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2011 J. Phys. A: Math. Theor. 44 305004

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Factorization of the Ising model form factors

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Received 4 March 2011, in final form 3 June 2011

Published 30 June 2011

Online at stacks.iop.org/JPhysA/44/305004

Abstract

We present a general method for analytically factorizing the n -fold form factor integrals $f_{N,N}^{(n)}(t)$ for the correlation functions of the Ising model on the diagonal in terms of the hypergeometric functions ${}_2F_1([1/2, N + 1/2]; [N + 1]; t)$ which appear in the form factor $f_{N,N}^{(1)}(t)$. New quadratic recursion and quartic identities are obtained for the form factors for $n = 2, 3$. For $n = 2, 3, 4$ explicit results are given for the form factors. These factorizations are proved for all N for $n = 2, 3$. These results yield the emergence of palindromic polynomials canonically associated with elliptic curves. As a consequence, understanding the form factors amounts to describing and understanding an infinite set of palindromic polynomials, canonically associated with elliptic curves. From an analytical viewpoint the relation of these palindromic polynomials with hypergeometric functions associated with elliptic curves is made very explicitly, and from a differential algebra viewpoint this corresponds to the emergence of direct sums of differential operators homomorphic to symmetric powers of a second order operator associated with elliptic curve.

PACS numbers: 05.50.+q, 05.10.-a, 02.30.Hq, 02.30.Gp, 02.40.Xx

Mathematics Subject Classification: 34M55, 47E05, 81Qxx, 32G34, 34Lxx, 34Mxx, 14Kxx

1. Introduction

The form factor expansion of Ising model correlation functions is essential for the study of the long distance behavior and the scaling limit of the model. This study was initiated in 1966 when Wu [1] computed the first term in the expansion of the row correlations both for $T > T_c$, where the result is a one dimensional integral, and for $T < T_c$, where the result is a 2 dimensional integral. By at least 1973 it was recognized [2] that the diagonal correlations and form factors are a specialization of the results for the row correlations. The extension to form factors for correlations in a general position and from the leading term to all terms was first made in 1976 [3]. This leads to the general result that for the two dimensional Ising model

with interaction energy $\mathcal{E} = -\sum_{j,k}\{E^v\sigma_{j,k}\sigma_{j+1,k} + E^h\sigma_{j,k}\sigma_{j,k+1}\}$, with $\sigma_{j,k} = \pm 1$, the form factor expansion for $T < T_c$ is

$$\langle\sigma_{0,0}\sigma_{M,N}\rangle = (1-t)^{1/4} \cdot \left\{1 + \sum_{n=1}^{\infty} f_{M,N}^{(2n)}\right\}, \tag{1}$$

where $t = (\sinh 2E^v/k_B T \sinh 2E^h/k_B T)^{-2}$, and for $T > T_c$

$$\langle\sigma_{0,0}\sigma_{M,N}\rangle = (1-t)^{1/4} \cdot \sum_{n=0}^{\infty} f_{M,N}^{(2n+1)}, \tag{2}$$

where $t = (\sinh 2E^v/k_B T \sinh 2E^h/k_B T)^2$, and where $f_{M,N}^{(n)}$ are n -fold integrals.

The form factor expansions (1) and (2) are of great importance for the study of the magnetic susceptibility of the Ising model

$$\chi(T) = \frac{1}{k_B T} \cdot \sum_{M,N} \{ \langle\sigma_{0,0}\sigma_{M,N}\rangle - \mathcal{M}^2 \}, \tag{3}$$

where $\mathcal{M} = (1-t)^{1/8}$ for $T < T_c$ and equals zero for $T > T_c$ is the spontaneous magnetization. The study of this susceptibility has been the outstanding problem in the field for almost 60 years. The susceptibility is expressed in terms of the form factor expansion as

$$k_B T \cdot \chi(T) = (1-t)^{1/4} \cdot \sum_m \chi^{(m)}(T), \tag{4}$$

where

$$\chi^{(m)}(T) = \sum_{M,N} f_{M,N}^{(m)}, \tag{5}$$

with $m = 2n$, for $T < T_c$, and $m = 2n + 1$, for $T > T_c$. In the last twelve years a large number of remarkable properties have been obtained for both $\chi^{(n)}(T)$ [4–13] and the specialization to the diagonal [14]

$$\chi_d^{(n)}(t) = \sum_{N,N} f_{N,N}^{(n)}. \tag{6}$$

These remarkable properties of $\chi^{(n)}$ and $\chi_d^{(n)}(t)$ must originate in properties of the $f_{M,N}^{(n)}$ themselves.

For 40 years after the first computations of Wu, the form factor integrals for $n \geq 2$ appeared to be intractable in the sense that they could not be expressed in terms of previously known special functions. However, in 2007 this intractability was shown to be false when Boukraa *et al* [15] discovered by means of differential algebra computations on Maple, using the form for the form factors proven in [16], many examples for n as large as nine that the form factors in the isotropic case $E^h = E^v$ can be written as sums of products of the complete elliptic integrals $K(t^{1/2})$ and $E(t^{1/2})$ with polynomial coefficients, where for the diagonal case ($M = N$) we may allow $E^v \neq E^h$.

These computer derived examples lead to the obvious

Conjecture 1. *All n -fold form factor integrals for Ising correlations may be expressed in terms of sums of products of one dimensional integrals with polynomial coefficients.*

The first discovery that the n -fold multiple integrals which arise in the study of integrable models can be decomposed into sums of products of one dimensional integrals (or sums) was made for the correlation functions of the XXZ spin chain

$$H_{XXZ} = - \sum_{j=-\infty}^{\infty} \{ \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z \}. \tag{7}$$

These correlations were expressed as multiple integrals for the massive regime ($\Delta < -1$) in 1992 [17] and in the massless regime ($-1 \leq \Delta \leq 1$) in 1996 [18]. In 2001 Boos and Korepin [19] discovered that for the case $\Delta = -1$, the special correlation function (called the emptiness probability)

$$P(n) = \left\langle \prod_{j=1}^n \left(\frac{1 + \sigma_j^z}{2} \right) \right\rangle, \tag{8}$$

for $n = 4$ could be expressed in terms of $\zeta(3)$, $\zeta(5)$, $\zeta^2(3)$ and $\ln 2$, and this decomposition in terms of sums of products of zeta functions of odd argument was extended to $P(5)$ in [20] and $P(6)$ in [21]. Similar decompositions of the correlation function $\langle \sigma_0^z \sigma_n^z \rangle$ were obtained for $n = 3$ in [22], for $n = 4$ in [23] and for $n = 5$ in [24]. The extension to the XXZ model chain (7) with $\Delta \neq -1$ of the decomposition of the integrals for the third neighbor correlation $\langle \sigma_0^i \sigma_3^i \rangle$ for $i = x, z$ was made in [25].

The discovery in [15] that a similar reduction takes place for Ising correlations thus leads to the more far reaching

Conjecture 2. *All multiple integral representations of correlations and form factors in all integrable models can be reduced to sums of products of one dimensional integrals.*

If correct this conjecture must rest upon a very deep and universal property of integrable models.

In [15] the form factors were reduced to sums of products of the complete elliptic integrals $K(t^{1/2})$ and $E(t^{1/2})$. However, the results become much more simple and elegant when expressed in terms of the hypergeometric functions F_N and F_{N+1} where

$$F_N = {}_2F_1([1/2, N + 1/2]; [N + 1]; t) \tag{9}$$

appears in the form factor for $n = 1$

$$f_{N,N}^{(1)}(t) = \frac{t^{N/2}}{\pi} \cdot \int_0^1 x^{N-1/2} (1-x)^{-1/2} (1-tx)^{-1/2} \cdot dx = \lambda_N \cdot t^{N/2} \cdot F_N, \tag{10}$$

where

$$\lambda_N = \frac{(1/2)_N}{N!}, \tag{11}$$

and $(a)_0 = 1$ and for $n \geq 1$ $(a)_n = a(a+1) \cdots (a+n-1)$ is Pochhammer's symbol. Note that $F_0 = \frac{2}{\pi} K(t^{1/2}) = f_{0,0}^{(1)}(t)$.

The expressions for $f_{N,N}^{(n)}(t)$ in terms of F_N and F_{N+1} are obtained from [15], rewritten by use of the contiguous relations for hypergeometric functions, and we give some of these expressions in appendix A. In all cases studied the form factors have the form

$$f_{N,N}^{(2n)}(t) = \sum_{m=0}^{n-1} K_m^{(2n)} \cdot f_{N,N}^{(2m)}(t) + \sum_{m=0}^{2n} C_m^{(2n)}(N; t) \cdot F_N^{2n-m} \cdot F_{N+1}^m, \tag{12}$$

$$\frac{f_{N,N}^{(2n+1)}(t)}{t^{N/2}} = \sum_{m=0}^{n-1} K_m^{(2n+1)} \cdot \frac{f_{N,N}^{(2m+1)}(t)}{t^{N/2}} + \sum_{m=0}^{2n+1} C_m^{(2n+1)}(N; t) \cdot F_N^{2n+1-m} \cdot F_{N+1}^m, \tag{13}$$

where $f_{N,N}^{(0)} = 1$. The degrees of the polynomials $C_m^{(j)}(N; t)$ are for $N \geq 1$

$$\deg C_m^{(2n)}(N; t) = \deg C_m^{(2n+1)}(N; t) = n \cdot (2N + 1), \tag{14}$$

with $C_m^{(n)}(N; t) \sim t^m$ as $t \sim 0$.

These polynomials are different from the corresponding polynomials in the K, E basis in that they have the palindromic property

$$C_m^{(2n)}(N; t) = t^{n(2N+1)+m} \cdot C_m^{(2n)}(N; 1/t), \tag{15}$$

$$C_m^{(2n+1)}(N; t) = t^{n(2N+1)+m} \cdot C_m^{(2n+1)}(N; 1/t). \tag{16}$$

We conjecture that these results are true generally.

In this paper we begin the analytic proof of conjecture 1 and the derivation and generalization of the results of [15] for the diagonal correlation $M = N$ by studying the three lowest order integrals $f_{N,N}^{(n)}(t)$ for $n = 2, 3, 4$. The results are summarized in section 2.

In section 3 we derive the results for $f_{N,N}^{(2)}(t)$. We proceed by first differentiating the integral $f_{N,N}^{(2)}(t)$ with respect to t , which removes the term proportional to $f_{N,N}^{(0)}(t)$ from the general form (12). The resulting two dimensional integral is then seen to factorize into a sum of products of one dimensional integrals. This factorized result is then compared with the derivative of (12) to give three coupled first order inhomogeneous equations for the three polynomials $C_m^{(2)}(N; t)$. These equations are decoupled to give inhomogeneous equations of degree three which are explicitly solved to find the unique polynomial solutions $C_m^{(2)}(N; t)$.

In section 4 we extend this method to $f_{N,N}^{(3)}(t)$. The first step is to apply to $f_{N,N}^{(3)}(t)$ the second order operator which annihilates $f_{N,N}^{(1)}(t)$. However, in this case we have not found the mechanism which factorizes the resulting three dimensional integral. Instead we use the property discovered in [15] that the resulting integral satisfies a fourth order homogeneous equation which is homomorphic to the symmetric cube of a second order operator and thus a factorized form is obtained. This form is then compared with the form obtained by applying the second order operator to the form (13), and from this comparison we obtain 4 coupled inhomogeneous equations for the 4 polynomials $C_m^{(3)}(t)$. These equations are then decoupled to give inhomogeneous equations of degree 5 for $C_3^{(3)}(N; t)$ and of degree 8 for the three remaining polynomials. We then solve these equations under the assumption that a polynomial solution exists.

The results for $f_{N,N}^{(n)}(t)$ with $n = 1, 2, 3$ have a great deal of structure which can be generalized to arbitrary arbitrary n . Of particular interest is the fact that $f_{N,N}^{(2n)}(t)$ vanishes as $t^{n(N+n)}$ and $f_{N,N}^{(2n+1)}(t)/t^{N/2}$ vanishes as $t^{n(N+n+1)}$ at $t \rightarrow 0$ while each individual term in the expansions (12) and (13) vanishes with a power (which may be zero) which is independent of N . This cancellation for $f_{N,N}^{(2)}(t)$ and $f_{N,N}^{(3)}(t)$ is demonstrated in section 5 and gives an interpretation of several features of the results obtained in sections 3 and 4. It also provides an alternative form (138) for $f_{N,N}^{(3)}(t)$ compared to the form (13). In section 6, in a differential algebra viewpoint, the canonical link between the 20th order ODEs associated with the $C_m^{(4)}(N; t)$ of $f_{N,N}^{(4)}(t)$ and the theory of elliptic curves is made very explicit with the emergence of direct sums of differential operators homomorphic to symmetric powers of a second order operator associated with elliptic curves, and in an analytical viewpoint, is made very explicit with exact expressions (given in appendix G), for the polynomials $C_m^{(4)}(N; t)$, valid for any N . We conclude in section 7 with a discussion of possible generalizations of our results.

2. Summary of formalism and results

The form factor integrals for the diagonal correlations are [15, 16] for $T < T_c$

$$f_{N,N}^{(2n)}(t) = \frac{t^{n(N+n)}}{(n!)^2 \pi^{2n}} \int_0^1 \prod_{k=1}^{2n} dx_k x_k^N \prod_{j=1}^n \left(\frac{(1-tx_{2j})(x_{2j}^{-1}-1)}{(1-tx_{2j-1})(x_{2j-1}^{-1}-1)} \right)^{1/2} \\ \times \prod_{1 \leq j \leq n} \prod_{1 \leq k \leq n} \left(\frac{1}{1-tx_{2k-1}x_{2j}} \right)^2 \prod_{1 \leq j < k \leq n} (x_{2j-1} - x_{2k-1})^2 (x_{2j} - x_{2k})^2, \quad (17)$$

and for $T > T_c$

$$f_{N,N}^{(2n+1)}(t) = \frac{t^{(n+1/2)N+n(n+1)}}{n!(n+1)! \pi^{2n+1}} \int_0^1 \prod_{k=1}^{2n+1} dx_k x_k^N \prod_{j=1}^{n+1} x_{2j-1}^{-1} [(1-tx_{2j-1})(x_{2j-1}^{-1}-1)]^{-1/2} \\ \times \prod_{j=1}^n x_{2j} [(1-tx_{2j})(x_{2j}^{-1}-1)]^{1/2} \prod_{1 \leq j \leq n+1} \prod_{1 \leq k \leq n} \left(\frac{1}{1-tx_{2j-1}x_{2k}} \right)^2 \\ \times \prod_{1 \leq j < k \leq n+1} (x_{2j-1} - x_{2k-1})^2 \prod_{1 \leq j < k \leq n} (x_{2j} - x_{2k})^2. \quad (18)$$

When $t = 0$ the integrals in (17) and (18) reduce to a special case of the Selberg integral [26, 27]

$$f_{N,N}^{(2n)}(t) \sim \frac{t^{n(N+n)}}{(n!)^2 \pi^{2n}} \frac{\Gamma(N+n+1/2)\Gamma(n+1/2)}{\Gamma(N+1/2)\Gamma(1/2)} \prod_{j=0}^{n-1} \left[\frac{\Gamma(N+j+1/2)\Gamma(j+1/2)\Gamma(j+2)}{\Gamma(N+n+j+1)} \right]^2 \quad (19)$$

and

$$f_{N,N}^{(2n+1)}(t) \sim \frac{t^{N(n+1/2)+n(n+1)}}{n! \pi^{2n+1}} \frac{\Gamma(N+1/2)\Gamma(1/2)}{\Gamma(N+n+1)} \prod_{j=0}^{n-1} \left[\frac{\Gamma(N+j+3/2)\Gamma(j+3/2)\Gamma(j+2)}{\Gamma(N+n+j+2)} \right]^2. \quad (20)$$

In particular

$$f_{N,N}^{(2)}(t) = t^{N+1} \cdot \frac{\lambda_{N+1}^2}{(2N+1)} + O(t^{N+2}), \quad (21)$$

$$f_{N,N}^{(3)}(t) = t^{3N/2+2} \cdot \frac{\lambda_{N+1}^3}{2(2N+1)(N+2)^2} + O(t^{3N/2+3}). \quad (22)$$

2.1. General formalism

For the special case $f_{N,N}^{(2)}(t)$ we will analytically derive the form (12) without making any assumptions. However, for the general case we will proceed by assuming the forms (12) and (13) as an ansatz and with this as a conjecture, we will derive inhomogeneous Fuchsian equations for the polynomials $C_m^{(n)}(N; t)$

$$\Omega_m^{(n)}(N; t) \cdot C_m^{(n)}(N; t) = I_m^{(n)}(N; t), \quad (23)$$

where $\Omega_m^{(n)}(N; t)$ is a linear differential operator and $I_m^{(n)}(N; t)$ a polynomial.

