International Journal of Modern Physics B, Vol. 8, No. 16 (1994) 2157–2201

DETERMINANTAL IDENTITIES ON INTEGRABLE MAPPINGS*

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Received 23 November 1993

We describe birational representations of discrete groups generated by involutions, having their origin in the theory of exactly solvable vertex-models in lattice statistical mechanics. These involutions correspond respectively to two kinds of transformations on $q \times q$ matrices: the inversion of the $q \times q$ matrix and an (involutive) permutation of the entries of the matrix. In a case where the permutation is a particular elementary transposition of two entries, it is shown that the iteration of this group of birational transformations yield algebraic elliptic curves in the parameter space associated with the (homogeneous) entries of the matrix. It is also shown that the successive iterated matrices do have remarkable factorization properties which yield introducing a series of canonical polynomials corresponding to the greatest common factor in the entries. These polynomials do satisfy a simple nonlinear recurrence which also yields algebraic elliptic curves, associated with biquadratic relations. In fact, these polynomials not only satisfy one recurrence but a whole hierarchy of recurrences. Remarkably these recurrences are universal: they are independent of q, the size of the matrices. This study provides examples of infinite dimensional integrable mappings.

1. Introduction

This paper belongs to a set of papers[1–3] analyzing birational representations of discrete groups generated by involutions, having their origin in the theory of exactly solvable vertex-models in lattice statistical mechanics. These mappings were actually introduced in previous publications[4,5] as symmetries of the Yang–Baxter equations and, more generally, as nontrivial symmetries of phase diagrams of lattice models in statistical mechanics.[6] More precisely these birational mappings are generated by simple transformations on 4×4 (or even $q \times q$) matrices, namely *the matricial inverse* combined with an arbitrary permutation of two entries. Six classes of transpositions, or of their associated mappings, have emerged from this study.[3] Only *three* of these classes correspond to *integrable* mappings.

We concentrate here on a particular transposition belonging to one of these three classes (denoted class I in the framework of the classification introduced in

PACS NOs.: 05.50, 05.20, 02.10, 02.20.
*work supported by CNRS.

Ref. 3). It gives a simple heuristic example of permutation which actually yields integrable mappings *for arbitrary $q \times q$ matrices* and, *amazingly, happens to be related with the sixteen-vertex model.* We will show that the iteration of these birational mappings exhibits remarkable factorization properties, *for any values of q*. These factorization properties explain why the complexity of the iterations, instead of having the exponential growth one expects at first sight, actually has a *polynomial growth.*[7] If one can imagine that the *integrability* of mappings yields a *polynomial growth* of the "complexity", the reciprocal statement is not obvious.[7]

It will also be shown that the polynomial factors, occurring in these factorizations, *do satisfy remarkable nonlinear recurrences.* Since these factors are related to *determinants* of $q \times q$ matrices, these nonlinear recurrences can also be seen as remarkable nonlinear identities bearing on determinants of matrices *of arbitrary size.* Moreover, these nonlinear recurrences in one variable will be shown to be related to *biquadratic* relations associated with *elliptic curves.* These *elliptic curves* are actually *the same* as the one generated by the birational transformations on an arbitrary number of variables previously mentioned. Let us recall that the equations of such an elliptic curve have been obtained as intersections of *quadrics* for birational transformations in $\mathbb{C}P_{15}$, for instance for all the integrable mappings studied in Ref. 8, as well as in the example of the sixteen-vertex model.[9a]

We will try here, on this simple example of permutation, to give a list as large as possible of structures, properties, formulas associated to these concepts, and their mutual connections: in particular the relation between the *polynomial growth* of the complexity[7] and the *integrability.* For heuristic reasons we will first present the different results and demonstrate them later on in Sec. 7.

2. The Notations

To set up the notations, let us consider the $q \times q$ matrix

$$R_q = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} & \cdots \\ m_{21} & m_{22} & m_{23} & m_{24} & \cdots \\ m_{31} & m_{32} & m_{33} & m_{34} & \cdots \\ m_{41} & m_{42} & m_{43} & m_{44} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} . \tag{2.1}$$

We introduce the following transformations, the matrix inverse $\widehat{I}$, the "homogeneous" matrix inverse I and the permutation of the entries m_{12} and m_{21}, we denote t:

$$t : \ m_{12} \leftrightarrow m_{21} \tag{2.2}$$

[a]More generally in $\mathbb{C}P_{q^2-1}$, these equations can be obtained as intersections of algebraic varieties simply deduced by sums and differences of minors of $q \times q$ matrices.

$$\widehat{I}: \; R_q \longrightarrow R_q^{-1} \tag{2.3}$$

$$I: \; R_q \longrightarrow R_q^{-1} \cdot \det(R_q) \; . \tag{2.4}$$

The "homogeneous" inverse I is a ***polynomial transformation*** on each of the entries m_{ij}. It associates to each entry m_{ij} its corresponding cofactor. The two transformations t and $\widehat{I}$ are ***involutions*** i.e. $\widehat{I}^2 = t^2 = \mathcal{I}d$, while I verifies $I^2 = (\det(R_q))^{q-2} \cdot \mathcal{I}d$, where $\mathcal{I}d$ denotes the identity transformation.

We also introduce the (generically infinite order) transformations $K = t \cdot I$ and $\widehat{K} = t \cdot \widehat{I}$. Transformation $\widehat{K}$ is clearly a ***birational*** transformation on the entries m_{ij} since its inverse transformation is $\widehat{I} \cdot t$, which is obviously a rational transformation. K is a ***homogeneous*** polynomial transformation on the entries m_{ij}.

We will first analyze some remarkable factorization properties of the iteration of the homogeneous transformation K, considering first, for heuristic reasons, the case of 3×3 matrices.

3. Factorization Properties

3.1. *The left-action of K*

Let us now consider a generic $q \times q$ matrix (2.1) denoted here M_0. Let f_1 denote the determinant of the initial matrix: $f_1 = \det(M_0)$, and let M_1 denote the matrix deduced from M_0 by transformation K, $M_1 = K(M_0)$.

For $q \geq 4$, the determinant of matrix M_1 ***amazingly factorizes*** some power of the determinant of matrix M_0, more precisely f_1^{q-3}. This enables us to introduce a new polynomial f_2:

$$f_2 = \frac{\det(M_1)}{f_1^{q-3}} \; . \tag{3.1}$$

One also remarks that f_1^{q-4} factorizes in all the entries of the matrix $K(M_1)$, leading to the definition of a new matrix M_2:

$$M_2 = \frac{K(M_1)}{f_1^{q-4}} \; . \tag{3.2}$$

One then considers the successive matrices obtained by iteration of transformation K and their corresponding determinants, and gets the factorizations

$$f_3 = \frac{\det(M_2)}{f_1^3 \cdot f_2^{q-3}} \, , \qquad M_3 = \frac{K(M_2)}{f_1^2 \cdot f_2^{q-4}} \, ,$$

$$f_4 = \frac{\det(M_3)}{f_1^{q-1} \cdot f_2^3 \cdot f_3^{q-3}} \, , \quad M_4 = \frac{K(M_3)}{f_1^{q-2} \cdot f_2^2 \cdot f_3^{q-4}} \; .$$

The factorization properties are now stabilized and they reproduce in a similar way shifting n by one at each step. Generally, one has the following recurrences for

$n \geq 1$ and $q \geq 4$:

$$M_{n+3} = \frac{K(M_{n+2})}{f_n^{q-2}\, f_{n+1}^2\, f_{n+2}^{q-4}} \tag{3.3}$$

$$f_{n+3} = \frac{\det(M_{n+2})}{f_n^{q-1}\, f_{n+1}^3\, f_{n+2}^{q-3}} \tag{3.4}$$

and the following relation *independent of q*:

$$\widehat{K}(M_{n+2}) = \frac{K(M_{n+2})}{\det(M_{n+2})} = \frac{M_{n+3}}{f_n f_{n+1} f_{n+2} f_{n+3}} \,. \tag{3.5}$$

It should be noticed that, for a given initial matrix M_0, the successive reduced matrices M_n belong to two matrix vector spaces. More precisely transformation K^2 preserves, for each initial matrix M_0, a six-dimensional vector space. Since this property is also satisfied for matrix M_1, one obtains two different vector spaces corresponding respectively to n even or odd (see Appendix A).

3.2. *The right-action of K*

In the previous section, the successive matrices M_n were transformed under the (left) action of transformation K. One can also introduce a "right-action" of K on the matrices M_n, on the entries of M_n, and on any polynomial expressions of these entries (such as the f_n's for instance), replacing the entries m_{ij} of M_0 by the corresponding entries of $K(M_0)$, i.e. $(K(M_0))_{ij}$.

