) \cdot \tilde{K}\Big), \qquad \text{(A.5)}$$

which expand in the form (13) as:

$$g_\pm(2,3) \;=\; 1 \;+ k^{10} \cdot \Big(\frac{7}{2^{17}} \;+ \frac{3 \cdot 5^2 \cdot 7}{2^{22}}\,k^2 \;+ \frac{3^3 \cdot 5 \cdot 7^2}{2^{25}}\,k^4 \\ + \frac{3 \cdot 5 \cdot 7 \cdot 11 \cdot 61}{2^{28}}\,k^6 \;+ \frac{5 \cdot 11 \cdot 107 \cdot 233}{2^{32}}\,k^8 \;\; + \cdots\Big) \\ \pm\, k^4 \cdot \Big(\frac{5}{2^7} \;+ \frac{5 \cdot 7}{2^{10}}\,k^2 \;+ \frac{3^3 \cdot 5 \cdot 7}{2^{15}}\,k^4 \;+ \frac{3 \cdot 7^2 \cdot 11}{2^{16}}\,k^6 \;+ \frac{3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13}{2^{21}}\,k^8 \\ + \frac{3^4 \cdot 5 \cdot 11^2 \cdot 13}{2^{25}}\,k^{10} \;+ \frac{3 \cdot 5 \cdot 7 \cdot 11^2 \cdot 13^2 \cdot 17}{2^{31}}\,k^{12} \\ + \frac{5 \cdot 7^2 \cdot 257 \cdot 1049}{2^{32}}\,k^{14} \;\; + \cdots\Big). \qquad \text{(A.6)}$$

Remark: Recalling the previous variable x (see (27)) corresponding to the square of the Landen modulus k_L (see (27)), one can actually also rewrite $C(2,3;t)$ as the product of *two other factors*:

$$C(2,3;t) \;=\; \frac{16}{9\,x^4} \cdot \Big(4 \cdot (x-1) \cdot (x-2) \;+ (x^2 - 8\,x + 8) \cdot \sqrt{1-x}\Big)^{1/2} \\ \times \Big((x\,-2) \cdot \tilde{E}_L \;\; -2 \cdot (x\,-1) \cdot \tilde{K}_L\Big) \\ \times \Big(3 \cdot \tilde{E}_L^2 \;\; -(x\,-1) \cdot \tilde{K}_L^2 \;\; +2 \cdot (x\,-2) \cdot \tilde{E}_L\,\tilde{K}_L\Big) \qquad \text{(A.7)}$$

where

$$\tilde{K}_L \;=\; \frac{2}{\pi} \cdot K\Big(\frac{2\sqrt{k}}{1+k}\Big) \;=\; {}_2F_1\Big([\frac{1}{2}, \frac{1}{2}],\, [1],\, x\Big),$$

$$\tilde{E}_L \;=\; \frac{2}{\pi} \cdot E\Big(\frac{2\sqrt{k}}{1+k}\Big) \;=\; {}_2F_1\Big(\frac{1}{2}, -\frac{1}{2}],\, [1],\, x\Big), \qquad \text{(A.8)}$$

The corresponding σ's for these two factors also verify Okamoto sigma forms of the Painlevé VI equation. We have similar results for all the (low-temperature) correlations $C(M,\,N)$ when $\nu = -k$.

Appendix A.2.2. Factors for $C(2,5)$

The two factors of $C(2,5)$ read

$$g_+(2,5) \; = \; -\frac{16}{45} \cdot \frac{(1-k^2)^{-1/8}}{k^6} \cdot \Big((7k^4 - 22k^2 + 7) \cdot \tilde{E}^3 \; - 5 \cdot (1-k^2)^3 \cdot \tilde{K}^3 \\ - (11k^2 - 17) \cdot (1-k^2)^2 \cdot \tilde{E}\tilde{K}^2 \; - (1-k^2) \cdot (2k^4 - 33k^2 + 19) \cdot \tilde{E}^2\tilde{K} \Big), \qquad \text{(A.9)}$$

$$g_-(2,5) \; = \; -\frac{16}{45} \cdot \frac{(1-k^2)^{-1/8}}{k^6} \cdot (1-k^2)^{1/2} \cdot \Big((2k^4 + 13k^2 + 2) \cdot \tilde{E}^2 \\ + (7k^4 - 15k^2 - 4) \cdot \tilde{E}\tilde{K} \; + 2 \cdot (1+2k^2) \cdot (1-k^2) \cdot \tilde{K}^2 \Big), \qquad \text{(A.10)}$$

which expand in the form (13) as:

$$g_\pm(2,5) \; = \; 1 \; + k^{14} \cdot \Big(\frac{3^2 \cdot 5}{2^{25}} \; + \frac{3^2 \cdot 7^2 \cdot 11}{2^{30}} k^2 \; + \frac{3 \cdot 5^2 \cdot 7^2 \cdot 11}{2^{32}} k^4 \\ + \frac{3^2 \cdot 7 \cdot 11 \cdot 13 \cdot 239}{2^{37}} k^6 \; + \frac{3^2 \cdot 5 \cdot 7 \cdot 13 \cdot 12289}{2^{41}} k^8 \; + \cdots \Big) \\ \pm \; k^6 \cdot \Big(\frac{7}{2^{10}} \; + \frac{3^2 \cdot 5 \cdot 7}{2^{15}} k^2 \; + \frac{3^2 \cdot 7 \cdot 11}{2^{16}} k^4 \; + \frac{3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13}{2^{22}} k^6 \\ + \frac{3^2 \cdot 5^2 \cdot 11^2 \cdot 13}{2^{25}} k^8 \; + \frac{3^2 \cdot 7 \cdot 11^2 \cdot 13^2 \cdot 17}{2^{31}} k^{10} \\ + \frac{5 \cdot 7 \cdot 11 \cdot 13^2 \cdot 17 \cdot 19}{2^{31}} k^{12} \; + \frac{3^2 \cdot 7 \cdot 11 \cdot 13^2 \cdot 17^2 \cdot 19}{2^{36}} k^{14} \\ + \frac{3^2 \cdot 5 \cdot 7 \cdot 13 \cdot 17^2 \cdot 19^2 \cdot 23}{2^{40}} k^{16} \; + \frac{3 \cdot 5 \cdot 7^2 \cdot 11 \cdot 9284039}{2^{43}} k^{18} \; + \cdots \Big).$$

Appendix A.2.3. Factors for $C(4,5)$

The two factors of $C(4,5)$ read

$$g_+(4,5) \; = \; \frac{16}{1575} \cdot \frac{(1-k^2)^{-1/8}}{k^6} \cdot \Big((2k^8 + 111k^6 - 34k^4 + 111k^2 + 2) \cdot \tilde{E}^2 \\ - (1-k^2) \cdot (43k^6 - 34k^4 + 179k^2 + 4) \cdot \tilde{K}\tilde{E} \\ - 2 \cdot (1-k^2)^2 \cdot (11k^4 - 34k^2 - 1) \cdot \tilde{K}^2 \Big), \qquad \text{(A.11)}$$

$$g_-(4,5) \; = \; \frac{16}{4725} \cdot \frac{(1-k^2)^{3/8}}{k^6} \cdot \Big((25k^6 - 825k^4 - 825k^2 + 25) \cdot \tilde{E}^3 \\ + 3 \cdot (2k^8 - 219k^6 + 121k^4 + 631k^2 - 23) \cdot \tilde{E}^2\tilde{K} \\ + 3 \cdot (1-k^2) \cdot (47k^6 - 121k^4 - 459k^2 + 21) \cdot \tilde{E}\tilde{K}^2 \\ + (1-k^2)^2 \cdot (69k^4 + 334k^2 - 19) \cdot \tilde{K}^3 \Big), \qquad \text{(A.12)}$$

which expand in the form of (13) as:

$$\begin{aligned} g_\pm(4,5) \;=\;& 1 \; + k^{14}\cdot \Big(\frac{3^3\cdot 11}{2^{25}} + \frac{3^3\cdot 7\cdot 11\cdot 13}{2^{30}}\,k^2 \\ & \quad + \frac{7\cdot 11\cdot 13\cdot 197}{2^{32}}\,k^4 + \frac{3^3\cdot 7\cdot 11\cdot 13\cdot 349}{2^{37}}\,k^6 + \cdots\Big) \\ & \pm k^6\cdot \Big(\frac{3\cdot 7}{2^{10}} + \frac{3^2\cdot 7\cdot 11}{2^{15}}\,k^2 + \frac{3^2\cdot 11\cdot 13}{2^{16}}\,k^4 + \frac{3\cdot 5^2\cdot 7\cdot 11\cdot 13}{2^{22}}\,k^6 \\ & \quad + \frac{3^2\cdot 5^2\cdot 11\cdot 13\cdot 17}{2^{25}}\,k^8 + \frac{3^2\cdot 7\cdot 11^2\cdot 13\cdot 17\cdot 19}{2^{31}}\,k^{10} \\ & \quad + \frac{7^2\cdot 11\cdot 13^2\cdot 17\cdot 19}{2^{31}}\,k^{12} + \frac{3^2\cdot 7\cdot 11\cdot 13^2\cdot 17\cdot 19\cdot 23}{2^{36}}\,k^{14} \\ & \quad + \frac{3^2\cdot 5^3\cdot 7\cdot 13\cdot 17^2\cdot 19\cdot 23}{2^{40}}\,k^{16} + \frac{3\cdot 11\cdot 13\cdot 29\cdot 7757221}{2^{43}}\,k^{18} + \cdots\Big). \end{aligned} \tag{A.13}$$

Appendix B. Sum decompositions of sigma functions

We give here examples of factors $\sigma_\pm(M,N;t)$ of (16).

