FORMAL CONSTRAINTS ON SERIES ANALYSIS ON THE POTTS MODEL

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Abstract: It is shown that the low temperature expansion of the partition function, magnetization and nearest neighbour correlation functions of the q-state checkerboard Potts model in a magnetic field drastically simplify on a very simple algebraic variety.These four formal constraints on the expansions are also analyzed in the framework of the resummed low temperature expansions and the large q expansions. These exact results are generalized straightforwardly to higher dimensional hypercubic lattices and also to some random problems.

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I. Introduction.

It has been shown in a previous paper[1] (denoted hereafter paper I) that the low temperature expansions of the partition function, spontaneous magnetization and nearest neighbour correlation functions of the checkerboard Potts model in zero magnetic field drastically simplify on the dual of the disorder variety of the model (for a review on disorder solutions see for instance Rùjan [2,3]). As this variety lies in the non physical domain of the parameter space , these results must be considered as exact formal constraints bearing on the low anisotropic temperature expansion of the model. In paper I these exact constraints have been verified on the low temperature expansions of the checkerboard Potts model obtained up to order twelve. In this work it is shown that these results can actually be generalized even in a non zero magnetic field, leading to new constraints .

In the first part of this paper, it is shown that, restricted to some algebraic varieties, the four quantities previously mentioned simplify to give rational expressions in the low temperature variables.An heuristic but simple argument is also given in order to understand these results These formal exact results are checked up to order twelve on the low temperature expansion of the model obtained in paper I. They are also checked on the resummed low temperature expansions and on the large q expansions. Finally, one shows that these results can be generalized straightforwardly to the Potts models on higher dimensional hypercubic lattices and also to some random field Potts models.

II. Formal constraints on the low temperature expansion of the checkerboard Potts model in a magnetic field.

The partition function per site Z of the q-state checkerboard scalar Potts model in a magnetic field is given by:

$$Z^N (a,b,c,d;h) = \sum_{\{\sigma\}} \prod_{<ij>} a^{\delta_{\sigma_i,\sigma_j}} \prod_{<jk>} b^{\delta_{\sigma_j,\sigma_k}} \prod_{<kl>} c^{\delta_{\sigma_k,\sigma_l}} \prod_{<li>} d^{\delta_{\sigma_l,\sigma_i}} \prod_{m} h^{\delta_{\sigma_m,0}} \qquad (1)$$

a,b,c,d denote the exponential of the four coupling constants of the model (see figure 1) and h is the exponential of the magnetic field. The spins σ belong to Z_q .One defines the normalized low temperature partition function per site by:

$$Z(a,b,c,d;h)= (abcdh^2)^{1/2} \Lambda(a,b,c,d;h)$$

The parameters of the low temperature expansion of the model will be denoted by: A=1/a, B=1/b,C=1/c,D=1/d and z=1/h. The expansion of $\ln \Lambda$ was established up to order twelve in paper I. Let us recall its first terms:

$$\ln\Lambda(A,B,C,D;z) = (q-1)ABCDz+(q-1)(A^2B^2C^2+A^2B^2D^2+A^2C^2D^2+B^2C^2D^2)\, z^2/2$$
$$+1/2(q-1)(q-2)(AB^2C^2D^2+A^2BC^2D^2+ A^2B^2CD^2 + A^2B^2C^2D)\, z^2 + ... \tag{2}$$

As a generalization of the variety studied in paper I (the dual of the disorder variety) we claim here that restricted to the algebraic variety :

$$D + ABC\,z + (q-2)\, ABCD\, z = 0 \tag{3}$$

the following results hold:

$$\ln \Lambda \Big|_{(3)}=1/2 \ \ln \ (1+(q-1)ABCD \ \ z \ )=1/2 \ \ln \ (1-(q-1) \ \ D^2 / \ (1+(q-2) \ D) \ ) \tag{4}$$

