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Ising n-fold integrals as diagonals of rational functions and integrality of series expansions

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Abstract

We show that the n-fold integrals $\chi^{(n)}$ of the magnetic susceptibility of the Ising model, as well as various other n-fold integrals of the 'Ising class', or n-fold integrals from enumerative combinatorics, like lattice Green functions, correspond to a distinguished class of functions generalizing algebraic functions: they are actually *diagonals of rational functions*. As a consequence, the power series expansions of the, analytic at $x = 0$, solutions of these linear differential equations 'derived from geometry' are *globally bounded*, which means that after just one rescaling of the expansion variable, they can be cast into series expansions with *integer coefficients*. We also give several results showing that the unique analytical solution of Calabi–Yau ODEs and, more generally, Picard–Fuchs linear ODEs with solutions of maximal weights are always diagonals of rational functions. Besides, in a more enumerative combinatorics context, generating functions whose coefficients are expressed in terms of nested sums of products of binomial terms can also be shown to be *diagonals of rational functions*. We finally address the question of the relations between the notion of *integrality* (series with integer coefficients, or, more generally, globally bounded series) and the *modularity* of ODEs.

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Contents

1. Introduction

The series expansions of many magnetic susceptibilities (or many other quantities like the spontaneous magnetization) of the Ising model on various lattices in arbitrary dimensions are actually series with *integer coefficients* [1–3]. This is a consequence of the fact that in a van der Waerden-type expansion of the susceptibility, all the contributing graphs are the ones with exactly two odd-degree vertices and the number of such graphs is an integer. When series expansions in theoretical physics, or mathematical physics, do not have such an obvious counting interpretation, the puzzling emergence of series with *integer coefficients* is a strong indication that some fundamental structure, symmetry, concept has been overlooked and that a deeper understanding of the problem remains to be discovered[6]. Algebraic functions are known to produce series with *integer coefficients*. Eisenstein's theorem [5] states that the Taylor series of a (branch of an) algebraic function can be recast into a series with integer coefficients, up to a rescaling by a constant (Eisenstein constant). An intriguing result due to [6] (see pp 368–73) states that a power series with *integer* coefficients and radius of convergence (at least) one is either rational or transcendental. This result also appears in Pólya and Szegö's famous Aufgaben book [7] (see problem VIII-167). Pólya [8] conjectured a stronger result, namely a power series with integer coefficients which converges in the open unit disc is either rational or admits the *unit circle as a natural boundary* (i.e. it has no analytic continuation beyond the unit disc). This was eventually proved[7] by Carlson [10]. Along this natural boundary line, it is worth recalling [11–13, 15, 16] that the series expansions of the full magnetic susceptibility of the 2D Ising model [17] correspond to a power series with *integer* coefficients[8]. For them, the unit circle certainly arises as a natural boundary [18] (with respect to the modulus variable k), but, unfortunately, this cannot be justified by Carlson's theorem[9].

A series with natural boundaries *cannot be D-finite*[10], i.e. solution of a linear differential equation with polynomial coefficients [22, 23]. For simplicity, let us restrict to series with *integer coefficients* (or series that have integer coefficients up to a variable rescaling) that are series expansions of D-finite functions. Wu *et al* [24] have shown that the previous full magnetic susceptibility of the 2D Ising model can be expressed (up to a normalization factor $(1-s)^{1/4}/s$, see [13, 25]) as an infinite sum of n-fold integrals, denoted by $\tilde{\chi}^{(n)}$, which are *actually D-finite*[11]. We found out that the corresponding (minimal-order) differential operators are Fuchsian [11, 13], and, in fact, 'special' Fuchsian operators: the critical exponents for *all* their singularities are *rational numbers*, and their Wronskians are Nth roots of *rational functions* [26]. Furthermore, it has been shown later that these $\tilde{\chi}^{(n)}$ are, in fact, solutions of *globally nilpotent* operators [27], or G-operators [28, 29]. It is worth noting that the series expansions, at the origin, of the $\tilde{\chi}^{(n)}$, in a well-suited variable [13, 25] w, actually have *integer coefficients*, even if this result does not have an immediate proof[12] for all integers n (in contrast with the full susceptibility). From the first truncated series expansions of $\tilde{\chi}^{(n)}$, the coefficients

[6] The emergence of *positive integer* coefficients corresponds to the existence of some underlying measure [4].

[7] The Pólya–Carlson result can be used to prove that some integer sequences, such as the sequence of prime numbers (p_n) [9], do not satisfy any linear recurrence relation with polynomial coefficients.

[8] In some variable w [11–13, 15]. In the modulus variable k, one needs to perform a simple rescaling by a factor 2 or 4 according to the type of (high- or low-temperature) expansions.

[9] The radius of convergence is 1 with respect to the modulus variable k in which the series *does not have* integer coefficients, being *globally bounded* only (this means that it can be recast into a series with integer coefficients by one rescaling of the variable k). If one considers the series expansion with respect to another variable (such as w) in which the series *does have* integer coefficients, then the radius of convergence is not 1.

[10] D-finite series are sometimes called *holonomic*. *A priori*, for multivariate functions, these notions differ [19]. The equivalence of these notions is proved by deep results of Bernšteĭn [20] and Kashiwara [21].

[11] For Ising models on higher dimensional lattices [1–3], no such decomposition of susceptibilities, as an infinite sum of D-finite functions, should be expected at first sight.

[12] We are interested in this paper in the emergence of *integers* as coefficients of D-finite series. In general, this emergence is *not obvious*: it cannot be simply explained at the level of the linear recurrence satisfied by the coefficients, as illustrated by the case of Apéry's calculations (see also section 6.2 and appendix D).

for generic n can be inferred [27]:

$$\tilde{\chi}^{(n)}(w) = 2^n \cdot w^{n^2} \cdot \big(1 + 4n^2 \cdot w^2 + 2 \cdot (4n^4 + 13n^2 + 1) \cdot w^4 + \tfrac{8}{3} \cdot (n^2+4)(4n^4 + 23n^2 + 3) \cdot w^6 + \tfrac{1}{3} \cdot (32n^8 + 624n^6 + 4006n^4 + 8643n^2 + 1404) \cdot w^8 + \tfrac{4}{15} \cdot (n^2+8) \cdot (32n^8 + 784n^6 + 6238n^4 + 16\,271n^2 + 3180) \cdot w^{10} + \cdots\big). \quad (1)$$

Note that the coefficients of $\tilde{\chi}^{(n)}(w)/2^n$, which depend on n^2, are *integer coefficients* when n is *any integer*, this integrality property of the coefficients for any integer n being not straightforward (see [19]). These coefficients are valid up to w^2 for $n \geqslant 3$, w^4 for $n \geqslant 5$, w^6 for $n \geqslant 7$, w^8 for $n \geqslant 9$ and w^{10} for $n \geqslant 11$ (in particular, it should be noted that $\tilde{\chi}^{(n)}$ is an even function of w only for even n). Further studies on these $\tilde{\chi}^{(n)}$ showed the fundamental role played by the theory of elliptic functions[13] (elliptic integrals, *modular forms*) and, much more unexpectedly, *Calabi–Yau ODEs* [30, 31]. These recent structure results thus suggest to see the occurrence of series with *integer coefficients* as a consequence of *modularity* [32] (modular forms, mirror maps [30–33], etc) in the Ising model.

Along this line, many other examples of series with *integer coefficients* emerged in mathematical physics (differential geometry, lattice statistical physics, enumerative combinatorics, *replicable functions*[14], etc). One must, of course, also recall Apéry's results [39]. Appendix A gives a list of *modular forms* and their associated series with integer coefficients, corresponding to various lattice Green functions [40–43], which are often expressed in terms of HeunG functions[15], which can be written as hypergeometric functions with *two alternative pullbacks* (see also sections 6.1 and 6.2). Let us underline, in appendix A, the *Green function for the diamond lattice* [43], the *Green function for the face-centred cubic (fcc) lattice* (see equation (19) in [43]) and more examples corresponding to the spanning tree generating functions [44] (and Mahler measures). This *integrality* is also seen in the *nome* and in other quantities like the *Yukawa coupling* [30].

In this paper, we restrict to series with *integer coefficients*, or, more generally, *globally bounded* [45] series of *one complex variable*, but it is clear that this integrality property does also occur in physics with *several complex variables*: they can, for instance, be seen for the previous (D-finite[16]) n-fold integrals $\tilde{\chi}^{(n)}$ for the anisotropic Ising model [46] (or for the Ising model on the checkerboard lattice), or the example of the lattice Ising models with a magnetic field[17] (see, for instance, [4]).

We take here a learn-by-example approach: on such quite technical questions, one often gets a much deeper understanding from highly non-trivial examples than from sometimes too general, or slightly obfuscated, mathematical demonstrations.

The main result of this paper will be to show that the $\tilde{\chi}^{(n)}$ are *globally bounded* series, as a consequence of the fact that they are actually *diagonals of rational functions for any value of the integer n*. We will generalize these ideas and show that an extremely large class of problems of mathematical physics can be interpreted in terms of *diagonals of rational functions*: n-fold integrals with an algebraic integrand of a certain type that we will characterize, Calabi–Yau

[13] Which is not a surprise for Yang–Baxter integrability specialists.

[14] The concept of replicable functions is closely related to *modular functions* [34] (see the replicability of Hauptmoduls), Calabi–Yau threefolds and more generally the concept of *modularity* [32, 35–38].

[15] Generically, HeunG functions are far from being modular forms.

[16] For several complex variables, the ODEs of the paper are replaced by Picard–Fuchs systems.

[17] Along this line, original alternative representations of the partition function of the Ising model in a magnetic field are also worth recalling [47].

ODEs, maximal unipotent monodromy (MUM) linear ODEs [48], series whose coefficients are *nested sums of products of binomials*, etc.

Another purpose of this paper is to 'disentangle' the notion of series with integer coefficients (*integrality*) and the notion of *modularity* [32, 35–38, 49, 50]. In this 'down-to-earth' paper, we essentially restrict to Picard–Fuchs ODEs and to a 'Calabi–Yau' framework; therefore, modularity[18] will just mean that the series solutions of Picard–Fuchs ODEs, *as well* as the corresponding nome series and the Yukawa series have *integer* coefficients.

This paper is organized as follows. Section 2 introduces the main concepts we need in this very paper, namely the concept of *diagonals of rational or algebraic functions*, and the concept of *globally bounded series*, recalling that diagonals of rational or algebraic functions are necessarily globally bounded series. Section 3 shows the main result of this paper, namely the n-fold Ising integrals $\tilde{\chi}^{(n)}$ are *diagonals of rational functions* for any value of the integer n, the corresponding series being, thus, globally bounded. Section 4 shows that series with (nested sums of products of) *binomial* coefficients are diagonals of rational functions. Section 5 discusses, in the most general framework, the conjecture that D-finite globally bounded series could be necessarily diagonals of rational functions. Section 6 provides a set of *modular form* examples (in particular *lattice Green functions*; see appendix A). Beyond modular forms, using new determinantal identities on the Yukawa couplings, and focusing on Hadamard products of modular forms, section 7 analyses the difference between integrality and modularity showing that the two concepts are actually quite different. Section 8 addresses, more specifically, the Calabi–Yau modularity, and the difference between integrality and modularity, underlining that the *integrality of the nome* series is crucial for modularity, the integrality of the Yukawa series being *not* sufficient. The conclusion, section 9, emphasizes the difference between the 'special properties' of *geometrical* nature and those of *arithmetic* nature, emerging in theoretical physics. Several large appendices provide detailed examples illustrating pedagogically the previous sections. In particular, appendix A provides many modular form examples associated with *lattice Green functions* and appendix E provides new representations of *Yukawa couplings as ratios of determinants*.

2. Series integrality, diagonal of rational functions

Let us recall some concepts that will be fundamental in this paper: first the notion of *globally bounded series*, and, then, the concept of the *diagonal* of a function[19], and some of its most important properties. The main reason to introduce this concept of the *diagonal* of a function, not very familiar to physicists, is that it enables consideration of *diagonals of rational functions*, this class of functions filling the gap between algebraic functions and G-series: they can be seen as *generalizations of algebraic functions*. Thus, this class of functions can play a *key role in deciphering the complexity of functions occurring in theoretical physics*.

2.1. Globally bounded series

Let us first recall the definition of being *globally bounded* [45] for a series. Consider a series expansion with rational coefficients, with a non-zero radius of convergence[20]. The series is

[18] Modularity is a wider concept than this 'Calabi–Yau' modularity (see modular up to a Tate twist, modular Galois representations [51]). Modular forms provide the simplest examples (see appendix A) of modularity (see also Serre's modularity conjecture and the Taniyama–Shimura conjecture). For a first introduction to these ideas, see [52].

[19] The functions are in fact defined by *series* of several complex variables: they have to be Taylor, or Laurent, series (no Puiseux series).

[20] A series like the Euler series $\sum_{n=0}^{\infty} n! \cdot x^n$ which has integer coefficients is excluded.

said to be globally bounded if there exists an integer N such that the series can be recast into a series with integer coefficients with just one rescaling $x \rightarrow Nx$.

A necessary condition for being globally bounded is that only a finite number of primes occurs as factors of the denominators of the rational number series coefficients. There is also a condition on the growth of these denominators that must be bounded exponentially [45] in such a way that the series has a non-zero p-adic radius of convergence for all primes p. When this is the case, it is easy to see that these series can be recast, with just one rescaling, into series with *integer coefficients*[21].

2.2. Definition of the diagonal of a rational function

Assume that $\mathcal{F}(z_1, \ldots, z_n) = P(z_1, \ldots, z_n)/Q(z_1, \ldots, z_n)$ is a rational function, where P and Q are polynomials of $z_1, \ldots, z_n$ with *rational coefficients* such that $Q(0, \ldots, 0) \neq 0$. This assumption implies that $\mathcal{F}$ can be expanded at the origin as a Taylor series with rational number coefficients

$$\mathcal{F}(z_1, z_2, \ldots, z_n) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} F_{m_1,\ldots,m_n} \cdot z_1^{m_1} \cdots z_n^{m_n}. \tag{2}$$

The *diagonal of* $\mathcal{F}$ is defined as the series of *one variable*

$$\mathsf{Diag}(\mathcal{F}(z_1, z_2, \ldots, z_n)) = \sum_{m=0}^{\infty} F_{m,m,\ldots,m} \cdot z^m. \tag{3}$$

More generally, one can define, in a similar way, the diagonal of any multivariate power series $\mathcal{F}$, with rational number coefficients, or with coefficients in a finite field[22].

2.3. Main properties of diagonals

The concept of the diagonal of a function has a lot of interesting properties (see, for instance, [54]). Let us recall, through examples, some of the most important ones.

The study of diagonals goes back, at least, to [55], in a combinatorial context, and to [56] in an analytical context *related to Hadamard products* [57]. Pólya showed that the diagonal of a rational function in *two variables* is always an *algebraic function*. The most basic example is $\mathcal{F} = 1/(1 - z_1 - z_2)$ for which

$$\mathsf{Diag}(\mathcal{F}) = \mathsf{Diag}\left(\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\binom{m_1+m_2}{m_1} \cdot z_1^{m_1} z_2^{m_2}\right)$$
$$= \sum_{m=0}^{\infty}\binom{2m}{m} \cdot z^m = \frac{1}{\sqrt{1-4z}}. \tag{4}$$

The proof of Pólya's result is based on the simple observation that the diagonal $\mathsf{Diag}(\mathcal{F})$ is equal to the coefficient of z_1^0 in the expansion of $\mathcal{F}(z_1, z/z_1)$. Therefore, by Cauchy's integral theorem, $\mathsf{Diag}(\mathcal{F})$ is given by the contour integral

$$\mathsf{Diag}(\mathcal{F}) = \frac{1}{2\pi \mathrm{i}} \oint_{\gamma} \mathcal{F}(z_1, z/z_1) \frac{\mathrm{d}z_1}{z_1}, \tag{5}$$

where the contour γ is a small circle around the origin. Therefore, by Cauchy's residue theorem, $\mathsf{Diag}(\mathcal{F})$ is the sum of the residues of the rational function $\mathcal{G} = \mathcal{F}(z_1, z/z_1)/z_1$ at all

[21] For a first set of series with integer coefficients, see appendix A, where a set of such series with integer coefficients corresponding to *modular forms* is displayed. See also sections 6.1 and 6.2.
[22] The definition even extends to multivariate Laurent power series; see e.g. [53].

its singularities $s(z)$ with a zero limit at $z = 0$. Since the residues of a rational function of two variables are algebraic functions, $\mathsf{Diag}(\mathcal{F})$ is itself an algebraic function.

For instance, when $\mathcal{F} = 1/(1 - z_1 - z_2)$, then $\mathcal{G} = \mathcal{F}(z_1, z/z_1)/z_1$ has two poles at $s = \frac{1}{2}(1 \pm \sqrt{1-4z})$. The only one approaching zero when $z \to 0$ is $s_0 = \frac{1}{2}(1 - \sqrt{1-4z})$. If $p(s)/q(s)$ has a simple pole at s_0, then its residue at s_0 is $p(s_0)/q'(s_0)$. Therefore,

$$\mathsf{Diag}(\mathcal{F}) = \frac{1}{2\pi \mathrm{i}} \oint_{\gamma} \frac{\mathrm{d}z_1}{z_1 - z_1^2 - z} = \frac{1}{1 - 2s_0} = \frac{1}{\sqrt{1-4z}}. \tag{6}$$

2.3.1. Diagonals of rational functions of more than two variables. When passing from *two to more variables*, diagonalization may still be interpreted using contour integration of a multiple complex integral over a so-called *vanishing cycle* [58]. However, the result *is not* an algebraic function anymore. A simple example is $\mathcal{F} = 1/(1 - z_2 - z_3 - z_1z_2 - z_1z_3)$, for which

$$\mathsf{Diag}(\mathcal{F}) = 1 + 4z + 36z^2 + 400z^3 + 4900z^4 + 63\,504z^5 + \cdots \tag{7}$$

is equal to the complete elliptic integral of the first kind

$$\mathsf{Diag}(\mathcal{F}) = \sum_{m \geqslant 0} \binom{2m}{m}^2 \cdot z^m = {}_2F_1\left(\left[\frac{1}{2}, \frac{1}{2}\right], [1]; 16z\right), \tag{8}$$

which is a *transcendental* function. A less obvious example (see [59] for a related example with a combinatorial flavour) is

$$\mathsf{Diag}\left(\frac{1}{1 - z_1 - z_2 - z_3 - z_1z_2 - z_2z_3 - z_3z_1 - z_1z_2z_3}\right) = \frac{1}{1-z} \cdot {}_2F_1\left(\left[\frac{1}{3}, \frac{2}{3}\right], [1]; \frac{54z}{(1-z)^3}\right). \tag{9}$$

It was shown by Christol [60–62] that the diagonal $\mathsf{Diag}(\mathcal{F})$ of *any* rational function $\mathcal{F}$ is *D-finite* in the sense that it satisfies a linear differential equation with polynomial coefficients[23]. Moreover, the diagonal of any algebraic power series with rational coefficients is a G-function *coming from geometry*, i.e. it satisfies the Picard–Fuchs-type differential equation associated with some one-parameter family of algebraic varieties. Diagonals of algebraic power series thus appear to be a *distinguished class* of G-functions[24]. It will be seen below (see section 2.5) that algebraic functions with n variables can be seen as diagonals of rational functions with $2n$ variables. Thus, diagonals of rational functions also appear to be a *distinguished class* of G-functions. It is worth noting that this distinguished class is stable by the Hadamard product: the *Hadamard product of two diagonals of rational functions is the diagonal of a rational function*.