Appendix B.1. Decomposition of $\sigma(M,N)$ with $M+N$ odd and N even

Appendix B.1.1. Decomposition for $\sigma(1,2)$

$$\begin{aligned} \sigma_\pm(1,2) \;=\;& \pm k^3\cdot \Big(\frac{3}{2^5} + \frac{3}{2^7}\,k^2 + \frac{3^2\cdot 5}{2^{12}}\,k^4 + \frac{3\cdot 37}{2^{14}}\,k^6 \\ & \quad + \frac{3\cdot 11\cdot 19}{2^{17}}\,k^8 + \frac{3^2\cdot 5^2\cdot 17}{2^{20}}\,k^{10} + \cdots\Big) \\ & + k^6\cdot \Big(\frac{3}{2^9} + \frac{3\cdot 7}{2^{12}}\,k^2 + \frac{3^3\cdot 5}{2^{15}}\,k^4 + \frac{3\cdot 73}{2^{16}}\,k^6 + \frac{3\cdot 7741}{2^{23}}\,k^8 + \cdots\Big). \end{aligned} \tag{B.1}$$

Appendix B.1.2. Decomposition for $\sigma(1,4)$

$$\begin{aligned} \sigma_\pm(1,4) \;=\;& \pm k^5\cdot \Big(\frac{3\cdot 5}{2^9} + \frac{3^2\cdot 5}{2^{11}}\,k^2 + \frac{3\cdot 5^2\cdot 7}{2^{15}}\,k^4 + \frac{3^2\cdot 5^2\cdot 7}{2^{17}}\,k^6 \\ & \quad + \frac{3^4\cdot 5^2\cdot 7\cdot 11}{2^{24}}\,k^8 + \frac{3^2\cdot 5\cdot 23\cdot 479}{2^{26}}\,k^{10} + \cdots\Big) \\ & + k^{10}\cdot \Big(\frac{3^2\cdot 5}{2^{17}} + \frac{3\cdot 5\cdot 23}{2^{19}}\,k^2 + \frac{3\cdot 5^2\cdot 7^2}{2^{22}}\,k^4 + \frac{3^2\cdot 5^2\cdot 7\cdot 43}{2^{26}}\,k^6 \\ & \quad + \frac{3^3\cdot 5^2\cdot 7\cdot 491}{2^{31}}\,k^8 + \frac{3^3\cdot 5\cdot 11\cdot 3209}{2^{32}}\,k^{10} + \cdots\Big). \end{aligned} \tag{B.2}$$

Appendix B.1.3. Decomposition for $\sigma(3,4)$

$$\begin{aligned} \sigma_\pm(3,4) \;=\;& \pm k^5\cdot \Big(\frac{5\cdot 7}{2^9} + \frac{7^2}{2^{11}}\,k^2 + \frac{3^2\cdot 7^2}{2^{15}}\,k^4 + \frac{3\cdot 5\cdot 7\cdot 11}{2^{17}}\,k^6 \\ & \quad + \frac{3\cdot 5\cdot 7^2\cdot 11\cdot 13}{2^{24}}\,k^8 + \frac{5\cdot 7^2\cdot 1301}{2^{26}}\,k^{10} + \cdots\Big) \\ & + k^{10}\cdot \Big(\frac{5\cdot 7^2}{2^{17}} + \frac{5^2\cdot 7^2}{2^{19}}\,k^2 + \frac{3^2\cdot 7\cdot 157}{2^{22}}\,k^4 + \frac{3\cdot 7\cdot 7129}{2^{26}}\,k^6 \\ & \quad + \frac{3\cdot 5\cdot 7^2\cdot 11\cdot 547}{2^{31}}\,k^8 + \frac{5\cdot 7^2\cdot 19\cdot 37\cdot 47}{2^{32}}\,k^{10} + \cdots\Big). \end{aligned} \tag{B.3}$$

Appendix B.2. Decomposition of $\sigma(M,N)$ with $M+N$ odd, N odd, $M \neq 0$

Appendix B.2.1. Decomposition for $\sigma(2,3)$

$$\begin{aligned}\sigma_{\pm}(2,3) \;=\;& \pm k^4 \cdot \Big(\frac{5}{2^6} + \frac{5^2}{2^{10}}\, k^2 + \frac{3\cdot 5\cdot 7}{2^{13}}\, k^4 + \frac{3\cdot 5^2\cdot 7}{2^{16}}\, k^6 \\ &+ \frac{5^2\cdot 59}{2^{18}}\, k^8 + \frac{5\cdot 28579}{2^{25}}\, k^{10} + \cdots \Big) \\ &+ k^8 \cdot \Big(\frac{5^2}{2^{13}} + \frac{5\cdot 11}{2^{14}}\, k^2 + \frac{3\cdot 5\cdot 7\cdot 31}{2^{20}}\, k^4 + \frac{3\cdot 5^2\cdot 7\cdot 11}{2^{21}}\, k^6 \\ &+ \frac{5^2\cdot 17\cdot 1531}{2^{28}}\, k^8 + \cdots \Big). \qquad \text{(B.4)}\end{aligned}$$

Appendix B.2.2. Decomposition for $\sigma(2,5)$

$$\begin{aligned}\sigma_{\pm}(2,5) \;=\;& \pm k^6 \cdot \Big(\frac{3\cdot 7}{2^{10}} + \frac{3\cdot 7^2}{2^{13}}\, k^2 + \frac{3^3\cdot 5\cdot 7}{2^{16}}\, k^4 + \frac{3^2\cdot 5\cdot 7^2\cdot 11}{2^{21}}\, k^6 \\ &+ \frac{3^2\cdot 5\cdot 7^2\cdot 11\cdot 13}{2^{25}}\, k^8 + \frac{3^3\cdot 7^2\cdot 11^2\cdot 13}{2^{28}}\, k^{10} + \frac{3\cdot 7^2\cdot 94823}{2^{31}}\, k^{12} + \cdots \Big) \\ &+ k^{12} \cdot \Big(\frac{3\cdot 7^2}{2^{20}} + \frac{3\cdot 7\cdot 31}{2^{21}}\, k^2 + \frac{3^3\cdot 5\cdot 7\cdot 131}{2^{28}}\, k^4 + \frac{3^3\cdot 5\cdot 7^2\cdot 47}{2^{29}}\, k^6 \\ &+ \frac{3^3\cdot 5\cdot 7^2\cdot 11\cdot 157}{2^{34}}\, k^8 + \frac{3^3\cdot 7^2\cdot 11\cdot 1709}{2^{35}}\, k^{10} \\ &+ \frac{3\cdot 7^2\cdot 131\cdot 293\cdot 1187}{2^{43}}\, k^{12} + \cdots \Big). \qquad \text{(B.5)}\end{aligned}$$

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Appendix B.2.3. Decomposition for $\sigma(4,5)$

$$\begin{aligned}\sigma_{\pm}(4,5) \;=\;& \pm k^6 \cdot \Big(\frac{3^2\cdot 7}{2^{10}} + \frac{3^3\cdot 7}{2^{13}}\, k^2 + \frac{3^4\cdot 11}{2^{16}}\, k^4 + \frac{3^3\cdot 5\cdot 11\cdot 13}{2^{21}}\, k^6 \\ &+ \frac{3^2\cdot 5^2\cdot 7\cdot 11\cdot 13}{2^{25}}\, k^8 + \frac{3^4\cdot 7\cdot 11\cdot 13\cdot 17}{2^{28}}\, k^{10} + \frac{3^3\cdot 7\cdot 23\cdot 43\cdot 47}{2^{31}}\, k^{12} + \cdots \Big) \\ &+ k^{12} \cdot \Big(\frac{3^3\cdot 7^2}{2^{20}} + \frac{3^3\cdot 7\cdot 19}{2^{21}}\, k^2 + \frac{3^4\cdot 7\cdot 11\cdot 79}{2^{28}}\, k^4 + \frac{3^2\cdot 7\cdot 11\cdot 1409}{2^{29}}\, k^6 \\ &+ \frac{3^2\cdot 5\cdot 11\cdot 13\cdot 4651}{2^{34}}\, k^8 + \frac{3^4\cdot 11\cdot 13\cdot 4871}{2^{35}}\, k^{10} \\ &+ \frac{3^3\cdot 7\cdot 37\cdot 1932439}{2^{43}}\, k^{12} + \cdots \Big). \qquad \text{(B.6)}\end{aligned}$$

Continuing in this fashion we obtain the form (19).

Note that the first $N+1$ terms of the $\pm k^{N+1}$ terms are proportional to ${}_2F_1([\frac{N+M}{2}, \frac{N-M}{2}], [N+1], k^2)$.

The series following $\pm k^{N+1}$ developes large primes at order $k^{3(N+1)}$ and the series following $k^{2\,(N+1)}$ developes much larger primes at order $k^{4\,(N+1)}$. This is expected from the result (28) by making a recursive expansion of the sigma form of Painlevé VI function $h(x)$ which satisfies (29) with parameters (31).

Appendix C. Reduction to Okamoto form: the $C(2, 3)$ example.

Let us illustrate the reduction of section 2 of the non-linear differential equation (23) to the Okamoto sigma-form of Painlevé VI (32), using the (Landen) substitution (26) together with transformation (28), on a simple example associated with the two factors of the low-temperature correlation function $C(2, 3)$. The two factors $g_+(2, 3)$ and $g_-(2, 3)$ have been given previously (see (A.5)):

$$g_+(2,3) \; = \; -{{2} \over {3}} \cdot {{(1\, -t)^{3/8}} \over {t}} \cdot \Bigl(3 \cdot \, \tilde{E}^2 \, \, + (t\, -5) \cdot \, \tilde{E} \tilde{K} \, \, \, -2 \cdot \, (t\, -1) \cdot \, \tilde{K}^2 \Bigr),$$

$$g_-(2,3) \; = \; {{2} \over {3}} \cdot {{(1\, -t)^{-1/8}} \over {t}} \cdot \Bigl((t\, +1) \cdot \, \tilde{E} \, \, + (t\, -1) \cdot \, \tilde{K} \Bigr), \qquad \text{(C.1)}$$

It is straightforward to see that the corresponding $\sigma_\pm(2, 3)$, deduced from formula (15), verify the non-linear equation (23) for $M = 2$ and $N = 3$. The expansion of $g_+(2, 3)$ and $\sigma_\pm(2, 3)$ are given previously (see (A.6), (B.4)). One can rewrite (28):

$$h(x) \; = \; {{x} \over {2}} \cdot {{1 \, + \sqrt{1-x}} \over {1 \, -\sqrt{1-x} }} \cdot \sigma(t) \; \; - {{ 3 M^2 \, -N^2 +3} \over {16}} \; \; + {{ M^2 \, -N^2 +1} \over { 8 \cdot \, (1 \, -\sqrt{1\, -x})}} \cdot x. \qquad \text{(C.2)}$$

The exact expressions of $\sigma_\pm(2, 3)$ are rational expressions in terms of the complete elliptic integrals $\tilde{E}$ and $\tilde{K}$ (see (10)):

$$\tilde{K}(t) \; = \; {}_2F_1\Bigl([{{1} \over {2}}, {{1} \over {2}}], \, [1], \, t\Bigr), \qquad \qquad \tilde{E}(t) \; = \; {}_2F_1\Bigl({{1} \over {2}}, -{{1} \over {2}}], \, [1], \, t\Bigr). \qquad \text{(C.3)}$$