Amazingly, the right-action of K on the f_n's and matrices M_n's also yields remarkable factorizations, moreover in this case f_1, and only f_1 does factorize:

$$(f_n)_K = f_{n+1} \cdot f_1^{\mu_n}$$

and

$$(M_n)_K = M_{n+1} \cdot f_1^{\nu_n} \,. \tag{3.7}$$

To shed some light on the relation between the right and left action of transformation K, one can also introduce the successive (non-factorized) matrices $\widehat{M_n}$ corresponding to n-times the left or right-action of K on M_0:

$$\widehat{M_n} = K^n(M_0) = (M_0)_{K^n} \,. \tag{3.8}$$

These matrices $\widehat{M_n}$ can be expressed in terms of the M_n's and f_n's:

$$\widehat{M_n} = K(\widehat{M}_{n-1}) = \left(f_1^{(q-1)^{n-4}} \cdot f_2^{(q-1)^{n-5}} \cdot f_3^{(q-1)^{n-6}} \cdots f_{n-4}^{(q-1)} \cdot f_{n-3}\right)^{(q-2)^3} \cdot f_{n-2}^{(q-2)(q-3)} \cdot f_{n-1}^{q-4} \cdot M_n \,. \tag{3.9}$$

Introducing the "natural" homogeneous polynomials $k_n = \det(\widehat{M_n})$, these polynomials are related to the "elementary" polynomials f_n:

$$k_n = \left(f_1^{(q-1)^{n-4}} \cdot f_2^{(q-1)^{n-5}} \cdot f_3^{(q-1)^{n-6}} \cdots f_{n-4}^{(q-1)} \cdot f_{n-3}\right)^{n_1} \cdot f_{n-2}^{n_2} \cdot f_{n-1}^{n_3} \cdot f_n^{n_4} \cdot f_{n+1} \ . \tag{3.10}$$

where

$$n_1 = q\,(q-2)^3 \ , \qquad n_2 = (q-1)^2\,(q-3)+2 \ , \qquad n_3 = (q-1)\,(q-3) \ , \qquad n_4 = q-3 \ . \tag{3.11}$$

3.3. *Polynomial growth of the degrees and exponents*

Denoting α_n the degree of the determinant of the matrix M_n, and β_n the degree of the polynomial f_n, one immediately gets, from the previous relations, the following linear recurrences on α_n, β_n:

$$\alpha_{n+2} = (q-1)\,\beta_n + 3\,\beta_{n+1} + (q-3)\,\beta_{n+2} + \beta_{n+3} \tag{3.12}$$

$$(q-1)\,\alpha_{n+2} = \alpha_{n+3} + q\,(q-2)\,\beta_n + 2\,q\,\beta_{n+1} + q\,(q-4)\,\beta_{n+2} \tag{3.13}$$

yielding relations between the two corresponding generating functions $\alpha(x)$ and $\beta(x)$:

$$x\,\alpha(x) = \left((q-1)\,x^3 + 3\,x^2 + (q-3)\,x + 1\right)\,\beta(x) \tag{3.14}$$

$$((q-1)\,x-1)\cdot\alpha(x) + q = \left(q\,(q-2)\,x^3 + 2q\,x^2 + q\,(q-4)\,x\right)\cdot\beta(x) \ . \tag{3.15}$$

From these relations between the two generating functions $\alpha(x)$ and $\beta(x)$, one deduces their expressions in terms of q ($q \geq 4$):

$$\alpha(x) = \frac{q\,(1+(q-3)x+3x^2+(q-1)x^3)}{(1+x)(1-x)^3} = q\sum_{n=0}^{\infty}\left(\frac{q\,n^2}{2} + \frac{q}{4} - \frac{(-1)^n\,q}{4} + (-1)^n\right)\,x^n \tag{3.16}$$

$$\beta(x) = \frac{q\,x}{(1+x)(1-x)^3} = \frac{q}{8}\sum_{n=0}^{\infty}(2\,n\,(n+2)+1-(-1)^n)\,x^n \ . \tag{3.17}$$

Recalling the definition of μ_n and ν_n, one also gets, from the right-action of transformation K (see (3.6) and (3.7)), the two linear relations

$$(q-1)\,\alpha_n = \alpha_{n+1} + q^2\,\nu_n \qquad (q-1)\,\beta_n = \beta_{n+1} + q\,\mu_n \tag{3.18}$$

yielding on the generating functions

$$((q-1)\,x-1)\cdot\alpha(x) = q^2\,x\,\nu(x) - q$$
$$((q-1)\,x-1)\cdot\beta(x) = q\,x\,\mu(x) - q\,x\,. \tag{3.19}$$

Equations (3.19), combined with (3.16), enable us to deduce the two generating functions $\mu(x)$ and $\nu(x)$:

$$\mu(x) = \frac{x\,((q-3)+2\,x^2-x^3)}{(1+x)(1-x)^3}$$
$$= \sum_{n=1}^{\infty}\left(\frac{q\,n^2}{4}+\frac{q\,n}{2}+\frac{q}{8}-\frac{(-1)^n\,q}{8}-\frac{n^2}{2}-\frac{3\,n}{2}-1\right)\,x^n \tag{3.20}$$

$$\nu(x) = \frac{x\,((q-4)+2\,x+(q-2)\,x^2)}{(1+x)(1-x)^3}$$
$$= \sum_{n=1}^{\infty}\left(\frac{q\,n^2}{2}+\frac{q}{4}-\frac{(-1)^n\,q}{4}+(-1)^n-n^2-1-n\right)\,x^n\,. \tag{3.21}$$

Remark: All these results are slightly modified if one considers $q=3$.
The left-factorizations read:

$$M_{n+3} = \frac{K(M_{n+2})}{f_n} \quad \text{and} \quad f_{n+3} = \frac{\det(M_{n+2})}{f_n^2} \tag{3.22}$$

yielding, for the inhomogeneous transformation:

$$\widehat{K}(M_{n+2}) = \frac{K(M_{n+2})}{\det(M_{n+2})} = \frac{M_{n+3}}{f_n\,f_{n+3}}\,.$$

As far as the right-factorizations are concerned, (3.6) and (3.7) are still valid. The generating functions $\beta(x)$ and $\mu(x)$ are directly obtained, replacing q by 3 respectively in (3.17) and (3.20). On the contrary, the generating functions $\alpha(x)$ and $\nu(x)$ are not given by (3.16) and (3.21), they read

$$\alpha(x) = \frac{3\,(1+2\,x^3)}{(1-x)^3(1+x)} \quad \text{and} \quad \nu(x) = \frac{x^3\,(1+x)^2}{(1-x^2)^3} = \frac{x^3}{(1+x)(1-x)^3}\,. \tag{3.23}$$

Because of all these remarkable factorization properties, the degree of these various expressions (entries of the matrices M_n's and polynomials f_n's) *does not grow exponentially* with n, as one could expect at first sight: it does grow *polynomially with* n. A proof of this statement will be given in Sec. 6.2. This polynomial growth is closely related to the integrability of these iterations, as will be seen later.

3.4. *Orbits of K for $q \times q$ matrices*

An efficient way to see the possible integrability of transformation K is to look at its iterations.[2,3] Let us, for instance, consider the iteration of transformation K acting on a 5×5 matrix. Figure 1 represents the projection (on two variables among the 24 others) of an orbit of such an iteration. On Fig. 1, and also on the other projections, one does see that these orbits are actually *curves*. If one takes any other initial point (another initial matrix) in the parameter space, namely $\mathbb{C}P_{24}$, one also sees curves. These curves *do foliate the whole parameter space* $\mathbb{C}P_{24}$. It is possible to prove that these curves are actually *elliptic curves*.