An immediate but important property of diagonals of rational functions, with rational number coefficients, is that they are *globally bounded*, which means that they have *integer coefficients* up to a simple change of variable $z \to Nz$, where $N \in \mathbb{Z}$.

2.3.2. Diagonals of rational functions modulo primes. Furstenberg [65] showed that the diagonal of *any multivariate rational power series* with coefficients *in a field of positive characteristic is algebraic*. Deligne [58] and Adamczewski and Bell [53] extended this result

[23] A more general result was proved by Lipshitz [63]: *the diagonal of any D-finite series is D-finite*; see also [64].
[24] Such diagonals are solutions of G-operators. They are functions that are *always algebraic modulo any prime p*. They fill the gap between algebraic functions and G-series: they can be seen as *generalizations of algebraic functions*.

to diagonals of algebraic functions. For instance, when $\mathcal{F} = 1/(1 - z_2 - z_3 - z_1z_2 - z_1z_3)$, one obtains modulo 7

$$\mathsf{Diag}(\mathcal{F}) \bmod 7 = 1 + 4z + z^2 + z^3 + 4z^7 + 2z^8 + 4z^9 + \cdots = \frac{1}{\sqrt[6]{1 + 4z + z^2 + z^3}} \mod 7. \qquad (10)$$

More generally, in this example, for any prime p, one obtains

$$\mathsf{Diag}(\mathcal{F}) = P(z)^{1/(1-p)} \mod p, \qquad (11)$$

where the polynomial $P(z)$ is nothing but [66–68]

$$P(z) = {}_2F_1\left(\left[\frac{1}{2}, \frac{1}{2}\right], [1]; 16z\right)^{1-p} \mod p = \sum_{n=0}^{(p-1)/2} \binom{p - 1/2}{n}^2 \cdot (16z)^n. \qquad (12)$$

Note, however, that the Furstenberg–Deligne result [65, 58], which we illustrate here with $\mathcal{F} = 1/(1 - z_2 - z_3 - z_1z_2 - z_1z_3)$, goes *far beyond the case of hypergeometric functions* for which simple closed formulae can be displayed.

2.4. Hadamard product and other products of series

Let us also recall the notion of the *Hadamard product* [57, 69] of two series, which we will denote by a star.

$$\text{If} \quad f(x) = \sum_{n=0}^{\infty} a_n \cdot x^n, \qquad g(x) = \sum_{n=0}^{\infty} b_n \cdot x^n, \quad \text{then}$$

$$f(x) \star g(x) = \sum_{n=0}^{\infty} a_n \cdot b_n \cdot x^n. \qquad (13)$$

The notion of the diagonal of a function and the notion of the Hadamard product are obviously related:

$$\mathsf{Diag}(f_1(x_1) \cdot f_2(x_2) \cdots f_n(x_n)) = f_1(x) \star f_2(x) \star \cdots \star f_n(x). \qquad (14)$$

In other words, the *diagonal of a product* of functions with separate variables is equal to the *Hadamard product* of these functions in a common variable. In particular, the Hadamard product of n rational (either algebraic or even D-finite) power series is D-finite[25].

The Hadamard product of two series with integer coefficients is straightforwardly a series with integer coefficients. Furthermore, the *Hadamard product of two operators*, annihilating two series, defined as the (minimal-order, monic) linear differential operator annihilating the Hadamard product of these two series, is a *product compatible with a large number of structures and concepts*[26] that naturally occur in lattice statistical mechanics. We have a similar compatibility property between the diagonal and the Hurwitz product [19, 70].

2.5. Furstenberg's result on algebraic functions

It was shown by Furstenberg [65] that *any algebraic series* in one variable can be written as the *diagonal of a rational function of two variables*. The basis of Furstenberg's result is the

[25] The Hadamard product of rational power series is still rational, but the Hadamard product of algebraic series is in general transcendental.
[26] For instance, the Hadamard product of two globally nilpotent [27] operators is *also globally nilpotent*.

fact that if $f(x)$ is a power series without a constant term and is a root of a polynomial $P(x, y)$ such that $P_y(0, 0) \neq 0$, then

$$f(x) = \mathrm{Diag}\left(y^2 \cdot \frac{P_y(xy, y)}{P(xy, y)}\right), \qquad \text{where} \qquad P_y = \frac{\partial P}{\partial y}. \tag{15}$$

When $P_y(0, 0) = 0$, formula (15) is not true anymore. However, Furstenberg's result still holds [19].

Note that this representation as the diagonal of a rational function is by no means unique as can be seen in the algebraic function[27]

$$f = \frac{x}{\sqrt{1-x}} = x + \frac{1}{2}x^2 + \frac{3}{8}x^3 + \frac{5}{16}x^4 + \frac{35}{128}x^5 + \frac{63}{256}x^6 + \cdots \tag{16}$$

which is the diagonal of $(2xy - cx + cy)/(x + y + 2)$ for *any rational number* c.

Furstenberg's proof *does not necessarily produce the simplest rational function* (see [19]).

Furstenberg's result has been generalized to power series expansions of algebraic functions in an *arbitrary number of variables* n: any *algebraic power series*[28] with rational coefficients is the diagonal of a rational function with $2n$ variables (see [71]).

3. Selected *n*-fold integrals are diagonals of rational functions

Among many multiple integrals that are important in various domains of mathematical physics, and before considering other n-fold integrals of the 'Ising class[29]', let us first consider the n-particle contribution to the magnetic susceptibility of the Ising model which we denote $\tilde{\chi}^{(n)}(w)$. They are given by $(n-1)$-dimensional integrals [11, 72]:

$$\tilde{\chi}^{(n)}(w) = \frac{(2w)^n}{n!}\left(\prod_{j=1}^{n-1}\int_0^{2\pi}\frac{\mathrm{d}\Phi_j}{2\pi}\right)\cdot Y \cdot \frac{1+X}{1-X}\cdot X^{n-1}\cdot G, \tag{17}$$

where, defining Φ_0 by $\sum_{i=0}^{n-1}\Phi_i = 0$, we set

$$X = \prod_{i=0}^{n-1} x_i, \qquad x_i = \frac{2w}{A_i + \sqrt{A_i^2 - 4w^2}}, \qquad Y = \prod_{i=0}^{n-1} y_i, \qquad y_i = \frac{1}{\sqrt{A_i^2 - 4w^2}},$$

$$G = \prod_{0\leqslant i<j\leqslant n-1}\frac{2 - 2\cos(\Phi_i - \Phi_j)}{(1 - x_i x_j)^2}, \qquad \text{where} \qquad A_i = 1 - 2w\cos(\Phi_i). \tag{18}$$

The integrality property (1) had been checked [12] for the first $\tilde{\chi}^{(n)}$ and inferred [27] for generic n. We are going to *prove it*[30] for any integer n, showing a very fundamental result, namely all the $(n-1)$-fold integrals $\tilde{\chi}^{(n)}$ *are very special*: they are actually *diagonals of rational functions*.

3.1. $\tilde{\chi}^{(3)}$ *as a toy example*

At first sight, the $\tilde{\chi}^{(n)}$ are involved transcendental holonomic functions. Could it be possible that they correspond to the *distinguished class* [53] of G-functions, generalizing algebraic functions, which have an interpretation as diagonals of multivariate algebraic functions

[27] Here, f is annihilated by $P(x, y) = (1-x)y^2 - x^2$, which is precisely such that $P_y(0, 0) = 0$.
[28] In the one-variable case, Puiseux series could be considered but only after ramifying the variable.
[29] Using the terminology introduced by Bailey *et al* [14], see also [15].
[30] For $\tilde{\chi}^{(n)}$, the rescaling factor ('Eisenstein constant') is 2 or 4 according to the fact that one considers high- or low-temperature series [12, 26].

(and consequently diagonals of rational functions with twice as many variables)? If this is the case, then the series of $\tilde{\chi}^{(n)}$ must *necessarily reduce modulo any prime to an algebraic function* (see section 2.3.2). $\tilde{\chi}^{(1)}$ and $\tilde{\chi}^{(2)}$ contributions being too degenerate (a rational function and too simple an elliptic function), let us consider the first non-trivial case, namely $\tilde{\chi}^{(3)}$. Its series expansion has already been displayed in [11]. It reads $\tilde{\chi}^{(3)}/8 = w^9 \cdot F(w)$ with

$$F(w) = 1 + 36w^2 + 4w^3 + 884w^{13} + 196w^5 + 18\,532w^6 + 6084w^7 + \cdots$$

Since we have obtained the exact ODE satisfied by $\tilde{\chi}^{(3)}$, we can produce as many coefficients as we want in its series expansion. Let us consider this series modulo the prime $p = 2$. It now reads as the lacunary series

$$F(w) \bmod 2 = 1 + w^8 + w^{24} + w^{56} + w^{120} + w^{248} + w^{504} + w^{1016} + \cdots,$$

solution of the functional equations on $F(w)$ or, with $z = w^8$, on $G(z) = 1 + w^8 \cdot F(w)$

$$F(w) = 1 + w^8 \cdot F(w^2), \qquad G(z) = z + G(z^2), \tag{19}$$

where one recognizes, with equation $G(z) = z + G(z^2)$, Furstenberg's example [65] of the simplest algebraic function in characteristic 2.[31] In fact, $H(w) = w^9 F(w)$ is the solution of the quadratic equation

$$H(w)^2 + w \cdot H(w) + w^{10} = 0 \bmod 2. \tag{20}$$

The calculations are more involved modulo $p = 3$. Indeed, $H(w) = \tilde{\chi}^{(3)}(w)/8$ satisfies, modulo 3, the polynomial equation of degree 9

$$p_9 \cdot H(w)^9 + w^6 \cdot p_3 \cdot H(w)^3 + w^{10} \cdot p_1 \cdot H(w) + w^{19} \cdot p_0^{(1)} \cdot p_0^{(2)} = 0, \tag{21}$$

where

$$\begin{aligned}
p_9 &= (w+1)^3(w^2+1)^{18}(w-1)^{24},\\
p_3 &= (w^2+1)^{18}(1-w)^{15}(w^4-w^2-1), \qquad p_1 = (w^2+1)^{20}(1-w)^{13},\\
p_0^{(1)} &= w^6 + w^5 + w^4 - w^2 - w + 1,\\
p_0^{(2)} &= w^{37} - w^{36} + w^{35} - w^{33} + w^{31} - w^{30} + w^{28} + w^{27} + w^{24} - w^{23} + w^{22}\\
&\quad - w^{21} - w^{18} - w^{16} + w^{14} - w^{12} - w^{11} - w^{10} + w^7 - w^5 - w^3 - 1.
\end{aligned} \tag{22}$$

The calculations are even more involved modulo larger primes. The (minimal-order) linear differential operator annihilating the $\tilde{\chi}^{(3)}$ series mod 5, reads[32]

$$\begin{aligned}
&(x+1)(x^2+x+1)(x+2) \cdot x^4 \cdot D_x^4 + 2x^3 \cdot (x^3 + 2x^2 + 4x + 4)(x+4) \cdot D_x^3\\
&\quad + x^2 \cdot (x^4 + 3x^3 + 4) \cdot D_x^2 + 4 \cdot (x^4 + 3) \cdot x \cdot D_x + 3.
\end{aligned} \tag{23}$$

If one can easily obtain this linear differential operator, finding the minimal polynomial of $\tilde{\chi}^{(3)}$ modulo 5, generalizing (20) or (21) or rather the polynomial $\tilde{P}(\kappa, w)$, where $\kappa = \tilde{\chi}^{(3)}(w)/w^9$, such that $\tilde{P}(\kappa, w) = 0 \bmod 5$, requires a *very large* number of coefficients. The polynomial [19] $\tilde{P}(\kappa, w)$ is of degree 50 in κ, of degree 832 in x and is the sum of 4058 monomials.

One can imagine, in a first step, that the $\tilde{\chi}^{(3)}$ series mod *any prime p* are *also algebraic functions*, and, in a second step, that $\tilde{\chi}^{(3)}$ may be the diagonal of a rational function. In fact, we are going to show, in the next section, a stronger result: the $\tilde{\chi}^{(n)}$ are *actually diagonals of rational functions, for any integer n*.

[31] Modulo the prime $p = 2$, the previous functional equation becomes $G(z) = z + G(z)^2$.
[32] This operator is of zero 5-curvature [27].

3.2. The $\tilde{\chi}^{(n)}$ are diagonals of rational functions

Let us, now, consider the general case where n is an arbitrary integer. With the change of variable $z_k = \exp(\mathrm{i}\Phi_k)$ (where $\mathrm{i}^2 = -1$), one clearly obtains

$$\prod_{k=0}^{n-1} z_k = 1, \qquad \frac{\mathrm{d}z_j}{z_j} = \mathrm{i}\mathrm{d}\Phi_j,$$
$$2\cos(\Phi_k) = z_k + \frac{1}{z_k}, \qquad 2\cos(\Phi_k - \Phi_j) = \frac{z_k}{z_j} + \frac{z_j}{z_k}, \tag{24}$$

and (17) becomes

$$\tilde{\chi}^{(n)}(w) = \frac{(2w)^n}{n!}\left(\prod_{j=1}^{n-1} \frac{1}{2\mathrm{i}\pi}\oint_{\mathcal{C}} \frac{\mathrm{d}z_j}{z_j}\right) \cdot F(w, z_1, \ldots, z_{n-1}), \tag{25}$$

where $\mathcal{C}$ is the path 'turning once counterclockwise around the unit circle' and F is algebraic over $\mathbb{Q}(w, z_1, \ldots, z_{n-1})$ and reads

$$F(w, z_1, \ldots, z_{n-1}) = Y \cdot X^{n-1} \cdot \frac{1+X}{1-X} \cdot G. \tag{26}$$

Now, let us suppose that F *is analytic*[33] *at the origin*, namely it has a Taylor expansion (2). Then applying $(n-1)$ times the residue formula, one finds

$$\tilde{\chi}^{(n)}(w) = \mathsf{Diag}\left(\frac{(2z_0\cdots z_{n-1})^n}{n!} \cdot F(z_0\cdots z_{n-1}, z_1, \ldots, z_{n-1})\right). \tag{27}$$

To check that this is actually true, we introduce an auxiliary set, namely $\mathcal{T}_n$ being the subset of Laurent series $\mathbb{Q}\big[z_1, \ldots, z_{n-1}, z_1^{-1}, \ldots, z_{n-1}^{-1}\big][[w]]$, consisting of series

$$f(w, z_1, \ldots, z_{n-1}) = \sum_{m=0}^{\infty} P_m \cdot w^m,$$

where P_m belongs to $\mathbb{Q}\big[z_1, \ldots, z_{n-1}, z_1^{-1}, \ldots, z_{n-1}^{-1}\big]$ and is such that the degree of P_m, in each of z_k^{-1}, is at most m.

Then to prove that $F(z_0z_1\ldots z_{n-1}, z_1, \ldots, z_{n-1})$ has a Taylor expansion, we only have to verify that $F(w, z_1, \ldots, z_{n-1})$ belongs to this auxiliary set $\mathcal{T}_n$. Checking this is a straightforward step-by-step computation on auxiliary functions:

$$A_k = 1 - w\cdot\left(z_k + \frac{1}{z_k}\right), \qquad \text{for} \quad 1 \leqslant i \leqslant n-1,$$
$$A_k = 1 - w\cdot\left(\frac{1}{z_1\cdots z_{n-1}} + z_1\cdots z_{n-1}\right), \qquad \text{for} \quad i = 0.$$

Hence, A_k belongs to this auxiliary set $\mathcal{T}_n$. So to be sure that the inverse or the square root of some function in this auxiliary set $\mathcal{T}_n$ is also in this auxiliary set $\mathcal{T}_n$, we only have to check that its first Taylor coefficient is actually 1, w, w^2 or w^n, n integer. It is straightforward to see that

$$A_k^2 - 4w^2 = 1 - 2w\cdot\left(z_k + \frac{1}{z_k}\right) + w^2\cdot\left(z_k - \frac{1}{z_k}\right)^2,$$
$$\text{hence} \quad \sqrt{A_k^2 - 4w^2} = 1 + \cdots, \tag{28}$$

[33] One could consider Laurent, instead of Taylor, expansions, but this is a slight generalization [53, 73, 74].

$$y_k = \frac{1}{\sqrt{A_k^2 - 4w^2}} = 1 + \cdots, \qquad Y = 1 + \cdots,$$

$$x_k = \frac{2w}{A_k + \sqrt{A_k^2 - 4w^2}} = w + \cdots, \qquad x_k x_j = w^2 + \cdots,$$

$$X = w^n + \cdots, \qquad \frac{1+X}{1-X} = 1 + \cdots,$$

$$G = \prod_{0 \leqslant k < j \leqslant n-1} \left(\frac{-(z_k - z_j)^2}{(1 - x_k x_j)^2 \cdot z_k z_j} \right) = \prod_{0 \leqslant k < j \leqslant n-1} \left(\frac{-(z_k - z_j)^2}{z_k z_j} \right) + \cdots.$$

Thus, this shows that F belongs to the auxiliary set $\mathcal{T}_n$:

$$F(w, z_1, \ldots, z_{n-1}) = \sum_{m=0}^{\infty} \sum_{m_1=-m}^{M_1(m)} \cdots \sum_{m_{n-1}=-m}^{M_{n-1}(m)} a_{m,m_1,\ldots,m_{n-1}} \cdot w^m \cdot z_1^{m_1+m} \ldots z_{n-1}^{m_{n-1}+m}. \qquad (29)$$

Consequently, it makes sense to take its diagonal. The residue theorem requires searching for the terms not containing z_j, i.e. such as $m_1 = \cdots = m_{n-1} = 0$. One therefore obtains

$$\tilde{\chi}^{(n)} = \sum_{m=0}^{\infty} a_{m,0,\ldots,0} \cdot w^m. \qquad (30)$$

In particular, we find

$$\tilde{\chi}^{(n)} = \mathsf{Diag}\left(\sum_{m=0}^{\infty} \sum_{m_1=-m}^{M_1(m)} \cdots \sum_{m_{n-1}=-m}^{M_{n-1}(m)} a_{m,m_1,\ldots,m_{n-1}} \cdot z_0^m z_1^{m_1+m} \ldots z_{n-1}^{m_n-1+m} \right)$$
$$= \mathsf{Diag}(F(z_0 z_1 \ldots z_{n-1}, z_1, \ldots, z_{n-1})). \qquad (31)$$

As F is algebraic, $\tilde{\chi}^{(n)}$ *is the diagonal of an algebraic function* of n variables and, consequently, *the diagonal of a rational function* of $2n$ variables.