Performing the (Landen) substitution (26) in these exact expressions of $\sigma_\pm(2, 3)$ and using (C.2) for $M = 2$ and $N = 3$, it is straightforward to verify (in Maple) that the corresponding function $h(x)$ *actually verifies the Okamoto sigma-form of Painlevé VI* (32) for $M = 2$ and $N = 3$. The expansions in x of the corresponding $h(x)$ read respectively

$$h_+ \; = \; -{{11} \over {8}} \; + {{1} \over {4}} \, x \, + {{5} \over {64}} \, x^2 \; + {{5} \over {128}} \, x^3 \; + {{205} \over {8192}} \, x^4 + {{295} \over {16384}} \, x^5 + {{29065} \over {2097152}} \, x^6 \; + \; \cdots \qquad \text{(C.4)}$$

and:

$$h_- \; = \; -{{11} \over {8}} \; + {{1} \over {4}} \, x \, + {{5} \over {64}} \, x^2 \; + {{5} \over {128}} \, x^3 \; + {{195} \over {8192}} \, x^4 + {{265} \over {16384}} \, x^5 + {{24695} \over {2097152}} \, x^6 \; + \; \cdots \qquad \text{(C.5)}$$

to be compared with the expansion of the algebraic solution (42) of (32) for $M = 2$ and $N = 3$:

$$h_0 \; = \; -{{11} \over {8}} \; + {{1} \over {4}} \, x \, + {{5} \over {64}} \, x^2 \; + {{5} \over {128}} \, x^3 \; + {{25} \over {1024}} \, x^4 \; + {{35} \over {2048}} \, x^5 \; + {{105} \over {8192}} \, x^6 \; \; + \; \cdots \qquad \text{(C.6)}$$

Appendix D. Painlevé VI transcendentals as deformations of elliptic functions and the crucial role of modular correspondences

Let us first recall (R. Fuchs [34], 1907) that the Painlevé VI equation (22) can be written (see (1.1) in [23], here $X = y$ in (22)):

$$t \cdot \, (1 \, -t) \cdot \, L_2 \cdot \, \int_{\infty}^{(X, \, Y)} \, {{dx} \over { \sqrt{ x \cdot \, (x \, -1) \cdot \, (x \, -t)}}}$$

$$= \; \; Y \cdot \, \Bigl(\alpha \; \, + \beta \cdot \, {{t} \over {X^2}} \; \, + \gamma \cdot \, {{t-1} \over {(X-1)^2}} \; \; + \delta \cdot \, {{t \cdot \, (t-1)} \over { (X \, -t)^2}} \Bigr), \qquad \text{(D.1)}$$

where $Y^2 = X \cdot (X-1) \cdot (X-t)$, and where:

$$L_2 = t \cdot (1-t) \cdot \frac{d^2}{dt^2} + (1-2t) \cdot \frac{d}{dt} - \frac{1}{4}. \tag{D.2}$$

Equation (D.1) provides a clear illustration of the fact that the Painlevé VI transcendentals *can be seen as a deformation of elliptic functions.* The crucial role played by the second derivative with respect to τ (the ratio of periods) displayed below in the equations (D.7), (D.8), (D.11) of Appendix D.1, is illustrated by the relation (see (1.18) in [23])

$$\frac{\Pi_{i>j}(e_i - e_j)^2}{9 \cdot (e_1 e_2' - e_2 e_1')^2} \cdot (e_2 - e_1)^{-3/2} \cdot \frac{d^2}{d\tau^2} = t \cdot (1-t) \cdot L_2 \cdot (e_2 - e_1)^{1/2}, \tag{D.3}$$

where the order-two linear differential operator L_2 is given by (D.2), and where the e_i's read:

$$e_i = \mathcal{P}\Big(\frac{T_i}{2}, \tau\Big) \qquad \text{where:} \qquad (T_0, T_1, T_2, T_3) = (0, 1, \tau, 1+\tau), \tag{D.4}$$

This means that, up to some dressing, *the second derivative with respect to τ is essentially the second order linear differential operator* L_2 which annihilates the simplest elliptic function, namely the complete elliptic integral of the first kind ${}_2F_1([1/2, 1/2], [1], t)$.

Appendix D.1. Modular correspondences and Painlevé VI transcendentals: the crucial role of the Landen transformations

Along a modular correspondence-line it is worth recalling Manin's idea [23] that the Painlevé VI equation for a particular choice of the four Okamoto parameters, can be written extremely simply in terms of the ratio of periods τ. Let us denote $\mathcal{P}(z, \tau)$ the $\mathcal{P}$-Weierstrass function and

$$\mathcal{P}_z(z, \tau) = \frac{\partial \mathcal{P}(z, \tau)}{\partial z}. \tag{D.5}$$

The fundamental role of the *Landen transformation* [17] is illustrated by the following identity† on the $\mathcal{P}$-Weierstrass function:

$$\mathcal{P}_z\Big(z, \frac{\tau}{2}\Big) = \mathcal{P}_z(z, \tau) + \mathcal{P}_z(z + \frac{\tau}{2}, \tau). \tag{D.6}$$

Manin's result means that the Painlevé VI equation can be written in a form (see equation (1.16) in [23]):

$$\frac{d^2 z(\tau)}{d\tau^2} = \Big(\frac{1}{2\pi i}\Big)^2 \cdot \sum_{i=0}^{3} \alpha_i \cdot \mathcal{P}_z(z + \frac{T_i}{2}, \tau). \tag{D.7}$$

Switching from the t variable to the τ variable, which is a (differentially algebraic) transcendental change of variable, changes the non-linear Painlevé VI equation into another equation *superficially simpler but where all the nonlinearity is encapsulated in the Weiertrass function* $\mathcal{P}_z$. Recalling [23], we see that if $z(\tau)$ is solution of the Painlevé VI equation with parameters $(\alpha_0, \alpha_1, \alpha_0, \alpha_1)$ one has

$$\begin{aligned}\frac{d^2 z(\tau)}{d\tau^2} &= \alpha_0 \cdot \Big(\mathcal{P}_z(z, \tau) + \mathcal{P}_z(z + \frac{\tau}{2}, \tau)\Big) \\ &\quad + \alpha_1 \cdot \Big(\mathcal{P}_z(z + \frac{1}{2}, \tau) + \mathcal{P}_z(z + \frac{1+\tau}{2}, \tau)\Big),\end{aligned} \tag{D.8}$$

† See the first equation without a number in section 1.6 of [23].

which can be rewritten using the identity (D.6), as:

$$\frac{d^2z(\tau)}{d\tau^2} \; = \; \frac{1}{4} \cdot \frac{d^2z(\tau)}{d(\tau/2)^2} \; = \; \alpha_0 \cdot \Bigl(\mathcal{P}_z(z, \, \frac{\tau}{2})\Bigr) \; + \alpha_1 \cdot \Bigl(\mathcal{P}_z(z + \frac{1}{2}, \, \frac{\tau}{2})\Bigr), \qquad (D.9)$$

or

$$\frac{d^2z(2\,\tau)}{d\tau^2} \; = \; 4 \cdot \alpha_0 \cdot \Bigl(\mathcal{P}_z(z, \, \tau)\Bigr) \; + 4 \cdot \alpha_1 \cdot \Bigl(\mathcal{P}_z(z + \frac{1}{2}, \, \tau)\Bigr), \qquad (D.10)$$

which means that $z(2\,\tau)$ *is also solution of the Painlevé VI equation but with parameters* $(4\,\alpha_0, \, 4\,\alpha_1, \, 0, \, 0)$. We thus see that we have a representation of the isogeny $\tau \, \rightarrow \, 2\,\tau$ on the Painlevé VI equations with a price to pay, *namely that the parameters are changed*. For $\alpha_1 \, = \, 0$ one gets the remarkable Hitchin's equation:

$$\frac{d^2 z}{d\tau^2} \; = \; -\frac{1}{2\,\pi^2} \cdot \frac{\partial \mathcal{P}(z, \, \tau)}{\partial z}. \qquad (D.11)$$

In that simple heuristic case $(\alpha_0, \, \alpha_1, \, \alpha_0, \, \alpha_1)$, the Landen transformation *preserves* the Gambier form (22) of Painlevé VI or the "master Painlevé equation" sigma-form of Painlevé VI. In general the Landen (or inverse Landen) transformation matches an Okamoto sigma-form of Painlevé VI onto a second order non-linear ODE like (23) with the Painlevé property which is *not* of the Okamoto sigma-form of Painlevé VI .

Appendix D.2. More modular correspondences and Painlevé VI transcendentals.

Let us recall Mazzocco and Vidunas paper on cubic and quartic transformations on Painlevé VI equation [35] (and also Vidunas and Kitaev paper [36]). In‡ equation (1.11) of [35], one has the following transformation (for the Tsuda, Okamoto, Sakai case [37]):

$$\tilde{t} \; = \; \frac{(1 \, + \sqrt{t})^2}{4\,\sqrt{t}}, \qquad \text{or:} \quad \frac{1}{\tilde{k}} \; = \; \frac{2\,\sqrt{k}}{1 \, + k}, \qquad \text{where:} \quad \tilde{t} \, = \, \tilde{k}^2, \quad t \, = \, k^2, \qquad (D.12)$$

which makes crystal clear that this transformation is, up to a Kramers-Wannier duality, again a *Landen transformation*.

In contrast, in the Picard's case, one has the algebraic transformation (see proposition 1.4 in [35])

$$\tilde{t} \; = \; \Bigl(\frac{t^{1/4} \, + \, 1}{t^{1/4} \, - \, 1}\Bigr)^4, \qquad (D.13)$$

which can be rewritten in a more symmetric way

$$\begin{aligned} t^4 \cdot \tilde{t}^4 \; & - 4 \cdot t^3 \cdot \tilde{t}^3 \cdot (t + \tilde{t}) \; + 2 \cdot (t^2 \cdot \tilde{t}^2 \, + 1) \cdot (3 \cdot t^2 - 376 \cdot t \cdot \tilde{t} + 3 \cdot \tilde{t}^2) \\ & - 4 \cdot (t \cdot \tilde{t} \, + 1) \cdot (t + \tilde{t}) \cdot (t^2 + 645 \cdot t \cdot \tilde{t} + \tilde{t}^2) \qquad (D.14) \\ & + t^4 + \tilde{t}^4 \, - 752 \cdot t \cdot \tilde{t} \cdot (t^2 \, + \tilde{t}^2) + 13348 \cdot t^2 \cdot \tilde{t}^2 \; - 4 \cdot (t + \tilde{t}) \; + 1 \; = \; 0. \end{aligned}$$

In order to have a relation between Hauptmoduls, let us perform the change of variables:

$$A \; = \; \frac{27 \cdot \, t^2 \cdot (1 - t)^2}{4 \cdot (t^2 - t + 1)^3}, \qquad B \; = \; \frac{27 \cdot \tilde{t}^2 \cdot (1 \, - \tilde{t})^2}{4 \cdot \bigl(\tilde{t}^2 - \tilde{t} + 1\bigr)^3}. \qquad (D.15)$$

The transformation (D.14) becomes the modular equation¶ which corresponds to $\tau \; \rightarrow \; 4 \cdot \tau$, or $\tau \; \rightarrow \; \tau/4$ (see section 5.1.1 of [24]). This modular equation can be obtained from the composition of the fundamental modular curve (corresponding to the Landen transformation) with itself (see section 5.1.1 of [24]).

‡ Or in proposition 3.1 of [35].

¶ It also corresponds to an isogeny of degree 3 of the underlying Legendre elliptic curve $w^2 \; = \; q \cdot (q-1) \cdot (q-t)$ (see page 5 of [35]).