3.5. *Recurrences on f_n's and k_n's*

Though the definition of the polynomials f_n depend on q (see (3.1), (3.2), ...), *they do satisfy the recurrences independently of q*:

$$\frac{f_n f_{n+3}^2 - f_{n+4} f_{n+1}^2}{f_{n-1} f_{n+3} f_{n+4} - f_n f_{n+1} f_{n+5}} = \frac{f_{n-1} f_{n+2}^2 - f_{n+3} f_n^2}{f_{n-2} f_{n+2} f_{n+3} - f_{n-1} f_n f_{n+4}} \,. \qquad (3.24)$$

The f_n's do satisfy *other recurrences also independent of q*. For example

$$\frac{f_{n+1} f_{n+4}^2 f_{n+5} - f_{n+2} f_{n+3}^2 f_{n+6}}{f_{n+2}^2 f_{n+3} f_{n+7} - f_n f_{n+4} f_{n+5}^2} = \frac{f_{n+2} f_{n+5}^2 f_{n+6} - f_{n+3} f_{n+4}^2 f_{n+7}}{f_{n+3}^2 f_{n+4} f_{n+8} - f_{n+1} f_{n+5} f_{n+6}^2} \qquad (3.25)$$

and

$$\frac{f_n f_{n+1} f_{n+5} - f_{n-1} f_{n+3} f_{n+4}}{f_n^2 f_{n+6} - f_{n-2} f_{n+4}^2} = \frac{f_{n+2} f_{n+3} f_{n+7} - f_{n+1} f_{n+5} f_{n+6}}{f_{n+2}^2 f_{n+8} - f_n f_{n+6}^2} \,. \qquad (3.26)$$

From (3.24) and (3.26), one immediately gets another recurrence

$$\frac{f_n f_{n+3}^2 - f_{n+4} f_{n+1}^2}{f_n^2 f_{n+6} - f_{n-2} f_{n+4}^2} = \frac{f_{n+2} f_{n+5}^2 - f_{n+6} f_{n+3}^2}{f_{n+2}^2 f_{n+8} - f_n f_{n+6}^2} \,. \qquad (3.27)$$

These recurrences can be written in terms of the k_n's, however they now depend on q. For instance, (3.24) becomes

$$\frac{k_{n+1}^{q-1} - k_{n+2}}{k_{n+2}^{q-2} k_{n+1}^{q-2} k_n^{q-1} - k_{n+3}} = \frac{k_{n+2}^{q-1} - k_{n+3}}{k_{n+3}^{q-2} k_{n+2}^{q-2} k_{n+1}^{q-1} - k_{n+4}} \cdot k_{n+1}^{q-1} k_{n+3}^{q-3} \,. \qquad (3.28)$$

Many other recurrences on the f_n's and the k_n's are also satisfied, as will be seen later.

3.6. *Recurrences on l_n and x_n*

Introducing also the variables l_n which are related to the *inhomogeneous* birational transformation $\widehat{K}$ by $l_n = \det(\widehat{K}^n(M_0))$, one has the following recurrence for any positive integer n:

$$\frac{l_{n+3}\, l_{n+2} - 1}{l_{n+1}\, l_{n+2}^2 l_{n+3}^2\, l_{n+4} - 1} = \frac{l_{n+2}\, l_{n+1} - 1}{l_n\, l_{n+1}^2 l_{n+2}^2\, l_{n+3} - 1} \cdot l_n\, l_{n+1}\, l_{n+2}\, l_{n+3} \; . \tag{3.29}$$

One notes that the variable l_n occurs, in this recurrence, *through the product*:

$$x_n = l_n l_{n+1} \; . \tag{3.30}$$

With these new variables, the previous equation on the l_n's can be written more simply as

$$\frac{x_{n+2} - 1}{x_{n+1}\, x_{n+2}\, x_{n+3} - 1} = \frac{x_{n+1} - 1}{x_n\, x_{n+1}\, x_{n+2} - 1} \cdot x_n\, x_{n+2} \; . \tag{3.31}$$

Similarly to what happened to the f_n's and k_n's, they also satisfy other recurrences, *independently of the value of* q. For instance, (3.25) becomes

$$\frac{l_{n+4} l_{n+3}^2 l_{n+2} - 1}{l_{n+5} l_{n+4}^3 l_{n+3}^4 l_{n+2}^3 l_{n+1} - 1} = \frac{l_{n+3} l_{n+2}^2 l_{n+1} - 1}{l_{n+4} l_{n+3}^3 l_{n+2}^4 l_{n+1}^3 l_n - 1} \cdot l_n l_{n+1} l_{n+2} l_{n+3} \tag{3.32}$$

which reads in terms of the x_n's:

$$\frac{x_{n+2} x_{n+3} - 1}{x_{n+1} x_{n+2}^2 x_{n+3}^2 x_{n+4} - 1} = \frac{x_{n+1} x_{n+2} - 1}{x_n x_{n+1}^2 x_{n+2}^2 x_{n+3} - 1} \cdot x_n x_{n+2} \; . \tag{3.33}$$

Similarly, (3.26) becomes

$$\frac{l_{n+3}\, l_{n+4}^2 l_{n+5}^2 l_{n+6} - 1}{l_{n+2}\, l_{n+3}^3 l_{n+4}^4 l_{n+5}^4 l_{n+6}^3 l_{n+7} - 1} = \frac{l_{n+1}\, l_{n+2}^2 l_{n+3}^2 l_{n+4} - 1}{l_n\, l_{n+1}^3 l_{n+2}^4 l_{n+3}^4 l_{n+4}^3 l_{n+5} - 1} \cdot l_n\, l_{n+1}^2 l_{n+2}^2 l_{n+3}^2 l_{n+4}^2 l_{n+5} \; . \tag{3.34}$$

which reads in terms of the x_n's:

$$\frac{x_{n+3} x_{n+4} x_{n+5} - 1}{x_{n+2} x_{n+3}^2 x_{n+4}^2 x_{n+5}^2 x_{n+6} - 1} = \frac{x_{n+1} x_{n+2} x_{n+3} - 1}{x_n x_{n+1}^2 x_{n+2}^2 x_{n+3}^2 x_{n+4} - 1} \cdot x_n x_{n+1} x_{n+2} x_{n+3} x_{n+4} \; . \tag{3.35}$$

These recurrences can be extended to n *any relative integer* as will be seen in Sec. 3.8.

3.7. *Relations between the k_n, f_n, l_n*

Let us give the relations between the various variables introduced in this paper (k_n, l_n, f_n, x_n). The k_n's can be written in terms of the l_n's:

$$k_n = l_0^{q(q-1)^{n-1}} \cdot l_1^{q(q-1)^{n-2}} \cdot l_2^{q(q-1)^{n-3}} \cdots l_{n-2}^{q(q-1)} \cdot l_{n-1}^{q} \cdot l_n \, . \tag{3.36}$$

Conversely, the l_n's can be written in terms of the k_n's:

$$l_n = k_n \cdot k_{n-1}^{-q} \cdot k_{n-2}^{q} \cdot k_{n-3}^{-q} \cdot k_{n-4}^{q} \cdots k_0^{\pm q} \, . \tag{3.37}$$

From the definition of the x_n's:

$$x_n = l_n \cdot l_{n+1} \tag{3.38}$$

one gets a very simple expression of the x_n's in terms of the k_n's:

$$x_n = k_{n+1} \cdot k_n^{1-q} \, . \tag{3.39}$$

The k_n's can be written in terms of the f_n's:

$$k_n = (f_1^{(q-1)^{n-4}} \cdot f_2^{(q-1)^{n-5}} \cdot f_3^{(q-1)^{n-6}} \cdots f_{n-4}^{(q-1)} \cdot f_{n-3})^{n_1} \cdot f_{n-2}^{n_2} \cdot f_{n-1}^{n_3} \cdot f_n^{n_4} \cdot f_{n+1} \tag{3.40}$$

where the n_i's are defined in (3.11).

From this equation, one gets the x_n's in terms of the f_n's, which amounts to taking into account the homogeneity of the recurrences on the f_n's (see Sec. 4.7):

$$\begin{aligned} x_n &= \frac{k_{n+1}}{k_n^{q-1}} \\ &= \frac{\left(f_1^{(q-1)^{n-3}} \cdot f_2^{(q-1)^{n-4}} \cdot f_3^{(q-1)^{n-5}} \cdots f_{n-3}^{(q-1)} \cdot f_{n-2}\right)^{n_1} \cdot f_{n-1}^{n_2} \cdot f_n^{n_3} \cdot f_{n+1}^{n_4} \cdot f_{n+2}}{\left[\left(f_1^{(q-1)^{n-4}} \cdot f_2^{(q-1)^{n-5}} \cdot f_3^{(q-1)^{n-6}} \cdots f_{n-4}^{(q-1)} \cdot f_{n-3}\right)^{n_1} \cdot f_{n-2}^{n_2} \cdot f_{n-1}^{n_3} \cdot f_n^{n_4} \cdot f_{n+1}\right]^{q-1}} \\ &= \frac{f_{n-1}^2 \, f_{n+2}}{f_{n+1}^2 \, f_{n-2}} \, . \end{aligned} \tag{3.41}$$

Let us note that one can deduce, from these previous recurrences on the f_n's, many more recurrences. The existence of such recurrences is closely related to the existence of an elliptic parameterization of the iterations. It is also remarkable that all the recurrences can be written in terms of the x_n's defined by (3.30). All these facts will be clearly understood when the proof of these recurrences will be given in Sec. 6.