In all cases which have been studied, the operator $\Omega_m^{(n)}(N; t)$, corresponding to the lhs of (23), has a *direct sum decomposition* where each term in the direct sum is *homomorphic to a either a symmetric power or a symmetric product for different values of N, of the second order operator*

$$O_2(N; t) = D_t^2 - \frac{1+N-Nt}{t(1-t)} \cdot D_t + \frac{4+4N-t-2Nt}{4t^2(1-t)}, \quad (24)$$

where $D_t = d/dt$. The operator $O_2(N; t)$ is equivalent to the operator $L_2(N; t)$ which annihilates $f_{N,N}^{(1)}(t)$ [15], as can be seen in the operator isomorphism

$$O_2(N; t) \cdot t^{N/2+1} = t^{N/2+1} \cdot L_2(N; t). \tag{25}$$

The solutions of $O_2(N; t)$ are expressed in terms of hypergeometric functions by noting that

$$t^2 \cdot (1 - t) \cdot O_2(N) = t \cdot (tD_t + a)(tD_t + b) - (tD_t - a')(tD_t - b'),$$

with

$$a = -N - 1/2, \quad b = -1/2, \quad a' = N + 1, \quad b' = 1, \tag{26}$$

which for $|t| < 1$ ³ has the two fundamental solutions [28, page 283]

$$t^{a'} \cdot {}_2F_1([a + a', b + a']; [a' - b' + 1]; t), \quad t^{b'} \cdot {}_2F_1([a + b', b + b']; [b' - a' + 1]; t). \tag{27}$$

Using (26) we have the two solutions of $O_2(N)$

$$u_1(N; t) = t^{N+1} \cdot {}_2F_1([1/2, 1/2 + N]; [N + 1]; t) = t^{N+1} \cdot F_N, \tag{28}$$

and:

$$t \cdot {}_2F_1([1/2, 1/2 - N]; [1 - N]; t). \tag{29}$$

The solution $u_1(N; t)$ in (28) is regular at $t = 0$ and has the expansion

$$u_1(N; t) = t^{N+1} \cdot \sum_{n=0}^{\infty} b_n(N) \cdot t^n, \tag{30}$$

with

$$b_n(N) = \frac{(1/2)_n(1/2 + N)_n}{(N + 1)_n n!}. \tag{31}$$

Since we will in this paper work with positive integer values of N , it is better to introduce as the second solution

$$t^{N+1} \cdot {}_2F_1([1/2, 1/2 + N]; [1]; 1 - t). \tag{32}$$

When N is *not an integer* the hypergeometric function (32) can be written as the following linear combination of the two previous solutions (28) and (29)

$$\frac{\Gamma(-N)}{\Gamma(1/2)\Gamma(1/2 - N)} \cdot t^{N+1} \cdot {}_2F_1([1/2, 1/2 + N]; [N + 1]; t) + \frac{\Gamma(N)}{\Gamma(1/2)\Gamma(1/2 + N)} \cdot t \cdot {}_2F_1([1/2, 1/2 - N]; [1 - N]; t). \tag{33}$$

The hypergeometric function (32) is not analytic at $t = 0$ but, instead, has a *logarithmic singularity*.

From [29, (2) on page 74 and (7) on page 75] we may choose to normalize the analytical part of the second solution to t as $t \rightarrow 0$. Denoting such a solution $u_2(N; t)$, it reads

$$u_2(N; t) = t \cdot \sum_{n=0}^{N-1} a_n(N) \cdot t^n + t^{N+1} \cdot N \cdot \lambda_N^2 \cdot \sum_{n=0}^{\infty} b_n(N)[k_n - \ln(t)] \cdot t^n, \tag{34}$$

³ For $|t| > 1$, we write $z = 1/t$ and the identical procedure is found to interchange a with a' and b with b' . Thus the two fundamental solutions valid near $t = \infty$ are $\tilde{u}_1(N; z) = z^{-1/2} \cdot {}_2F_1([1/2, 1/2 + N]; [1 + N]; z) = z^{-1/2} \cdot F_N$, $\tilde{u}_2(N; z) = z^{-N-1/2} \cdot {}_2F_1([1/2, 1/2 - N]; [1 - N]; z)$. The identification of the hypergeometric functions of (28) and (29) with these two solutions is a consequence of the palindromic property of the operator $O_2(N; t)$. However, we note that $\tilde{u}_j(N; z)$ is not the analytic continuation of $u_j(N; t)$.

with $a_0(N) = 1$ and for $n \geq 1$

$$a_n(N) = \frac{(1/2)_n(1/2 - N)_n}{(1 - N)_n n!} = \lambda_N \cdot \frac{(1/2)_n(N - n)!}{(1/2)_{N-n} n!} \tag{35}$$

and $k_n = H_n(1) + H_{n+N}(1) - H_n(1/2) - H_{n+N}(1/2)$, where

$$H_n(z) = \sum_{k=0}^{n-1} \frac{1}{z + k} \tag{36}$$

are the partial sums of the harmonic series. The series expansion (34) corresponds to the maximal unipotent monodromy structure of $O_2(N; t)$ which amounts to writing the second solution as:

$$u_2(N; t) = w_2(N; t) - N \cdot \lambda_N^2 \cdot u_1(N; t) \cdot \ln(t) \tag{37}$$

where $w_2(N; t) = t + \dots$ is analytical at $t = 0$. This function $w_2(N; t)$ is the solution analytic at $t = 0$, different from $u_1(N; t)$, of an order-four operator which factorizes as the product $\tilde{O}_2(N; t) \cdot O_2(N; t)$, where $\tilde{O}_2(N; t)$ and $O_2(N; t)$ are two order-two homomorphic operators

$$\tilde{O}_2(N; t) \cdot I_1 = J_1 \cdot O_2(N; t), \tag{38}$$

where one of the two order-one intertwiners I_1 and J_1 is quite simple, namely

$$I_1 = \frac{1}{t} \cdot D_t - \frac{t - 2}{2t^2 \cdot (t - 1)} - \frac{N}{2t^2}. \tag{39}$$

Finally, we note the relation which follows from the Wronskian of $O_2(N; t)$,

$$u_1(N) \cdot u_2(N + 1) - \beta_N \cdot u_2(N) \cdot u_1(N + 1) = t^{N+2}, \tag{40}$$

with

$$\beta_N = \frac{(2N + 1)^2}{4N(N + 1)}. \tag{41}$$

2.2. Explicit results for $f_{N,N}^{(2)}(t)$

For $f_{N,N}^{(2)}(t)$ the parameter $K_0^{(2)}$ and the polynomials $C_m^{(2)}(N; t)$ of the form (12) are explicitly computed in section 3 as

$$K_0^{(2)} = N/2, \tag{42}$$

and

$$C_m^{(2)}(N; t) = A_m^{(2)} \cdot t^m \cdot \sum_{n=0}^{2N+1-m} c_{m;n}^{(2)}(N) \cdot t^n, \tag{43}$$

with

$$A_n^{(2)} = (-1)^{n+1} \cdot \frac{N}{2} \cdot \binom{n}{2} \cdot \beta_N^n. \tag{44}$$

Using the notation that

$$[f]_n \equiv \text{the coefficient of } t^n \text{ in the expansion of } f \text{ at } t = 0 \tag{45}$$

we have for $0 \leq n \leq N - 1$

$$c_{2;n}^{(2)}(N) = c_{2;2N-1-n}^{(2)}(N) = [t^{-2}u_2(N)^2]_n = \sum_{k=0}^n a_k(N) \cdot a_{n-k}(N), \tag{46}$$

$$c_{1;n}^{(2)}(N) = c_{1;2N-n}^{(2)}(N) = [t^{-2}u_2(N)u_2(N+1)]_n = \sum_{k=0}^n a_k(N) \cdot a_{n-k}(N+1), \quad (47)$$

and

$$c_{1;N}^{(2)}(N) = \lambda_N^2 + c_{2;N-1}^{(2)}(N), \quad (48)$$

and where for $0 \leq n \leq N$

$$c_{0;n}^{(2)}(N) = c_{0;2N+1-n}^{(2)}(N) = [t^{-2}u_2^2(N+1)]_n = c_{2;n}^{(2)}(N+1), \quad (49)$$

where $a_n(N)$ is given by (35). We note that the sum (46) for $c_{2;N-1}^{(2)}$ may be written by use of the second form of $a_n(N)$ in (35) in the alternative form

$$c_{2;N-1}^{(2)} = \lambda_N^2 \cdot 2N \cdot H_N(1/2), \quad (50)$$

where $H_N(z)$ is given by (36).

We also derive the recursion relation for $N \geq 1$

$$f_{N,N}^{(2)}(t) = Nf_{1,1}^{(2)}(t) - \frac{N}{2}t^{1/2} \cdot \sum_{j=1}^{N-1} \frac{f_{j,j}^{(1)}(t) \cdot f_{j+1,j+1}^{(1)}(t)}{j(j+1)}. \quad (51)$$

2.3. Explicit results for $f_{N,N}^{(3)}(t)$

For $f_{N,N}^{(3)}(t)$ the parameter $K_0^{(3)}$ and the polynomials $C_m^{(3)}(N; t)$ of the form (13) are explicitly computed in section 4 as

$$K_0^{(3)} = \frac{3N+1}{6}, \quad (52)$$

and

$$C_m^{(3)}(N; t) = A_m^{(3)} \cdot t^m \cdot \sum_{n=0}^{2N+1-m} c_{m,n}^{(3)}(N) \cdot t^n + \frac{N-1}{N} \lambda_N \cdot C_m^{(2)}(N, t), \quad (53)$$

where we make the definition $C_3^{(2)}(N, t) = 0$ and

$$A_n^{(3)} = (-1)^{n+1} \cdot \frac{2}{3} \cdot \binom{n}{3} \cdot \lambda_N \cdot \beta_N^n. \quad (54)$$

The coefficients $c_{m;n}^{(3)}(N)$'s are given by a simple *quartic expression* of the a_n 's and b_n 's. For $0 \leq n \leq N-1$ they read

$$\begin{aligned} c_{3;n}^{(3)}(N) &= c_{3;2N-2-n}^{(3)}(N) = [t^{-N-4}u_2^3(N)u_1(N)]_n \\ &= \sum_{m=0}^n \sum_{l=0}^m \sum_{k=0}^l a_k(N) \cdot a_{l-k}(N) \cdot a_{m-l}(N) \cdot b_{n-m}(N), \end{aligned} \quad (55)$$

and

$$\begin{aligned} c_{2;n}^{(3)}(N) &= c_{2;2N-n-1}^{(3)}(N) = [t^{-N-4}u_2^2(N)u_2(N+1)u_1(N)]_n \\ &= \sum_{m=0}^n \sum_{l=0}^m \sum_{k=0}^l a_k(N) \cdot a_{l-k}(N) \cdot a_{m-l}(N+1) \cdot b_{n-m}(N), \end{aligned} \quad (56)$$

for $0 \leq n \leq N$

$$\begin{aligned} c_{0;n}^{(3)}(N) &= c_{0;2N-n+1}^{(3)}(N) = [t^{-N-4}u_2^3(N+1)u_1(N)]_n \\ &= \sum_{m=0}^n \sum_{l=0}^m \sum_{k=0}^l a_k(N+1) \cdot a_{l-k}(N+1) \cdot a_{m-l}(N+1) \cdot b_{n-m}(N), \end{aligned} \quad (57)$$

and for $0 \leq n \leq N - 1$

$$c_{1;n}^{(3)}(N) = c_{1;2N-n}^{(3)}(N) = [t^{-N-4}u_2(N)u_2^2(N+1)u_1(N)]_n$$

$$= \sum_{m=0}^n \sum_{l=0}^m \sum_{k=0}^l a_k(N) \cdot a_{l-k}(N+1) \cdot a_{k-m}(N+1) \cdot b_{n-m}(N), \quad (58)$$

with the middle term of $C_1^{(3)}(N; t)$ of order $N + 1$

$$c_{1;N}^{(3)} = \frac{\beta_N \cdot \lambda_N^2}{N} + \beta_N \cdot \lambda_N \cdot [(N-1) \cdot c_{2;N-1}^{(3)} + 4c_{2;N-1}^{(2)}]$$

$$- \frac{2}{3} \cdot \frac{\beta_N \cdot \lambda_N}{N^2} \cdot [2N^2 \cdot c_{3;N-2}^{(3)} + (N^2 - 1/4) \cdot c_{3;N-1}^{(3)}], \quad (59)$$

where $a_n(N)$ and $b_n(N)$ are given by (35) and (31).

3. The derivation of the results for $f_{N,N}^{(2)}(t)$

We begin our derivation of the results for $f_{N,N}^{(2)}$ of section 2.2 by integrating (17) (with $2n = 2$) by parts using

$$u = y^{N-1/2} \cdot (1-y)^{1/2} \cdot (1-ty)^{1/2}, \quad (60)$$

$$du = y^{N-3/2} \frac{[N \cdot (1-y) \cdot (1-ty) - 1/2(1-ty^2)]}{(1-y)^{1/2} \cdot (1-ty)^{1/2}} \cdot dy, \quad (61)$$

$$dv = \frac{dy}{(1-txy)^2}, \quad v = \frac{y}{1-txy}, \quad (62)$$

to find

$$f_{N,N}^{(2)}(t) = \int_0^1 dx \int_0^1 dy \frac{t^{N+1}}{2\pi^2} \frac{x^{N+1/2}y^{N-1/2}(1-ty^2)}{(1-x)^{1/2}(1-tx)^{1/2}(1-y)^{1/2}(1-ty)^{1/2}(1-txy)}$$

$$- N \int_0^1 dx \int_0^1 dy \frac{t^{N+1}}{\pi^2} \frac{x^{N+1/2}y^{N-1/2}(1-y)^{1/2}(1-ty)^{1/2}}{(1-x)^{1/2}(1-tx)^{1/2}(1-txy)}. \quad (63)$$

The first term in (63) is separated into two parts as

$$\int_0^1 dx \int_0^1 dy \frac{t \cdot x^N y^N x^{1/2}}{2\pi^2} \frac{1}{y^{1/2} (1-x)^{1/2}(1-tx)^{1/2}(1-y)^{1/2}(1-ty)^{1/2}(1-txy)}$$

$$- \int_0^1 dx \int_0^1 dy \frac{t \cdot x^N y^N}{2\pi^2} \frac{tx^{1/2}y^{3/2}}{(1-x)^{1/2}(1-tx)^{1/2}(1-y)^{1/2}(1-ty)^{1/2}(1-txy)}, \quad (64)$$

and in this second term we interchange $x \leftrightarrow y$. Then, recombining these two terms, we see that the factor $1 - txy$ cancels between the numerator and denominator in (64). Thus the first term in (63) factorizes and we find

$$f_{N,N}^{(2)}(t) = \int_0^1 dx \int_0^1 dy \frac{t^{N+1}}{2\pi^2} \frac{x^{N+1/2}y^{N-1/2}}{(1-x)^{1/2}(1-tx)^{1/2}(1-y)^{1/2}(1-ty)^{1/2}}$$

$$- N \int_0^1 dx \int_0^1 dy \frac{t^{N+1}}{\pi^2} \frac{x^{N+1/2}y^{N-1/2}(1-y)^{1/2}(1-ty)^{1/2}}{(1-x)^{1/2}(1-tx)^{1/2}(1-txy)}$$

$$= \frac{t^{1/2}}{2} \cdot f_{N,N}^{(1)} \cdot f_{N+1,N+1}^{(1)} - N \int_0^1 dx \int_0^1 dy \frac{t^{N+1}}{\pi^2} \frac{x^{N+1/2}y^{N-1/2}(1-y)^{1/2}(1-ty)^{1/2}}{(1-x)^{1/2}(1-tx)^{1/2}(1-txy)}. \quad (65)$$

From (17) we find for $N \geq 1$ that the integral in the second term of (65) is $f_{N,N}^{(2)}(t) - f_{N+1,N+1}^{(2)}(t)$ and thus we have

$$f_{N,N}^{(2)}(t) = \frac{t^{1/2}}{2} \cdot f_{N,N}^{(1)}(t) \cdot f_{N+1,N+1}^{(1)}(t) - N \cdot [f_{N,N}^{(2)}(t) - f_{N+1,N+1}^{(2)}(t)]. \quad (66)$$

From (66) we obtain the recursion relation

$$f_{N+1,N+1}^{(2)}(t) = \frac{N+1}{N} \cdot f_{N,N}^{(2)}(t) - \frac{t^{1/2}}{2N} \cdot f_{N,N}^{(1)}(t) \cdot f_{N+1,N+1}^{(1)}(t), \quad (67)$$

and thus for $N \geq 1$

$$f_{N,N}^{(2)}(t) = N f_{1,1}^{(2)}(t) - \frac{N}{2} t^{1/2} \cdot \sum_{j=1}^{N-1} \frac{f_{j,j}^{(1)}(t) \cdot f_{j+1,j+1}^{(1)}(t)}{j(j+1)}. \quad (68)$$

To proceed further we return to (65) which we write in terms of F_N as

$$f_{N,N}^{(2)}(t) = \frac{\lambda_N \lambda_{N+1}}{2} \cdot t^{N+1} \cdot F_N \cdot F_{N+1} - N \int_0^1 dx \int_0^1 dy \frac{t^{N+1}}{\pi^2} \frac{x^{N+1/2} y^{N-1/2} (1-y)^{1/2} (1-ty)^{1/2}}{(1-x)^{1/2} (1-tx)^{1/2} (1-txy)}. \quad (69)$$

The integral in (69) does not have a manifest factorization. However, if we compute $df_{N,N}^{(2)}(t)/dt$ in the contour integral form of (17), and note that

$$\frac{d}{dt} \left[\frac{(y-t^{1/2})(1-t^{1/2}y)}{(x-t^{1/2})(1-t^{1/2}x)} \right]^{1/2} = \frac{1}{t^{1/2}} \left[\frac{(y-t^{1/2})(1-t^{1/2}y)}{(x-t^{1/2})(1-t^{1/2}x)} \right]^{1/2} \times \frac{(xy-1)(x-y)(t-1)}{(y-t^{1/2})(1-t^{1/2}y)(x-t^{1/2})(1-t^{1/2}x)}, \quad (70)$$

the resulting integral does factorize and, introducing G_N , some well-suited linear combination of F_N and F_{N+1} ,

$$G_N = {}_2F_1([3/2, N+3/2]; [N+1]; t) = \frac{1+t}{(1-t)^2} \cdot F_N - \frac{t}{(1-t)^2} \cdot \frac{2N+1}{N+1} \cdot F_{N+1}, \quad (71)$$

we find

$$\frac{df_{N,N}^{(2)}(t)}{dt} = (1-t) \cdot t^N \cdot \frac{(2N+1)\lambda_N^2}{16(N+1)} \cdot [(2N+1)^2 \cdot F_{N+1} \cdot G_N - (2N-1)(2N+3) \cdot F_N \cdot G_{N+1}]. \quad (72)$$

It remains to integrate (72). However, in general, integrals of products of two hypergeometric functions with respect to the argument will not have the form of the product of two hypergeometric functions. We will thus proceed in the opposite direction by differentiating (12) for $2n=2$ with respect to t and equating the result to (72) to obtain differential equations for the $C_m^{(2)}(N; t)$ which we will then solve to obtain the final results (42)–(44).

