DISORDER SOLUTIONS ON SPIN MODELS

2 ND UC Conference on statistical mechanics.
March 26 - 29, 1986, Davis California

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A number of papers have appeared in the last fifteen years that deal with disorder points of particular two-dimensional lattice models (Stephenson 1970, Welberry and Galbraith 1973, Welberry and Miller 1978, Verhagen 1976, Enting 1977 and 1978, Rujan 1982, Peschel and Rys 1982, Dhar 1982). Very different techniques have been used to obtain these solutions : methods related to crystal growth (Enting 1978, Welberry and Miller 1978), the use of the statistical theory of Markov processes (Verhagen 1978) or else based on the transfer matrix method of statistical mechanics (Rujan 1982, Baxter 1984).

Because of these various technics, methods and presentations of these disorder solutions on different domains of (mathematical) physics (crystallography, statistical mechanics, ...) there can exist at first sight some feeling of confusion : there is a need to extract the most simple and intrinsic concepts to understand these solutions. Indeed we will try to show that all these methods rely on the same very simple mechanism : a certain local decoupling of the spin degrees of freedom which results in an effective reduction of dimensionality for the spin system. Such a property will be provided by a simple localcondition bearing on the Boltzmann weight of the elementary cell generating the lattice.

We should first explain this terminology of disorder points, solutions... .

Stephenson introduced this terminology on the simplest example of such disorder solution : the two-dimensional anisotropic triangular Ising model (for antiferromagnetic coupling constants).

Stephenson remarked on this model, where correlations can be calculated, that there is a special point, a special temperature, in the paramagnetic (disordered) phase at which the behaviour of correlation functions changes from monotonic to oscillatory.

Let us consider for instance a two point intra-row correlation function, for a temperature less than the disorder temperature T_D it has the usual exponential decrease((up to some power law); for a temperature higher than T_D this exponential decrease is modulated by some oscillatory factor, at the disorder point precisely one has a typical one-dimensional behaviour for the correlation function.

Actually this terminology isn't so bad; if you consider a specific two points correlation function. The critical temperature corresponds to a temperature where the long range order disappear: there remains only a short range order. The disorder temperature corresponds to a temperature where even that short range order vanishes.

In that framework we can say that this point corresponds to the highest disorder for the system. Of course it is possible to give a great number of definitions of the disorder points; here we will not deal with the physical but rather the mathematical aspects of these solutions. So our definition of disorder points will be the following: some special points in the parameter space of the model for which some effective reduction of dimensionality for the spin system occur that enable us eventually to calculate different quantities like partition function, correlation functions... .

For the simplicity of the exposal let us first concentrate on a specific (but very interesting) important model: the q-state checker board scalar Potts model (fig 1). The lattice can be seen as a chess-board with black and white elementary square. The Boltzmann weight of a black elementary cell is given by:

$$W(\sigma_i, \sigma_j, \sigma_k, \sigma_l) = e^{K_1 \delta_{\sigma_i, \sigma_j} + K_2 \delta_{\sigma_j, \sigma_k} + K_3 \delta_{\sigma_k, \sigma_l} + K_4 \delta_{\sigma_l, \sigma_i}}$$

σ_i denotes the spin of the Potts model, it can take q colours

δ denotes the usual Kronecker symbol

K_i denotes the four coupling constants of the model corresponding to nearest neighbour interactions : if two nearest neighbour spin σ_i and σ_j are in the same state, the same colour the Boltzmann weight corresponding to the bond $\langle ij \rangle$ is e^{K_i} if not it is normalized to 1 .

The partition of the model is given by $Z = \sum_{\{\sigma\}} \prod_{\blacksquare} W(\sigma_i, \sigma_j, \sigma_k, \sigma_l)$

where $\sum_{\{\sigma\}}$ denotes the sum over all the spin configuration on the lattice, $\prod_{\blacksquare}$ the product over all the elementary black squares of the previous Boltzmann weight.

Of course a similar definition can be given if one replaces the black by the white elementary cells (this corresponds to permute K_1 and K_3 and in the same time K_2. and K_4).