Appendix E. Factorization of $C(0,5)$ and $C(0,7)$

From [1] we can deduce the four factors $g_i(0,5)$ for $C(0,5)$. We display here the $\tilde{g}_i(0,N)$ related to the $g_i(0,N)$ by

$$g_i(0,N) \; = \; (1-t)^{N/16} \cdot t^{-N/8} \cdot \tilde{g}_i(0,N), \qquad i \; = \; 1,2,$$
$$g_i(0,N) \; = \; (1-t)^{-N/16} \cdot t^{N/8} \cdot \tilde{g}_i(0,N), \qquad i \; = \; 3,4. \tag{E.1}$$

These $\tilde{g}_i(0,N)$'s read† for $N = 5$:

$$\tilde{g}_1(0,5) \; = \; \frac{2}{3} \cdot (1-t)^{-3/8} \cdot t^{-1} \cdot \Big((2t-1)\cdot \tilde{E} \; -(t-1)\cdot \tilde{K}\Big), \tag{E.2}$$

$$\tilde{g}_2(0,5) \; = \; \frac{2}{3} \cdot (1-t)^{-1/8} \cdot t^{-1} \cdot \Big((t+1)\cdot \tilde{E} \; +(t-1)\cdot \tilde{K}\Big), \tag{E.3}$$

$$\tilde{g}_3(0,5) \; = \; -\frac{8}{3} \cdot (1-t)^{1/4} \cdot t^{-2} \cdot \Big((t-2)\cdot \tilde{E} \; -2\cdot (t-1)\cdot \tilde{K}\Big) \tag{E.4}$$

$$\tilde{g}_4(0,5) \; = -\frac{8}{3} \cdot (1-t)^{1/2} \cdot t^{-2} \cdot \Big(3\,\tilde{E}^2 \; +2\cdot (t-2)\cdot \tilde{E}\tilde{K} \; -(t-1)\cdot \tilde{K}^2\Big). \tag{E.5}$$

For $N = 5$, these $\tilde{g}_i(0,N)$'s have the following expansions near $t = 0$:

$$\tilde{g}_1(0,5) \; = \; 1 \; + \frac{5}{2^7}\, t^2 \; + \frac{3^2\cdot 5}{2^{10}}\, t^3 \; + \frac{3\cdot 5^2\cdot 19}{2^{15}}\, t^4 \; + \frac{5479}{2^{17}}\, t^5 \; + \frac{5\cdot 11\cdot 3041}{2^{22}}\, t^6$$
$$+ \frac{3^2\cdot 5\cdot 7\cdot 11\cdot 23}{2^{21}}\, t^7 \; + \frac{3^4\cdot 5\cdot 7\cdot 11^2\cdot 227}{2^{31}}\, t^8 \; + \frac{5\cdot 7\cdot 11^2\cdot 17581}{2^{31}}\, t^9 \; + \cdots \tag{E.6}$$

$$\tilde{g}_2(0,5) \; = \; 1 \; + \frac{5}{2^7}\, t^2 \; + \frac{5\cdot 7}{2^{10}}\, t^3 \; + \frac{3^3\cdot 5\cdot 7}{2^{15}}\, t^4 \; + \frac{7\cdot 463}{2^{17}}\, t^5 \; + \frac{3\cdot 5\cdot 7\cdot 863}{2^{22}}\, t^6$$
$$+ \frac{3^3\cdot 5\cdot 149}{2^{20}}\, t^7 \; + \frac{3^2\cdot 5\cdot 7\cdot 11\cdot 19\cdot 563}{2^{31}}\, t^8 \; + \frac{3\cdot 5\cdot 17\cdot 132199}{2^{31}}\, t^9 \; + \cdots \tag{E.7}$$

$$\tilde{g}_3(0,5) \; = \; 1 \; - \frac{5}{2^7}\, t^2 \; - \frac{5}{2^7}\, t^3 \; - \frac{5\cdot 113}{2^{14}} t^4 - \frac{5\cdot 7^2}{2^{13}}\, t^5 \; - \frac{5\cdot 7\cdot 3119}{2^{22}}\, t^6$$
$$- \frac{5\cdot 19163}{2^{22}}\, t^7 \; - \frac{5\cdot 7\cdot 11\cdot 56443}{2^{30}}\, t^8 \; - \frac{5^2\cdot 11\cdot 17657}{2^{23}}\, t^9 \; - \cdots \tag{E.8}$$

$$\tilde{g}_4(0,5) \; = \; 1 - \frac{5}{2^7}\, t^2 \; - \frac{5}{2^7}\, t^3 \; - \frac{3\cdot 5\cdot 19}{2^{13}}\, t^4 \; - \frac{5^3}{2^{12}}\, t^5 \; - \frac{5\cdot 22541}{2^{22}}\, t^6$$
$$- \frac{3^4\cdot 5\cdot 13\cdot 19}{2^{22}}\, t^7 \; - \frac{5^3\cdot 47\cdot 1951}{2^{29}}\, t^8 \; - \frac{5\cdot 517129}{2^{27}}\, t^8 \; - \cdots \tag{E.9}$$

The four factors $g_i(0,7)$ for $C(0,7)$ read respectively

$$\tilde{g}_1(0,7) \; = \; \frac{8}{15t^2} \cdot (1-t)^{-1/4} \cdot \Big(2\cdot (t^2-t+1)\cdot \tilde{E} \; -(t-2)\cdot (t-1)\cdot \tilde{K}\Big), \tag{E.10}$$

$$\tilde{g}_2(0,7) \; = \; \frac{8}{45t^2} \cdot (1-t)^{-1/2} \cdot \Big((4t^2+11t-11)\cdot \tilde{E}^2$$
$$+8\cdot (t-2)\cdot (t-1)\cdot \tilde{E}\tilde{K} \; -5\cdot (1-t)^2\cdot \tilde{K}^2\Big), \tag{E.11}$$

† This normalization is chosen to have $\tilde{g}_i(0,N)$ series normalised as follows: $\tilde{g}_i(0,N) \; = \; 1 + o(t^2)$.

$$\tilde{g}_3(0,7) \;=\; \frac{64}{45t^4} \cdot (1-t)^{5/8} \cdot \Bigl((4t^2-19t+4) \cdot \tilde{E}^2 - 2 \cdot (8t^2-15t+4) \cdot \tilde{E}\tilde{K} \; + (7t\,-4) \cdot (t\,-1) \cdot \tilde{K}^2\Bigr), \qquad (E.12)$$

$$\tilde{g}_4(0,7) \;=\; -\frac{64}{45t^4} \cdot (1-t)^{3/8} \cdot \Bigl((11t^2-11t-4) \cdot \tilde{E}^2 + 2 \cdot (t-1) \cdot (3t^2-7t-4) \cdot \tilde{E}\tilde{K} \; - (t-1)^2 \cdot (3t+4) \cdot \tilde{K}^2\Bigr), \qquad (E.13)$$

which have the following expansions at $t = 0$:

$$\tilde{g}_1(0,7) \;=\; 1 \;+\; \frac{7}{2^7}\,t^2 \;+\frac{7}{2^7}\,t^3 \;+\frac{3^2\cdot 7\cdot 13}{2^{14}}\,t^4 \;+\frac{7\cdot 53}{2^{13}}\,t^5 \;+\frac{3^4\cdot 5\cdot 7\cdot 61}{2^{22}}\,t^6 + \frac{3\cdot 7^2\cdot 13\cdot 83}{2^{22}}\,t^7 \;+\frac{3\cdot 5\cdot 7\cdot 357293}{2^{30}}\,t^8 \;+\frac{5\cdot 7\cdot 13\cdot 19\cdot 1009}{2^{28}}\,t^9 + \frac{3^3\cdot 7\cdot 13\cdot 17\cdot 29\cdot 3449}{2^{37}}\,t^{10} \;+\cdots \qquad (E.14)$$

$$\tilde{g}_2(0,7) \;=\; 1 \;+\frac{7}{2^7}\,t^2 \;+\frac{7}{2^7}\,t^3 \;+\frac{7\cdot 61}{2^{13}}\,t^4 \;+\frac{7\cdot 29}{2^{12}}\,t^5 \;+\frac{5^2\cdot 7\cdot 11\cdot 103}{2^{22}}\,t^6 + \frac{7\cdot 47\cdot 577}{2^{22}}\,t^7 \;+\frac{7\cdot 13\cdot 41\cdot 6257}{2^{29}}\,t^8 \;+\frac{7\cdot 803461}{2^{27}}\,t^9 + \frac{7\cdot 23\cdot 17281729}{2^{36}}\,t^{10} \;+\;\cdots \qquad (E.15)$$

$$\tilde{g}_3(0,7) \;=\; 1 \;-\frac{7}{2^7}\,t^2 \;-\frac{7}{2^7}\,t^3 \;-\frac{3^2\cdot 5^2\cdot 7}{2^{15}}\,t^4 \;-\frac{7\cdot 1553}{2^{18}}\,t^5 - \frac{5\cdot 7\cdot 13\cdot 331}{2^{22}}\,t^6 - \frac{3^2\cdot 11\cdot 10631}{2^{25}}\,t^7 \;-\frac{7\cdot 13\cdot 652831}{2^{31}}\,t^8 - \frac{5\cdot 7\cdot 11\cdot 13\cdot 42257}{2^{33}}\,t^9 \;-\frac{3^2\cdot 7\cdot 11\cdot 13\cdot 389\cdot 1733}{2^{38}}\,t^{10} \;-\cdots \qquad (E.16)$$

$$\tilde{g}_4(0,7) \;=\; 1 \;-\frac{7}{2^7}\,t^2 \;-\frac{7}{2^7}\,t^3 \;-\frac{3^2\cdot 5^2\cdot 7}{2^{15}}\,t^4 \;-\frac{3\cdot 7\cdot 11\cdot 47}{2^{18}}\,t^5 - \frac{3\cdot 5\cdot 7\cdot 1429}{2^{22}}\,t^6 - \frac{3^2\cdot 116131}{2^{25}}\,t^7 \;-\frac{3^2\cdot 7\cdot 11\cdot 89\cdot 953}{2^{31}}\,t^8 - \frac{5\cdot 7\cdot 11\cdot 283\cdot 1913}{2^{33}}\,t^9 \;-\frac{3^7\cdot 7^2\cdot 11\cdot 13\cdot 389}{2^{38}}\,t^{10} \;-\;\cdots \qquad (E.17)$$