From a straightforward use of the contiguous relations of hypergeometric functions [29], we introduce the following well-suited linear combination of F_N and F_{N+1}

$$\bar{F}_N = {}_2F_1([3/2, N+3/2]; [N+2]; t) = \frac{4(N+1)}{2N+1} \cdot \frac{dF_N}{dt} = \frac{1}{1-t} \cdot (2 \cdot (N+1) \cdot F_N - (2N+1) \cdot F_{N+1}). \quad (73)$$

The derivative of (12) with $2n = 2$ may be written in the quadratic form⁴

$$B_1 \cdot F_N^2 + B_2 \cdot F_N \cdot \bar{F}_N + B_3 \cdot \bar{F}_N^2, \tag{74}$$

with

$$B_1 = \frac{dC_0^{(2)}}{dt} - \frac{(N+1)}{2(N+1/2)t} \cdot C_1^{(2)} + \frac{(N+1)}{(N+1/2)} \cdot \frac{dC_1^{(2)}}{dt} - \frac{(N+1)^2}{(N+1/2)^2t} \cdot C_2^{(2)} + \frac{(N+1)^2}{(N+1/2)^2} \cdot \frac{dC_2^{(2)}}{dt}, \tag{75}$$

$$B_2 = \frac{(N+1/2)}{(N+1)} \cdot C_0^{(2)} + \left[1 + \frac{1}{2(N+1/2)} + \frac{1-2t+N(1-t)}{2(N+1/2)t} \right] \cdot C_1^{(2)} - \frac{(1-t)}{2(N+1/2)} \cdot \frac{dC_1^{(2)}}{dt} + \frac{(N+1)(3+2N-2t)}{2(N+1/2)^2t} \cdot C_2^{(2)} - \frac{(N+1)(1-t)}{(N+1/2)^2} \cdot \frac{dC_2^{(2)}}{dt}, \tag{76}$$

$$B_3 = -\frac{(1-t)}{4(N+1)} \cdot C_2^{(2)} - \frac{(2+2N-t)(1-t)}{4(N+1/2)^2t} \cdot C_2^{(2)} + \frac{(1-t)^2}{4(N+1/2)^2} \cdot \frac{dC_2^{(2)}}{dt}. \tag{77}$$

The derivative of $f_{N,N}^{(2)}(t)$ in (72) by use of contiguous relations [29] is expressed in terms of F_N and \bar{F}_N as

$$\frac{df_{N,N}^{(2)}(t)}{dt} = B_4 \cdot F_N^2 + B_5 \cdot F_N \cdot \bar{F}_N + B_6 \cdot \bar{F}_N^2, \tag{78}$$

where

$$B_4 = \frac{2N+1}{4} \cdot \lambda_N^2 t^N, \\ B_5 = \frac{[t - N(1-t)](2N+1)}{4(N+1)} \cdot \lambda_N^2 t^N, \\ B_6 = -\frac{N\beta_N \lambda_N^2}{4(N+1)} \cdot (1-t) \cdot t^{N+1}. \tag{79}$$

3.1. Linear differential equations for $C_m^{(2)}(N; t)$

To obtain the $C_m^{(2)}(N; t)$ we equate (74) with (78) and find the following first order system of equations for $C_m^{(2)}(N; t)$

$$\frac{(2N+1)}{4} \cdot \lambda_N^2 \cdot t^N = \frac{dC_0^{(2)}}{dt} - \frac{(N+1)}{2(N+1/2)t} \cdot C_1^{(2)} + \frac{(N+1)}{(N+1/2)} \cdot \frac{dC_1^{(2)}}{dt} - \frac{(N+1)^2}{(N+1/2)^2t} \cdot C_2^{(2)} + \frac{(N+1)^2}{(N+1/2)^2} \cdot \frac{dC_2^{(2)}}{dt}, \tag{80}$$

$$\frac{(2N+1) \cdot [t - N(1-t)]}{4(N+1)} \cdot \lambda_N^2 \cdot t^N = \frac{(N+1/2)}{(N+1)} \cdot C_0^{(2)} + \left[1 + \frac{1}{2(N+1/2)} + \frac{1-2t+N(1-t)}{2(N+1/2)t} \right] \cdot C_1^{(2)} - \frac{(1-t)}{2(N+1/2)} \cdot \frac{dC_2^{(2)}}{dt} + \frac{(N+1)(3+2N-2t)}{2(N+1/2)^2t} \cdot C_2^{(2)} - \frac{(N+1)(1-t)}{(N+1/2)^2} \cdot \frac{dC_2^{(2)}}{dt}, \tag{81}$$

⁴ For convenience the dependence of the $C_m^{(2)}$ on N and t is suppressed here and below (see (78)).

$$-\frac{(2N+1)^2}{16(N+1)^2} \cdot \lambda_N^2 \cdot (1-t) \cdot t^{N+1} = -\frac{(1-t)}{4(N+1)} \cdot C_1^{(2)} - \frac{(2+2N-t)(1-t)}{4(N+1/2)^2 t} \cdot C_2^{(2)} + \frac{(1-t)^2}{4(N+1/2)^2} \cdot \frac{dC_2^{(2)}}{dt}. \tag{82}$$

From this first order coupled system we obtain third order uncoupled equations for the $C_m^{(2)}(N; t)$

$$2(1-t)^2 \cdot t^2 \cdot \frac{d^3 C_0^{(2)}}{dt^3} - 6(N - (N-1)t)(1-t)t \cdot \frac{d^2 C_0^{(2)}}{dt^2} + 2[N + 2N^2 + (1+4N - 4N^2)t - (5N - 2N^2)t^2] \cdot \frac{dC_0^{(2)}}{dt} + (2N+1)(2Nt - 2N-1) \cdot C_0^{(2)} = \frac{N(N+1)(2N+1)^2}{2} \cdot \lambda_N^2 \cdot (1-t) \cdot t^N, \tag{83}$$

$$2(1-t)^2(1+t) \cdot t^3 \cdot \frac{d^3 C_1^{(2)}}{dt^3} - 2(1-t)[1+3N+4t+(1-3N)t^2] \cdot t^2 \cdot \frac{d^2 C_1^{(2)}}{dt^2} + 2[2+4N+2N^2+(3+4N-2N^2) \cdot t - (3+8N+2N^2) \cdot t^2 + 2N^2 \cdot t^3] \cdot t \cdot \frac{dC_1^{(2)}}{dt} - [4+8N+4N^2+(5+6N) \cdot t - (5+10N+4N^2) \cdot t^2] \cdot C_1^{(2)} = \frac{(2N+1)^2 \cdot [-2N^2(N+1) \cdot (t+1)^2 + (4N+1) \cdot t]}{(N+1)} \cdot \lambda_N^2 \cdot (1-t) \cdot t^{N+1}, \tag{84}$$

and

$$2(1-t)^2 \cdot t^3 \cdot \frac{d^3 C_2^{(2)}}{dt^3} - 6(1+N-Nt) \cdot (1-t) \cdot t^2 \cdot \frac{d^2 C_2^{(2)}}{dt^2} + 2[7+9N+2N^2 - (7+12N+4N^2) \cdot t + (1+3N+2N^2) \cdot t^2] \cdot t \cdot \frac{dC_2^{(2)}}{dt} - [16+24N+8N^2 - (15+28N+12N^2) \cdot t + (2+6N+4N^2) \cdot t^2] \cdot C_2^{(2)} = \frac{N^2(2N+1)^4 \cdot (1-t)}{8(N+1)^2} \cdot \lambda_N^2 \cdot t^{N+2}. \tag{85}$$

From (83)–(85) it follows that $C_m^{(2)}(N; t)$ and $t^{2N+m+1} \cdot C_m^{(2)}(N; 1/t)$ satisfy the same equation and thus, if $C_m^{(2)}(N; t)$ are polynomials they will satisfy the palindromic property (15). From (83) and (85) it follows that the polynomials $C_0^{(2)}(N; t)$ and $C_2^{(2)}(N; t)$ satisfy

$$C_0^{(2)}(N; t) = \frac{N}{(N+1) \cdot \beta_{N+1}^2 \cdot t^2} \cdot C_2^{(2)}(N+1; t). \tag{86}$$

We therefore may restrict our considerations to $C_1^{(2)}(N; t)$ and $C_2^{(2)}(N; t)$.

We will obtain the polynomial solutions for the differential equations (83)–(85) by demonstrating that the homogeneous parts of the equations are homomorphic to symmetric products or symmetric powers of the second order operator $O_2(N)$.

3.2. Polynomial solution for $C_2^{(2)}(N; t)$

Denote $\Omega_2^{(2)}(N, t)$ the order-three linear differential operator acting on $C_2^{(2)}(N, t)$ on the left hand side of (85). Then it is easy to discover that the operator $\Omega_2^{(2)}(N, t)$ is *exactly* the symmetric square of the second-order operator $O_2(N; t)$

$$\Omega_2^{(2)}(N, t) = \text{Sym}^2(O_2(N; t)), \tag{87}$$

which has the three linearly independent solutions

$$u_1(N; t)^2, \quad u_1(N; t) \cdot u_2(N; t), \quad u_2^2(N; t) \tag{88}$$

where the functions $u_j(N; t)$ for $j = 1, 2$ are defined by (30)–(36). The indicial exponents of (85) at $t = 0$ are

$$2N + 2, \quad N + 2, \quad 2, \tag{89}$$

which are the exponents respectively of the three solutions (88). Therefore, because the inhomogeneous term in (85) starts at t^{N+1} the coefficients $c_{2;n}^{(2)}$ in (43) for $0 \leq n \leq N - 1$ will be proportional to the first N coefficients in the expansion of $u_2^2(N; t)$ about $t = 0$.

Equation (85) is invariant under the substitution

$$C_2^{(2)}(N; t) \longrightarrow t^{2N+3} \cdot C_2^{(2)}(N; 1/t), \tag{90}$$

which maps one solution into another. Therefore if it is known that the solution $C_2^{(2)}(N; t)$ is a polynomial the palindromic property

$$c_{2;n}^{(2)} = c_{2;2N-1-n}^{(2)} \tag{91}$$

must hold and thus $C_2^{(2)}(N; t)$ is given by (46) where the normalizing constant $A_2^{(2)}$ remains to be determined.

However, the invariance (90) is, by itself, not sufficient to guarantee the existence of a polynomial solution with the palindromic property (15). To demonstrate that there is a polynomial solution we examine the recursion relation which follows from (85)

$$\begin{aligned} & A_2^{(2)} \cdot \{ 2n(2N - n)(N - n) \cdot c_{2;n}^{(2)}(N) \\ & \quad + (4Nn - 2N - 2n^2 + 2n - 1)(2n - 1 - 2N) \cdot c_{2;n-1}^{(2)}(N) \\ & \quad + 2(n - 1)(2N - n + 1)(N - n + 1) \cdot c_{2;n-2}^{(2)}(N) \} \\ & = (\delta_{n,N} - \delta_{n,N+1}) \cdot \frac{N^2(2N + 1)^4}{8(N + 1)^2} \cdot \lambda_N^2. \end{aligned} \tag{92}$$

where $c_{2;n}^{(2)}(N) = 0$ for $n \leq -1$ and we may set $c_{2;0}^{(2)} = 1$ by convention. By sending $n \rightarrow 2N - n + 1$ in (92) we see that $c_{2;n}^{(2)}(N)$ and $c_{2;2N-n-1}^{(2)}(N)$ do satisfy the same equation as required by (91).

To prove that the solution $C_2^{(2)}(N; t)$ is indeed a polynomial we examine the recursion relation (92) for $n = N$. If there were no inhomogeneous term then, because of the factor $N - n$ in front of $c_{2;n}^{(2)}$, the recursion relation (92) for $n = N$ would give a constraint on $c_{2;N-1}^{(2)}$ and $c_{2;N-2}^{(2)}$. This constraint does in fact not hold, which is the reason that the solution $u_2^2(N; t)$ is not analytic at $t = 0$ but instead has a term $t^{N+2} \ln t$. However, when there is a nonzero inhomogeneous term at order t^{N+2} the recursion equation (92) is satisfied with a nonzero $A_2^{(2)}$. The remaining coefficients $c_{2;n}^{(2)}$ for $N \leq 2N - 1$ are determined by the palindromy constraint (91).

For $C_2^{(2)}(N; t)$ to be a polynomial we must have $c_{2;n}^{(2)}(N) = 0$ for $n \geq 2N$. From the recursion relation (92) we see that because of the coefficient $2N - n$ in front of $c_{2;n}^{(2)}(N)$

the coefficient $c_{2;2N}^{(2)}(N)$ may be freely chosen. The choice of $c_{2;2N}^{(2)}(N) \neq 0$ corresponds to the solution of $\Omega_2^{(2)}(N; t)$ which has the indicial exponent $N + 2$ and clearly does not give a polynomial solution. However by setting $n = 2N + 1$ in (92) we obtain

$$2(N + 1)(2N + 1) \cdot c_{2;2N+1}^{(2)}(N) - (2N + 1)^2 \cdot c_{2;2N}^{(2)}(N) = 0. \tag{93}$$

and if we choose $c_{2;2N}^{(2)}(N) = 0$ we obtain $c_{2;2N+1}^{(2)}(N) = 0$ also. Therefore because (92) is a three term relation, it follows that $c_{2;n}^{(2)}(N) = 0$ for $n \geq 2N$ as required for a polynomial solution.

It remains to explicitly evaluate the normalization constant $A_2^{(2)}$ which satisfies (92) with $n = N$. A more efficient derivation is obtained if we return to the original inhomogeneous equation (85). Then we note that if we include the term with $n = 0$ in the second terms on the right-hand side of (34) in the computation of the term of order t^{N+2} in the left hand side of (85) we must get zero because u_2^2 is a solution of the homogeneous part of (85). Therefore when we use the extra term in u_2^2 of

$$-2t^{N+2} \cdot N \cdot \lambda_N^2 \cdot \ln t, \tag{94}$$

in the lhs of (85), and keep the terms which do not involve $\ln t$, we find

$$2(N^2 - 1) \cdot c_{2;N-2}^{(2)}(N) - (2N^2 - 1) \cdot c_{2;N-1}^{(2)}(N) = -4N^3 \cdot \lambda_N^2. \tag{95}$$

Thus, using (95) we evaluate (92) with $n = N$ as

$$-4A_2^{(2)}N^3\lambda_N^2 = \frac{N^2(2N + 1)^4}{8(N + 1)^2} \cdot \lambda_N^2, \tag{96}$$

and thus

$$A_2^{(2)} = -\frac{N}{2} \cdot \beta_N^2. \tag{97}$$

3.3. Polynomial solution for $C_1^{(2)}(N; t)$

The computation of $C_1^{(2)}(N; t)$ has features which are characteristic of $C_m^{(n)}(N; t)$ which are not seen in $C_2^{(2)}(N; t)$. Similarly to what has been done in the previous subsection we introduce $\Omega_1^{(2)}(N; t)$, the order-three linear differential operator acting on $C_1^{(2)}(N; t)$ in the lhs of (84).