Let us impose the following local condition on the previous "black" Boltzmann weight W :

$$\sum_{\sigma_j, \sigma_k} W(\sigma_i, \sigma_j, \sigma_k, \sigma_l, a, b, c, d) = \lambda(a, b, c, d)$$

when one sums over the spin configurations of the two spins at the top of the square σ_j and σ_k, the Boltzmann weight must be independent of the spin configurations of the two spins at the bottom of the square σ_i and σ_l.

Introducing the notations a, b, c, d for the exponential of the coupling constants e^{K_1}, ... e^{K_4}, it is a straightforward matter to see that condition (1) reads

$$\frac{\frac{1}{d} - 1}{\frac{1}{d} + q - 1} = \frac{a - 1}{a + q - 1} \; \frac{b - 1}{b + q - 1} \; \frac{c - 1}{c + q - 1} \qquad (2)$$

with the following expression for λ :

$$\lambda = \frac{(a - 1)(b - 1)(c - 1)}{\frac{1}{d} - 1}$$

The specialists of the Potts model have recognized the expression of the high-temperature variable (sometimes called transmissivity) $t = \frac{a - 1}{a + q - 1}$.

Let us now impose particular boundary conditions for the lattice ; on the upper layer , all K_4 interactions are missing, so that the spins of the upper layer only interact with those below. It immediately follows that if one sums over all the spins of the upper layer and if one requires the disorder condition (1) (that is (2)) the same boundary conditions reappear for the next layer : the disorder condition (1) means exactly that when one sums over the spin at the top of the black square, the two spins at the bottom no longer interact, therefore the coupling constants K_4 vanishes and we recover the same boundary conditions. One can iterate this procedure immediately and "eat" that way the whole lattice. What remains of this decimation procedure is just a multiplicative factor for each elementary black square. We get that way, immediately the partition function per site of the checkerboard Potts model restricted to the disorder condition (2)(fig1a)

$$Z_{site} = \lambda^{\frac{1}{2}}$$

Therefore we see that the partition function of that two dimensional lattice reduces to the partition function of an isolated elementary black square.

A similar decimation procedure can be performed to calculate correlation functions. Let us notice equation (1) also means that if one sums over the spin configurations of the two spins at the bottom of a white square, the Boltzmann weight of that white square must be independent of the spin configurations of the two spins at the top of that square. So we can integrate from the top row of spins downwards recursively "eating away" the black squares and from the bottom layer upwards by integrating over white squares. We put a cross on the square that are "eaten away" in that procedure (fig 1b.c). We see that two different cases must be distinguished : the light cone correlation functions for which the correlations are the same as those on a string and the other one which are the same as the one on a finite lattice. Therefore we see that an infinite number of correlation functions can be calculated exactly when we restrict ourselves to the disorder condition (2).

We thus have a fantastic amount of informations restricted to the disorder condition (2). We can illustrate this fact for an important subcase of the previous Potts model : the (q=2) checkerboard Ising model.

In that simple case an exact calculation of the susceptibility restricted (to the disorder condition) can even be performed : let us denote t_i (i=1, ... 4) the four high temperature variables of that checkerboard Ising model. One disorder condition for the checkerboard Ising model is given by $t_1 t_2 t_3 + t_4 = 0$. Some tedious algebra that will not be explained here leads to a rather compact expression for the susceptibility restricted to the previous disorder condition :

$$\chi = \frac{(1+t_1)(1+t_2)(1+t_3)(1+t_1 t_2 t_3)}{(1-t_1 t_2)(1-t_1 t_3)(1-t_2 t_3)}$$

This is a remarkably simple expression for a very complex quantity like the susceptibility : one remarks in particular an unexpected symmetry of permutation between t_1, t_2, t_3.

All these remarkable exact expressions restricted to the disorder points fully legitimate to build a perturbation theory around these disorder solutions.

For simplicity let us just consider the triangular limit of the checkerboard Ising model ($t_4 = 1$, $K_4 = \infty$).