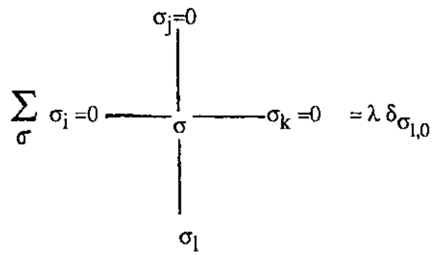
$$A\, (\partial/\partial A) \ln \Lambda \ (A,B,C,D; z) \Big|_{(3)} = 0 \tag{5}$$

$$D\, (\partial/\partial D) \ln \Lambda \ (A,B,C,D; z) \Big|_{(3)} = D/2\Big(d/dD\Big)\ln\, (1- (q-1)\, D^2 /(1+(q-2)D) \,) \tag{6}$$

$$z\, (\partial/\partial z) \ \ln \Lambda \ (A,B,C,D; z) \Big|_{(3)} = (1-M)(q-1)/q \ = 0 \tag{7}$$

where M denotes the magnetization of the model. One thus has four exact results bearing on the low temperature expansion of $\ln \Lambda$ and on its different derivatives. Using formal calculation program REDUCE [4] and the expansion of paper I we have actually verified, up to order twelve, equations (4),(5),(6),(7) .

These properties can be understood in an heuristic way with a decimation procedure very similar to the one introduced by Jaekel and Maillard[5] or Dhar and Maillard[6]. Indeed, let us consider the following local condition on the elementary cell of the model (see fig 2).



$$= \sum a^{\delta_{\sigma_i,\sigma}}\, b^{\delta_{\sigma_j,\sigma}}\ c^{\delta_{\sigma_k,\sigma}}\, d^{\delta_{\sigma_l,\sigma}}\, \delta_{\sigma_i,0}\, \delta_{\sigma_j,0}\, \delta_{\sigma_k,0} \tag{8}$$

This amounts to say that, when the three spins σ_i , σ_j , σ_k are fixed in the same state 0 the sum over the spin σ constrains the remaining spin σ_l to be in the same state 0. Analytically, this condition is expressed as (3). When it is met, λ is given by:

$$\lambda = \ abcdh \ (1+(q-1) \ \ ABCDz) \tag{9}$$

It is a straightforward matter to see that, using this local condition on the lattice with the boundary conditions depicted on figure 3, one can decimate the whole lattice. Indeed , this condition allows to"eat" the lattice from the top (or from the bottom) so that the partition function per site reduces to the one of an elementary cell (see figure 2) . It is also possible to argue in a similar way, eating the lattice from the top and the bottom, to show that the nearest neighbour correlation functions reduce also to the one of some one dimensional model [6]. The exact expressions thus obtained are actually in agreement with (4),(5),(6),(7).

This decimation procedure can be seen as the "dual" of the one described in Jaekel and Maillard [5] and Dhar and Maillard [6] In these decimation procedures well suited for the disorder solutions, the pure disordered state plays a special role, which is played here by the pure ordered state. These very simple and exact results are reminiscent of the one concerning the KDP model which can be solved in any dimension for $T < T_c$ [7] .This last solution also corresponds to a pure ordered state. However in the case of the KDP model (for $T < T_c$) the pure ordered state is the ground state of the model . In the transfer matrix formalism the pure ordered state is an eigenvector , the eigenvalue of which is the largest of the whole spectrum. For $T > T_c$, it is still an eigenvector but its eigenvalue is no longer the largest one. In the case of this paper, (3) is not a physical condition and (4) will in general no longer be the logarithm of the (physical) normalized partition function of the model but the analytical continuation to non physical values of the parameters of the low temperature expansion. Equations (3) to (7) are formal but exact conditions that are indeed satisfied by the (analytical continuation of the) low temperature expansion of the model.

III. Resummed low temperature expansions.

The resummed low (or high) temperature expansion is well suited for the analysis of the inversion relation on the checkerboard Potts model [8]. The two expansion parameters are for instance B and D while A , C and z are not necessarily small.