The indicial exponents at $t = 0$ of the operator $\Omega_1^{(2)}(N; t)$ are

$$1, \quad N + 1, \quad 2N + 2 \tag{98}$$

This order-three operator $\Omega_1^{(2)}(N; t)$ is found to be related to the *symmetric product* of $O_2(N)$ and $O_2(N + 1)$ by the direct sum decomposition

$$\text{Sym}(O_2(N), O_2(N + 1)) \cdot t = \Omega_1^{(2)} \oplus \left(D_t - \frac{N + 1}{t} \right). \tag{99}$$

The three linearly independent solutions of $\Omega_1^{(2)}(N; t)$ are to be found in the set of four functions

$$\begin{aligned} t^{-1} \cdot u_1(N; t) \cdot u_1(N + 1; t), & \quad t^{-1} \cdot u_2(N, t) \cdot u_1(N + 1; t), \\ t^{-1} \cdot u_1(N; t) \cdot u_2(N + 1; t), & \quad t^{-1} \cdot u_2(N; t) \cdot u_2(N + 1; t), \end{aligned} \tag{100}$$

where from the definitions of $u_1(N; t)$ in (30) and $u_2(N; t)$ in (34) the behaviors of these four solutions as $t \rightarrow 0$ are t^{2N+2} , t^{N+2} , t^{N+1} , t respectively.

Following the argument given above for $C_2^{(2)}(N; t)$ we conclude, because the inhomogeneous term in (83) is of order t^{N+1} , that the terms up through order t^N must be proportional to the solution of the homogeneous equation

$$t^{-1} \cdot u_2(N; t) \cdot u_2(N + 1; t), \tag{101}$$

which begins at order t . This observation determines the form (43) and the coefficients (47) $c_{1;n}^{(2)}(N)$ for $0 \leq n \leq N - 1$. The normalizing constant $A_1^{(2)}$ and the remaining coefficient $c_{1;N}^{(2)}(N)$ (48) are then obtained from the inhomogeneous equation (83). Finally, to prove that $C_1^{(2)}(N; t)$ is actually a palindromic polynomial the recursion relation for the coefficients $c_{1;n}^{(2)}(N)$ must be used. Details of these computations are given in appendix B.

3.4. The constant $K_0^{(2)}$

Finally, we need to evaluate the constant of integration $K_0^{(2)}$ in (12). This is easily done by noting, from the original integral expression (17), that $f_{N,N}^{(2)}(0) = 0$ for all N . From (43)–(44) we see that

$$C_0^{(2)}(N; 0) = -\frac{N}{2}, \quad C_1^{(2)}(N; 0) = C_2^{(2)}(N; 0) = 0, \tag{102}$$

and using this in (12) we obtain $K_0^{(2)} = N/2$ as desired.

4. The derivation of the results for $f_{N,N}^{(3)}(t)$

The form factor $f_{N,N}^{(3)}(t)$ is defined by the integral (18) with $2n + 1 = 3$, and if we are to follow the method of evaluation developed for $f_{N,N}^{(2)}(t)$, we need to demonstrate analytically that there is an operator which, when acting on the integral, will split it into three factors. Unfortunately we have not analytically obtained such a result.

However, we are able to proceed by using the methods of differential algebra and from [15] it is known computationally for integer N that $f_{N,N}^{(3)}$ is annihilated by the operator $L_4(N) \cdot L_2(N)$ where

$$L_2(N) = D_t^2 + \frac{2t - 1}{(t - 1)t} \cdot D_t - \frac{1}{4t} + \frac{1}{4(t - 1)} - \frac{N^2}{4t^2}, \tag{103}$$

and $L_2(N)$ annihilates $f_{N,N}^{(1)}(t)$, and where

$$\begin{aligned} L_4(N) = & D_t^4 + 10 \frac{(2t - 1)}{(t - 1)t} \cdot D_t^3 + \frac{(241t^2 - 241t + 46)}{2(t - 1)^2 t^2} \cdot D_t^2 \\ & + \frac{(2t - 1)(122t^2 - 122t + 9)}{(t - 1)^3 t^3} \cdot D_t + \frac{81(5t - 1)(5t - 4)}{16 t^3 (t - 1)^3} - \frac{5 N^2}{2 t^2} \cdot D_t^2 \\ & + \frac{(23 - 32t)N^2}{2(t - 1)t^3} \cdot D_t + \frac{9(8 - 17t)N^2}{8(t - 1)t^4} + \frac{9 N^4}{16 t^4}. \end{aligned} \tag{104}$$

Furthermore the operator $L_4(N)$ is homomorphic to the symmetric cube of $L_2(N)$ by the following relation,

$$L_4(N) \cdot Q(N) = R(N) \cdot \text{Sym}^3(L_2(N)), \tag{105}$$

where

$$\begin{aligned} Q(N) = & (t - 1) \cdot t \cdot D_t^3 + \frac{7}{2}(2t - 1) \cdot D_t^2 + \frac{(41t^2 - 41t + 6)}{4(t - 1)t} \cdot D_t \\ & + \frac{9(2t - 1)}{8(t - 1)t} - \frac{9(t - 1)N^2}{4 t} \cdot D_t - \frac{9(2t - 1)}{8 t^2} N^2, \end{aligned} \tag{106}$$

and

$$R(N) = (t - 1) \cdot t \cdot D_t^3 + \frac{23}{2}(2t - 1) \cdot D_t^2 + \frac{21}{4} \frac{6 - 29t + 29t^2}{(t - 1)t} \cdot D_t + \frac{9}{8} \frac{(2t - 1)(125t^2 - 125t + 16)}{(t - 1)^2 t^2} - \frac{9N^2}{4} \cdot \left(\frac{(t - 1)}{t} \cdot D_t + \frac{(10t - 9)}{2t^2} \right). \quad (107)$$

We therefore conclude, since $f_{N,N}^{(3)}(t)$ is regular at $t = 0$ and the solution of $L_2(N)$ which is regular at $t = 0$ is F_N , that

$$Q(N) \cdot B_0 \cdot t^{3N/2} \cdot F_N^3 = L_2(N) \cdot f_{N,N}^{(3)}, \quad (108)$$

where B_0 is a normalizing constant which is determined from the behavior at $t = 0$. From the integral (18) we find

$$f_{N,N}^{(3)} = \frac{N + 2}{4(N + 1/2)} \left(\frac{(1/2)_{N+1}}{(N + 2)!^3} \right)^3 \cdot t^{3N/2+2} + O(t^{3N/2+3}), \quad (109)$$

and from the expansion of F_N we have

$$Q(N) \cdot t^{3N/2} \cdot F_N^3 = \frac{3(2N + 1)^3}{8(N + 1)^2(N + 2)} \cdot t^{3N/2} + O(t^{3N/2+1}), \quad (110)$$

and thus

$$B_0 = \frac{1}{3} \cdot \lambda_N^3. \quad (111)$$

Operating $Q(N)$ on $t^{3N/2} F_N^3$, one can write the result in the basis F_N and \bar{F}_N . Similarly, one can operate on the form $f_{N,N}^{(3)}$ in (13) with $L_2(N)$ and write the result in the same basis F_N and \bar{F}_N . Then, matching powers of the hypergeometric functions on both sides of the relation (108) will yield four coupled inhomogeneous ODEs to be solved. The four coupled ODEs are given in appendix C.

For $C_m^{(3)}(N; t)$ with $m = 0, 1, 2$, the reduction of the four coupled second order equations leads to inhomogeneous 8th order uncoupled ODEs for each $C_m^{(3)}(N; t)$ separately, of the form

$$\sum_{j=0}^8 P_{m,j}(t) \cdot t^j \cdot \frac{d^j}{dt^j} C_m^{(3)}(N; t) = I_m(t), \quad (112)$$

where

$$I_0 = t^{N+1} \cdot \sum_{j=0}^{14} I_0(j) \cdot t^j, \quad I_1 = t^{N+1} \cdot \sum_{j=0}^{17} I_1(j) \cdot t^j, \quad (113)$$

$$I_2 = t^{N+2} \cdot \sum_{j=0}^{14} I_2(j) \cdot t^j,$$

where the $I_m(t)$ are antipalinromic and $P_{m,n}(t)$ are polynomials. In particular

$$P_{m,8}(t) = (1 - t)^9 \cdot P_m(t), \quad (114)$$

where $P_0(t)$ and $P_2(t)$ are order six and $P_1(t)$ is order eight.

However, for $C_3^{(3)}(N; t)$ a step-by-step elimination process in the coupled system terminates in a fifth order equation instead. We derive and present this 5th order equation in appendix D, but the eighth order equations given by Maple are too long to present.

4.1. Polynomial solution for $C_3^{(3)}(N; t)$

The homogeneous operator on the lhs of the ODE (D.1) for $C_3^{(3)}(N, t)$ is found on Maple to be isomorphic to $\text{Sym}^4(O_2(N)) \cdot t^{(N+1)}$, the symmetric fourth power of $O_2(N)$ multiplied by $t^{(N+1)}$. Therefore all five solutions of the homogeneous equation are given as $t^{-(N+1)}$ times products of the solutions $u_1(N; t)$ and $u_2(N; t)$. The fifth order ODE has at $t = 0$ the indicial exponents

$$-N + 3, \quad 3, \quad N + 3, \quad 2N + 3, \quad 3N + 3. \tag{115}$$

Therefore because the polynomial solution must by definition be regular at $t = 0$, the first $N + 1$ terms (from t^3 through t^{N+3}) in the solution

$$t^{-(N+1)} \cdot u_2^3(N) \cdot u_1(N), \tag{116}$$

which vanishes as t^3 , will solve the inhomogeneous equation (D.1), so that

$$C_3^{(3)}(N; t) = A_3^{(3)} \cdot t^3 \cdot \sum_{n=0}^{2N-2} c_{3;n}^{(3)} \cdot t^n, \tag{117}$$

where for $0 \leq n \leq N - 1$

$$c_{3;n}^{(3)} = \sum_{m=0}^n \sum_{l=0}^m \sum_{k=0}^l a_k(N) \cdot a_{l-k}(N) \cdot a_{m-l}(N) \cdot b_{n-m}(N). \tag{118}$$

The lowest order inhomogeneous term is t^{N+3} which is the next indicial exponent in (115) and therefore the normalizing constant $A_3^{(3)}$ is found from the first logarithmic term in the solution of the homogeneous equation by exactly the same argument used for $C_2^{(2)}(N; t)$. Thus we find

$$A_3^{(3)} = \frac{2}{3} \cdot \beta_N^3 \cdot \lambda_N. \tag{119}$$

The remaining demonstration that $C_3^{(3)}(N; t)$ is a palindromic polynomial follows from the recursion relation for the coefficients, as was done for $C_2^{(2)}(N; t)$, with the exception that because the inhomogeneous term in (D.1) is proportional to $t^{N+1}(t^2 - 1)$ instead of $t^{N+1}(t - 1)$, there is an identity which must be verified. Details are given in appendix D.

4.2. Polynomial solutions for $C_2^{(3)}(N; t)$ and $C_0^{(3)}(N; t)$.

A new feature appears in the computation of $C_2^{(3)}(N; t)$ and $C_0^{(3)}(N; t)$.

The indicial exponents at $t = 0$ of the 8-th order operator $\Omega_2^{(3)}(N; t)$

$$-N + 2, \quad 2, \quad 3, \quad N + 2, \quad N + 3, \quad 2N + 2, \quad 2N + 3, \quad 3N + 3, \tag{120}$$

and for $\Omega_0^{(3)}(N; t)$ are

$$-N, \quad 0, \quad 1, \quad N + 1, \quad N + 2, \quad 2N + 2, \quad 2N + 3, \quad 3N + 3. \tag{121}$$

From these exponents it might be expected that the solution of $\Omega_2^{(3)}(N; t)(\Omega_0^{(3)}(N; t))$ which is of order $t^2(t^0)$ could have a logarithmic term $t^3 \ln t(t \ln t)$ which would preclude the existence of a polynomial solution of the corresponding inhomogeneous equation. However, this does, in fact, not happen because there is a decomposition of the 8th order operators into a direct sum of the third order operators $\Omega_2^{(2)}(N; t)(\Omega_0^{(2)}(N; t))$ with exponents $2, N + 2, 2N + 2(0, N + 1, 2N + 2)$ and new fifth order operators $M_m^{(3)}(N; t)$

$$\Omega_m^{(3)}(N; t) = M_m^{(3)}(N; t) \oplus \Omega_m^{(2)}(N; t) \tag{122}$$

with exponents $-N+2, 2, N+2, 2N+2, 3N+2$ for $M_2^{(3)}(N; t)$ and $-N, 0, N+1, 2N+2, 3N+3$ for $M_0^{(3)}(N; t)$. Furthermore $M_2^{(3)}(N; t)$ is homomorphic to the symmetric fourth power of $O_2(N)$ and $M_0^{(3)}(N; t)$ is homomorphic to the symmetric fourth power of $O_2(N+1)$ (see appendix E for details). The inhomogeneous equation is solved in terms of a linear combination of the solutions of the third order and fifth order homogeneous equations.

However, a simpler form of the answer results if we notice the isomorphisms

$$\Omega_2^{(3)}(N; t) = \text{Sym}(O_2(N), O_2(N), O_2(N), O_2(N+1)) \cdot t^{N+2}, \quad (123)$$

$$\Omega_0^{(3)}(N; t) = \text{Sym}(O_2(N), O_2(N+1), O_2(N+1), O_2(N+1)) \cdot t^{N+4}. \quad (124)$$

The desired solutions for $\Omega_2^{(3)}(N; t)$ are constructed from the two solutions which have the exponents 2 and 3,

$$t^{-N-2} \cdot u_2^2(N) \cdot u_1(N) \cdot u_2(N+1), \quad t^{-N-2} \cdot u_2^3(N) \cdot u_1(N+1), \quad (125)$$

which, by use of the Wronskian condition (40), may be rewritten as a linear combination of two solutions each with the exponent of 2 as

$$A_2^{(3)} \cdot t^{-N-2} \cdot u_2^2(N) \cdot u_1(N) \cdot u_2(N+1) + B_2^{(3)} \cdot u_2^2(N), \quad (126)$$

and similarly for $C_0^{(3)}(N; t)$, we choose as the solution of the homogeneous equation the two solutions with exponent 0

$$A_0^{(3)} \cdot t^{-N-4} \cdot u_2^3(N+1) \cdot u_1(N) + B_0^{(3)} t^{-2} \cdot u_2^2(N+1). \quad (127)$$

This procedure determines the constants $c_{2;n}^{(3)}$ for $0 \leq n \leq N-1$ and $c_{0;n}^{(3)}$ for $0 \leq n \leq N$, with palindromy determining the remaining $c_{2;n}^{(3)}$ for $N \leq n \leq 2N-1$ (56) and $c_{0;n}^{(3)}$ for $N+1 \leq n \leq 2N+1$ (57).

The constants $A_2^{(3)}$ and $B_2^{(3)}$ in (53) are found by using (53) with (56) in the inhomogeneous equation for $C_2^{(3)}(N; t)$ and matching the first two terms in the inhomogeneous terms of orders t^{N+2} and t^{N+3} (which are the same orders as the corresponding indicial exponents (120)). This generalizes the determination of $A_m^{(2)}$ for $C_m^{(2)}$ above. Similarly the constants $A_0^{(3)}$ and $B_0^{(3)}$ are found using (53) with (57) in the inhomogeneous equation for $C_0^{(3)}$ and matching to the inhomogeneous terms t^{N+1} and t^{N+2} . Thus we obtain the results (53)–(54) summarized in section 2.3.

4.3. Polynomial solution for $C_1^{(3)}(N; t)$

The computation of $C_1^{(3)}(N; t)$ has further new features.

The 8th order homogeneous operator $\Omega_1^{(3)}(N; t)$ of the inhomogeneous equation for $C_1^{(3)}(N; t)$ has the eight indicial exponents at $t = 0$

$$-N+1, 1, 2, N+1, N+2, 2N+2, 2N+3, 3N+3, \quad (128)$$

and, as in the case of $\Omega_0^{(3)}(N; t)$ and $\Omega_2^{(3)}(N; t)$ has a decomposition into a direct sum of $\Omega_1^{(2)}(N; t)$ and a fifth order operator. However, simpler results are obtained by observing that $\Omega_1^{(3)}(N; t)$ is homomorphic to the symmetric product

$$\text{Sym}(O_2(N), O_2(N), O_2(N+1), O_2(N+1)) \cdot t^{N+3} = \Omega_1^{(3)}(N; t) \oplus \left(D_t - \frac{(N+1)}{t} \right), \quad (129)$$

which satisfies a 9-th order ODE with indicial exponents at $t = 0$

$$-N+1, 1, 2, N+1, N+2, N+3, 2N+2, 2N+3, 3N+3. \quad (130)$$

The solutions with exponents of 1 and 2 are respectively

$$t^{-N-3} \cdot u_2(N) \cdot u_1(N) \cdot u_2^2(N+1), \quad t^{-N-3} \cdot u_2^2(N) \cdot u_2(N+1) \cdot u_1(N+1). \quad (131)$$

Again, recalling the Wronskian relation (40), we may construct the polynomial $C_1^{(3)}(N; t)$, similar to the construction of $C_2^{(3)}(N; t)$, from the linear combination

$$A_1^{(3)} \cdot t^{-N-3} \cdot u_2(N) \cdot u_1(N) \cdot u_2^2(N+1) + B_1^{(3)} \cdot t^{-1} \cdot u_2(N) \cdot u_2(N+1), \quad (132)$$

which determines the coefficients $c_{1;n}^{(3)}$ for $0 \leq n \leq N-1$, with palindromy determining the remaining $c_{1;n}^{(3)}$ for $N+1 \leq n \leq 2N$ (58). The coefficients $A_1^{(3)}$ and $B_1^{(3)}$ (132) are determined in a manner similar to the determination of $A_2^{(3)}$ and $B_2^{(3)}$, by matching to the terms of order t^{N+1} and t^{N+2} .

