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Factorization properties of birational mappings ★

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Abstract

We analyse birational mappings generated by transformations on $q \times q$ matrices which correspond respectively to two kinds of transformations: the matrix inversion and a permutation of the entries of the $q \times q$ matrix. Remarkable factorization properties emerge for quite general involutive permutations.

It is shown that factorization properties do exist, even for birational transformations associated with noninvolutive permutations of entries of $q \times q$ matrices, and even for more general transformations which are rational transformations but no longer birational. The existence of factorization relations independent of q, the size of the matrices, is underlined.

The relations between the polynomial growth of the complexity of the iterations, the existence of recursions in a single variable and the integrability of the mappings, are sketched for the permutations yielding these properties.

All these results show that permutations of the entries of the matrix yielding factorization properties are not so rare. In contrast, the occurrence of recursions in a single variable, or of the polynomial growth of the complexity are, of course, less frequent but not completely exceptional.

Keywords: Birational transformations; Rational transformations; Discrete dynamical systems; Non-linear recursion relations; Iterations; Integrable mappings; Elliptic curves; Algebraic surfaces; Automorphisms of algebraic varieties; Complexity of iterations; Polynomial growth; Lattice statistical mechanics;

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1. Introduction

In the last few years, integrable discrete dynamical systems have attracted growing attention [1–6]. The developments in this truly interdisciplinary field are impressive. The large recent literature on *integrable* maps is the consequence of the discoveries of many results, concepts, structures, and of an accumulation of many new interesting examples [7,8]. These developments come after, or in parallel to the developments of *integrability* in classical or quantum field theory, lattice statistical mechanics, and many other domains of mathematical physics (the list of these domains being quite large: quantum groups, knot theory, combinatorics, number theory...).

Several concepts and structures crucial to understand integrability, like for instance the existence of Lax pair, or Bäcklund transformations [9], can be seen to exist in a *continuous framework as well as a discrete one.* Paradoxically many structures of the *discrete* dynamical systems are, by no means, simpler than the corresponding structures of their continuous counterparts. In this respect the "Deus ex Machina" of the integrability of the two-dimensional models of lattice statistical mechanics, namely the existence of *Yang–Baxter equations* [10,11], is intrinsiquely a *discrete concept* with no obvious natural counterpart. This intrinsiquely *discrete* concept enables, for instance, to see clearly why quantum integrability [12] on a lattice is, to some extend, "easier" to understand than classical integrability on a continuous space.

Among these various exact properties and structures some are directly related to the integrability (existence of Lax pair, of Bäcklund transformations, of bi-Hamiltonian structure [13], of various kinds of symplectic structure [14], ...), and some are less obviously related to integrability, namely the *reversibility* of the mappings (which can be analysed in a continuous or discrete framework) and a property *specific of discrete problems*, which has not yet deserved as much attention as the previously mentioned structures and properties, namely the occurrence of *factorization properties* for the iteration of the mappings.

The connection between *reversibility* and integrability has been already been analyzed by Quispel et al. [15,16]. We will concentrate here on the analysis of the occurrence of *factorization properties* for the iteration of the mappings, integrable or not. Some analysis of *factorization properties* of mappings have even been performed by Veselov for large classes of mappings but in the context of (birational) transformations of *two* variables (*Cremona transformations* [17–19]).

In this very paper we will try to better understand the relation between the *factorization properties* which occur in the iteration of particular (birational) mappings of many variables (in fact an arbitrary number of variables) associated with transformations on matrices, and the *reversibility* of the mappings, as well as their possible *integrability.* We will provide a large number of new examples of such birational mappings associated with matrices and will try, from this accumulation of examples, to better understand the "natural framework" for the occurrence of (more or less remarkable) factorization properties. Is the reversible character [20] of the mapping necessary to get factorizations? Is the existence of algebraic varieties preserved by the birational mappings necessary to get factorizations?

In previous papers birational mappings [21–23] generated by *involutive transformations on matrices* have been studied. They have their origin in the theory of exactly solvable models in lattice statistical mechanics [24–29]. These involutions respectively correspond to two kinds of transformations on $q \times q$ matrices: the inversion of the $q \times q$ matrix and an (*involutive*) *permutation of the entries* of the matrix. In these papers, *permutations of two entries* [21–23], as well as permutations corresponding to *discrete symmetries* of lattice models of statistical mechanics [24–29] were first analysed. For these permutations, it has actually been shown that the iteration of the associated birational transformations presents some remarkable factorization properties [21,22]. These factorization properties explain why the complexity of these iterations, instead of having the exponential growth one expects at first sight, may have a *polynomial growth* of the complexity [21,22,30]. It has also been shown that the polynomial factors occurring in these factorizations *may satisfy noteworthy non-linear recursion relations* and that some of these recursions were actually *integrable*, yielding elliptic curves [21,22].

We will consider here *more general examples of permutations of the entries* of a $q \times q$ matrix and we will analyse the iterations of the associated birational transformations in the parameter space associated with the (homogeneous) entries of the matrix. Again, one will analyse the relations between these various structures and properties (polynomial growth of the calculations, existence of recursions [22], integrability of the mappings, nature of the algebraic varieties preserved by these mappings...). It will be seen *that factorization properties actually exist for quite general permutations of the entries*. It will also be shown that some generating functions of the degree of the iterated transformations are often simple and satisfy remarkable functional equations in the quite general framework. Furthermore, it will be seen that, even for quite general permutations, polynomial growth, or the existence of (integrable) recursion relations, are not completely exceptional (see Sections 6.1.1 and 6.1.11 in the following).

We will consider three different classes of mappings: birational mappings corresponding to more general, but still involutive, permutations of the entries, birational mappings corresponding to permutations which are *no longer involutive*, and, finally, *rational* mappings which are *no longer invertible.*

Our aim is to provide tools and criterions to classify "chaotic" *birational*, or just *rational*, mappings of *many* variables. We want to identify structures that still "survive chaos".

Through various examples one will try to understand how these remarkable factorization properties can occur in such a very general framework.

2. General framework

Generalizing the analysis performed in [21,22], we consider here the following problem (which is interesting in itself for the theory of mappings of *many variables* disregarding the relation with the theory of integrable lattice models [22,23]): analyzing the iteration of *birational* transformations generated by the matrix inversion and a permutation of the entries of the matrix. One interesting problem one hopes to solve is to find *permutations* of the entries of the matrix for which the corresponding birational transformations yield integrable mappings or polynomial growth of the calculations.

Let us consider the following $q \times q$ matrix:

$$R_q = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} & \cdots \\ m_{21} & m_{22} & m_{23} & m_{24} & \cdots \\ m_{31} & m_{32} & m_{33} & m_{34} & \cdots \\ m_{41} & m_{42} & m_{43} & m_{44} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} . \tag{2.1}$$

We use the same notations as in [21,22], that is, we introduce the following transformations, the matrix inverse $\widehat{I}$, the homogeneous matrix inverse I and, in the following, t will denote a permutation of the entries of the $q \times q$ matrix (in the next section t will be the permutation t_{12-21} which permutes m_{12} and m_{21}):

$$\widehat{I} : R_q \longrightarrow R_q^{-1} \quad \text{or} \quad I : R_q \longrightarrow R_q^{-1} \cdot \det(R_q) . \tag{2.2}$$

The homogeneous inverse I is a polynomial transformation on each of the entries m_{ij} of R_q, which associates to each m_{ij} its corresponding cofactor. Transformation $\widehat{I}$ is an *involution*, whereas $I^2 = (\det(R_q))^{q-2} \cdot \mathcal{I}$, where $\mathcal{I}$ denotes the identity transformation and transformation t will often be, in the following, an involution. We also introduce the (generically infinite order) transformations,

$$K = t \cdot I \quad \text{and} \quad \widehat{K} = t \cdot \widehat{I} . \tag{2.3}$$

The transformation $\widehat{K}$ is clearly a *birational transformation* on the entries m_{ij} since its inverse transformation is $\widehat{I} \cdot t$, which is obviously a rational transformation. Transformation K is a *homogeneous polynomial transformation* on the entries m_{ij}.

3. Recalls

Let us recall the factorization properties and recursion relations associated with a set of simple permutations detailed in [21,22].

3.1. Iterations associated with permutation $m_{12} - m_{21}$

Let us first recall the factorization properties and recursion relations obtained for permutation of two entries [1] m_{12} and m_{21} [21–23]. We first recall some notable factorization properties of the iteration of the homogeneous transformation K, bearing on

[1] This permutation represents one among a set permutations which have been denoted class I in previous papers [22,23].

$q \times q$ matrices. This transformation corresponds to *integrable mappings* and yields *elliptic curves for arbitrary* q [21–23].

Let us consider a generic $q \times q$ matrix denoted M_0 (see (2.1)) with $q \geq 4$. Let us consider the successive matrices obtained by the iteration of the homogeneous transformation K. The first of these matrices and the first determinant one encounters are denoted, respectively,

$$M_1 = K(M_0) , \quad f_1 = \det(M_0) . \tag{3.1}$$

The determinant of matrix M_1 *factorizes remarkably* enabling the introduction of a homogeneous polynomial f_2 (which is a polynomial of the q^2 homogeneous entries of M_0),

$$f_2 = \frac{\det(M_1)}{f_1^{q-3}} . \tag{3.2}$$

Also, notably, f_1^{q-4} *factorizes in all the entries* of the matrix $K(M_1)$, leading to introduce a new matrix M_2,

$$M_2 = \frac{K(M_1)}{f_1^{q-4}} . \tag{3.3}$$

Again, $\det(M_2)$ factorizes permitting the introduction of a new polynomial f_3. Moreover, $f_1^2 \cdot f_2^{q-4}$ factorizes *in all the entries of* this matrix $K(M_2)$. Such factorizations occur at each step of the iteration, yielding a *polynomial growth* of the calculations (polynomial growth of the degree of all the quantities one encounters [22]). One can actually introduce the following polynomials f_n and matrices M_n corresponding to the "optimal" factorizations in the iterations:

$$\begin{aligned} f_3 &= \frac{\det(M_2)}{f_1^3 \cdot f_2^{q-3}} , \quad M_3 = \frac{K(M_2)}{f_1^2 \cdot f_2^{q-4}} , \\ f_4 &= \frac{\det(M_3)}{f_1^{q-1} \cdot f_2^3 \cdot f_3^{q-3}} , \quad M_4 = \frac{K(M_3)}{f_1^{q-2} \cdot f_2^2 \cdot f_3^{q-4}} , \quad \ldots , \end{aligned} \tag{3.4}$$

and generally, for $n \geq 1$ and $q \geq 4$,

$$M_{n+3} = \frac{K(M_{n+2})}{f_n^{q-2}\, f_{n+1}^2\, f_{n+2}^{q-4}} , \quad f_{n+3} = \frac{\det(M_{n+2})}{f_n^{q-1}\, f_{n+1}^3\, f_{n+2}^{q-3}} . \tag{3.5}$$

A relation *independent of the matrix size* q pops out immediately,

$$\widehat{K}(M_{n+2}) = \frac{K(M_{n+2})}{\det(M_{n+2})} = \frac{M_{n+3}}{f_n f_{n+1} f_{n+2} f_{n+3}} . \tag{3.6}$$

Conversely relation (3.6) enables one to get M_n as the nth iteration of $\widehat{K}$,

$$M_n = f_n \cdot f_{n-2} \cdot \widehat{K}^n(M_0) . \tag{3.7}$$

Taking the determinant of the left-hand side and the right-hand side of (3.7) and recalling the second relation in (3.5), one can eliminate $\det(M_n)$ to get a relation which does not depend on q,

$$\det(\widehat{K}^n(M_0)) = f_{n+1} \cdot f_n^{-3} \cdot f_{n-1}^3 \cdot f_{n-2}^{-1} \cdot \tag{3.8}$$

This defines the (left) action of transformation K on matrices M_n's and the set of polynomials f_n's closely related to determinants of these successive matrices. For $q = 3$ the factorization scheme is *slightly different* and simpler,

$$M_{n+3} = \frac{K(M_{n+2})}{f_n}, \quad f_{n+3} = \frac{\det(M_{n+2})}{f_n^2}. \tag{3.9}$$

One can also introduce a *right-action* of K on matrices M_n's, on the entries of the M_n's or on any polynomial expressions of these entries (such as the f_n's for instance), replacing the entries m_{ij} of M_0 with the corresponding entries of $K(M_0)$, i.e. $(K(M_0))_{ij}$. Amazingly, the *"right action"* of K on the f_n's and the matrices M_n's reads a *remarkable factorization* of f_1 and only f_1 [21,22],

$$(f_n)_K = f_{n+1} \cdot f_1^{\mu_n}, \quad (M_n)_K = M_{n+1} \cdot f_1^{\nu_n}. \tag{3.10}$$

Denoting α_n the degree of the determinant of matrix M_n, and β_n the degree of polynomial f_n, one immediately gets from Eqs. (3.5), (3.6), (3.10) the following linear relations (with integer coefficients):

$$\begin{aligned}
&\alpha_{n+2} = (q-1)\beta_n + 3\beta_{n+1} + (q-3)\beta_{n+2} + \beta_{n+3}, \\
&(q-1)\alpha_{n+2} = \alpha_{n+3} + q(q-2)\beta_n + 2q\beta_{n+1} + q(q-4)\beta_{n+2}, \\
&(q-1)\beta_n = \beta_{n+1} + q\mu_n, \quad (q-1)\alpha_n = \alpha_{n+1} + q^2\nu_n.
\end{aligned} \tag{3.11}$$

Let us introduce $\alpha(x)$, $\beta(x)$, $\mu(x)$ and $\nu(x)$ which are the generating functions of the α_n's, β_n's, μ_n's and ν_n's,

$$\begin{aligned}
&\alpha(x) = \sum_{n=0}^{\infty} \alpha_n \cdot x^n, \quad \beta(x) = \sum_{n=0}^{\infty} \beta_n \cdot x^n, \\
&\mu(x) = \sum_{n=0}^{\infty} \mu_n \cdot x^n, \quad \nu(x) = \sum_{n=0}^{\infty} \nu_n \cdot x^n.
\end{aligned} \tag{3.12}$$

Linear relations between these various generating functions are obtained from (3.11). For instance, Eqs. (3.5) yield for the generating functions $\alpha(x)$ and $\beta(x)$,

$$\begin{aligned}
&x\alpha(x) = \Big((q-1)x^3 + 3x^2 + (q-3)x + 1\Big) \cdot \beta(x), \\
&((q-1)x - 1) \cdot \alpha(x) + q = \Big(q(q-2)x^3 + 2qx^2 + q(q-4)x\Big) \cdot \beta(x),
\end{aligned} \tag{3.13}$$

and Eq. (3.6) yields

$$(1+x)\cdot\alpha(x) = q(1+x)\cdot(1+x^2)\cdot\beta(x) + q\,. \tag{3.14}$$

The explicit expressions of these generating functions read respectively

$$\alpha(x) = \frac{q(1+(q-3)x+3x^2+(q-1)x^3)}{(1+x)(1-x)^3}, \quad \beta(x) = \frac{qx}{(1+x)(1-x)^3},$$
$$\mu(x) = \frac{x((q-3)+2x^2-x^3)}{(1+x)(1-x)^3}, \quad \nu(x) = \frac{x((q-4)+2x+(q-2)x^2)}{(1+x)(1-x)^3}\,. \tag{3.15}$$

In the explicit expressions of exponents α_n's, β_n's, μ_n's and ν_n's, one sees that the iteration of transformation K yields, because of the above mentioned factorizations ((3.5), ...), a *polynomial growth* of the complexity of the calculations: the degree of all the expressions one encounters in the iterations (the α_n's, β_n's, μ_n's and ν_n's, the entries of the successive matrices M_n, ...) has a *quadratic growth* with n, for instance

$$\beta_n = \tfrac{1}{8}q(2n(n+2)+1-(-1)^n)\,. \tag{3.16}$$

In the generating functions $\alpha(x)$, $\beta(x)$, $\mu(x)$ and $\nu(x)$, this corresponds to the fact that *one only has* $x=\pm 1$ *singularities.*

A similar analysis can be performed considering the factorization properties of K^2 instead of K. For instance, one gets the following relations [2]:

$$M_{n+4} = \frac{K^2(M_{n+2})}{f_n^{(q-2)(q-1)}\, f_{n+1}^{3q-4}\, f_{n+2}^{q^2-5q+6}\, f_{n+3}^{q-4}}, \quad f_{n+3} = \frac{\det(M_{n+2})}{f_n^{q-1}\, f_{n+1}^{3}\, f_{n+2}^{q-3}}, \tag{3.17}$$

and

$$\frac{K^2(M_{n+2})}{(\det(M_{n+2}))^{q-2}} = M_{n+4}\cdot\left(\frac{f_{n+1}}{f_{n+3}}\right)^2. \tag{3.18}$$

From these factorization relations (3.17) linear relations are deduced on the α_n's and β_n's, or on the generating functions $\alpha(x)$ and $\beta(x)$, equivalent to (3.11) or (3.13), enabling one to recover the exact expressions (3.15). Of course, a similar analysis can be performed on K^4 (or any power of K), but will not be detailed here.

Similarly one can introduce the "right action" of transformation K^2,

$$(f_n)_{K^2} = f_{n+2}\cdot f_2^{\mu_n^{(2;2)}}\cdot f_1^{\mu_n^{(2;1)}}\,. \tag{3.19}$$

It can be noticed that $\mu^{(2;2)}(x)$ is actually equal to $\mu(x)$. It is easy to prove, for arbitrary q, that

$$\mu^{(2;1)}(x) = \frac{(x+(q-3))\cdot((q-1)-x)}{(1-x)^3(1+x)},$$
$$\mu^{(2;2)}(x) = \mu(x) = \frac{x((q-3)+2x^2-x^3)}{(1+x)(1-x)^3}, \tag{3.20}$$

[2] These results concerning the factorization properties of K^N are new and cannot be found in the previously mentioned series of papers [21,22], etc...

which is a consequence of

$$\mu(x) \cdot (1 + \mu_1 \cdot x) = x \cdot (\mu_1 + \mu^{(2;1)}(x)) \quad \text{together with} \quad \mu_1 = q - 3 . \tag{3.21}$$

Again, the "right action" of transformation $K^3, K^4, \ldots, K^8$ can be introduced: more details are given in Appendix A. From now on we also denote $\mu(x)$ by $\mu^{(1;1)}(x)$.

Similar results are obtained for the right action of K^N for arbitrary values of the integers q and N. The results read

$$(f_n)_{K^N} = f_{n+N} \cdot f_N^{\mu_n^{(N;N)}} \cdot f_{(N-1)}^{\mu_n^{(N;N-1)}} \cdot f_{(N-2)}^{\mu_n^{(N;N-2)}} \cdots f_1^{\mu_n^{(N;1)}} , \tag{3.22}$$

yielding

$$\begin{aligned} & x^{N-1} \cdot (\mu_{N-1} \cdot \mu^{(1;1)}(x) \cdot \mu^{(2;1)}(x) + \mu_{N-3} \cdot \mu^{(3;1)}(x) + \cdots \\ & \quad + \mu_1 \cdot \mu^{(N-1;1)}(x)) \\ & \quad = (\mu_1 \cdot x + \mu_2 \cdot x^2 + \cdots + \mu_{N-1} \cdot x^{N-1}) + x^{N-1} \cdot \mu^{(N;1)}(x)) - \mu^{(1;1)}(x) . \end{aligned} \tag{3.23}$$

From relation (3.23) one deduces a remarkably simple expression for $\mu^{(N;1)}(x)$ valid for $N \geq 4$,

$$\mu^{(N;1)}(x) = \frac{q \cdot (q-2)^3 \cdot (q-1)^{(N-4)} \cdot x}{(1-x)^3 (1+x)} . \tag{3.24}$$

To see that this exact expression of $\mu^{(N;1)}(x)$ is a consequence of relation (3.23), for arbitrary q and for $N \geq 4$, is a bit tedious: it is proved in Appendix A. One can however get this simple result directly from factorization (3.22), performing the right action of $K^{(N+1)}$ as the (right) action of K^N on factorization (3.10),

$$(f_n)_{K^{N+1}} = \Big(\big(f_n \big)_K \Big)_{K^N} = \Big(f_{n+1} \cdot f_1^{\mu_n} \Big)_{K^N} = \Big(f_{n+1} \Big)_{K^N} \cdot \Big(f_1^{\mu_n} \Big)_{K^N} , \tag{3.25}$$

which directly yields for the generating functions,

$$\begin{aligned} & x \cdot \mu^{(N+1;1)}(x) - \mu^{(N;1)}(x) + x \cdot \Big(1 - \mu^{(1;1)}(x)\Big) \cdot \mu_1^{(N;1)} \\ & \quad = x \cdot \mu^{(N+1;1)}(x) - \mu^{(N;1)}(x) \\ & \qquad + \Big(q(q-1)^{(N-4)}(q-3)^3\Big) \cdot \Big(1 - \mu^{(1;1)}(x)\Big) \cdot x = 0 . \end{aligned} \tag{3.26}$$

These factorizations allow to introduce the (optimal) factorization polynomials f_n. Remarkably, these polynomials *do satisfy, independently of q, a whole hierarchy of non-linear recursion relations* [21,22] as, for example,

$$\frac{f_n f_{n+3}^2 - f_{n+4} f_{n+1}^2}{f_{n-1} f_{n+3} f_{n+4} - f_n f_{n+1} f_{n+5}} = \frac{f_{n-1} f_{n+2}^2 - f_{n+3} f_n^2}{f_{n-2} f_{n+2} f_{n+3} - f_{n-1} f_n f_{n+4}} , \tag{3.27}$$

or, among many other recursions [21,22],

$$\frac{f_{n+1}f_{n+4}^2f_{n+5} - f_{n+2}f_{n+3}^2f_{n+6}}{f_{n+2}^2f_{n+3}f_{n+7} - f_nf_{n+4}f_{n+5}^2} = \frac{f_{n+2}f_{n+5}^2f_{n+6} - f_{n+3}f_{n+4}^2f_{n+7}}{f_{n+3}^2f_{n+4}f_{n+8} - f_{n+1}f_{n+5}f_{n+6}^2} . \qquad (3.28)$$

These recursions have been shown to yield *elliptic curves* [21,23].

These recursions can also be analysed in terms of the inhomogeneous birational transformation $\widehat{K}$ instead of the homogeneous transformation K (see (2.3)). Let us introduce the variables l_n's by

$$l_n = \det(\widehat{K}(\widehat{K}(\cdots(M_0)\cdots))) = \det(\widehat{K}^n(M_0)) . \qquad (3.29)$$

Recursions (3.27) yield

$$\frac{l_{n+3}\,l_{n+2} - 1}{l_{n+1}\,{l_{n+2}}^2{l_{n+3}}^2\,l_{n+4} - 1} = \frac{l_{n+2}\,l_{n+1} - 1}{l_n\,{l_{n+1}}^2{l_{n+2}}^2\,l_{n+3} - 1} \cdot l_n\,l_{n+1}\,l_{n+2}\,l_{n+3} . \qquad (3.30)$$

Noting that the variables l_n's always occur, in these recursions, through the product $x_n = l_nl_{n+1}$, the previous equation on the l_n's (see (3.30)) becomes

$$\frac{x_{n+2} - 1}{x_{n+1}\,x_{n+2}\,x_{n+3} - 1} = \frac{x_{n+1} - 1}{x_n\,x_{n+1}\,x_{n+2} - 1} \cdot x_n\,x_{n+2} . \qquad (3.31)$$

Similarly to the f_n's, there is a *whole hierarchy of recursion relations* satisfied by the l_n's or the x_n's [21,22]. Let us just give here another example of a recursion bearing on the x_n's,

$$\frac{x_{n+2}\,x_{n+3} - 1}{x_{n+1}\,x_{n+2}^2\,x_{n+3}^2\,x_{n+4} - 1} = \frac{x_{n+1}\,x_{n+2} - 1}{x_n\,x_{n+1}^2\,x_{n+2}^2\,x_{n+3} - 1} \cdot x_n\,x_{n+2} . \qquad (3.32)$$

The analysis of this hierarchy of *compatible* non-linear recursions has been performed in [21] and will not be detailed here. All these factorizations and recursions *can be proved even for arbitrary* q [21,22].

Of course relations between these various variables exist. The x_n's can, for instance, be written [21,22] as simple expressions in terms of the f_n's,

$$x_n = \frac{f_{n-1}^2\,f_{n+2}}{f_{n+1}^2\,f_{n-2}} . \qquad (3.33)$$

Introducing (homogeneous) variables q_n's,

$$x_n = q_{n+1}/q_n , \qquad (3.34)$$

recursions (3.31) (as well as all the recursions of the hierarchy [21]) can be written in the following form:

$$R(q_n, q_{n+1}, \ldots, q_{n+p}) = R(q_{n+s}, q_{n+s+1}, \ldots, q_{n+s+p}) , \qquad (3.35)$$

where $R(q_n, q_{n+1}, \ldots, q_{n+p})$ denotes a rational expression of $q_n, q_{n+1}, \ldots, q_{n+p}$. A recursion of the form (3.35) can easily be "integrated" to get a set of biquadratic relations. Examples of such integrations for the (integrable) recursions of the previously mentioned hierarchy have been given in [21]. They typically read as follows [21]:

$$(\rho_n q_{n+1} - 1) \cdot (\rho_{n+1} q_n - 1) = \lambda \cdot q_n q_{n+1} \cdot (\mu + q_n + q_{n+1}) \,. \qquad (3.36)$$

It should be noted that one can also integrate these recursions in terms of the x_n's, using the integration performed with the appropriate variables q_n's, but the results are more involved. Let us, for instance, consider the integration of one of our recursions in terms of two biquadratics [21], denoted as $B_1(q_n, q_{n+1})$ and $B_2(q_{n+1}, q_{n+2})$. Using the very relation between the x_n's and the q_n's, the system of these two biquadratic relations reads

$$B_1(q_n, q_n \cdot x_n) = 0\,, \quad B_2(q_n \cdot x_n, q_n \cdot x_n \cdot x_{n+1}) = 0\,. \qquad (3.37)$$

Eliminating the homogeneous variable q_n, one immediately gets a relation between x_n and x_{n+1}. Let us consider, for instance, the simplest example of integrable recursion, that is (3.31). For this example the two q_n-biquadratics, B_1 and B_2, *identify* and the resultant between them yields a *bicubic*,

$$\begin{aligned} \mathcal{B}(x_n, x_{n+1}) = & A \cdot (1 + x_n^3 x_{n+1}^3) + B \cdot x_n^3 + (2A + B) \cdot x_n \cdot (1 + x_n^2 x_{n+1}^2) \\ & + C \cdot x_n x_{n+1}^2 + (A + 2B) \cdot x_n^2 \cdot (1 + x_n x_{n+1}) \\ & + (3B - C) \cdot x_n x_{n+1} \cdot (1 + x_n x_{n+1}) + D x_n^2 x_{n+1} = 0\,. \end{aligned} \qquad (3.38)$$

Let us now consider recursion (3.32) which, in terms of the variables q_n's, can be integrated and yields *two biquadratic relations*. It also gives *two bicubics* of the form

$$\begin{aligned} & \widehat{A} \cdot (1 + x_n^3 x_{n+1}^3) + \widehat{B} \cdot x_n^3 + \widehat{C} \cdot x_n \cdot (1 + x_n^2 x_{n+1}^2) + \widehat{D} \cdot x_n^2 \cdot (1 + x_n x_{n+1}) \\ & + \widehat{E} \cdot x_n x_{n+1}^2 + \widehat{F} \cdot x_n x_{n+1} \cdot (1 + x_n x_{n+1}) + \widehat{G} \cdot x_n^2 x_{n+1} = 0\,, \end{aligned} \qquad (3.39)$$

with some involved relations between the coefficients of (3.39) that will not be written here. More details[3] are given in Appendix B.