Using the definition of sigma (71) we find from (E.14)-(E.16) that:

$$\sigma_1(0,7) \;=\; \frac{7}{8} \;-\frac{7}{16}\,t \;-\frac{7}{2^6}\,t^2 \;-\frac{7}{2^7}\,t^3 \;-\frac{5\cdot 7^2}{2^{13}}\,t^4 \;-\frac{7\cdot 41}{2^{14}}\,t^5 \;-\frac{7\cdot 3251}{2^{21}}\,t^6 \;-\frac{7\cdot 41\cdot 103}{2^{22}}\,t^7 - \frac{7\cdot 22853}{2^{25}}\,t^8 \;-\frac{7\cdot 32027}{2^{26}}\,t^9 \;-\frac{7\cdot 11848691}{2^{35}}\,t^{10} \;+\cdots \qquad (E.18)$$

$$\sigma_2(0,7) \;=\; \frac{7}{8} \;-\frac{7}{16}\,t \;-\frac{7}{2^6}\,t^2 \;-\frac{7}{2^7}\,t^3 \;-\frac{3^2\cdot 5\cdot 7}{2^{13}}\,t^4 \;-\frac{7\cdot 71}{2^{14}}\,t^5 \;-\frac{7\cdot 13\cdot 577}{2^{21}}\,t^6 - \frac{7\cdot 19\cdot 23\cdot 29}{2^{22}}\,t^7 \;-\frac{3\cdot 7\cdot 115908}{2^{27}}\,t^8 \;-\frac{7\cdot 211\cdot 2857}{2^{28}}\,t^9 - \frac{7\cdot 13\cdot 89\cdot 58321}{2^{35}}\,t^{10} \;+\cdots \qquad (E.19)$$

$$\sigma_3(0,7) \;=\; -\frac{7}{8} \,+\frac{7}{16}\,t \,+\frac{7}{2^6}\,t^2 \,+\frac{7}{2^7}\,t^3 \,+\frac{5\cdot 7}{2^{10}}\,t^4 \,+\frac{7\cdot 17\cdot 53}{2^{18}}\,t^5 \,+\frac{7\cdot 11\cdot 31}{2^{17}}\,t^6 \,+\frac{7\cdot 34679}{2^{24}}\,t^7 \,+\frac{7\cdot 28517}{2^{24}}\,t^8 \,+\frac{7\cdot 673\cdot 915}{2^{32}}\,t^9 \,+\frac{7\cdot 163\cdot 520129}{2^{36}}\,t^{10} \,+\cdots \qquad \text{(E.20)}$$

$$\sigma_4(0,7) \;=\; -\frac{7}{8} \,+\frac{7}{16}\,t \,+\frac{7}{2^6}t^2 \,+\frac{7}{2^7}\,t^3 \,+\frac{5\cdot 7}{2^{10}}\,t^4 \,+\frac{3^4\cdot 7\cdot 11}{2^{18}}\,t^5 \,+\frac{7\cdot 331}{2^{17}}\,t^6 \,+\frac{5\cdot 7\cdot 6381}{2^{24}}\,t^7 \,+\frac{5\cdot 7\cdot 5279}{2^{24}}\,t^8 \,+\frac{3\cdot 7\cdot 53\cdot 83\cdot 421}{2^{32}}\,t^9 \,+\frac{7\cdot 61\cdot 311\cdot 3929}{2^{36}}\,t^{10} \,+\cdots \qquad \text{(E.21)}$$

Appendix E.1. Lambda extensions for $C(0,7)$

The recursion procedure on the solution for the sigma function defined by (71) are expressed in the forms (107)-(108) for $j = 1, 2$ as

$$\sigma_j(0,7;\lambda_j) \;=\; \frac{7}{8}\cdot\sqrt{1-t} \,+\lambda_j\,t^4 \,+\frac{3}{2}\,t^5\,\lambda_j \,+\frac{5^2\cdot 17}{2^8}\,t^6\,\lambda_j \,+\frac{5\cdot 13^2}{2^9}\,t^7\,\lambda_j \,+\Big(\frac{197\cdot 1301}{5\cdot 2^{15}}\,\lambda_j \,+\frac{1}{4}\,\lambda_j^2\Big)\cdot t^8 \,+\Big(\frac{7\cdot 73\cdot 929}{5\cdot 2^{16}}\,\lambda_j \,+\frac{7}{8}\,\lambda_j^2\Big)\cdot t^9 \,+\Big(\frac{3\cdot 7\cdot 61\cdot 21713}{5\cdot 2^{22}}\,\lambda_j \,+\frac{29\cdot 41}{5\cdot 2^7}\,\lambda_j^2\Big)\cdot t^{10} \,+\cdots \qquad \text{(E.22)}$$

and for $j = 3, 4$

$$\sigma_j(0,7;\lambda_j) \;=\; -\frac{7}{8}\cdot\sqrt{1-t} \,+\lambda_j\,t^5 \,+2\,t^6\,\lambda_j \,+\frac{887}{5\cdot 2^6}\,t^7\,\lambda_j \,+\frac{1061}{5\cdot 2^6}\,t^8\,\lambda_j \,+\frac{7\cdot 43049}{5\cdot 2^{14}}\,t^9\,\lambda_j \,+\Big(\frac{3\cdot 7\cdot 37\cdot 103}{5\cdot 2^{12}}\lambda_j \,+\frac{1}{5}\,\lambda_j^2\Big)\cdot t^{10} \,+\cdots \qquad \text{(E.23)}$$

The algebraic functions $\sigma_A^{(4)}(t;7)$ and $\sigma_A^{(1)}(t;7)$ (see (98))

$$\sigma_A^{(4)}(t;7) \;=\; \frac{7}{8}\,\sqrt{1-t}, \qquad\qquad \sigma_A^{(1)}(t;7) \;=\; -\frac{7}{8}\,\sqrt{1-t}, \qquad \text{(E.24)}$$

are two solutions of the sigma form of Painlevé VI (81) for $N = 7$. Setting

$$\lambda_2 \;=\; -\lambda_1 \;=\; \frac{5\cdot 7}{2^{13}}, \qquad\qquad \lambda_4 \;=\; -\lambda_3 \;=\; \frac{5\cdot 7}{2^{18}}, \qquad \text{(E.25)}$$

we see that (E.22) and (E.23) reproduce (E.18)-(E.20).

Appendix F. Computation of $B_1^{(1)}(0,N;t)$ and $B_1^{(4)}(0,N;t)$

To compute $B_1^{(1)}(0,N;t)$ and $B_1^{(4)}(0,N;t)$ in the expansion (107) and (108) we put $\sigma_2(0,N;t)$ and $\sigma_4(0,N;t)$ into the Okamoto equation (74) using the relations (80) and set the coefficient of each power of λ separately to zero. This gives the following linear differential equations for $B_1^{(1)}(0,N;t)$ and $B_1^{(4)}(0,N;t)$. The D-finite function $B_1^{(4)}(0,N;t)$ is solution of the second-order linear differential operator:

$$L_2^{(+)} \; = \; 4 \cdot t^2 \cdot (1-t)^2 \cdot \frac{d^2}{dt^2} \; + 2 \cdot t \cdot (t-1) \cdot \Big((2N+1) \cdot t \; - 2N\Big) \cdot \frac{d}{dt}$$
$$+ N \cdot \Big((t-1) \cdot \Big((N-1) \cdot t + 2\Big) \; - (t-2) \cdot \sqrt{1-t}\Big). \qquad \text{(F.1)}$$

The D-finite function $B_1^{(1)}(0,N;t)$ is solution of the second-order linear differential operator:

$$M_2^{(+)} \; = \; 4 \cdot t^2 \cdot (1-t)^2 \cdot \frac{d^2}{dt^2} \; + 2 \cdot t \cdot (t-1) \cdot \Big((2N+5) \cdot t \; - 2N \; - 4\Big) \cdot \frac{d}{dt}$$
$$+ \Big((t-1) \cdot \Big((N+1) \cdot (N+2) \cdot t \; - 2N\Big) \; + N \cdot (t-2) \cdot \sqrt{1-t}\Big). \qquad \text{(F.2)}$$

In order to get rid of the $\sqrt{1-t}$ terms we do the following trick: we introduce the companion operators of $L_2^{(+)}$ (resp. of $M_2^{(+)}$) which amount to changing the sign of $\sqrt{1-t}$. We denote $L_2^{(-)}$ (resp. of $M_2^{(-)}$) these linear differential operators. We calculate the LCLM (direct sum) of these two linear differential operators: $L_4 \; = \; LCLM(L_2^{(+)} \oplus L_2^{(-)}) \; = \; L_2^{(+)} \oplus L_2^{(-)}$, which is an order-four linear differential operator with polynomial coefficients (no square roots anymore). Using the LCLM-DFactorisation of Maple we find another LCLM (direct sum) for the order-four linear differential operator L_4

$$L_4 \; = \; \mathcal{L}_2^A \oplus \mathcal{L}_2^B \; = \; LCLM(\mathcal{L}_2^A, \, \mathcal{L}_2^B), \qquad \text{(F.3)}$$

where the two order-two linear differential operators $\mathcal{L}_2^A$ and $\mathcal{L}_2^B$ are, now, linear differential operators with rational coefficients, reading respectively:

$$\mathcal{L}_2^A \; = \; \frac{d^2}{dt^2} \; + \frac{N+1}{t} \cdot \frac{d}{dt} \; - \frac{N^2}{4 \cdot t \cdot (1-t)}, \qquad \text{(F.4)}$$

$$\mathcal{L}_2^B \; = \; \frac{d^2}{dt^2} \; + \frac{N+1}{t} \cdot \frac{d}{dt} \; + \frac{N^2 \cdot (t-1) \; - (t-2)}{4 \cdot t \cdot (1-t)^2}. \qquad \text{(F.5)}$$

The solution $f_1(t)$ of (F.4), which is analytic at $t = 0$, reads:

$$f_1(t) \; = \; {}_2F_1\Big([\frac{N}{2}, \frac{N}{2}], \, [N+1], \, t\Big). \qquad \text{(F.6)}$$

The solution $f_2(t)$ of (F.5), which is analytic at $t = 0$, reads:

$$f_2 \; = \; \sqrt{1-t} \cdot {}_2F_1\Big([\frac{N}{2}, \frac{N}{2}+1], \, [N+1], \, t\Big). \qquad \text{(F.7)}$$

The solution of (F.1) is a linear combination of f_1 and f_2

$$c_1 \cdot f_1 \; + c_2 \cdot f_2, \qquad \text{(F.8)}$$

which is determined using (F.8) in (F.1). This way we find that the solution $B_1^{(4)}(0,N;t)$, which is normalized to unity at $t = 0$, has $c_1 = c_2 = 1/2$ and reads:

$$B_1^{(4)}(0,N;t) \; = \qquad \text{(F.9)}$$
$$= \; \frac{1}{2} \cdot \Big({}_2F_1\Big([\frac{N}{2}, \frac{N}{2}], \, [N+1], \, t\Big) \; + \sqrt{1-t} \cdot {}_2F_1\Big([\frac{N}{2}, \frac{N}{2}+1], \, [N+1], \, t\Big)\Big)$$
$$= \; 1 \; + \Big(\frac{N-1}{4}\Big) \cdot t \; + \Big(\frac{N^3 + 2N^2 - 2N - 2}{32 \cdot (N+1)}\Big) \cdot t^2 \; + \; \cdots$$