Finally the term $c_{1;N}^{(3)}$ is computed by using the previously determined results for $C_2^{(3)}(N; t)$ and $C_3^{(3)}(N; t)$ in the coupled differential equation (C.3), giving the result (59).

4.4. Determination of $K_0^{(3)}$

It remains to determine the constant $K_0^{(3)}$ (52), which is easily done by setting $t = 0$ in (13) to obtain

$$0 = K_0^{(3)} \cdot \lambda_N + A_0^{(3)} + \frac{N-1}{N} \cdot \lambda_N \cdot A_0^{(2)}. \quad (133)$$

Using (44) and (54).

5. The Wronskian cancellation for $f_{N,N}^{(2)}(t)$ and $f_{N,N}^{(3)}(t)$

The polynomials $C_m^{(n)}(N; t)$ are of order t^m as $t \rightarrow 0$. However, from (21) and (22) we see that $f_{N,N}^{(2)}(t)$ vanishes as t^{N+1} and $f_{N,N}^{(3)}(t)$ vanishes as t^{N+2} . Therefore for $t \rightarrow 0$, a great deal of cancellation must occur in (12) and (13). This cancellation is an important feature of the structure of the results of section 2.2 and 2.3.

To prove the cancellations we note that the n th power of the Wronskian relation (40) is

$$t^{-n(N+2)} \cdot \sum_{j=0}^n (-1)^j \cdot \binom{n}{j} \cdot \beta_N^j \cdot [u_2(N+1)u_1(N)]^{n-j} \cdot [u_2(N)u_1(N+1)]^j = 1, \quad (134)$$

or alternatively,

$$\sum_{j=0}^n (-1)^j \cdot \binom{n}{j} \cdot \beta_N^j \cdot \left[\frac{u_2(N+1)}{t} \right]^{n-j} \cdot u_2(N)^j \cdot F_N^{n-j} F_{N+1}^j = 1. \quad (135)$$

Thus, by defining $\stackrel{N}{\equiv}$ to mean equality up though and including terms of order t^N we see immediately from the form (12) with (42) for $K_0^{(2)}$ and (46), (47) and (49) for the $c_{m,n}^{(2)}$ with $0 \leq n \leq N$ that the terms though order t^N in $f_{N,N}^{(2)}(t)$ are

$$f_{N,N}^{(2)} \stackrel{N}{\equiv} \frac{N}{2} \cdot \left\{ 1 - \sum_{j=0}^2 (-1)^j \binom{2}{j} \cdot \beta_N^j \cdot \left[\frac{u_2(N+1)}{t} \right]^{2-j} \cdot u_2(N)^j F_N^{2-j} F_{N+1}^j \right\}, \quad (136)$$

which vanishes by use of (135). This derivation has made no use of $c_{1;N}^{(2)}$. This term contributes only to order t^{N+1} and may be determined from the normalization amplitude (21). This provides an alternative to the derivation of (48) of appendix B.

To prove the cancellation for $f_{N,N}^{(3)}(t)$ we note that, because of the term $C_m^{(2)}(N; t)$ in $C_m^{(3)}(N; t)$ for $m = 0, 1, 2$ in (53), we may use the expression (12) and (42)–(49) for $f_{N,N}^{(2)}(t)$ in the form

$$\sum_{m=0}^2 C_m^{(2)}(N; t) \cdot F_N^{2-m} F_{N+1}^m = f_{N,N}^{(2)}(t) - \frac{N}{2}. \tag{137}$$

Thus from (13), (10) and (137) we obtain an alternative form for $f_{N,N}^{(3)}(t)$ of

$$f_{N,N}^{(3)}(t) = \left\{ \frac{2}{3} + \frac{N-1}{N} f_{N,N}^{(2)}(t) \right\} \cdot f_{N,N}^{(1)}(t) + t^{N/2} \cdot \sum_{m=0}^3 \bar{C}_m^{(3)}(N; t) \cdot F_N^{3-m} F_{N+1}^m, \tag{138}$$

where

$$\bar{C}_m^3(N; t) = (-1)^{n+1} \cdot \frac{2}{3} \cdot \binom{n}{3} \cdot \beta_N^n \cdot \lambda_N \cdot \sum_{n=0}^{2N+1-m} c_{m;n}^{(3)} t^n. \tag{139}$$

We have already demonstrated by use of (136) that $f_{N,N}^{(2)}(t)$ vanishes though order t^N . Therefore, using the expressions (55)–(58) for $c_{m;n}^{(3)}$ which are all valid through (at least) order t^N and the definition (10) of $f_{N,N}^{(1)}(t)$ we find

$$\frac{f_{N,N}^{(3)}(t)}{t^{N/2}} \stackrel{N}{=} \frac{2}{3} \lambda_N F_N \cdot \left\{ 1 - \sum_{j=0}^3 (-1)^j \cdot \binom{3}{j} \cdot \beta_N^j \cdot \left[\frac{u_2(N+1)}{t} \right]^{3-j} \cdot u_2(N)^j \cdot F_N^{3-j} F_{N+1}^j \right\}, \tag{140}$$

which vanishes by use of the Wronskian relation (135) with $n = 3$.

We have thus demonstrated that $f_{N,N}^{(3)}(t)/t^{N/2}$ vanishes to order t^N as $t \rightarrow 0$. However we see, from the original integral (18), that in fact $f_{N,N}^{(3)}(t)/t^{N/2}$ is of order t^{N+2} . Therefore the coefficient of t^{N+1} must also vanish. This is not proven by (140). However the coefficient $c_{1,N}^{(3)}$ has not been used in the derivation of (140) and the choice of $c_{1,N}^{(3)}$ to make the coefficient of t^{N+1} vanish provides an alternative derivation of (59).

6. Factorization for $f_{N,N}^{(n)}$ with $n \geq 4$

In principle the methods of differential algebra of the previous sections can be extended to form factors $f_{N,N}^{(n)}(t)$ with $n \geq 4$. However, the complexity of the calculations rapidly increases.

For $f_{N,N}^{(2n)}(t)$ there are $2n + 1$ polynomials $C_m^{(2n)}(N; t)$ and since from [15] we find that for $N \geq 1$

$$L_{2n+1} \cdots L_3 \cdot L_1 \cdot f_{N,N}^{(2n)}(t) = 0, \tag{141}$$

where L_k is a linear differential operator of order k , the polynomials $C_m^{(2n)}(N; t)$ will satisfy a system of $2n + 1$ coupled differential equations where the maximum derivative order is n^2 . These equations can be decoupled into $2n + 1$ Fuchsian ODEs which generically have order $n^2(2n + 1)$.

Similarly for $f_{N,N}^{(2n+1)}(t)$ we found in [15] that

$$L_{2n+2} \cdots L_4 \cdot L_2 \cdot f_{N,N}^{(2n+1)}(t) = 0, \tag{142}$$

and thus the $2n + 2$ polynomials $C_m^{(2n+1)}(N; t)$ satisfy inhomogeneous coupled equations of maximum differential order $n(n + 1)$ which for $N \geq 1$ are generically decoupled into Fuchsian equations of order $2n(n + 1)^2$.

We have obtained for $f_{N,N}^{(4)}(t)$ the 20th order ODEs for $C_m^{(4)}(N; t)$ in the cases $N = 1, \dots, 10$ and will illustrate the new features which arise by considering the case $m = 4$.

We find by use of Maple that (at least for low values of N) the operator $\Omega_4^{(4)}(N; t)$ has a direct sum decomposition

$$\Omega_4^{(4)}(N; t) = M_7^{(4)}(N) \oplus M_{5;1}^{(4)}(N) \oplus M_{5;2}^{(4)}(N) \oplus M_3^{(4)}(N), \quad (143)$$

where $M_{k;n}^{(4)}(N)$ is order k and is homomorphic to the symmetric $k - 1$ power of $O_2(N)$

$$M_7^{(4)}(N) \cdot J_2^{(4)}(N; t) = G_2^{(4)}(N; t) \cdot \text{Sym}^6(O_2(N)), \quad (144)$$

$$M_{5;1}^{(4)}(N) \cdot J_1^{(4)}(N; t) = G_1^{(4)}(N; t) \cdot \text{Sym}^4(O_2(N)), \quad (145)$$

$$M_{5;2}^{(4)}(N) = \text{Sym}^4(O_2(N)), \quad (146)$$

$$M_3^{(4)}(N) \cdot J_0(N; t) = G_0^{(4)}(N; t) \cdot \text{Sym}^2(O_2(N)), \quad (147)$$

where the intertwiners $J_m^{(4)}(2; t)$ and $G_m^{(4)}(2; t)$ are linear differential operator of order m . The intertwiners $J_m^{(4)}(2; t)$ in (144)–(147), are explicitly given in appendix F. Further examples of intertwiners are given in appendix F. These differential algebra exact results (in particular (144)–(147)) are the illustration of the *canonical link between the palindromic polynomials and the theory of elliptic curves*.

Direct sum decompositions⁵ have been obtained for $\Omega_m^{(2)}(N; t)$, $\Omega_m^{(3)}(N; t)$ and $\Omega_m^{(4)}(N; t)$ and we conjecture that this occurs generically for all $\Omega_m^{(n)}(N; t)$. Taking into account the homomorphism of $O_2(N; t)$ and $O_2(N + 1; t)$, and recalling, for instance, subsections 4.2 and 4.3, it may be easier to write direct sum decomposition formulae in terms of sum of symmetric products of $O_2(N; t)$ and $O_2(N + 1; t)$. In order to extend these results, beyond these few special cases of $\Omega_m^{(4)}(N; t)$, a deeper and systematic study of the homomorphisms is still required.

From an analytical viewpoint, a complication which needs to be understood is how to use the solutions of the homogeneous operators $\Omega_m^{(n)}(N; t)$ to obtain the polynomial solution of the inhomogeneous equations. The first difficulty here is that for $C_m^{(4)}(N; t)$ the inhomogeneous terms are large polynomials, of order 100 and higher. Moreover, the orders of palindromy point of the $C_4^{(4)}(N; t)$ with $N = 1, \dots, 10$ are all larger than the order t^{N+4} where the solutions of the homogeneous operators $M_{k;n}^{(4)}(N; t)$ have their first logarithmic singularity. Consequently linear combinations of solutions must be made which cancel these logarithmic singularities at t^{N+4} to give sets of solutions to $\Omega_4^{(4)}(N; t)$ which are analytic up to the order of the first inhomogeneous terms. Thus the determination of the correct linear combination of solutions of the operators $M_{k;n}^{(4)}(N; t)$ is significantly more complex than was the case for $C_m^{(3)}(N; t)$. Exact results for the $C_j^{(4)}$'s, based on the Wronskian cancellation method of section 5, and valid for any value of N , are displayed in appendix G. These are exact results for the palindromic polynomials in terms of F_N and $u_2(N)$, namely two *hypergeometric functions associated with elliptic curves*. Thus, these analytical results can also be seen as an illustration of the canonical link between our palindromic polynomials and the theory of elliptic curves. They confirm the deep relation we find, algebraically and analytically, on these structures with the theory of elliptic curves. In a forthcoming publication we will show that the relation is in fact, more specifically, a close relation with *modular forms*.

⁵ Note that in direct sum decomposition like (143), some ambiguity may occur with terms like $M_{5;1}^{(4)}(N) \oplus M_{5;2}^{(4)}(N)$ where $M_{5;1}^{(4)}(N)$ and $M_{5;2}^{(4)}(N)$ are both homomorphic to a same operator (here $\text{Sym}^4(O_2(N))$).

7. Conclusions

In this paper we have proven the factorization, *for all* N , of the diagonal form factor $f_{N,N}^{(n)}(t)$ for $n = 2, 3$ previously seen in [15] for $N \leq 4$ and provided a conjecture for $n = 4$. Besides new results like the quadratic recursion (51), or non trivial quartic identities (like (55)–(58)), one of the main result of the paper is the fact that, introducing the selected hypergeometric functions F_N , which are also elliptic functions, and are simply related to the (simplest) form factor $f_{N,N}^{(1)}$, the form factors actually become polynomials of these F_N 's with palindromic polynomial coefficients. The complexity of the form factors, is, thus, reduced to some encoding in terms of palindromic polynomials. As a consequence, understanding the form factors amounts to describing and understanding an infinite set of palindromic polynomials, canonically associated with elliptic curves.

We also observe that all of these palindromic polynomials are built from the solutions of the operator $O_2(N)$, and, therefore, are all properties of the basic elliptic curve which underlies all computations of the Ising model. There is a deep structure here which needs to be greatly developed. The differential algebra approach of the linear differential operators associated with these palindromic polynomials is found to be a surprisingly rich structure canonically associated with elliptic curves. In a forthcoming publication, we will show that such rich structures are *closely related to modular forms*.

Analytically, the conjecture and the Wronskian method of logarithm cancellation can be extended to large values of n , but the method of proof by differential equations becomes prohibitively cumbersome for $n \geq 4$. This is very similar to the situation which occurred for the factorization of correlations in the XXZ model where the factorizations of [19–25] done for small values of the separation of the spins by means of explicit computations on integrals was proven for all separations in [30] by means of the qKZ equation satisfied by the correlations and not by the explicit integrals which are the solution of this equation. This suggests that our palindromic polynomials may profitably be considered as a specialization of polynomials of n variables. Moreover, if the two conjectures presented in the introduction are indeed correct, then such kind of structures could also have relevance to the 8 vertex model and to the higher genus curves which arise in the chiral Potts model. Consequently the computations presented here could be a special case of a much larger modularity phenomenon. This could presumably generalize the relations which the Ising model has with modular forms and Calabi–Yau structures [31].

Acknowledgment

This work was supported in part by the National Science Foundation grant PHY-0969739.

Appendix A. Form factors in the basis F_N and F_{N+1}

By use of the contiguous relations for hypergeometric functions the examples given in [15] of $f_{N,N}^{(n)}(t)$ expressed in terms of the elliptic integrals $K(t^{1/2})$ and $E(t^{1/2})$ may be re-expressed in terms of the functions F_N and F_{N+1} . Several examples are as follows

$$f_{0,0}^{(2)} = \frac{t}{4} \cdot F_0 \cdot F_1, \quad (\text{A.1})$$

$$f_{1,1}^{(2)} = \frac{1}{2} - \frac{1}{4}(t+1)(2t^2+t+2) \cdot F_1^2 + \frac{3^2}{2^5} \cdot t \cdot (4t^2+5t+4) \cdot F_1 \cdot F_2 - \frac{3^4}{2^7} t^2(t+1) \cdot F_2^2, \tag{A.2}$$

$$f_{2,2}^{(2)} = 1 - \frac{1}{2^6}(t+1)(64t^4+16t^3+99t^2+16t+64) \cdot F_2^2 + \frac{5^2}{2^8 \cdot 3} \cdot t \cdot (64t^4+88t^3+105t^2+88t+64) \cdot F_2 \cdot F_3 - \frac{5^4}{2^7 \cdot 3^2} \cdot t^2 \cdot (t+1)(2t^2+t+2) \cdot F_3^2, \tag{A.3}$$

$$f_{3,3}^{(2)} = \frac{3}{2} - \frac{1}{2^7 \cdot 3} \cdot (t+1)(576t^6+96t^5+730t^4+425t^3+730t^2+96t+576) \cdot F_3^2 + \frac{7^2}{2^{12} \cdot 3} \cdot t(768t^6+928t^5+1240t^4+1455t^3+1240t^2+928t+768) \cdot F_3 F_4 - \frac{7^4}{2^{15} \cdot 3} \cdot t^2(t+1)(64t^4+16t^3+99t^2+16t+64) \cdot F_4^2. \tag{A.4}$$