3.8. *The reversibility of the recurrences*

It is important to note that all these recurrences on the l_n's or x_n's *are actually valid for n being a relative integer* and not only for $n \geq 0$. One can in fact directly extend the definition of the l_n's to n being any relative integer:

$$l_n = \det(\widehat{K}^n(M_0)) , \quad n = -\infty, \cdots , +\infty , \quad \text{with :} \ \widehat{K} = t \cdot \widehat{I} , \quad \widehat{K}^{-1} = \widehat{I} \cdot t .$$

One has for instance,

$$l_0 = \det(M_0) , \quad l_1 = \det(\widehat{K}(M_0))$$

$$l_{-1} = \det(\widehat{K}^{-1}(M_0)) = \det(\widehat{I} \cdot t(M_0)) = \frac{1}{\det(t(M_0))}$$

$$l_2 = \det(\widehat{K}^2(M_0))$$

$$l_{-2} = \frac{1}{\det(\widehat{K}(t(M_0)))} = \det(\widehat{K}^{-2}(M_0)) .$$

The previous recurrences are valid for n positive or negative, for example

$$\frac{l_2 l_1 - 1}{l_3 l_2^2 l_1^2 l_0 - 1} = \frac{l_1 l_0 - 1}{l_2 l_1^2 l_0^2 l_{-1} - 1} \cdot l_{-1} l_0 l_1 l_2 \tag{3.42}$$

or

$$\frac{l_1 l_0 - 1}{l_2 l_1^2 l_0^2 l_{-1} - 1} = \frac{l_0 l_{-1} - 1}{l_1 l_0^2 l_{-1}^2 l_{-2} - 1} \cdot l_{-2} l_{-1} l_0 l_1 \tag{3.43}$$

or even

$$\frac{l_3 l_2^2 l_1 - 1}{l_4 l_3^3 l_2^4 l_1^3 l_0 - 1} = \frac{l_2 l_1^2 l_0 - 1}{l_3 l_2^3 l_1^4 l_0^3 l_{-1} - 1} \cdot l_{-1} l_0 l_1 l_2 . \tag{3.44}$$

Moreover, *one should note that these recurrences are reversible. They are invariant under transformation* $l_n \longrightarrow 1/l_{-n}$. The fact that all these recurrences are *reversible* is not surprising since one has representations of *birational transformations* and that the group of transformations one deals with is clearly reversible:

$$t \cdot \widehat{I} \longrightarrow \widehat{I} \cdot t . \tag{3.45}$$

4. Hierarchy of Recurrences and its Symmetries

The variables x_n satisfy many more recurrences, valid *for arbitrary values of q*, and *for n any relative integer*. Let us show, for instance, that all the recurrences on the

x_n's can be written in the general form

$$\frac{x_{n+1}^{i_1} x_{n+2}^{i_2} \cdots x_{n+t}^{i_t} - 1}{x_n^{j_0} x_{n+1}^{j_1} x_{n+2}^{j_2} \cdots x_{n+t+1}^{j_{t+1}} - 1} x_n^{k_0} x_{n+1}^{k_1} x_{n+2}^{k_2} \cdots x_{n+t+s}^{k_{t+s}}$$

$$= \frac{x_{n+1+s}^{i_1} x_{n+2+s}^{i_2} \cdots x_{n+t+s}^{i_t} - 1}{x_{n+s}^{j_0} x_{n+1+s}^{j_1} x_{n+2+s}^{j_2} \cdots x_{n+t+s+1}^{j_{t+1}} - 1} \tag{4.1}$$

with the first and the last exponents i_1, i_t, j_0, j_{t+1}, k_0 and k_{t+s}, being equal to 1. It can be written in a more symbolic way, with obvious notations as

$$\frac{x_n^{(i)} - 1}{x_n^{(j)} - 1} . x_n^{(k)} = \frac{x_{n+s}^{(i)} - 1}{x_{n+s}^{(j)} - 1} \ . \tag{4.2}$$

Up to a simple multiplicative factor $x_n^{(k)} = x_0^{k_0} x_1^{k_1} x_2^{k_2} \ldots$, one has the same rational expression on the left-hand side and the right-hand side of Eq. (4.1) up to a shift s $(n \longrightarrow n+s)$.

Let us also associate to a recurrence of the form (4.1) the symbolic "coding" sequence

$$((s), (i_1, i_2, \cdots, i_t), (j_0, j_1, j_2, \cdots, j_{t+1}), (k_0, k_1, k_2, \cdots, k_{t+s})) \ . \tag{4.3}$$

Let us first recall recurrence (3.31), which corresponds to the sequence ((1), (1), (1, 1, 1), (1, 0, 1)):

$$\frac{x_{n+2} - 1}{x_{n+1}\, x_{n+2}\, x_{n+3} - 1} = \frac{x_{n+1} - 1}{x_n\, x_{n+1}\, x_{n+2} - 1} \cdot x_n\, x_{n+2} \ . \tag{4.4}$$

This recurrence will be established in the following (see Sec. (6.1)). At first it will be shown, in the next section, that recurrence (4.4) generates a *whole hierarchy of recurrences* (which are actually verified for the birational transformations associated to the transposition analyzed in this paper). In order to classify these recurrences we will introduce a short notation $S_{N\,r}^{(s)}$ for the coding sequences or the recurrences. The index s corresponds to the shift of recurrence (4.1), the indices N and r being two other indices enabling us to distinguish between various recurrences with a given shift s.

The first recurrence of this hierarchy, (4.4), will be denoted $S_{1\,1}^{(1)}$.

4.1. *Equivalence of two recurrences*

Among the various recurrences satisfied by the x_n's some of them simply *identify*. It is a straightforward calculation to see that recurrence (4.4) is exactly the same relation between successive x_n's as an apparently different recurrence of the form (4.1). This new equivalent recurrence is one of the most simple recurrences satisfied by the x_n's:

$$\frac{x_{n+2}-1}{x_{n+1}\,x_{n+3}-1} = \frac{x_{n+1}-1}{x_n\,x_{n+2}-1}\,x_n\,x_{n+2}^2\,. \tag{4.5}$$

This new sequence ((1), (1), (1, 0, 1), (1, 0, 2)) will be denoted $S^{(1)}_{1\,2}$. More generally, recalling the general form for the recurrences described here (see (4.1)), it is a straightforward calculation to show that two recurrences exactly identify if one has the following relations:

$$x_{n+1}^{i_1}x_{n+2}^{i_2}\cdots x_{n+t}^{i_t}\cdot\, x_n^{k_0}x_{n+1}^{k_1}x_{n+2}^{k_2}\cdots x_{n+t+s}^{k_{t+s}} = x_n^{j_0}x_{n+1}^{j_1}x_{n+2}^{j_2}\cdots x_{n+t+1}^{j_{t+1}} \tag{4.6}$$

or

$$x_n^{k_0}x_{n+1}^{k_1}x_{n+2}^{k_2}\cdots x_{n+t+s}^{k_{t+s}} = x_n^{j_0}x_{n+1}^{j_1}x_{n+2}^{j_2}\cdots x_{n+t+1}^{j_{t+1}}\cdot x_{n+1+s}^{i_1}x_{n+2+s}^{i_2}\cdots x_{n+t+s}^{i_t}\,. \tag{4.7}$$

These two conditions can be rewritten in a more compact way using the symbolic notations (4.2)

$$x_n^{(i)}.\, x_n^{(k)} = x_n^{(j)} \tag{4.8}$$

or

$$x_n^{(k)} = x_n^{(j)}.\, x_{n+s}^{(i)}\,. \tag{4.9}$$

More precisely, *if condition* (4.8) *is satisfied, recurrence* (4.1) *is equivalent to recurrence*

$$\frac{x_n^{(i)}-1}{x_n^{(k)}-1}\cdot x_n^{(k)}.\, x_{n+s}^{(i)} = \frac{x_{n+s}^{(i)}-1}{x_{n+s}^{(k)}-1} \tag{4.10}$$

and, conversely, *if condition* (4.9) *is satisfied, recurrence* (4.1) *is equivalent to recurrence*

$$\frac{x_n^{(i)}-1}{x_n^{(j)}.x_n^{(i)}-1}\cdot\; x_n^{(j)} = \frac{x_{n+s}^{(i)}-1}{x_{n+s}^{(j)}.x_{n+s}^{(i)}-1}\,. \tag{4.11}$$

4.2. *Procedure* Π

When one of the two conditions (4.8) or (4.9) are satisfied, other recurrences, *with the same shift*, can actually be deduced from the previous ones by the following transformation which we will call "procedure Π ":

$$\Pi:\; x_n \longrightarrow x_n\cdot\, x_{n+s} \tag{4.12}$$

acting on the left and right-hand side of (4.2), the factor $x_n^{(k)}$ being changed in a different way. More precisely, *if condition* (4.8) *is satisfied, recurrence* (4.2) *implies*

$$\frac{x_n^{(i)}.x_{n+s}^{(i)}-1}{x_n^{(j)}.x_{n+s}^{(j)}-1}\cdot\ x_n^{(k)}=\frac{x_{n+s}^{(i)}.x_{n+2s}^{(i)}-1}{x_{n+s}^{(j)}.x_{n+2s}^{(j)}-1} \tag{4.13}$$

and, conversely, *if condition* (4.9) *is satisfied, recurrence* (4.2) *yields*

$$\frac{x_n^{(i)}.x_{n+s}^{(i)}-1}{x_n^{(j)}.x_{n+s}^{(j)}-1}\cdot\ x_n^{(j)}.x_{n+2s}^{(i)}=\frac{x_{n+s}^{(i)}.x_{n+2s}^{(i)}-1}{x_{n+s}^{(j)}.x_{n+2s}^{(j)}-1}\,. \tag{4.14}$$