An alternative form for $B_1^{(4)}(0, N; t)$ is obtained by use of the identities (64) on page 64 and (38) on page 103 of [38] to write

$$
{}_2F_1\Bigl([\frac{N}{2}, \frac{N}{2}], \, [N+1], \, t\Bigr) \; = \; (1\,-t) \cdot {}_2F_1\Bigl([\frac{N}{2}+1, \frac{N}{2}+1], \, [N+1], \, t\Bigr) \qquad \text{(F.10)}
$$

$$
= \; {}_2F_1\Bigl([\frac{N}{2}, \frac{N}{2}+1], \, [N+1], \, t\Bigr) \; - \frac{N \cdot t}{2 \cdot \, (N+1)} \cdot {}_2F_1\Bigl([\frac{N}{2}+1, \frac{N}{2}+1], \, [N+2], \, t\Bigr),
$$

where using

$$
\frac{d}{dt}\, {}_2F_1\Bigl([\frac{N}{2}, \frac{N}{2}], \, [N+1], \, t\Bigr)
$$

$$
= \; \frac{N^2}{4 \cdot \, (N+1)} \cdot {}_2F_1\Bigl([\frac{N}{2}+1, \frac{N}{2}+1], \, [N+2], \, t\Bigr), \qquad \text{(F.11)}
$$

we obtain the alternative expression (118):

$$
B_1^{(4)}(0, N; t) \; = \; \frac{1}{2\,N} \cdot \Bigl(2 \cdot \, t \cdot \, \sqrt{1-t} \cdot \, \frac{d}{dt}\, {}_2F_1\Bigl([\frac{N}{2}, \frac{N}{2}], \, [N+1], \, t\Bigr)
$$

$$
+ \, N \cdot \; (1 \, + \sqrt{1-t}) \cdot {}_2F_1\Bigl([\frac{N}{2}, \frac{N}{2}], \, [N+1], \, t\Bigr)\Bigr).
$$

This representation can be used to find a direct verification of (F.1) which does not require the use of Maple.

Appendix F.1. Computation of $B_1^{(1)}(0, N; t)$

The derivation of the exact expression (119) of $B_1^{(1)}(0, N; t)$ from (F.2) is done in a similar manner. The LCLM-DFactorisation of the LCLM (direct-sum) of $M_2^{(+)}$ and $M_2^{(-)}$ gives two order-two linear differential operators $\mathcal{M}_2^A$ and $\mathcal{M}_2^B$ which are, now, linear differential operators with rational coefficients, reading respectively:

$$
\mathcal{M}_2^A \; = \; \frac{d^2}{dt^2} \; + \frac{N+3}{t} \cdot \frac{d}{dt} \; + \frac{N^2 \cdot \, t \; + 4 \cdot \, (N+1) \cdot \, (t\,-1)}{,} \qquad \text{(F.12)}
$$

$$
\mathcal{M}_2^B \; = \; \frac{d^2}{dt^2} \; + \frac{N+3}{t} \cdot \frac{d}{dt}
$$

$$
+ \frac{1 \; + (t\,-1) \cdot \, (N^2\, t \, + (4\,N \, + 3) \cdot \, (t-1))}{4 \cdot \, t^2 \cdot \, (t\,-1)^2}. \qquad \text{(F.13)}
$$

We find in analogy with (F.9)

$$
B_1^{(1)}(0, N; t) \; = \; \frac{2 \cdot \, (N+1)}{t} \cdot \; {}_2F_1\Bigl([\frac{N}{2}, \frac{N}{2}], \, [N+1], \, t\Bigr)
$$

$$
- \, \frac{2 \cdot \, (N+1)}{t} \cdot \sqrt{1-t} \cdot {}_2F_1\Bigl([\frac{N}{2}, \frac{N}{2}+1], \, [N+1], \, t\Bigr), \qquad \text{(F.14)}
$$

and using (F.11) the alternative expression (119):

$$
B_1^{(1)}(0, N; t) \; = \; -\frac{2 \cdot \, (N+1)}{N\,t} \cdot \Bigl(2 \cdot \, t \cdot \, \sqrt{1-t} \cdot \, \frac{d}{dt}\, {}_2F_1\Bigl([\frac{N}{2}, \frac{N}{2}], \, [N+1], \, t\Bigr)
$$

$$
- \, N \cdot \, \Bigl(1 \, - \sqrt{1-t}\Bigr) \cdot {}_2F_1\Bigl([\frac{N}{2}, \frac{N}{2}], \, [N+1], \, t\Bigr)\Bigr) \qquad \text{(F.15)}
$$

$$
= \; 2 \cdot \, (N\,+1) \cdot \frac{1-t}{t} \cdot \, {}_2F_1\Bigl([\frac{N}{2}\,+1, \frac{N}{2}\,+1], \, [N+1], \, t\Bigr)
$$

$$
- \, 2 \cdot \, (N\,+1) \cdot \, \frac{\sqrt{1-t}}{t} \cdot \, {}_2F_1\Bigl([\frac{N}{2}, \frac{N}{2}\,+1], \, [N+1], \, t\Bigr) \qquad \text{(F.16)}
$$

Table G1. Determinant parameters for $C(M,N)$

n_1	n_2	$\tilde{N}$	n_3	n_4	η	p	p'
$N/2$	$N/2$	N	$-M/2$	$M/2$	0	$\frac{M-N}{2}$	$\frac{M-N}{2}$
$N/2$	$N/2$	N	$-M/2$	$M/2$	0	$-\frac{M+N}{2}$	$-\frac{M+N}{2}$

$$= \quad 1 \; + \Big(\frac{N+1}{4}\Big) \cdot t \; + \Big(\frac{N^3 \, + 8\,N^2 \, + 20\,N \, + 12}{32 \cdot (N \, + 3)}\Big) \cdot t^2 \; + \; \cdots$$

Appendix G. Forrester-Witte determinants

In this appendix we display the parameters of the Forrester-Witte determinants for the *two* factors of $C(M,N)$ with $M+N$ odd and the *four* factors of $C(0,N)$ with N odd, and give quite remarkable identities between Toeplitz determinants related directly to the factorizations of $C(M,N)$ analysed in this paper.

Appendix G.1. Determinant parameters for $C(M,N)$ with $M+N$ odd

Let us first recall form eq. (125) of [1] that the low-temperature correlations $C(M,N)$ with $M+N$ odd can also be written in terms of Forrester-Witte determinants with the Okamoto parameters given in table G1 as

$$C(M,N) \;=\; (1-k^2)^{((N-M)^2+1)/4} \cdot \, D\Big(N, 0, \frac{M-N}{2}, \frac{M-N}{2}, k\Big), \qquad \text{(G.1)}$$

where we use the notation $D\Big(\tilde{N}, \eta, p, p', k\Big)$ to mean the Toeplitz determinant obtained from the $\tilde{N} \times \tilde{N}$ matrix with elements $A_n^{(p,p',\eta)}$ (see eqs. (140) and (144)).

One can verify that the other choice of parameters in table G1 gives a similar expression since:

$$D\Big(N, 0, \frac{M-N}{2}, \frac{M-N}{2}, k\Big) \\ = \; (1-k^2)^{MN} \cdot \, D\Big(N, 0, -\frac{M+N}{2}, -\frac{M+N}{2}, k\Big). \qquad \text{(G.2)}$$

Appendix G.2. Determinant parameters for the two factors of $C(M,N)$ with $M+N$ odd

For the two factors of $C(M,N)$ with $M+N$ odd the Okamoto parameters are chosen from the set in (31). For N even both factors must have $\tilde{N} = N/2$ and for N odd we must have $\tilde{N} = (N+1)/2$ for one factor and $\tilde{N} = (N-1)/2$ for the other. There are many choices for n_i from the set (31) which are given in table G2 for N even and table G3 for N odd.

However, one can make the following remarks

Table G2. Determinant parameters for N even, $\tilde{N} = N/2$ and $\eta > 0$

n_1	n_2	$\tilde{N}$	n_3	n_4	η	p	p'
$\frac{M+N+1}{4}$	$-\frac{M-N+1}{4}$	$N/2$	$\frac{M+N-1}{4}$	$-\frac{M-N-1}{4}$	$N/2$	$-\frac{N-1}{2}$	$-\frac{M}{2}$
$\frac{M+N+1}{4}$	$-\frac{M-N+1}{4}$	$N/2$	$-\frac{M-N-1}{4}$	$\frac{M+N-1}{4}$	$N/2$	$\frac{M-N}{2}$	$-\frac{1}{2}$
$-\frac{M-N+1}{4}$	$\frac{M+N+1}{4}$	$N/2$	$\frac{M+N-1}{4}$	$-\frac{M-N-1}{4}$	$N/2$	$-\frac{M+N}{2}$	$\frac{1}{2}$
$-\frac{M-N+1}{4}$	$\frac{M+N+1}{4}$	$N/2$	$-\frac{M-N-1}{4}$	$\frac{M+N-1}{4}$	$N/2$	$-\frac{N+1}{2}$	$\frac{M}{2}$
$\frac{M+N-1}{4}$	$-\frac{M-N-1}{4}$	$N/2$	$\frac{M+N+1}{4}$	$-\frac{M-N+1}{4}$	$N/2$	$-\frac{N+1}{2}$	$-\frac{M}{2}$
$\frac{M+N-1}{4}$	$-\frac{M-N-1}{4}$	$N/2$	$-\frac{M-N+1}{4}$	$\frac{M+N+1}{4}$	$N/2$	$\frac{M-N}{2}$	$\frac{1}{2}$
$-\frac{M-N-1}{4}$	$\frac{M+N-1}{4}$	$N/2$	$\frac{M+N+1}{4}$	$-\frac{M-N+1}{4}$	$N/2$	$-\frac{M+N}{2}$	$-\frac{1}{2}$
$-\frac{M-N-1}{4}$	$\frac{M+N-1}{4}$	$N/2$	$-\frac{M-N+1}{4}$	$\frac{M+N+1}{4}$	$N/2$	$-\frac{N-1}{2}$	$\frac{M}{2}$