Let us write the Boltzmann weight associated with each elementary cell of the triangular lattice as

$$W = e^{K_1 \sigma_2 \sigma_3 + K_2 \sigma_1 \sigma_3 + K_3 \sigma_1 \sigma_2} = \lambda \cdot (1 + T_1 \sigma_2 \sigma_3 + T_2 \sigma_1 \sigma_3 + T_3 \sigma_1 \sigma_2)$$

σ_i is an Ising variable ($\sigma_i = \pm 1$).

The three variables $T_i = \dfrac{t_i + t_j t_k}{1 + t_1 t_2 t_3}$ (i,j,k=1,2,3) are three new variables well suited to an expansion in the vicinity of the disorder varieties. Indeed the equations of the three disorder varieties are given by the vanishing condition of these T_i, ($T_i = 0$) .

A new diagrammatic expansion originates from the expansion of the product over all elementary cells . The partition function (or the correlation functions) are thus given as the sum of closed diagrams connected or not , which pass at most once on each hatched triangle (fig 2) .

Because of this constraint and these new variables , this new diagrammatic differs in nature from the usual hight temperature one. In particular no self-intersection is allowed. A complete resummation of all these diagrams, making use of the well-known Vdovitchenko - Feynmann counting rule gives the following expression for the partition function per site :

$$\ln \lambda + \frac{1}{8\pi^2} \int_0^{2\pi}\!\!\int_0^{2\pi} dq_1\, dq_2 \ln \Big[1 + T_1^2 + T_2^2 + T_3^2 + 2(T_2 T_3 - T_1)\cos q_1 + 2(T_1 T_3 - T_2)\cos q_2 + 2(T_1 T_2 - T_3)\cos(q_1 + q_2) \Big] \qquad (3)$$

it is straightforward but tedious matter to check that this expression identifies with the exact expression known for the partition function of the anisotropic triangular Ising model. One verifies easily that the critical variety is given by the simple equation : $T_1 + T_2 + T_3 = 1$

This new diagrammatic is the natural one for the study of the vicinity of the disorder varieties. Indeed remarkably the expansion of (3) in powers of T_3 leads to algebraic expressions at all orders in T_3 ; This is in contrast with the expansion of the partition function in the high temperature variable t_3 which leads, even at small orders, to complicated elliptic functions.

From a diagrammatic point of view let us consider the class of diagrams corresponding to a fixed order in T_3. Remarkably enough it is possible to exhibit closed recursion relations in each of these classes separately and also to give a physical interpretation to these classes of diagrams. For the first order in T_3 this corresponds to the discretized time evolution of two random walkers on a one-dimensional lattice : at each step the walkers must either jump to a neighbouring site with probability T_1 for a left jump, T_2 for a right jump, or die with probability $1 - (T_1 + T_2)$. A particular history for the two walkers corresponds to a diagram of the previous class on the triangular lattice (fig 2b). The generating function of the diagrams of this class satisfies a simple algebraic equation deduced from some closed recursion relations :

$$T_1T_2 \cdot g^2(T_1, T_2) + (T_1^2 + T_2^2 - 1) \cdot g(T_1, T_2) + T_1T_2 = 0$$

This generating function g which can also be obtained from the expansion of (3) at first order in T_3, has a non analytic behaviour when $T_1 + T_2 \longrightarrow 1$:

$$1 - g \sim (1 - T_1 - T_2)^{1-\alpha} \qquad \text{with} \qquad \alpha = \frac{1}{2}$$

In fact the occurence and the location of this transition are nothing but a consequence of the original transition of the Ising model itself: when $T_3 \longrightarrow 0$ the critical variety reduces to $T_1 + T_2 = 1$.

Secondly the $\alpha = \frac{1}{2}$ algebraic singularity was already known to occur for the triangular Ising model (Blöte and Hilhorst 1982). In fact the vicinity of the disorder varieties, asymptotic to the critical one, is the only region of the parameter space where the usual logarithmic singularity of the Ising model ($\alpha = 0$) is replaced by another singular behaviour (namely an hidden $\alpha = \frac{1}{2}$ singularity).