Coming back to our previous examples, namely recurrences (4.4) and (4.5), procedure II generates, from (4.5), the new recurrence corresponding to the sequence ((1), (1, 1), (1, 1, 1, 1), (1, 0, 1, 1)), namely

$$\frac{x_{n+2}\,x_{n+3}-1}{x_{n+1}\,x_{n+2}\,x_{n+3}\,x_{n+4}-1}=\frac{x_{n+1}\,x_{n+2}-1}{x_n\,x_{n+1}\,x_{n+2}\,x_{n+3}-1}\cdot x_n\,x_{n+2}\,x_{n+3}\,. \tag{4.15}$$

We denote this sequence $S_{2\,2}^{(1)}$, the increment of the value of N in $S_{N\,r}^{(s)}$ $(1\to 2)$ corresponding to the action of procedure II on sequence $S_{1\,2}^{(1)}$. Similarly procedure II generates from (4.4) the new recurrence denoted $S_{2\,1}^{(1)}=((1),(1,1),(1,2,2,1),(1,0,1,0))$:

$$\frac{x_{n+2}\,x_{n+3}-1}{x_{n+1}\,x_{n+2}^2\,x_{n+3}^2\,x_{n+4}-1}=\frac{x_{n+1}\,x_{n+2}-1}{x_n\,x_{n+1}^2\,x_{n+2}^2\,x_{n+3}-1}\cdot x_n\,x_{n+2}\,. \tag{4.16}$$

Let us note that the two recurrences (namely (4.16) and (4.15)), deduced from procedure II *from two equivalent recurrences* (4.4) and (4.5), *are not equivalent. They do not satisfy either* (4.8) or (4.9).

4.3. *Shift procedure*

In order to get other recurrences in this "hierarchy" it is worth noticing that one can always deduce new recurrences using a *"shift" procedure.* Starting from recurrence (4.2) one immediately gets, combining this very recurrence with itself where n has been shifted by s:

$$\frac{x_n^{(i)}-1}{x_n^{(j)}-1}.\,x_n^{(k)}.\,x_{n+s}^{(k)}=\frac{x_{n+2s}^{(i)}-1}{x_{n+2s}^{(j)}-1}\,. \tag{4.17}$$

Let us, for instance, consider recurrence (4.5). The relation we deduce from this "shift-procedure" corresponds to the sequence $S_{1\,2}^{(2)}=((2),(1),(1,0,1),$

(1, 1, 2, 2)) and reads

$$\frac{x_{n+3}-1}{x_{n+2}\,x_{n+4}-1} = \frac{x_{n+1}-1}{x_n\,x_{n+2}-1}\cdot x_n\,x_{n+1}\,x_{n+2}^2\,x_{n+3}^2\,. \tag{4.18}$$

Similarly one gets, from the first recurrence we introduced, namely (4.4), another recurrence with a shift of 2 associated with the sequence $S^{(2)}_{1\,1}$ = ((2), (1), (1, 1, 1), (1, 1, 1, 1)):

$$\frac{x_{n+3}-1}{x_{n+2}\,x_{n+3}\,x_{n+4}-1} = \frac{x_{n+1}-1}{x_n\,x_{n+1}\,x_{n+2}-1}\cdot x_n\,x_{n+1}\,x_{n+2}\,x_{n+3}\,. \tag{4.19}$$

Let us also consider recurrences (4.15) and (4.16). One easily gets, using the "shift procedure", two new recurrences with a shift of 2, respectively $S^{(2)}_{2\,2}$ = ((2), (1, 1), (1, 1, 1, 1), (1, 1, 1, 2, 1)):

$$\frac{x_{n+3}\,x_{n+4}-1}{x_{n+2}\,x_{n+3}\,x_{n+4}\,x_{n+5}-1} = \frac{x_{n+1}\,x_{n+2}-1}{x_n\,x_{n+1}\,x_{n+2}\,x_{n+3}-1}\cdot x_n\,x_{n+1}\,x_{n+2}\,x_{n+3}^2\,x_{n+4} \tag{4.20}$$

and $S^{(2)}_{2\,1}$ = ((2), (1, 1), (1, 2, 2, 1), (1, 1, 1, 1, 0)):

$$\frac{x_{n+3}\,x_{n+4}-1}{x_{n+2}\,x_{n+3}^2\,x_{n+4}^2\,x_{n+5}-1} = \frac{x_{n+1}\,x_{n+2}-1}{x_n\,x_{n+1}^2\,x_{n+2}^2\,x_{n+3}-1}\cdot x_n\,x_{n+1}\,x_{n+2}\,x_{n+3}\,. \tag{4.21}$$

4.4. *More recurrences*

At this point one can recall the previous symmetries (procedure Π or equivalence when conditions (4.8) or (4.9) are satisfied ...) to see the possible relations between the four last recurrences (4.18), (4.19), (4.20) and (4.21). We first note that recurrences (4.20) and (4.21) *are actually equivalent*. We also note that (4.20) *satisfies condition* (4.9) and that (4.21) *satisfies condition* (4.8). Recurrence (4.18) neither satisfies (4.8) nor (4.9). On the contrary, recurrence (4.19) *actually verifies condition* (4.9), and is thus equivalent to another recurrence, $S^{(2)}_{1\,3}$ = ((2), (1), (1, 2, 1), (1, 1, 1, 0)):

$$\frac{x_{n+3}-1}{x_{n+2}\,x_{n+3}^2\,x_{n+4}-1} = \frac{x_{n+1}-1}{x_n\,x_{n+1}^2\,x_{n+2}-1}\cdot x_n\,x_{n+1}\,x_{n+2}\,. \tag{4.22}$$

This recurrence satisfies condition (4.8).

Since recurrence (4.19) satisfies condition (4.9), procedure II yields another recurrence, $S_{2\,3}^{(2)} = ((2), (1, 0, 1), (1, 1, 2, 1, 1), (1, 1, 1, 0, 0, 1))$:

$$\frac{x_{n+3}\, x_{n+5} - 1}{x_{n+2}\, x_{n+3}\, x_{n+4}^2\, x_{n+5}\, x_{n+6} - 1} = \frac{x_{n+1}\, x_{n+3} - 1}{x_n\, x_{n+1}\, x_{n+2}^2\, x_{n+3}\, x_{n+4} - 1} \cdot x_n\, x_{n+1}\, x_{n+2}\, x_{n+5}\,. \tag{4.23}$$

Moreover, let us recall that (4.9) is deduced from (4.4) by procedure II, that (4.19) is deduced from (4.4) by a shift procedure and that (4.21) is also deduced from (4.16) by a shift procedure. These three relations induce a relation between (4.19) and (4.21), similar to procedure II, denoted Π_1, corresponding to the transformation $x_n \to x_n . x_{n+1}$ (on the left hand side and right hand side of recurrence like (4.2) but not on factor $x_n^{(k)}$).

Similarly, recurrence (4.18), which does not yield any recurrence by procedure II, and which *does not satisfy condition* (4.9) nor (4.8), can be seen to be related to (4.20) by the "induced" procedure Π_1: we have a situation similar to the previous one, namely recurrence (4.15) is deduced from (4.5) by procedure II, recurrence (4.18) is deduced from recurrence (4.5) by a shift procedure and recurrence (4.20) is also deduced from (4.15) by a shift procedure ... Note that procedure II yields other recurrences. For instance, from (4.20) which satisfies condition (4.9), and from (4.21) and (4.22), which verify (4.8), one gets new recurrences.