The relations between these various properties and structures (factorization properties, existence of recursions on a *single* variable, *integrability*, ...) have been detailed in [21,22]. The fact that products of a *fixed* number of f_n's occur in relation (3.27) is related to the fact that products of a fixed number of f_n's occur in the factorizations previously detailed (3.5). The *polynomial growth* of the complexity of these iterations is also related to the occurrence of products of fixed numbers of polynomials like (3.27). However, it is important to note that *polynomial growth can actually occur even with more involved factorization properties where the number of* f_n*'s occurring in the factorizations is not fixed but grows like* n (see Section 4.2 in the following). Of course the existence of factorization properties is a necessary condition for a polynomial growth of the calculations.

[3] These results are new and cannot be found in previous publications.

3.2. *Iterations associated with elementary permutations: the six classes*

In order to clarify the relation between all these various properties and structures, all the possible permutations *of two entries* of the $q \times q$ matrix have been considered [22]. It has been shown that *all the permutations of two entries can actually be reduced to six different classes of transpositions* [22], denoted as class I, class II, ..., class VI [22,23]. These classes are defined for $q \times q$ matrices ($q \geq 4$). Class VI can be represented, for instance, as transposition $m_{11} \leftrightarrow m_{12}$, class V can be represented as $m_{11} \leftrightarrow m_{23}$, class IV can be represented as $m_{12} \leftrightarrow m_{13}$, class III can be represented as $m_{12} \leftrightarrow m_{23}$, class I can be represented as $m_{12} \leftrightarrow m_{21}$ and class II can be represented as $m_{12} \leftrightarrow m_{34}$ (see [22,23]).

The analysis of the iteration of the homogeneous transformation K for the permutations of class III (and class II but only for $q = 4$) yields *the same factorizations* as for class I [22]. However, *the corresponding polynomials f_n's do not satisfy recursion* (3.27) *or any other simple recursion in a single variable*[4].

For all these various classes, as well as for the other birational transformations analysed in this paper, one has no factorization of $K(M_0)$ and one will denote

$$M_1 = K(M_0)\,, \quad f_1 = \det(M_0)\,. \tag{3.40}$$

Remark. For all these classes (and this is also true for all the examples analysed in this paper) the "right-action" of K on the f_n's or the M_n's factorizes f_1 *and only* f_1 (see relations (3.10)), yielding the same linear relations for the exponents α_n's, β_n's, μ_n's and ν_n's as for class I (see relations (3.11)) and thus the following linear relations for the corresponding generating functions:

$$((q-1)x-1)\cdot\beta(x) = qx\mu(x) - qx\,,$$
$$((q-1)x-1)\cdot\alpha(x) = q^2x\nu(x) - q\,. \tag{3.41}$$

3.2.1. *Class V*

The factorizations corresponding to the iterations of the homogeneous transformation K (see Eq. (3.4) for class I), read for class V,

$$\det(M_{n+2}) = f_n^{q-1}\cdot f_{n+1}\cdot f_{n+2}^{q-3}\cdot f_{n+3}\,, \quad K(M_{n+2}) = f_n^{q-2}\cdot f_{n+2}^{q-4}\cdot M_{n+3}\,. \tag{3.42}$$

As well as for classes I and III, two relations *independent of q*, are actually verified, namely Eq. (3.6), and

$$\det(\widehat{K}^n(M_0)) = f_{n+1}\cdot f_n^{-3}\cdot f_{n-1}\cdot f_{n-2}^{-1}\,. \tag{3.43}$$

Again, a similar analysis can be performed considering the factorization properties of K^2 instead of K. One gets

[4] Nevertheless, there actually exist recursions on a *finite* set of variables which enable, after elimination, to get an (involved) algebraic relation on the variables l_n's or x_n's (see [22]).

$$M_{n+4} = \frac{K^2(M_{n+2})}{f_n^{(q-2)(q-1)} f_{n+1}^{q-2} f_{n+2}^{(q-1)(q-2)} f_{n+3}^{q-4}},$$

$$\frac{K^2(M_{n+2})}{(\det(M_{n+2}))^{q-2}} = \frac{M_{n+4}}{(f_{n+2} f_{n+3})^2}. \tag{3.44}$$

Of course, a similar analysis can be performed for K^4 but will not be detailed here.

Factorizations (3.42) yield linear relations for the degrees of the polynomials $\det(M_n)$ and f_n (the α_n's and β_n's), for instance

$$\alpha_{n+2} = (q-1)\beta_n + \beta_{n+1} + (q-3)\beta_{n+2} + \beta_{n+3}. \tag{3.45}$$

The generating functions $\alpha(x)$ and $\beta(x)$ read

$$\alpha(x) = \frac{q((q-1)x^3 + x^2 + (q-3)x + 1)}{(1+x)(1-3x+x^2-x^3)},$$

$$\beta(x) = \frac{qx}{(1+x)(1-3x+x^2-x^3)}. \tag{3.46}$$

In these generating functions, it is clear that one has an *exponential growth of the complexity of the calculations*, since the degree of all the polynomials one encounters grows exponentially. The exponents α_n's, β_n's, μ_n's and ν_n's grow exponentially like λ^n where $\lambda \sim 2.769\cdots$ This exponential growth seems to be incompatible with the existence of recursions: actually there is no simple recursion on a single variable like (3.27) on the f_n's.

3.2.2. class VI

The iterations of transformation K for class VI yield, for arbitrary n, *"string-like" factorizations*:

$$K(M_n) = M_{n+1} \cdot (f_n \cdot f_{n-2} \cdot f_{n-4} \cdot f_{n-6} \cdots f_{\xi_n})^{q-3}, \tag{3.47}$$

$$\det(M_n) = f_{n+1} \cdot f_n^{q-2} \cdot f_{n-1} \cdot f_{n-2}^{q-2} \cdot f_{n-3} \cdot f_{n-4}^{q-2} \cdots f_1^{\zeta_n}, \tag{3.48}$$

where $\xi_n = 1$ for n odd and $\xi_n = 2$ for n even, and $\zeta_n = 1$ for n even and $\zeta_n = q-2$ for n odd.

Let us note that one has the following simple "string-like" relation independent of q:

$$\widehat{K}(M_n) = \frac{K(M_n)}{\det(M_n)} = \frac{M_{n+1}}{f_1 \cdot f_2 \cdot f_3 \cdots f_n \cdot f_{n+1}}. \tag{3.49}$$

Relation (3.49) enable one to get M_n as the nth iteration of $\widehat{K}$,

$$M_n = f_n \cdot f_{n-2} \cdot f_{n-4} \cdot f_{n-6} \cdots \widehat{K}^n(M_0). \tag{3.50}$$

Taking the determinant of both sides of (3.50), and recalling relation in (3.48), one can eliminate $\det(M_n)$ to get a relation *which does not depend on q*,

$$\det(\widehat{K}^n(M_0)) = f_{n+1} \cdot (f_n^{-2} \cdot f_{n-1} \cdot f_{n-2}^{-2} \cdot f_{n-3}) \cdot (f_{n-4}^{-2} \cdot f_{n-5} \cdot f_{n-6}^{-2} \cdot f_{n-7}) \cdots \tag{3.51}$$

By introducing γ_n the degree[5] of $\det(\widehat{K}^n(M_0))$, one gets from (3.51),

$$\gamma_n = \beta_{n+1} - (2\beta_n - \beta_{n-1} + 2\beta_{n-2} - \beta_{n-3}) \\ -(2\beta_{n-4} - \beta_{n-5} + 2\beta_{n-6} - \beta_{n-7}) - \cdots , \tag{3.52}$$

and

$$\alpha_n = \gamma_n + q \cdot (\beta_n + \beta_{n-2} + \beta_{n-4} + \beta_{n-6} + \cdots) , \tag{3.53}$$

which yields

$$\alpha(x) = \gamma(x) + \frac{q \cdot \beta(x)}{1 - x^2} , \quad \text{where} \quad \gamma(x) = \frac{q}{1+x} . \tag{3.54}$$

Again, a similar analysis can be performed considering the factorization properties of K^2 instead of K,

$$M_{n+2} = \frac{K^2(M_n)}{\left(f_{n+1} \cdot f_n^{q-1} \cdot f_{n-1} \cdot f_{n-2}^{q-1} \cdot f_{n-3} \cdot f_{n-4}^{q-1} \cdots f_1^{\zeta_n}\right)^{(q-3)}} , \tag{3.55}$$

where $\zeta_n = 1$ for n even and $\zeta_n = q - 1$ for n odd, and

$$\frac{K^2(M_n)}{(\det(M_n))^{q-2}} = \frac{M_{n+2}}{f_1 \cdot f_2 \cdot f_3 \cdots f_n \cdot f_{n+1}} . \tag{3.56}$$

Eqs. (3.48) and (3.49) yield linear relations for the α_n's and β_n's,

$$\alpha_n = \beta_{n+1} + (q-2)\beta_n + \beta_{n-1} + (q-2)\beta_{n-2} + \beta_{n-3} \\ +(q-2)\beta_{n-4} + \cdots + \zeta_n \beta_1 , \tag{3.57}$$

and

$$q(\beta_1 + \beta_2 + \cdots + \beta_{n+1}) = \alpha_n + \alpha_{n+1} , \tag{3.58}$$

or for the generating functions $\alpha(x)$ and $\beta(x)$,

$$x \cdot \alpha(x) = \frac{1}{1 - x^2} \cdot (1 + (q-2) \cdot x) \cdot \beta(x) , \tag{3.59}$$

and from (3.58),

$$\frac{q\beta(x)}{(1-x)} = (1+x) \cdot \alpha(x) - q . \tag{3.60}$$

The generating functions $\alpha(x)$ and $\beta(x)$ read

$$\alpha(x) = \frac{q}{1+x} + \frac{q^2 x}{(1+x)(1-2x)} , \quad \beta(x) = \frac{qx(1-x)}{1-2x} . \tag{3.61}$$

The α_n's and β_n's grow exponentially like 2^n.

[5] It is clear that $\gamma_n = (-1)^n \cdot q$, see also Appendix F.

3.2.3. Class IV

When $q \geq 3$, the factorizations corresponding to the iterations of K for class IV read

$$\det(M_n) = f_{n+1} \cdot (f_n^{q-2} \cdot f_{n-1} \cdot f_{n-2}^{q-1} \cdot f_{n-3}^2) \cdot (f_{n-4}^{q-2} \cdot f_{n-5} \cdot f_{n-6}^{q-1} \cdot f_{n-7}^2) \cdots f_1^{\delta_n}, \tag{3.62}$$

$$K(M_n) = M_{n+1} \cdot (f_n^{q-3} \cdot f_{n-2}^{q-2} \cdot f_{n-3}) \cdot (f_{n-4}^{q-3} \cdot f_{n-6}^{q-2} \cdot f_{n-7}) \cdots f_1^{\zeta_n}, \tag{3.63}$$

where $\zeta_n = q-3$ for $n = 1 \pmod 4$, $\zeta_n = 0$ for $n = 2 \pmod 4$, $\zeta_n = q-2$ for $n = 3 \pmod 4$ and $\zeta_n = 1$ for $n = 0 \pmod 4$ and δ_n also depends on the truncation. *As well as for class* VI, *factorization relations* (3.49) and (3.50) *independent of q, occur.* Similarly to Section 3.2.2, one can eliminate $\det(M_n)$ between relation (3.50) and relation (3.62), to get a q-independent relation for $\widehat{K}$,

$$\det(\widehat{K}^n(M_0)) = f_{n+1} \cdot (f_n^{-2} \cdot f_{n-1} \cdot f_{n-2}^{-1} \cdot f_{n-3}^2) \cdot (f_{n-4}^{-2} \cdot f_{n-5} \cdot f_{n-6}^{-1} \cdot f_{n-7}^2) \cdots. \tag{3.64}$$

The exact expressions for $\alpha(x)$ and $\beta(x)$ read

$$\alpha(x) = \frac{q}{1+x} + \frac{q^2 x\,(1+x^2)}{(1-x)(1+x)(1-x-x^3)}, \qquad \beta(x) = \frac{qx\,(1+x^2)}{1-x-x^3}. \tag{3.65}$$

Again relations (3.53) and (3.54) are still valid. It is clear that one has an exponential growth of exponents α_n's, β_n's, μ_n's and ν_n's: these coefficients grow like λ^n where $\lambda \sim 1.465\cdots$. Therefore, there is no recursion involving products of *fixed* numbers of f_n's like (3.27). The f_n's do not satisfy simple recursions like (3.27), they *do satisfy "pseudo-recursions" where products from f_1 to f_n occur,*

$$\frac{(f_{n+2} - f_{n-1} f_{n+1})}{(f_n - f_{n-3} f_{n-1})} \cdot \frac{f_{n-6} f_{n-10} f_{n-14} \cdots}{f_{n-4} f_{n-8} f_{n-12} \cdots} = \frac{f_n (f_{n-1} f_{n-5} f_{n-9} \cdots) - (f_{n+1} f_{n-3} f_{n-7} \cdots)}{f_{n-2}(f_{n-3} f_{n-7} f_{n-11} \cdots) - (f_{n-1} f_{n-5} f_{n-9} \cdots)}. \tag{3.66}$$

However, introducing again the "determinantal" variables l_n's (see Eq. (3.29)), class IV *actually satisfies simple recursion relations* on these l_n's, *independent of q*, reading, for instance,

$$\frac{l_n\, l_{n+1}\, l_{n+2}\, l_{n+3} - 1}{l_{n+1}\, l_{n+2} - 1} = \frac{l_{n+2}\, l_{n+3}\, l_{n+4}\, l_{n+5} - 1}{l_{n+3}\, l_{n+4} - 1} \cdot l_n\, l_{n+1}\, l_{n+3}\, l_{n+4}. \tag{3.67}$$

Generically, these recursions *are not integrable*, except on codimension-one algebraic varieties [22,23]. For many different initial matrices M_0, they can however be quite regular, corresponding to weak chaos, and have been called "almost integrable" [23].

3.2.4. An integrable subcase for class IV

There does exist an algebraic condition bearing on the entries of the matrix for which the birational transformations associated with class IV correspond to *integrable* mappings [22,23]. This integrability condition has been written elsewhere [22]. The factorizations corresponding to the iterations of K for class IV, restricted to this integrable subcase, read for arbitrary n (denoting $M_n^{\rm int}$ and f_n the matrices and polynomials associated to this integrable subcase),

$$\det(M_n^{\rm int}) = f_{n+1} \cdot f_n^2 \cdot f_{n-1} \cdot f_{n-2}^3 \cdot f_{n-3}^2 \cdot f_{n-4}^3 \,, \tag{3.68}$$

$$K(M_n^{\rm int}) = M_{n+1}^{\rm int} \cdot f_n \cdot f_{n-2}^2 \cdot f_{n-3} \cdot f_{n-4}^2 \,, \tag{3.69}$$

and

$$\widehat{K}(M_n^{\rm int}) = \frac{K(M_n^{\rm int})}{\det(M_n^{\rm int})} = \frac{M_{n+1}^{\rm int}}{f_{n-4} \cdot f_{n-3} \cdot f_{n-2} \cdot f_{n-1} f_n \cdot f_{n+1}} \,. \tag{3.70}$$

From these equations one gets linear relations on the degrees of polynomials $\det(M_n^{\rm int})$'s and f_n's (the α_n's and β_n's), and their generating functions $\alpha(x)$ and $\beta(x)$,

$$\alpha(x) = \frac{4(1+x-x^2+3x^3)}{(1+x)(1-x)^3}, \quad \beta(x) = \frac{4x}{(1+x)(1-x)^3(1+x+x^2)} \,. \tag{3.71}$$

These relations clearly show that the additional factorizations, occurring for this integrable subcase, yield factorizations, like (3.68) or (3.70), bearing on a *fixed* number of polynomials f_n and even more, to a *polynomial growth* of the calculations instead of the exponential growth previously described (see Section 3.2.3), since all the poles are *roots of unity*.

The new polynomials f_n's *do satisfy, in this integrable subcase, some recursions bearing on a fixed number of polynomials*, as, for example,

$$\frac{f_{n+2}\, f_{n+7}\, f_{n+9} - f_{n+3}\, f_{n+5}\, f_{n+10}}{f_{n+3}\, f_{n+7}\, f_{n+8} - f_{n+4}\, f_{n+5}\, f_{n+9}} = \frac{f_{n+1}\, f_{n+6}\, f_{n+8} - f_{n+2}\, f_{n+4}\, f_{n+9}}{f_{n+2}\, f_{n+6}\, f_{n+7} - f_{n+3}\, f_{n+4}\, f_{n+8}} \,. \tag{3.72}$$

Introducing the well-suited variable $q_n^{\rm int}$,

$$q_n^{\rm int} = \frac{f_n\, f_{n+5}}{f_{n+2}\, f_{n+3}} \,, \tag{3.73}$$

recursion (3.72) can be integrated into a biquadratic relation [23],

$$(1+\lambda q_{n+1}^{\rm int}) \cdot (1+\lambda q_{n+2}^{\rm int}) \cdot (q_{n+1}^{\rm int} + q_{n+2}^{\rm int} - \rho) = \mu q_{n+1}^{\rm int} q_{n+2}^{\rm int} \,. \tag{3.74}$$

Introducing the variable $x_n = q_{n+1}^{\rm int}/q_n^{\rm int}$, recursion (3.72) reads

$$\frac{x_{n+2}-1}{x_{n+1}x_{n+3}-1} = \frac{x_{n+1}-1}{x_n x_{n+2}-1}\,\frac{x_n x_{n+2}}{x_{n+1}} \,. \tag{3.75}$$

This recursion is integrable: it yields elliptic curves [22].

4. Birational transformations associated with vertex models

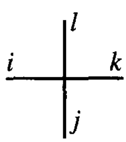
Let us now consider permutations of entries of $q \times q$ matrices which are associated to symmetries of vertex models.

4.1. Iterations associated with the sixteen-vertex model

In the case of 4×4 matrices, a particular permutation of the entries of the matrix, t_1, has been introduced in the framework of the symmetries of the *sixteen-vertex model* [29]. This permutation corresponds to

$$t_1 : m_{13} \leftrightarrow m_{31}\,, \quad m_{14} \leftrightarrow m_{32}\,, \quad m_{23} \leftrightarrow m_{41}\,, \; m_{24} \leftrightarrow m_{42}\,, \tag{4.1}$$

which amounts to permuting the two 2×2 (off-diagonal) submatrices of the 4×4 matrix M_0. This transposition t_1 corresponds to a partial transposition of one direction (say the horizontal one denoted by "1", the other transposition t_2 corresponding to the other direction denoted by "2") of a two-dimensional vertex model [26,27,29],



(4.2)

The action of t_1 and t_2 on the R-matrix is given by [29]

$$(t_1 R)^{ij}_{kl} = R^{kj}_{il}, \quad (t_2 R)^{ij}_{kl} = R^{il}_{kj}, \quad t = t_1 \cdot t_2\,. \tag{4.3}$$

Remarkably, the symmetry group generated by the matrix inverse $\widehat{I}$ and transformation t_1, or the infinite generator $K_{t_1} = t_1 \cdot \widehat{I}$, has been shown to yield *elliptic curves* [27,29] which foliate the whole parameter space of the sixteen vertex model.

Let us consider a 4×4 matrix M_0 and the successive matrices obtained by iteration of transformation $K_{t_1} = t_1 \cdot I$, where t_1 is defined by (4.1). Similarly to the factorizations described in (3.1), one has, for arbitrary n, the following factorizations for the iterations of K_{t_1}:

$$M_{n+2} = \frac{K_{t_1}(M_{n+1})}{F_n^2}\,, \quad F_{n+2} = \frac{\det(M_{n+1})}{F_n^3}\,, \tag{4.4}$$

$$\widehat{K}_{t_1}(M_{n+2}) = \frac{K_{t_1}(M_{n+2})}{\det(M_{n+2})} = \frac{M_{n+3}}{F_{n+1}F_{n+3}}\,. \tag{4.5}$$

From these factorizations, one can easily get linear relations on the exponents α_n, β_n, μ_n and ν_n, on their generating functions, for instance

$$(1 + x) \cdot \alpha(x) = 4(1 + x^2) \cdot \beta(x) + 4\,, \tag{4.6}$$

and exact expressions for their generating functions and for the exponents α_n's and β_n's,

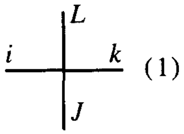
$$\alpha(x) = \frac{4(1+3x^2)}{(1-x)^3}, \quad \beta(x) = \frac{4x}{(1-x)^3},$$
$$\alpha_n = 4(2n^2+1), \quad \beta_n = 2n(n+1). \tag{4.7}$$

Amazingly, the F_n's, which correspond to K_{t_1}, do satisfy recursions which are *exactly the same as* (3.27) *where the* f_n*'s are replaced by the* F_n*'s* [23],

$$\frac{F_n F_{n+3}^2 - F_{n+4} F_{n+1}^2}{F_{n-1} F_{n+3} F_{n+4} - F_n F_{n+1} F_{n+5}} = \frac{F_{n-1} F_{n+2}^2 - F_{n+3} F_n^2}{F_{n-2} F_{n+2} F_{n+3} - F_{n-1} F_n F_{n+4}}. \tag{4.8}$$

4.2. "Self-similar" generalization of transposition t_1 for $2m \times 2m$ matrices

Let us now consider a more general vertex model where one direction, denoted as direction (1), is singled out. Pictorially this can be represented as follows:



(4.9)

where i and k (corresponding to direction (1)) can take m values, while J and L take m values.

The action of t_1, the "partial" transposition on direction (1), is given by [29]

$$(t_1 R)_{kL}^{iJ} = R_{iL}^{kJ}, \quad \text{that is} \quad t_1 : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \longrightarrow \begin{pmatrix} A & C \\ B & D \end{pmatrix}, \tag{4.10}$$

where A, B, C and D are $m \times m$ matrices.

Denoting $q = 2m$ the size of the matrices, the analysis of the corresponding factorizations yields for arbitrary n "string-like" factorizations [31],

$$K(M_n) = M_{n+1} \cdot f_n^{q-5} \cdot f_{n-1}^5 \cdot f_{n-2}^{2(q-5)} \cdot f_{n-3}^6 \cdot f_{n-4}^{2(q-5)} \cdot f_{n-5}^6 \cdots, \tag{4.11}$$

$$\det(M_n) = f_{n+1} \cdot f_n^{q-4} \cdot f_{n-1}^7 \cdot f_{n-2}^{2(q-4)} \cdot f_{n-3}^8 \cdot f_{n-4}^{2(q-4)} \cdot f_{n-5}^8 \cdot f_{n-6}^{2(q-4)} \cdots, \tag{4.12}$$

and a relation *independent of* the matrix size q,

$$\widehat{K}(M_n) = \frac{K(M_n)}{\det(M_n)} = \frac{M_{n+1}}{(f_1 \cdot f_2 \cdots f_{n-1})^2 \cdot f_n \cdot f_{n+1}}. \tag{4.13}$$

The equivalents of relations (3.50) and (3.51) read, respectively, two q-independent relations,

$$M_n = f_n \cdot (f_{n-2} \cdot f_{n-4} \cdot f_{n-6} \cdots)^2 \cdot \widehat{K}^n(M_0),$$
$$\det(\widehat{K}^n(M_0)) = f_{n+1} \cdot f_n^{-4} \cdot f_{n-1}^7 \cdot (f_{n-2}^{-8} \cdot f_{n-3}^8)$$
$$\cdot (f_{n-4}^{-8} \cdot f_{n-5}^8) \cdot (f_{n-6}^{-8} \cdot f_{n-7}^8) \cdots. \tag{4.14}$$

Eq. (4.13) gives a linear relation *valid for arbitrary* q,

$$(1+x)\cdot\alpha(x) - \frac{q\left(1+x^2\right)\beta(x)}{1-x} - q = 0\,. \tag{4.15}$$

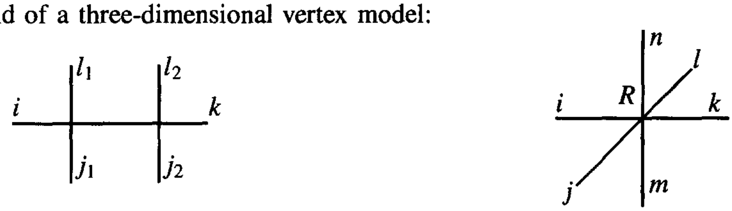
One gets

$$\alpha(x) = \frac{q}{1+x} + q^2\frac{x(1+x^2)}{(1+x)(1-x)^4}, \quad \beta(x) = \frac{qx}{(1-x)^3},$$
$$\alpha_n = \tfrac{1}{3}q(2n+1)(2n^2+2n+3)\,, \quad \beta_n = \tfrac{1}{2}qn(n+1)\,. \tag{4.16}$$

The α_n's and β_n's are, respectively, *cubic and quadratic functions* of n. *One notes that such a polynomial growth actually occurs with involved "string-like" factorizations, such as* (4.11) and (4.13).

Let us however note that this generalization of the discrete symmetries of the sixteen vertex model does not enable us to recover the results of Section 4.1 as particular subcases taking $m = 2$.

This framework enables us *to take into account the analysis of* N-site monodromy matrices [31] (take $m = 2^N$) of two-dimensional models, as well as the analysis of d-dimensional 2^d-state *vertex models* (take $m = 2^{d-1}$). Let us just give here a pictorial representation of the two sites ($N = 2$) monodromy matrix of a two-dimensional model and of a three-dimensional vertex model:



(4.17)

One can introduce similar "partial" transposition t_1 for d-dimensional vertex models ($d = 3,4,\ldots$) [28]. If one represents the R-matrix of this vertex-model as a $2^d \times 2^d$ matrix, symmetry t_1 amounts to permuting two off-diagonal $2^{d-1} \times 2^{d-1}$ submatrices B and C of this $2^d \times 2^d$ matrix (see [27,28] for $d = 3$) where A, B, C and D are $2^{d-1} \times 2^{d-1}$ matrices. For d-dimensional vertex models, there are other "partial" transpositions $t_2, t_3, \ldots, t_{d-1}$ [27,28] which can also be represented as (4.10) after some relabeling of rows and columns.

For a *three-dimensional cubic vertex model* [27,28], a "partial" transposition t_1 associated with one of the three directions of the cubic lattice has also been introduced [27,28]: the analysis of the factorizations corresponding to the iterations of transformation K for t_1 for a (general[6]) three-dimensional model (8×8 matrix) [26,27] gives the previous *"string-like" factorizations* (4.11) and (4.13), and the same generating functions (4.16), but, of course, for $q = 8$, i.e. $m = 4$ (see also [31]).