In conclusion we would like to underline the fact that very important and drastic physical or analytical consequences can be deduced from the extreme simplification of the model on (or in the vicinity) of these disorder varieties Let us consider the phase diagram in the parameter space of a model; the different phases are separated by the critical varieties on which the partition function (and other quantities) are non analytical. In the case of the anisotropic triangular Potts model for instance, we have represented the critical variety of the model (fig 3) given by an algebraic equation in the three high temperature variables of the model. For that specific model the partition function can even be calculated exactly and is given by some infinite eulerian product (some elliptic function). But we have also exhibited for that very model some algebraic varieties (in that parameter space) called the disorder varieties on which (and in the vicinity of which) the partition function is analytical and has a very simple analytical expression. It is difficult to imagine what could be the analytical behaviour of the partition function on the intersection of these two algebraic varieties : in fact it can be seen that both varieties avoid remarkably and intersect only for trivial values of the parameters of the model.

It should be noted that all the presently known cases of such regular exact solutions correspond either to no intersection as previously seen or to a multicritical (tricritical for instance) intersection for which a remarkable cancellation of the different singular part occur. In any cases this regular exact solutions give extremely precious insights on the phase diagram of the model (fig 4). Many other constraints or consequences can be deduced from these solutions

In particular it can be interesting to combine these exact solutions with the other exact informations we have at our disposal on a model : for instance the partition function of the checkerboard Potts model satisfies an exact functional relation, the inversion relation :

$$Z (a, b, c, d) . Z (2 - q - a, \frac{1}{b}, 2 - q - c, \frac{1}{d}) = (a-1)(1-q-a)(c-1)(1-q-c)$$

and some obvious symmetries deduced from the symmetry of the lattice :

$$Z (a, b, c, d) = Z (c, b, a, d) = \ldots$$

Because of these functional equations the partition function can be seen as some automorphic function with respect to an infinite discrete group of symmetries G acting in the parameter space (a, b, c, d) :

$$G \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus Z$$

Another interesting example is the checkerboard Ising model which is an exactly solvable (free fermion model) in terms of some complicated elliptic functions: the modulus of these elliptic functions is

$$k = \frac{\prod_{i=1}^{4} t_i \left(1 - t_i^{2} \right) \left(t_i^{*} + t_j^{*} t_k^{*} t_l^{*} \right)}{\prod_{i=1}^{4} t_i^{*} \left(1 - t_i^{*2} \right) \left(t_i + t_j t_k t_l \right)}$$

where $t_i^{*} = e^{-2K_i}$ denote the dual variables of the high-temperature variables of the model. One remarks easily that this modulus trivializes on the disorder varieties of the model and this explains how quite sophisticated elliptic expressions trivialize to the previously noticed simple rational expressions.

These disorder varieties are deeply related to the "good variables" one has to introduce on an exactly solvable model.

Finally we should underline the fact that all the results and ideas developped here are not restricted to the two dimensional models with nearest neighbour interactions (and without magnetic field).

We give to the reader as an exercice to find disorder solutions for the checkerboard Ising model with a magnetic field, a cubic Ising model with three parameters or a face centered cubic Ising model with two parameters, for instance, (see fig 5): all one has to do is to consider the appropriate elementary cell and introduce a disorder condition on that elementary cell by saying that when one sums over the spin configurations at the top of the cell the Boltzmann weight is independent of the spin configurations at the bottom. That is your turn to play on your favorite model !

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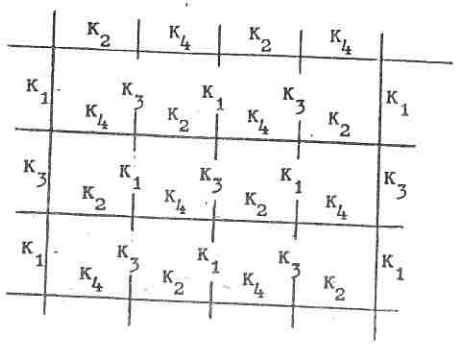
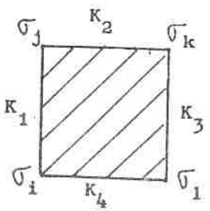
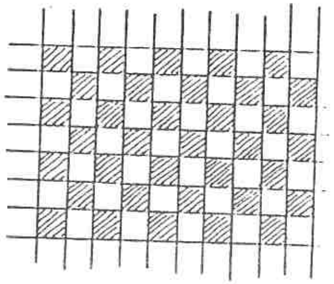
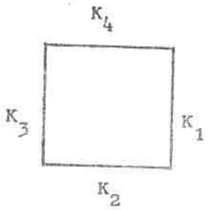
Checkerboard Q-state scalar Potts model