Let us now recall recurrence (3.35), that is $S_{1\,4}^{(2)} = ((2), (1, 1, 1), (1, 2, 2, 2, 1), (1, 1, 1, 1, 1))$:

$$\frac{x_{n+3}x_{n+4}x_{n+5} - 1}{x_{n+2}x_{n+3}^2x_{n+4}^2x_{n+5}^2x_{n+6} - 1} = \frac{x_{n+1}x_{n+2}x_{n+3} - 1}{x_n x_{n+1}^2 x_{n+2}^2 x_{n+3}^2 x_{n+4} - 1} \cdot x_n x_{n+1} x_{n+2} x_{n+3} x_{n+4}\,. \tag{4.24}$$

This recurrence does satisfy condition (4.8) which is a condition to be equivalent to another recurrence. This new equivalent recurrence, $S_{1\,5}^{(2)} = ((2), (1, 1, 1), (1, 1, 1, 1, 1), (1, 1, 1, 2, 2, 1))$ reads

$$\frac{x_{n+3}x_{n+4}x_{n+5} - 1}{x_{n+2}x_{n+3}x_{n+4}x_{n+5}x_{n+6} - 1} = \frac{x_{n+1}x_{n+2}x_{n+3} - 1}{x_n x_{n+1} x_{n+2} x_{n+3} x_{n+4} - 1} \cdot x_n x_{n+1} x_{n+2} x_{n+3}^2 x_{n+4}^2 x_{n+5}\,. \tag{4.25}$$

It is important to note that neither recurrence (3.35), nor (4.25), are equivalent, or related by a II procedure, or by a shift procedure, to one of the previously mentioned recurrences ((4.4), (4.5), (4.19), (4.18), (4.16), (4.15), (4.21), (4.20), (4.22) ...). Recurrences (3.35) and (4.25) are *consequences of recurrences* (4.5) and (4.4) *and only of these recurrences.*

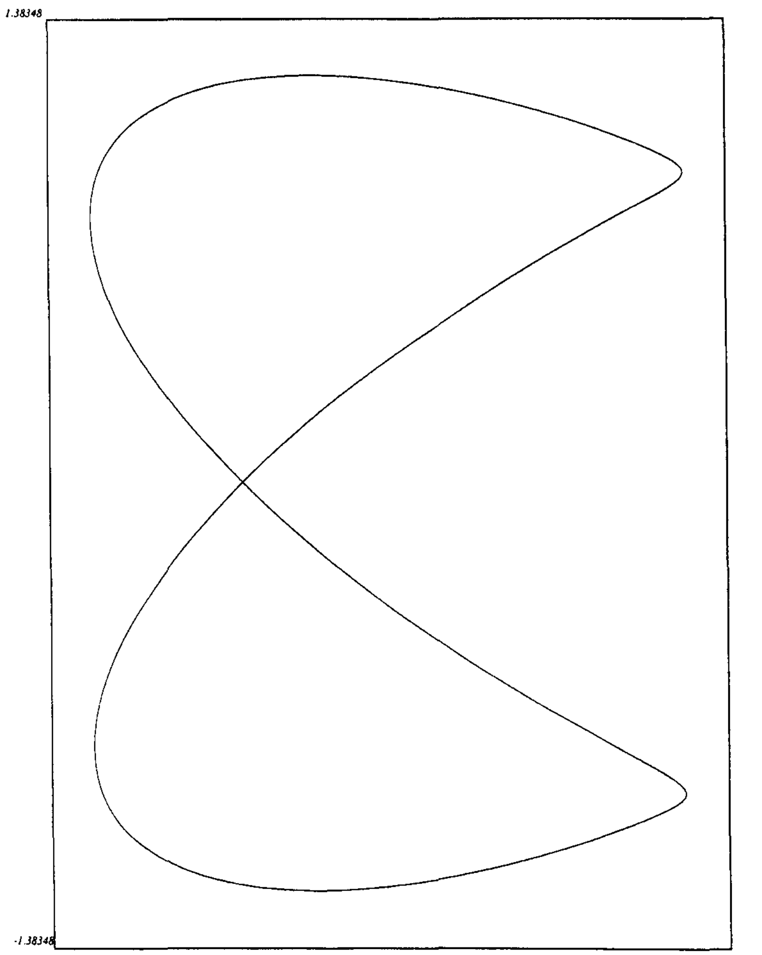


Fig. 1. Projection of the iteration of transformation K acting on a 5×5-matrix.

4.5. *Figures*

Having simple nonlinear recurrences in one variable (the variable x_n for instance), it is tempting to visualize the iteration corresponding to these recurrences, *considered now for themselves, without referring to the matrix framework of our birational transformation anymore.* Let us consider, for instance, recurrence (4.4) associated to $S_{1\,1}^{(1)}$. It reads

$$x_{n+3} = \frac{(1 - x_{n+2}) - x_n \, x_{n+2} \, (1 - x_{n+2} \, x_{n+1})}{x_n \, x_{n+1} \, x_{n+2}^2 \, (x_{n+1} - 1)} \; . \tag{4.26}$$

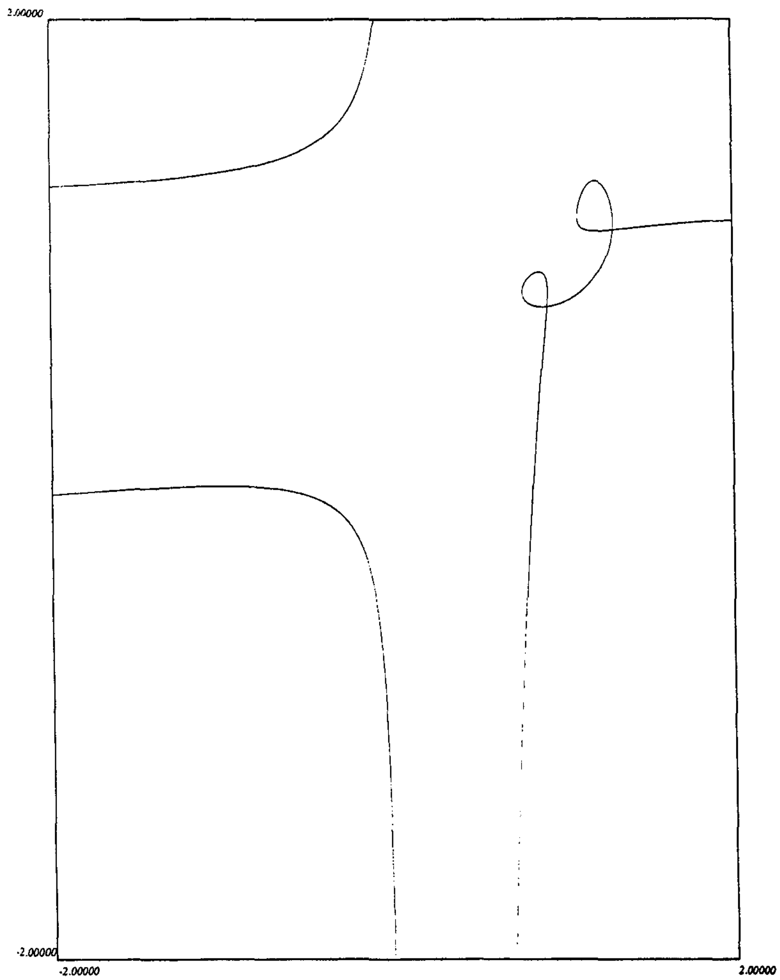


Fig. 2. Iteration of recurrence (4.26) in the $(x_n,\ x_{n+1})$-plane.

The iteration of recurrence (4.26) yields Fig. 2 in the $(x_n,\ x_{n+1})$-plane. It is clear on Fig. 2, that one gets a *curve*. It will be shown in the next section, that this curve is actually an *algebraic elliptic curve*.

Similarly, one can iterate recurrence (4.15) to get, in the $(x_n,\ x_{n+1})$-plane, Fig. 3. Iterating recurrence (4.20) one gets Fig. 4. Again it will be shown in the next section that the curves corresponding to Figs. 3 and 4 are also *algebraic elliptic curves.*

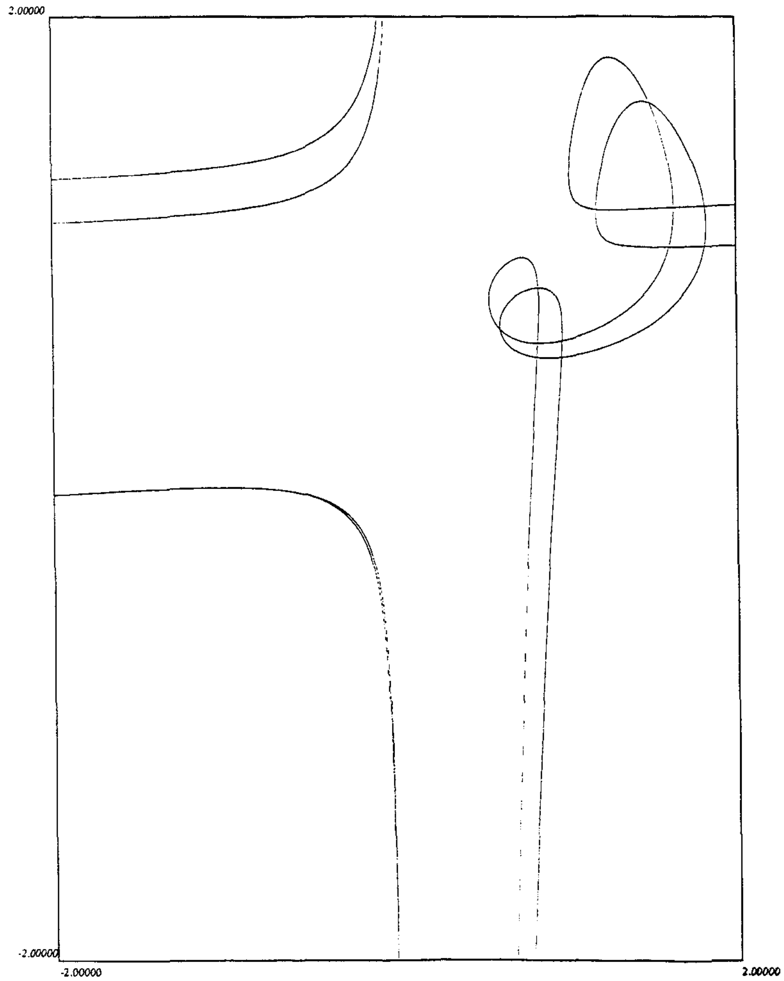


Fig. 3. Iteration of recurrence (4.15) in the (x_n, x_{n+1})-plane.