We thus see that we can actually *find explicitly* the algebraic function such that its diagonal is the n-fold integral $\tilde{\chi}^{(n)}$: it is *nothing but the integrand* of the n-fold integral, up to simple transformations (namely $F(w, z_1, \ldots, z_{n-1}) \rightarrow F(z_0 z_1 \ldots z_{n-1}, z_1, \ldots, z_{n-1})$).

Remark. $\tilde{\chi}^{(n)}$ is a solution of a linear differential equation and has a radius of convergence equal to $1/4$ in w. Among the other solutions of this equation, there is the function obtained by changing the square root appearing in x_k into its opposite. *A priori,* there are 2^n ways to do this, hence 2^n new solutions but not all distinct. At first sight, for these new solutions, x_k are no longer in $\mathcal{T}_n$.

In fact, we find some quite interesting structure. Let us consider, for instance, the case of $\tilde{\chi}^{(3)}$. If one considers other choices of sign in front of the nested square roots in the integrand, the series expansions of the corresponding n-fold integrals read

$$w + 6w^2 + 28w^3 + 124w^4 + 536w^5 + 2280w^6 + 9604w^7 + 40\,164w^8 + 167\,066w^9 + 692\,060w^{10} + 2857\,148w^{11} + \cdots$$

$$w^2 + 6w^3 + 30w^4 + 140w^5 + 628w^6 + 2754w^7 + 11\,890w^8 + 50\,765w^9 + 214\,958w^{10} + 904\,286w^{11} + \cdots.$$

One remarks that *all these alternative series* are, as $\tilde{\chi}^{(3)}$, series with *integer coefficients* [19].

3.3. More n-fold integrals of the Ising class and a simple integral of the Ising class

It is clear that the demonstration we have performed on the $\chi^{(n)}$ can also be performed straightforwardly, *mutatis mutandis*, with other n-fold integrals of the 'Ising class[34] [14, 15]' like the n-fold integrals Φ_H in [72], which amounts to getting rid of the fermionic term G (see (26)), the $\chi_d^{(n)}$ corresponding to n-fold integrals associated with the diagonal[35] susceptibility [16, 31] (the magnetic field is located on a diagonal of the square lattice), the $\Phi_D^{(n)}$ in [15] which are simple integrals, and also for all the lattice Green functions displayed in [43, 48], and the list is far from being exhaustive. For instance, the simple integral $\Phi_D^{(n)}$ is the diagonal of the algebraic function

$$\frac{2}{n!} \cdot (1-t^2)^{-1/2} \cdot \frac{G_n F_n^{n-1}}{G_n F_n^{n-1} - (2wt)^n} - \frac{1}{n!}, \quad \text{where} \tag{32}$$

$$F_n = 1 - 2w + (1 - 4w + 4w^2 - 4w^2t^2)^{1/2},$$
$$G_n = 1 - 2wt \cdot T_{n-1}\left(\frac{1}{t}\right) + \left(\left(1 - 2wt \cdot T_{n-1}\left(\frac{1}{t}\right)\right)^2 - 4w^2 \cdot t^2\right)^{1/2}, \tag{33}$$

and $T_{n-1}(t)$ is the $(n-1)$th Chebyshev polynomial of the first kind. The way we have obtained these Chebyshev results (32) is displayed in [19].

As opposed to the $\chi^{(n)}$, the integral $\Phi_D^{(n)}(w)$ is the diagonal of an algebraic function of *two* variables (and, thus, the diagonal of a rational function of *four* variables) *independently of the actual value of n*.

If the $\chi^{(n)}$ are fundamental to understand the Ising model [17, 24] or the $\chi_d^{(n)}$ have a physical meaning associated with the diagonal susceptibility [16, 31] for the Ising model, most of the n-fold integrals of the 'Ising class [14]' do not have that importance or even that physical meaning (even if they have played a crucial role in understanding the singularities of the Ising model [15]). What we see here with, for instance, the Φ_H [15, 72] is that the demonstration they are diagonals of rational functions is exactly the same as for the $\chi^{(n)}$ (see section 3.2) because a key analyticity assumption of the integrand is also fulfilled.

3.4. More general n-fold integrals as diagonals

More generally, the demonstration we have performed on the $\tilde{\chi}^{(n)}$ can be performed for *any* n-fold integral that can be recast in the following form:

$$\int_C \int_C \cdots \int_C \frac{dz_1}{z_1} \frac{dz_2}{z_2} \cdots \frac{dz_n}{z_n} \cdot \mathcal{A}(x, z_1, z_2, \ldots, z_n), \tag{34}$$

where the subscript C denotes the unit circle and $\mathcal{A}$ denotes an algebraic function of the n variables, which (this is the crucial ingredient), as a function of several variables x and z_k, has an *analytical* expansion at $(x, z_1, z_2, \ldots, z_n) = (0, 0, 0, \ldots, 0)$:

$$\mathcal{A}(x, z_1, z_2, \ldots, z_n) = \sum_{m=0}^{\infty} \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} A_{m,m_1,m_2,\ldots,m_n} \cdot z_1^{m_1} z_2^{m_2} \cdots z_n^{m_n} \cdot x^m. \tag{35}$$

Consequently, an extremely large set of n-fold integrals occurring in theoretical physics (lattice statistical mechanics, enumerative combinatorics, number theory, differential geometry, etc) can actually be seen to be *diagonals of rational functions*. These n-fold integrals

[34] For the purpose of this section, n-fold integrals of the 'Ising class' will mean n-fold integrals that are known to arise in the study of the two-dimensional Ising model susceptibility.
[35] Of course, this 'diagonal [16, 31] wording' should not be confused with the notion of diagonal of a function.

correspond to series expansions (in the variation parameter x) that are *globally bounded* (can be written after one rescaling into series with integer coefficients) and are solutions of *globally nilpotent* [27] linear differential operators.

Such a general n-fold integral is, thus, the diagonal of an algebraic function (or of a rational function with twice as many variables [71]) which is essentially the *integrand* of such an n-fold integral. Furthermore, such a general n-fold integral is the solution of a (globally nilpotent [27]) linear differential operator, which can be obtained exactly from the integrand, using the creative telescoping method [19].

Finally, in the case of Calabi–Yau ODEs (see below), these functions can be interpreted as periods of Calabi–Yau varieties, these algebraic varieties being essentially the integrand of such n-fold integrals. The *integrand* is thus the *key ingredient* to wrap, in the same bag, the *algebraic geometry viewpoint, the differential geometry viewpoint and the analytic and arithmetic approaches* (series with integer coefficients).

4. Calabi–Yau ODE solutions and series with binomials seen as diagonals

4.1. Recalling Calabi–Yau ODEs

Calabi–Yau ODEs have been defined in [75] as order-4 linear differential ODEs that satisfy the following conditions: they are MUM [76, 77], they satisfy a 'Calabi–Yau condition' which amounts to imposing that the exterior squares of these order-4 operators are of order 5 (instead of the order 6 one expects in the generic case), the series solution, analytic at $x = 0$, is globally bounded (can be reduced to integer coefficients) and the series of their nome and Yukawa coupling are globally bounded. In the literature, one also finds a cyclotomic condition on the monodromy at the point at $x = \infty$ and/or the conifold[36] character of one of the singularities [79].

Let us recall that a linear ODE has MUM [30, 78] if all the exponents at (for instance) $x = 0$ are zero. In a hypergeometric framework, the MUM condition amounts to restricting to hypergeometric functions of the type ${}_{n+1}F_n([a_1, a_2, \cdots a_n], [1, 1, \cdots 1], x)$ since the indicial exponents at $x = 0$ are the solutions of $\rho(\rho + b_1 - 1) \cdots (\rho + b_n - 1) = \rho^{n+1} = 0$, where b_j are the lower parameters which are here all equal to 1.

Let us consider a MUM order-4 linear differential operator. The four solutions y_0, y_1, y_2 and y_3 of this order-4 linear differential operator read

$$y_0, \quad y_1 = y_0 \cdot \ln(x) + \tilde{y}_1, \qquad y_2 = y_0 \cdot \frac{\ln(x)^2}{2} + \tilde{y}_1 \cdot \ln(x) + \tilde{y}_2,$$
$$y_3 = y_0 \cdot \frac{\ln(x)^3}{6} + \tilde{y}_1 \cdot \frac{\ln(x)^2}{2} + \tilde{y}_2 \cdot \ln(x) + \tilde{y}_3,$$

where $y_0, \tilde{y}_1, \tilde{y}_2$ and $\tilde{y}_3$ are analytical at $x = 0$ (with also $\tilde{y}_1(0) = \tilde{y}_2(0) = \tilde{y}_3(0) = 0$).

The nome of this linear differential operator reads

$$q(x) = \exp\left(\frac{y_1}{y_0}\right) = x \cdot \exp\left(\frac{\tilde{y}_1}{y_0}\right). \tag{36}$$

Calabi–Yau ODEs have been defined as being MUM, thus having one solution analytical at $x = 0$. As far as Calabi–Yau ODEs are concerned, the fact that this solution, analytical at $x = 0$, has an integral representation, and, furthermore, an integral representation of the form (34) together with (35), is far from clear, even if one may have a 'geometry prejudice' that

[36] The local exponents are 0, 1, 1, 2. For the cyclotomic condition on the monodromy at ∞, see proposition 3 in [78].

this solution, analytical at $x = 0$, can be interpreted as a 'period' and 'derived from geometry' [28, 29, 80].

Large tables of Calabi–Yau ODEs have been obtained by Almkvist *et al* [78, 81, 82]. It is worth noting that the coefficients A_n of the series corresponding to the solution analytical at $x = 0$ are, most of the time, *nested sums of products of binomials*, less frequently nested sums of products of binomials and of harmonic numbers[37] H_n, and, in rare cases, no 'closed formula' is known for these coefficients.

Let us show, in the case of A_n coefficients being *nested sums of products of binomials*, that the solution of the Calabi–Yau ODE, analytical at $x = 0$, which is by construction a series with integer coefficients, is *actually a diagonal of a rational function* and, furthermore, that this rational function can actually be easily built.

4.2. Calculating the rational function for the nested product of binomials

For pedagogical reasons, we will just consider here a very simple example[38] of a series $\mathcal{S}(x)$, with integer coefficients, given by a sum of products of binomials

$$\mathcal{S}(x) = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}^3 \cdot x^n = \mathrm{HeunG}(-1/8, 1/4, 1, 1, 1, 1; -x)$$
$$= 1 + 2x + 10x^2 + 56x^3 + 346x^4 + 2252x^5 + 15\,184x^6 + \cdots. \qquad (37)$$

This is the generating function of sequence **A** in Zagier's tables of binomial coefficient sums (see p 354 in [83]).

The calculations of this section can straightforwardly (sometimes tediously) be generalized to more complicated [84] nested sums of products of binomials[39].

Finding that a series is a diagonal of a rational function amounts to framing it into a residue form like (34). In order to achieve this, we write the binomial $\binom{n}{k}$ as the residue

$$\binom{n}{k} = \frac{1}{2i\pi}\cdot\int_C \frac{(1+z)^n}{z^k}\cdot\frac{dz}{z}, \qquad (38)$$

and, thus, we can rewrite $\mathcal{S}(x)$ as

$$(2i\pi)^3\cdot\mathcal{S}(x) = \sum_{n=0}^{\infty}\iiint\sum_{k=0}^{n}\frac{1}{(z_1z_2z_3)^k}\cdot((1+z_1)(1+z_2)(1+z_3)\cdot x)^n\cdot\frac{dz_1\,dz_2\,dz_3}{z_1z_2z_3}$$
$$= \iiint\sum_{n=0}^{\infty}\frac{1-(1/(z_1z_2z_3))^{(n+1)}}{1-(1/(z_1z_2z_3))}\cdot((1+z_1)(1+z_2)(1+z_3)\cdot x)^n\cdot\frac{dz_1\,dz_2\,dz_3}{z_1z_2z_3}$$
$$= -\iiint\sum_{n=0}^{\infty}\frac{z_1z_2z_3}{1-z_1z_2z_3}\cdot((1+z_1)(1+z_2)(1+z_3)\cdot x)^n\cdot\frac{dz_1\,dz_2\,dz_3}{z_1z_2z_3}$$
$$+ \iiint\sum_{n=0}^{\infty}\frac{1}{1-z_1z_2z_3}\cdot\left(\frac{(1+z_1)(1+z_2)(1+z_3)\cdot x}{z_1z_2z_3}\right)^n\cdot\frac{dz_1\,dz_2\,dz_3}{z_1z_2z_3}$$
$$= \iiint R(x; z_1, z_2, z_3)\cdot\frac{dz_1\,dz_2\,dz_3}{z_1z_2z_3}, \qquad (39)$$

where $R(x; z_1, z_2, z_3)$ reads

$$\frac{z_1z_2z_3}{(1 - x\cdot(1+z_1)(1+z_2)(1+z_3))(z_1z_2z_3 - x\cdot(1+z_1)(1+z_2)(1+z_3))}.$$

[37] The generating function of harmonic numbers is $H(x) = \sum H_n \cdot x^n = -\ln(1-x)/(1-x)$.
[38] See proposition 7.3.2 in [77].
[39] Not necessarily corresponding to modular forms as can be seen in (48) and (49).

From this last result, one deduces immediately that (37) is actually the diagonal of

$$\frac{1}{(1 - z_0 \cdot (1 + z_1)(1 + z_2)(1 + z_3)) \cdot (1 - z_0 z_1 z_2 z_3 (1 + z_1)(1 + z_2)(1 + z_3))}.$$

Note that, as a consequence of a combinatorial identity due to Strehl and Schmidt [85–87], $\mathcal{S}(x)$ can also be written as

$$\mathcal{S}(x) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{n} \cdot x^n = \sum_{n=0}^{\infty} \sum_{k=[n/2]}^{n} \binom{n}{k}^2 \binom{2k}{n} \cdot x^n. \tag{40}$$

Calculations similar to (39) on this alternative binomial representation (40) enable the expression of (37) as the diagonal of an alternative rational function

$$\frac{1}{(1 - z_0 \cdot (1 + z_1)(1 + z_2)(1 + z_3)^2) \cdot (1 - z_0 z_1 z_2 \cdot (1 + z_1)(1 + z_2))}. \tag{41}$$

We thus see that we can actually *obtain explicitly*, from straightforward calculations, the rational function (40) for the Calabi–Yau-like ODEs (occurring from *differential geometry* or *enumerative combinatorics*) when series with nested sums of binomials take place, and, more generally, for enumerative combinatorics problems (related or not to Calabi–Yau manifolds) where series with *nested sums of binomials* take place.

These effective calculations are actually algorithmic and guarantee to obtain an *explicit expression* for the rational function (40). However, the rational function is far from being unique, and worse, the number of variables the rational function depends on is far from being the smallest possible number. Finding the 'minimal' rational function (whatever the meaning of 'minimal' may be) is a very difficult problem. Appendix B provides a non-trivial illustration of this fact with explicit calculations on the well-known Apéry series and its rewriting due to Strehl and Schmidt [85–87]. We see in a crystal clear way in appendix B that when a given function is a diagonal of a rational function, the rational function is far from being unique, the 'simplest' representation (minimal number of variables, lowest degree polynomials, etc) being hard to find. Similar computations[40] show that the generating functions of sequences **B** and **E** in Zagier's list [83] are both diagonals of rational function in four variables.

All these calculations can systematically be performed on any series defined by *nested sums of products of binomials*. We have performed such calculations on a large number of the series corresponding to the list of Almkvist *et al* [78], which are given by such *nested sums of products of binomials*.

5. Comments and speculations

5.1. A theorem of [45]

In [45] (p 61 theorem 12 and theorem IX.4.2 of [29]), it is proved that any power series with an *integral representation* and of *maximal weight W* for the corresponding *Picard–Fuchs linear differential equation* is the *diagonal of a rational function* and, in particular, is *globally bounded*.

The technical nature of the original papers is such that the result itself is difficult to find. This paragraph is devoted to explaining, in down-to-earth terms, the somewhat esoteric expressions used in its wording and to explaining what it means in explicit examples. As the original proof is very obfuscated, its principle is sketched in [19].

[40] These results are given in section 5.1 of [19].

Disappointingly, when applied to a hypergeometric ${}_{n+1}F_n$, this result becomes somewhat trivial. More precisely, the hypergeometric function is of maximal weight if and only if $b_j = 1$ for all j (there are only $n!$ in the denominator of coefficients). In that case, it is obviously the Hadamard product of algebraic functions, therefore the diagonal of a rational function:

$$ {}_nF_{n-1}([\alpha_1, \alpha_2, \ldots, \alpha_n], [1, 1, \ldots 1], x) = (1-x)^{-\alpha_1} \star (1-x)^{-\alpha_2} \cdots \star (1-x)^{-\alpha_n}. \qquad (42) $$

Therefore, we now have (at least) three sets of problems yielding the diagonals of rational functions: the n-fold integrals of the form (34) with (35), the Picard–Fuchs linear ODEs with the solution of maximal monodromy weight and, finally, the problems of enumerative combinatorics where nested sums of products of binomials take place.