$\sigma_i \in Z_q$



$Z = \sum_{\{\sigma\}} \prod_{▨} W(\sigma_i, \sigma_j, \sigma_k, \sigma_l)$

$W(\sigma_i, \sigma_j, \sigma_k, \sigma_l) = e^{K_1\delta_{\sigma_i\sigma_j} + K_2\delta_{\sigma_j\sigma_k} + K_3\delta_{\sigma_k\sigma_l} + K_4\delta_{\sigma_l\sigma_i}}$

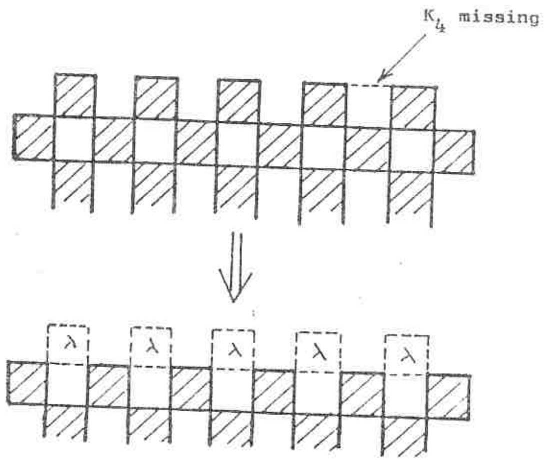
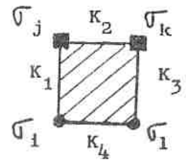
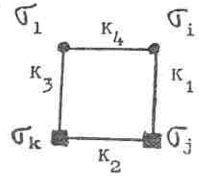


fig 1a

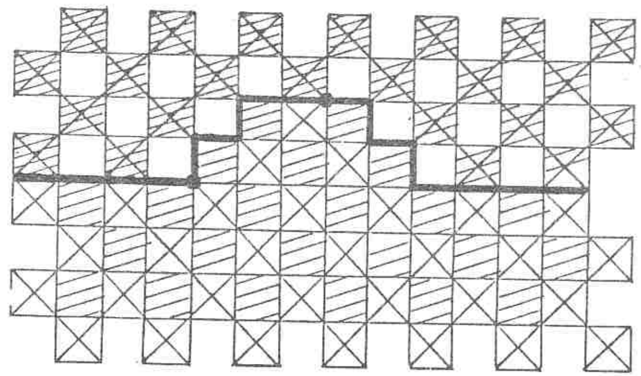
A similar decimation procedure can be performed to calculate correlation functions.



$= \lambda$ independent of σ_i, σ_l $=$



■ summation over the spin configuration



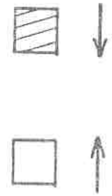
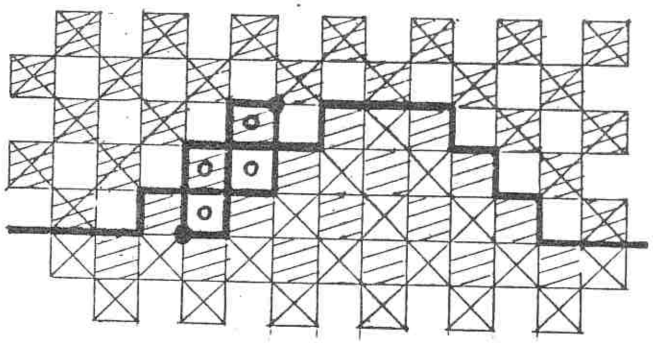