In contrast, the iteration of recurrences (4.25), (4.19), (4.18) (or (3.35) ...) yields (generically) figures like Fig. 5 which corresponds to iterate the "almost" integrable[8] recurrence (4.25).

These figures make very clear that recurrences (4.16), (4.15), (4.21) and (4.20) *are all integrable mappings* and that recurrences (4.19), (4.18), (3.35) or (4.25) *are not integrable mappings.* Of course these recurrences, which are consequences of integrable recurrences also yield curves when the initial conditions are in agreement with recurrences (4.16), (4.15), (4.21) and (4.20) (see Sec. 3.4) or, equivalently,

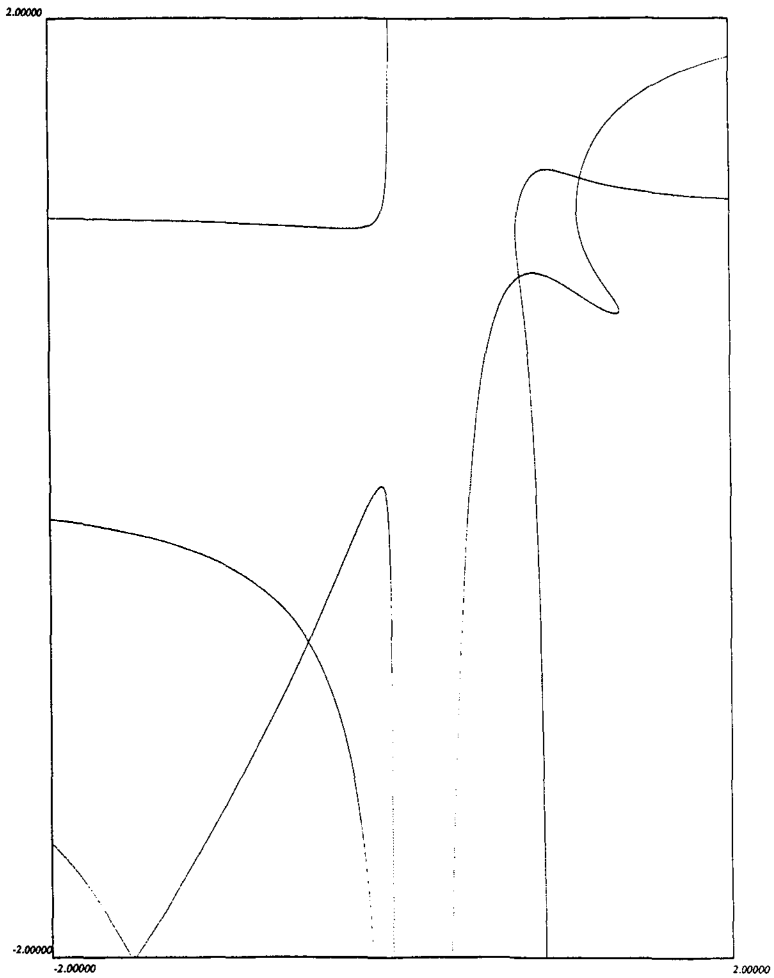


Fig. 4. Iteration of recurrence (4.20) in the (x_n, x_{n+1})-plane.

when the x_n's are defined, as we did in Sec. 3.7, as determinants of the iteration of a matrix under the action of transformation $\widehat{K}$.

4.6. *Integrability versus non-integrability in the hierarchy*

Recurrence (4.4) has been integrated in Ref. 8 and yields *biquadratic relations* in terms of some new variables q_n defined by $x_n = q_{n+1}/q_n$.

$$(\rho - q_n - q_{n+1})(q_n\, q_{n+1} + \lambda) = \mu \; . \tag{4.27}$$

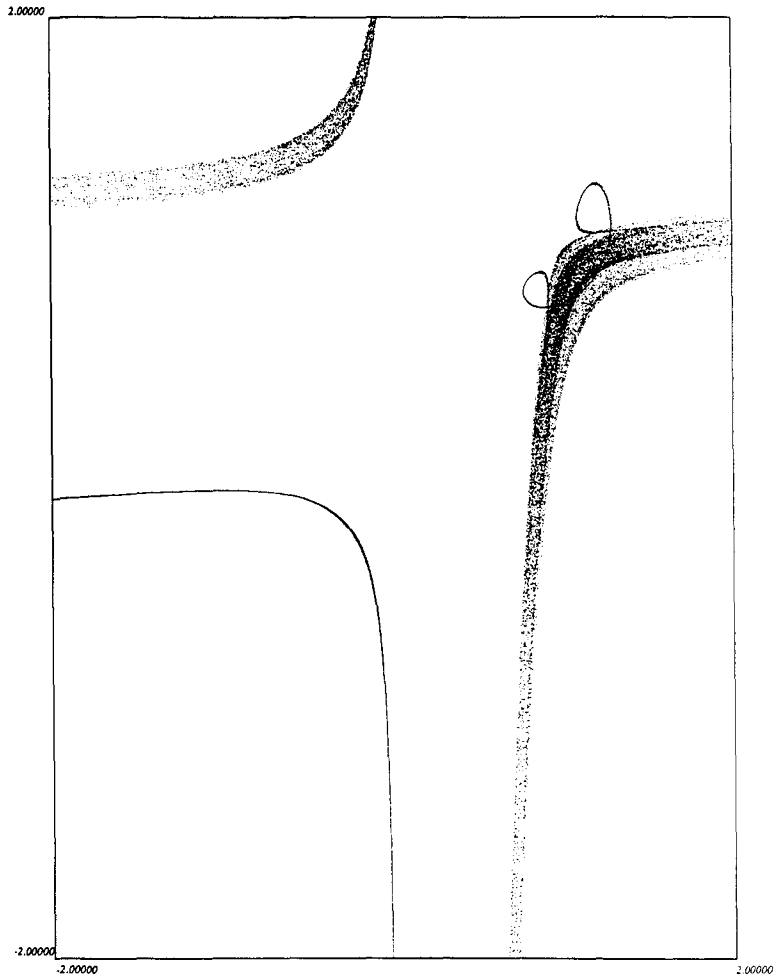


Fig. 5. Iteration, in the (x_n, x_{n+1})-plane, of a nonintegrable recurrence: recurrence (4.25).

This equation is reminiscent of Eq. (1) in Ref. 10 and similar, or more general, equations have been analyzed in detail by many authors.[10–16]

Recurrence (4.5) has been seen to be equivalent to recurrence (4.4), and thus correspond to the same biquadratic relation (4.27).

4.6.1. *An example of integration*

Let us now consider some recurrences which are not equivalent to (4.4) or (4.5), but are, however, consequences of these two recurrences. In particular let us show that they can also be integrated in a similar way. Let us introduce the variables q_n such that $x_n = q_{n+1}/q_n$. Recurrence (4.16) can also directly be written as

$$\frac{(q_{n+3} - q_{n+1})}{q_{n+2} \cdot (q_{n+3}\, q_{n+4} - q_n\, q_{n+1})} = \text{same expression with } n \longrightarrow n+1 \; . \qquad (4.28)$$

This form for recurrence (4.28) yields a straightforward integration introducing a first integration parameter λ which enables us to write (4.28) as

$$q_{n+3} - q_{n+1} = \lambda \cdot (q_{n+3}\, q_{n+4} - q_n\, q_{n+1}) \cdot q_{n+2} \qquad (4.29)$$

and to introduce two new constants of integration ρ_1 and ρ_2:

$$\lambda \cdot q_n\, q_{n+1}\, q_{n+2} - q_{n+1} = \rho_n = \rho_{n+2} \; . \qquad (4.30)$$

From the two relations (4.30) one easily gets

$$\rho_n \cdot q_{n+3} - q_{n+1}\, q_{n+3} = \rho_{n+1} \cdot q_n - q_n\, q_{n+2} \; . \qquad (4.31)$$

This last relation enables us to introduce a last constant of integration:

$$\rho_{n+1} \cdot q_n + \rho_n \cdot q_{n+1} + \rho_{n+1} \cdot q_{n+2} - q_n\, q_{n+2} = \mu \; . \qquad (4.32)$$

Eliminating q_{n+2} between (4.32) and (4.30) finally gives *two* biquadratic relations, depending on the parity of n:

$$\lambda \cdot q_n\, q_{n+1} \cdot (\rho_{n+1} \cdot q_n + \rho_n \cdot q_{n+1} - \mu) + (\rho_n - q_{n+1}) \cdot (\rho_{n+1} - q_n) = 0 \; . \qquad (4.33)$$

Integration of other recurrences are detailed in Appendix B.