Particular subcases of this three-dimensional vertex model providing natural three-dimensional generalizations of the Baxter model, that is to say particular K-invariant patterns for the initial matrix M_0 have been analysed in [31]. For a sixteen-parameter

[6] A general 8×8 matrix means 64 homogeneous entries.

model [28,32], the "string-like" factorizations are changed *into factorizations with a fixed number of terms* (see [31]),

$$K(M_n) = M_{n+1} \cdot f_n^3 \cdot f_{n-1}^6 , \quad \det(M_n) = f_{n+1} \cdot f_n^4 \cdot f_{n-1}^7 ,$$
$$\widehat{K}(M_n) = \frac{M_{n+1}}{f_{n-1} \cdot f_n \cdot f_{n+1}} . \tag{4.18}$$

The generating functions $\alpha(x)$, $\beta(x)$, and the α_n's and β_n's read

$$\alpha(x) = \frac{8(1+4x+7x^2)}{(1-x)^3} , \quad \beta(x) = \frac{8x}{(1-x)^3} ,$$
$$\alpha_n = 8(6n^2+1) , \quad \beta_n = 4n(n+1) . \tag{4.19}$$

The α_n's and β_n's are *both quadratic functions* of *n*,

$$\alpha_n = 8(6n^2+1) , \quad \beta_n = 4n(n+1) . \tag{4.20}$$

One remarks that $\beta(x)$ (and, therefore, $\mu(x)$) *is the same as for the general* 8×8 matrix (see (4.16)), the only difference being on the α_n's (or equivalently on the generating function $\alpha(x)$): the *cubic* growth of the α_n's being replaced by a *quadratic growth* (see (4.20)). Let us note that relation (4.14) becomes

$$f_{n-3} \cdot f_{n-5} \cdot f_{n-7} \cdots M_n = f_n \cdot f_{n-2} \cdot f_{n-4} \cdot f_{n-6} \cdots \widehat{K}^n(M_0) . \tag{4.21}$$

It is also worth recalling that for a particular pattern of this three-dimensional generalization of the Baxter model (see [31]), the iteration of K (or $\widehat{K}$) actually *yields elliptic curves* (see [27,28]). In this remarkable subcase, the factorization relations (4.18) are again slightly modified. The new polynomials f_n's defined in this restricted (integrable) subcase can actually *be shown to verify a hierarchy of non-linear recursions* in [7] $\sqrt{f_n}$ [31], *which actually identify exactly with recursions* of class I (3.27) and (3.28) [21] or with recursions (4.8). The factorizations are modified as follows for arbitrary *n*:

$$K(M_n) = M_{n+1} \cdot f_n^4 \cdot f_{n-1}^2 \cdot f_{n-2}^6 , \quad \det(M_n) = f_{n+1} \cdot f_n^5 \cdot f_{n-1}^3 \cdot f_{n-2}^7 , \tag{4.22}$$

yielding

$$\widehat{K}(M_n) = \frac{K(M_n)}{\det(M_n)} = \frac{M_{n+1}}{f_{n-2} \cdot f_{n-1} \cdot f_n \cdot f_{n+1}} . \tag{4.23}$$

As a consequence of the identification with the factorizations detailed in Section 3.1, the generating functions $\alpha(x)$, $\beta(x)$, $\mu(x)$ and $\nu(x)$ are exactly the same as the ones given in (3.15), ..., but for $q = 8$. The integrability of this subcase is associated with the *occurrence of one more singularity* for $\alpha(x)$ (compare (3.15) with (4.16)).

[7] For this model the f_n's are perfect square.

4.3. t_1 for Q^4-state vertex models

Let us suppose here that the indices i, j, k, l (see Fig. 4.2) can take Q colors (Q^4-state vertex model). The R-matrix is a $q \times q$ matrix (with $q = Q^2$) which can be seen as Q^2 blocks $A[i,j]$,

$$R = \begin{pmatrix} A[1,1] & A[1,2] & A[1,3] & \cdots & A[1,Q] \\ A[2,1] & A[2,2] & A[2,3] & \cdots & A[2,Q] \\ A[3,1] & A[3,2] & A[3,3] & \cdots & A[3,Q] \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A[Q,1] & A[Q,2] & A[Q,3] & \cdots & A[Q,Q] \end{pmatrix}, \tag{4.24}$$

where the $A[i,j]$'s are $Q \times Q$ matrices. With these notations the partial transposition t_1 amounts to permuting matrices $A[\alpha,\beta]$ and $A[\beta,\alpha]$.

Similarly to factorizations described in (4.1), one has, for arbitrary n, the following factorizations for the iterations of K_{t_1} acting on $q \times q$ R-matrices like (4.24):

$$M_{n+2} = \frac{K_{t_1}(M_{n+1})}{F_n^{q-2}}, \quad F_{n+2} = \frac{\det(M_{n+1})}{F_n^{q-1}}, \quad \frac{K_{t_1}(M_{n+2})}{\det(M_{n+2})} = \frac{M_{n+3}}{F_{n+1}F_{n+3}}. \tag{4.25}$$

It is clear that these factorizations generalize the one described in Section 4.1, namely (4.4) and (4.5). For instance, one recovers relation (4.5) of the sixteen vertex model, which is independent of q. From these factorizations, one can easily get linear relations on exponents α_n's, β_n's, μ_n's and ν_n's, linear relations for their generating functions, for instance

$$(1+x) \cdot \alpha(x) = q \cdot (1+x^2) \cdot \beta(x) + q, \tag{4.26}$$

and exact expressions for their generating functions,

$$\alpha(x) = \frac{q \cdot (1 + (q-1)x^2)}{(1-x)(1-(q-2)x+x^2)},$$
$$\beta(x) = \frac{qx}{(1-x)(1-(q-2)x+x^2)}. \tag{4.27}$$

The α_n's, β_n's, μ_n's and ν_n's have an *exponential growth in terms of n* when Q is no longer equal to 2 (or 0...). The Q^4-state vertex models are, therefore, not generically good candidates for integrability when the number of colors Q is no longer 2. However, integrability cases when Q is different from 2 are not completely ruled out: the occurrence of *Yang-Baxter integrability together with polynomial growth* for some subcases of this Q^4-state vertex model has been analysed in [31]

Section 4.3, together with Section 4.2, shows that *there exist, at least, two kinds of generalizations of transposition* t_1 of the sixteen vertex model (4×4 matrices) and that these two generalizations *yield drastically different results as far as the factorization properties are concerned* (polynomial versus exponential growth). In contrast with class I and class VI (or class IV), the equivalent of relations (3.7), or (3.50), is more involved than the simple factorizations (4.25),

$$f_{n-1} \cdot (f_{n-3} \cdot f_{n-5} \cdot f_{n-7} \cdots)^2 \cdot M_n = f_n \cdot (f_{n-2} \cdot f_{n-4} \cdot f_{n-6} \cdots)^2 \cdot \widehat{K}^n(M_0) . \tag{4.28}$$

The equivalent of relations (3.51) or (4.14) are also complex. This yields a more intricate dependence of $\alpha(x)$ and $\beta(x)$ in terms of q (see also Appendix C).

5. "Straight" generalization of t_1

Another way to generalize transposition t_1 for $q \times q$ matrices, amounts to performing the following permutation of the entries:

$$t_1 \,:\, m_{12} \leftrightarrow m_{21} , \quad m_{32} \leftrightarrow m_{41} , \quad m_{23} \leftrightarrow m_{14} , \quad m_{43} \leftrightarrow m_{34} , \tag{5.1}$$

the other entries of the $q \times q$ matrix being unchanged. This is the way transposition $m_{12} \leftrightarrow m_{21}$ has been generalized to $q \times q$ matrices in Section 3.1 (see also [21]). This corresponds to transformation t_1 of (4.1) in the upper left 4×4 submatrix and the identity transformation elsewhere.

Before considering the general case of $q \times q$ matrices, let us first note that for $q = 5$, the analysis of the factorizations *does not yield any factorization* on the determinants of the iterates of the initial matrix by K or any factorization on the corresponding matrices. However, in calculating $M_n = K^n(M_0)$, *one sees that a subset of the entries of this matrix actually factorizes*, namely

$$\begin{aligned} &(M_n)_{11} , \quad (M_n)_{13} , \quad (M_n)_{22} , \quad (M_n)_{24} , \\ &(M_n)_{31} , \quad (M_n)_{33} , \quad (M_n)_{42} , \quad (M_n)_{44} \end{aligned} \tag{5.2}$$

factorize f_{n-1}. It is clear that these entries are singled out by t_1 (see (5.1)).

The analysis of the factorizations can actually be performed for $q \geq 6$, yielding the following "string-like" factorizations for arbitrary n:

$$\begin{aligned} K(M_n) \; &= M_{n+1} \cdot f_n^{q-6} \cdot f_{n-1}^4 \cdot f_{n-2}^{q-6} \cdot f_{n-3}^4 \cdot f_{n-4}^{q-6} \cdots , \\ \det(M_n) &= f_{n+1} \cdot f_n^{q-5} \cdot f_{n-1}^5 \cdot f_{n-2}^{q-5} \cdot f_{n-3}^5 \cdot f_{n-4}^{q-5} \cdots , \end{aligned} \tag{5.3}$$

and

$$\widehat{K}(M_n) = \frac{K(M_n)}{\det(M_n)} = \frac{M_{n+1}}{f_{n+1} \cdot f_n \cdot f_{n-1} \cdots f_1} , \tag{5.4}$$

which gives for the generating functions $\alpha(x)$ and $\beta(x)$,

$$(1 + x) \cdot \alpha(x) - \frac{q\beta(x)}{1 - x} - q = 0 . \tag{5.5}$$

These two generating functions read

$$\alpha(x) = q\frac{(1+4x^2+(q-5)x)}{(1-x)(1+x)(1-4x)}, \quad \beta(x) = \frac{qx}{1-4x}. \tag{5.6}$$

These factorizations can actually be proved for arbitrary q in a similar way as the demonstration performed for transposition $m_{12} \leftrightarrow m_{21}$ for $q \times q$ matrices [21]. This will not be performed here.

5.1. *Integrable subcase of the "straight" generalization of t_1*

The f_n's *do not satisfy simple recursions* (like (3.27), ...). However, let us introduce an initial matrix M_0 of a particular form, namely

$$M_0 = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}, \tag{5.7}$$

where A denotes a 4×4 submatrix, B a $(q-4) \times 4$ submatrix, C a $(q-4) \times (q-4)$ submatrix and "0" the $4 \times (q-4)$ submatrix with zero entries. All the successive matrices occurring in the iteration of K are of the same form (5.7). It is clear that this "restricted factorization problem" [31] corresponding to the initial matrix M_0 yields recursions which can be deduced from the ones associated with t_1 for a 4×4 matrix (see Section 4.1). For instance, relation

$$\frac{\left({f_3}^3{f_2}^3{f_1}^4 - f_4\right)\left({f_3}^4 - f_4\right){f_2}^8{f_3}^3 f_4}{\left({f_4}^3{f_3}^3{f_2}^4 - f_5\right)\left({f_2}^4 - f_3\right)} = 1 \tag{5.8}$$

is valid for $q \times q$ matrices of the form (5.7) but *is not satisfied for generic $q \times q$ matrices*. One easily gets from the "semi-direct product" form for M_0 (see (5.7)), the following relations enabling one to write the polynomials $f_n^{(A)}$'s corresponding to the action of K_{t_1} on the 4×4 submatrix A in (5.7) (see subsection (4.1)) in terms of the polynomials f_n's given by (5.3),

$$f_1^{(A)} = \frac{f_1}{\det(C)}, \quad f_2^{(A)} = \frac{f_2}{f_1^{(A)}(\det(C))^4},$$
$$f_3^{(A)} = \frac{f_3}{f_2^{(A)}(f_1^{(A)})^7(\det(C))^{16}} \cdots, \tag{5.9}$$

$$\cdots \; f_n^{(A)} = \frac{f_n}{f_{n-1}^{(A)}(f_{n-2}^{(A)})^7(f_{n-3}^{(A)})^{27}(f_{n-4}^{(A)})^{108}\cdots(f_1^{(A)})^{27\cdot 4^{n-4}}(\det(C))^{4^{n-1}}}. \tag{5.10}$$

With relations (5.9), recursion (5.8) reads

$$\frac{f_4^{(A)}\cdot(f_2^{(A)})^3 - (f_3^{(A)})^3\cdot f_1^{(A)}}{f_3^{(A)}(f_1^{(A)})^3 - (f_2^{(A)})^3} = \frac{f_5^{(A)}\cdot f_3^{(A)}\cdot(f_1^{(A)})^2 - (f_4^{(A)})^2\cdot f_2^{(A)}}{f_4^{(A)}f_2^{(A)} - f_1^{(A)}(f_3^{(A)})^2}, \tag{5.11}$$

which is actually one of the recursions occurring in the analysis of the transformations K_{t_1} (see [21]). Similarly, all the recursions on the $f_n^{(A)}$'s (see (4.8) and [21]) can be written in terms of the f_n's. Introducing $\beta_n^{(A)}$ as the degree of the $f_n^{(A)}$'s, relations (5.9) yield when $n \geq 3$,

$$\beta_n = \beta_n^{(A)} + \beta_{n-1}^{(A)} + 7\beta_{n-2}^{(A)} + 27\beta_{n-3}^{(A)} + 108\beta_{n-4}^{(A)} + \cdots + z_{n-1}\beta_1^{(A)} \\ +4^{n-1} \cdot (q-4) , \tag{5.12}$$

where $z_{n-1} = 27 \cdot 4^{n-4}$. The generating function of the z_n's reads

$$z(x) = \frac{(1-x)^3}{1-4x} . \tag{5.13}$$

Relation (5.12) immediately yields simple relations between the generating functions $z(x)$, $\beta(x)$ and $\beta^{(A)}(x)$ (where $\beta^{(A)}(x)$ denotes the generating function of the $\beta_n^{(A)}$'s, which is known (see (4.7) in Section 4.1),

$$\beta(x) = \frac{q-4}{1-4x} + z(x) \cdot \beta^{(A)}(x) = z(x) \cdot \frac{4x}{(1-x)^3} + \frac{q-4}{1-4x} = \frac{qx}{1-4x} . \tag{5.14}$$

This relation allows us to understand how the $(1-x)$ *singularity for* $\beta^{(A)}(x)$, which is closely *related to the integrability of the recursions on the* f_n's (4.8), *can actually be replaced by* $(1-4x)$ *singularity for* $\beta(x)$.

6. More general permutations

In the previous examples it has been seen that notable factorizations occur for birational transformations originating from *involutive* transformations on matrices. In particular, it has been seen that the factorization schemes either involve a fixed number of polynomials, or are "periodic" (see, for instance, factorization (5.3)). In both cases this yields *rational* expressions for the generating functions of the successive degrees of the calculations. Through these examples one can see clearly how the factorization schemes can be modified when one is restricted to particular (K-invariant) subcases. It is obvious that, for a subcase, the growth of the calculations can only be smaller than for the generic case: one can only expect *additional* factorizations. A polynomial growth can even occur for subcases, but this does not mean that the birational transformations *are integrable* [8].

Several examples of *polynomial growth* of birational transformations *yielding algebraic surfaces* [31] exist (one can think of these transformations as "shifts on a torus" [31]). Furthermore the occurrence of *recursions in a single variable* (like (3.67)) does not mean that the orbits of these birational transformations are necessarily *curves* (see recursion (3.67) in Section 3.2.3).

[8] The *polynomial growth* is related to the fact that transformation can be represented as a *shift of some Abelian variety*. At the present moment all the examples of iterations of birational transformations which exhibit polynomial growth correspond to transformations which can be represented as a *shift of some Jacobian variety* [31].

There is a need to analyse more examples in order to see if the *birational* character of the transformations, or the fact that they originate from *product of two involutions*, is crucial in order to get such factorizations where either a "fixed" number of polynomials, or "periodicity", occur.

Many transformations corresponding to more general permutations are given in the following examples. *Some of them are no longer involutions.* In all the examples detailed from now on, two quite different kinds of permutations on $q \times q$ matrices will be encountered, which have already been illustrated in the previously detailed examples of Sections 4.2 and 4.3.

"Straight" generalizations

The first kind of permutations on $q \times q$ matrices simply generalizes any permutation introduced on, for instance, a 4×4 matrix. Let us write the $q \times q$ matrix in blocks,

$$M_0 = \begin{pmatrix} A_{4,4} & B_{4,q-4} \\ C_{q-4,4} & D_{q-4,q-4} \end{pmatrix} . \tag{6.1}$$

Submatrices $A_{4,4}$, $B_{4,q-4}$, $C_{q-4,4}$ and $D_{q-4,q-4}$ are respectively 4×4, $4 \times (q-4)$, $(q-4) \times 4$ and $(q-4) \times (q-4)$ matrices.

A simple extension amounts to permuting the entries of the submatrix $A_{4,4}$ according to the permutation introduced on the 4×4 matrix, and to leaving the others submatrices $B_{4,q-4}$, $C_{q-4,4}$, $D_{q-4,q-4}$ unchanged. We will call such extensions "*straight*" generalizations.

In order to understand why the expression of $\beta(x)$ depends so simply on the matrix size q in the case of "straight" generalization, let us consider a particular limit of the initial matrix, namely $B_{4,q-4} = 0$. It is possible for these "straight" generalizations to understand, from this limit, that the q-dependence of the generating function $\beta(q,x)$ is simple and reads

$$a_n = \frac{\beta_n(4)}{4} = \frac{\beta_n(q)}{q} , \tag{6.2}$$

where the a_n's do not depend on q. More details are given in Appendix C.

"Self-similar" generalizations

The second kind of permutations on $q \times q$ matrices corresponds to permutations which extend the permutation introduced on, for instance, a 4×4 matrix in such a way that *the number of entries which are permuted on* $q \times q$ matrices *grows like* q. Permutation (4.10), and the corresponding birational transformations K detailed in Section 4.2, illustrate such a situation. We will call such extensions "*self-similar*" generalizations. In the previous examples of "*self-similar*" generalizations of transformation t_1, it has been seen that there exist (at least ...) *two different kinds* of "*self-similar*" generalizations, namely the generalization detailed in Section 4.3 on $q^2 \times q^2$ matrices and the one detailed in Section 4.2 on $2m \times 2m$ matrices. The expression of $\beta(x)$ also depends very simply on the matrix size q for the "*self-similar*" generalizations introduced in

Section 4.2: $\beta(x)$ is proportional to q like in (6.2). *Note that this is not the case for the "self-similar" generalizations introduced in Section* 4.3. In order to understand why the expression of $\beta(x)$ depends so simply on the matrix size q for the *"self-similar"* generalizations introduced in Section 4.2, one can again consider a particular limit [9] of the initial matrix,

$$M_0 = \begin{pmatrix} A_1 & 0 \\ B & A_2 \end{pmatrix}, \tag{6.3}$$

on which one has the action of the *same* transposition on each of the two $m \times m$ submatrices: A_1 and A_2. One immediately gets

$$\beta_{2q}(x) = \beta_q(x) + \beta_q(x)\,, \quad \text{yielding} \quad \beta_q(x) = \tfrac{1}{4} q \beta_4(x)\,. \tag{6.4}$$

More details are given in Appendix C.

Let us first give a list of birational transformations corresponding to *involutive* permutations.

6.1. Birational transformations corresponding to involutive permutations

Let us first consider examples of birational transformations corresponding to *involutive* permutations which are slightly more involved than the one of Sections 3.1 or 3.2.

6.1.1. Permutation [12–21,34–43]

Let us now consider, for a 4×4 matrix, permutation t which amounts to permuting m_{12} and m_{21}, and, at the same time, m_{34} and m_{43},

$$t: \; m_{12} \leftrightarrow m_{21}\,, \quad m_{43} \leftrightarrow m_{34}\,. \tag{6.5}$$

Surprisingly enough, one has exactly the *same factorizations and relations as the one detailed in the previous section for transposition* t_1 (see relations (4.4) and (4.5)).

Introducing the following pattern of the 4×4 matrix, which depends on eight homogeneous entries (this pattern is invariant under the action of the group generated by the matrix inverse I and permutation t (see (6.5)) [24,33]:

$$R_A = \begin{pmatrix} R_1 & R_2 \\ R_2 & R_1 \end{pmatrix} \quad \text{with} \quad R_i = \begin{pmatrix} r_1^{(i)} & r_2^i \\ r_3^i & r_4^i \end{pmatrix}, \quad i = 1,2. \tag{6.6}$$

One can consider factorization analysis as the one detailed in Section 3 for initial matrices M_0, corresponding to particular patterns such as (6.6), as soon as the form of the matrices obtained by iteration of $K = t \cdot I$ are of the "same form" as M_0 [33] ("restricted factorization problem" [23]).

It is important to note that, for matrices M_n's of the form (6.6), *polynomials F_n's defined by* (4.4) and (4.5) *actually satisfy the same recursions as* (4.8) *and, furthermore, the same hierarchy described in* [22]. However, for matrices M_n of the general

[9] This cannot be done for the "self-similar" generalizations of Section 4.3.

form (sixteen homogeneous entries), one no longer has any recursion on the F_n's. The non-existence of recursions on the F_n's can be understood considering the image of the successive iterations of K (or equivalently $\widehat{K}$) corresponding to permutation (6.5) in the parameter space of all the entries of matrices M_n's (eight homogeneous parameters for pattern (6.6) and sixteen homogeneous parameters in the general case).

Fig. 1 shows the projection of the orbits generated [10] by the iteration of K for pattern (6.6). It corresponds to the following initial values: $r_1^{(1)} = 1$, $r_2^{(1)} = -4$, $r_3^{(1)} = 0.2$, $r_4^{(1)} = 1.2$, $r_1^{(2)} = 1.3$, $r_2^{(2)} = -0.15$, $r_3^{(2)} = 0.3$, $r_4^{(2)} = 0.9$. It is clear in Fig. 1 that the orbits lie on *curves* which are certainly elliptic curves [23]. This *suites with the existence of recursion relations* in the F_n's.

Figs. 2a, 2b, 2c and 2d correspond to the general case (*sixteen* homogeneous parameters). These figures seem to indicate that the orbits lie on *surfaces*: it is shown in [23] that this is actually the case and that these surfaces are actually *algebraic surfaces* given, *by intersection of quadrics.* This *suites with the non-existence of recursions* in the F_n's in the general case.

One can consider the successive iterates $\widehat{K}^n(M_n)$ of an initial matrix of the form (6.6) seeking for the smallest affine space containing these successive points (see [21]). It can actually be shown that

$$\widehat{K}^n(M_n) = a_0^{(n)} \cdot M_0 + a_1^{(n)} \cdot M_2 + a_2^{(n)} \cdot M_4 + a_3^{(n)} \cdot M_6 . \tag{6.7}$$

Relation (6.7) shows that the orbits lie on a *three-dimensional affine space* (in the general case the dimension is larger).

Note that for pattern (6.6), one has an obvious factorization of f_1 as follows: $f_1 = \det(M_0) = \frac{1}{16} \cdot \det(R_1 + R_2) \cdot \det(R_1 - R_2)$. The problem of the relations between such additional factorizations, *consequence of the particular form of the matrix*, and the factorizations, consequence of the specificity of the non-linear homogeneous transformation K, (see (3.4)), will not be detailed here.

One should note that the "straight" generalizations of this permutation to $q \times q$ ($q \geq 5$) *give exactly the same results as the one detailed in Section 5.*

Let us now give a list of other (increasingly complex) examples. Visualizations of the orbits (like Figs. 1, 2a, 2b, 2c and 2d) will not be performed under: in most of these examples one encounters an exponential growth of the calculations which yields quite "chaotic" orbits.

6.1.2. A second example

Let us consider t the following permutation of the entries of a 4×4 matrix:

$$t : \; m_{11} \leftrightarrow m_{44} , \quad m_{24} \leftrightarrow m_{31} , \quad m_{23} \leftrightarrow m_{32} . \tag{6.8}$$

The analysis of the factorizations of the iterations of transformation $K = t \cdot I$ yields, for arbitrary n, "string-like" factorizations,

[10] In the plane made of two (inhomogeneous) variables among all the (inhomogeneous) coordinates of the parameter space.

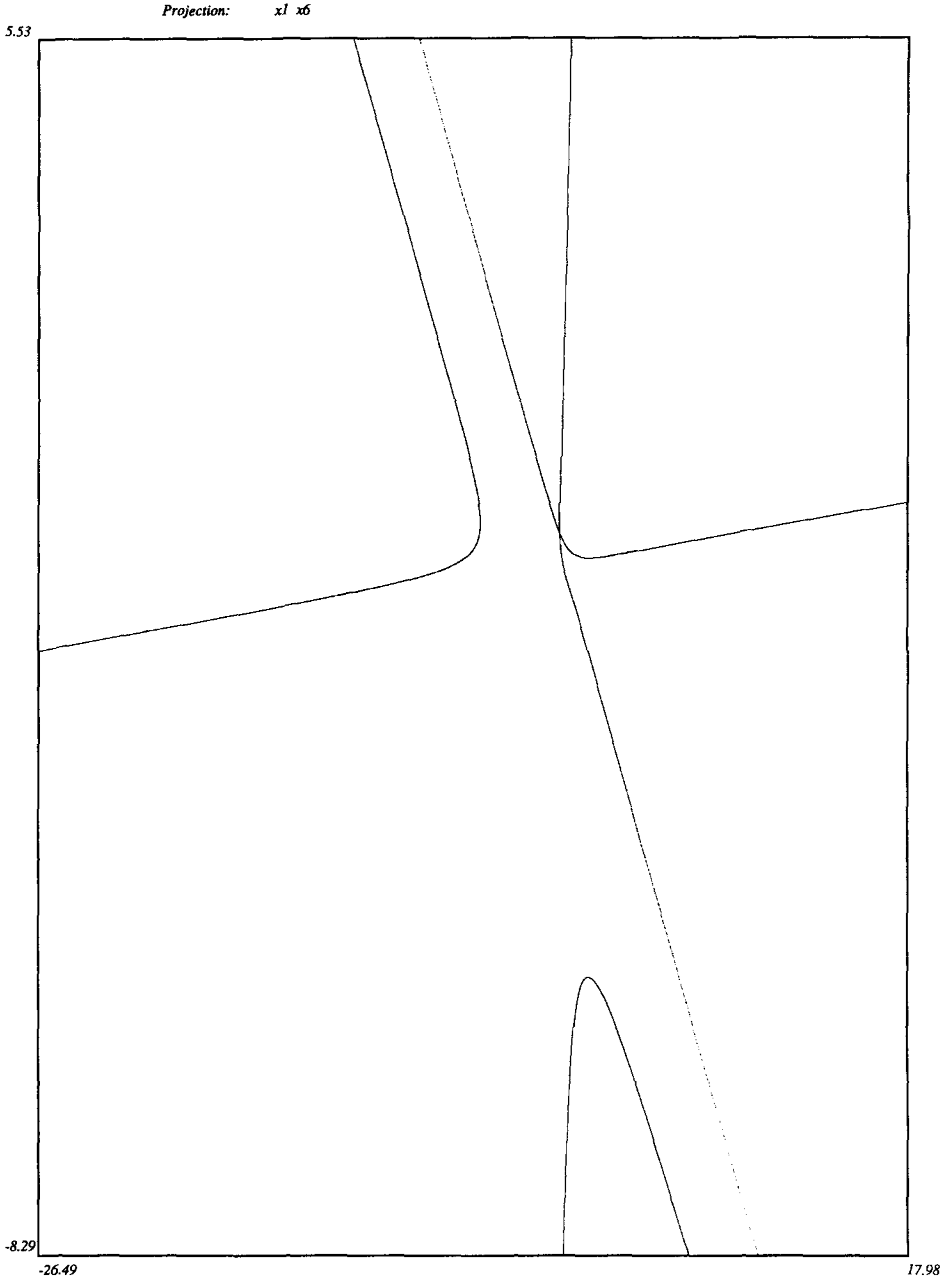


Fig. 1. Projection of the orbit corresponding to the iteration of transformation $\widehat{K}$ of an initial matrix of the form (6.6).