fig 1b



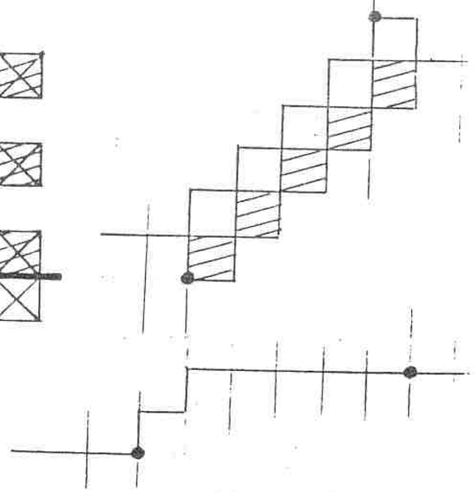
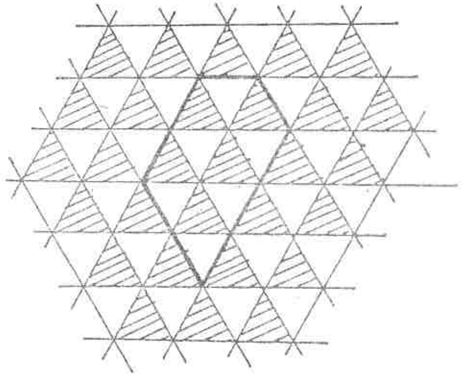
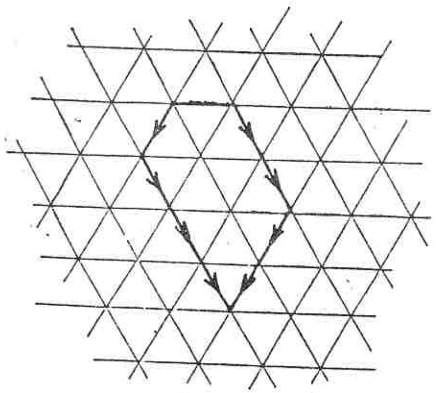


fig 1c



closed diagrams which pass at most once on each hatched triangle

fig 2



a particular history for the two walkers corresponds to a diagram of the first order in T_3