Diagonals of rational functions thus occur in quite a large set of problems of theoretical physics. At first sight, one can see the frequent appearance of diagonals of rational functions in physics just as a mathematical curiosity[41], and be surprised that, for instance, so many series in physics are, modulo a prime, algebraic functions. Being diagonals of rational functions is not just a mathematical curiosity: it corresponds (see the next section) to G-operators, and their *rational number exponents*, and can be seen as a first step to modularity properties (see the sections below) in some work-in-progress integrability.

5.2. A conjecture of [45]

The diagonal of a rational function is globally bounded (i.e. it has a non-zero radius of convergence and integer coefficients up to one rescaling) and *D-finite* (i.e. the solution of a linear differential equation with polynomial coefficients)[42].

The converse statement is the conjecture in [45] saying that *any D-finite, globally bounded series is necessarily the diagonal of a rational function*.

A remarkable result of Chudnovski's ([89] chapter VIII) asserts that the minimal linear differential operator of a G-function (and in particular of a D-finite globally bounded series) is a G-operator (i.e. at least a globally nilpotent operator) [27–29]. The conjecture in [45] amounts to saying something more: if the solution of this globally nilpotent linear differential operator is not only a G-series but a *globally bounded series, then it is the diagonal of a rational function*.

Conversely, the solution, analytical at $x = 0$, of a globally nilpotent linear differential operator is necessarily a G-function [28, 29]. Moreover, a 'classical' conjecture, with numerous avatars, claims that any G-function comes from geometry i.e. roughly speaking, it has an integral representation[43].

To test the validity of the conjecture of [45], we look for counterexamples not contradicting classical conjectures. For instance, we search D-finite power series with *integer coefficients* which are *not algebraic* but have an integral representation and are not of maximal weight for the corresponding Picard–Fuchs linear ODE.

As a first step, let us limit ourselves to hypergeometric functions ${}_{n+1}F_n$. The monodromy weight W denotes the number of ones among the b_i.

When ${}_{n+1}F_n$ is globally bounded and has $W = 0$, its minimal ODE has a p-curvature zero for almost all primes p. However, a Grothendieck conjecture proved for ${}_3F_2$ in [90] and generalized to ${}_{n+1}F_n$ in [91] asserts that, under these circumstances, the hypergeometric

[41] In 1944, the occurrence of elliptic functions in Onsager's solution of the Ising model was also seen as a mathematical curiosity.

[42] The series expansion of the susceptibility of the isotropic 2D Ising model can be recast into a series with *integer coefficients* (see [12, 18, 26, 88]), but it *cannot be the diagonal of a rational function* since the full susceptibility is *not a D-finite function* [88].

[43] Bombieri–Dwork conjecture; see, for instance, [29].

function is *algebraic*. We display in appendix C a set of ${}_{n+1}F_n$ hypergeometric functions which yield, naturally, series with *integer coefficients*, many of them corresponding to such *algebraic hypergeometric functions*. Even if such examples are quite non-trivial, the purpose of our paper is to focus on *transcendental* (non-algebraic) functions.

So we are looking for *globally bounded* hypergeometric functions satisfying $1 \leqslant W \leqslant n-1$. In general, such hypergeometric functions are G-series but are very far from being globally bounded. The hypergeometric world extends largely outside the world of diagonals of rational functions.

Such an example in the first case $n = 2, W = 1$ was given in [45]:

$$
{}_3F_2\left(\left[\tfrac{1}{9}, \tfrac{4}{9}, \tfrac{5}{9}\right], \left[\tfrac{1}{3}, 1\right]; 3^6 x\right) = 1 + 60x + 20\,475x^2 + 9373\,650x^3 + 4881\,796\,920x^4 + 2734\,407\,111\,744x^5 + 1605\,040\,007\,778\,900x^6 + \cdots. \tag{43}
$$

The integer coefficients read with the rising factorial (or Pochhammer) symbol

$$
\frac{(1/9)_n \cdot (4/9)_n \cdot (5/9)_n}{(1/3)_n \cdot (1)_n \cdot n!} \cdot 3^{6n} = \frac{\rho(n)}{\rho(0)}, \tag{44}
$$

where

$$
\rho(n) = \frac{\Gamma(1/9+n)\Gamma(4/9+n)\Gamma(5/9+n)}{\Gamma(1/3+n)\Gamma(1+n)\Gamma(1+n)} \cdot 3^{6n}. \tag{45}
$$

Note that, at first sight, it is *far from clear*[44] in (45), or on the simple recursion on the $\rho(n)$ coefficients (with the initial value $\rho(0) = 1$)

$$
\frac{\rho(n+1)}{\rho(n)} = 3 \cdot \frac{(1+9n)(4+9n)(5+9n)}{(1+3n)(1+n)^2}, \tag{46}
$$

to see that the $\rho(n)$ are actually integers. A sketch of the (quite arithmetic) proof that the $\rho(n)$ are actually integers is given in appendix D.

Because of the $1/3$ in the right (lower) parameters of (43), the hypergeometric function (43) is not an obvious Hadamard product of algebraic functions (and thus a diagonal of a rational function) and one can see that it is not an algebraic hypergeometric function by calculating its p-curvature and finding that it is not zero [80] (see also [91, 92]). Proving that an algebraic function is the diagonal of a rational function and proving that a solution of maximal weight for a Picard–Fuchs equation is the diagonal of a rational function use two entirely distinct ways. The hope is to combine both techniques to conclude in the intermediate situation.

This example remained for 20 years the only 'blind spot' of the conjecture in [45]. We have recently found many other ${}_3F_2$ examples[45], such that their series expansions have *integer coefficients* but are not obviously diagonals of rational functions. Some of these new hypergeometric examples[46] read, for instance,

$$
{}_3F_2\left(\left[\tfrac{1}{9}, \tfrac{2}{9}, \tfrac{7}{9}, \right], \left[\tfrac{2}{3}, 1\right], 3^6 x\right) = 1 + 21x + 5544x^2 + 2194\,500x^3 + 1032\,711\,750x^4 + 535\,163\,031\,270x^5 + 294\,927\,297\,193\,620x^6 + 169\,625\,328\,357\,359\,160x^7 + 100\,668\,944\,872\,954\,458\,000x^8 + \cdots
$$

or

$$
{}_3F_2\left(\left[\tfrac{1}{7}, \tfrac{2}{7}, \tfrac{4}{7}, \right], \left[\tfrac{1}{2}, 1\right], 7^4 x\right), \qquad {}_3F_2\left(\left[\tfrac{1}{11}, \tfrac{2}{11}, \tfrac{6}{11}, \right], \left[\tfrac{1}{2}, 1\right], 11^4 x\right). \tag{47}
$$

Unfortunately, these hypergeometric examples are on the same 'frustrating footing' as Christol's example (43): we are not able to show that one of them is actually a diagonal of a

[44] In contrast with cases where binomial (and thus integer) expressions take place.
[45] ${}_2F_1$ cases are straightforward and cannot provide counterexamples to the conjecture in [45].
[46] See also [19].

rational function, or, conversely, to show that one of them cannot be the diagonal of a rational function.

6. Integrality versus modularity: learning by examples

A large number of examples of integrality of series solutions comes from modular forms. Let us just display two such modular forms associated with HeunG functions of the form HeunG$(a, q, 1, 1, 1, 1; x)$. Many more similar examples can be found in [19].

6.1. First modular form example

One can, for instance, rewrite the example (37) of subsection 4.2, namely HeunG$(-1/8, 1/4, 1, 1, 1, 1; -x)$, as a hypergeometric function with *two rational pullbacks*:

$$\begin{aligned}
\mathrm{HeunG}(-1/8, 1/4, 1, 1, 1, 1; -x) &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}^3 x^n \\
&= ((1+4x)\cdot(1+228x+48x^2+64x^3))^{-1/4} \\
&\quad\times {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1]; \frac{1728\cdot(1-8x)^6\cdot(1+x)^3\cdot x}{(1+228x+48x^2+64x^3)^3\cdot(1+4x)^3}\right) \\
&= ((1-2x)\cdot(1-6x+228x^2-8x^3))^{-1/4} \\
&\quad\times {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1]; \frac{1728\cdot(1-8x)^3\cdot(1+x)^6\cdot x^2}{(1-6x+228x^2-8x^3)^3\cdot(1-2x)^3}\right).
\end{aligned} \tag{48}$$

The relation between the two pullbacks that are related by the 'Atkin' involution[47] $x \leftrightarrow -1/8/x$ gives the modular curve:

$$\begin{aligned}
&1953\,125y^3z^3 - 187\,500y^2z^2\cdot(y+z) + 375yz\cdot(16z^2-4027yz+16y^2) \\
&\quad - 64\cdot(z+y)\cdot(y^2+z^2+1487yz) + 110\,592\cdot yz = 0.
\end{aligned} \tag{49}$$

Series (48) is the solution of the (exactly) *self-adjoint* linear differential operator Ω where $(\theta = x\cdot D_x)$

$$x\cdot\Omega = \theta^2 - x\cdot(7\theta^2+7\theta+2) - 8x^2\cdot(\theta+1)^2. \tag{50}$$

6.2. Second modular form example

The integrality of series solutions can be quite non-trivial like the solution of the Apéry-like operator

$$\begin{aligned}
&\Omega = x\cdot(1-11x-x^2)\cdot D_x^2 + (1-22x-3x^2)\cdot D_x - (x+3), \\
&\text{or:}\quad x\cdot\Omega = \theta^2 - x\cdot(11\theta^2+11\theta+3) - x^2\cdot(\theta+1)^2,
\end{aligned} \tag{51}$$

which can be written as a HeunG function. This (at first sight involved) HeunG function reads

$$\begin{aligned}
&\mathrm{HeunG}\left(-\frac{123}{2}+\frac{55}{2}\cdot 5^{1/2}, -\frac{33}{2}+\frac{15}{2}\cdot 5^{1/2}, 1, 1, 1, 1; \left(\frac{11-5^{3/2}}{2}\right)\cdot x\right) \\
&\quad = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}\cdot x^n = 1 + 3\cdot x + 19\cdot x^2 + 147\cdot x^3 + \cdots
\end{aligned}$$

[47] In previous papers [93, 30], with some abuse of language, we called such an involution an *Atkin–Lehner involution*. In fact this terminology is commonly used in the mathematical community for an involution $\tau \to -N/\tau$, on τ, the ratio of periods, and *not* for our x-involution. This is why we switch to the wording *'Atkin' involution*.

but *actually corresponds to a modular form*, which can be written in two different ways using *two pullbacks*:

$$(x^4 + 12x^3 + 14x^2 - 12x + 1)^{-1/4} \, {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1]; \frac{1728 \cdot x^5 \cdot (1 - 11x - x^2)}{(x^4 + 12x^3 + 14x^2 - 12x + 1)^3}\right)$$
$$= (1 + 228x + 494x^2 - 228x^3 + x^4)^{-1/4}$$
$$\times \, {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1]; \frac{1728 \cdot x \cdot (1 - 11x - x^2)^5}{(1 + 228x + 494x^2 - 228x^3 + x^4)^3}\right). \quad (52)$$

Modular form examples of series with integer coefficients displayed in appendix A correspond to lattice Green functions [48]. Therefore, they have *n-fold integral representations*[48] and, *after* section 3.4, can be seen to *be diagonals of rational functions*.

7. Integrality versus modularity

7.1. Diffeomorphisms of unity pullbacks

Let us consider a first simple example of a hypergeometric function which is the solution of a Calabi–Yau ODE and which occurred at least twice in the study of the Ising susceptibility n-fold integrals [30, 31] $\chi^{(n)}$ and $\chi_d^{(n)}$, namely ${}_4F_3([1/2, 1/2, 1/2, 1/2], [1, 1, 1]; 256x)$, where we perform a (diffeomorphism of unity) pullback:

$${}_4F_3\left(\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right], [1, 1, 1]; \frac{256x}{1 + c_1x + c_2x^2 + \cdots}\right) = 1 + 16 \cdot x$$
$$+ 16 \cdot (81 - c_1) \cdot x^2 + 16 \cdot \left(10\,000 + c_1^2 - c_2 - 162c_1\right) \cdot x^3 + \cdots. \quad (53)$$

If the pullback in (53) is such that the coefficients c_n, at its denominator, are integers, one finds that the series expansion is actually a series with integer coefficients, for *every such pullback* (i.e. for every integer coefficient c_n). Furthermore, a straightforward calculation of the corresponding nome $q(x)$ and its compositional inverse (mirror map) $x(q)$ *also yields series with integer coefficients*:

$$q(x) = x + (64 - c_1) \cdot x^2 + \left(c_1^2 + 7072 - c_2 - 128c_1\right) \cdot x^3 + \cdots, \quad (54)$$

$$x(q) = q + (c_1 - 64) \cdot q^2 + \left(c_1^2 + 1120 + c_2 - 128c_1\right) \cdot q^3 + \cdots, \quad (55)$$

when its Yukawa coupling [30], seen as a function of the nome q, $K(q)$ is also a series with integer coefficients and is *independent of the pullback*:

$$K(q) = 1 + 32 \cdot q + 4896 \cdot q^2 + 702\,464 \cdot q^3 + \cdots. \quad (56)$$

This independence of the Yukawa coupling with regard to pullbacks is a known property and has been proven in [75] for any pullback of the diffeomorphism of unity form $p(x) = x + \cdots$

Seeking for Calabi–Yau ODEs, Almkvist *et al* have obtained [78] quite a large list of fourth-order ODEs, which are MUM by definition and have, by construction, the *integrality* for the solution series analytic at $x = 0$. Looking at the Yukawa coupling of these ODEs is a way to define *equivalence classes up to pullbacks* of ODEs sharing the same Yukawa coupling. This 'wraps in the same bag' all the linear ODEs that are the same *up to pullbacks*. Let us recall how difficult it is to see if a given Calabi–Yau ODE has, up to operator equivalence and up to pullback, a hypergeometric function solution [30, 31] because finding the pullback is extremely difficult [30, 31]. We may have, for the Ising model, some ${}_{n+1}F_n$ hypergeometric

[48] In contrast, the modular form examples displayed in appendix H of [19] correspond to differential geometry examples discovered by Golyshev and Stienstra [94], where no n-fold integral representation is available at first sight.

function prejudice [30, 31]: it is, then, important to have an invariant that is independent of this pullback that we cannot find most of the time.

Finally, let us remark that the Yukawa coupling is *not preserved by the operator equivalence*. Two linear differential operators, which are homomorphic, *do not necessarily have the same Yukawa coupling* (see appendix E).

7.2. Yukawa couplings in terms of determinants

Another way to understand this fundamental *pullback invariance* amounts to rewriting the Yukawa coupling [75, 95] not from the definition usually given in the literature (second derivative with respect to the ratio of periods) but in terms of determinants of solutions (Wronskians, etc) that naturally present nice covariance properties with respect to pullback transformations (see appendix E).

We have the alternative definition for the *Yukawa coupling* given in appendix E:

$$K(q) = \left(q \cdot \frac{\mathrm{d}}{\mathrm{d}q}\right)^2 \left(\frac{y_2}{y_0}\right) = \frac{W_1^3 \cdot W_3}{W_2^3}, \tag{57}$$

where the determinantal variables W_m are determinants built from the four solutions of the MUM differential operator. This alternative definition, in terms of these W_m, enables to understand the *remarkable invariance of the Yukawa coupling by pullback transformations* [31]. These determinantal variables W_m quite naturally, and canonically, yield to introduce another 'Yukawa coupling' (which, in fact, *corresponds to the Yukawa coupling of the adjoint operator* (see E.12)). This 'adjoint Yukawa coupling' is *also invariant by pullbacks*. It has, for the previous example, the following series expansion with integer coefficients:

$$K^\star(q) = 1 + 32 \cdot q + 4896 \cdot q^2 + 702\,464 \cdot q^3 + \cdots \tag{58}$$

which can actually be identified with (56). The equality of the Yukawa coupling for this order-4 operator, and for its (formal) adjoint operator, is a straightforward consequence of the fact that the order-4 operator annihilating ${}_4F_3\big(\big[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\big], [1, 1, 1]; 256x\big)$ is exactly *self-adjoint*, and, more generally, of the fact that the order-4 operator, annihilating (53), is conjugated to its adjoint by a simple function.

7.3. Modularity

This example with its corresponding relations (53), (54), (56), (58) may suggest a quite wrong prejudice that the *integrality of the solution* of an order-4 linear differential operator automatically yields to the integrality of the nome, mirror map and Yukawa coupling, which we will call, for short, '*modularity*'. This is *far from being the case* as can be seen, for instance, in the following interesting example, where the nome and Yukawa coupling $K(q)$ *do not correspond to globally bounded series*, when the ${}_4F_3$ solution of the order-4 operator and the Yukawa coupling *seen as a function of* x, $K(x)$, are, actually, both *series with integer coefficients*.

Let us consider the following ${}_4F_3$ hypergeometric function which is clearly a Hadamard product of algebraic functions and, thus, the diagonal of a rational function:

$$\begin{aligned} {}_4F_3\big(\big[\tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \tfrac{3}{4}\big], [1, 1, 1]; x\big) &= (1-x)^{-1/3} \star (1-x)^{-1/2} \star (1-x)^{-1/4} \star (1-x)^{-3/4} \\ &= \mathsf{Diag}((1-z_1)^{-1/3}(1-z_2)^{-1/2}(1-z_3)^{-1/4}(1-z_4)^{-3/4}). \end{aligned}$$

It is therefore globally bounded:

$$\begin{aligned} {}_4F_3\big(\big[\tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \tfrac{3}{4}\big], [1, 1, 1]; 2304x\big) &= 1 + 72x + 45\,360x^2 + 46\,569\,600x^3 \\ &\quad + 59\,594\,535\,000x^4 + 86\,482\,063\,571\,904x^5 + \cdots. \end{aligned} \tag{59}$$

Its Yukawa coupling, seen as a function of x, is actually a *series with integer coefficients* in x:

$$K(x) = 1 + 480x + 872\,496x^2 + 1728\,211\,968x^3 + 3566\,216\,754\,432x^4 + 7536\,580\,798\,814\,208x^5 + 16\,177\,041\,308\,360\,579\,328x^6 + \cdots . \quad (60)$$

However, do note that the series, in terms of the nome, is *not globally bounded*:

$$K(q) = 1 + 480q + 653\,616q^2 + 942\,915\,456q^3 + 1408\,019\,875\,200q^4 + \cdots + 571\,436\,303\,929\,319\,146\,711\,343\,817\,202\,689\,132\,288\frac{q^{12}}{11} + \cdots . \quad (61)$$

In fact, the nome $q(x)$ and the mirror map $x(q)$ are *also not globally bounded*. Note that in this example, the non-integrality appears at order 12 (for $x(q)$, $q(x)$ and $K(q)$). If the prime 11 in the denominator in (61) was the only one, one could recast the series into a series with integer coefficients introducing another rescaling $2304x \to 11 \times 2304x$. But, in fact, we see the appearance of an *infinite number of other primes* at higher order denominators in $x(q)$, $q(x)$ and $K(q)$.