For $f_{N,N}^{(3)}$ with $N = 0, \dots, 4$

$$f_{0,0}^{(3)} = \frac{1}{2 \cdot 3} \cdot f_{0,0}^{(1)} - \frac{1}{2 \cdot 3}(1+t) \cdot F_0^3 + \frac{1}{2^2} t \cdot F_0^2 \cdot F_1, \tag{A.5}$$

$$\frac{f_{1,1}^{(3)}}{t^{1/2}} = \frac{2}{3} \cdot \frac{f_{1,1}^{(1)}}{t^{1/2}} - \frac{1}{2^3 \cdot 3}(1+t)(2^3 t^2+13t+2^3) \cdot F_1^3 + \frac{3^2}{2^6} t(8t^2+15t+8) \cdot F_1^2 \cdot F_2 - \frac{3^4}{2^6} t^2(t+1) F_1 F_2^2 + \frac{3^5}{2^9} t^3 \cdot F_2^3, \tag{A.6}$$

$$\frac{f_{2,2}^{(3)}}{t} = \frac{7}{2 \cdot 3} \cdot \frac{f_{2,2}^{(1)}}{t} - \frac{1}{2^{10} \cdot 3}(1+t)(2^6 \cdot 3 \cdot 7t^4+1136t^3+3229t^2+1136t+1344) \cdot F_2^3 + \frac{5^2}{2^{11} \cdot 3} t(2^5 \cdot 3^2 t^4+596t^3+859t^2+596t+2^5 \cdot 3^2) \cdot F_2^2 \cdot F_3 - \frac{5^5}{2^{10} \cdot 3^2} (t+1)(3t^2+4t+3)t^2 \cdot F_2 \cdot F_3^2 + \frac{5^6}{2^{11} \cdot 3^4} t^3(3t^2+8t+3) \cdot F_3^3, \tag{A.7}$$

$$\frac{f_{3,3}^{(3)}}{t^{3/2}} = \frac{5}{3} \cdot \frac{f_{3,3}^{(1)}}{t^{3/2}} - \frac{1}{2^{11} \cdot 3^4}(t+1)(2^7 \cdot 3^3 \cdot 5^2 t^6+49680t^5+153306t^4+160427t^3 + 153306t^2+49680t+2^7 \cdot 3^3 \cdot 5^2) \cdot F_3^3 + \frac{7^2}{2^{16} \cdot 3^3} t(2^{10} \cdot 3^2 \cdot 5t^6+79200t^5+128104t^4+168593t^3 + 128104t^2+79200t+2^{10} \cdot 3^2 \cdot 5) \cdot F_3^2 \cdot F_4 - \frac{7^4}{2^{16} \cdot 3^3} (t+1)t^2(2^4 \cdot 3^2 \cdot 5t^4+670t^3+1763t^2+670t+2^4 \cdot 3^2 \cdot 5) \cdot F_3 \cdot F_4^2 + \frac{7^6}{2^{21} \cdot 3^4} t^3(2^6 \cdot 5t^4+740t^3+1407t^2+740t+2^6 \cdot 5) \cdot F_4^3, \tag{A.8}$$

$$\begin{aligned}
 \frac{f_{4,4}^{(3)}}{t^2} = & \frac{13}{6} \cdot \frac{f_{4,4}^{(1)}}{t^2} - \frac{1}{222 \cdot 3} (t+1)(2^{14} \cdot 5 \cdot 7 \cdot 13t^8 + 3254 \cdot 272t^7 + 11 \cdot 474 \cdot 624t^6 + 8672 \cdot 032t^5 \\
 & + 20 \cdot 423 \cdot 231t^4 + 8672 \cdot 032t^3 + 11 \cdot 474 \cdot 624t^2 + 3254 \cdot 272t + 2^{14} \cdot 5 \cdot 7 \cdot 13) \cdot F_4^3 \\
 & + \frac{3^3}{2^{23} \cdot 5} t(2^{12} \cdot 3 \cdot 5^2 \cdot 7t^8 + 3334 \cdot 912t^7 + 4845 \cdot 120t^6 + 7068 \cdot 720t^5 \\
 & + 8865 \cdot 649t^4 + 7068 \cdot 720t^3 + 4845 \cdot 120t^2 + 3334 \cdot 912t + 2^{12} \cdot 3 \cdot 5^2 \cdot 7) \cdot F_4^2 \cdot F_5 \\
 & - \frac{3^6}{2^{20} \cdot 5^3} t^2(t+1)(2^4 \cdot 3^2 \cdot 5 \cdot 7^2t^6 + 26 \cdot 292t^5 + 69 \cdot 377t^4 + 78 \cdot 580t^3 \\
 & + 69 \cdot 377t^2 + 26 \cdot 292t + 2^4 \cdot 3^2 \cdot 5 \cdot 7^2) \cdot F_4 \cdot F_5^2 \\
 & + \frac{3^8}{2^{20} \cdot 5^3} t^3(2^3 \cdot 3^3 \cdot 5 \cdot 7t^6 + 2^3 \cdot 3^3 \cdot 7 \cdot 11t^5 + 28 \cdot 413t^4 + 46 \cdot 432t^3 \\
 & + 28 \cdot 413t^2 + 2^3 \cdot 3^3 \cdot 7 \cdot 11t + 2^3 \cdot 3^3 \cdot 5 \cdot 7) \cdot F_5^3.
 \end{aligned} \tag{A.9}$$

The coefficients which are not given in factored form all contain large prime factors.

For $f_{N,N}^{(4)}$ with $N = 0, 1, 2, 3$

$$f_{0,0}^{(4)} = \frac{1}{3} \cdot f_{0,0}^{(2)} - \frac{1}{2^2 \cdot 3} \cdot t \cdot F_0^4 + \frac{1}{2^5} \cdot t \cdot F_0^2 \cdot F_1^2, \tag{A.10}$$

$$\begin{aligned}
 f_{1,1}^{(4)} = & -\frac{1}{2^3 \cdot 3} + \frac{5}{2 \cdot 3} \cdot f_{1,1}^{(2)} + \frac{1}{2^5 \cdot 3} (4t^4 + 4t^3 + 15t^2 + 4t + 4)(t+1)^2 \cdot F_1^4 \\
 & - \frac{3}{2^7} t(t+1)(8t^4 + 18t^3 + 35t^2 + 18t + 8) \cdot F_1^3 \cdot F_2 \\
 & + \frac{3^4}{2^{11}} \cdot t^2(8t^4 + 28t^3 + 45t^2 + 28t + 8) \cdot F_1^2 \cdot F_2^2 \\
 & - \frac{3^5}{2^{12}} \cdot t^3 \cdot (t+1) \cdot (4t^2 + 11t + 4) \cdot F_1 \cdot F_2^3 + \frac{3^7}{2^{15}} t^4(t^2 + 4t + 1) \cdot F_2^4,
 \end{aligned} \tag{A.11}$$

$$\begin{aligned}
 f_{2,2}^{(4)} = & -\frac{1}{3} + \frac{2^2}{3} \cdot f_{2,2}^{(2)} + \frac{1}{2^{14} \cdot 3} (2^{14}t^{10} + 40 \cdot 960t^9 + 84 \cdot 480t^8 + 136 \cdot 640t^7 + 176 \cdot 180t^6 \\
 & + 201 \cdot 075t^5 + 176 \cdot 180t^4 + 136 \cdot 640t^3 + 84 \cdot 480t^2 + 40 \cdot 960t + 2^{14}) \cdot F_2^4 \\
 & - \frac{5^2}{2^{14} \cdot 3^2} t(t+1)(2^{13}t^8 + 13 \cdot 312t^7 + 29 \cdot 504t^6 + 36 \cdot 320t^5 + 45 \cdot 337t^4 \\
 & + 36 \cdot 320t^3 + 29 \cdot 504t^2 + 13 \cdot 312t + 2^{13}) \cdot F_2^3 \cdot F_3 \\
 & + \frac{5^4}{2^{17} \cdot 3^2} t^2(2^{12}t^8 + 11 \cdot 264t^7 + 21 \cdot 760t^6 + 31 \cdot 576t^5 + 36 \cdot 209t^4 \\
 & + 31 \cdot 576t^3 + 21 \cdot 760t^2 + 11 \cdot 264t + 2^{12}) \cdot F_2^2 \cdot F_3^2 \\
 & - \frac{5^6}{2^{15} \cdot 3^4} \cdot t^3(t+1)(2^8t^6 + 480t^5 + 906t^4 + 979t^3 + 906t^2 + 480t + 2^8) \cdot F_2 \cdot F_3^3 \\
 & + \frac{5^8}{2^{15} \cdot 3^4} t^4(2^5t^6 + 96t^5 + 177t^4 + 224t^3 + 177t^2 + 96t + 2^5) \cdot F_3^4,
 \end{aligned} \tag{A.12}$$

$$\begin{aligned}
 f_{3,3}^{(4)} = & -\frac{7}{2^3} + \frac{11}{2 \cdot 3} \cdot f_{3,3}^{(2)} + \frac{1}{2^{16} \cdot 3^4} (2^{13} \cdot 3^4 \cdot 7t^{14} + 10 \cdot 838 \cdot 016t^{13} + 19 \cdot 643 \cdot 904t^{12} \\
 & + 34 \cdot 169 \cdot 856t^{11} + 50 \cdot 403 \cdot 584t^{10} + 62 \cdot 791 \cdot 680t^9 + 73 \cdot 309 \cdot 425t^8 + 79 \cdot 935 \cdot 700t^7
 \end{aligned}$$

$$\begin{aligned}
 &+ 73\,309\,425t^6 + 62\,791\,680t^5 + 50\,403\,584t^4 + 34\,169\,856t^3 + 19\,643\,904t^2 \\
 &+ 10\,838\,016t + 2^{13} \cdot 3^4 \cdot 7) \cdot F_3^4 \\
 &- \frac{7^2}{2^{19} \cdot 3^4} t(t+1)(2^{14} \cdot 3^3 \cdot 7t^{12} + 4257\,792t^{11} + 9547\,776t^{10} + 13\,813\,120t^9 \\
 &+ 19\,341\,120t^8 + 21\,399\,090t^7 + 24\,976\,435t^6 + 21\,399\,090t^5 + 19\,341\,120t^4 \\
 &+ 13\,813\,120t^3 + 9547\,776t^2 + 4257\,792t + 2^{14} \cdot 3^3 \cdot 7) \cdot F_3^3 \cdot F_4 \\
 &+ \frac{7^4}{2^{25} \cdot 3^5} t^2 \cdot (2^{15} \cdot 3^2 \cdot 7t^{12} + 4988\,928t^{11} + 9680\,384t^{10} + 15\,992\,320t^9 \\
 &+ 21\,863\,120t^8 + 26\,325\,960t^7 + 28\,527\,015t^6 + 26\,325\,960t^5 + 21\,863\,120t^4 \\
 &+ 15\,992\,320t^3 + 9680\,384t^2 + 4988\,928t + 2^{15} \cdot 3^2 \cdot 7) \cdot F_3^2 \cdot F_4^2 \\
 &- \frac{7^6}{2^{27} \cdot 3^4} t^3 \cdot (t+1)(2^{14} \cdot 3 \cdot 7t^{10} + 501\,760t^9 + 1191\,680t^8 + 1548\,640t^7 \\
 &+ 2065\,400t^6 + 2169\,745t^5 + 2065\,400t^4 + 1548\,640t^3 \\
 &+ 1191\,680t^2 + 501\,760t + 2^{14} \cdot 3 \cdot 7) \cdot F_3 \cdot F_4^3 \\
 &+ \frac{7^8}{2^{31} \cdot 3^4} t^4 \cdot (2^{12} \cdot 7t^{10} + 71\,680t^9 + 147\,840t^8 + 235\,040t^7 + 299\,555t^6 \\
 &+ 339\,180t^5 + 299\,555t^4 + 235\,040t^3 + 147\,840t^2 + 71\,680t + 2^{12} \cdot 7) \cdot F_4^4. \tag{A.13}
 \end{aligned}$$

For $f_{N,N}^{(5)}$ with $N = 1, 2, 3$

$$\begin{aligned}
 \frac{f_{1,1}^{(5)}}{t^{1/2}} &= -\frac{2^2}{5} \cdot \frac{f_{1,1}^{(1)}}{t^{1/2}} + \frac{f_{1,1}^{(3)}}{t^{1/2}} + \frac{1}{2^6 \cdot 3 \cdot 5} (t+1)^2 (2^6 + 136t^3 + 159t^2 + 136t + 2^6) \cdot F_1^5 \\
 &- \frac{3}{2^8} t(t+1)(2^5 t^4 + 80t^3 + 99t^2 + 80t + 2^5) \cdot F_1^4 \cdot F_2 \\
 &+ \frac{3^3}{2^{12}} t^2 (2^7 t^4 + 368t^3 + 483t^2 + 368t + 2^7) \cdot F_1^3 \cdot F_2^2 \\
 &- \frac{3^5}{2^{10}} t^3 (t+1)(4t^2 + 5t + 4) \cdot F_1^2 \cdot F_2^3 \\
 &+ \frac{3^7}{2^{15}} t^4 (8t^2 + 13t + 8) \cdot F_1 \cdot F_2^4 - \frac{3^9}{2^{15} \cdot 5} (t+1) \cdot t^5 \cdot F_2^5, \tag{A.14}
 \end{aligned}$$

$$\begin{aligned}
 \frac{f_{2,2}^{(5)}}{t} &= -\frac{137}{2^3 \cdot 5} \cdot \frac{f_{2,2}^{(1)}}{t} + \frac{3}{2} \cdot \frac{f_{2,2}^{(3)}}{t} \\
 &+ \frac{1}{2^{18} \cdot 3 \cdot 5} (8t^2 + 7t + 8)(2^9 \cdot 3 \cdot 61t^8 + 241\,856t^7 + 508\,200t^6 + 708\,609t^5 \\
 &+ 780\,244t^4 + 708\,609t^3 + 508\,200t^2 + 241\,856t + 2^9 \cdot 3 \cdot 61) \cdot F_2^5 \\
 &- \frac{5^2}{2^{18} \cdot 3^2} t(t+1)(92\,160t^8 + 239\,360t^7 + 540\,576t^6 + 723\,924t^5 \\
 &+ 868\,861t^4 + 723\,924t^3 + 540\,576t^2 + 239\,360t + 92\,160) \cdot F_2^4 \cdot F_3 \\
 &+ \frac{5^4}{2^{20} \cdot 3^3} t^2 (90\,624t^8 + 338\,816t^7 + 743\,304t^6 + 1122\,432t^5 + 1278\,697t^4 \\
 &+ 1122\,432t^3 + 743\,304t^2 + 338\,816t + 90\,624) \cdot F_2^3 \cdot F_3^2 \\
 &- \frac{5^6}{2^{17} \cdot 3^4} t^3 (t+1)(1392t^6 + 4010t^5 + 6983t^4 + 8136t^3
 \end{aligned}$$

$$\begin{aligned}
 &+ 6983t^2 + 4010t + 1392) \cdot F_2^2 \cdot F_3^3 \\
 &+ \frac{5^8}{2^{20} \cdot 3^5} t^4 (684t^6 + 2752t^5 + 5161t^4 + 6240t^3 + 5161t^2 + 2752t \\
 &+ 684) \cdot F_2 \cdot F_3^4 \\
 &- \frac{5^9}{2^{19} \cdot 3^6} t^5 (t + 1)(42t^4 + 133t^3 + 167t^2 + 133t + 42) \cdot F_3^5, \tag{A.15}
 \end{aligned}$$

$$\begin{aligned}
 \frac{f_{3,3}^{(5)}}{t^{3/2}} = & - \frac{127}{3 \cdot 5} \cdot \frac{f_{3,3}^{(1)}}{t^{3/2}} + 2 \cdot \frac{f_{3,3}^{(3)}}{t^{3/2}} \\
 & + \frac{1}{2^{20} \cdot 3^5 \cdot 5} (2^{16} \cdot 3^4 \cdot 5 \cdot 17t^{14} + 1377976320t^{13} + 3016452096t^{12} \\
 & + 5930641920t^{11} + 9308313280t^{10} + 12328157240t^9 + 14834544515t^8 \\
 & + 15849843292t^7 + 14834544515t^6 + 12328157240t^5 + 9308313280t^4 \\
 & + 5930641920t^3 + 3016452096t^2 + 1377976320t + 2^{16} \cdot 3^4 \cdot 5 \cdot 17) \cdot F_3^5 \\
 & - \frac{7^2}{2^{22} \cdot 3^5} t(t + 1)(2^{19} \cdot 3^3 \cdot 5t^{12} + 151511040t^{11} + 351000576t^{10} \\
 & + 605214208t^9 + 835692208t^8 + 1025976166t^7 + 1112168875t^6 \\
 & + 1025976166t^5 + 835692208t^4 + 605214208t^3 + 351000576t^2 \\
 & + 151511040t + 2^{19} \cdot 3^3 \cdot 5) \cdot F_3^4 \cdot F_4 \\
 & + \frac{7^4}{2^{29} \cdot 3^5} t^2 (2^{18} \cdot 3^3 \cdot 5^2 t^{12} + 572129280t^{11} + 1334317056t^{10} \\
 & + 2446757888t^9 + 3545541888t^8 + 4425343776t^7 + 4784608975t^6 \\
 & + 4425343776t^5 + 3545541888t^4 + 2446757888t^3 + 1334317056t^2 \\
 & + 572129280t + 2^{18} \cdot 3^3 \cdot 5^2) \cdot F_3^3 \cdot F_4^2 \\
 & - \frac{7^6}{2^{27} \cdot 3^5} t^3 (t + 1)(2^{13} \cdot 3 \cdot 5 \cdot 7t^{10} + 2007040t^9 + 4885888t^8 + 7228048t^7 \\
 & + 9666130t^6 + 10423545t^5 + 9666130t^4 + 7228048t^3 + 4885888t^2 \\
 & + 2007040t + 2^{13} \cdot 3 \cdot 5 \cdot 7) \cdot F_3^2 \cdot F_4^3 \\
 & + \frac{7^8}{2^{34} \cdot 3^5} t^4 (2^{14} \cdot 5 \cdot 13t^{10} + 3665920t^9 + 9078784t^8 + 15185664t^7 \\
 & + 20375540t^6 + 22605185t^5 + 20375540t^4 + 15185664t^3 + 9078784t^2 \\
 & + 3665920t + 2^{14} \cdot 5 \cdot 13) \cdot F_3 \cdot F_4^4 \\
 & - \frac{7^{10}}{2^{35} \cdot 3^5 \cdot 5} t^5 (t + 1)(2^{13} \cdot 5t^8 + 104960t^7 + 267136t^6 + 319904t^5 \\
 & + 436441t^4 + 319904t^3 + 267136t^2 + 104960t + 2^{13} \cdot 5) \cdot F_4^5. \tag{A.16}
 \end{aligned}$$

Appendix B. Polynomial solution calculations for $C_1^{(2)}(N; t)$

We here give explicitly the calculational details for $C_1^{(2)}(N; t)$.