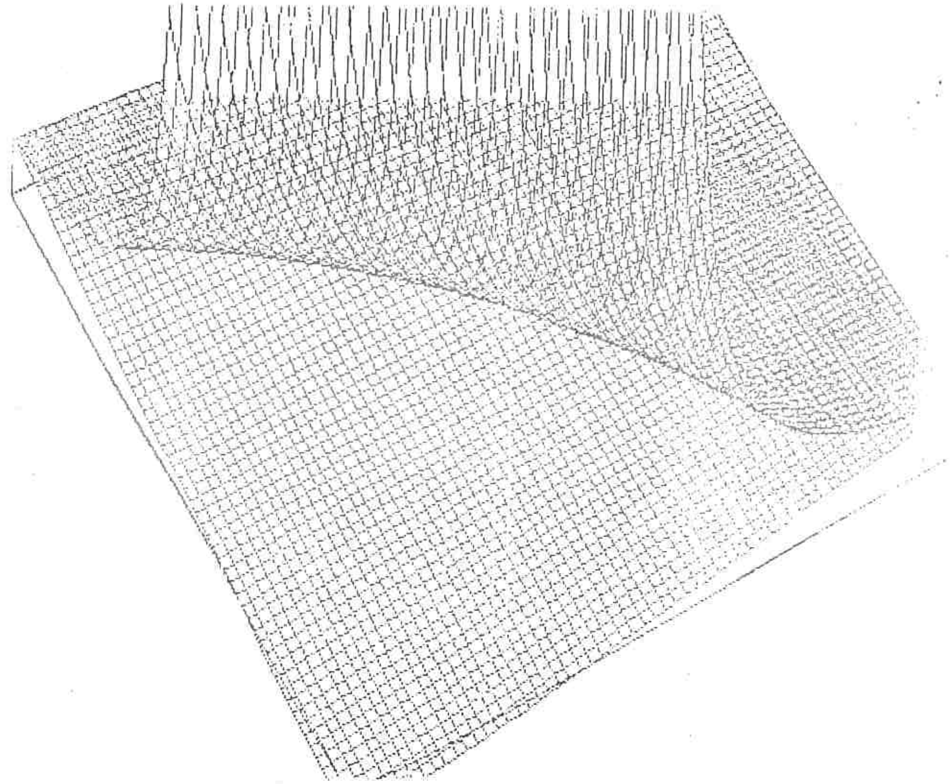
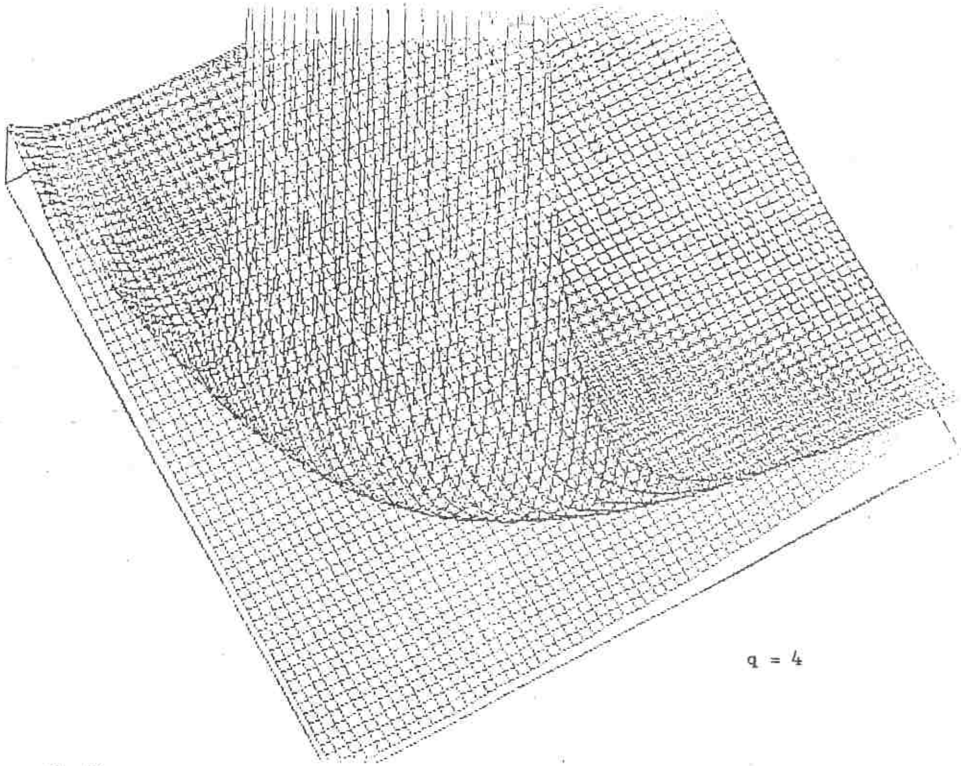