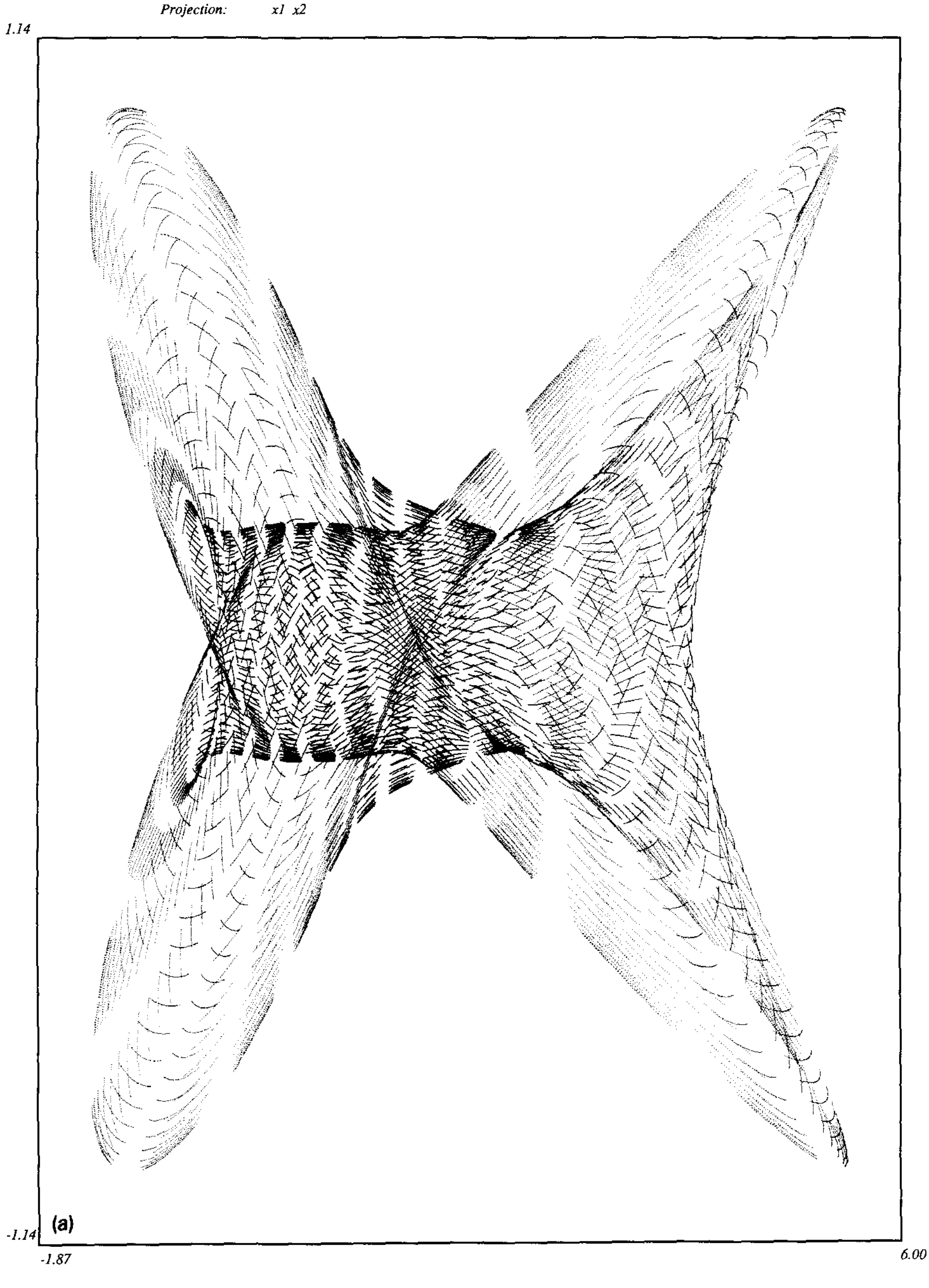


Fig. 2. (a)–(d) Projections of the orbit corresponding to the iteration of transformation $\widehat{K}$ of a generic 4×4 matrix.

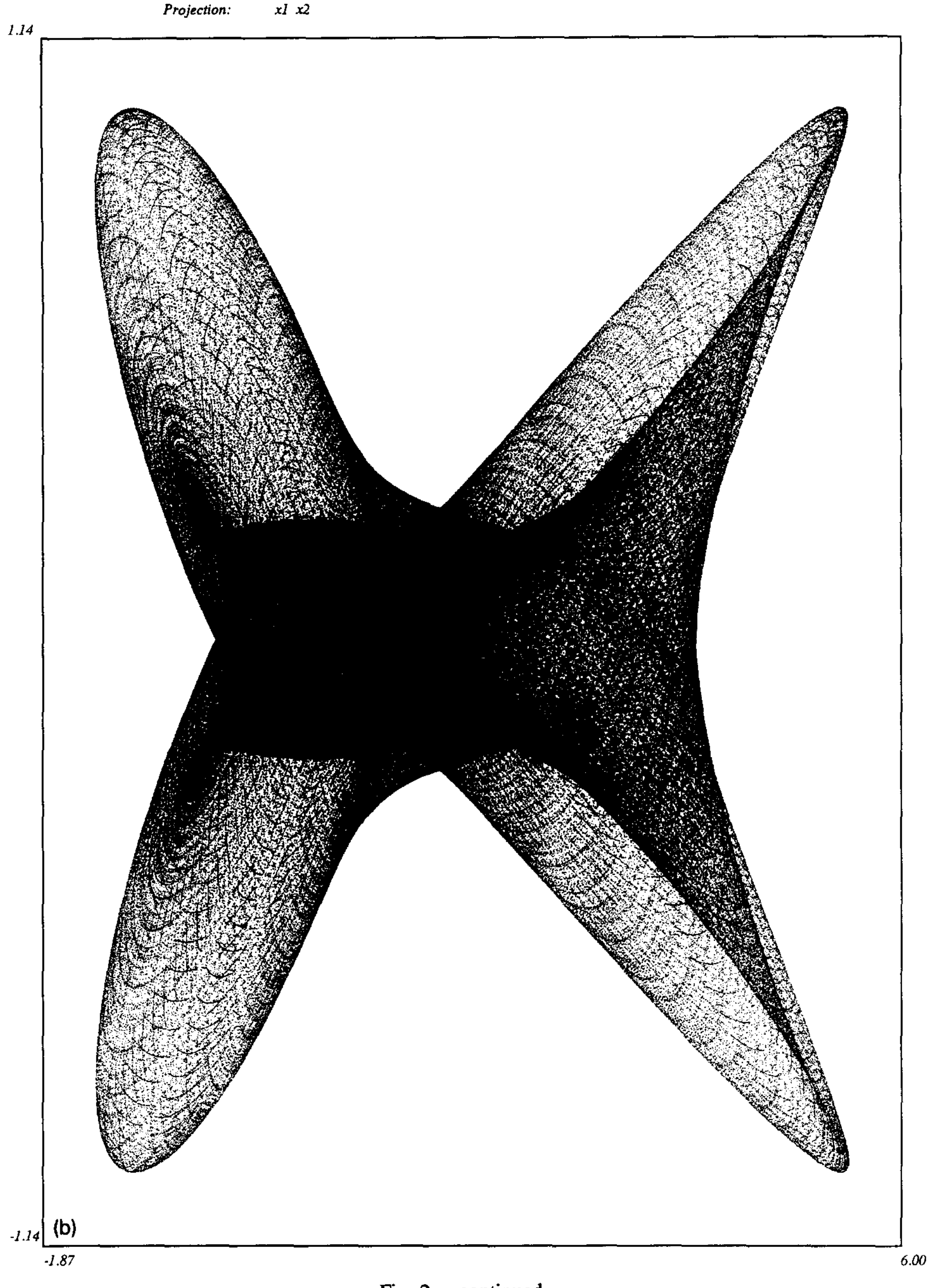


Fig. 2 — continued.

$$K(M_n) = M_{n+1} \cdot f_{n-2}^2 \cdot f_{n-5}^2 \cdot f_{n-7}^2 \cdots ,$$
$$\det(M_n) = f_{n+1} \cdot f_{n-1} \cdot f_{n-2}^3 \cdot f_{n-4} \cdot f_{n-5}^3 \cdots , \qquad (6.9)$$

and

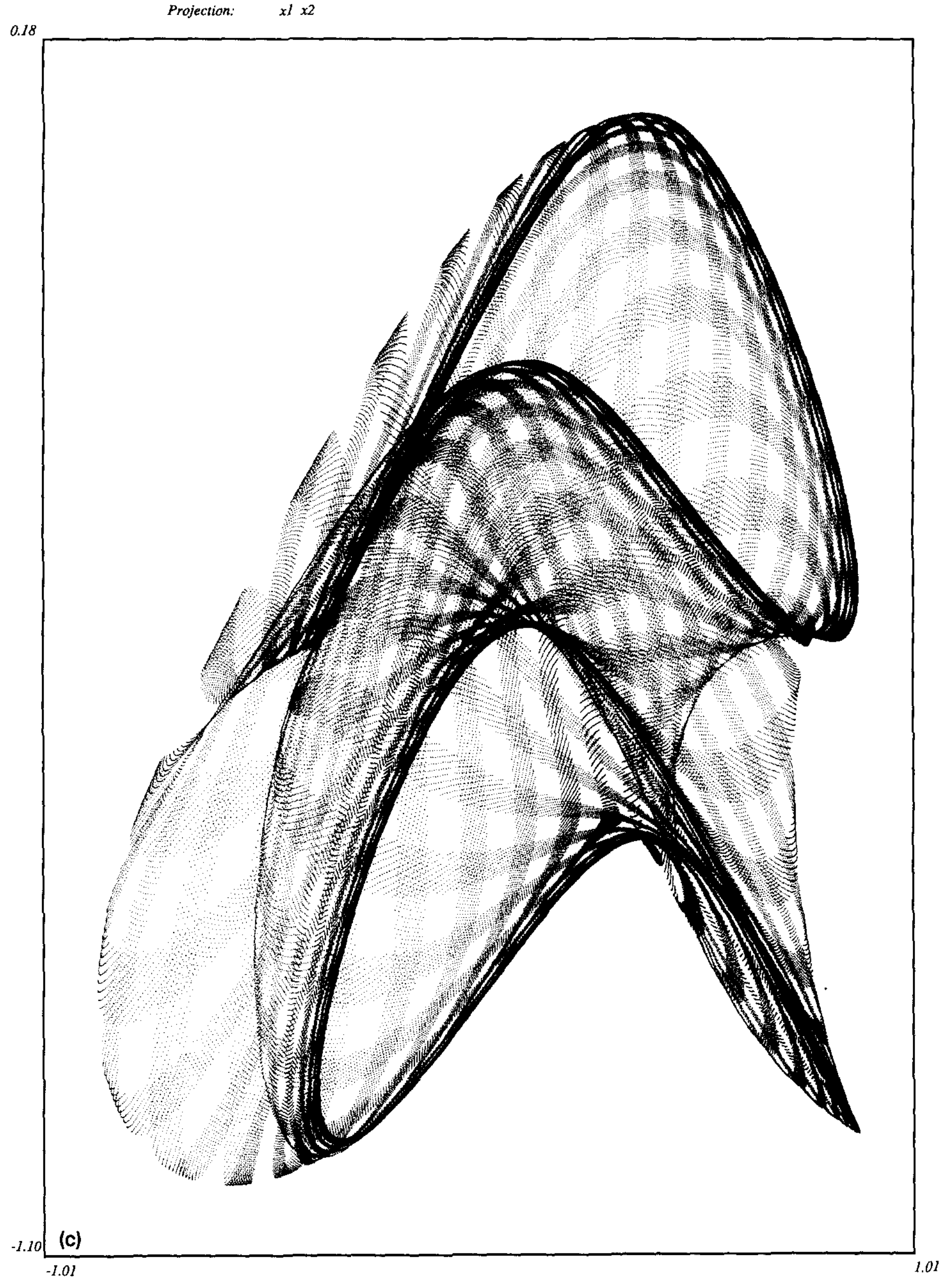


Fig. 2 — continued.

$$\widehat{K}(M_n) = \frac{K(M_n)}{\det(M_n)} = \frac{M_{n+1}}{(f_{n+1}f_{n-1}) \cdot (f_{n-2}f_{n-4}) \cdot (f_{n-5}f_{n-7}) \cdots} . \tag{6.10}$$

Again, the "right action" of K on the f_n's and the matrices M_n's reads factorizations of f_1 like (3.10), leading to functional Eqs. (3.41) on $\alpha(x)$, $\beta(x)$, $\mu(x)$ and $\nu(x)$.

Eqs. (6.9) and (6.10) give

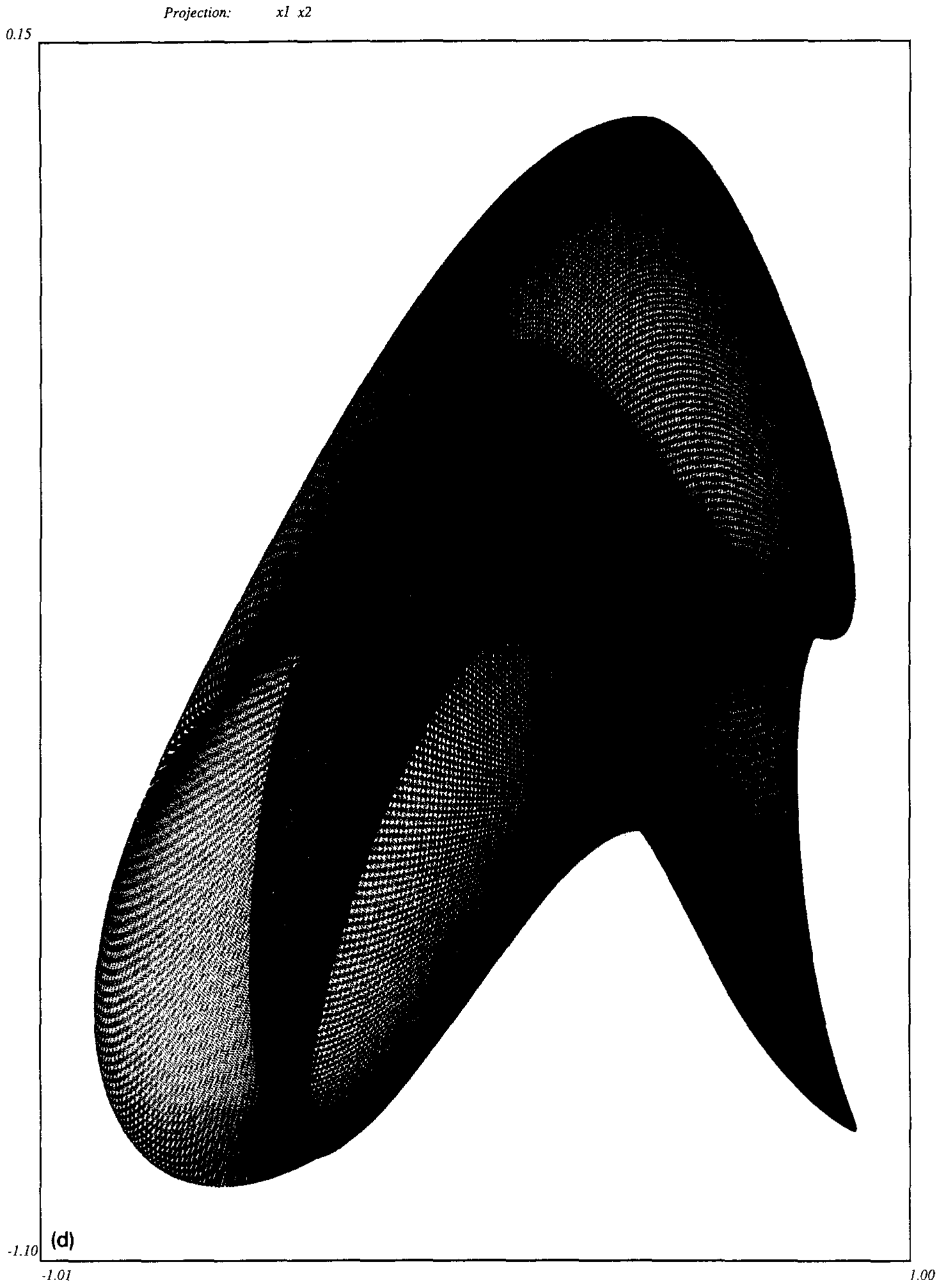


Fig. 2 — continued.

$$(1+x)\cdot\alpha(x)-\frac{4\beta(x)}{1-x}+\frac{4x\beta(x)}{1-x^3}-4=0\,,$$

$$3x\alpha(x)-\alpha(x)-\frac{8\beta(x)x^3}{1-x^3}+4=0\,, \tag{6.11}$$

$$(1+x)\cdot\alpha(x) - \frac{4\left(1+x^2\right)\cdot\beta(x)}{1-x^3} - 4 = 0\,, \tag{6.12}$$

leading to

$$\alpha(x) = \frac{4\,(1+x)\left(1-x+2x^2\right)}{(1-x)\left(1-2x-x^2-2x^3\right)}\,, \quad \beta(x) = \frac{4x\left(1+x+x^2\right)}{1-2x-x^2-2x^3}\,,$$
$$\mu(x) = \frac{x^2\,(1+x)}{1-2x-x^2-2x^3}\,, \quad \nu(x) = \frac{2x^3}{(1-x)\cdot\left(1-2x-x^2-2x^3\right)}\,. \tag{6.13}$$

The α_n's, β_n's, μ_n's and ν_n's grow exponentially like λ^n with $\lambda \sim 2.658\cdots$.

6.1.3. A third example

Let us consider t the following permutation of the entries of a 4×4 matrix:

$$t:\; m_{11} \leftrightarrow m_{44}\,, \quad m_{12} \leftrightarrow m_{43}\,, \quad m_{14} \leftrightarrow m_{41}\,, \quad m_{13} \leftrightarrow m_{42}\,. \tag{6.14}$$

The iterations of the associated transformation $K = t\cdot I$ give, for arbitrary n, "string-like" factorizations,

$$K(M_n) = M_{n+1}\cdot f_n\cdot f_{n-3}\cdot f_{n-4}\cdot f_{n-7}\cdot f_{n-8}\cdots\,, \tag{6.15}$$
$$\det(M_n) = f_{n+1}\cdot\Big(f_n\cdot f_{n-3}\cdot f_{n-4}\cdot f_{n-7}\cdot f_{n-8}\cdots\Big)^2\,, \tag{6.16}$$

and

$$\widehat{K}(M_n) = \frac{K(M_n)}{\det(M_n)} = \frac{M_{n+1}}{(f_{n+1}\cdot f_n)\cdot(f_{n-3}\cdot f_{n-4})\cdot(f_{n-7}\cdot f_{n-8})\cdots}\,. \tag{6.17}$$

Again, the "right action" of K on the f_n's and the matrices M_n's reads factorizations of f_1 like (3.10), leading to functional equations on $\alpha(x)$, $\beta(x)$, $\mu(x)$ and $\nu(x)$ like (3.41).

Eq. (6.16) suggests

$$(1-2x+x^4)\cdot\beta(x) - 4x + 4x^2 - 4x^3 + 4x^4 = 0\,, \tag{6.18}$$

which is actually verified, while one obtains from (6.17),

$$(1+x)\cdot\alpha(x) - \frac{4(1+x)\cdot\beta(x)}{1-x^4} - 4 = 0\,. \tag{6.19}$$

From Eqs. (6.18) and (6.19) one gets the exact expression of these generating functions,

$$\alpha(x) = \frac{4(1+x-x^2+x^3)}{\left(1-x-x^2-x^3\right)(1-x)}\,, \quad \beta(x) = \frac{4\left(1+x^2\right)x}{1-x-x^2-x^3}\,,$$
$$\mu(x) = \frac{2x\left(1-x+x^2\right)}{1-x-x^2-x^3}\,, \quad \nu(x) = \frac{x\left(1-x+x^2\right)}{\left(1-x-x^2-x^3\right)(1-x)}\,. \tag{6.20}$$

The α_n's, β_n's, μ_n's and ν_n's grow exponentially like λ^n with $\lambda \sim 1.839\cdots$.

6.1.4. "Straight" generalizations of the third example

Let us consider a "straight" generalization of the same permutation of the entries as (6.14) for $q \times q$ matrices ($q \geq 5$), the other entries being unchanged. The iterations of transformation K yields, for arbitrary n, "string-like" factorizations,

$$K(M_n) = M_{n+1} \cdot f_n^{q-4} \cdot f_{n-1} \cdot f_{n-2}^{q-3} \cdot f_{n-3} \cdot f_{n-4}^{q-3} \cdots ,$$
$$\det(M_n) = f_{n+1} \cdot f_n^{q-3} \cdot f_{n-1}^2 \cdot f_{n-2}^{q-2} \cdot f_{n-3}^2 \cdot f_{n-4}^{q-2} \cdots , \tag{6.21}$$

and the same relation as (3.49) or (5.4),

$$\widehat{K}(M_n) = \frac{K(M_n)}{\det(M_n)} = \frac{M_{n+1}}{f_1 \cdot f_2 \cdot f_3 \cdots f_n \cdot f_{n+1}} , \tag{6.22}$$

from which one deduces again the linear relation (3.60) on $\alpha(q,x)$ and $\beta(q,x)$. The generating functions read

$$\alpha(q,x) = \frac{q\left(1+(q-3)x+x^2+x^3\right)}{(1-x)\,(1+x)\,\left(1-2x-x^2\right)} , \quad \beta(q,x) = \frac{qx}{1-2x-x^2} ,$$
$$\mu(q,x) = \frac{x\,(q-3-x)}{1-2x-x^2} , \quad \nu(q,x) = \frac{x\left(q-4+x+x^2\right)}{(1-x)\,(1+x)\,\left(1-2x-x^2\right)} . \tag{6.23}$$

The α_n's, β_n's, μ_n's and ν_n's grow exponentially like λ^n with $\lambda = 1+\sqrt{2} \sim 2.414\cdots$.

6.1.5. A "self-similar" generalization of the third example

Let us consider t the following permutation of the entries for $q \times q$ matrices ($q \geq 5$):

$$t: \; m_{11} \leftrightarrow m_{qq} , \quad m_{12} \leftrightarrow m_{q(q-1)} , \quad \ldots \; m_{1(q-1)} \leftrightarrow m_{q2} , \quad m_{1q} \leftrightarrow m_{q1} , \tag{6.24}$$

and the associated transformation $K = t \cdot I$. The iterations of the associated transformation K yield, for arbitrary n, "string-like" factorizations,

$$K(M_n) = M_{n+1} \cdot f_n^{q-3} \cdot f_{n-4}^{q-3} \cdot f_{n-8}^{q-3} \cdot f_{n-12}^{q-3} \cdots , \tag{6.25}$$

$$\det(M_n) = (f_{n+1} \cdot f_n^{q-2}) \cdot (f_{n-3} \cdot f_{n-4}^{q-2}) \cdot (f_{n-7} \cdot f_{n-8}^{q-2}) \cdots , \tag{6.26}$$

and the same equation as (6.17),

$$\widehat{K}(M_n) = \frac{K(M_n)}{\det(M_n)} = \frac{M_{n+1}}{(f_{n+1} \cdot f_n) \cdot (f_{n-3} \cdot f_{n-4}) \cdot (f_{n-7} \cdot f_{n-8}) \cdots} . \tag{6.27}$$

One remarks that there is a *shift of four* of the index n in the factorization scheme ((6.25), (6.26), (6.27)). From relation (6.27), a linear relation on the generating functions $\alpha(q,x)$ and $\beta(q,x)$ is deduced,

$$(1+x) \cdot \alpha(q,x) - \frac{q(1+x)\beta(q,x)}{1-x^4} - q = 0 . \tag{6.28}$$

The explicit expressions of the generating functions $\alpha(q,x)$, $\beta(q,x)$, $\mu(q,x)$ and $\nu(q,x)$ read

$$\alpha(q,x) = \frac{q\,(1+(q-2)x)}{(1+x)\,(1-2x)}, \quad \beta(q,x) = \frac{qx\,(1-x)\,\left(1+x^2\right)}{1-2x},$$
$$\mu(q,x) = \frac{\left((q-2) - qx + qx^2 - (q-1)x^3\right)x}{1-2x}, \quad \nu(q,x) = \frac{(q-3)\,x}{(1+x)\,(1-2x)}. \tag{6.29}$$

The α_n's, β_n's, μ_n's and ν_n's grow exponentially like 2^n.

This self-similar generalization (see (6.24)) gives different results compared to the one for $q=4$. The relation independent of q, namely (6.27) *is actually valid for* $q=4$ (see (6.17)). However the q-dependent factorizations are different for $q \geq 5$ and $q=4$ (compare Eq. (6.25) and (6.15) or (6.26) and (6.16)).

6.1.6. The "self-similar" generalization of the third example for $q=3$

Let us examine the"self-similar" generalization introduced in the previous section, but for $q=3$. This amounts to considering the involutive permutation t given by

$$t:\ m_{11} \leftrightarrow m_{33}, \quad m_{12} \leftrightarrow m_{32}, \quad m_{13} \leftrightarrow m_{31}. \tag{6.30}$$

The factorizations of the associated transformation $K = t \cdot I$ are "string-like" factorizations as for $q=4$,

$$f_2 = \frac{\det(M_1)}{f_1}, \quad M_2 = K(M_1), \quad f_3 = \frac{\det(M_2)}{f_1 \cdot f_2},$$
$$M_3 = K(M_2), \quad f_4 = \frac{\det(M_3)}{f_1^2 \cdot f_2 \cdot f_3}, \quad \cdots, \tag{6.31}$$

and, for arbitrary n, the "string-like" factorizations,

$$K(M_n) = M_{n+1} \cdot f_{n-2} \cdot f_{n-3} \cdot f_{n-6} \cdot f_{n-7} \cdots,$$
$$\det(M_n) = f_{n+1} \cdot f_n \cdot f_{n-1} \cdot f_{n-2}^2 \cdot f_{n-3}^2 \cdot f_{n-4}$$
$$\cdot f_{n-5} \cdot f_{n-6}^2 \cdot f_{n-7}^2 \cdot f_{n-8} \cdot f_{n-9} \cdots, \tag{6.32}$$

again yielding relation (3.49). The linear relation (3.60) is deduced from relation (3.49), and one obtains from (6.32),

$$(2x-1)\cdot\alpha(q,x) + 3 - \frac{3x^3(1+x)\beta(q,x)}{1-x^4} = 0, \tag{6.33}$$

and

$$x\cdot\alpha(q,x) - \frac{2x^3(1+x)\beta(q,x)}{1-x^4} - \frac{\beta(q,x)}{1-x} = 0. \tag{6.34}$$

These equations give the expressions of $\alpha(q,x)$ and $\beta(q,x)$,

$$\alpha(q,x) = \frac{3(1+x^2+x^3)}{(1-x)\cdot(1-x-x^3)}, \quad \beta(q,x) = \frac{3x\cdot(1+x^2)}{1-x-x^3}. \tag{6.35}$$

These generating functions correspond to an exponential growth, like λ^n with $\lambda \sim 1.465\cdots$.