Using the form (43) in the inhomogeneous equation (83) we find the recursion relation for the coefficients $c_{1;n}^{(2)}$ for $n \neq N, N + 1, N + 2, N + 3$

$$2n \cdot (n - N)(n - 2N - 1) \cdot c_{1;n}^{(2)} - \{2n^3 - 6Nn^2 - 2(4 + N - 2N^2)n + 5 + 6N\} \cdot c_{1;n-1}^{(2)}$$

$$\begin{aligned}
 & - \{2n^3 - 6(3 + N)n^2 + (46 + 34N + 4N^2)n - 35 - 38N - 8N^2\} \cdot c_{1;n-2}^{(2)} \\
 & + 2(n - 2)(n - 2N - 3)(n - N - 3) \cdot c_{1;n-3}^{(2)} = 0,
 \end{aligned}
 \tag{B.1}$$

where by definition $c_{1;n}^{(2)} = 0$ for $n \leq -1$. This recursion relation has four terms instead of the three terms in the corresponding relation (92) for $c_{2;n}^{(2)}$. We note that, if we send $n \rightarrow 2N - n + 3$ in (B.1), we see that $c_{1;n}^{(2)}$ and $c_{2N-n}^{(2)}$ satisfy the same equation. Since the coefficient of $c_{1;n}^{(2)}$ vanishes for $n = 0$, the term $c_{1;0}^{(2)}$ is not determined from (B.1) and by convention we set $c_{1;0}^{(2)} = 1$.

Following the procedure used for $C_2^{(2)}(N; t)$ we note that equation (84) will be satisfied to order t^N if we choose the $c_{1;n}^{(2)}$ for $0 \leq n \leq N - 1$ to be the corresponding coefficients in $t^{-1} \cdot u_2(N; t) \cdot u_2(N + 1; t)$ and hence (47) follows.

The inhomogeneous recursion relations for $n = N, N + 1$ are

$$\begin{aligned}
 A_1^{(2)} \{ & -(2N^2 + 2N - 5) \cdot c_{1;N-1}^{(2)} + (8N^2 + 8N - 35) \cdot c_{1;N-2}^{(2)} \\
 & - 6(N - 2)(N + 3) \cdot c_{1;N-3}^{(2)} \} = -2N^2(2N + 1)^2 \lambda_N^2,
 \end{aligned}
 \tag{B.2}$$

$$\begin{aligned}
 A_1^{(2)} \{ & -2N(N + 1) \cdot c_{1;N+1}^{(2)} - (2N + 1)^2 \cdot c_{1;N}^{(2)} \\
 & + (6N^2 + 6N - 5) \cdot c_{1;N-1}^{(2)} + 4(N + 2)(N - 1) \cdot c_{1;N-2}^{(2)} \} \\
 = & - \frac{(2N + 1)^2(4N^3 + 4N^2 - 4N - 1)\lambda_N^2}{4(N + 1)},
 \end{aligned}
 \tag{B.3}$$

and the relations for $N+2, N+3$ are identical with $N, N+1$, respectively, with the (palindromic) replacement

$$c_{1;N-m}^{(2)} \longrightarrow c_{1;N+m}^{(2)}.
 \tag{B.4}$$

If there were no inhomogeneous term (B.2) would be a new constraint in the coefficients $c_{1;n}^{(2)}$ for $n = N - 1, N - 2, N - 3$. However this constraint does not hold (because the solution to the homogeneous equation has a term $t^{N+1} \ln t$).

The normalizing constant $A_1^{(2)}$ can be evaluated from (B.2) and the sum on the LHS of (B.2) is evaluated the same way the corresponding sum was for $C_2^{(2)}(N; t)$, by comparing with the full solution $t^{-1}u_2(N; t)u_2(N + 1; t)$ of the homogeneous equation. Thus we find

$$\begin{aligned}
 & -(2N^2 + 2N - 5) \cdot c_{1;N-1}^{(2)}(N) + (8N^2 + 8N - 35) \cdot c_{1;N-2}^{(2)}(N) \\
 & - 6(N + 3)(N - 2) \cdot c_{1;N-3}^{(2)}(N) = -2N^2(N + 1)\lambda_N^2,
 \end{aligned}
 \tag{B.5}$$

and, hence, we find from (B.2)

$$A_1^{(2)} = N\beta_N.
 \tag{B.6}$$

It remains to compute $c_{1;N}^{(2)}$ from (B.3). We obtain the palindromic solution by requiring that $c_{1;N+1}^{(2)} = c_{1;N-1}^{(2)}$ and thus (B.3) reduces to

$$\begin{aligned}
 & (2N + 1)^2 \cdot c_{1;N}^{(2)} + (4N^2 + 4N - 5) \cdot c_{1;N-1}^{(2)} + 4(N + 2)(N - 1) \cdot c_{1;N-2}^{(2)} \\
 = & - \frac{(2N + 1)^2(4N^3 + 4N^2 - 4N - 1)\lambda_N^2}{4(N + 1)}.
 \end{aligned}
 \tag{B.7}$$

An equivalent and more efficient method for evaluating $c_{1;N}^{(2)}$, which avoids the need to evaluate the sums on the lhs of (B.7), is to directly evaluate $C_1^{(2)}(N; t)$ in terms of $C_2^{(2)}(N; t)$ by use of the coupled equation (82). From this we find

$$c_{1;N}^{(2)}(N) = \lambda_N^2 + \sum_{k=0}^{N-1} a_k(N) \cdot a_{N-1-k}(N), \tag{B.8}$$

and, by explicitly evaluating the sum in (B.8), we obtain the result (48). Finally, the $c_{1;n}^{(2)}$ for $N + 1 \leq n \leq 2N$ are determined from the palindromy of (B.1).

Appendix C. Coupled differential equations for $C_m^{(3)}(N; t)$

The four coupled differential equations for $C_m^{(3)}(N; t)$ are

$$\begin{aligned} &-\frac{2N+1}{2 \cdot (t-1)t} \cdot C_0^{(3)}(t) - \frac{(N+1)(2tN+1+t)}{t^2 \cdot (2N+1)(t-1)} \cdot C_1^{(3)}(t) \\ &- 2 \frac{(N+1)^2(2tN+3)}{t^2(2N+1)^2(t-1)} \cdot C_2^{(3)}(t) - 8 \frac{(N+1)^3(tN-t+3)}{(t-1)(2N+1)^3t^2} \cdot C_3^{(3)}(t) \\ &+ \frac{tN+2t-N-1}{(t-1)t} \cdot \frac{d}{dt} C_0^{(3)}(t) + 2 \frac{(N+1)(tN-N+t)}{t(t-1)(2N+1)} \cdot \frac{d}{dt} C_1^{(3)}(t) \\ &+ 4 \frac{(N+1)^2(tN-N+1)}{t(t-1)(2N+1)^2} \cdot \frac{d}{dt} C_2^{(3)}(t) + 8 \frac{(N+1)^3(tN-N+2-t)}{t(2N+1)^3(t-1)} \\ &\cdot \frac{d}{dt} C_3^{(3)}(t) + \frac{d^2}{dt^2} C_0^{(3)}(t) + 2 \frac{N+1}{2N+1} \cdot \frac{d^2}{dt^2} C_1^{(3)}(t) + 4 \frac{(N+1)^2}{(2N+1)^2} \cdot \frac{d^2}{dt^2} C_2^{(3)}(t) \\ &+ 8 \frac{(N+1)^3}{(2N+1)^3} \cdot \frac{d^2}{dt^2} C_3^{(3)}(t) = \frac{3}{4} t^{N-1} \cdot (2N+1) \cdot B_0(N), \end{aligned} \tag{C.1}$$

$$\begin{aligned} &-2 \frac{(N+1)(2N+6tN+3t+2)}{t^2(2N+1)^2} \cdot C_1^{(3)}(t) - 8 \frac{(N+1)^2(4tN+4N+5+t)}{t^2(2N+1)^3} \cdot C_2^{(3)}(t) \\ &- 24 \frac{(N+1)^3(6N+2tN+9-2t)}{(2N+1)^4t^2} \cdot C_3^{(3)}(t) + 6 \frac{d}{dt} C_0^{(3)}(t) \\ &+ 4 \frac{(N+1)(5tN+N+3t+1)}{t(2N+1)^2} \cdot \frac{d}{dt} C_1^{(3)}(t) \\ &+ 8 \frac{(N+1)^2(2N+4tN+t+4)}{t(2N+1)^3} \cdot \frac{d}{dt} C_2^{(3)}(t) \\ &+ 48 \frac{(N+1)^3(tN+N+3-t)}{t(2N+1)^4} \cdot \frac{d}{dt} C_3^{(3)}(t) + 4 \frac{(t-1)(N+1)}{(2N+1)^2} \cdot \frac{d^2}{dt^2} C_1^{(3)}(t) \\ &+ 16 \frac{(N+1)^2(t-1)}{(2N+1)^3} \cdot \frac{d^2}{dt^2} C_2^{(3)}(t) + 48 \frac{(N+1)^3(t-1)}{(2N+1)^4} \cdot \frac{d^2}{dt^2} C_3^{(3)}(t) \\ &= \frac{3}{2} t^{N-1} \cdot (2N^2t+t+4tN-2N-2N^2) \cdot B_0(N), \end{aligned} \tag{C.2}$$

$$\begin{aligned} &C_1^{(3)}(t) + 4 \frac{(N+1)(2N+2-t)}{t(2N+1)^2} \cdot C_2^{(3)}(t) + 4 \frac{(t-1)(N+1)}{(2N+1)^2} \cdot \frac{d}{dt} C_2^{(3)}(t) \\ &+ 4 \frac{(N+1)^2(12N^2-16tN-2t^2N+30N+2t^2+18-17t)}{(2N+1)^4t^2} \cdot C_3^{(3)}(t) \end{aligned}$$

$$\begin{aligned}
 &+ 8 \frac{(t-1)(N+1)^2(tN+5N-t+5)}{t(2N+1)^4} \cdot \frac{d}{dt} C_3^{(3)}(t) \\
 &+ 8 \frac{(t-1)^2(N+1)^2}{(2N+1)^4} \cdot \frac{d^2}{dt^2} C_3^{(3)}(t) = \frac{3}{4} \frac{(2N+1)^2}{(N+1)} \cdot t^{N+1} \cdot B_0(N), \tag{C.3}
 \end{aligned}$$

$$\begin{aligned}
 6 \cdot C_0^{(3)}(t) &+ 4 \frac{(N+1)(4N+2tN+4-t)}{t(2N+1)^2} \cdot C_1^{(3)}(t) \\
 &+ 8 \frac{(N+1)^2(4N^2+8N^2t-10t^2N+10tN+12N-t-4t^2+8)}{(2N+1)^4t^2} \cdot C_2^{(3)}(t) \\
 &+ 48 \frac{(N+1)^3(4N^2-2t^2N-10tN+16N+2t^2+13-14t)}{(2N+1)^5t^2} \cdot C_3^{(3)}(t) \\
 &+ 16 \frac{(t-1)(N+1)}{(2N+1)^2} \cdot \frac{d}{dt} C_1^{(3)}(t) \\
 &+ 16 \frac{(t-1)(N+1)^2(5tN+3N+2t+3)}{t(2N+1)^4} \cdot \frac{d}{dt} C_2^{(3)}(t) \\
 &+ 96 \frac{(N+1)^3(t-1)(tN+3N+4-t)}{t(2N+1)^5} \cdot \frac{d}{dt} C_3^{(3)}(t) \\
 &+ 16 \frac{(t-1)^2(N+1)^2}{(2N+1)^4} \cdot \frac{d^2}{dt^2} C_2^{(3)}(t) + 96 \frac{(t-1)^2(N+1)^3}{(2N+1)^5} \cdot \frac{d^2}{dt^2} C_3^{(3)}(t) \\
 &= 3t^N \cdot (3N(t-1) + 2t - 1) \cdot B_0(N), \tag{C.4}
 \end{aligned}$$

where $B_0(N)$ is given by (111).

Appendix D. The ODE and recursion relation for $C_3^{(3)}(N; t)$

The ODE for $C_3^{(3)}(N; t)$ can be found by carefully using the four coupled ODEs (C.1)–(C.4). First use (C.3) to solve for $C_1^{(3)}(N; t)$ and then use this is in equation (C.4) in order to solve for $C_0^{(3)}(N; t)$. Next, use both $C_0^{(3)}(N; t)$ and $C_1^{(3)}(n; t)$ in (C.1) and (C.2) to produce ODEs of orders four in $C_2^{(3)}(N, t)$ and five in $C_3^{(3)}(N; t)$ in (C.1) and orders three in $C_2^{(3)}(N; t)$ and four in $C_3^{(3)}(N; t)$ in (C.2).

In the new (C.1), the fourth derivative of $C_2^{(3)}(N; t)$ can be solved in terms of the other derivatives, and likewise in the new (C.2), the third derivative of $C_2^{(3)}(N; t)$ can be solved in terms of the other derivatives. Taking the derivative of the expression for the third derivative of $C_2^{(3)}(N; t)$ and equating it to the expression for the fourth derivative of $C_2^{(3)}(N; t)$ we find an alternate expression for the third derivative of $C_2^{(3)}(N; t)$. Finally, equating the two expressions for the third derivative of $C_2^{(3)}(N; t)$, a full cancellation of all of the derivatives of $C_2^{(3)}(N; t)$ takes place, leaving a fifth order ODE in terms of only $C_3^{(3)}(t)$

$$\begin{aligned}
 &4[2(N-1)(2N+1)(3N+1)(N+1)t^4 - (2N+3)(36N^3 - 7N^2 - 69N - 32)t^3 \\
 &+ 4(N+2)(36N^3 - 10N^2 - 116N - 69)t^2 \\
 &- (2N+5)(60N^3 - 23N^2 - 275N - 188)t \\
 &+ 18(2N+3)(N+3)(N-3)(N+1)] \cdot C_3^{(3)}(t) \\
 &- 8[(N-1)(2N+1)(3N+1)(N+1)t^4 \\
 &- (-130N + 40N^3 - 47 + 24N^4 - 73N^2)t^3 \\
 &+ 2(18N^4 - 129 + 45N^3 - 113N^2 - 270N)t^2
 \end{aligned}$$

$$\begin{aligned}
 & - (-253N^2 - 422 + 24N^4 - 740N + 80N^3)t \\
 & + (N + 1)(6N^3 + 19N^2 - 114N - 211)] \cdot t \cdot \frac{d}{dt} C_3^{(3)}(t) \\
 & + 20(t - 1)[2N(N - 1)(N + 1) \cdot t^3 - 3(2N^3 - 4N^2 - 8N - 3) \\
 & \cdot t^2 + 3(-13 + 2N^3 - 8N^2 - 24N)t] \\
 & - 2(N - 9)(N + 2)(N + 1)] \cdot t^2 \cdot \frac{d^2}{dt^2} C_3^{(3)}(t) \\
 & + 40(t - 1)^2[(N - 1)^2 t^2 - (4N + 1 + 2N^2)t \\
 & + (N + 5)(N + 1)] \cdot t^3 \cdot \frac{d^3}{dt^3} C_3^{(3)}(t) \\
 & - 40(t - 1)^3[(N - 1)t - N - 1]t^4 \cdot \frac{d^4}{dt^4} C_3^{(3)}(t) + 8(t - 1)^4 \cdot t^5 \cdot \frac{d^5}{dt^5} C_3^{(3)}(t) \\
 = & - \frac{3(t^2 - 1) \cdot N^2(2N + 1)^6}{(N + 1)^3} \cdot t^{N+3} \cdot B_0(N). \tag{D.1}
 \end{aligned}$$

From this differential equation we obtain the recursion relation for the coefficients $c_{3;n}^{(3)}$ and the normalization constant $A_3^{(3)}$ defined by the form (53), where by definition $c_{3;n}^{(3)} = 0$ for $n \leq -1$

$$\begin{aligned}
 A_3^{(3)} \cdot \{ & 8n(2N - n)(N - n)(N + n)(3N - n) \cdot c_{3;n}^{(3)} + 4(2N + 1 - 2n)(2 - 7n \\
 & + 7N - N^2 + 4n^4 - 12N^3 - 8n^3 + 24N^3n \\
 & - 4N^2n + 11n^2 + 4N^2n^2 - 16Nn^3 + 24Nn^2 - 22Nn) \cdot c_{3;n-1}^{(3)} \\
 & - 16(N + 1 - n)(9 - 22n + 22N + N^2 + 3n^4 - 18N^3 - 12n^3 + 18N^3n \\
 & - 6N^2n + 23n^2 + 3N^2n^2 - 12Nn^3 + 36Nn^2 - 46Nn) \cdot c_{3;n-2}^{(3)} \\
 & + 4(2N + 3 - 2n)(32 - 69n + 69N + 7N^2 + 4n^4 - 36N^3 - 24n^3 + 24N^3n \\
 & - 12N^2n + 59n^2 + 4N^2n^2 - 16Nn^3 + 72Nn^2 - 118Nn) \cdot c_{3;n-3}^{(3)} \\
 & - 8(n - 2)(2N + 2 - n)(N + 2 - n)(N - 2 + n)(3N + 2 - n) \cdot c_{3;n-4}^{(3)} \} \\
 = & (\delta_{n,N} - \delta_{n,N+2}) \cdot \frac{3(2N + 1)^6}{(N + 1)^3} \cdot B_0. \tag{D.2}
 \end{aligned}$$

We note by sending $n \rightarrow 2N - n + 2$ that $c_{3;n}^{(3)}$ and $c_{3;2N-2-n}^{(3)}$ satisfy the same equation.