This expansion has already been established for the model with zero magnetic field up to order four in B and D [8]. This result can be generalized when the magnetic field is non zero. This expansion reads:

$$\ln \Lambda (A,B,C,D\,;z) = (q-1)\frac{BX+DY}{2} + (q-1)(q-2)\frac{BX^2+DY^2}{2} + (q-1)\frac{A^2z^2+C^2z^2+2A^2C^2z^4}{2(1-A^2C^2z^4)}$$
$$(BD+BY+DX)^2 + (q-1)(q-2)\frac{ACz^2(A+C+2A^2C^2z^4)}{2(1-A^3C^3z^4)}(BD+BY+DX)^2 +$$
$$(q-1)(q-2)^2\frac{ACz}{2(1-A^2C^2z^2)}(2BDXY + AC(B^2X^2+D^2Y^2))+$$
$$(q-1)^2\left\{\left[-\frac{1}{2}-\frac{3}{(1-A^2C^2z^2)}\right]\left(\frac{A^2C^2z^2}{1-A^2C^2z^2}\right)^2 (B^4+D^4)/2 + 4\left[1-\frac{3ACz}{1-A^2C^2z^2}\right]\left(\frac{ACz}{1-A^2C^2z^2}\right)^2\right.$$
$$\left.(B^3D+BD^3) + 2\left[\frac{1}{2}-\frac{3}{1-A^2C^2z^2}\right](1+2A^2C^2z^2)\left(\frac{ACz}{1-A^2C^2z^2}\right)^2 B^2D^2\right\} + \ldots\ldots \tag{10}$$

where :

$$X = \frac{ACz}{1-A^2C^2z^2}(D+ABCz) \quad \text{and} \quad Y = \frac{ACz}{1-A^2C^2z^2}(B+ADCz)$$

On this resummed expansion one easily verifies the inversion relation of the model[8].

$$\ln \Lambda(A,B,C,D\,;z) + \ln \Lambda\left(\frac{1}{A}, \frac{-B}{1+(q-2)B}, \frac{1}{C}, \frac{-D}{1+(q-2)D}; \frac{1}{z}\right) =$$
$$\frac{1}{2}\ln\left(1-\frac{(q-1)B^2}{1+(q-2)B}\right) + \frac{1}{2}\ln\left(1-\frac{(q-1)D^2}{1+(q-2)D}\right) \tag{11}$$

Note that the exact formal results (4),(5),(6) and (7) can also be verified on the resummed expansion . Indeed, (3) is actually compatible with B and D both small. Equations (4),(5),(6),(7) lead to much more severe constraints on the resummed expansions than on the classic low temperature expansions. Indeed they have to be verified for B and D small *whatever* A,C and z are. One actually verifies these equations up to order four in B and D. Conversely one can use equations (4),(5),(6),(7) together with the inversion relation (1) and the low temperature expansion of paper I up to order twelve in order to determine the resummed expansion up to order five (and eventually to higher orders). However, this requires the detailed analysis of some specific classes of diagrams which is pretty tedious. Work is in progress to get the fifth order that way.

IV. Large q expansions on the checkerboard Potts model.

Large q expansions have already been obtained on the checkerboard Potts model [9] even in a magnetic field [10]. They are useful to analyze the vicinity of the critical variety of the model where a great number of exact results are available for zero magnetic field (partition function, internal energy, spontaneous magnetization, latent heat, critical exponents... see for a review the paper by Wu [11]). Equations (4),(5),(6), and (7) can also be checked on the large q expansion in a magnetic field. In this expansion A,B,C have to be considered as order $q^{-1/2}$ while D when satisfying (3) must be considered as order three in $q^{-1/2}$. Indeed , we have verified all this set of equations up to order six in $q^{-1/2}$ using the expansion given by Maillard and Rammal [10]. Work is in progress to derive order seven in $q^{-1/2}$ from these constraints.

V. Exact results on a cubic Potts model.

The results of the previous sections can be extended to a generalized cubic Potts model (depending on six coupling constants) in a magnetic field. The elementary cell of the model is depicted in figure 4.