6.1.7. Fourth example

Let us look at t the following permutation of the entries of a 4×4 matrix and the associated transformation $K = t \cdot I$:

$$t: \; m_{11} \leftrightarrow m_{31}, \quad m_{21} \leftrightarrow m_{41}, \quad m_{12} \leftrightarrow m_{32}, \quad m_{22} \leftrightarrow m_{42}. \tag{6.36}$$

The iterations of transformation K yield, for arbitrary n, "string-like" factorizations,

$$K(M_n) = M_{n+1} \cdot f_n \cdot (f_{n-5} \cdot f_{n-6}) \cdot (f_{n-11} \cdot f_{n-12}) \cdots, \tag{6.37}$$

$$\det(M_n) = f_{n+1} \cdot \Big(f_n \cdot f_{n-5} \cdot f_{n-6} \cdot f_{n-11} \cdot f_{n-12} \cdots \Big)^2, \tag{6.38}$$

and

$$\widehat{K}(M_n) = \frac{K(M_n)}{det(M_n)} = \frac{M_{n+1}}{(f_{n+1} \cdot f_n) \cdot (f_{n-5} \cdot f_{n-6}) \cdot (f_{n-11} \cdot f_{n-12}) \cdots}. \tag{6.39}$$

One notes that there is a *shift of six* of the index n in the factorization scheme. Eq. (6.38) suggests some simple equation such as

$$\big(1 - 2x + x^6\big) \cdot \beta(x) - 4x + 4x^2 - 4x^3 + 4x^4 - 4x^5 + 4x^6 = 0, \tag{6.40}$$

which is actually verified, while one gets from (6.39)

$$(1 + x) \cdot \alpha(x) - \frac{4\,(1 + x) \cdot \beta(x)}{1 - x^6} - 4 = 0. \tag{6.41}$$

Eqs. (6.40) and (6.41), as well as Eqs. (3.41), yield

$$\alpha(x) = \frac{4(1 + x - x^2 + x^3 - x^4 + x^5)}{(1 - x)\,\big(1 - x - x^2 - x^3 - x^4 - x^5\big)},$$

$$\beta(x), = \frac{4x\,\big(1 - x + x^2\big)\,\big(1 + x + x^2\big)}{1 - x - x^2 - x^3 - x^4 - x^5},$$

$$\mu(x) = \frac{2x\,\big(1 - x + x^2 - x^3 + x^4\big)}{1 - x - x^2 - x^3 - x^4 - x^5},$$

$$\nu(x) = \frac{x\,\big(1 - x + x^2 - x^3 + x^4\big)}{(1 - x)\,\big(1 - x - x^2 - x^3 - x^4 - x^5\big)}. \tag{6.42}$$

The α_n's, β_n's, μ_n's and ν_n's grow exponentially like λ^n with $\lambda \sim 1.965\cdots$.

6.1.8. Fourth example for 5×5 matrices

Let us consider the "straight" generalization of the previous example to 5×5 matrices. For 5×5 matrices the analysis of the factorizations becomes different from the previous one. It yields, for arbitrary n, factorizations where *a product of a fixed number of* f_n's occurs,

$$K(M_n) = M_{n+1} \cdot f_n \cdot f_{n-1}^3 \cdot f_{n-2}^4, \quad \det(M_n) = f_{n+1} \cdot f_n^3 \cdot f_{n-1}^5 \cdot f_{n-2}^5, \tag{6.43}$$

yielding

$$\widehat{K}(M_n) = \frac{K(M_n)}{\det(M_n)} = \frac{M_{n+1}}{f_{n+1} \cdot (f_n \cdot f_{n-1})^2 \cdot f_{n-2}} . \tag{6.44}$$

From these relations one gets the linear relations

$$\begin{aligned} &\alpha_{n+2} = 5\beta_n + 5\beta_{n+1} + 3\beta_{n+2} + \beta_{n+3} , \\ &4\alpha_{n+2} = \alpha_{n+3} + 5(\beta_{n+2} + 3\beta_{n+1} + 4\beta_n) , \\ &\alpha_n + \alpha_{n+1} = 5(\beta_{n+1} + 2\beta_n + 2\beta_{n-1} + \beta_{n-2}) , \end{aligned} \tag{6.45}$$

and the relations on the generating functions,

$$\begin{aligned} &x \cdot \alpha(x) = (1 + 3x + 5x^2 + 5x^3) \cdot \beta(x) , \\ &(4x-1) \cdot \alpha(x) + 5 = 5(x + 3x^2 + 4x^3) \cdot \beta(x) , \\ &(1+x) \cdot \alpha(x) = 5(1+x) \cdot (1 + x + x^2) \cdot \beta(x) + 5 , \end{aligned} \tag{6.46}$$

which give

$$\begin{aligned} &\alpha(x) = \frac{5(1 + 3x + 5x^2 + 5x^3)}{(1+x) \cdot (1-2x)} , \quad \beta(x) = \frac{5x}{(1+x) \cdot (1-2x)} , \\ &\mu(x) = \frac{x \cdot (3-2x)}{(1+x) \cdot (1-2x)} , \quad \nu(x) = \frac{x \cdot (1 + 3x + 4x^2)}{(1+x) \cdot (1-2x)} . \end{aligned} \tag{6.47}$$

6.1.9. "Straight" generalization of the fourth example for $q \geq 6$

For $q \times q$ matrices ($q \geq 6$) the analysis of the factorizations is *different from the two previous cases*, $q = 4$ and $q = 5$. It yields, for arbitrary n,

$$K(M_n) = M_{n+1} \cdot f_n^{q-3} , \quad \det(M_n) = f_{n+1} \cdot f_n^{q-2} , \quad \widehat{K}(M_n) = \frac{M_{n+1}}{f_{n+1} \cdot f_n} . \tag{6.48}$$

From these relations, the following linear relations are derived:

$$\begin{aligned} &\alpha_n = (q-2) \cdot \beta_n + \beta_{n+1} , \quad (q-1) \cdot \alpha_n = \alpha_{n+1} + q(q-3) \cdot \beta_n , \\ &\alpha_n + \alpha_{n+1} = q(\beta_{n+1} + \beta_n) , \\ &\text{and} \\ &x \cdot \alpha(x) = (1 + (q-2)x) \cdot \beta(x) , \\ &((q-1)x - 1) \cdot \alpha(x) + q = q(q-3)x \cdot \beta(x) , \\ &(1+x) \cdot \alpha(x) = q(1+x) \cdot \beta(x) + q , \end{aligned} \tag{6.49}$$

which give

$$\begin{aligned} &\alpha(q,x) = \frac{q(1 + (q-2)x)}{(1+x) \cdot (1-2x)} , \quad \beta(q,x) = \frac{qx}{(1+x) \cdot (1-2x)} , \\ &\mu(q,x) = \frac{x((q-2) - 2x)}{(1+x) \cdot (1-2x)} , \quad \nu(q,x) = \frac{(q-3)x}{(1+x) \cdot (1-2x)} . \end{aligned} \tag{6.50}$$

One can see that, for $q = 5$, $\beta(q,x)$ and $\mu(q,x)$ are identical to the expressions obtained for 5×5 matrices. Moreover, the expressions for $\alpha(q,x)$ and $\nu(q,x)$ *are* *actually identical to the one given for* class VI (see (3.15)). *They are also identical to the one given for the "self-similar" third example of section* (6.1.5) (see (6.29)).

6.1.10. Fifth example

Let us consider a 4×4 matrix and the two following permutations of the entries:

$$t: \ m_{11} \leftrightarrow m_{44}, \quad m_{21} \leftrightarrow m_{34}, \quad m_{24} \leftrightarrow m_{31},$$

$$t: \ m_{23} \leftrightarrow m_{32}.$$

The analysis of the factorization of the corresponding homogeneous transformation K yields exactly the *same factorizations as for the sixteen vertex model* (permutation t_1 for a 4×4 matrix, see Section 4.1). However, one no longer gets recursions like (4.8). However, some polynomials like $(F_6 - F_3 F_5^2)$, $(F_4 - F_3^2)$,... do factorize.

6.1.11. Towards most general permutations: from 3×3 *to* $q \times q$ *matrices*

Of course, it is easy to accumulate examples of involutive permutations and their associated birational transformations K, and to analyse the corresponding factorizations properties. We have seen that the previous examples are often organized in a family of transformations sharing the same, or similar, factorization properties (straight generalizations, self-similar generalizations...). The previous examples show that *factorization properties are not so rare*: however, most of the previous examples yield exponential growth of the calculations. A set of particularly interesting examples are the ones which yield *polynomial growth* since integrable mappings have this property: it would be interesting to get more examples of this kind.

A systematic search of these examples with polynomial growth cannot be envisaged, however it has been seen that the factorization schemes of these families depend very simply on q, the matrix size, while the recursion relations (when they exist) *are independent of* q (see, for instance, (3.27)). It is thus tempting to look systematically *at all the possible permutations of the entries of a simple* 3×3 matrix, seeking recursions on the f_n's, the l_n's or the x_n's (see (3.30) and (3.31)).

We have obtained the following miscellaneous results:

– The following permutations *do satisfy the same integrable recursions on the* l_n's, or the x_n's, (see Eq. (3.30)) *as the one for class* I (permutation $m_{12} \leftrightarrow m_{21}$ for example):

$$p_1: \ m_{11} \leftrightarrow m_{33}, \quad m_{13} \leftrightarrow m_{31}, \quad p_2: \ m_{12} \leftrightarrow m_{32}, \quad m_{21} \leftrightarrow m_{23},$$

$$p_3: \ m_{12} \leftrightarrow m_{21}, \quad m_{23} \leftrightarrow m_{32}, \quad p_4: \ m_{12} \leftrightarrow m_{21}, \quad m_{13} \leftrightarrow m_{31},$$

$$p_5: \ m_{11} \leftrightarrow m_{22}, \quad m_{12} \leftrightarrow m_{21}, \quad p_6: \ m_{12} \leftrightarrow m_{21}, \quad m_{13} \leftrightarrow m_{23}, \quad m_{31} \leftrightarrow m_{32},$$

$$p_7: m_{12} \leftrightarrow m_{21}, \quad m_{11} \leftrightarrow m_{33}, \quad m_{32} \leftrightarrow m_{23}.$$

Of course this list is far from being exhaustive...

– The following permutation *does satisfy the same recursions on the* l_n's, or the x_n's, (see Eq. (3.67)) *as the one for class* IV (permutation $m_{13} \leftrightarrow m_{23}$ for example):

$$p_8: \ m_{11} \leftrightarrow m_{22}, \quad m_{12} \leftrightarrow m_{21}, \quad m_{13} \leftrightarrow m_{23}. \tag{6.51}$$

The factorizations associated with these various permutations *are exactly the same as for class* I, for 3×3 matrices for p_1, p_2, p_3, p_4, p_5, p_6 and p_7 (see (3.9)), and the *same factorizations as for class* IV for 3×3 matrices for p_8 (see (3.62) and (3.63) for $q=3$). The f_n's associated with permutations p_i's ($i=1,\cdots 7$) and permutation p_8 *satisfy the same recursions as the* f_n's associated respectively with class I (namely (3.27) or (3.28)) and class IV (namely (3.66) but for p_8 only).

Let us recall that, for class I, the factorization schemes for the 3×3 matrices are actually different from the ones for the $q\times q$ matrices for $q\geq 4$ (see relation (3.9)). Therefore, we have reconsidered all these examples of permutations considering their ("straight") generalizations to $q\times q$ matrices for $q\geq 4$.

For $q\times q$ matrices, permutations p_1, p_2 and p_5 do satisfy the *same factorization relations as the ones of class* I (see (3.5) and (3.6)) and also verify the *same recursion relations* on the f_n's as the ones of class I (see (3.27) and (3.28)). *Permutations* p_1, p_2 and p_5, *therefore, provide new examples of (involutive) permutations associated with integrable recursion relations* and yielding *polynomial growth.*

Permutations p_3 and p_4 do satisfy the *same factorization relations as the ones of class* I, and, therefore, have a *polynomial growth* of the calculations. However they *do not verify the same recursion relations* on the f_n's as the one for class I. The situation is similar to the ones described in Section 6.1.1 where *polynomial growth occurs, the orbits being algebraic surfaces* (see Figs. 2a, 2b, 2c and 2d, see also [31]).

Permutations p_6 and p_7 *factorize less than class* I. For p_6, for 4×4 matrices, one has "string-like" factorizations,

$$K(M_n) = M_{n+1}\cdot\Big(f_{n-1}\cdot f_{n-3}\cdot f_{n-5}\cdot f_{n-7}\cdots\Big)^2,$$
$$\det(M_n) = f_{n+1}\cdot f_n\cdot f_{n-1}^3\cdot f_{n-2}\cdot f_{n-3}^3\cdot f_{n-4}\cdot f_{n-5}^3\cdots, \tag{6.52}$$

and relation (3.49),

$$M_{n+1} = (f_{n+1}\cdot f_n\cdot f_{n-1}\cdots f_2\cdot f_1)\cdot\widehat{K}(M_n). \tag{6.53}$$

From these factorizations it can easily be verified that one has an exponential growth like 2^n: $\beta_n = 2^{n+1}$.

For p_7, for 4×4 matrices, there are "string-like" factorizations,

$$K(M_n) = M_{n+1}\cdot f_{n-3}^4\cdot f_{n-5}^4\cdot f_{n-7}^4\cdot f_{n-9}^4\cdots,$$
$$\det(M_n) = f_{n+1}\cdot f_{n-1}\cdot f_{n-2}^2\cdot f_{n-3}^6\cdot f_{n-4}^2\cdot f_{n-5}^6\cdot f_{n-6}^2\cdot f_{n-7}^6 \cdot f_{n-8}^2\cdot f_{n-9}^6\cdot f_{n-10}^2\cdots, \tag{6.54}$$

and a new and "non standard" (note the absence f_n in (6.55)) "string-like" relation [11],

$$M_{n+1} = f_{n+1}\cdot f_{n-1}\cdot f_{n-2}^2\cdot f_{n-3}^2\cdot f_{n-4}^2\cdot f_{n-5}^2\cdot f_{n-6}^2\cdot f_{n-7}^2\cdot f_{n-8}^2$$

[11] A similar but simpler relation for transformation $\widehat{K}$ has already been seen in the analysis of the symmetries of a three-dimensional vertex model see relation (4.5) in [31].

$$\cdot f_{n-9}^2 \cdots \widehat{K}(M_n) \,. \tag{6.55}$$

In fact, we have not yet completely reached the stabilization of the "factorization scheme". It would be necessary to perform iterations beyond M_{13} and f_{13} which become quite large formal calculations given that one has an exponential growth like λ^n with $\lambda \sim 2.769 \cdots$.

At these orders of iteration a good *approximation* of the generating function $\alpha(x)$ and $\beta(x)$ reads

$$\alpha(x) = \frac{4 \cdot (1 + 2x^3 + 5x^4)}{(1 - 3x + x^2 - x^3) \cdot (1 - x) \cdot (1 + x)} \,, \quad \beta(x) = \frac{4x}{1 - 3x + x^2 - x^3} \,. \tag{6.56}$$

On the other hand, for 4×4 matrices, p_8 *does not satisfy the same factorizations and recursions on the* f_n's as the one for class IV (see (3.62), (3.63) and (3.66)). The factorizations are more involved and read

$$M_{n+2} = \frac{K_{t_1}(M_{n+1})}{f_n^2 \cdot f_{n-2}^2 \cdot f_{n-4}^4 \cdot f_{n-6}^4 \cdot f_{n-8}^6 \cdot f_{n-10}^6 \cdots} \,,$$
$$f_{n+2} = \frac{\det(M_{n+1})}{f_{n+1} \cdot f_n^3 \cdot f_{n-1} \cdot f_{n-2}^4 \cdot f_{n-3}^2 \cdot f_{n-4}^6 \cdot f_{n-5}^2 \cdot f_{n-6}^7 \cdot f_{n-7}^3 \cdot f_{n-8}^9 \cdot f_{n-9}^3 \cdot f_{n-10}^{10} \cdot f_{n-11}^4 \cdots} \,, \tag{6.57}$$

and

$$\widehat{K}(M_n) = \frac{M_{n+1}}{\cdots f_{n-11}^4 \cdot (f_{n-10} \cdot f_{n-9} \cdot f_{n-8} \cdot f_{n-7})^3 \cdot (f_{n-6} \cdot f_{n-5} \cdot f_{n-4} \cdot f_{n-3})^2 \cdot f_{n-2} \cdot f_{n-1} \cdot f_n \cdot f_{n+1}} \,, \tag{6.58}$$

which yield the linear relations

$$\begin{aligned}
\alpha_{n+1} &= \beta_{n+2} + \beta_{n+1} + 3\beta_n + \beta_{n-1} + 4\beta_{n-2} + 2\beta_{n-3} + 6\beta_{n-4} + 2\beta_{n-5} \\
&\quad + 7\beta_{n-6} + 3\beta_{n-7} + \cdots \,, \\
3\alpha_{n+1} &= \alpha_{n+2} + 4 \cdot (2\beta_n + 2\beta_{n-2} + 4\beta_{n-4} + 4\beta_{n-6} + 6\beta_{n-8} + 6\beta_{n-10} \cdots) \,, \\
\alpha_n + \alpha_{n+1} &= 4 \cdot (\beta_{n+1} + \beta_n + \beta_{n-1} + \beta_{n-2}) \\
&\quad + 8 \cdot (\beta_{n-3} + \beta_{n-4} + \beta_{n-5} + \beta_{n-6}) \\
&\quad + 12 \cdot (\beta_{n-7} + \beta_{n-8} + \beta_{n-9} + \beta_{n-10}) + 16 \cdot (\beta_{n-11} + \cdots) + \cdots \,.
\end{aligned} \tag{6.59}$$

Again, we have not completely reached the value of n for which the factorization scheme is stabilized. Despite the fact that this stability regime is not completely reached, it seems, however, that there is a 2^n exponential growth of the calculations. A tentative exact expression for the generating functions (see Appendix F) is for instance

$$\beta(x) = \frac{4x \cdot (1 - x^4)}{1 - 2x} \,, \tag{6.60}$$

and

$$\alpha(x) = \frac{4}{1-2x} + \frac{4x\,(1+3x)}{(1+x)\cdot(1-x)\cdot(1-2x)}$$
$$= \frac{4\left(1+x+2x^2\right)}{(1+x)\cdot(1-x)\cdot(1-2x)}\,. \qquad (6.61)$$

The birational transformations associated with permutations p_i's show that, quite often, one encounters nice properties and structures for 3×3 matrices, and, to some extent, similarly with the birational transformations of class I, that these properties *can sometimes survive* "straight" generalizations to $q \times q$ matrices.

Let us note that there are several ways one can generalize to $q \times q$ matrices permutations p_i's introduced for 3×3 matrices. Let us consider, for instance, for 4×4 matrices, permutation p'_5, which is the product of permutation p_5 and of permutation $m_{34} \leftrightarrow m_{43}$. Permutation p'_5 $(m_{11} \leftrightarrow m_{22},\, m_{12} \leftrightarrow m_{21},\, m_{34} \leftrightarrow m_{43})$ yields the *same factorizations as* t_1 for 4×4 matrices (sixteen vertex model see Section 4.1) and, therefore, also *yields a polynomial growth of the calculations*, with the same generating functions as the one detailed in Section 4.1. However, the f_n's do not verify any recursion or "pseudo-recursion" (see (3.66)): the situation is again similar to the one detailed in Section 6.1.1 with the example of permutation [12–21,34–43]. The "straight" generalization of p'_5 yields results similar to the "straight" generalization of t_1 (see Section 5).

– When the permutations of the entries are *no longer involutions* it becomes difficult, for 3×3 matrices, to get any nice factorization properties: however, we have been able in the following to find many examples of permutations which *are not involutions*, but exhibit factorization properties for $q \times q$ matrices with $q \geq 4$.

6.2. *Breaking the involutive framework*

Let us now examine *birational* transformations corresponding to permutations of the entries which are *no longer involutive permutations.*

6.2.1. *A 3-cycle permutation and its "straight" generalization*

For all the previous examples, permutation t was an *involutive* permutation [12]. Let us, however, note that even if t is a noninvolutive permutation, for instance a 3-cycle, transformation $K = t \cdot I$ is still a *birational* transformation (in the case of a 3-cycle, the inverse of K corresponds to $I \cdot t^2$, which is also a homogeneous rational transformation). At first sight one expects the corresponding transformations K to be more involved with less structure and properties (less factorization properties, no more polynomial growth of the complexity, or integrability, ...).

Let us consider the following 3-cycle permutation for $q \times q$ matrices:

$$m_{11} \longrightarrow m_{12} \longrightarrow m_{21} \longrightarrow m_{11}\,. \qquad (6.62)$$

[12] This has many consequences: the group of birational transformations generated by t and I is isomorphic to the infinite dihedral group.

The results given here *are valid for* $q \geq 4$. Let us look at permutation (6.62) for $q \times q$ matrices. The factorizations now read

$$f_2 = \frac{\det(M_1)}{f_1^{q-3}}, \quad M_2 = \frac{K(M_1)}{f_1^{q-4}}, \quad f_3 = \frac{\det(M_2)}{f_1^3 \cdot f_2^{q-3}},$$
$$M_3 = \frac{K(M_2)}{f_2^{q-4} f_1^2}, \quad f_4 = \frac{\det(M_3)}{f_1^{q-3} \cdot f_2^3 \cdot f_3^{q-3}}, \quad \ldots \tag{6.63}$$

and, for arbitrary n, *"string-like" factorizations*,

$$K(M_n) = M_{n+1} \cdot (f_n^{q-4} \cdot f_{n-1}^2 \cdot f_{n-2}^{q-4} \cdot f_{n-3}^2 \cdot f_{n-4}^{q-4} \cdots),$$
$$\det(M_n) = f_{n+1} \cdot f_n^{q-3} \cdot f_{n-1}^3 \cdot f_{n-2}^{q-3} \cdot f_{n-3}^3 \cdot f_{n-4}^{q-3} \cdots f_1^{\zeta_n}, \tag{6.64}$$

where $\zeta_n = 2$ for n even and $\zeta_n = 3$ for n odd, yielding again *the simple "string-like" relation independent of* q (3.49) and relation (3.60). Eqs. (3.49) and (6.64) yield linear relations for the α_n's and β_n's,

$$\alpha_n = \beta_{n+1} + (q-3) \cdot \beta_n + 3\beta_{n-1} + (q-3) \cdot \beta_{n-2} + 3\beta_{n-3},$$
$$+ (q-3) \cdot \beta_{n-4} + \cdots + \zeta_n \beta_1$$
$$(q-1) \cdot \alpha_n = \alpha_{n+1} + 2q(\beta_{n-1} + \beta_{n-3} + \beta_{n-5} + \cdots)$$
$$+ q(q-4)(\beta_n + \beta_{n-2} + \beta_{n-4} + \cdots),$$
$$\text{and} \quad \alpha_n + \alpha_{n+1} = q(\beta_1 + \beta_2 + \cdots + \beta_{n+1}), \tag{6.65}$$

and for the generating functions $\alpha(x)$ and $\beta(x)$,

$$q + ((q-1) \cdot x - 1) \cdot \alpha(x) = \frac{qx \cdot (2x + q - 4)}{1 - x^2} \cdot \beta(x),$$
$$x\alpha(x) = \frac{1 + 2x^2 + (q-3)x}{1 - x^2} \cdot \beta(x), \tag{6.66}$$

together with relation (3.49). These linear relations enable us to get

$$\alpha(x) = \frac{q \cdot (1 + (q-3)x + 2x^2)}{(1-2x) \cdot (1-x^2)}, \quad \beta(x) = \frac{qx}{1-2x},$$
$$\mu(x) = \frac{(q-3)x}{1-2x}, \quad \nu(x) = \frac{x \cdot (2x + (q-4))}{(1-x^2) \cdot (1-2x)}. \tag{6.67}$$

6.2.2. N-cycles

Let us consider a permutation of order *eight*, namely the 8-cycle,

$$m_{2,1} \longrightarrow m_{2,2} \longrightarrow m_{2,3} \longrightarrow m_{2,4} \longrightarrow m_{2,5} \longrightarrow m_{2,6}$$
$$\longrightarrow m_{2,7} \longrightarrow m_{2,8} \longrightarrow m_{2,1}. \tag{6.68}$$

The factorization properties read for an 8×8 matrix,

$$K(M_n) = M_{n+1} \cdot (f_n^5 \cdot f_{n-2}^5 \cdot f_{n-4}^5 \cdot f_{n-6}^5 \cdots),$$
$$\det(M_n) = f_{n+1} \cdot f_n^6 \cdot f_{n-1} \cdot f_{n-2}^6 \cdot f_{n-3} \cdot f_{n-4}^6 \cdot f_{n-5} \cdots . \quad (6.69)$$

One has again the simple "string-like" relation independent of q (3.49),

$$\widehat{K}(M_n) = \frac{K(M_n)}{\det(M_n)} = \frac{M_{n+1}}{f_1 \cdot f_2 \cdot f_3 \cdots f_n \cdot f_{n+1}} . \quad (6.70)$$

This yields linear recursions for the α_n's and β_n's, namely (6.65), but for $q = 8$ and

$$\alpha_n = \beta_{n+1} + 6\beta_n + \beta_{n-1} + 6\beta_{n-2} + \beta_{n-3} + 6\beta_{n-4} + \cdots , \quad (6.71)$$

giving for $\alpha(x)$ and $\beta(x)$,

$$x\alpha(x) = \frac{(1+6x) \cdot \beta(x)}{1 - x^2} \quad \text{together with (3.60)} . \quad (6.72)$$

The generating functions $\alpha(x)$ and $\beta(x)$ read

$$\alpha(x) = \frac{8(1+6x)}{(1-2x) \cdot (1+x)}, \quad \beta(x) = \frac{8x(1-x)}{1-2x}, \quad (6.73)$$

which is nothing but expressions $\alpha(x)$ and $\beta(x)$ given, for class VI, for $q = 8$. This corresponds again to a 2^n exponential growth.

Let us now consider a permutation of order *seven*, namely the 7-cycle still acting on the entries of an 8×8 matrix,

$$m_{2,1} \longrightarrow m_{2,2} \longrightarrow m_{2,3} \longrightarrow m_{2,4} \longrightarrow m_{2,5} \longrightarrow m_{2,6} \longrightarrow m_{2,7} \longrightarrow m_{2,1} . \quad (6.74)$$

The following result is obtained: the factorization properties are exactly the same as the one described by (6.69) and (6.70) and, *therefore, the generating functions are the same as the one given in* (6.73).

Amazingly, the factorization properties are also exactly the same for the N-cycles of the form

$$m_{2,1} \longrightarrow m_{2,2} \longrightarrow m_{2,3} \cdots \longrightarrow m_{2,N} \longrightarrow m_{2,1} , \quad (6.75)$$

where $N = 6, 5, 4, 3, 2$. For $N = 2$ one recovers exactly the elementary permutation $m_{2,1} \leftrightarrow m_{2,2}$ of class VI and its corresponding factorizations (see Section 3.2.2): this is compatible with the fact that *the results for all these* N-cycles *are the same as the one given for class* VI for $q = 8$.

In fact, these results are not restricted to 8×8 *matrices: considering* N-cycles *acting on* $q \times q$ *matrices, one does have the same factorizations as the one given by relations* (3.47) and (3.49) *which correspond to class* VI (that is $N = 2$). *In particular, these factorizations are independent of* N.