We do not have modularity because we do not have (up to rescaling) the nome integrality: the nome series is not globally bounded.

7.4. Order-2 differential operators ω_n associated with modular forms

After Maier [96], let us underline that modular forms can be written as hypergeometric functions with *two different pullbacks* and, consequently, one can associate order-2 differential operators with these modular forms.

Let us consider the two order-2 operators

$$\omega_2 = D_x^2 + \frac{(96x+1)}{(64x+1)\cdot x}\cdot D_x + \frac{4}{(64x+1)x}, \quad (62)$$

$$\omega_3 = D_x^2 + \frac{(45x+1)}{(27x+1)\cdot x}\cdot D_x + \frac{3}{(27x+1)x}, \quad (63)$$

which are associated with two modular forms corresponding, on their associated nomes q, to the transformations $q \to q^2$ and $q \to q^3$, respectively (multiplication of τ, the ratio of their periods by 2 and 3), as can be seen in their respective solutions:

$$\begin{aligned} {}_2F_1\left(\left[\frac{1}{4}, \frac{1}{4}\right], [1]; -64x\right) &= (1+256x)^{-1/4} \cdot {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1]; \frac{1728x}{(1+256x)^3}\right) \\ &= (1+16x)^{-1/4} \cdot {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1]; \frac{1728x^2}{(1+16x)^3}\right) \\ &= 1 - 4x + 100x^2 - 3600x^3 + 152\,100x^4 - 7033\,104x^5 \\ &\quad + 344\,622\,096x^6 - 17\,582\,760\,000x^7 + 924\,193\,822\,500x^8 \\ &\quad - 49\,701\,090\,010\,000x^9 + \cdots , \end{aligned} \quad (64)$$

and

$$((1+27x)(1+243x)^3)^{-1/12} \cdot {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1]; \frac{1728x}{(1+243x)^3(1+27x)}\right)$$
$$= ((1+27x)(1+3x)^3)^{-1/12} \cdot {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1]; \frac{1728x^3}{(1+3x)^3(1+27x)}\right)$$

$$= {}_2F_1\left(\left[\frac{1}{3}, \frac{1}{3}\right], [1], -27x\right) = 1 - 3x + 36x^2 - 588x^3 + 11\,025x^4 - 223\,587x^5 + 4769\,856x^6 - 105\,423\,552x^7 + 2391\,796\,836x^8 - 55\,365\,667\,500x^9 + \cdots. \quad (65)$$

The relation between the two Hauptmodul pullbacks in (64),

$$u = \frac{1728x}{(1+256x)^3}, \qquad v = \frac{1728x^2}{(1+16x)^3} = u\left(\frac{1}{2^{12}x}\right), \quad (66)$$

corresponds to the (genus-0) fundamental modular curve:

$$5^9 \cdot u^3v^3 - 12 \cdot 5^6 \cdot u^2v^2 \cdot (u+v) + 375uv \cdot (16u^2 + 16v^2 - 4027uv) - 64(u+v) \cdot (v^2 + 1487uv + u^2) + 2^{12}3^3 \cdot uv = 0. \quad (67)$$

The relation between the two Hauptmodul pullbacks in (65),

$$u = \frac{1728x}{(1+243x)^3(1+27x)}, \qquad v = \frac{1728x^3}{(1+3x)^3(1+27x)} = u\left(\frac{1}{3^6x}\right), \quad (68)$$

corresponds to the (genus-0) modular curve:

$$2^{27}5^9 \cdot u^3v^3 \cdot (u+v) + 2^{18}5^6u^2v^2 \cdot (27v^2 + 27u^2 - 45\,946uv) + 2^93^55^3uv \cdot (u+v) \cdot (v^2 + 241\,433uv + u^2) + 729(u^4 + v^4) - 39\,628 \cdot 3^9 \cdot (u^2 + v^2) \cdot uv + 15\,974\,803 \cdot 2 \cdot 3^{10} \cdot u^2v^2 + 31 \cdot 2^93^{11}uv \cdot (u+v) - 2^{12}3^{12}uv = 0. \quad (69)$$

Similarly, one can consider the order-2 operators ω_n associated with other modular forms corresponding to $\tau \to n \cdot \tau$. The ω_n can be simply deduced from [96], for modular forms corresponding to *genus-0* curves, i.e. for $n = 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 16, 18, 25$. Since the solutions can be written as ${}_2F_1$ hypergeometric up to *rational pullbacks*, these genus-0 ω_n are obviously *order-2* operators. After a simple rescaling, the solutions, analytic at $x = 0$, can be rewritten as the series with *integer coefficients*.

One can also consider the other ω_n corresponding to *higher genus* modular curves. In these cases, one does not have a rational parametrization like (66) or (68), but *one still has an identity of the same hypergeometric function with two different pullbacks*, these two pullbacks being *algebraic functions and not rational functions* (see (66) or (68)). These algebraic functions correspond to the so-called *modular polynomials* [19].

For instance, for $\tau \to 11 \cdot \tau$, one has a *genus-1* modular curve; the modular polynomial reads

$$P_{11}^*(x, H) = (1 + 228x + 486x^2 - 540x^3 + 225x^4)^3 \cdot H^2 - 1728 \cdot \mathcal{Q}_1(x) \cdot x \cdot H + 1728^2x^{12}, \quad (70)$$

with

$$\mathcal{Q}_1(x) = 1 - 55x + 1188x^2 - 12\,716x^3 + 69\,630x^4 - 177\,408x^5 + 133\,056x^6 + 132\,066x^7 - 187\,407x^8 + 40\,095x^9 + 24\,300x^{10} - 6750x^{11}.$$

One has the identity

$${}_2F_1\left(\left[\tfrac{1}{12}, \tfrac{5}{12}\right], [1]; H_1\right) = A(x) \cdot {}_2F_1\left(\left[\tfrac{1}{12}, \tfrac{5}{12}\right], [1]; H_2\right), \quad (71)$$

where the two Hauptmoduls H_1 and H_2 are the two solutions of $P_{11}^*(x, H) = 0$ and $A(x)$ is the algebraic function such that

$$\frac{A(x)^4}{11^2} + \frac{11^2}{A(x)^4} = \frac{2}{11^2} \cdot \frac{7321 - 87\,612x + 73\,206x^2 + 21\,060x^3 - 23\,175x^4}{1 + 228x + 486x^2 - 540x^3 + 225x^4},$$

this last relation between $A(x)^4$ and x corresponding to a genus-1 curve with the *same* j*-invariant* as the genus-1 curve $P_{11}^*(x, H) = 0$, namely $j = -496^3/11^5$. The two hypergeometric functions in (71) are *actually series with integer coefficients*:

$$_2F_1([1/12, 5/12], [1]; H_1) = 1 + 60x - 4560x^2 + 614\,400x^3 - 95\,660\,400x^4 + 16\,231\,863\,060x^5 - 2905\,028\,387\,700x^6 + \cdots \quad (72)$$

$$_2F_1([1/12, 5/12], [1]; H_2) = 1 + 60x^{11} + 3300x^{12} + 110\,220x^{13} + 2904\,660x^{14} + 66\,599\,940x^{15} + 1394\,683\,620x^{16} + 27\,425\,371\,380x^{17} + \cdots. \quad (73)$$

More details are given for this $\tau \rightarrow 11 \cdot \tau$ case in appendix I of [19]. Note that although $_2F_1([1/12, 5/12], [1]; H_1)$ and $_2F_1([1/12, 5/12], [1]; H_2)$ are solutions of the *same order-4* operator [19], one can find an appropriate algebraic function $\mathcal{A}(x)$, such that $\mathcal{A}(x) \cdot {_2F_1}([1/12, 5/12], [1], H_1)$ is the solution of an *order-2* operator ω_{11} (see [19] for more details).

The other ω_n, corresponding to higher genus modular curves [97], are actually *also order-2 operators*. The explicit expressions of ω_n for the elliptic values $n = 17, 19$ and the *hyperelliptic* values [97] $n = 23, 29, 31, 41, 47, 59, 71$ are given in [19]. The geni of the associated modular curves [97] are, respectively, [19] *genus-1* for $\tilde{\omega}_{17}(x)$, $\tilde{\omega}_{19}(x)$, *genus-2* for $\tilde{\omega}_{23}(x)$, $\tilde{\omega}_{29}(x)$, $\tilde{\omega}_{31}(x)$, *genus-3* for $\tilde{\omega}_{41}(x)$, *genus-4* for $\tilde{\omega}_{47}(x)$, *genus-5* for $\tilde{\omega}_{59}(x)$ and *genus-6* for $\tilde{\omega}_{71}(x)$.

7.5. Hadamard products of the ω_n

The two operators ω_2 and ω_3 have a 'modularity' property: their series expansions analytic at $x = 0$, (64) and (65), *as well as* the corresponding nomes and mirror maps are series with integer coefficients. The Hadamard product is quite a natural operation to introduce because *it preserves the global nilpotence of the operators*, *preserves the integrality of series solutions* and is a *natural operation to introduce when seeking for diagonals of rational functions*[49]. Let us perform the Hadamard product of these two operators. With some abuse of language [31], the Hadamard product of the two order-2 operators (62) and (63),

$$H_{2,3} = \omega_2 \star \omega_3 = D_x^4 + 6\frac{(2064x - 1)}{(1728x - 1) \cdot x} \cdot D_x^3 + \frac{(19\,020x - 7)}{(1728x - 1) \cdot x^2} \cdot D_x^2 + \frac{(4788x - 1)}{(1728x - 1) \cdot x^3} \cdot D_x + \frac{12}{(1728x - 1) \cdot x^3}, \quad (74)$$

is defined as the (minimal-order) linear differential operator having, as a solution, the Hadamard product of the solution series (64) and (65), which is, by construction, a series with integer coefficients. This series is, of course, nothing but the expansion of the hypergeometric function:

$$_4F_3\left(\left[\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{3}, \tfrac{1}{3}\right], [1, 1, 1]; 1728x\right) = {_2F_1}\left(\left[\tfrac{1}{4}, \tfrac{1}{4}\right], [1]; -64x\right) \star {_2F_1}\left(\left[\tfrac{1}{3}, \tfrac{1}{3}\right], [1]; -27x\right). \quad (75)$$

In a similar way, one can consider (see [19]) $H_{2,2} = \omega_2 \star \omega_2$ (resp. $H_{3,3} = \omega_3 \star \omega_3$) the Hadamard product of the order-2 operator (62) (resp. (63)) with itself (Hadamard square). These two operators have, respectively, the hypergeometric solutions

$$_4F_3\left(\left[\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}\right], [1, 1, 1]; 4096x\right), \qquad {_4F_3}\left(\left[\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right], [1, 1, 1]; 729x\right), \quad (76)$$

corresponding to series expansions with *integer coefficients*. These operators $H_{2,2}$ and $H_{3,3}$ are MUM operators. We can, therefore, define, without any ambiguity, the nome (and mirror map)

[49] And, consequently, has been heavily used to build Calabi–Yau-like ODEs (see [75]).

and Yukawa coupling of this order-4 operator [31]. One finds that the nome[50] and the mirror map (and the Yukawa coupling as a function of the x variable) are *not globally bounded*: they *cannot* be reduced, by one rescaling, to series with integer coefficients.

The three linear differential operators $H_{2,3}$, $H_{2,2}$ and $H_{3,3}$ are MUM and of order 4; however, they *are not of Calabi–Yau type*.

7.6. Hadamard products versus Calabi–Yau ODEs

The occurrence of Calabi–Yau type operators, which we could imagine, at first sight, to be extremely rare, is in fact quite frequent among such Hadamard products, as can be seen with other values of n and m. For instance, one can introduce[51] $H_{4,4} = \omega_4 \star \omega_4$, the Hadamard square of ω_4, which is an irreducible order-4 linear differential operator and has the hypergeometric solution already encountered for some n-fold integrals of the decomposition of the full magnetic susceptibility of the Ising model [30, 31] (see also subsections 7.1 and 7.2):

$$ {}_4F_3\big(\big[\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\big], [1, 1, 1]; 256x\big) = {}_2F_1\big(\big[\tfrac{1}{2}, \tfrac{1}{2}\big], [1]; -16x\big) \star {}_2F_1\big(\big[\tfrac{1}{2}, \tfrac{1}{2}\big], [1]; -16x\big). \qquad (77) $$

The associated operator having (77) as a solution obeys the 'Calabi–Yau condition' such that its exterior square is of *order 5*.

Let us give in a table the orders (which go from 4 to 20) of the various $H_{m,n} = H_{n,m}$ Hadamard products of the order-2 operators associated with the (genus-0) modular forms operators ω_n and ω_m:

n\m	2	3	4	5	6	7	8	9	10	12	13	16	18	25
2	4	4	4	6	4	6	4	4	10	8	10	8	12	14
3		4	4	6	4	6	4	4	10	8	10	8	12	14
4			4∗	6	4∗	6	4∗	4∗	10	8	10	8	12	14
5				6	6	8	6	6	12	10	12	10	14	16
6					4∗	6	4∗	4∗	10	8	10	8	12	14
7						6	6	6	12	10	12	10	14	16
8							4∗	4∗	10	8	10	8	12	14
9								4∗	10	8	10	8	12	14
10									10	14	16	14	18	20
12										8	14	12	16	18
13											10	14	18	20
16												8	16	18
18													12	20
25														14

where the star (∗) denotes Calabi–Yau ODEs[52].

The following operators are of order 4: $H_{2,2}$, $H_{2,3}$, $H_{2,4}$, $H_{2,6}$, $H_{2,8}$, $H_{2,9}$, $H_{3,3}$, $H_{3,4}$, $H_{3,6}$, $H_{3,8}$, $H_{3,9}$, etc. Their exterior squares, which are of order 6, do not have rational solutions[53].

[50] The nome of the Hadamard product of two operators has no simple relation with the nome of these two linear differential operators.

[51] To obtain the Hadamard product of two linear differential operators, use, for instance, Maple's command gfun[hadamardproduct].

[52] Recall that Calabi–Yau ODEs are defined by a list of constraints [75], the most important ones being, besides being MUM, that their *exterior square are of order 5*. There are more exotic conditions like the cyclotomic condition on the monodromy at ∞; see proposition 3 in [78].

[53] They cannot be homomorphic to Calabi–Yau ODEs.

The order-4 operators $H_{3,3}$ and $H_{3,4}$ are all MUM operators[54], but, similarly to the situation encountered with $H_{2,2}$, their nome, mirror map and Yukawa couplings are *not globally bounded*.

The following operators are of order 6: $H_{2,5}$, $H_{2,7}$, $H_{3,5}$, $H_{3,7}$, $H_{4,5}$, $H_{4,7}$, $H_{5,5}$, $H_{5,6}$, $H_{5,8}$, $H_{5,9}$, $H_{6,7}$, $H_{7,7}$, $H_{7,8}$, $H_{7,9}$. Their exterior squares, which are of order 15, do not have rational solutions (and cannot be homomorphic to higher order Calabi–Yau linear ODEs).

Remarkably, the following ten order-4 operators $H_{4,4}$, $H_{4,6}$, $H_{4,8}$, $H_{4,9}$, $H_{6,6}$, $H_{6,8}$, $H_{6,9}$, $H_{8,8}$, $H_{8,9}$ and $H_{9,9}$ (with a star in the previous table) are all MUM and *are such that their exterior squares are of order 5*[55]: they are *Calabi–Yau ODEs*. Actually, the nome, mirror map and Yukawa coupling series are *series with integer coefficients for all these order-4 Calabi–Yau operators*. The Yukawa couplings of all these order-4 Calabi–Yau operators identify with the Yukawa couplings of the corresponding adjoint operators. The Yukawa coupling series of these Calabi–Yau operators are, respectively, for $H_{4,4}$

$$K(q) = K^{\star}(q) = 1 + 32 \cdot q + 4896 \cdot q^2 + 702\,464 \cdot q^3 + 102\,820\,640 \cdot q^4 + 15\,296\,748\,032 \cdot q^5 + 2302\,235\,670\,528 \cdot q^6 + \cdots, \quad (78)$$

which is no. 3 in Almkvist *et al* 's large tables of Calabi–Yau ODEs [78] and is the well-known one for ${}_4F_3([1/2, 1/2, 1/2, 1/2], [1, 1, 1]; 256x)$ and for $H_{4,6}$

$$K(q) = K^{\star}(q) = 1 + 20 \cdot q + 36 \cdot q^2 + 15\,176 \cdot q^3 + 486\,564 \cdot q^4 + 21\,684\,020 \cdot q^5 + 1209\,684\,456 \cdot q^6 + \cdots, \quad (79)$$

which is no. 137 in tables [78].

We give, in [19], the expansion of the Yukawa coupling for a set of another $H_{m,n}$ such that their exterior squares are of order 5 (not 6 as one could expect for a generic irreducible order-4 operator) that *actually are Calabi–Yau* operators. Actually, operator $H_{4,8}$ is no. 36 in Almkvist *et al* 's large tables of Calabi–Yau ODEs [78]. Operators $H_{4,9}$, $H_{6,6}$, $H_{6,8}$ and $H_{6,9}$ are, respectively, [78] nos 133, 144, 176 and 178. Furthermore, operators $H_{8,8}$, $H_{8,9}$ and $H_{9,9}$ are, respectively, [78] nos 107, 163 and 165.

It will be shown, in a forthcoming publication, that the occurrence of an order 5 for the exterior power (the 'Calabi–Yau condition') means that these operators are necessarily *conjugated* (by an algebraic function) to their adjoints. Thus, the 'adjoint Yukawa coupling' $K^{\star}(q)$ is necessarily equal to the Yukawa coupling $K(q)$ for these operators.