The generalization of condition (3) for this model reads:

$$\sum_{\sigma} e^{K_1\sigma_i\sigma} e^{K_2\sigma_j\sigma} e^{K_3\sigma_k\sigma} e^{K_4\sigma_l\sigma} e^{K_5\sigma_m\sigma} e^{K_6\sigma_n\sigma} e^{H\sigma} \cdot \delta_{\sigma_i,0}\,\delta_{\sigma_j,0}\,\delta_{\sigma_k,0}\,\delta_{\sigma_l,0}\,\delta_{\sigma_m,0} = \mu\,\delta_{\sigma_n,0} \qquad (14)$$

This condition means that when the five spins σ_i , ... ,σ_m , are in the same state (say 0) the summation over the central spin σ constrains the remaining spin to be also in the state 0. The suitable boundary conditions for that cubic lattice are depicted in figure 5. Introducing the low temperature variables $A = e^{-K_1}$, $B = e^{-K_2}$, $C=e^{-K_3}$, $D=e^{-K_4}$, $E=e^{-K_5}$, $F=e^{-K_6}$ and $z=e^{-h}$ one gets, as a generalization of equations (1) to (5), on the algebraic variety:

$$F+ABCDEz+(q-2)ABCDEFz = 0 \qquad (15)$$

$$\ln\Lambda(A,B,C,D,E,F;\,z) = 1/2\,\ln\left(1-(q-1)F^2/(1+(q-2)F)\right) \qquad (16)$$

$$A\,(\partial/\partial A)\,\ln\Lambda\Big|_{(15)} = 0 \qquad (17)$$

$$F\,(\partial/\partial F)\,\ln\Lambda\Big|_{(15)} = F/2\;\; d/df\,\ln\left(1-(q-1)F^2/(1+(q-2)F)\right) \qquad (18)$$

$$z\,(\partial/\partial z)\,\ln\Lambda\Big|_{(15)} = 0 \qquad (19)$$

One can easily verify these equations on the lowest orders of the low temperature expansion of this cubic model.

Note that, in the subcase of the six parameters cubic Ising model with zero magnetic field, equation (15) reads :

$$e^{-K_6}+e^{-\sum_{i=1}^{5}K_i} = 0 \qquad (20)$$

This condition leads to non physical values of the coupling constants. However it is worth noticing that, if one considers the dual of the cubic Ising model which is a pure gauge model , condition (20) becomes a physical condition bearing on the six coupling constants associated to each plaquette of the cubic elementary cell :

$$-\mathrm{th}\,K_6+\prod_{i=1}^{5}\mathrm{th}\,K_i = 0 \qquad (21)$$

One recovers that way a disorder condition obtained by Neuberger on this pure gauge model [12]. This result is one of the few exact results on a three dimensional gauge model.

VI. Random models.

These exact formal results can also be generalized to some random models . Let us for instance consider the previous checkerboard Potts model. Let us suppose also that a random magnetic field is located on half of the spins corresponding to a sublattice of the square lattice (see figure 6).The decimation procedure introduced in section II holds provided the appropriate boundary conditions depicted on figure 6.
One thus has the following formal results for the quenched partition function restricted to the variety :

$$D + ABC + (q-2)\,ABCD = 0 \qquad (22)$$

One can actually verify this result at the lowest orders :

$$\langle \ln \Lambda\ (A,B,C,D;\ \{z\})\rangle \Big|_{(22)} = (q-1)\ ABCD\ (1+\langle z\rangle)/2 + 1/2\ (q-1)(1+\langle z\rangle)(B^2C^2D^2 + \ldots)$$
$$+ 1/2\ (q-1)(q-2)(1+\langle z\rangle)(AB^2C^2D^2 + \ldots) + \ldots + (-5/2)\ (q-1)^2 \langle z^2\rangle\ A^2B^2C^2D^2 + \ldots$$
$$= 1/2 \ln \big(1 - (q-1)D^2/(1+(q-2)D)\big) \qquad (23)$$

Also note that condition (22) and the exact formal constraints (23) on the quenched partition function are independent of the distribution of the random field.