6.3. Summary of the "birational" results

We have seen in the previous examples that the *involutive* character of the permutations is *not a necessary condition* for factorization properties of transformations K:

factorization properties do occur for permutations that *are no longer involutions* (N-cycles ...). Let us however note that transformations K are still *birational.*

Among these various examples, *some factorizations happen, unexpectedly, to be exactly the same as the ones for permutations of quite a different type* (for instance, elementary permutations of two entries, like class VI and the N-cycle of the previous Section 6.2.2, or class I and some of the permutations p_i's).

It is also worth noting that one can have the *same factorization schemes and not necessarily verify the same recursions* (on the f_n's, see Section 6.1.10).

There is a quite frequent occurrence of "string-like" factorizations: factorizations involving a fixed number of polynomials are more exceptional. Note however that this opposition is not very relevant: it has been seen in Sections 3.2.1 and 4.2 that these two types of factorizations, "string-like" versus "fixed number", do not yield systematically to drastically different factorization schemes: one can have "string-like" factorizations *together with polynomial growth* (see Section 4.2) or, on the contrary, factorizations with a fixed number of polynomials and exponential growth (see Section 3.2.1).

Section 6.1.11 shows that the set of *integrable* birational transformations, or of transformations associated with *polynomial growth,* or even of transformations *associated with recursions on the* f_n's (or on the x_n's) seems to be larger than one could expect at first sight for 3×3 matrices, and *even for* the "straight" generalizations of these examples to $q \times q$ matrices.

7. Leaving the birational framework

Considering permutations of the entries, which are *no longer involutive* (N-cycle ...), does not break the *birational* character of transformations K and can even yield factorizations similar, or identical, to the one corresponding to elementary involutive permutations of two entries (for example class VI and the N-cycles). Let us now try to see the consequences of *relaxing the birational character* of the transformations, analyzing the factorizations properties of *noninvertible, but still rational,* transformations *slightly different* from the previous birational ones.

Instead of a permutation of the entries, let us consider a transformation T which is the product of an involutive permutation of the entries t, for instance

$$t: \ m_{11} \leftrightarrow m_{31}, \quad m_{12} \leftrightarrow m_{32}, \quad m_{21} \leftrightarrow m_{41}, \quad m_{22} \leftrightarrow m_{42}, \tag{7.1}$$

introduced in the *fourth example* (Section 6.1.7), together with a *projection* transformation P ($P^2 = P$).

We will consider various examples of such projections. For instance we will consider projection P_1 defined by

$$P_1: \ m(i,j) \longrightarrow m(i,j) \quad \text{when} \quad j \neq 4$$
$$\text{and} \quad (m_{14}, m_{24}, m_{34}, m_{44}) \longrightarrow (m_{14}, m_{24}, m_{34}, m_{34}), \tag{7.2}$$

projection P_2 defined by

$$P_2: \ m(i,j) \longrightarrow m(i,j) \quad \text{when} \quad j \neq 5$$
$$\text{and} \quad (m_{15}, m_{25}, m_{35}, m_{45}, m_{55}) \longrightarrow (m_{15}, m_{25}, m_{35}, m_{45}, m_{35}), \tag{7.3}$$

and projection P_3 defined by

$$P_3: \ m(i,j) \longrightarrow m(i,j) \quad \text{when} \quad i \neq 6$$
$$\text{and} \quad (m_{61}, m_{62}, m_{63}, m_{64}, m_{65}, m_{66}) \longrightarrow (m_{51}, m_{52}, m_{53}, m_{54}, m_{55}, m_{66}). \tag{7.4}$$

Let us now examine the iteration of transformation $K = T \cdot I = P \cdot t \cdot I$, where P is one of these projections P_i's. Transformation K *is clearly not a birational transformation anymore* but it is *still rational.*

7.1. *Projection P_2 and 5×5 matrices*

Let us first consider transformation $K = T \cdot I = P_2 \cdot t \cdot I$, where P_2 is projection (7.3) on 5×5 matrices, and t is the (straight generalization of) the permutation of the fourth example namely (7.1). For arbitrary n one gets the factorizations

$$K(M_n) = M_{n+1} \cdot f_n \cdot f_{n-1}, \quad \det(M_n) = f_{n+1} \cdot f_n^2 \cdot f_{n-1},$$
$$\widehat{K}(M_n) = \frac{M_{n+1}}{f_n \cdot f_{n+1}}. \tag{7.5}$$

These factorizations give linear relations on the α_n's and the β_n's or on $\alpha(x)$ and $\beta(x)$,

$$\alpha_n = \beta_{n-1} + 2\beta_n + \beta_{n+1}, \quad 4 \cdot \alpha_n = \alpha_{n+1} + 5 \cdot (\beta_n + \beta_{n-1}),$$
$$x \cdot \alpha(x) = \beta(x) \cdot (1+x)^2, \quad 5 + (4x-1) \cdot \alpha(x) = 5x(1+x)\beta(x),$$
$$5(1+x) \cdot \beta(x) + 5 = (1+x) \cdot \alpha(x). \tag{7.6}$$

The expressions of the generating functions are

$$\alpha(x) = \frac{5(1+x)}{1-3x+x^2}, \quad \beta(x) = \frac{5x}{(1+x) \cdot (1-3x+x^2)}. \tag{7.7}$$

The α_n's or β_n's grow like λ^n with $\lambda \sim 2.618\cdots$ These factorizations have to be compared with the one of transformation $K = t \cdot I$, where t denotes the (involutive) permutation (7.1) of the fourth example.

One should note that relations (3.10), *corresponding to the "right action" of K, are still valid for this transformation which is no longer birational.*

At first sight, one expects transformation K to be quite a "chaotic" transformation since K is not birational anymore. Remarkable factorization properties are not expected. In fact, one can see that *many parts of the factorization analysis performed in* [21] *are still valid*[13] *for transformations which are no longer birational.*

[13] Such demonstrations are based on the fact that some matrix $U = R \cdot K(R)$ (see [21,22]) is, up to a finite number of entries, the identity matrix.

7.2. *Projection P_1 and 4×4 matrices*

Let us now look at transformation $K = T \cdot I = P_1 \cdot t \cdot I$, where P_1 is projection (7.2) on 4×4 matrices and t is the permutation of the fourth example namely (7.1). The following factorizations are obtained:

$$
\begin{aligned}
& f_2 = \frac{\det(M_1)}{f_1}, \quad M_2 = K(M_1), \quad f_3 = \frac{\det(M_2)}{f_1^2 \cdot f_2}, \quad M_3 = \frac{K(M_2)}{f_1}, \\
& f_4 = \frac{\det(M_3)}{f_1^2 \cdot f_2^2 \cdot f_3}, \quad M_4 = \frac{K(M_3)}{f_2 f_1}, \quad f_5 = \frac{\det(M_4)}{f_1 \cdot f_2^2 \cdot f_3^2 \cdot f_4}, \quad M_5 = \frac{K(M_4)}{f_2 f_3}, \\
& f_6 = \frac{\det(M_5)}{f_1^2 \cdot f_2 \cdot f_3^2 \cdot f_4^2 \cdot f_5}, \quad M_6 = \frac{K(M_5)}{f_4 f_3 f_1}, \\
& f_7 = \frac{\det(M_6)}{f_1^3 \cdot f_2^2 \cdot f_3 \cdot f_4^2 \cdot f_5^2 \cdot f_6}, \quad M_7 = \frac{K(M_6)}{f_5 f_4 f_2 f_1}, \\
& f_8 = \frac{\det(M_7)}{f_1^5 \cdot f_2^3 \cdot f_3^2 \cdot f_4 \cdot f_5^2 \cdot f_6^2 \cdot f_7}, \quad M_8 = \frac{K(M_7)}{f_6 f_5 f_3 f_2 f_1^3}, \\
& f_9 = \frac{\det(M_8)}{f_1^2 \cdot f_2^5 \cdot f_3^3 \cdot f_4^2 \cdot f_5 \cdot f_6^2 \cdot f_7^2 \cdot f_8}, \quad M_9 = \frac{K(M_8)}{f_7 f_6 f_4 f_3 f_2^3 f_1}, \\
& f_{10} = \frac{\det(M_9)}{f_1^2 \cdot f_2^2 \cdot f_3^5 \cdot f_4^3 \cdot f_5^2 \cdot f_6 \cdot f_7^2 \cdot f_8^2 \cdot f_9}, \quad M_{10} = \frac{K(M_9)}{f_8 f_7 f_5 f_4 f_3^3 f_2 f_1}, \\
& f_{11} = \frac{\det(M_{10})}{f_1^2 \cdot f_2^2 \cdot f_3^2 \cdot f_4^5 \cdot f_5^3 \cdot f_6^2 \cdot f_7 \cdot f_8^2 \cdot f_9^2 \cdot f_{10}}, \quad M_{11} = \frac{K(M_{10})}{f_9 f_8 f_6 f_5 f_4^3 f_3 f_2}, \\
& f_{12} = \frac{\det(M_{11})}{f_1^2 \cdot f_2^2 \cdot f_3^2 \cdot f_4^2 \cdot f_5^5 \cdot f_6^3 \cdot f_7^2 \cdot f_8 \cdot f_9^2 \cdot f_{10}^2 \cdot f_{11}}, \quad \ldots,
\end{aligned}
\tag{7.8}
$$

and, for arbitrary n,

$$
\begin{aligned}
K(M_n) &= M_{n+1} \cdot \Big(f_{n-1} \cdot f_{n-2} \cdot f_{n-4} \cdot f_{n-5} \cdot f_{n-6}^3 \cdot f_{n-7} \cdot f_{n-8} \\
&\qquad \cdot f_{n-9} \cdot f_{n-10} \cdot f_{n-11} \cdots \Big), \\
\det(M_n) &= f_{n+1} \cdot \Big(f_n \cdot f_{n-1}^2 \cdot f_{n-2}^2 \cdot f_{n-3} \cdot f_{n-4}^2 \cdot f_{n-5}^3 \cdot f_{n-6}^5 \cdot f_{n-7}^2 \\
&\qquad \cdot f_{n-8}^2 \cdot f_{n-9}^2 \cdot f_{n-10}^2 \cdot f_{n-11}^2 \cdot f_{n-12}^2 \cdots \Big),
\end{aligned}
\tag{7.9}
$$

and

$$
\widehat{K}(M_n) = \frac{M_{n+1}}{\cdots f_{n-11} \cdot f_{n-10} \cdot f_{n-9} \cdot f_{n-8} \cdot f_{n-7} \cdot f_{n-6}^2 \cdot f_{n-5}^2 \cdot f_{n-4} \cdot f_{n-3} \cdot f_{n-2} \cdot f_{n-1} \cdot f_n \cdot f_{n+1}}. \tag{7.10}
$$

Again, relations (3.10), *corresponding to the "right action" of K, are still valid.*

These factorizations are more involved than the other examples detailed in this paper. In fact *it is not even clear that the generating functions corresponding to these factorizations are rational functions*, since *no obvious "periodicity"* occurs in the factorization scheme. Factorizations (7.10) yield a linear relation between the α_n's and the β_n's,

$$\alpha_n + \alpha_{n+1} = 4(\beta_1 + \beta_2 + \cdots + \beta_{n+1}) + \beta_{n-5} + \beta_{n-6} + \cdots \,, \tag{7.11}$$

or on the generating functions,

$$(1+x)\cdot\alpha(x) - 4 = \frac{4\beta(x)}{(1-x)} + 4\beta(x)\cdot\rho_1(x)\,, \tag{7.12}$$

where $\rho_1(x) = x^6 + x^7 + \cdots$. One also has

$$(3x-1)\cdot\alpha(x) + 4 = 4\beta(x)\cdot(x^2 + x^3 + x^5 + x^6 + 3x^7 + x^8 + x^9 + x^{10} + x^{11} + x^{12} + \cdots)\,, \tag{7.13}$$

and

$$x\cdot\alpha(x) = 4\beta(x)\cdot(1 + x + 2x^2 + 2x^3 + x^4 + 2x^5 + 3x^6 + 5x^7 + 2x^8 + 2x^9 + 2x^{10} + 2x^{11} + 2x^{12} + 2x^{13} + \cdots)\,. \tag{7.14}$$

Let us give the first coefficients of the generating functions $\alpha(x)$, $\beta(x)$, $\mu(x)$ and $\nu(x)$,

$$\begin{aligned}
\alpha(x) &= 4 + 12x + 36x^2 + 92x^3 + 228x^4 + 572x^5 + 1428x^6 + 3564x^7 \\
&\quad + 8852x^8 + 21996x^9 + 54660x^{10} + 135836x^{11} + 337556x^{12} \\
&\quad + 838812x^{13} + \cdots\,, \\
\beta(x) &= 4x + 8x^2 + 20x^3 + 48x^4 + 120x^5 + 300x^6 + 744x^7 + 1848x^8 \\
&\quad + 4592x^9 + 11412x^{10} + 28360x^{11} + 70472x^{12} \\
&\quad + 175120x^{13} + 435168x^{14} + \cdots\,, \\
\mu(x) &= x + x^2 + 3x^3 + 6x^4 + 15x^5 + 39x^6 + 96x^7 + 238x^8 + 591x^9 \\
&\quad + 1469x^{10} + 3652x^{11} + \cdots\,, \\
\nu(x) &= 1 + x^2 + 3x^3 + 7x^4 + 18x^5 + 45x^6 + 115x^7 + 285x^8 + 708x^9 \\
&\quad + 1759x^{10} + \cdots\,.
\end{aligned} \tag{7.15}$$

There is an exponential growth of the calculations like λ^n with $\lambda \sim 2.484\cdots$. A good approximation for $\beta(x)$, compatible with expansions (7.15) and yielding positive integer coefficients, is, for instance,

$$\beta(x) = \frac{4(1-x)(1+x+x^2)(1-x+x^3-x^4+x^5-x^7+x^8)}{1-3x+x^2+x^3-x^4+x^5-x^6+x^{10}-x^{11}}\,. \tag{7.16}$$

If factorizations (7.9) were "periodic" (like in the "string-like" factorizations previously detailed where shifts of four, six, ... occur) one could expect $\rho_1(x)$ to be in the following form:

$$\rho_1(x) = \frac{(1+x)\cdot x^6}{1-x^P}\,. \tag{7.17}$$

where the integer P denotes the period of these factorizations ($P \geq 7$). Some speculations concerning this very example are given in Appendix F.

7.3. Non-birational deformation of class I

Let us now consider transformation $K = T \cdot I = P_1 \cdot t \cdot I$, where P_1 is a projection (7.2) on $q \times q$ matrices ($q \geq 4$), and t is a transposition of class I, namely $m_{12} \leftrightarrow m_{21}$. The factorizations now read

$$f_2 = \frac{\det(M_1)}{f_1^{q-4}}, \quad M_2 = \frac{K(M_1)}{f_1^{q-5}}, \quad f_3 = \frac{\det(M_2)}{f_1^4 \cdot f_2^{q-4}}, \quad M_3 = \frac{K(M_2)}{f_1^3 \cdot f_2^{q-5}},$$
$$f_4 = \frac{\det(M_3)}{f_1^{q-4} \cdot f_2^4 \cdot f_3^{q-4}}, \quad \ldots, \tag{7.18}$$

and, for arbitrary n,

$$K(M_n) = M_{n+1} \cdot \Big(f_n^{q-5} \cdot f_{n-1}^3 \cdot f_{n-2}^{q-5} \cdot f_{n-3}^3 \cdot f_{n-4}^{q-5} \cdot f_{n-5}^3 \cdot f_{n-6}^{q-5} \cdot f_{n-7}^3 \cdots\Big),$$
$$\det(M_n) = f_{n+1} \cdot \Big(f_n^{q-4} \cdot f_{n-1}^4 \cdot f_{n-2}^{q-4} \cdot f_{n-3}^4 \cdot f_{n-4}^{q-4} \cdot f_{n-5}^4 \cdot f_{n-6}^{q-4} \cdot f_{n-7}^4$$
$$\cdot f_{n-8}^{q-4} \cdot f_{n-9}^4 \cdots\Big), \tag{7.19}$$

and again relation (3.49),

$$\widehat{K}(M_n) = \frac{K(M_n)}{\det(M_n)} = \frac{M_{n+1}}{f_1 \cdot f_2 \cdots f_{n-1} \cdot f_n \cdot f_{n+1}}. \tag{7.20}$$

Again one gets from (6.65) relation (3.60) and *also the general relations between* $\alpha(x)$, $\beta(x)$, $\mu(x)$ and $\nu(x)$, corresponding to the "right action" of K, namely (3.41).

Eqs. (7.19) and (7.20) yield linear recursions for the α_n's and β_n's, namely (6.65), and

$$\alpha_n = \beta_{n+1} + (q-4) \cdot \beta_n + 4\beta_{n-1} + (q-4) \cdot \beta_{n-2}$$
$$+4\beta_{n-3} + (q-4) \cdot \beta_{n-4} + \cdots + \zeta_n \cdot \beta_1, \tag{7.21}$$

and for the generating functions $\alpha(x)$ and $\beta(x)$,

$$x \cdot \alpha(x) = \beta(x) + \frac{1}{1-x^2} \cdot \Big((q-4)x\beta(x) + 4x^2\beta(x)\Big),$$
$$q + ((q-1)x - 1) \cdot \alpha(x) = \frac{1}{1-x^2} \cdot \Big(q(q-5)x + 3qx^2\Big) \cdot \beta(x). \tag{7.22}$$

The generating functions $\alpha(x)$, $\beta(x)$, $\mu(x)$ and $\nu(x)$ read

$$\alpha(x) = \frac{q(1 + (q-4)x + 3x^2)}{(1-x^2) \cdot (1-3x)}, \quad \beta(x) = \frac{qx}{1-3x}, \quad \mu(x) = \frac{(q-4)x}{1-3x},$$
$$\nu(x) = \frac{x((q-5) + 3x)}{(1-x^2) \cdot (1-3x)}. \tag{7.23}$$

7.4. Non-birational deformation of N-cycles

Let us consider 6×6 matrices and the N-cycle defined in Section 6.2.2 (see (6.75)), combined with projection P_3 ($K = P_3 \cdot t \cdot I$). One obtains *the same factorizations independently of N* ($N = 2, 3, ..., 6$), namely for arbitrary n, *"string-like" factorizations,*

$$K(M_n) = M_{n+1} \cdot (f_n^2 \cdot f_{n-1} \cdot f_{n-2}^2 \cdot f_{n-3} \cdot f_{n-4}^2 \cdot f_{n-5} \cdots),$$
$$\det(M_n) = f_{n+1} \cdot f_n^3 \cdot f_{n-1}^2 \cdot f_{n-2}^3 \cdot f_{n-3}^2 \cdot f_{n-4}^3 \cdots . \tag{7.24}$$

Let us note that one has again the simple "string-like" relation (3.49), *which is independent of q,*

$$\widehat{K}(M_n) = \frac{K(M_n)}{\det(M_n)} = \frac{M_{n+1}}{f_1 \cdot f_2 \cdots f_n \cdot f_{n+1}} . \tag{7.25}$$

This yields linear relations for the α_n's and β_n's, namely (6.65) for $q = 6$, and

$$\alpha_n = \beta_{n+1} + 3\beta_n + 2\beta_{n-1} + 3\beta_{n-2} + 2\beta_{n-3} + 3\beta_{n-4} + \cdots . \tag{7.26}$$

and for the generating functions $\alpha(x)$ and $\beta(x)$,

$$x\alpha(x) = \beta(x) + \frac{(3x + 2x^2) \cdot \beta(x)}{1 - x^2} . \tag{7.27}$$

The generating functions $\alpha(x)$ and $\beta(x)$ read

$$\alpha(x) = \frac{6(1 + 3x + x^2)}{1 - 3x + x^2}, \quad \beta(x) = \frac{6x(1 - x)}{1 - 3x + x^2} . \tag{7.28}$$

These generating functions correspond to an exponential growth λ^n with $\lambda = 3/2 + \sqrt{5}/2 \sim 2.618 \cdots$.

The analysis of transformations, which are products of the matrix inversion I, of quite general (*no longer involutive*) permutations of the entries and of projection transformations (like P_1 see (7.2)), clearly opens a very large class of transformations. A systematic study seems hard to perform, but there is a need to accumulate far more examples in order to get some hint about this huge domain of investigation.

8. More relations on $\alpha(x)$, $\beta(x)$, $\mu(x)$ and $\nu(x)$

For all the various birational, or just rational, transformations described here, one remarks that the factorization relations always take the following general form at the nth step of the iterations:

$$\det(M_n) = f_{n+1} \cdot f_n^{\zeta_1} \cdot f_{n-1}^{\zeta_2} \cdot f_{n-2}^{\zeta_3} \cdot f_{n-3}^{\zeta_4} \cdot f_{n-4}^{\zeta_5} \cdots f_1^{\zeta_n}, \tag{8.1}$$

$$K(M_n) = M_{n+1} \cdot f_n^{\eta_0} \cdot f_{n-1}^{\eta_1} \cdot f_{n-2}^{\eta_2} \cdot f_{n-3}^{\eta_3} \cdot f_{n-4}^{\eta_4} \cdots f_1^{\eta_{n-1}}, \tag{8.2}$$

$$\widehat{K}(M_n) = \frac{K(M_n)}{\det(M_n)} = \frac{M_{n+1}}{f_{n+1} \cdot f_n^{\rho_1} \cdot f_{n-1}^{\rho_2} \cdot f_{n-2}^{\rho_3} \cdot f_{n-3}^{\rho_4} \cdots f_1^{\rho_n}}, \tag{8.3}$$

the exponents η_n's, ζ_n's and ρ_n's being positive integers. Eq. (8.1) yields a bilinear relation between the α_n's, β_n's and ζ_n's,

$$\alpha_n = \beta_{n+1} + \zeta_1\beta_n + \zeta_2\beta_{n-1} + \zeta_3\beta_{n-2} + \cdots + \zeta_n\beta_1 . \tag{8.4}$$

Introducing a new generating function for the ζ_n's,

$$\zeta(x) = 1 + \zeta_1 x + \zeta_2 x^2 + \zeta_3 x^3 + \cdots , \tag{8.5}$$

relation (8.4) simply reads

$$x\alpha(x) = \zeta(x) \cdot \beta(x) . \tag{8.6}$$

Similarly, if one introduces generating functions for the η_n's and ρ_n's,

$$\eta(x) = \eta_0 + \eta_1 x + \eta_2 x^2 + \eta_3 x^3 + \cdots , \tag{8.7}$$

$$\rho(x) = 1 + \rho_1 x + \rho_2 x^2 + \rho_3 x^3 + \cdots , \tag{8.8}$$

a bilinear relation between the α_n's, β_n's and η_n's is immediately obtained from relation (8.2),

$$(q-1)\alpha_n = \alpha_{n+1} + q(\eta_0\beta_n + \eta_1\beta_{n-1} + \eta_2\beta_{n-2} + \eta_3\beta_{n-3} + \cdots + \eta_{n-1}\beta_1) \tag{8.9}$$

leading to a relation between the three generating functions $\alpha(x)$, $\beta(x)$ and $\eta(x)$,

$$\alpha(x) + qx \cdot \eta(x) \cdot \beta(x) = q + (q-1) \cdot x\alpha(x) . \tag{8.10}$$

The absence of factorizations corresponds to $\eta(x) = 0$, or $\zeta(x) = 1$, or $\rho(x) = 1$, that is

$$\beta(x) = x \cdot \alpha(x) = \frac{q \cdot x}{1 - (q-1) \cdot x} . \tag{8.11}$$

A relation between $\alpha(x)$, $\beta(x)$ and $\rho(x)$ results from relation (8.3),

$$q + q\rho(x)\beta(x) = (1+x) \cdot \alpha(x) , \tag{8.12}$$

which generalizes Eqs. (4.15), (5.5), (6.19), (6.41)... It is seen in Appendix C that other relations relating these generating functions also exist. The three new generating functions $\zeta(x)$, $\eta(x)$ and $\rho(x)$ are simply related,

$$\zeta(x) = x\eta(x) + \rho(x) . \tag{8.13}$$

There is also

$$qx\nu(x) + \eta(x) = 1 + \eta(x) \cdot \mu(x) . \tag{8.14}$$

Relation (8.14) has been shown in [21]. Appendix D proves another relation between the three generating functions $\eta(x)$, $\nu(x)$ and $\mu(x)$,

$$\frac{x\eta(x)\mu(x)}{1-(q-1)x} + \nu(x) = \frac{x\eta(x)}{1-(q-1)x}. \tag{8.15}$$

Taking into account relations (3.41), it is quite clear to see that Eq. (8.14) is actually compatible with relation (8.10).

The generating functions $\alpha(x)$, $\beta(x)$, $\mu(x)$ and $\nu(x)$ are related. From relations (8.10), (8.12), (8.13) one can get all these various generating functions *from only two of them.* Among all the various generating functions corresponding to various permutations, *it appears that two generating functions are particularly simple namely* $\beta(x)$ *and especially* $\rho(x)$. However class VI and class IV share the *same* generating function $\rho(x)$, since Eq. (8.3) is satisfied for both classes, but $\zeta(4,x)$, or $\zeta(q,x)$, are different for these two classes (see relation (8.1)). *Therefore, in a general framework, one should not seek for any additional relation* (for instance between $\rho(x)$ and $\zeta(4,x)$ or $\zeta(q,x)$).

Explicit expressions of $\rho(x)$ *and of the generating function* $\zeta(q,x)$, *for various permutations considered here* are given in Appendix F. The amazing simplicity of $\rho(x)$ may suggest to try to look directly at the factorization properties of $\widehat{K}$ (see relation (E.4) in Appendix E).

8.1. Comments on the explicit expressions of the generating functions

From the examples of the previous sections it is remarkable to notice the following relations between $\alpha(q,x)$ (or $\beta(q,x)$), $\zeta(q,x)$ and another function $\sigma(q,x)$ (*which often identifies with* $\zeta(0,x)$):

$$\alpha(q,x) = \frac{q\zeta(q,x)}{(1+x)\sigma(q,x)}, \quad \beta(q,x) = \frac{qx}{(1+x)\sigma(q,x)}. \tag{8.16}$$

These relations are actually proved in Appendix E. From Eq. (8.12) combined with Eq. (8.16), a relation between $\rho(q,x)$ and $\zeta(q,x)$ is obtained,

$$\frac{qx}{(1+x)} \cdot \rho(q,x) = \zeta(q,x) - \sigma(q,x). \tag{8.17}$$

From relations (8.12) one can immediately calculate $\rho(q,x)$ for various generalizations to $q \times q$ matrices,

$$\rho(q,x) = \frac{(1+x)\cdot\alpha(x) - q}{q\beta(x)}. \tag{8.18}$$

Let us recall the various generalizations of transformations t_1 (namely, one "straight" generalization and two different self-similar generalizations), to better understand the relations between $\rho(x)$, $\zeta(q,x)$, $\zeta(0,x)$ and $\sigma(q,x)$. One has

$$\rho(q,x) = 1 + x^2 \tag{8.19}$$

for t_1 (see Section 4.1) for 4×4 matrices and for the self-similar generalizations of t_1 detailed in Section 4.3 (corresponding to q^4-state vertex models). In contrast, one has

$$\rho(q,x) = \frac{1+x^2}{1-x} \tag{8.20}$$

for the "self-similar" generalizations of t_1 for $2^d \times 2^d$ or for $2m \times 2m$ matrices (see Eq. (4.15)). In this last case, the generating function $\zeta(q,x)$ reads

$$\begin{aligned}\zeta(q,x) &= \frac{1+6x^2+x^4+(q-4)(x+x^3)}{(1+x)\,(1-x)} \\ &= \frac{(1-x)^3}{(1+x)} + \frac{qx(1+x^2)}{(1-x)(1+x)} = \zeta(0,x) + \frac{qx(1+x^2)}{(1-x)(1+x)} \,. \end{aligned} \tag{8.21}$$

There are two different kinds of "self-similar" generalizations of t_1 associated with 4×4 matrices. The expressions of $\sigma(q,x)$ for these two "self-similar" generating functions are also different,

$$\sigma(q,x)^{Q^4-\text{state}} = \sigma(q,x)^{2m\times 2m} - (q-4)\cdot\frac{x(1-x)}{1+x} \,. \tag{8.22}$$

For the "straight" generalizations of t_1 of Section 5, one gets

$$\rho(q,x) = \frac{1}{1-x} \,. \tag{8.23}$$

In this case, the generating function $\zeta(q,x)$ reads

$$\begin{aligned}\zeta(q,x) &= \frac{1+4x^2+(q-5)x}{(1-x)\,(1+x)} = \frac{1-4x}{(1+x)} + \frac{qx}{(1-x)(1+x)} \\ &= \zeta(0,x) + \frac{qx}{(1-x)(1+x)} \,. \end{aligned} \tag{8.24}$$

It is interesting to compare these three generalizations of t_1. Two, which *do not contain* $q=4$ *as a subcase*, actually verify

$$\sigma(q,x) = \zeta(0,x) \,, \tag{8.25}$$

while the third one, which contains $q=4$ *as a subcase, does not satisfy relation* (8.25) and, therefore, yields more involved expressions for $\alpha(q,x)$ and $\beta(q,x)$ (in particular in terms of q, see relation (4.27)).