Table G3. Determinant parameters for N odd, $\tilde{N} = (N \pm 1)/2$ and $\eta > 0$

n_1	n_2	$\tilde{N}$	n_3	n_4	η	p	p'
$\frac{M+N+1}{4}$	$-\frac{M-N-1}{4}$	$(N+1)/2$	$\frac{M+N-1}{4}$	$-\frac{M-N+1}{4}$	$(N-1)/2$	$-\frac{N}{2}$	$-\frac{M+1}{2}$
$\frac{M+N+1}{4}$	$-\frac{M-N-1}{4}$	$(N+1)/2$	$-\frac{M-N+1}{4}$	$\frac{M+N-1}{4}$	$(N-1)/2$	$\frac{M-N}{2}$	$-\frac{1}{2}$
$-\frac{M-N-1}{4}$	$\frac{M+N+1}{4}$	$(N+1)/2$	$\frac{M+N-1}{4}$	$-\frac{M-N+1}{4}$	$(N-1)/2$	$-\frac{M+N}{2}$	$-\frac{1}{2}$
$-\frac{M-N-1}{4}$	$\frac{M+N+1}{4}$	$(N+1)/2$	$-\frac{M-N+1}{4}$	$\frac{M+N-1}{4}$	$(N-1)/2$	$-\frac{N}{2}$	$\frac{M-1}{2}$
$\frac{M+N-1}{4}$	$-\frac{M-N+1}{4}$	$(N-1)/2$	$\frac{M+N+1}{4}$	$-\frac{M-N-1}{4}$	$(N+1)/2$	$-\frac{N}{2}$	$-\frac{M-1}{2}$
$\frac{M+N-1}{4}$	$-\frac{M-N+1}{4}$	$(N-1)/2$	$-\frac{M-N-1}{4}$	$\frac{M+N+1}{4}$	$(N+1)/2$	$\frac{M-N}{2}$	$\frac{1}{2}$
$-\frac{M-N+1}{4}$	$\frac{M+N-1}{4}$	$(N-1)/2$	$\frac{M+N+1}{4}$	$-\frac{M-N-1}{4}$	$(N+1)/2$	$-\frac{M+N}{2}$	$\frac{1}{2}$
$-\frac{M-N+1}{4}$	$\frac{M+N-1}{4}$	$(N-1)/2$	$-\frac{M-N-1}{4}$	$\frac{M+N+1}{4}$	$(N+1)/2$	$-\frac{N}{2}$	$\frac{M+1}{2}$

(i) The FW-determinants associated to the first (resp. last) four rows of table G2, when N is even, are all related. For example, one has

$$D\Big(\frac{N}{2}, \frac{N}{2}, \frac{M-N}{2}, -\frac{1}{2}, k\Big) \\ = (-1)^{N/2} \cdot (1-k^2)^{N\,(M-1)/4} \cdot D\Big(\frac{N}{2}, \frac{N}{2}, -\frac{M+N}{2}, \frac{1}{2}, k\Big), \qquad (G.3)$$

and:

$$D\Big(\frac{N}{2}, \frac{N}{2}, \frac{M-N}{2}, \frac{1}{2}, k\Big) \\ = (-1)^{N/2} \cdot (1-k^2)^{N\,(M+1)/4} \cdot D\Big(\frac{N}{2}, \frac{N}{2}, -\frac{M+N}{2}, -\frac{1}{2}, k\Big). \qquad (G.4)$$

So we can use any one of them to represent the two factors appearing in $C(M, N)$ when $(M + N)$ is odd. We have choosen in the following to use the Okamoto parameters of row 2 and 6 in table G2.

(ii) Similarly, the FW-determinants associated to the first (resp. last) four rows of table G3, when N is odd, are also all related. For example, one has

$$D\Big(\frac{N-1}{2}, \frac{N+1}{2}, \frac{M-N}{2}, \frac{1}{2}, k\Big) \\ = (1-k^2)^{M\,(N-1)/4} \cdot D\Big(\frac{N-1}{2}, \frac{N+1}{2}, -\frac{M+N}{2}, \frac{1}{2}, k\Big), \qquad (G.5)$$

and

$$D\Big(\frac{N+1}{2}, \frac{N-1}{2}, \frac{M-N}{2}, -\frac{1}{2}, \, k\Big) \\ = \, (1-k^2)^{M \, (N+1)/4} \, \cdot \, D\Big(\frac{N+1}{2}, \frac{N-1}{2}, -\frac{M+N}{2}, -\frac{1}{2}, \, k\Big), \qquad (G.6)$$

So we can use any one of them to represent the two factors appearing in $C(M,N)$ when $(M+N)$ is odd, $M \, \le \, N$. We have choosen in the following to use the Okamoto parameters of row 2 and 6 in table G3.

(iii) We can now summarize the factorisations in two factors seen on $C(M,N)$ when $(M+N)$ is odd by the following two identities on Toeplitz determinants (we denote by $k_L \, = \, 2\sqrt{k}/(1+k)$ the Landen tranform of k)

- when M is odd and N is even

$$D\Big(N, 0, \frac{M-N}{2}, \frac{M-N}{2}, \, k\Big) \\ = \, (-1)^{N/2} \cdot \, 2^{N(N-2)/2} \cdot \, k^{-N^2/4} \cdot \, (1+k)^{N(2M-N)/2} \qquad (G.7) \\ \times \, D\Big(\frac{N}{2}, \frac{N}{2}, \frac{M-N}{2}, \frac{1}{2}, k_L\Big) \, \cdot \, D\Big(\frac{N}{2}, \frac{N}{2}, \frac{M-N}{2}, -\frac{1}{2}, \, k_L\Big).$$

- when M is even and N is odd

$$D\Big(N, 0, \frac{M-N}{2}, \frac{M-N}{2}, k\Big) \\ = \, 2^{(N-1)^2/2} \cdot \, k^{-(N^2-1)/4} \cdot \, (1+k)^{(2MN-N^2-1)/2} \qquad (G.8) \\ \times \, D\Big(\frac{N-1}{2}, \frac{N+1}{2}, \frac{M-N}{2}, \frac{1}{2}, \, k_L\Big) \cdot D\Big(\frac{N+1}{2}, \frac{N-1}{2}, \frac{M-N}{2}, -\frac{1}{2}, \, k_L\Big).$$

Replacing these relations in eq. (G.1), we obtain the factorisation of $C(M,N)$, with $(M+N)$ even, in two factors.

These factorisation relations can be seen as a consequence of the symmetry of the $N \times N$ Toeplitz matrix associated to $\eta = 0$ and $p = p' = (M-N)/2$ and Wilf relations (8) when N is even and (9) when N is odd [5].

Appendix G.3. Expressions of $g_\pm(M,N,t)$ in $C(M,N,t)$ with $(M+N)$ odd, in terms of Toeplitz determinants

- when M is odd and N is even, the factors $g_\pm(M,N,t)$ of $C(M,N,t)$ (see (11)) are given by:

$$(-1)^{N/2} \cdot \, D\Big(\frac{N}{2}, \frac{N}{2}, \frac{M-N}{2}, \frac{1}{2}, \, k_L\Big) \, \cdot \, \Big(\frac{1+k}{1-k}\Big)^{(N-M)/4} \, \cdot \, S_1,$$

$$D\Big(\frac{N}{2}, \frac{N}{2}, \frac{M-N}{2}, -\frac{1}{2}, \, k_L\Big) \, \cdot \, \Big(\frac{1+k}{1-k}\Big)^{(M-N)/4} \, \cdot \, S_1,$$

with

$$S_1 \, = \, (-1)^{E((N+2)/4)} \cdot \, 2^{N(N-2)/4} \cdot \, \frac{(1-k)^{(M-N)^2/8}}{k^{N^2/8}} \, \cdot \, (1+k)^{(2MN+M^2-N^2)/8},$$

where $E(x)$ denotes the integer part of x.

Table G4. Determinant parameters for $C(0,N)$ with $N = 4m+1$

n_1	n_2	$\tilde{N}$	n_3	n_4	η	p	p'
$m+\frac{1}{2}$	$-\frac{1}{2}$	$\frac{N-1}{4}$	m	0	$\frac{N-1}{4}$	$-m+\frac{1}{2}$	$-m-\frac{1}{2}$
$m+\frac{1}{2}$	$-\frac{1}{2}$	$\frac{N-1}{4}$	0	m	$\frac{N-1}{4}$	$\frac{1}{2}$	$-\frac{1}{2}$
$-\frac{1}{2}$	$m+\frac{1}{2}$	$\frac{N-1}{4}$	m	0	$\frac{N-1}{4}$	$-2m-\frac{1}{2}$	$\frac{1}{2}$
$-\frac{1}{2}$	$m+\frac{1}{2}$	$\frac{N-1}{4}$	0	m	$\frac{N-1}{4}$	$-m-\frac{1}{2}$	$m+\frac{1}{2}$
$m+\frac{1}{2}$	$\frac{1}{2}$	$\frac{N+3}{4}$	m	0	$\frac{N-1}{4}$	$-m-\frac{1}{2}$	$-m-\frac{1}{2}$
$m+\frac{1}{2}$	$\frac{1}{2}$	$\frac{N+3}{4}$	0	m	$\frac{N-1}{4}$	$-\frac{1}{2}$	$-\frac{1}{2}$
$\frac{1}{2}$	$m+\frac{1}{2}$	$\frac{N+3}{4}$	m	0	$\frac{N-1}{4}$	$-2m-\frac{1}{2}$	$-\frac{1}{2}$
$\frac{1}{2}$	$m+\frac{1}{2}$	$\frac{N+3}{4}$	0	m	$\frac{N-1}{4}$	$-m-\frac{1}{2}$	$m-\frac{1}{2}$
m	0	$\frac{N-1}{4}$	$m+\frac{1}{2}$	$-\frac{1}{2}$	$\frac{N-1}{4}$	$-m-\frac{1}{2}$	$-m-\frac{1}{2}$
0	m	$\frac{N-1}{4}$	$m+\frac{1}{2}$	$-\frac{1}{2}$	$\frac{N-1}{4}$	$-2m-\frac{1}{2}$	$-\frac{1}{2}$
m	0	$\frac{N-1}{4}$	$-\frac{1}{2}$	$m+\frac{1}{2}$	$\frac{N-1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$
0	m	$\frac{N-1}{4}$	$-\frac{1}{2}$	$m+\frac{1}{2}$	$\frac{N-1}{4}$	$-m+\frac{1}{2}$	$m-\frac{1}{2}$
m	0	$\frac{N-1}{4}$	$m+\frac{1}{2}$	$\frac{1}{2}$	$\frac{N+3}{4}$	$-m-\frac{1}{2}$	$-m+\frac{1}{2}$
0	m	$\frac{N-1}{4}$	$m+\frac{1}{2}$	$\frac{1}{2}$	$\frac{N+3}{4}$	$-2m-\frac{1}{2}$	$\frac{1}{2}$
m	0	$\frac{N-1}{4}$	$\frac{1}{2}$	$m+\frac{1}{2}$	$\frac{N+3}{4}$	$-\frac{1}{2}$	$\frac{1}{2}$
0	m	$\frac{N-1}{4}$	$\frac{1}{2}$	$m+\frac{1}{2}$	$\frac{N+3}{4}$	$-m-\frac{1}{2}$	$m+\frac{1}{2}$

- when M is even and N is odd, one has

$$D\Big(\frac{N-1}{2}, \frac{N+1}{2}, \frac{M-N}{2}, \frac{1}{2}, k_L\Big) \cdot \Big(\frac{1+k}{1-k}\Big)^{(N-M)/4} \cdot S_2,$$

$$D\Big(\frac{N+1}{2}, \frac{N-1}{2}, \frac{M-N}{2}, -\frac{1}{2}, k_L\Big) \cdot \Big(\frac{1+k}{1-k}\Big)^{(M-N)/4} \cdot S_2,$$

with

$$S_2 \;=\; (-1)^{E((N+2)/4)} \cdot 2^{(N-1)^2/4} \cdot \frac{(1-k)^{(M-N)^2/8}}{k^{(N^2-1)/8}} \cdot (1+k)^{(2MN+M^2-N^2-2)/8}.$$

All these expressions are compatible with the series expansions of Appendix A. From the above relations, we can also obtain closed expressions for $f_1(t)$ and $f_2(t)$ appearing in Appendix A.