VII. Conclusion.

It has been shown that new stringent exact constraints bearing on the low temperature expansions of two and three (or higher) dimensional Potts models in a magnetic field (even random fields) can be derived when one restricts the parameter

space to some algebraic varieties. Let us stress that these constraints are formal . Nevertheless , they constitute surprisingly simple but new results on the analytical structure of the Potts model. In particular, these results are helpful to check (or even to get) different expansions of this model.

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Figure captions.

figure 1: The checkerboard lattice.

figure 2: The elementary cell of the checkerboard Potts model.

figure 3: Appropriate boundary conditions for the decimation procedure on the checkerboard lattice . The spins with a cross are fixed to be in the same state zero.

figure 4: The elementary cell of the generalized cubic Potts model.

figure 5: Appropriate boundary conditions for the decimation procedure on the generalized cubic lattice . The spins with a cross are fixed to be in the same state zero.

figure 6: The random field is located on spins with a cross.

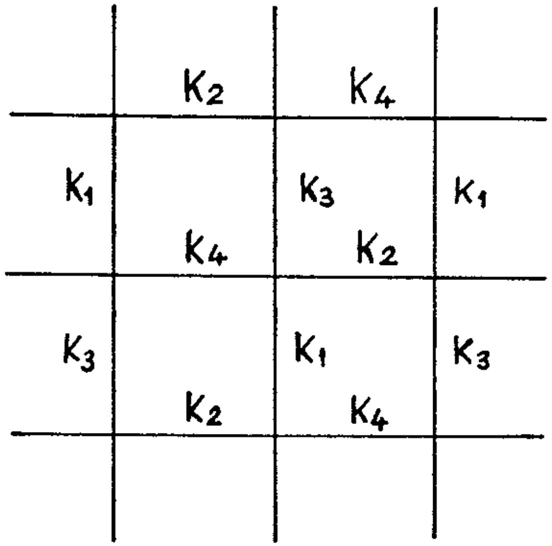

Figure 1

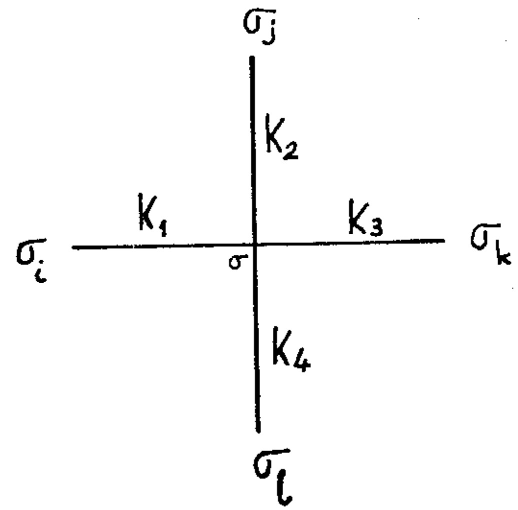

Figure 2

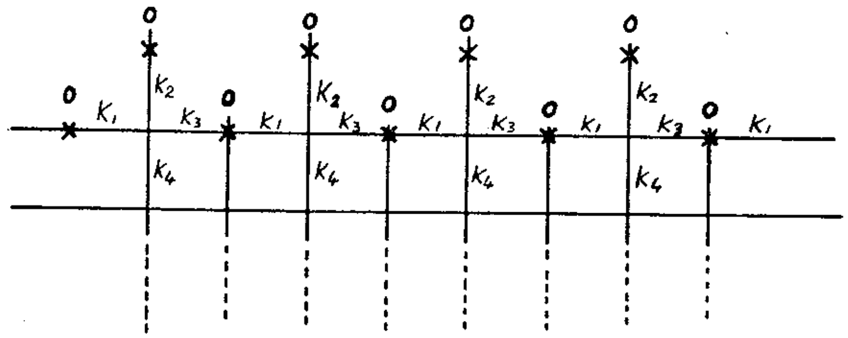

Figure 3

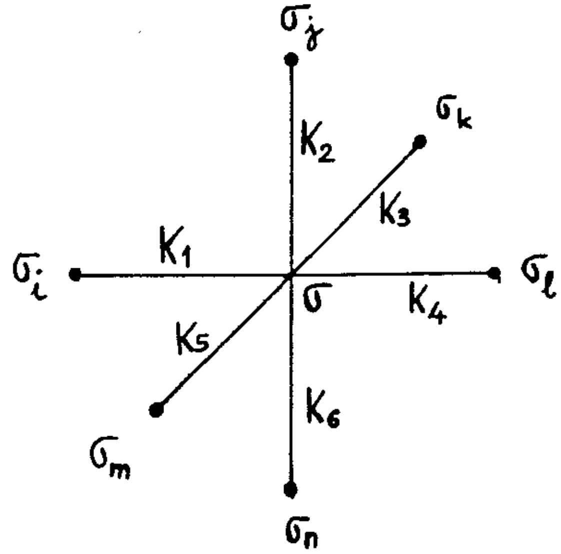

Figure 4

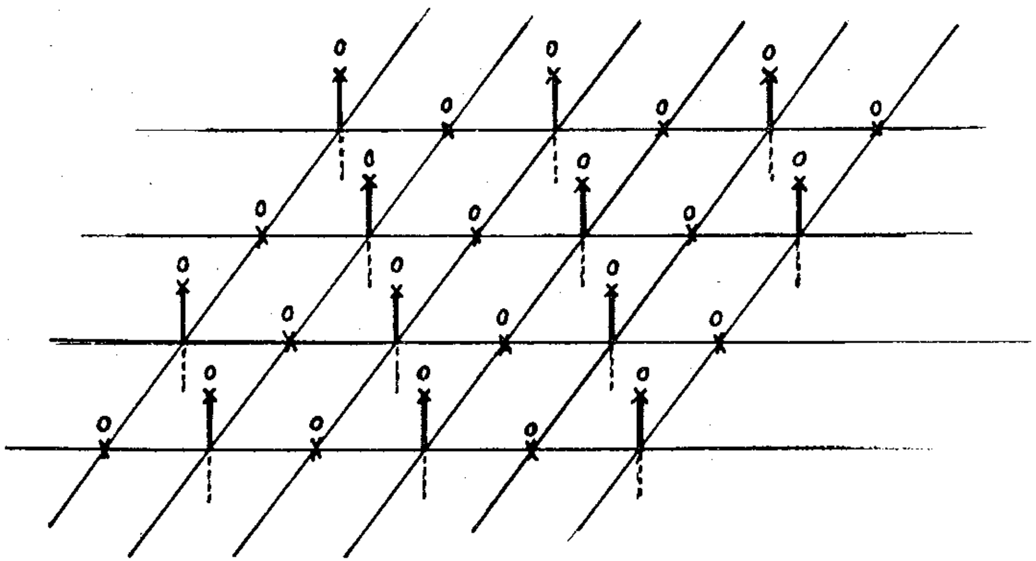

Figure 5

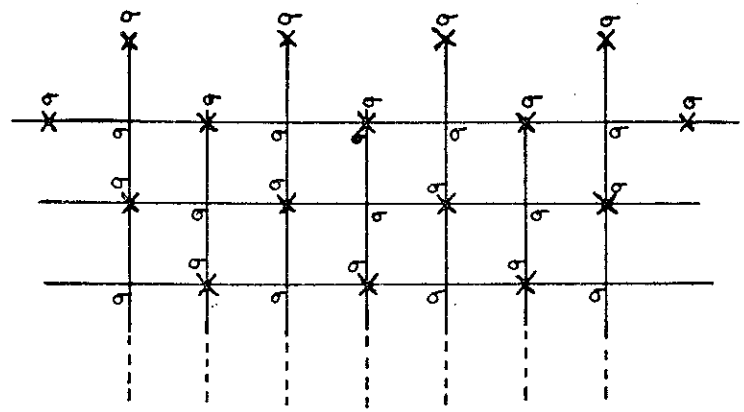

Figure 6