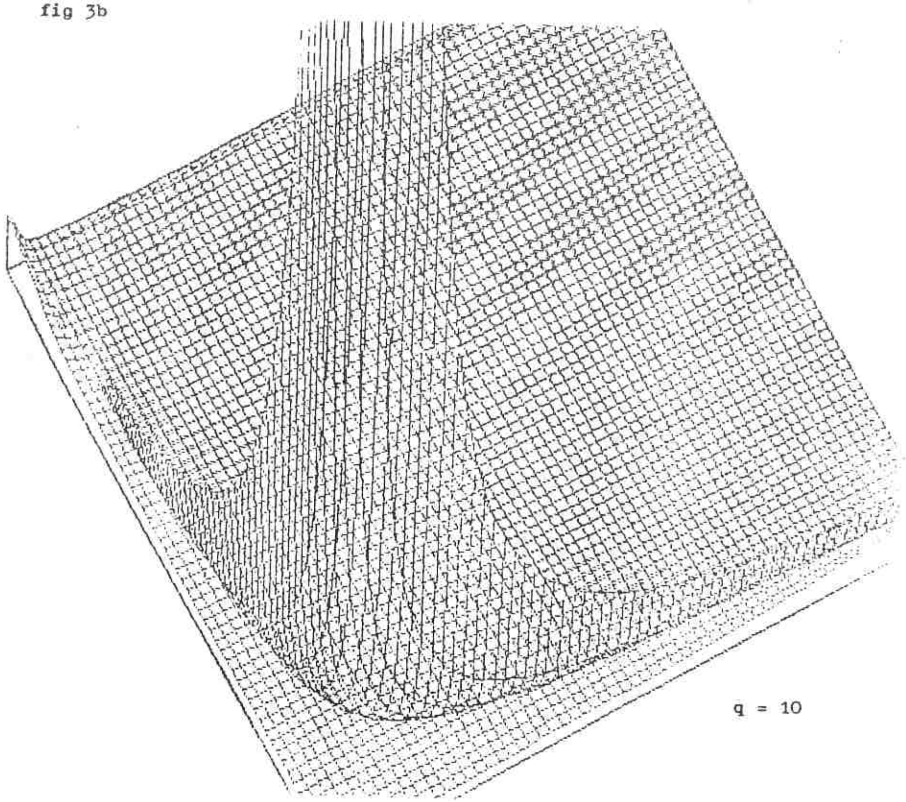
fig 3a $q = 2$

Parameter space of the anisotropic triangular q state scalar Potts model : the three high temperature variables are in $[-1/q, 1]$. One has represented the critical variety and (below) the disorder variety of the model for $q = 2$, $q = 4$ and $q = 10$.



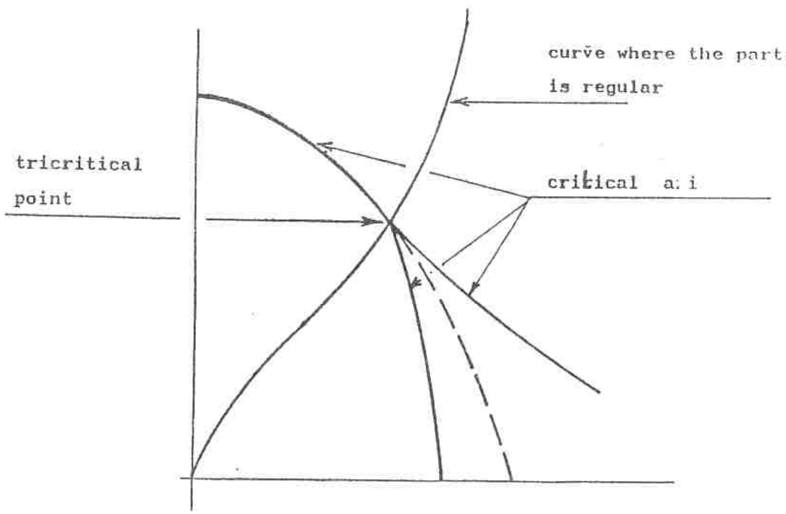
q = 4

fig 3b



q = 10

multicritical intersection.

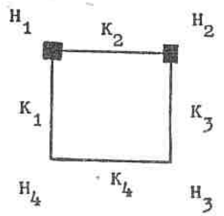


Example : Nishimori line on the spin-glass problem

fig 4

Exercise: ■ sommation over the spin configuration

i)

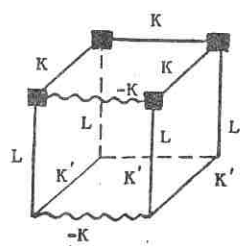


Checkerboard Ising model with a magnetic field

$H = H_1 + H_3 = H_2 + H_4$

H_i auxiliary magnetic fields

ii)

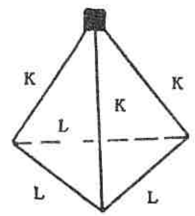


cubic Ising model with three coupling constants

disorder condition: $\text{th}2K' + \text{th}^2 L . \text{th}2K = 0$

〰 : negative bounds

iii)



Ising model on the f. c. c. lattice

hint: use the star triangle relation

fig 5