Appendix G.4. Determinant parameters for the four factors of $C(0,N)$ with N odd

For the four factors of $C(0,N)$ with N odd the Okamoto parameters are chosen from the set (73). The two cases must be considered separately $N = 4\,m \pm 1$. For $N = 4\,m + 1$ the values of $\tilde{N}$ for the factors are

$$\tilde{N} \;=\; m, \;\; m, \;\; m, \;\; m+1, \qquad\qquad (G.9)$$

and for $n = 4m - 1$ the values of $\tilde{N}$ are:

$$\tilde{N} \;=\; m, \;\; m, \;\; m, \;\; m-1. \qquad\qquad (G.10)$$

The choices of n_i which give integer $\tilde{N}$ for $N = 4\,m + 1$ are given in table G4 and for $N = 4\,m - 1$ in table G5.

One can make similar remarks as in the previous section:

Table G5. Determinant parameters for $C(0,N)$ with $N = 4m-1$

n_1	n_2	$\tilde{N}$	n_3	n_4	η	p	p'
$m-\frac{1}{2}$	$\frac{1}{2}$	$\frac{N+1}{4}$	m	0	$\frac{N+1}{4}$	$-m-\frac{1}{2}$	$-m+\frac{1}{2}$
$m-\frac{1}{2}$	$\frac{1}{2}$	$\frac{N+1}{4}$	0	m	$\frac{N+1}{4}$	$-\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	$m-\frac{1}{2}$	$\frac{N+1}{4}$	m	0	$\frac{N+1}{4}$	$-2m-\frac{1}{2}$	$-\frac{1}{2}$
$\frac{1}{2}$	$m-\frac{1}{2}$	$\frac{N+1}{4}$	0	m	$\frac{N+1}{4}$	$-m+\frac{1}{2}$	$-\frac{1}{2}+m$
$m-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{N-3}{4}$	m	0	$\frac{N+1}{4}$	$-m+\frac{1}{2}$	$-m+\frac{1}{2}$
$m-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{N-3}{4}$	0	m	$\frac{N+1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$
$-\frac{1}{2}$	$m-\frac{1}{2}$	$\frac{N-3}{4}$	m	0	$\frac{N+1}{4}$	$-2m+\frac{1}{2}$	$\frac{1}{2}$
$-\frac{1}{2}$	$m-\frac{1}{2}$	$\frac{N-3}{4}$	0	m	$\frac{N+1}{4}$	$-m+\frac{1}{2}$	$m+\frac{1}{2}$
m	0	$\frac{N+1}{4}$	$m-\frac{1}{2}$	$\frac{1}{2}$	$\frac{N+1}{4}$	$-m+\frac{1}{2}$	$-m+\frac{1}{2}$
0	m	$\frac{N+1}{4}$	$m-\frac{1}{2}$	$\frac{1}{2}$	$\frac{N+1}{4}$	$-2m+\frac{1}{2}$	$\frac{1}{2}$
m	0	$\frac{N+1}{4}$	$\frac{1}{2}$	$m-\frac{1}{2}$	$\frac{N+1}{4}$	$-\frac{1}{2}$	$-\frac{1}{2}$
0	m	$\frac{N+1}{4}$	$\frac{1}{2}$	$m-\frac{1}{2}$	$\frac{N+1}{4}$	$-m-\frac{1}{2}$	$m-\frac{1}{2}$
m	0	$\frac{N+1}{4}$	$m-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{N-3}{4}$	$-m+\frac{1}{2}$	$-m-\frac{1}{2}$
0	m	$\frac{N+1}{4}$	$m-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{N-3}{4}$	$-2m+\frac{1}{2}$	$-\frac{1}{2}$
m	0	$\frac{N+1}{4}$	$-\frac{1}{2}$	$m-\frac{1}{2}$	$\frac{N-3}{4}$	$\frac{1}{2}$	$-\frac{1}{2}$
0	m	$\frac{N+1}{4}$	$-\frac{1}{2}$	$m-\frac{1}{2}$	$\frac{N-3}{4}$	$-m+\frac{1}{2}$	$m-\frac{1}{2}$

(i) The Okamoto parameters of the FW-determinants are displayed in tables G4 and G5 in four groups of four rows. We can use any row in each group to represent the four factors appearing in $C(0,N)$ when N is odd.

(ii) As in the previous section, we can summarize the factorisations in four factors seen on $C(0,N)$ when N is odd by the following identities on Toeplitz determinants

- when $N = 1 \mod 4$, i.e. $N = 1, 5, 9, 13, \cdots$

$$D\Big(\frac{N-1}{2}, \frac{N+1}{2}, -\frac{N}{2}, \frac{1}{2}, k_L\Big) = (-1)^{(N-1)/4} \cdot \Big(\frac{1+k}{1-k}\Big)^{(N-1)^2/8} \times D\Big(\frac{N-1}{4}, \frac{N-1}{4}, \frac{1}{2}, -\frac{1}{2}, k\Big) \cdot D\Big(\frac{N-1}{4}, \frac{N+3}{4}, -\frac{1}{2}, \frac{1}{2}, k\Big), \tag{G.11}$$

and:

$$D\Big(\frac{N+1}{2}, \frac{N-1}{2}, -\frac{N}{2}, -\frac{1}{2}, k_L\Big) = (1+k) \cdot \Big(\frac{1+k}{1-k}\Big)^{(N-1)(N+3)/8} \cdot \times D\Big(\frac{N-1}{4}, \frac{N-1}{4}, \frac{1}{2}, \frac{1}{2}, k\Big) \cdot D\Big(\frac{N+3}{4}, \frac{N-1}{4}, -\frac{1}{2}, -\frac{1}{2}, k\Big). \tag{G.12}$$

- when $N = 3 \mod 4$, i.e. $N = 3, 7, 11, 15, \cdots$

$$D\Big(\frac{N+1}{2}, \frac{N-1}{2}, -\frac{N}{2}, -\frac{1}{2}, k_L\Big) = (-1)^{(N+1)/4} \cdot \Big(\frac{1+k}{1-k}\Big)^{(N+1)^2/8} \times D\Big(\frac{N+1}{4}, \frac{N+1}{4}, -\frac{1}{2}, \frac{1}{2}, k\Big) \cdot D\Big(\frac{N+1}{4}, \frac{N-3}{4}, \frac{1}{2}, -\frac{1}{2}, k\Big), \tag{G.13}$$

and:

$$D\Big(\frac{N-1}{2}, \frac{N+1}{2}, -\frac{N}{2}, \frac{1}{2}, k_L\Big)$$
$$= \quad (1+k) \cdot \left(\frac{1+k}{1-k}\right)^{(N+1)(N-3)/8} \tag{G.14}$$
$$\times \, D\Big(\frac{N+1}{4}, \frac{N+1}{4}, -\frac{1}{2}, -\frac{1}{2}, k\Big) \cdot D\Big(\frac{N-3}{4}, \frac{N+1}{4}, \frac{1}{2}, \frac{1}{2}, k\Big).$$

Replacing these relations in eqs. (G.1) and using (G.7) and (G.8), we obtain the factorisation of $C(0,N)$, with N even, in four factors.

Appendix G.5. Expressions of $g_i(M,N,t)$ in $C(0,N,t)$ with N odd, in terms of Toeplitz determinants

If we denote $S = (-1)^{E((N+4)/8)}$, then the factors appearing in eq. (70), solutions of the nonlinear equation (81), with the coefficient of their leading term normalised to one, are given by

- for $N = 1 \mod 4$, i.e. $N = 1, 5, 9, 13, \cdots$

$$D\Big(\frac{N-1}{4}, \frac{N-1}{4}, \frac{1}{2}, -\frac{1}{2}, k\Big) \cdot \frac{(1-k^2)^{-1/16}}{k^{(N+1)^2/16}} \cdot 2^{(N-1)(N-3)/8} \cdot S,$$
$$(-1)^{(N-1)/4} \cdot D\Big(\frac{N-1}{4}, \frac{N-1}{4}, \frac{1}{2}, \frac{1}{2}, k\Big) \cdot \frac{(1-k^2)^{3/16}}{k^{(N+1)^2/16}} \cdot 2^{(N-1)(N-3)/8} \cdot S,$$
$$D\Big(\frac{N-1}{4}, \frac{N+3}{4}, -\frac{1}{2}, \frac{1}{2}, k\Big) \cdot \frac{(1-k^2)^{-1/16}}{k^{(N+1)(N-3)/16}} \cdot 2^{(N^2-1)/8} \cdot S,$$
$$D\Big(\frac{N+3}{4}, \frac{N-1}{4}, -\frac{1}{2}, -\frac{1}{2}, k\Big) \cdot \frac{(1-k^2)^{3/16}}{k^{(N+1)(N-3)/16}} \cdot 2^{(N^2-1)/8} \cdot S.$$

- for $N = 3 \mod 4$, i.e. $N = 3, 7, 11, 15, \cdots$

$$D\Big(\frac{N-3}{4}, \frac{N+1}{4}, \frac{1}{2}, \frac{1}{2}, k\Big) \cdot \frac{(1-k^2)^{3/16}}{k^{(N-1)(N+3)/16}} \cdot 2^{(N-1)(N-3)/8} \cdot S.$$
$$D\Big(\frac{N+1}{4}, \frac{N-3}{4}, \frac{1}{2}, -\frac{1}{2}, k\Big) \cdot \frac{(1-k^2)^{-1/16}}{k^{(N-1)(N+3)/16}} \cdot 2^{(N-1)(N-3)/8} \cdot S,$$
$$(-1)^{(N+1)/4} \cdot D\Big(\frac{N+1}{4}, \frac{N+1}{4}, -\frac{1}{2}, -\frac{1}{2}, k\Big) \cdot \frac{(1-k^2)^{3/16}}{k^{(N-1)^2/16}} \cdot 2^{(N^2-1)/8} \cdot S,$$
$$D\Big(\frac{N+1}{4}, \frac{N+1}{4}, -\frac{1}{2}, \frac{1}{2}, k\Big) \cdot \frac{(1-k^2)^{-1/16}}{k^{(N-1)^2/16}} \cdot 2^{(N^2-1)/8} \cdot S,$$

All these expressions are compatible with the series expansions of Appendix E.

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