For $n = 0$ (D.2) is identically zero for any $c_{3;0}^{(3)}$ which we set equal to unity by convention. For $0 \leq n \leq N - 1$ the rhs of (D.2) vanishes and hence the $c_{3;n}^{(3)}$ are identical with the coefficients (118) of the solution (116) of the homogeneous equation.

For $n = N$ the coefficient of $c_{3;N}^{(3)}$ vanishes, and thus if there were no inhomogeneous term, the coefficients $c_{3;n}^{(3)}$ for $n = N - 4, N - 3, N - 2N - 1$ would have to satisfy a non trivial constraint. This constraint does not, in fact, hold and is the reason that the homogeneous equation has a term $t^{N+3} \ln t$. However, with a nonvanishing inhomogeneous term, the equation for $n = N$ determines the normalization constant.

For $n = N + 1$ the equation (D.2) reduces to

$$\begin{aligned}
 & 8(N + 1)(N - 1)(2N + 1)(2N - 1) \cdot (c_{3;N+1}^{(3)} - c_{3;N-4}^{(3)}) \\
 & - 8(2N + 1)(2N - 1)(2N^2 - 1) \cdot (c_{3;N}^{(3)} - c_{3;N-2}^{(3)}) = 0 \tag{D.3}
 \end{aligned}$$

which will be satisfied by the palindromic property

$$c_{3;n}^{(3)} = c_{3;2N-2-n}^{(3)} \tag{D.4}$$

with $n = N + 1$ and $n = N$. Finally, the $c_{3;n}^{(3)}$ for $N \leq n \leq 2N - 2$ are determined from the palindromy of (D.2).

Appendix E. Homomorphisms for $C_0^{(3)}(N; t)$ and $C_2^{(3)}(N; t)$

The fifth order operator $M_0^{(3)}(N; t)$ in the direct sum decomposition (122) of $\Omega_0^{(3)}(N; t)$ has the homomorphism (in terms of the operator $L_2(N)$)

$$M_0^{(3)}(N) \cdot J_0^{(3)}(N; t) = G_0^{(3)}(N; t) \cdot \text{Sym}^4(L_2(N + 1)), \tag{E.1}$$

where the intertwiners $J_0^{(3)}(N; t)$ and $G_0^{(3)}(N; t)$ are:

$$\begin{aligned} J_0^{(3)}(N; t) &= t^{N+1} \cdot (t - 1) \cdot t \cdot \left(D_t - \frac{d \ln(R_N^A)}{dt} \right) \\ &= t^{N+1} \cdot ((t - 1) \cdot t \cdot D_t - (2N + 2(N + 1))), \end{aligned} \tag{E.2}$$

$$G_0^{(3)}(N; t) = t^{N+1} \cdot (t - 1) \cdot t \cdot \left(D_t - \frac{d \ln(R_N^B)}{dt} \right), \tag{E.3}$$

where

$$R_N^A = (t - 1)^{2(2N+1)} \cdot t^{-2(N+1)}, \tag{E.4}$$

$$R_N^B = \frac{(t + 1)(t - 1)^{4N-3}}{t^{2N+6}} \cdot P_N, \tag{E.5}$$

$$\begin{aligned} P_N &= (4N + 3) \cdot (3N + 2) \cdot (t^2 + 1) + 2(20N^2 + 15N + 2) \cdot t \\ &= (4N + 3) \cdot (3N + 2) \cdot (t + 1)^2 + 4(2(2N + 1)(N - 1) + N) \cdot t. \end{aligned} \tag{E.6}$$

The homomorphism for $M_2^{(3)}(N; t)$ is

$$M_2^{(3)}(N; t) \cdot J_2^{(3)}(N; t) = G_2^{(3)}(N; t) \cdot \text{Sym}^4(L_2(N + 1)), \tag{E.7}$$

where the intertwiners $J_2^{(3)}(N; t)$ and $G_2^{(3)}(N; t)$ are

$$J_2^{(3)}(N; t) = t^{N+2} \cdot ((t - 1) \cdot t \cdot D_t + 2(N + 1) \cdot t + 2N), \tag{E.8}$$

$$G_2^{(3)}(N; t) = t^{N+2} \cdot (t - 1) \cdot t \cdot \left(D_t - \frac{d \ln(R_N^B(-(N + 1)))}{dt} \right), \tag{E.9}$$

where R_N^B is exactly the R_N^B in (E.5).

Appendix F. Homomorphisms for $\Omega_4^{(4)}(N; t)$

Many exact results have been obtained on the intertwiners occurring in (144), (145), (146), (147). Let us display the simplest ones.

For $J_0^{(4)}(N; t)$ we have

$$\begin{aligned} J_0^{(4)}(2; t) &= t^2 \cdot (t + 1) \cdot (2t^2 + t + 2), \\ J_0^{(4)}(3; t) &= t^2 \cdot (t + 1) \cdot (64t^4 + 16t^3 + 99t^2 + 16t + 64), \\ J_0^{(4)}(4; t) &= t^2 \cdot (t + 1) \cdot (576t^6 + 96t^5 + 730t^4 + 425t^3 + 730t^2 + 96t + 576), \\ J_0^{(4)}(5; t) &= t^2 \cdot (t + 1) \cdot (16384t^8 + 2048t^7 + 19264t^6 \\ &\quad + 6608t^5 + 28861t^4 + 6608t^3 + 19264t^2 + 2048t + 16384). \end{aligned} \tag{F.1}$$

For $J_1^{(4)}(N; t)$ we have

$$\begin{aligned}
 2t^3 \cdot J_1^{(4)}(2; t) &= (t - 1) \cdot J_0^{(4)}(2; t) \cdot D_t - 2t \cdot (10t^4 + 2t^3 - 5t - 4), \\
 64t^4 \cdot J_1^{(4)}(3; t) &= (t - 1) \cdot J_0^{(4)}(3; t) \cdot D_t \\
 &\quad - 2t \cdot (448t^6 + 32t^5 + 95t^4 - 220t^2 - 112t - 128), \\
 576 \cdot t^5 \cdot J_1^{(4)}(4; t) &= (t - 1) \cdot J_0^{(4)}(4; t) \cdot D_t \\
 &\quad - 2t \cdot (5184t^8 + 192t^7 + 406t^6 + 1148t^5 - 2471t^3 - 1288t^2 - 864t - 1152), \\
 16384 \cdot t^6 \cdot J_1^{(4)}(5; t) &= (t - 1) \cdot J_0^{(4)}(5; t) \cdot D_t \\
 &\quad - 2t \cdot (180224t^{10} + 4096t^9 + 7488t^8 + 15168t^7 + 41307t^6 \\
 &\quad - 83454t^4 - 44112t^3 - 29952t^2 - 22528t - 32768). \tag{F.2}
 \end{aligned}$$

Finally, the simplest $J_2^{(4)}(N; t)$, namely $J_2^{(4)}(2; t)$ reads:

$$\begin{aligned}
 16t^6 \cdot J_2^{(4)}(2; t) &= 8(t - 1)^2 \cdot J_0^{(4)}(2; t) \cdot D_t^2 - t \cdot (t - 1) \\
 &\quad \cdot (432t^4 + 80t^3 - 99t^2 - 240t - 208) \cdot D_t \\
 &\quad + 3(1040t^5 - 1176t^4 - 233t^3 - 100t^2 + 168t + 256). \tag{F.3}
 \end{aligned}$$

Appendix G. Exact results for the $C_m^{(4)}$'s

The $f_{N,N}^{(4)}(t)$'s have a new feature not previously seen. The inhomogeneous terms on the ODE's for $C_m^{(2)}(N; t)$ and $C_m^{(3)}(N; t)$ begin at t^{N+a} where a is 0,1 or 2 depending on the values of m . Therefore, to the order needed for the polynomial solution, the logarithms in the solution $u_2(N)$ never can contribute. However, for $f_{N,N}^{(4)}(N; t)$ the order of the inhomogeneous terms grows as t^{2N} instead of t^N . Therefore, since logarithms occur in $u_2(N)$ at order t^{N+1} in order to find the polynomial solution to the 20th order inhomogeneous equation in terms of the solutions of $u_2(N)$ and $u_1(N)$, we need to find linear combinations of solutions of the terms in the direct sum decomposition which cancel these logarithms.

This procedure for solving the inhomogeneous equations is too cumbersome by itself to obtain explicit results as was done for $f_{N,N}^{(2)}(t)$ and $f_{N,N}^{(3)}(t)$. However, when the cancellation of logarithms is combined with the Wronskian cancellation method of section 5, it is possible to conjecture results for $C_m^{(4)}(N; t)$ which have been verified to satisfy the 20th order inhomogeneous equations through $N = 10$:

$$\begin{aligned}
 C_0^{(4)} \stackrel{2N+1}{=} -\bar{K}_0^{(4)} \cdot \frac{u_2^4(N+1)}{t^4} - \bar{K}_0^{(4)} \cdot \frac{4}{N} \cdot C_0^{(2)} \cdot \frac{u^2(N+1)}{t^2} + \frac{2}{3} \cdot \left[C_0^{(2)} \cdot \frac{u_2^2(N+1)}{t^2} - 2\beta_N \right. \\
 \left. \cdot C_0^{(2)} \cdot \frac{u_2^2(N+1) \cdot u_2(N)}{t^2} \cdot F_{N+1} - C_1^{(2)} \cdot \frac{u_2^3(N+1)}{t^3} \cdot F_{N+1} \right] \\
 + \frac{N\lambda^2}{3} \cdot \beta_N \cdot \frac{u_2^3(N+1)}{t^4} \cdot t^{N+2} \cdot F_{N+1}, \tag{G.1}
 \end{aligned}$$

$$\begin{aligned}
 C_1^{(4)} \stackrel{2N+1}{=} 4 \cdot \bar{K}_0^{(4)} \cdot \beta_N \frac{u_2^3(N+1)u_2(N)}{t^3} \\
 + \bar{K}_0^{(4)} \cdot \frac{4}{N} \cdot \left[2\beta_N \cdot C_0^{(2)} \cdot \frac{u_2(N+1) \cdot u_2(N)}{t} - C_1^{(2)} \cdot \frac{u_2^2(N+1)}{t^2} \right] \\
 + \frac{2}{3} \cdot \left[6\beta_N^2 \cdot C_0^{(2)} \cdot \frac{u_2(N+1) \cdot u_2^2(N)}{t} \cdot F_{N+1} + 2C_1^{(2)} \right]
 \end{aligned}$$

$$\begin{aligned} & \cdot \frac{u_2^3(N+1)}{t^3} \cdot F_N - 2C_2^{(2)} \cdot \frac{u_2^3(N+1)}{t^3} \cdot F_{N+1} \Big] \\ & - \frac{N\lambda^2}{3} \left[\beta_N \cdot \frac{u_2^3(N+1)}{t^3} \cdot t^{N+1} \cdot F_N + 3\beta_N^2 \cdot \frac{u_2^2(N+1) \cdot u_2(N)}{t^3} \cdot t^{N+2} \cdot F_{N+1} \right], \end{aligned} \tag{G.2}$$

$$\begin{aligned} C_2^{(4)} \stackrel{2N+1}{=} & -6\bar{K}_0^{(4)} \cdot \beta_N^2 \cdot \frac{u_2^2(N+1) \cdot u_2^2(N)}{t^2} - \left(\frac{4}{N} \cdot \bar{K}_0^{(4)} + 2 \right) \\ & \cdot \left[\beta_N^2 \cdot C_0^{(2)} \cdot u_2^2(N) - 2\beta_N \cdot C_1^{(2)} \frac{u_2(N+1) \cdot u_2(N)}{t} + C_2^{(2)} \frac{u_2^2(N+1)}{t^2} \right] \\ & + \frac{2}{3} \cdot \left[-6\beta_N^3 \cdot C_0^{(2)} \cdot u_2^3(N) \cdot F_{N+1} - 9\beta_N \cdot C_1^{(2)} \right. \\ & \cdot \frac{u_2(N+1) \cdot u_2(N)}{t} + 6C_2^{(2)} \cdot \frac{u_2^3(N+1)}{t^3} \cdot F_N \Big] \\ & + \frac{N\lambda^2}{3} \cdot \left[3\beta_N^2 \cdot \frac{u_2^2(N+1) \cdot u_2(N)}{t^2} \cdot t^{N+1} \cdot F_N \right. \\ & \left. + 3 \cdot \beta_N^3 \cdot \frac{u_2(N+1) \cdot u_2^2(N)}{t^2} \cdot t^{N+2} \cdot F_{N+1} \right], \end{aligned} \tag{G.3}$$

$$\begin{aligned} C_3^{(4)} \stackrel{2N+2}{=} & \bar{K}_0^{(4)} 4 \cdot \beta_N^3 \cdot \frac{u_2(N+1) \cdot u_2^3(N)}{t} \\ & - \bar{K}_0^{(4)} \cdot \frac{4}{N} \cdot \left[\beta_N^2 \cdot C_1^{(2)} \cdot u_2^2(N) - 2\beta_N \cdot C_2^{(2)} \cdot \frac{u_2(N+1)u_2(N)}{t} \right] \\ & + \frac{2}{3} \cdot \left[2\beta_N^3 \cdot C_0^{(2)} \cdot u_2^3(N) \cdot F_N - 2\beta_N^3 \cdot C_1^{(2)} \cdot u_2^3(N) \right. \\ & \cdot F_{N+1} - 6\beta_N \cdot C_2^{(2)} \cdot \frac{u_2^2(N+1) \cdot u_2(N)}{t^2} \cdot F_N \Big] \\ & - \frac{N\lambda^2}{3} \cdot \left[3\beta_N^3 \cdot \frac{u_2(N+1) \cdot u_2^2(N)}{t} \cdot t^{N+1} \cdot F_N + \beta_N^4 \cdot \frac{u_2^3(N)}{t} t^{N+2} \cdot F_{N+1} \right], \end{aligned} \tag{G.4}$$

$$\begin{aligned} C_4^{(4)} \stackrel{2N+3}{=} & -\bar{K}_0^{(4)} \cdot \beta_N^4 \cdot u_2^4(N) - \bar{K}_0^{(4)} \cdot \frac{4}{N} \cdot \beta_N^2 \cdot C_2^{(2)} \cdot u_2^2(N) \\ & + \frac{2}{3} \cdot \left[\beta_N^3 \cdot C_1^{(2)} \cdot u_2^3(N) \cdot F_N + 2\beta_N^2 \cdot C_2^{(2)} \cdot \frac{u_2(N+1) \cdot u_2^2(N)}{t} \right. \\ & \left. \cdot F_N + \beta_N^2 \cdot C_2^{(2)} \cdot u_2^2(N) \right] + \frac{N\lambda^2}{3} \cdot \beta_N^4 \cdot u_2^3(N) \cdot t^{N+1} \cdot F_N. \end{aligned} \tag{G.5}$$

In order to construct the full $C_m^{(4)}$, the expressions above are series expanded up to the order of palindromy, with palindromy determining the rest of the terms. The palindromy points of the $C_m^{(4)}$ are given as follows: $m = 0 : 2N + 1$, $m = 1 : 2N + 1$, $m = 2 : 2N + 2$, $m = 3 : 2N + 2$, $m = 4 : 2N + 3$. Therefore, the expressions above give all terms to all $C_m^{(4)}$ except for the middle term of $C_2^{(4)}$ at order $2N + 2$, which is determined such that all terms in $f_N^{(4)}$ cancel up to and including $2N + 3$.

Note that while these $C_m^{(4)}$ guarantee that all terms will vanish up to and including $2N + 3$, it is not obvious that the expansion at order $2N + 4$ will match the expansion of $f_N^{(4)}$, even though it is the case.

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