On the other hand, *the ten linear differential operators denoted by a star* $(*)$ *in the previous table all share the same property*: they have, as a solution, the Hadamard product of two HeunG function solutions of the form $\mathrm{HeunG}(a, q, 1, 1, 1, 1; x)$. Note, however, that this HeunG viewpoint of the most interesting $H_{m,n}$ does not really help. Even inside this restricted set of HeunG function solutions of the form $\mathrm{HeunG}(a, q, 1, 1, 1, 1; x)$, it is hard to find exhaustively the values of the parameter a and of the accessory parameter q, such that the series $\mathrm{HeunG}(a, q, 1, 1, 1, 1; x)$ is globally bounded, or, just, such that the order-2 operator, having $\mathrm{HeunG}(a, q, 1, 1, 1, 1; x)$ as a solution, is globally nilpotent [27].

Many $H_{m,n}$ are not MUM, for instance, the order-8 operator $H_{12,12}$, or the order-6 operator $H_{3,7}$, is *not MUM*. Concerning $H_{3,7}$, and as far as its six solutions are concerned, they are structured 'like' the four solutions of an order-4 MUM operator, together with the two solutions of another order-2 MUM operator, but the order-6 operator $H_{3,7}$ is not a direct sum of an order-4 and order-2 operator. We have two solutions analytical at $x = 0$ (with no logarithmic terms) and two solutions involving $\ln(x)$. A linear combination of these two solutions analytical at $x = 0$ is, by construction, a series with *integer coefficients* (the Hadamard product of the two

[54] Note that the Hadamard product of two MUM ODEs is not necessarily a MUM ODE: the order-6 operator $H_{3,7}$ is *not* MUM.

[55] They are conjugated to their (formal) adjoint by a function.

series with integer coefficients which are the initial ingredients in this calculation), when the other linear combinations are *not globally bounded*.

8. Calabi–Yau modularity

The previous examples correspond to a 'modularity' inherited from elliptic curves, more precisely Hadamard products of modular forms. Let us consider here two Calabi–Yau examples that do not seem to be reducible[56] to ${}_4F_3$ hypergeometric functions.

A first order-4 Calabi–Yau operator, found by Batyrev and van Straten [77], which is self-adjoint and also corresponds to Hadamard products of simple hypergeometric functions (see (F.2)), is given in appendix F. All the associated series (solution (F.2), nome, Yukawa coupling) are the series expansion with *integer coefficients*. We do not have a representation of the solution (F.2), as an n-fold integral of the form (34). However, since (F.2) can be expressed as a sum of products of binomials (see (F.3)), we can conclude, again, that (F.2) is *actually a diagonal of a rational function*.

8.1. A Batyrev and van Straten Calabi–Yau ODE [77]

A second example of an order-4 operator, corresponding to Calabi–Yau threefolds in $P_1 \times P_1 \times P_1 \times P_1$, has been found by Batyrev and van Straten [77] (see[57] p 34):

$$B_2 = \theta^4 - 4x \cdot (5\theta^2 + 5\theta + 2) \cdot (2\theta + 1)^2 + 64x^2 \cdot (2\theta + 3) \cdot (2\theta + 1) \cdot (2\theta + 2)^2. \qquad (80)$$

It corresponds to the series solution with coefficients:

$$\binom{2n}{n} \cdot \sum_{k=0}^{n} \binom{n}{k}^2 \cdot \binom{2k}{k} \cdot \binom{2n-2k}{n-k}. \qquad (81)$$

Its Wronskian W_4 is a rational function such that

$$x^3 \cdot W_4^{1/2} = \frac{1}{(1-64x)(1-16x)}. \qquad (82)$$

This operator is also a Calabi–Yau operator: it is MUM and it is such that its exterior square is *order 5*. This order-5 property is a consequence of B_2 being conjugated to its adjoint: $B_2 \cdot x = x \cdot \text{adjoint}(B_2)$.

The series solution of (80) can be written as a Hadamard product

$$\mathcal{S} = (1-4x)^{-1/2} \star \text{HeunG}(4, 1/2, 1/2, 1/2, 1, 1/2; 16x)^2$$
$$= 1 + 8x + 168x^2 + 5120x^3 + 190\,120x^4 + 7939\,008x^5 + 357\,713\,664x^6 + \cdots, \qquad (83)$$

the modular form character of HeunG(4, 1/2, 1/2, 1/2, 1, 1/2; 16x) being illustrated with identities (A.4) in appendix A of [19]. Its nome reads

$$q = x + 20x^2 + 578x^3 + 20\,504x^4 + 826\,239x^5 + 36\,224\,028x^6 + 1684\,499\,774x^7$$
$$+ 81\,788\,693\,064x^8 + 4104\,050\,140\,803x^9 + 211\,343\,780\,948\,764x^{10} + \cdots. \qquad (84)$$

The *mirror map* of (80) reads

$$x(q) = q - 20q^2 + 222q^3 - 2704q^4 + 21\,293q^5 - 307\,224q^6 + 80\,402q^7$$
$$- 67\,101\,504q^8 - 1187\,407\,098q^9 - 37\,993\,761\,412q^{10} + \cdots. \qquad (85)$$

[56] The possibility that a solution of an order-4 operator, non-trivially equivalent to these Calabi–Yau operators [77], could be written as a ${}_4F_3$ hypergeometric function, *up to an involved algebraic pullback*, is not totally excluded. However, it is extremely difficult to rule out such a highly non-trivial hypergeometric scenario.

[57] There is a small misprint in [77, p 34]: $(2\theta + 1)$ must be replaced by $(2\theta + 1)^2$ in the $4x$ term.

The Yukawa coupling of (80) reads

$$K(q) = K^\star(q) = 1 + 4q + 164q^2 + 5800q^3 + 196\,772q^4 + 6564\,004q^5 + 222\,025\,448q^6 + 7574\,684\,408q^7 + 259\,866\,960\,036q^8 + \cdots. \quad (86)$$

The equality of the Yukawa coupling with the 'adjoint' Yukawa coupling, $K(q) = K^\star(q)$, is a straight consequence of relation $B_2 \cdot x = x \cdot \text{adjoint}\,(B_2)$.

Recalling Batyrev and van Straten [77], and following Morrison [76], do note that one can also write the Yukawa coupling as

$$K(q) = \frac{x(q)^3 \cdot W_4^{1/2}}{y_0^2} \cdot \left(\frac{q}{x(q)} \cdot \frac{\mathrm{d}x(q)}{\mathrm{d}q}\right)^3 = \frac{W_4^{1/2}}{y_0^2} \cdot \left(q \cdot \frac{\mathrm{d}x(q)}{\mathrm{d}q}\right)^3, \quad (87)$$

where W_4 is the Wronskian (82). From this alternative expression for the Yukawa coupling, *valid when the operator is conjugated to its adjoint* (see (E.7)), it is obvious that if the analytic series $y_0(x)$, *as well as* the nome (84) are series with *integer coefficients*, then the mirror map (85) is also a series with integer coefficients, and, therefore, y_0 seen as a function of the nome q, as well as $x^3W_4^{1/2}$ (since it is a *rational function*) are also series with integer coefficients. *Consequently, the Yukawa coupling is a series with integer coefficients* (as a series in q or in x).

More generally, if one assumes that a linear differential operator has a globally bounded solution series, one knows that this operator is a G-operator, necessarily globally nilpotent and, consequently, its Wronskian, or the square root of the Wronskian (see $W_4^{1/2}$ in (87)), will be an Nth root of a rational function and, thus, *will correspond to a globally bounded series*. *Thus, the globally bounded character of the analytic series $y_0(x)$ together with the nome yields the globally bounded character of the mirror map, Yukawa coupling, which we associate with the modularity*[58]. In contrast, the globally bounded character of the analytic series $y_0(x)$, *together with* the globally bounded character of the Yukawa coupling (seen, for instance, as a series in x), *does not imply* that the nome, or the mirror map, is globally bounded as can be seen in example (59) (see (60) and (61)).

8.2. An operator non-trivially homomorphic to B_2

Let us now consider the order-4 operator

$$\mathcal{B}_2 = 256x^2 \cdot \theta^2(2\theta+3)(2\theta+1) - 4x \cdot (2\theta+1)(2\theta-1)(5\theta^2 - 5\theta + 2) + (\theta-1)^4. \quad (88)$$

This operator is non-trivially[59] homomorphic to the Calabi–Yau operator (80):

$$\mathcal{B}_2 \cdot x \cdot (2\theta+1) = x \cdot (2\theta+1) \cdot B_2. \quad (89)$$

As a consequence of the previous intertwining relation, one immediately finds that the series solution analytic at $x = 0$ of this new MUM operator (88) is nothing but the action of the order-1 operator $x \cdot (2\theta + 1)$ on the series (83) and reads

$$x \cdot (2\theta+1)[\mathcal{S}] = x + 24x^2 + 840x^3 + 35\,840x^4 + 1711\,080x^5 + 87\,329\,088x^6 + 4650\,277\,632x^7 + 254\,905\,896\,960x^8 + \cdots. \quad (90)$$

It is obviously also a series with *integer coefficients* (the action of $x \cdot (2\theta + 1)$ on the series with integer coefficients is straightforwardly a series with integer coefficients). More generally, the globally bounded series remain globally bounded series by non-trivial operator equivalence,

[58] Similar results can be found in Delaygue's thesis [98] in a framework where the coefficients of hypergeometric series are *ratio of factorials* (see appendix C).

[59] The intertwiners between $\mathcal{B}_2$ and B_2 are operators not simple functions.

namely homomorphisms between operators (generically, the intertwiner operators are not simple functions).

The exterior square of the order-4 operator (88) is an order-6 operator which is, in fact, the direct sum of an order-5 operator $\mathcal{E}_5$ and an order-1 operator.

Operator $\mathcal{B}_2$ is non-trivially homomorphic to its adjoint:

$$\mathcal{B}_2 \cdot x^3 \cdot (2\theta + 3) \cdot (2\theta + 5) = x^3 \cdot (2\theta + 3) \cdot (2\theta + 5) \cdot \text{adjoint}\,(\mathcal{B}_2). \qquad (91)$$

The Yukawa coupling of this order-4 operator (88), non-trivially homomorphic to (80), reads

$$K(q) = 1 - 4q - 140q^2 - 4040q^3 - 64\,436\frac{q^4}{3} + 1889\,332\frac{q^5}{3} + 88\,331\,368\frac{q^6}{5} + 1652\,707\,624\frac{q^7}{9} - 69\,295\,027\,684\frac{q^8}{63} + \cdots. \qquad (92)$$

The Yukawa coupling series (92) is *not globally bounded*.

The 'adjoint Yukawa coupling' of this order-4 operator (88) reads

$$K^\star(q) = 1 + 12q + 564q^2 + 20\,440q^3 + 865\,732q^4 + 37\,162\,444q^5 + 8\,255\,346\,664\frac{q^6}{5} + 1121\,762\,648\,248\frac{q^7}{15} + 72\,336\,859\,374\,772\frac{q^8}{21} + \cdots. \qquad (93)$$

Again, the adjoint Yukawa coupling series (93) is *not globally bounded*.

In this example, one sees that the Yukawa couplings of two non-trivially homomorphic operators are *not necessarily equal*. The Yukawa couplings of two homomorphic operators are equal *when* the two operators are *conjugated by a function* (trivial homomorphism). The modularity property is *not preserved* by (non-trivial) operator equivalence: it may depend on a condition that the exterior square of the order-4 operators is of order *5*. The Calabi–Yau property is not preserved by operator equivalence.

To sum up. All these examples show that the *integrality* (globally bounded series) is *far from identifying with modularity*.

9. Conclusion

Seeking for the linear differential operators for $\chi^{(n)}$, we discovered, some years ago, that they were Fuchsian operators [11, 13], and, in fact, 'special' Fuchsian operators, namely Fuchsian operators with rational exponents for all their singularities, and with Wronskians that are Nth roots of rational functions. Then, we discovered that they were G-operators (or equivalently globally nilpotent [27]), and more recently, we accumulated results [31] indicating that they are 'special' G-operators. There are, in fact, *two quite different kinds of 'special features'* of these G-operators. On the one side, we have the fact that one of their solutions is not only G-series but is a *globally bounded* series. This special character has been addressed in this very paper, and we have seen that in fact, this '*integrality* property [99]' is a consequence of *quite general mathematical assumptions* often satisfied in physics (the integrand is not only algebraic but has an expansion at the origin of the form[60] (35)). However, we have also seen another special property of these G-operators, namely the fact that they seem to be quite systematically *homomorphic to their adjoints* [31]. We will show, in a forthcoming publication, that this last property amounts, on the associated linear differential systems, to having *special differential Galois groups* and that their exterior or symmetric squares have *rational solutions*. This last property is a property of a more 'physical' nature than the previous one, related to an

[60] Puiseux series are excluded.

underlying *Hamiltonian structure* [100], or as this is the case, for instance, in the Ising model, related to the underlying isomonodromic structure in the problem, which yields the occurrence of some underlying Hamiltonian structure [100]. In general, the *integrality* of *G*-operators *does not* imply the operator to be homomorphic to its adjoint, and conversely being homomorphic to its adjoint *does not* imply[61] integrality (and even does not imply[62] the operator to be Fuchsian). Interestingly, the $\chi^{(n)}$, as well as many important problems of theoretical physics, correspond to *G*-operators that present these two complementary 'special characters' (integrality and, up to homomorphisms, self-adjointness) and, quite often, this is seen in the framework of the emergence of 'modularity'.

Nomes, mirror maps and Yukawa couplings are *not D-finite* functions: they are solutions of quite involved *nonlinear* (higher order Schwarzian) ODEs (see, for instance, appendix D in [27]). Therefore, the question of the series integrality of the nomes, mirror maps, Yukawa couplings and other pullback invariants (see appendix E) requires addressing the very difficult question of series integrality for (involved) *nonlinear* ODEs. Note, however, as seen in section 8.1, in particular in (87), that the integrality of the series $y_0(x)$ and of the nome $q(x)$ is *sufficient to ensure*, *provided that the operator is conjugated to its adjoint* (see (E.7)), the integrality of the other quantities such as the Yukawa coupling and mirror maps. However, the integrality of the nome remains an involved problem. These questions will certainly remain open for some time.

In contrast, and more modestly, we have shown that a *very large set of problems in mathematical physics* (see sections 3.4, 4 and 5.1) *actually corresponds to diagonals of rational functions*. In particular, we have been able to show that $\chi^{(n)}$ *n*-fold integrals of the susceptibility of the two-dimensional Ising model are actually *diagonals of rational functions for any value of the integer n*, thus proving that the $\chi^{(n)}$ are *globally bounded for any value of the integer n*. As can be seen in the 'ingredients' of our simple demonstration (see section 3.4), no elliptic curves and their modular forms [102], no Calabi–Yau [103] or Frobenius manifolds [100], or Shimura curves, or arithmetic lattice assumption [104, 105] are required to prove the result. We just need to have an *n*-fold integral such that its integrand is *not only algebraic* but has an expansion at the origin of the form (35).

The integrality of all the $\chi^{(n)}$, a consequence of the remarkable result that all the $\chi^{(n)}$ are diagonals of rational functions, raises the question of the *modularity* of the $\chi^{(n)}$. Now, the full susceptibility can, formally, be seen as the diagonal of an *infinite sum of rational functions*. This also raises the question of defining, and addressing, modularity for non-holonomic functions[63] like the full susceptibility.

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[61] See appendices M and O in [19] which give an example of a (hypergeometric) family of order-4 operators satisfying the Calabi–Yau condition that their exterior square is of order 5 and, even, a family of self-adjoint order-4 operators, the corresponding hypergeometric solution series being *not globally bounded*.
[62] For instance, the operator $D_x^n - xD_x - 1/2$ (see p 74 of [101]) with an irregular singularity is self-adjoint.
[63] Along this line, recall Chazy's equations [106] and their (circle) natural boundaries, and, especially, Harnad and McKay's paper [107] on *modular* solutions to equations of generalized Halphen type.

Appendix A. Modular forms and series integrality

First example. The generating function of the integers

$$\sum_{k=0}^{n} \binom{n}{k}^2 \cdot \binom{2k}{k} \cdot \binom{2n-2k}{n-k} = \binom{2n}{n} \cdot {}_2F_1\left(\left[\frac{1}{2}, -n, -n, -n\right], \left[1, 1, -\frac{2n-1}{2}\right]; 1\right) \tag{A.1}$$

is nothing else but the expansion of the square of a HeunG function

$$\begin{aligned} \text{HeunG}\left(4, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, 1, \tfrac{1}{2}; 16 \cdot x\right) &= 1 + 2x + 12x^2 + 104x^3 \\ &+ 1078x^4 + 12\,348x^5 + 150\,528x^6 + 1914\,432x^7 + \cdots \end{aligned} \tag{A.2}$$

solution of the order-2 operator

$$H_{\text{diam}} = \theta^2 - 2 \cdot x \cdot (10\theta^2 + 5\theta + 1) + 16x^2 \cdot (2\theta + 1)^2, \tag{A.3}$$

which corresponds to the *diamond lattice* [43]. This HeunG function (A.2) is *actually a modular form* which can be written in two different ways:

$$\begin{aligned} \text{HeunG}\left(4, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}; 16x\right) &= (1-4x)^{-1/2} \cdot {}_2F_1\left(\left[\frac{1}{6}, \frac{1}{3}\right], [1]; \frac{108x^2}{(1-4x)^3}\right) \\ &= (1-16x)^{-1/2} \cdot {}_2F_1\left(\left[\frac{1}{6}, \frac{1}{3}\right], [1]; -\frac{108x}{(1-16x)^3}\right). \end{aligned} \tag{A.4}$$

These two pullbacks are *related by an 'Atkin' involution* $x \leftrightarrow 1/64/x$. The associated modular curve relating these two pullbacks (A.4) yielding *the modular curve*:

$$\begin{aligned} 4 \cdot y^3z^3 - 12y^2z^2 \cdot (y+z) + 3yz \cdot (4y^2 + 4z^2 - 127yz) \\ - 4 \cdot (y+z) \cdot (y^2 + z^2 + 83yz) + 432yz = 0, \end{aligned} \tag{A.5}$$

which is (y, z)-symmetric and is *exactly the rational modular curve* in equation (27) already found for the order-3 operator F_3 in [31] for the five-particle contribution $\tilde{\chi}^{(5)}$ of the magnetic susceptibility of the Ising model.