Conversely, considering t_1 for $q=4$, for which $\rho(x) = 1+x^2$, one can see that relation (8.25), together with (8.17), would give

$$\zeta(q,x) = \zeta(0,x) + \frac{qx}{(1+x)}\cdot\rho(x) = \zeta(0,x) + \frac{qx(1+x^2)}{(1+x)} \,, \tag{8.26}$$

which yields negative coefficients ζ_n's in the series expansion. A *necessary condition* in order to get a generalization to $q\times q$ matrices, *such that* $\sigma(q,x) = \zeta(0,x)$, is, therefore, that all the coefficients of the series expansion of

$$\frac{x\rho(x)}{1+x} \tag{8.27}$$

are actually positive.

One can actually verify relation (8.25) for all the *the "straight" generalizations given here* and for a great number of other ("self-similar") generalizations given here.

It should be noted that in all the examples which can be generalized to $q \times q$ matrices, *$\rho(x)$ is independent of q*. Moreover $\rho(x)$ is remarkably simple in all these examples since it has *zeros or poles only on the unit circle.*

The generating functions corresponding to all these examples of permutations are actually rational functions. This is closely related to the simplicity of relation (8.3), more precisely to *the "periodicity" occurring in these factorization relations* (see factorizations (6.10) or (6.17)) and, more generally, to the occurrence of a "shift" in the factorization scheme (see for instance factorization (6.58) where a shift of four occurs). *Where does this periodicity comes from?* In fact, this can be understood if one assumes some regularity order by order in the factorizations of the iterations. However, it is not clear that these regularities should be verified for quite general permutations of the entries (see (6.55) and (7.10)).

When leaving the birational framework it is not even clear that the generating functions are still rational functions (see Section 7.2).

Remark. In the case of "straight" generalizations, and also of the "self-similar" generalizations *satisfying relation* (8.25), this very relation enables one to very quickly get the singularities of the generating functions. Relations (8.16) and (8.17), for $q = 4$ only, allow one to quickly get the singularities of the generating functions $\alpha(x)$, $\beta(x)$,... They are simply obtained from the *numerator* of

$$\zeta(0,x) = \zeta(4,x) - \frac{4x}{(1+x)} \cdot \rho(x)\,. \tag{8.28}$$

For instance, for class IV one gets

$$\zeta(4,x) = \frac{1+2x+x^2+3x^3+x^4}{1-x^4} \quad \text{and} \quad \rho(x) = \frac{1}{1-x}\,, \tag{8.29}$$

and

$$\zeta(0,x) = \frac{1-2x+x^2-x^3+x^4}{1-x^4} = \frac{1-x-x^3}{(1-x)(1+x^2)}\,. \tag{8.30}$$

One gets for class I, II, II,

$$\zeta(0,x) = (1-x)^3\,, \quad \zeta(4,x) = 1+x+6x^2+3x^3\,, \quad \rho(x) = 1+x^2\,, \tag{8.31}$$

for class V:

$$\zeta(0,x) = 1-3x+x^2-x^3\,, \quad \rho(x) = 1+x^2\,, \tag{8.32}$$

and for class I,

$$\zeta(0,x) = \frac{1-2x}{1-x^2}\,, \quad \zeta(4,x) = \frac{1+2x}{1-x^2}\,, \quad \rho(x) = \frac{1}{1+x}\,. \tag{8.33}$$

This provides a condition for the *polynomial growth* of the calculations, which can easily be checked, *provided condition* (8.25) *is actually satisfied.*

9. Conclusion

In previous papers [21,22], a classification of birational transformations associated with elementary *permutations* of two entries was performed. It led to six different classes [22]. The analysis of the factorizations corresponding to these six sets of permutations (see Section 3) has shed some light on the relations between different properties and structures related to integrable mappings such as the *polynomial growth* of the complexity of the iterations and the existence of nontrivial recursions bearing on the factorized polynomials f_n's.

For more general permutations, simple factorization schemes have been seen to exist: they can be seen as many (unexpected) regularities and structures in a framework which should, at first glance, be more related to the theory of chaos than to integrable mappings and their associated structures. Actually, a large number of structures and properties which emerge here, originate from the analysis of *integrable* mappings [21,22] and it is striking to see *these structures survive in a much more general framework*, that is, mappings which are far from being integrable.

The occurrence of elliptic curves (i.e. integrable mappings), of polynomial growth of the calculations, or of recursion relations in a *single* variable such as (3.27), or even (3.66), is, of course, less frequent *but not exceptional* (see Section 6.1.11). *This makes room for the analysis of very large new classes of mappings presenting remarkable properties and structures.*

For all the examples of permutations which can be generalized to $q \times q$ matrices, remarkable factorization relations *independent of* q occur (see (6.27) (6.10) and (6.39)).

Among the different types of generalizations that can be introduced, ("straight" generalizations or various "self-similar" generalizations...), one has to distinguish the ones which verify relation (8.25) and the ones which do not verify relation (8.25) (see Section 4.3).

In all the examples detailed here it has been seen that the generating functions are often quite simple expressions, satisfying remarkable functional relations. Some of these remarkable properties can actually be proved (see in particular Appendix E). However, in a more general framework, several questions need to be better understood. In particular: *where does the rationality of the generating functions, and in particular* $\rho(x)$, *comes from? Where do the periodicity, and more generally, the regularities of the factorizations come from?*

A large number of open problems still remains. We have just sketched the analysis of the factorizations corresponding to transformations K associated with *quite general permutations* in Section 6, or even rational, but not birational, transformations (see Section 7). To some extent it is possible to understand, with some assumptions, the factorization properties of these various, quite general, transformations generalizing the demonstrations of factorizations performed in [21]. The analysis of the iteration of transformations associated with quite general permutations, like the birational one described in Section 6, or even of the *rational but not birational* transformations (see Section 7), opens a very large domain of research, which merits further investigations.

Appendix A. Right action of K^n and generating functions

Let us introduce the "right action" of transformation K^3,

$$(f_n)_{K^3} = f_{n+3} \cdot f_3^{\mu_n^{(3;3)}} \cdot f_2^{\mu_n^{(3;2)}} \cdot f_1^{\mu_n^{(3;1)}} . \tag{A.1}$$

$\mu^{(3;3)}(x)$ is equal to $\mu^{(2;2)}(x)$ or $\mu(x)$ and $\mu^{(3;2)}(x)$ is equal to $\mu^{(2;1)}(x)$ and it is easy to prove, for arbitrary q, that

$$\mu^{(3;1)}(x) = \frac{x((q^3 - 5q^2 + 7q - 1) - x)}{(1-x)^3(1+x)} . \tag{A.2}$$

One remarks that $\mu^{(3;3)}(x)$ is actually equal to $\mu^{(2;2)}(x) = \mu(x)$, which is a consequence of

$$\mu^{(1;1)}(x) \cdot (1 + \mu_2 x^2) + \mu_1 \cdot x^2 \cdot \mu^{(2;1)}(x) - (\mu_1 \cdot x + \mu_2 \cdot x^2) = x^2 \cdot \mu^{(3;1)}(x))$$
$$\text{together with} \quad \mu_1 = \mu_1^{(1;1)} = q - 3 , \quad \mu_2 = \mu_2^{(1;1)} = 2(q-3) . \tag{A.3}$$

Introducing the "right action" of transformation K^4, the result is

$$(f_n)_{K^4} = f_{n+4} \cdot f_4^{\mu_n^{(4;4)}} \cdot f_3^{\mu_n^{(4;3)}} \cdot f_2^{\mu_n^{(4;2)}} \cdot f_1^{\mu_n^{(4;1)}} . \tag{A.4}$$

Again, one notes that $\mu^{(4;4)}(x)$ is equal to $\mu^{(3;3)}(x)$ or $\mu^{(2;2)}(x)$ or $\mu^{(1;1)}(x) = \mu(x)$, that $\mu^{(4;3)}(x)$ is equal to $\mu^{(3;2)}(x)$ or $\mu^{(2;1)}(x)$ and that $\mu^{(4;2)}(x)$ is equal to $\mu^{(3;1)}(x)$. The only new generating function is

$$\mu^{(4;1)}(x) = \frac{q(q-2)^3 x}{(1-x)^3(1+x)} . \tag{A.5}$$

One can easily show from (A.1), (A.4) and from the right action of K (see (3.10)), the following relation on the coefficients of $\mu^{(4;1)}(x)$, $\mu^{(3;1)}(x)$, $\mu^{(2;1)}(x)$ and $\mu^{(1;1)}(x)$:

$$\mu_n^{(4;1)} = \mu_{n+3} + \mu_3 \cdot \mu_n^{(1;1)} + \mu_2 \cdot \mu_n^{(2;1)} + \mu_1 \cdot \mu_n^{(3;1)} , \tag{A.6}$$

or equivalently,

$$\mu^{(1;1)}(x) \cdot (1 + \mu_2 x^2) + \mu_1 \cdot x^2 \cdot \mu^{(2;1)}(x) - (\mu_1 \cdot x + \mu_2 \cdot x^2) = x^2 \cdot \mu^{(3;1)}(x))$$
$$\text{together with} \quad \mu_1 = \mu_1^{(1;1)} = q - 3 , \quad \mu_2 = \mu_2^{(1;1)} = 2(q-3) . \tag{A.7}$$

The same calculations performed for the right action of K^5, K^6, K^7 and K^8 yield similar results with

$$\mu^{(5;1)}(x) = \frac{q(q-1)(q-2)^3 x}{(1-x)^3(1+x)}, \quad \mu^{(6;1)}(x) = \frac{q(q-1)^2(q-2)^3 x}{(1-x)^3(1+x)},$$

$$\mu^{(7;1)}(x) = \frac{q(q-1)^3(q-2)^3 x}{(1-x)^3(1+x)}, \quad \mu^{(8;1)}(x) = \frac{q(q-1)^4(q-2)^3 x}{(1-x)^3(1+x)} \cdots. \tag{A.8}$$

Let us now recall relation (3.23) and expression (3.24), given in Section 3.1 for transformation K^N,

$$\begin{aligned} & x^{N-1}\cdot\Big(\mu_{N-1}\cdot\mu^{(1;1)}(x)+\mu_{N-2}\cdot\mu^{(2;1)}(x) \\ & +\mu_{N-2}\cdot\mu^{(3;1)}(x)+\cdots+\mu_1\cdot\mu^{(N-1;1)}(x)\Big) \\ & = x^{N-1}\cdot\mu^{(N;1)}(x))+(\mu_1\cdot x+\mu_2\cdot x^2+\cdots+\mu_{N-1}\cdot x^{N-1})-\mu^{(1;1)}(x), \end{aligned} \tag{A.9}$$

and the remarkably simple expression for $\mu^{(M;1)}(x)$,

$$\mu^{(M;1)}(x) = \frac{q\cdot(q-2)^3\cdot(q-1)^{(M-4)}\cdot x}{(1-x)^3(1+x)}. \tag{A.10}$$

Let us prove recursively that relation (A.9) has (A.10) for a solution for an arbitrary value of N. Let us assume that (A.10) is valid for $N=1$ to $N=M-1$. The left-hand side of (3.23) can be written as

$$\begin{aligned} & x^{N-1}\cdot\Big(\mu_{N-1}\cdot\mu^{(1;1)}(x)+\mu_{N-2}\cdot\mu^{(2;1)}(x)+\mu_{N-3}\cdot\mu^{(3;1)}(x)\Big) \\ & +x^{N-1}\cdot\Big(\mu_{N-4}\cdot\mu^{(4;1)}(x)+\mu_{N-5}\cdot\mu^{(5;1)}(x)++\cdots+\mu_1\cdot\mu^{(N-1;1)}(x)\Big) \\ & = x^{N-1}\cdot\Big(\mu_{N-1}\cdot\mu^{(1;1)}(x)+\mu_{N-2}\cdot\mu^{(2;1)}(x)+\mu_{N-3}\cdot\mu^{(3;1)}(x)\Big) \\ & +x^{N-1}\cdot\Big(\sum_{r=1\ldots N-4}\mu_r\cdot(q-1)^{(N-4-r)}\Big)\cdot\frac{q\cdot(q-2)^3\cdot x}{(1-x)^3(1+x)}. \end{aligned} \tag{A.11}$$

Expanding $\mu(x)$ one can easily get the expression of the μ_n's,

$$\mu_n = \frac{(q-2)n^2}{4}+\frac{(q-3)n}{2}+\frac{(q-8)}{8}+(-1)^{n+1}\cdot\frac{q}{8}. \tag{A.12}$$

The expression of the sum can also be calculated simply. One can, for instance, get it as the coefficient of x^{N-4} of the function

$$\frac{\mu^{(1;1)}(x)}{1-(q-1)x} = \frac{x\cdot((q-3)+2x^2-x^3)}{(1+x)(1-x)^3(1-(q-1)x)}, \tag{A.13}$$

which reads

$$W_N = \frac{5}{8}-\frac{(-1)^N}{8}-\frac{N}{4}-\frac{(N-3)\,(N-2)}{4}+(q-1)^{N-4}. \tag{A.14}$$

From (A.9) one obtains the expression of $x^{N-1}\cdot\mu^{(N;1)}(x)$ as

$$x^{N-1} \cdot \mu^{(N;1)}(x)) = x^{N-1} \cdot \Big(\mu_{N-1} \cdot \mu^{(1;1)}(x) + \mu_{N-2} \cdot \mu^{(2;1)}(x) + \mu_{N-3} \cdot \mu^{(3;1)}(x)\Big) + x^N \cdot W_N \cdot \frac{q \cdot (q-2)^3}{(1-x)^3(1+x)} + \Big(\mu^{(1;1)}(x) - (\mu_1 \cdot x + \mu_2 \cdot x^2 + \cdots + \mu_{N-1} \cdot x^{N-1})\Big) . \tag{A.15}$$

There is the equality

$$\mu^{(1;1)}(x) - (\mu_1 \cdot x + \mu_2 \cdot x^2 + \cdots + \mu_{N-1} \cdot x^{N-1}) = x^N \cdot \frac{R(x)}{8(1-x)^3(1+x)} , \tag{A.16}$$

where $R(x)$ reads

$$R(x) = (-1)^N \cdot q \cdot (1-x)^3 + q \cdot (x^3 - 3x^2 - 5x - 1) + 8(1+x) + 2 \cdot \Big((2-q) \cdot (1 - x - x^2 + x^3) + qx\Big) \cdot N^2 + 4 \cdot \Big((x^3 - 3x^2 - x + 12) + q(x^2 - 1)\Big) \cdot N .$$

Remarkably, the terms

$$x^{N-1} \cdot \Big(\frac{5}{8} - \frac{(-1)^{N-4}}{8} - \frac{N}{4} - \frac{(N-3)\,(N-2)}{4}\Big) \cdot \frac{q \cdot (q-2)^3 \cdot x}{(1-x)^3(1+x)} + \Big(\mu^{(1;1)}(x) - (\mu_1 \cdot x + \mu_2 \cdot x^2 + \cdots + \mu_{N-1} \cdot x^{N-1})\Big) + x^{N-1} \cdot \Big(\mu_{N-1} \cdot \mu^{(1;1)}(x) + \mu_{N-2} \cdot \mu^{(2;1)}(x) + \mu_{N-3} \cdot \mu^{(3;1)}(x)\Big) \tag{A.17}$$

cancel out and one finally gets

$$\mu^{(N;1)}(x) = \frac{q \cdot (q-2)^3 \cdot (q-1)^{(N-4)} \cdot x}{(1-x)^3(1+x)} . \tag{A.18}$$

Let us finally mention that many more relations between the generating functions can be deduced from relation (3.23). Let us, for instance, mention the following relation between the $\mu^{(M;1)}(x)$'s and $\beta(x)$, valid for any N:

$$\Big(1 - (q-1)^N \cdot x^N\Big) \cdot \beta(x) + x^N \cdot \Big(\sum_{r=1,N} \beta_r \cdot \mu^{(N+1-r;1)}(x)\Big) = \beta_1 \cdot x + \beta_2 \cdot x^2 + \beta_3 \cdot x^3 + \cdots + \beta_N \cdot x^N . \tag{A.19}$$

Appendix B. Comments on the integration of the recursions on the x_n's

Recursion (3.31) has been integrated in [23] and yields *biquadratic relations* in terms of some new variables q_n defined by $x_n = q_{n+1}/q_n$,

$$(\rho - q_n - q_{n+1}) \cdot (q_n q_{n+1} + \lambda) = \mu . \tag{B.1}$$

The elliptic curve (B.1) can be rewritten, after several transformations, in the *canonical Weierstrass's form* [29,33],

$$y^2 = 4x^3 - g_2 x - g_3 \quad \text{where} \quad 12g_2 = 16\lambda^2 + 8\lambda\rho^2 - 24\rho\mu + \rho^4$$
$$\text{and} \quad 216g_3 = -48\lambda^2\rho^2 - 64\lambda^3 + 144\lambda\rho\mu - 216\mu^2 - 12\lambda\rho^4 + 36\rho^3\mu - \rho^6\,, \tag{B.2}$$

and the discriminant reads

$$\Delta = g_2^3 - 27g_3^2 = -\mu^2(16\lambda^3 - \rho^3\mu + 8\lambda^2\rho^2 + \lambda\rho^4 - 36\lambda\rho\mu + 27\mu^2)\,. \tag{B.3}$$

It should be noted that our recursions can also be integrated in terms of the x_n's, using the integration performed with the well-suited variables q_n's (see (3.34) in Section 3.1). Let us, for instance, consider the integration of one of our recursion relations in terms of two biquadratics [21] (see for instance (3.36)), denoted $B_1(q_n, q_{n+1})$ and $B_2(q_{n+1}, q_{n+2})$ (*in our previous examples it has been seen that one has* $B_2(q_{n+1}, q_{n+2}) = B_1(q_{n+2}, q_{n+1})$). Using the very relation between the x_n's and the q_n's, the system of these two biquadratic relations reads

$$B_1(q_n, q_n \cdot x_n) = 0\,, \quad B_2(q_n \cdot x_n, q_n \cdot x_n \cdot x_{n+1}) = 0\,. \tag{B.4}$$

Eliminating the homogeneous variable q_n, one immediately obtains a relation between x_n and x_{n+1}. Let us consider, for instance, the simplest example of integrable recursion, that is (3.31). In this example the two q_n-biquadratics, B_1 and B_2, identify (see (B.1)) and the resultant between them yields a *bicubic*,

$$\begin{aligned}\mathcal{B}(x_n, x_{n+1}) = A \cdot \Big(1 + x_n^3 x_{n+1}^3\Big) + B \cdot x_n^3 + \Big(2A + B\Big) \cdot x_n \cdot \Big(1 + x_n^2 x_{n+1}^2\Big) \\ + C \cdot x_n x_{n+1}^2 + \Big(A + 2B\Big) \cdot x_n^2 \cdot \Big(1 + x_n x_{n+1}\Big) \\ + \Big(3B - C\Big) \cdot x_n x_{n+1} \cdot \Big(1 + x_n x_{n+1}\Big) + D x_n^2 x_{n+1} = 0\,,\end{aligned} \tag{B.5}$$

where A, B, C and D read

$$A = (\lambda\rho - \mu)^2\,, \quad B = \mu^2 - \lambda\mu\rho + \lambda^3\,, \quad C = \lambda^3\,,$$
$$D = 4\lambda^3 - 7\mu\rho\lambda + 6\mu^2 - \mu\rho^3 + \lambda\rho^4\,. \tag{B.6}$$

It is worth noting that this involved expression for D can in fact be simply related to the expression of the discriminant Δ given in (B.3) in terms of λ, μ and ρ. This relation comes from the following observation: the discriminant Δ is equal, up to a multiplicative factor $-\mu^2$, to $\mathcal{B}(x_n, x_{n+1})$ for $x_n = 1$ and $x_{n+1} = 1$,

$$\begin{aligned}\Delta = -\mu^2 \cdot \mathcal{B}(1,1) = -\mu^2 \cdot \Big(8A + 13B - C + D\Big) \\ = \frac{(C-B)^2(BC^2 + C^2A - 11CA^2 - 14CBA - 2B^2C + B^3 - 3AB^2 - A^3 + 3A^2B)}{CA^2}\,.\end{aligned} \tag{B.7}$$

or, equivalently, D can be expressed in terms of A, B, C (using $\mu^2 = (C-B)^2/A$),

$$D = -\frac{BC^2 - 2B^2C - CBA - 3CA^2 + B^3 + 3A^2B - A^3 - 3AB^2}{AC}. \tag{B.8}$$

The other recursions can be integrated similarly in terms of the variables x_n's, and one also get *bicubic relations.*

Let us, for instance, consider recursion (3.32) which, in terms of the variables q_n's, can be integrated and yields *two biquadratic relations*. It also gives *two bicubics* in the following form:

$$\widehat{A} \cdot \left(1 + x_n^3 x_{n+1}^3\right) + \widehat{B} \cdot x_n^3 + \widehat{C} \cdot x_n \cdot \left(1 + x_n^2 x_{n+1}^2\right) + \widehat{D} \cdot x_n^2 \cdot \left(1 + x_n x_{n+1}\right)$$
$$+\widehat{E} \cdot x_n x_{n+1}^2 + \widehat{F} \cdot x_n x_{n+1} \cdot \left(1 + x_n x_{n+1}\right) + \widehat{G} \cdot x_n^2 x_{n+1} = 0, \tag{B.9}$$

with some involved relations between the coefficients of (B.9) that will not be written here.

It is interesting to see what kind of relations can be deduced from the previous procedure. The result is the following: the elimination of q_n between $B(q_n, q_n \cdot x_n)$ and $B(q_n \cdot x_n \cdot x_{n+1}, q_n \cdot x_n)$ [14] for a general biquadratic B with its nine coefficients, yields a *biquartic* of a particular form, namely

$$\mathcal{A} \cdot (1 + x_n^4 x_{n+1}^4) + (\mathcal{B} \cdot x_n + \mathcal{C} \cdot x_{n+1}) \cdot (1 + x_n^3 x_{n+1}^3) + \mathcal{D} \cdot x_n x_{n+1} (1 + x_n^2 x_{n+1}^2)$$
$$+\mathcal{E} \cdot x_n^2 x_{n+1}^4 + \mathcal{F} \cdot x_n^4 x_{n+1}^2 + (\mathcal{G} \cdot x_n^2 x_{n+1} + \mathcal{H} \cdot x_n x_{n+1}^2) \cdot (1 + x_n x_{n+1})$$
$$+\mathcal{I} \cdot x_n^2 x_{n+1}^2 + \mathcal{J} \cdot x_n x_{n+1}^3 + \mathcal{K} \cdot x_n^3 x_{n+1} + \mathcal{L} \cdot x_n^2 + \mathcal{M} \cdot x_{n+1}^2 = 0. \tag{B.10}$$

In the limit corresponding to recursion (3.31) one has $\mathcal{A} = 0, \mathcal{B} = 0, \mathcal{L} = 0, \mathcal{M} = 0, \mathcal{E} = 0, \mathcal{F} = 0$ and one recovers (B.9) from (B.10). The bicubic (B.9) is obtained from (B.10) with the following correspondence: $\widehat{A} = \mathcal{C}, \widehat{B} = \mathcal{K}, \widehat{C} = \mathcal{D}, \widehat{D} = \mathcal{G}, \widehat{E} = \mathcal{J}, \widehat{F} = \mathcal{H}, \widehat{G} = \mathcal{I}$.

Let us however note that the elimination[15] of q_n between $B(q_n, q_n \cdot x_n)$ and $B(q_n \cdot x_n, q_n \cdot x_n \cdot x_{n+1})$, for a general biquadratic B, yields much higher degree relations: $A \cdot x_n^8 x_{n+1}^8 + \cdots = 0$.

Appendix C. Comments on the "straight" generalization

Let us consider transformation K for a "straight" generalization and let us assume that the initial matrix M_0 take the particular form (5.7). The successive "reduced" matrices M_n's then read

[14] Note the permutation of the arguments for the second biquadratic: $B(q_n \cdot x_n \cdot x_{n+1}, q_n \cdot x_n)$ instead of $B(q_n \cdot x_n, q_n \cdot x_n \cdot x_{n+1})$.

[15] This elimination procedure yields quite different results from a direct elimination of q_{n+1} between $B(q_n, q_{n+1})$ and $B(q_{n+2}, q_{n+1})$, which remarkably results in a *symmetric biquadratic* relation between q_n and q_{n+2}, while the elimination of q_{n+1} between $B(q_n, q_{n+1})$ and $B(q_{n+1}, q_{n+2})$ yields a *biquartic* relation between q_n and q_{n+2}.

$$M_n = \begin{pmatrix} A_n & 0 \\ B_n & C_n \end{pmatrix} , \tag{C.1}$$

where A_n denotes a 4×4 submatrix, B_n a $(q-4)\times 4$ submatrix and "0" the $4\times(q-4)$ submatrix with zero entries. C_n is a $(q-4)\times(q-4)$ submatrix equal, up to some homogeneous polynomial, to matrix C (or C^{-1}) of relation (5.7), depending of the parity of n. The $(q-4)\times 4$ submatrix B_n can be very complex, however it does not modify the factorization properties of the matrices M_n's, nor the determinants $\det(M_n)$'s. Therefore we will represent matrices M_n's by (A_n, C_n), forgetting submatrix B_n. Let us study the successive action of K on the $q\times q$ matrices M_n's, on the corresponding factorized polynomials f_n's and the 4×4 matrices A_n, and the corresponding factorized polynomials $\tilde{f}_n$'s. With obvious notations one has the relations

$$K\Big((A_n, C_n)\Big) = \Big(\det(C_n)\cdot K(A_n), \det(C_n)\cdot C_n^{-1}\Big)$$
$$\text{or}\quad \widehat{K}\Big((A_n, C_n)\Big) = \Big(\widehat{K}(A_n), C_n^{-1}\Big) . \tag{C.2}$$

Let us denote $K_{(4)}$ and $\widehat{K}_{(4)}$, transformations K and $\widehat{K}$ restricted to 4×4 matrices. Let us also consider matrices m_n's, corresponding to the transformations $K_{(4)}$ and $\widehat{K}_{(4)}$ acting on the initial 4×4 matrix A in (5.7) and the corresponding factorized polynomials $\tilde{f}_n$'s. The following relations which amount to replacing M_n by m_n, f_n by $\tilde{f}_n$ and q by 4 are obtained:

$$\det(m_n) = \tilde{f}_{n+1}\cdot(\tilde{f}_n^2\cdot\tilde{f}_{n-1}\cdot\tilde{f}_{n-2}^3\cdot\tilde{f}_{n-3}^2)\cdot(\tilde{f}_{n-4}^2\cdot\tilde{f}_{n-5}\cdot\tilde{f}_{n-6}^3\cdot\tilde{f}_{n-7}^2)\cdots\tilde{f}_1^{\delta_n} , \tag{C.3}$$

$$m_{n+1} = \tilde{f}_{n+1}\cdot\tilde{f}_n\cdot\tilde{f}_{n-1}\cdot\tilde{f}_{n-2}\cdots\tilde{f}_2\cdot\tilde{f}_1\cdot\widehat{K}(m_n) . \tag{C.4}$$

The A_n's are proportional to the m_n's up to homogeneous polynomial factors. From relation (C.2) one immediately gets

$$\widehat{K}^n(M_0) = \Big(\widehat{K}^n_{(4)}(A_0), C^{(-1)^n}\Big) . \tag{C.5}$$

Recalling relation (3.50) for $q\times q$ and 4×4 matrices, one clearly gets from (C.5), relation

$$M_n = f_n\cdot f_{n-2}\cdot f_{n-4}\cdots\Big(\frac{m_n}{\tilde{f}_n\cdot\tilde{f}_{n-2}\cdot\tilde{f}_{n-4}\cdots}, C^{(-1)^n}\Big) . \tag{C.6}$$

For the sake of simplicity, let us consider the example of class IV. Recalling (3.64), one immediately gets

$$f_{n+1}\cdot(f_n^{-2}\cdot f_{n-1}\cdot f_{n-2}^{-1}\cdot f_{n-3})\cdot(f_{n-4}^{-2}\cdot f_{n-5}\cdot f_{n-6}^{-1}\cdot f_{n-7})\cdots$$
$$= \det(C)^{(-1)^n}\cdot\det\Big(\widehat{K}^n_{(4)}(A_0)\Big) . \tag{C.7}$$

From (3.64), for $q=4$, one also has

$$\tilde{f}_{n+1} \cdot (\tilde{f}_n^{-2} \cdot \tilde{f}_{n-1} \cdot \tilde{f}_{n-2}^{-1} \cdot \tilde{f}_{n-3}) \cdot (\tilde{f}_{n-4}^{-2} \cdot \tilde{f}_{n-5} \cdot \tilde{f}_{n-6}^{-1} \cdot \tilde{f}_{n-7}) \cdots = \det\Big(\widehat{K}^n_{(4)}(A_0)\Big) . \tag{C.8}$$

Eliminating $\det\Big(\widehat{K}^n_{(4)}(A_0)\Big)$, one obtains

$$f_n = \tilde{f}_n \cdot \det(C_0)^{\pm 1} \cdot \left(\left(\frac{f_{n-1}}{\tilde{f}_{n-1}}\right)^2 \cdot \left(\frac{\tilde{f}_{n-2}}{f_{n-2}}\right) \cdot \left(\frac{f_{n-3}}{\tilde{f}_{n-3}}\right) \cdot \left(\frac{\tilde{f}_{n-1}}{f_{n-1}}\right)^2\right) \cdot \left(\left(\frac{f_{n-5}}{\tilde{f}_{n-5}}\right)^2 \cdot \left(\frac{\tilde{f}_{n-6}}{f_{n-6}}\right) \cdots\right) \cdots , \tag{C.9}$$

which yields recursively,

$$f_n = \tilde{f}_n \cdot \det(C_0)^{a_n} . \tag{C.10}$$

From relation (C.9) one gets a linear recursion on the degrees of $\det(M_n)$, $\det(m_n)$, f_n, $\tilde{f}_n$, (namely $\alpha_n(q)$, $\alpha_n(4)$, $\beta_n(q)$, $\beta_n(4)$),

$$\alpha_n(q) - \alpha_n(4) = q\Big(\beta_n(q) + \beta_{n-2}(q) + \beta_{n-4}(q) + \cdots\Big) - 4\Big(\beta_n(4) + \beta_{n-2}(4) + \beta_{n-4}(4) + \cdots\Big) + (-1)^n \cdot (q-4) . \tag{C.11}$$

This linear relation (C.11) yields the following relation between the generating functions $\alpha(q,x)$, $\alpha(4,x)$, $\beta(q,x)$, $\beta(4,x)$:

$$\alpha(q,x) - \alpha(4,x) = \frac{1}{1-x^2} \cdot \Big(q\beta(q,x) - 4\beta(4,x)\Big) + \frac{q-4}{1+x} . \tag{C.12}$$

The following is also obtained from relation (C.10):

$$\beta_n(q) = \beta_n(4) + (q-4) \cdot a_n \quad \text{or} \quad \beta(q,x) = \beta(4,x) + (q-4) \cdot a(x)$$
yielding (6.2) . (C.13)

The fact that $\beta_n(q)$ is a *linear* function of q (see (6.2)) can be seen to be related to the homogeneity of the initial matrix,

$$M_0 \longrightarrow \lambda \cdot M_0 . \tag{C.14}$$

Let us now consider another example, namely, class V. The calculations are similar to the previous one. One now has for arbitrary n,

$$M_n = f_n \cdot f_{n-2} \cdot \left(\frac{m_n}{\tilde{f}_n \tilde{f}_{n-2}}, C^{(-1)^n}\right) , \tag{C.15}$$

yielding on the generating functions,

$$\alpha(q,x) - \alpha(4,x) = (1+x^2) \cdot \Big(q\beta(q,x) - 4\beta(4,x)\Big) + \frac{q-4}{1+x} , \tag{C.16}$$

and from (3.43),

$$f_n = \tilde{f}_n \cdot \det(C_0)^{\pm 1} \cdot \left(\frac{f_{n-1}}{\tilde{f}_{n-1}}\right)^3 \cdot \left(\frac{\tilde{f}_{n-2}}{f_{n-2}}\right) \cdot \left(\frac{f_{n-3}}{\tilde{f}_{n-3}}\right). \tag{C.17}$$

Of course, one has again relations (C.13) from (C.10).

Let us now consider class I (or III). One recovers the same relation as for class V (namely (C.15)), and therefore (C.16). One also gets from (3.8),

$$f_n = \tilde{f}_n \cdot \det(C_0)^{\pm 1} \cdot \left(\frac{f_{n-1}}{\tilde{f}_{n-1}}\right)^3 \cdot \left(\frac{\tilde{f}_{n-2}}{f_{n-2}}\right)^3 \cdot \left(\frac{f_{n-3}}{\tilde{f}_{n-3}}\right), \tag{C.18}$$

which yields again relation (C.10) where

$$a_n = \frac{n(n+2)}{4} \quad \text{for } n \text{ even and} \quad a_n = \frac{(n+1)^2}{4} \quad \text{for } n \text{ odd.}$$

Recalling recursions (3.27) or (3.28), satisfied by polynomials f_n's and also polynomials $\tilde{f}_n$'s, relation (C.10) can be seen to be closely related to *symmetries of the recursion*. Recursions (3.27) are invariant under (C.10) together with (C.19) and closely related to the (three parameters) symmetries previously described in [21],

$$f_n \longrightarrow a^{n^2} \cdot b^n \cdot c \cdot f_n . \tag{C.19}$$

In fact, these calculations can be performed, quite generally, for "straight" generalizations, giving M_n's in terms of the m_n's, the f_n's and $\tilde{f}_n$'s. Such calculations yield a relation between $\alpha(q,x)$, $\alpha(4,x)$, $\rho(x)$, $\beta(q,x)$ and $\beta(4,x)$,

$$\alpha(q,x) - \alpha(4,x) = \frac{\rho(x)}{1+x} \cdot \Big(q\beta(q,x) - 4\beta(4,x)\Big) + \frac{q-4}{1+x}, \tag{C.20}$$

which is nothing but relation (8.12).

Appendix D. More relations on the generating functions

Let us recall the general form (given in Section 8), for the successive factorizations of transformation K which define the ζ_n's, the η_n's and the ρ_n's, namely (8.1), (8.2) and (8.3).

Factorization (8.1) yields the following bilinear relation between the α_n's, β_n's and ζ_n's:

$$\alpha_n = \beta_{n+1} + \zeta_1\beta_n + \zeta_2\beta_{n-1} + \zeta_3\beta_{n-2} + \cdots + \zeta_n\beta_1 . \tag{D.1}$$

From relation (8.2) one directly gets the following bilinear relation between the α_n's, β_n's and η_n's:

$$(q-1) \cdot \alpha_n = \alpha_{n+1} + q \cdot \Big(\eta_0\beta_n + \eta_1\beta_{n-1} + \eta_2\beta_{n-2} + \eta_3\beta_{n-3} + \cdots + \eta_{n-1}\beta_1\Big), \tag{D.2}$$

leading to a relation between the three generating functions $\alpha(x)$, $\beta(x)$ and $\eta(x)$,

$$\alpha(x) + qx\eta(x)\cdot\beta(x) = q + (q-1)\cdot x\alpha(x)\,. \tag{D.3}$$

One can introduce the (non-factorized) matrices $\widehat{M}_n$, which are related to the successive (homogeneous) matrices M_n's as follows:

$$\widehat{M}_n = K(\widehat{M}_{n-1}) = K^n(M_0) = \Big(M_0\Big)_{K^n}\,. \tag{D.4}$$

The following relations on $\widehat{M}_n$'s are obtained:

$$\begin{aligned}
&\widehat{M}_1 = K(\widehat{M}_0) = K(M_0) = M_1\,, \quad \widehat{M}_2 = K(\widehat{M}_1) = f_1^{\eta_0}\cdot M_2\,,\\
&\widehat{M}_3 = K(\widehat{M}_2) = f_2^{\eta_0}\cdot f_1^{\eta_1}\cdot f_1^{\eta_0(q-1)}\cdot M_3\,,\\
&\widehat{M}_4 = K(\widehat{M}_3) = f_3^{\eta_0}\cdot f_2^{\eta_1}\cdot f_1^{\eta_2}\cdot\Big(f_2^{\eta_0}\cdot f_1^{\eta_1}\cdot f_1^{\eta_0(q-1)}\Big)^{(q-1)}\cdot M_4\,,\\
&\quad\vdots\\
&\widehat{M}_n = K(\widehat{M}_{n-1}) = K^n(M_0) = \Big(f_1^{\eta_0\cdot(q-1)^{n-1}+\eta_1\cdot(q-1)^{n-2}+\eta_2\cdot(q-1)^{n-3}+\cdots}\Big)\\
&\qquad\cdot\Big(f_2^{\eta_0\cdot(q-1)^{n-2}+\eta_1\cdot(q-1)^{n-3}+\eta_2\cdot(q-1)^{n-4}+\cdots}\Big)\cdots f_n^{\eta_0}\cdot M_n\\
&\quad\vdots
\end{aligned}$$

After some calculations the following is reached from (D.5):

$$\begin{aligned}
&\Big(\eta_0\cdot(q-1)^{n-1}+\eta_1\cdot(q-1)^{n-2}+\eta_2\cdot(q-1)^{n-3}+\cdots+\eta_{n-1}\Big)\cdot\mu_1+\cdots\\
&\quad+\Big(\eta_0\cdot(q-1)^2+\eta_1\cdot(q-1)+\eta_2\Big)\cdot\mu_{n-2}\\
&\quad+\Big(\eta_0\cdot(q-1)+\eta_1\Big)\cdot\mu_{n-1}+\eta_0\cdot\mu_n+\nu_{n+1}\\
&=\eta_0\cdot(q-1)^n+\eta_1\cdot(q-1)^{n-1}+\eta_2\cdot(q-1)^{n-2}+\cdots+\eta_n\,,
\end{aligned} \tag{D.5}$$

which yields the following relations between the three generating functions $\eta(x)$, $\nu(x)$ and $\mu(x)$:

$$\frac{x\eta(x)\mu(x)}{1-(q-1)x} + \nu(x) = \frac{x\eta(x)}{1-(q-1)x}\,. \tag{D.6}$$

This relation is nothing but relation (8.14), which can also be obtained by performing the "right action" of K on factorization (8.2), and using (3.10), one gets

$$\nu_{n+1} = (q-1)\nu_n + \eta_n - (\eta_0\mu_n + \eta_1\mu_{n-1} + \cdots + \eta_{n-1}\mu_1)\,. \tag{D.7}$$

Similarly, performing the right action of K on factorization (8.1), and using (3.10), one obtains

$$\mu_{n+1} = \zeta_{n+1} + q\nu_n - (\zeta_n\mu_1 + \zeta_{n-1}\mu_2 + \cdots + \mu_n\zeta_1)\,. \tag{D.8}$$

Moreover, it can be shown that (8.2), the factorization relation on $K(M_n)$, necessarily yields the factorization of the determinant, namely (8.1) (and also the inequalities $\zeta_n \geq$

$1+\eta_{n-1}$ when $\eta_{n-1} \neq 0$). Factorizations (8.1) and (8.2) and the "right" factorizations (3.10) are equivalent when assuming (D.7) and (D.8). The proof is given in [21]. Relation (D.8) yields

$$q x \nu(x) + \zeta(x) = 1 + \zeta(x) \cdot \mu(x) . \tag{D.9}$$

Appendix E. Proof of the q-independent relations

From relations (8.1) and (8.3), one can get the action of $\widehat{K}$ on the initial matrix M_0,

$$\widehat{K}^n(M_0) = M_n \cdot f_n^{-1} \cdot f_{n-1}^{(\eta_0-\zeta_1)+1} \cdot f_{n-2}^{(\eta_1-\zeta_2)-(\eta_0-\zeta_1)-1} \cdot f_{n-3}^{(\eta_2-\zeta_3)-(\eta_1-\zeta_2)+(\eta_0-\zeta_1)+1} \cdot f_{n-4}^{(\eta_3-\zeta_4)-(\eta_2-\zeta_3)+(\eta_1-\zeta_2)-(\eta_0-\zeta_1)-1} \cdots = M_n \cdot f_n^{\kappa_1} \cdot f_{n-1}^{\kappa_2} \cdot f_{n-2}^{\kappa_3} \cdots . \tag{E.1}$$

The action of $\widehat{K}$ on (E.1) yields

$$\widehat{K}^{n+1}(M_0) = \widehat{K}(M_n) \cdot f_n^{-\kappa_1} \cdot f_{n-1}^{-\kappa_2} \cdot f_{n-2}^{-\kappa_3} \cdots f_1^{-\kappa_n}$$
$$= \frac{M_{n+1} \cdot f_n^{-\kappa_1} \cdot f_{n-1}^{-\kappa_2} \cdot f_{n-2}^{-\kappa_3} \cdots f_1^{-\kappa_n}}{f_{n+1} f_n^{\zeta_1-\eta_0} \cdot f_{n-1}^{\zeta_2-\eta_1} \cdot f_{n-2}^{\zeta_3-\eta_2} \cdots f_1^{\zeta_n-\eta_{n-1}}} = M_{n+1} \cdot f_{n+1}^{\kappa_1} \cdot f_n^{\kappa_2} \cdot f_{n-1}^{\kappa_3} \cdots f_1^{\kappa_{n+1}} \tag{E.2}$$

From (E.2) one gets the following linear relations:

$$\kappa_1 = -1 , \quad \kappa_1 + \kappa_2 = \eta_0 - \zeta_1 , \quad \kappa_2 + \kappa_3 = \eta_1 - \zeta_2 ,$$
$$\kappa_3 + \kappa_4 = \eta_2 - \zeta_3 , \quad \cdots . \tag{E.3}$$

Defining l_n as the determinant of the left-hand side of the previous equation, one gets from (E.1) the expression of l_n as

$$l_n = f_{n+1} \cdot f_n^{\zeta_1-q} \cdot f_{n-1}^{\zeta_2+q((\eta_0-\zeta_1)+1)} \cdot f_{n-2}^{\zeta_3+q((\eta_1-\zeta_2)-(\eta_0-\zeta_1)-1)} \cdot f_{n-3}^{\zeta_4+q((\eta_2-\zeta_3)-(\eta_1-\zeta_2)+(\eta_0-\zeta_1)+1)} \cdot f_{n-4}^{\zeta_5+q((\eta_3-\zeta_4)-(\eta_2-\zeta_3)+(\eta_1-\zeta_2)-(\eta_0-\zeta_1)-1)} \cdots$$
$$= f_{n+1} \cdot f_n^{\sigma_1} \cdot f_{n-1}^{\sigma_2} \cdot f_{n-2}^{\sigma_3} \cdots . \tag{E.4}$$

Let us introduce the two new generating functions,

$$\sigma(x) = 1 + \sigma_1 x + \sigma_2 x^2 + \sigma_3 x^3 + \cdots ,$$
$$\kappa(x) = \kappa_1 x + \kappa_2 x^2 + \kappa_3 x^3 + \cdots . \tag{E.5}$$

From the very definition of l_n ($l_n = \det(\widehat{K}^n(M_0))$), as well as from the definition of the κ_n's and the σ_n's (see (E.2) and (E.4)) and from (8.1), one gets the linear relations

$$\sigma_1 = \zeta_1 - q , \quad \sigma_2 = \zeta_2 + q\kappa_1 , \quad \sigma_3 = \zeta_3 + q\kappa_2 \cdots , \tag{E.6}$$

which yields

$$\sigma(x) = \zeta(x) + qx \cdot \kappa(x) , \tag{E.7}$$

and (E.3) yields

$$(1 + x) \cdot \kappa(x) = x^2 \cdot \eta(x) - x \cdot \zeta(x) . \tag{E.8}$$

The right hand side of (E.8) is nothing more than $-x \cdot \rho(x)$, therefore a very simple relation between $\rho(x)$ and $\kappa(x)$ is obtained,

$$\kappa(x) = -\frac{x\rho(x)}{1+x} . \tag{E.9}$$

From (E.7) and (E.9) one finally reaches

$$\sigma(x) = \zeta(x) - q\frac{x\rho(x)}{1+x} . \tag{E.10}$$

It is obvious that the degree of the l_n's is $\pm q$, therefore, one immediately gets from Eq. (E.4) the linear relation

$$q \cdot (-1)^n = \beta_{n+1} + (\sigma_1 \cdot \beta_n + \sigma_2 \cdot \beta_{n-1} + \sigma_3 \cdot \beta_{n-2} + \cdots) , \tag{E.11}$$

which yields

$$\frac{qx}{1+x} = \beta(q,x) \cdot \sigma(x) \quad \text{or} \quad \beta(q,x) = \frac{qx}{(1+x)\sigma(x)} \tag{E.12}$$

and from factorization (E.1),

$$\alpha(q,x) = \frac{q}{(1+x)} \cdot \left(1 - \frac{q\kappa(x)}{\sigma(x)}\right) = \frac{q\zeta(q,x)}{(1+x)\sigma(x)} . \tag{E.13}$$

All these relations do not yet prove that for any generalization ("straight" generalizations, self-similar generalizations,...) the generating functions $\rho(x)$ (or equivalently $\kappa(x)$) and $\sigma(x)$ *are actually independent of* q. However, we have actually proved in [21] that $u_1 = q - N$ and $v_1 = q - N + 1$, which show that κ_1 and σ_1 *are actually independent of* q. Moreover relations (E.1) and (E.4), together with the particular form of $M \cdot K(M)$ or $M \cdot \widehat{K}(M)$ *in the case of the "straight" generalizations*, are certainly sufficient to argue that $\kappa(x)$) and $\sigma(x)$ are actually independent of q in the case of "straight" generalizations (see also Appendix A).

Appendix F. Some explicit expressions of $\rho(x)$

Let us give here explicit expressions of $\rho(x)$ and, for instance, of the generating function $\zeta(q,x)$, for various permutations considered here.

For class I (and class III see Section 3.1, and for class V (see subsection (3.2.1)), one has

$$\rho(x) = (1 + x) \cdot (1 + x^2) . \tag{F.1}$$

For class I (and class III and also class II but *only* for $q = 4$), and for class V, $\zeta(q,x)$ reads, respectively,

$$\zeta(q,x) = (1-x)^3 + q(x+x^3)\,,$$
$$\zeta(q,x) = 1 - 3x + x^2 - x^3 + q(x+x^3)\,. \tag{F.2}$$

For class VI one also has

$$\rho(x) = \frac{1}{1-x} \tag{F.3}$$

(see Section 3.2.2) and for class IV (see Section 3.2.3) and for class II for $q \geq 5$. For class IV, and class VI, $\zeta(q,x)$ reads, respectively,

$$\zeta(q,x) = \frac{1+(q-2)x+x^2+(q-1)x^3+x^4}{1-x^4} = \zeta(0,x) + \frac{qx}{(1-x)(1+x)}\,,$$
$$\zeta(q,x) = \frac{1+(q-2)x}{1-x^2} = \zeta(0,x) + \frac{qx}{(1-x)(1+x)}\,. \tag{F.4}$$

For class II for $q \geq 5$, $\zeta(q,x)$ reads

$$\begin{aligned}\zeta(q,x) &= \left(1 + \frac{(q-3)x+2x^2+(q-2)x^3+3x^4}{1-x^4}\right)\\ &= \zeta(0,x) + \frac{qx}{(1-x)\cdot(1+x)}\\ &= \frac{1-2x-2x^3}{(1+x)\cdot(1+x^2)} + \frac{qx}{(1-x)\cdot(1+x)}\,.\end{aligned} \tag{F.5}$$

For the second example of Section 6.1.2, one has

$$\rho(x) = \frac{1+x^2}{1-x^3}\,, \tag{F.6}$$

and for the "self-similar" generalization of the third example of Section 6.1.3, (see Eq. (6.19)) one has

$$\rho(x) = \frac{1+x}{1-x^4}\,. \tag{F.7}$$

For this "self-similar" generalization of the third example defined as in Section 6.1.5, $\zeta(q,x)$ reads

$$\zeta(q,x) = \frac{1+(q-2)x}{1-x^4} = \zeta(0,x) + \frac{qx}{(1-x^4)}\,. \tag{F.8}$$

For the "straight" generalization of the third example defined as in Section 6.1.4, the generating function $\zeta(q,x)$ reads

$$\zeta(q,x) = \frac{1+(q-3)x+x^2+x^3}{1-x^2} = \zeta(0,x) + \frac{qx}{(1-x)(1+x)} \tag{F.9}$$

In this example one sees that $\rho(x)$ is *different* for the "straight" and "self-similar" generalizations.

For the fourth example (see Section 6.1.7, Eq. (6.41)) for $q = 4$ *only*, one has

$$\rho(x) = \frac{1+x}{1-x^6}\,. \tag{F.10}$$

In this example for $q = 5$ one has $\rho(x) = (1+x)\cdot(1+x+x^2)$ and $\rho(x) = 1+x$ for the "straight" generalization for $q \geq 6$. In this last case, $\kappa(x)$ is remarkably simple, ($\kappa(x) = -1$). Recalling the very definition of the κ_n's, this means a simple result for the action of $\widehat{K}$,

$$\widehat{K}^n(M_0) = \frac{M_n}{f_n}\,. \tag{F.11}$$

For permutation p_7 for 4×4 matrices (see Section 6.1.11), an *approximation* of the generating function $\rho(x)$, compatible with the approximations of $\alpha(x)$ and $\beta(x)$ (see (6.56)) reads

$$\rho(x) = \frac{1-x+x^2+x^3}{1-x}\,. \tag{F.12}$$

In fact, with the number of iterations performed here (namely, ten iterations) one can only be confident about the first ten coefficients in the expansion of $\rho(x)$,

$$\rho(x) = 1+x^2+2x^3+2x^4+2x^5+2x^6+2x^7+2x^8+2x^9+2x^{10}+\cdots\,. \tag{F.13}$$

The zeroes of expression (F.12) are not on the unit circle, but with the number of iterations performed here one cannot rule out the fact that the zeroes of $\rho(x)$ *could well be on the unit circle.*

For permutation p_8 for 4×4 matrices (see Section 6.1.11) the expansion of $\rho(x)$, which corresponds to factorizations (6.57) read

$$\rho(x) = 1+x+x^2+x^3+2x^4+2x^5+2x^6+2x^7+3x^8+3x^9 \\ +3x^{10}+3x^{11}+4x^{12}+\cdots\,. \tag{F.14}$$

If the stability regime for the factorization scheme is actually reached one can write $\rho(x)$ as

$$\rho(x) = \frac{1}{(1-x)\cdot(1-x^4)}\,, \tag{F.15}$$

and the relations on the generating functions $\alpha(x)$ and $\beta(x)$,

$$x\cdot\alpha(x) = \beta(x)\cdot\frac{1+x+2x^2}{(1-x^4)\cdot(1-x^2)}\,, \\ 4+(3x-1)\cdot\alpha(x) = \beta(x)\cdot\Big(\frac{8x^2}{(1-x^2)\cdot(1-x^4)}\Big)\,, \tag{F.16}$$

which give expressions (6.60) and (6.61) for $\alpha(x)$ and $\beta(x)$.

Let us make the following comment: up to M^{12}, f^{13} one cannot rule out other expressions of $\rho(x)$. For instance, with the prejudice that the zeroes of $\rho(x)$ have to be on the unit circle, the "Eulerian" product,

$$\rho(x) = \frac{(1+x^4)\cdot(1+x^8)\cdot(1+x^{12})\cdot(1+x^{16})\cdots(1+x^{4\cdot n})\cdots}{1-x}, \tag{F.17}$$

which expands as

$$\rho(x) = 1 + x + x^2 + x^3 + 2x^4 + 2x^5 + 2x^6 + 2x^7 + 3x^8 + 3x^9 + 3x^{10} + 3x^{11} + 5x^{12} + 5x^{13} + 5x^{14} + 5x^{15} + 7x^{16} + 7x^{17} + 7x^{18} + 7x^{19} + \cdots, \tag{F.18}$$

or (without this prejudice)

$$\rho(x) = \frac{1+x^4+x^8}{1-x}, \tag{F.19}$$

which expands as

$$\rho(x) = 1 + x + x^2 + x^3 + 2x^4 + 2x^5 + 2x^6 + 2x^7 + 3x^8 + 3x^9 + 3x^{10} + 3x^{11} + 3x^{12} + 3x^{13} + 3x^{14} + 3x^{15} + 3x^{16} + 3x^{17} + 3x^{18} + 3x^{19} + \cdots, \tag{F.20}$$

are good approximations which are ruled out in favor of (F.15) only with the calculation of M_{13}.

Finally, let us consider the rational but not birational transformation of Section 7.2 ($K = T\cdot I = P_1 \cdot t \cdot I$, where P_1 is projection (7.2)) on 4×4 matrices. With a prejudice that $\rho(x)$ should be a *rational* function with *zeroes on the unit circle* one rules out the integers period P introduced in (7.17) up to $P = 14$, leading for a first approximation for $\rho(x)$,

$$\rho(x) = \frac{(1+x^2)^2\cdot(1+x^4)\cdot(1-x^2+x^4)}{(1-x)\cdot(1+x+x^2+x^3+x^4+x^5+x^6)\cdot(1-x+x^2-x^3+x^4-x^5+x^6)}. \tag{F.21}$$

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