This result in [41, 43] can be rephrased as follows. One introduces the order-3 operator which has the following ${}_3F_2$ solution:

$$\frac{1}{(4-x^2)^3} \cdot {}_3F_2\left(\left[\frac{1}{3}, \frac{1}{2}, \frac{2}{3}\right], [1, 1], \frac{27x^4}{(4-x^2)^3}\right), \tag{A.6}$$

associated with the *Green function of the diamond lattice*. Along a *modular form line,* let us note that this hypergeometric function actually has *two* pullbacks:

$${}_3F_2\left(\left[\frac{1}{3}, \frac{1}{2}, \frac{2}{3}\right], [1, 1], \frac{27x^4}{(4-x^2)^3}\right) = \frac{x^2-4}{4 \cdot (x^2-1)} \cdot {}_3F_2\left(\left[\frac{1}{3}, \frac{1}{2}, \frac{2}{3}\right], [1, 1], \frac{27x^2}{4 \cdot (x^2-1)^3}\right). \tag{A.7}$$

These two pullbacks are related by the 'Atkin' involution $x \to 2/x$:

$$u(x) = \frac{27x^4}{(4-x^2)^3}, \qquad v(x) = u\left(\frac{2}{x}\right) = \frac{27x^2}{4 \cdot (x^2-1)^3}, \tag{A.8}$$

corresponding, again, to the modular curve (A.5).

Second example. The HeunG function

$$\mathrm{HeunG}(-3, 0, 1/2, 1, 1, 1/2; 12\cdot x)$$
$$= (1+4x)^{-1/4}\cdot \mathrm{HeunG}\left(4, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, \frac{16x}{1+4x}\right)$$
$$= 1 + 6x^2 + 24x^3 + 252x^4 + 2016x^5 + 19\,320x^6 + 183\,456x^7 + 1823\,094x^8 + 18\,406\,752x^9 + 189\,532\,980x^{10} + \cdots \qquad (A.9)$$

is the solution of the order-2 operator

$$H_{\mathrm{fcc}} = \theta^2 - 2x\cdot\theta\cdot(4\theta+1) - 24\cdot x^2\cdot(2\theta+1)\cdot(\theta+1). \qquad (A.10)$$

The square of (A.9) is actually the solution of an order-3 operator (see equation (19) in [43]) emerging for lattice Green functions of the face-centred cubic (fcc) lattice, which is thus the symmetric square of (A.10). This hypergeometric function with a polynomial pullback can also be written as

$$\mathrm{HeunG}(-3, 0, 1/2, 1, 1, 1/2; 12\cdot x)$$
$$= {}_2F_1\left(\left[\frac{1}{6}, \frac{1}{3}\right], [1]; 108\cdot x^2\cdot(1+4x)\right)$$
$$= (1-12x)^{-1/2}\cdot {}_2F_1\left(\left[\frac{1}{6}, \frac{1}{3}\right], [1]; -\frac{108\cdot x\cdot(1+4x)^2}{(1-12x)^3}\right), \qquad (A.11)$$

where the involution $x \leftrightarrow -1/4\cdot(1+4x)/(1-12x)$ takes place. The modular curve relating these two pullbacks is *exactly the rational curve* (A.5) already obtained in [31].

Third example. The HeunG function $\mathrm{HeunG}(1/9, 1/12, 1/4, 3/4, 1, 1/2; 4x)$ is the solution of the order-2 operator corresponding to the simple cubic lattice Green function:

$$H_{sc} = \theta^2 - x\cdot(40\theta^2 + 20\theta + 3) + 9\cdot x^2\cdot(4\theta+3)\cdot(4\theta+1).$$

The square of this HeunG function is a series with *integer* coefficients which can be identified with the Hadamard product of $(1-4x)^{-1/2}$ with a modular form:

$$\mathrm{HeunG}(1/9, 1/12, 1/4, 3/4, 1, 1/2; 4x)^2$$
$$= (1-4x)^{-1/2} \star \mathrm{HeunG}(1/9, 1/3, 1, 1, 1, 1; x)$$
$$= 1 + 6x + 90x^2 + 1860x^3 + 44\,730x^4 + 1172\,556x^5 + 32\,496\,156x^6 + 936\,369\,720x^7 + 27\,770\,358\,330x^8 + 842\,090\,474\,940x^9 + \cdots. \qquad (A.12)$$

The HeunG function $\mathrm{HeunG}(1/9, 1/12, 1/4, 3/4, 1, 1/2; 4x)$ is globally bounded: the series of $\mathrm{HeunG}(1/9, 1/12, 1/4, 3/4, 1, 1/2; 8x)$ is a series with integer coefficients. One can also write this HeunG function in terms of a ${}_2F_1([1/6, 1/3], [1], x)$ hypergeometric function up to a simple algebraic pullback (with a square root), or in terms of a ${}_2F_1([1/8, 3/8], [1], x)$ hypergeometric function:

$$\mathrm{HeunG}(1/9, 1/12, 1/4, 3/4, 1, 1/2; 4x) = C_2^{1/4}\cdot {}_2F_1([1/8, 3/8], [1]; P_2),$$

with

$$C_2 = \frac{1}{9\cdot(1+12x)^2}\cdot(5 - 36x + 4\cdot(1-36x)^{1/2}), \qquad P_2 = \frac{128\cdot x}{(1+12x)^4}\cdot p_2,$$
$$p_2 = (1 - 42x + 352x^2 - 288x^3) + (1-4x)\cdot(1-20x)\cdot(1-36x)^{1/2}.$$

Do note that taking the Galois conjugate (changing $(1-36x)^{1/2}$ into $-(1-36x)^{1/2}$) gives the series expansion of $3^{-1/2}\cdot \mathrm{HeunG}(1/9, 1/12, 1/4, 3/4, 1, 1/2; 4x)$. This shows that there exists an identity for ${}_2F_1([1/8, 3/8], [1], x)$ with *two different pullbacks*, namely the previous

P_2 and its Galois conjugate, these two pullbacks being related by a (symmetric genus-0) modular curve:

$$5308\,416 \cdot y^4z^4 + 442\,368 \cdot y^3z^3 \cdot (y+z) + 512y^2z^2 \cdot (27y^2 + 27z^2 - 27\,374xy) \\ + 192yz \cdot (y+z) \cdot (y^2 + z^2 + 10\,718yz) + y^4 + z^4 + 3622\,662y^2z^2 \\ - 19\,332 \cdot yz \cdot (y^2 + z^2) + 79\,872 \cdot yz \cdot (y+z) - 65\,536 \cdot yz = 0. \qquad (A.13)$$

Revisiting the examples. In a recent paper [44] corresponding to spanning tree generating functions and Mahler measures, a result from Rogers (equation (36) in [44]) is given where the following two ${}_5F_4$ hypergeometric functions take place:

$$ {}_5F_4\left(\left[\frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 1, 1\right], [2, 2, 2, 2], \frac{256x^3}{9 \cdot (x+3)^4}\right), \\ {}_5F_4\left(\left[\frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 1, 1\right], [2, 2, 2, 2], \frac{256x}{9 \cdot (1+3x)^4}\right). \qquad (A.14)$$

The corresponding order-5 linear differential operators (annihilating these two ${}_5F_4$ hypergeometric functions) are actually homomorphic (the intertwiners being order-4 operators). The relation between these two pullbacks $y = 256x^3/9/(x+3)^4$ and $z = 256x/9/(1+3x)^4$ remarkably *gives, again, the previous* (y, z)*-symmetric modular curve* (A.13).

The order-5 linear differential operator, corresponding to the first ${}_5F_4$ hypergeometric function, factorizes into an order-1 operator, an order-3 operator and an order-1 operator, the order-3 operator being, in fact, exactly the symmetric square of an order-2 operator:

$$L_1 \cdot \mathrm{Sym}^2(W_2) \cdot \frac{x^4}{(x-9)(x+3)^4} \cdot R_1,$$

where the order-1 operators read, respectively,

$$L_1 = D_x - \frac{\mathrm{d}}{\mathrm{d}x} \ln\left(\frac{x-9}{(9x^2+14x+9) \cdot (x+3)^4}\right), \qquad R_1 = D_x - \frac{\mathrm{d}}{\mathrm{d}x} \ln\left(\frac{(x+3)^4}{x^3}\right),$$

and the order-2 operator W_2 reads

$$W_2 = D_x^2 + 3\frac{(6 \cdot x^2 + 7x + 3)}{(9x^2+14x+9) \cdot x} \cdot D_x + \frac{3}{4} \cdot \frac{3x+2}{(9x^2+14x+9) \cdot x}. \qquad (A.15)$$

We have a similar result for the order-5 linear differential operator corresponding to the second ${}_5F_4$ hypergeometric function.

Another solution of this order-5 linear differential operator reads

$$\frac{(x+3)^4}{x^3} \cdot \int \frac{x-9}{(x+3) \cdot x} \cdot {}_3F_2\left(\left[\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\right], [1, 1], \frac{256x^3}{9 \cdot (x+3)^4}\right) \cdot \mathrm{d}x. \qquad (A.16)$$

The expansion of the ${}_3F_2$ hypergeometric function in (A.16) is globally bounded (change $x \to 9x$ to obtain a series with integer coefficients).

Recalling the two previous pullbacks, we have, in fact, the following identity:

$$3 \cdot (1+3x) \cdot {}_3F_2\left(\left[\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\right], [1, 1], \frac{256x^3}{9 \cdot (x+3)^4}\right) \\ = (x+3) \cdot {}_3F_2\left(\left[\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\right], [1, 1], \frac{256x}{9 \cdot (1+3x)^4}\right). \qquad (A.17)$$

However, this ${}_3F_2$ hypergeometric function is nothing but the square of a ${}_2F_1$ hypergeometric function:

$${}_3F_2\left(\left[\tfrac{1}{4}, \tfrac{1}{2}, \tfrac{3}{4}\right], [1, 1], x\right) = {}_2F_1\left(\left[\tfrac{1}{8}, \tfrac{3}{8}\right], [1], x\right)^2. \qquad (A.18)$$

Thus, the previous identity (A.17) is nothing but the identity on a ${}_2F_1$ hypergeometric function with *two different pullbacks*:

$$(1+3x)^{1/2} \cdot {}_2F_1\Big(\Big[\frac{1}{8}, \frac{3}{8}\Big], [1], \frac{256x^3}{9 \cdot (x+3)^4}\Big) = \Big(1+\frac{x}{3}\Big)^{1/2} \cdot {}_2F_1\Big(\Big[\frac{1}{8}, \frac{3}{8}\Big], [1], \frac{256x}{9 \cdot (1+3x)^4}\Big). \qquad (A.19)$$

The expansion of (A.19) is globally bounded. One obtains a series with *positive integer* coefficients using the simple rescaling $x \to 36 \cdot x$. Note that the two pullbacks can be exchanged by the simple 'Atkin' involution $x \leftrightarrow 1/x$, being related by the modular curve occurring for the simple cubic lattice, namely (A.13).

We have a similar result for the other ${}_5F_4$ hypergeometric functions popping out in [44].

For instance, for the diamond lattice, one obtains an expression (see equation (50) in [44]) where the following two ${}_5F_4$ hypergeometric functions take place[64]:

$$ {}_5F_4\Big(\Big[\frac{5}{3}, \frac{3}{2}, \frac{4}{3}, 1, 1\Big], [2, 2, 2, 2], \frac{-27x^2}{4 \cdot (1-x^2)^3}\Big), \quad {}_5F_4\Big(\Big[\frac{5}{3}, \frac{3}{2}, \frac{4}{3}, 1, 1\Big], [2, 2, 2, 2], \frac{27x^4}{(4-x^2)^3}\Big). \qquad (A.20)$$

These two pullbacks can be exchanged by the simple 'Atkin' involution $x \leftrightarrow 2/x$. These two pullbacks have been seen to be related by the (genus-0) (y, z)-symmetric modular curve (A.5):

$$4y^3z^3 - 12y^2z^2 \cdot (y+z) + 3yz(4y^2 + 4z^2 - 127yz) - 4 \cdot (y+z) \cdot (y^2 + z^2 + 83yz) + 432yz = 0. \qquad (A.21)$$

Similarly to (A.17), we have an identity between two ${}_3F_2$ hypergeometric functions (namely ${}_3F_2([2/3, 1/2, 1/3], [1, 1], z)$) with the two pullbacks (A.20), and these ${}_3F_2$ hypergeometric functions being the square of ${}_2F_1$ hypergeometric functions, one finds that the '*deus ex machina*' is the identity similar to (A.19):

$$(1-x^2)^{1/2} \cdot {}_2F_1\Big(\Big[\frac{1}{3}, \frac{1}{6}\Big], [1], \frac{27x^4}{(4-x^2)^3}\Big) = \Big(1-\frac{x^2}{4}\Big)^{1/2} \cdot {}_2F_1\Big(\Big[\frac{1}{3}, \frac{1}{6}\Big], [1], \frac{-27x^2}{4 \cdot (1-x^2)^3}\Big). \qquad (A.22)$$

The series expansion of (A.22) is globally bounded. Rescaling the x variable as $x \to 4x$, the series expansion becomes a series with *positive integer* coefficients (up to the first constant term).

For the fcc lattice, one obtains an expression (see equation (52) in [44]) where the following two ${}_5F_4$ hypergeometric functions take place[65]:

$$ {}_5F_4\Big(\Big[\frac{5}{3}, \frac{3}{2}, \frac{4}{3}, 1, 1\Big], [2, 2, 2, 2], \frac{x \cdot (x+3)^2}{(x-1)^3}\Big), \quad {}_5F_4\Big(\Big[\frac{5}{3}, \frac{3}{2}, \frac{4}{3}, 1, 1\Big], [2, 2, 2, 2], \frac{x^2 \cdot (x+3)}{4}\Big). \qquad (A.23)$$

This example is nothing but the previous diamond lattice example (A.20) with the change of variable $x \to -3x^2/(x^2-4)$ in (A.23). Therefore, the two pullbacks in (A.23) are, again,

[64] Note a small misprint in equation (50) of [44]: one should read $-27z^2/4/(1-z^2)^3$ instead of $-27z^4/4/(1-z^2)^3$.
[65] There is one more misprint in [44]: the pullback $-x(x+3)/(x-1)^3$ must be changed into $x(x+3)/(x-1)^3$.

related by the modular curve (A.5). The two pullbacks in (A.23) can actually be seen directly in the following identity (equivalent to (A.22)):

$$ {}_2F_1\left(\left[\frac{1}{3}, \frac{1}{6}\right], [1], \frac{x\cdot(x+3)^2}{(x-1)^3}\right) = (1-x^2)^{1/2}\cdot {}_2F_1\left(\left[\frac{1}{3}, \frac{1}{6}\right], [1], \frac{x^2\cdot(x+3)}{4}\right). $$

Finally, in the equation (17) of [44] on Mahler measures, the two following ${}_4F_3$ hypergeometric functions take place:

$$ {}_4F_3\left(\left[\frac{5}{3}, \frac{4}{3}, 1, 1\right], [2,2,2], \frac{27x}{(x-2)^3}\right), $$
$$ {}_4F_3\left(\left[\frac{5}{3}, \frac{4}{3}, 1, 1\right], [2,2,2], \frac{27x^2}{(x+4)^3}\right). \qquad \text{(A.24)} $$

These two previous pullbacks can be exchanged by an 'Atkin' involution $x \leftrightarrow -8/x$ and are related by the (genus-0) (y, z)-symmetric modular curve:

$$ 8y^3z^3 - 12y^2z^2\cdot(y+z) + 3yz\cdot(2y^2+2z^2+13yz) - (y+z)\cdot(y^2+z^2+29yz) + 27yz = 0. \qquad \text{(A.25)} $$

The underlying identity on ${}_2F_1$ hypergeometric functions with the two pullbacks (A.24) reads

$$ -2\cdot(x-2)\cdot {}_2F_1\left(\left[\frac{1}{3}, \frac{2}{3}\right], [1], \frac{27x^2}{(x+4)^3}\right) = (x+4)\cdot {}_2F_1\left(\left[\frac{1}{3}, \frac{2}{3}\right], [1], \frac{27x}{(x-2)^3}\right). \qquad \text{(A.26)} $$

The series expansion of (A.26) is globally bounded. Rescaling the x variable as $x \to -8x$, the series expansion becomes a series with *positive integer* coefficients.

Appendix B. Seeking for the 'minimal' rational function

The effective calculations of section 4.2 guarantee to obtain an *explicit expression* for the rational function associated with (40); however, the rational function is far from being unique. Recalling the well-known Apéry series $\mathcal{A}(x)$, and its rewriting due to Strehl and Schmidt [85–87],

$$ \mathcal{A}(x) = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2\cdot x^n = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{n}{k}\binom{n+k}{k}\binom{k}{j}^3\cdot x^n $$
$$ = 1 + 5x + 73x^2 + 1445x^3 + 33\,001x^4 + \cdots, \qquad \text{(B.1)} $$

$\mathcal{A}(x)$ is known to be the diagonal of the rational function in five variables $1/R_1/R_2$, where R_1 and R_2 read [60]

$$ R_1 = 1 - z_0, \qquad R_2 = (1-z_1)(1-z_2)(1-z_3)(1-z_4) - z_0z_1z_2, $$

as well as the diagonal of the rational function in five variables $1/Q_1/Q_2$, where Q_1 and Q_2 read [61, 45]

$$ Q_1 = 1 - z_1z_2z_3z_4, \qquad Q_2 = (1-z_3)(1-z_4) - z_0\cdot(1+z_1)(1+z_2), $$

and *also* the diagonal of the rational function in six variables $1/P_1/P_2/P_3$, where P_1, P_2 and P_3 read [60]

$$ P_1 = 1 - z_0z_1, \qquad P_2 = 1 - z_2 - z_3 - z_0z_2z_3, \qquad P_3 = 1 - z_4 - z_5 - z_1z_4z_5. $$

A yet different diagonal representation for the Apéry series, due to Delaygue[66], is provided by the diagonal of the rational function in eight variables:

$$\frac{1}{(1 - z_4z_5z_6z_7)\cdot(1 - z_0\cdot(1+z_4))\cdot(1-z_1\cdot(1+z_5))\cdot(1-z_2-z_6)\cdot(1-z_3-z_7)}.$$

Calculations similar to (39) on these new binomial expressions provide two new rational functions such that (B.1) can be written as the diagonal of one of these two rational functions. One is a rational function of five variables of the form $1/Q_1^{(5)}/Q_2^{(5)}$

$$Q_1^{(5)} = 1 - z_0z_1z_2z_3z_4\cdot(1+z_1)(1+z_2)(1+z_3)(1+z_4),$$
$$Q_2^{(5)} = 1 - z_0\cdot(1+z_1)(1+z_2)(1+z_3)^2(1+z_4)^2, \qquad \text{(B.2)}$$

and another one is a rational function of six variables of the form $1/Q_1^{(6)}/Q_2^{(6)}/Q_3^{(6)}$

$$Q_1^{(6)} = 1 - z_0z_3z_4z_5\cdot(1+z_1)(1+z_2)^2(1+z_3)(1+z_4)(1+z_5),$$
$$Q_2^{(6)} = 1 - z_0z_1z_2z_3z_4z_5\cdot(1+z_1)(1+z_2), \qquad \text{(B.3)}$$
$$Q_3^{(6)} = 1 - z_0\cdot(1+z_1)(1+z_2)^2(1+z_3)(1+z_4)(1+z_5).$$

Appendix C. Hypergeometric series with coefficients as ratios of factorials

As a consequence of the classification by Beukers and Heckman [91] of all algebraic ${}_nF_{n-1}$, the ${}_8F_7$ hypergeometric series

$${}_8F_7\left(\left[\tfrac{1}{30},\tfrac{7}{30},\tfrac{11}{30},\tfrac{13}{30},\tfrac{17}{30},\tfrac{19}{30},\tfrac{23}{30},\tfrac{29}{30}\right],\left[\tfrac{1}{5},\tfrac{2}{5},\tfrac{3}{5},\tfrac{4}{5},\tfrac{1}{2},\tfrac{2}{3},\tfrac{1}{3}\right],2^{14}3^95^5x\right)$$

has *integer coefficients* and is an *algebraic function*. The Galois group belonging to this function is the Weyl group $W(E_8)$, which has 696 729 600 elements [108]. It is an *algebraic series* of degree 483 840. More precisely, it was noted by Rodriguez-Villegas [109] that the previous power series reads

$$\sum_{n=0}^{\infty}\frac{(30n)!n!}{(15n)!(10n)!(6n)!}\cdot x^n, \qquad \text{(C.1)}$$

which is closely related to the series introduced by Chebyshev in his work [110] on the distribution of prime numbers to establish the estimate [19] on the prime counting function $\pi(x)$.

Considering hypergeometric series such that their coefficients are ratios of factorials, Rodriguez-Villegas [109] gives the conditions of these factorials for the hypergeometric series to be algebraic (all the coefficients are thus integers). A simple example is, for instance, the algebraic function:

$${}_3F_2\left(\left[\frac{1}{4},\frac{1}{2},\frac{3}{4}\right],\left[\frac{1}{3},\frac{2}{3}\right];\frac{2^8}{3^3}\cdot x\right) = \sum_{n=0}^{\infty}\binom{4n}{n}\cdot x^n. \qquad \text{(C.2)}$$

Along this line, it is worth recalling Delaygue's thesis [98] (see also [111]) which gives some results[67] for series expansions[68] such that their coefficients are *ratios of factorials*, namely

[66] Private communication.
[67] Necessary and sufficient conditions for the integrality of the mirror map series.
[68] These series are not algebraic functions.

${}_2F_1([1/3, 2/3], [1]; 27x)$ and ${}_4F_3([1/2, 1/2, 1/2, 1/2], [1, 1, 1]; 256x)$, giving, respectively, the series

$$\sum_{n=0}^{\infty} \frac{(3n)!}{(n!)^3} \cdot x^n, \qquad \sum_{n=0}^{\infty} \frac{((2n)!)^4}{(n!)^8} \cdot x^n \quad \text{and}$$

$${}_4F_3\left(\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{6}, \frac{5}{6}\right], [1, 1, 1]; 2^8 3^3 \cdot x\right) = \sum_{n=0}^{\infty} \frac{(6n)!(2n)!}{(3n)!(n!)^5} \cdot x^n. \qquad \text{(C.3)}$$

These ratios of factorials are all *integer numbers*.

Appendix D. Proof of integrality of series (43)

Let us sketch the proof of the integrality of series (43), namely the integrality of coefficients (45). For each power of the integer number $q = p^n$, a term like $4 + 9n$ is periodically divisible (period p) by q. In order to have the ratio (46) an integer, one needs the numerator to be divisible by this factor q *before* the denominator. The case $p = 3$ is an easy one. Another prime p does not divide 9. One needs to find the first case of divisibility, namely the first integer n such that $4 + 9n = kq$ (this corresponds to the smallest k). If $dq = 1$, mod 9, then $k = 4d$, mod 9. In other words, the smallest k is the rest of $4d$, mod 9. Consequently, we have replaced the calculations, for every integer q, by a *finite set of* calculations for $d = 1, 2, 4, 5, 7, 8$. Let us use this approach for the ratio (46).

Remark. The terms $n + 1$ are always the last to be divisible by q. Hence, one can forget its factors. However, one needs as many factors at the numerator as at the denominator. For the other terms, the following table of the rest of $d \cdot a$ gives the complete proof:

.	1	2	4	5	7	8
1	1	2	4	5	7	8
4	4	8	7	2	1	5
5	5	1	2	7	8	4
3	3	6	3	6	3	6

(D.1)

One finds that this is always a factor of the numerator, before the occurrence of a factor at the denominator.

Appendix E. Yukawa couplings

E.1. Yukawa couplings as the ratio of determinants

Consider an order-4 MUM linear differential operator. Let us introduce the determinantal variables $W_m = \det(M_m)$ which are the determinants[69] of the following $m \times m$ matrices M_m, $m = 1, \ldots, 4$, with entries expressed in terms of derivatives of the four solutions $y_0(x)$, $y_1(x)$, $y_2(x)$ and $y_3(x)$ of the MUM linear differential operator (see section 4.1 for the definitions). One takes $W_1(x) = y_0(x)$ and

$$M_2 = \begin{bmatrix} y_0 & y_1 \\ y_0' & y_1' \end{bmatrix}, \quad M_3 = \begin{bmatrix} y_0 & y_1 & y_2 \\ y_0' & y_1' & y_2' \\ y_0'' & y_1'' & y_2'' \end{bmatrix}, \quad M_4 = \begin{bmatrix} y_0 & y_1 & y_2 & y_3 \\ y_0' & y_1' & y_2' & y_3' \\ y_0'' & y_1'' & y_2'' & y_3'' \\ y_0''' & y_1''' & y_2''' & y_3''' \end{bmatrix},$$

$$\text{where} \qquad y_i' = \frac{\mathrm{d}}{\mathrm{d}x} y_i, \qquad y_i'' = \frac{\mathrm{d}^2}{\mathrm{d}x^2} y_i, \qquad y_i''' = \frac{\mathrm{d}^3}{\mathrm{d}x^3} y_i. \qquad \text{(E.1)}$$

[69] For an order-4 operator, the Wronskian is W_4.

Since q is equal to $q = \exp(y_1/y_0)$ and its derivative verifies

$$q \cdot \frac{d}{dq} = \frac{W_1^2}{W_2} \cdot \frac{d}{dx} = \frac{y_0^2}{W_2} \cdot \frac{d}{dx}, \qquad (E.2)$$

we have that

$$\left(q \cdot \frac{d}{dq}\right)^2 = \frac{y_0^4}{W_2^2} \cdot \frac{d^2}{dx^2} + 2\frac{y_0^3}{W_2^2}\frac{dy_0}{dx} \cdot \frac{d}{dx} - \frac{y_0^4}{W_2^3}\frac{dW_2}{dx} \cdot \frac{d}{dx}. \qquad (E.3)$$

We deduce, after some simple algebra, an alternative definition for the *Yukawa coupling*:

$$K(q) = \left(q \cdot \frac{d}{dq}\right)^2 \left(\frac{y_2}{y_0}\right) = \frac{W_1^3 \cdot W_3}{W_2^3} = \frac{y_0^3 \cdot W_3}{W_2^3}, \qquad (E.4)$$

to be compared with another previous alternative expression previously given (87) for the *Yukawa coupling*

$$K(q) = \frac{x(q)^3 \cdot W_4^{1/2}}{y_0^2} \cdot \left(\frac{q}{x(q)} \cdot \frac{dx(q)}{dq}\right)^3 = \frac{W_4^{1/2}}{y_0^2} \cdot \left(q \cdot \frac{dx(q)}{dq}\right)^3. \qquad (E.5)$$

In fact, from (E.2), we deduce

$$\left(q \cdot \frac{dx(q)}{dq}\right)^3 = \frac{W_1^6}{W_2^3} = \frac{y_0^6}{W_2^3}, \qquad (E.6)$$

and so (E.5) is compatible with (E.4) *if the following identity is verified*:

$$W_3^2 = W_4 \cdot y_0^2 = W_4 \cdot W_1^2. \qquad (E.7)$$

This identity is in fact specific for *order-4 operators conjugated to their adjoints* (see (E.17)). Therefore, we prefer to use definition (E.4) for the Yukawa coupling, instead of the more restricted definition (E.5).

Let us assume that the pullback $p(x)$ has a series expansion of the form

$$p(x) = \lambda \cdot x^r \cdot A(x), \qquad (E.8)$$

where the exponent r is an integer, λ is a constant and $A(x)$ is a function analytic at $x = 0$ with the series expansion:

$$A(x) = 1 + \alpha_1 \cdot x + \alpha_2 \cdot x^2 + \cdots.$$

The determinantal variables W_m transform very nicely under pullbacks $p(x)$ of the form (E.8):

$$(W_1(x), W_2(x), W_3(x), W_4(x))$$
$$\longrightarrow \left(W_1(p(x)), \frac{p'}{r} \cdot W_2(p(x)), \frac{p'^3}{r^3} \cdot W_3(p(x)), \frac{p'^6}{r^6} \cdot W_4(p(x))\right), \quad p' = \frac{dp(x)}{dx}. \qquad (E.9)$$

One can show that the nome (36) of an order-N operator transforms under a pullback $p(x)$:

$$q(x) \longrightarrow Q(x) \qquad \text{with} \qquad \lambda \cdot Q(x)^r = q(p(x)). \qquad (E.10)$$

From the covariance property (E.9), and from the previous transformation $q \to \lambda \cdot q^r$ for the nome, one easily obtains the transformation of the Yukawa coupling seen as a function of the nome $K(q) \to K(\lambda \cdot q^r)$:

$$K(q(x)) = \frac{W_1(x)^3 \cdot W_3(x)}{W_2(x)^3}$$
$$\longrightarrow \frac{W_1(p(x))^3 \cdot W_3(p(x))}{W_2(p(x))^3} = K(q(p(x))) = K(\lambda \cdot Q(x)^r). \qquad (E.11)$$

For $\lambda = 1$ and $r = 1$ (i.e. when the pullback is a deformation of the identity transformation), one recovers the known invariance of the Yukawa coupling by pullbacks (see proposition 3 in [112]).

One finds *another* pullback invariant ratio, namely

$$K^{\star} = \frac{W_1 \cdot W_3^3}{W_4 \cdot W_2^3}, \qquad \text{(E.12)}$$

which is, in fact, *nothing but the Yukawa coupling for the adjoint* of the original operator.

Another invariance property is worth noting. Let us consider two linear differential operators Ω_1 and Ω_2 of order N that are equivalent in the sense of the equivalence of linear differential operators. This means that there exist linear differential operator intertwiners I_1, I_2, J_1, J_2, of order at most $N - 1$ such that

$$\Omega_1 \cdot I_1 = I_2 \cdot \Omega_2 \qquad \text{and} \qquad J_1 \cdot \Omega_1 = \Omega_2 \cdot J_2. \qquad \text{(E.13)}$$

Let us assume that one of these intertwiners is a linear differential operator of order 0 (a function), then the previous homomorphism between operators amounts to saying that the two operators are conjugated by a function

$$\Omega_2 = \rho(x) \cdot \Omega_1 \cdot \rho(x)^{-1}, \qquad \text{(E.14)}$$

which corresponds to changing the four solutions as follows: $y_i \rightarrow \rho(x) \cdot y_i$. In such a case, the previous determinant variables transform, again, very nicely under the 'gauge' function $\rho(x)$:

$$(W_1, W_2, W_3, W_4) \rightarrow (\rho(x) \cdot W_1,\ \rho(x)^2 \cdot W_2,\ \rho(x)^3 \cdot W_3,\ \rho(x)^4 \cdot W_4). \qquad \text{(E.15)}$$

It is straightforward to see that the Yukawa coupling and the 'dual Yukawa' $K^{\star}$ are *invariant by such a transformation*[70]. Two conjugated operators (E.14) automatically have the same Yukawa coupling.

Do note that the Yukawa couplings for two operators, which are non-trivially homomorphic to each other (intertwiners of order 1, 2, etc), are actually different. *The (pullback invariant) Yukawa coupling is not preserved by operator equivalence* (see subsection 8.2).

Remark. The definition of these determinantal variables W_i heavily relies on the MUM structure of the operator between the four solutions and the definition of W_i, in particular the log ordering of the solutions. It is worth noting that if one permutes the four solutions y_i, one would obtain 24 other sets of W_1, W_2, W_3, W_4 which are actually *also nicely covariant by pullbacks*, thus yielding a finite set of other 'Yukawa couplings' or adjoint Yukawa coupling $K^{\star}$ *also invariant by pullbacks*.

In fact, these 'Yukawa couplings' (E.4), and other adjoint Yukawa $K^{\star}$ (E.12), can even be defined when the linear differential operator is not MUM, and they are still invariant by pullbacks.

E.2. Pullback invariants for higher order ODEs

These simple calculations can straightforwardly be generalized to higher order linear differential equations. We give here the invariants for higher order linear differential operators.

Let us give for the nth order linear differential operator the list of the K_n invariants by pullback transformations:

$$K_3 = \frac{W_1^3 \cdot W_3}{W_2^3}, \qquad K_4 = \frac{W_1^8 \cdot W_4}{W_2^6}, \qquad K_5 = \frac{W_1^{15} \cdot W_5}{W_2^{10}}, \qquad K_6 = \frac{W_1^{24} \cdot W_6}{W_2^{15}}, \cdots$$

$$K_n = \frac{W_1^{a_n} \cdot W_n}{W_2^{b_n}}, \qquad \text{with:} \quad a_n = n \cdot (n-2), \qquad b_n = \frac{n \cdot (n-1)}{2}. \qquad \text{(E.16)}$$

[70] K and $K^{\star}$ (and their combinations) are the only monomials $W_1^{n_1} W_2^{n_2} W_3^{n_3} W_4^{n_4}$ to be invariant by (E.9) and (E.15).

An nth-order linear differential operator has K_n as an invariant by pullback transformation, as well as all K_m with $m \leqslant n$. K_3 is the Yukawa coupling and one remarks, for the order-4 operators, that another pullback invariant $K^\star$ (see (E.12)), which is actually also the Yukawa coupling of the adjoint operator, is nothing but K_3^3/K_4.

For order-4 operators conjugated to their adjoint (see (E.14)) (i.e. operators homomorphic to their adjoint, the intertwiner being an order-0 differential operator, a function), one has the equality

$$K_4 = K_3^2, \qquad \text{i.e.} \qquad K_3 = K^\star, \quad \text{or} \quad W_3^2 = W_1^2 \cdot W_4, \tag{E.17}$$

to be compared with the equality in Almkvist *et al* (see proposition 2 in [75])

$$y_0 y_3' - y_3 y_0' = y_1 y_2' - y_2 y_1', \tag{E.18}$$

which is satisfied when the Calabi–Yau condition that the exterior square is of order 5 is satisfied.

If a linear differential operator Ω_4 verifies condition (E.17), its conjugate by a function, $\rho(x) \cdot \Omega_4 \cdot \rho(x)^{-1}$ also verifies condition (E.17) (their Yukawa couplings are equal).

Condition (E.17) is not satisfied for linear differential operators homomorphic to their adjoint with *non-trivial* intertwiner (of order greater than 0). For instance, the order-4 operator (88) does not satisfy condition (E.17).

Appendix F. More Hadamard products: a Batyrev and van Straten Calabi–Yau ODE

An order-4 operator has been found by Batyrev and van Straten [77] associated with a Calabi–Yau threefold on $P_2 \times P_2$

$$B_1 = \theta^4 - 3x \cdot (7\theta^2 + 7\theta + 2) \cdot (3\theta + 1) \cdot (3\theta + 2) - 72x^2 \cdot (3\theta + 5) \cdot (3\theta + 4) \cdot (3\theta + 2) \cdot (3\theta + 1). \tag{F.1}$$

This operator is conjugated to its adjoint: $B_1 \cdot x = x \cdot \text{adjoint}\,(B_1)$.

Operator (F.1) is a *Calabi–Yau* operator [77]: it is MUM and is such that its exterior square is of *order* 5. It has a solution analytical at $x = 0$ which is actually the Hadamard product of the previous selected hypergeometric ${}_2F_1$:

$${}_2F_1\left(\left[\frac{1}{3}, \frac{2}{3}\right], [1]; 27x\right) \star \left(\frac{1}{1+4x} \cdot {}_2F_1\left(\left[\frac{1}{3}, \frac{2}{3}\right], [1]; \frac{27 \cdot x}{(1+4x)^3}\right)\right). \tag{F.2}$$

The coefficients of the series expansion of (F.2) are integers: they can actually be written as a sum of *products of binomials*:

$$\frac{(3n)!}{(n!)^3} \cdot \sum_{k=0}^{n} \binom{n}{k}^3 = \binom{3n}{n} \cdot \binom{2n}{n} \cdot \sum_{k=0}^{n} \binom{n}{k}^3. \tag{F.3}$$

The series expansions of the nome and the Yukawa coupling of this Calabi–Yau operator are series with *integer coefficients*, reading, respectively,

$$q = x + 48x^2 + 4626x^3 + 549\,304x^4 + 74\,589\,735x^5 + 11\,014\,152\,048x^6 + \cdots$$
$$K(q) = 1 + 21q + 3861q^2 + 429\,159q^3 + 57\,224\,661q^4 + 7337\,893\,896q^5 + \cdots$$
$$K(x) = 1 + 21x + 4869x^2 + 896\,961x^3 + 175\,176\,657x^4 + 34\,770\,008\,997x^5 + \cdots.$$

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