4.6.2. *Towards integration*

Let us show here how the remarkable form of recurrence (4.1) enables us to perform *two integrations* for recurrences satisfying either (4.8) or (4.9), thus introducing two constants of integration.

Before considering this general demonstration, let us briefly analyze the non-integrable recurrence (4.19). Introducing again the variables q_n's such that $x_n = q_{n+1}/q_n$, one can easily see that one can only perform one integration step. Recurrence (4.19) can be written as

$$\frac{q_{n+2} - q_{n+1}}{(q_{n+3} - q_n) \cdot q_{n+1}\, q_{n+2}} = \text{same expression with } n \longrightarrow n+2 \qquad (4.34)$$

which yields an integration "dead end":

$$q_{n+2} - q_{n+1} = \lambda_n \cdot (q_{n+3} - q_n) \cdot q_{n+1}\, q_{n+2} \quad \text{with } \lambda_n = \lambda_{n+2} \,. \tag{4.35}$$

In contrast, if one introduces *other homogeneous variables* q_n *taking into account the shift of* 2 in (4.34), that is $x_n = q_{n+2}/q_n$, one can easily see that more integration steps can be performed. Recurrence (4.19) is now written as

$$\frac{q_{n+3} - q_{n+1}}{(q_{n+3}\, q_{n+4} - q_n\, q_{n+1}) \cdot q_{n+1}\, q_{n+2}\, q_{n+3}} = \text{same expression with } n \longrightarrow n+2 = \lambda_n \,, \quad \text{with } \lambda_n = \lambda_{n+2} \tag{4.36}$$

which gives

$$\lambda_n \cdot q_n\, q_{n+1}\, q_{n+2} + \frac{1}{q_{n+1}} = \rho_n \quad \text{with } \rho_n = \rho_{n+2} \,. \tag{4.37}$$

Let us now consider a recurrence of the general form (4.1) written in a symbolic way as (4.2). Introducing some variables q_n's, well-suited for such a recurrence with a shift of s, namely $x_n = q_{n+s}/q_n$, recurrence (4.2) becomes, with obvious symbolic notations:

$$\frac{q^{(i)}_{n+s} - q^{(i)}_n}{q^{(j)}_{n+s} - q^{(j)}_n} \cdot \frac{q^{(j)}_n}{q^{(i)}_n\, q^{(k)}_n} = \text{same expression with } n \rightarrow n+s = \lambda_n \text{ with } \lambda_n = \lambda_{n+s} \,. \tag{4.38}$$

Let us now assume that condition (4.9) is verified. Rewritten in terms of the q_n's, condition (4.9) reads

$$\frac{q^{(k)}_{n+s}}{q^{(k)}_n} = \frac{q^{(j)}_{n+s}\, q^{(i)}_{n+2s}}{q^{(j)}_n\, q^{(i)}_{n+s}} \,. \tag{4.39}$$

This condition on the q_n's can be rewritten as

$$\frac{q^{(k)}_n}{q^{(j)}_n \cdot q^{(i)}_{n+s}} = \alpha_n \quad \text{with } \alpha_n = \alpha_{n+s} \tag{4.40}$$

but in fact one can actually verify that α_n is actually equal to 1. Relation (4.40) means that the cofactor $q^{(j)}_n / q^{(i)}_n\, q^{(k)}_n$ occurring in recurrence (4.38) is actually equal to $1/(q^{(i)}_n \cdot q^{(i)}_{n+s})$, which allows us to integrate (4.38) as follows:

$$\frac{1}{q^{(i)}_n} - \frac{1}{q^{(i)}_{n+s}} = \lambda_{n+s}\, q^{(j)}_{n+s} - \lambda_n\, q^{(j)}_n \,. \tag{4.41}$$

This last relation enables us to perform one more integration, introducing a new set of constants

$$\lambda_n\, q_n^{(j)} + \frac{1}{q_n^{(i)}} = \rho_n \quad \text{with } \rho_n = \rho_{n+s} \tag{4.42}$$

or equivalently

$$\lambda_n\, q_n^{(j)}\, q_n^{(i)} = \rho_n\, q_n^{(i)} - 1 \quad \text{with } \rho_n = \rho_{n+s} \text{ and } \lambda_n = \lambda_{n+s}\ . \tag{4.43}$$

Let us now assume that condition (4.8) is verified. Rewritten in terms of the q_n's, condition (4.8) reads

$$\frac{q_{n+s}^{(j)}}{q_n^{(j)}} = \frac{q_{n+s}^{(i)}\, q_{n+s}^{(k)}}{q_n^{(i)} q_n^{(k)}}\ . \tag{4.44}$$

This condition on the q_n's also reads

$$\frac{q_n^{(i)} \cdot q_n^{(k)}}{q_n^{(j)}} = \alpha_n \quad \text{with} \quad \alpha_n = \alpha_{n+s} \tag{4.45}$$

and, again, one verifies that α_n is equal to 1. Relation (4.40) means that the cofactor, occurring in recurrence (4.38), is actually equal to 1, which enables one to integrate (4.38) as

$$q_{n+s}^{(i)} - q_n^{(i)} = \lambda_{n+s}\, q_{n+s}^{(j)} - \lambda_n\, q_n^{(j)}\ . \tag{4.46}$$

This last relation enables us to perform one more integration, introducing a new set of constants

$$\lambda_n\, q_n^{(j)} - q_n^{(i)} = \rho_n \quad \text{with} \quad \rho_n = \rho_{n+s} \tag{4.47}$$

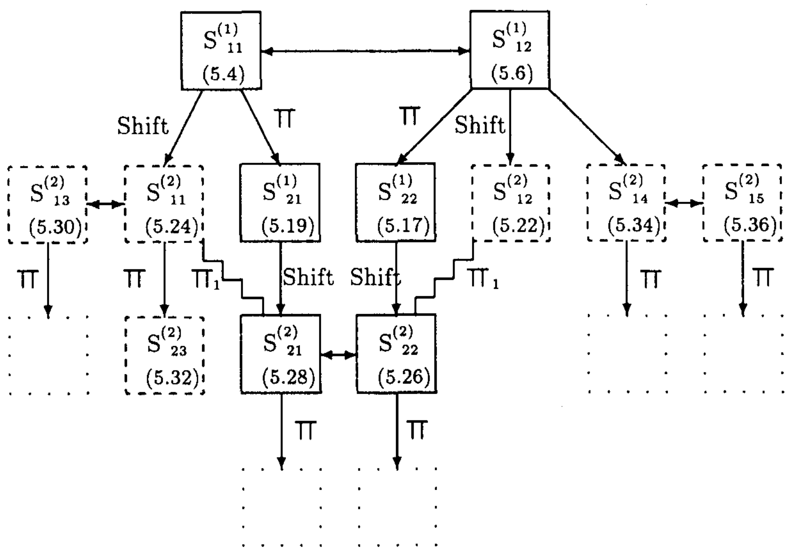
or equivalently

$$\lambda_n\, q_n^{(j)} = \rho_n + q_n^{(i)} \quad \text{with} \quad \rho_n = \rho_{n+s} \text{ and } \lambda_n = \lambda_{n+s}\ . \tag{4.48}$$

All these constants of integration may, or may not, be enough to completely integrate recurrence (4.1): for instance it has been seen that many of the recurrences in the hierarchy ((4.19) for example) *are simply not integrable*, even when satisfying (4.8) or (4.9).

4.6.3. *Hierarchy diagram*

Let us sum up the connections between these different recurrences and their respective integrability, or nonintegrability, by the following diagram.



where □ corresponds to an integrable recurrence

and ⬚ corresponds to a non-integrable recurrence (4.49)

This diagram is of course a "truncated" diagram: it goes to infinity since one can always transform any recurrence into a new recurrence using the shift procedure of Sec. 4.3.

4.7. *Three-parameter symmetry group*

Let us try here to understand why the homogeneous variables q_n's are well-suited variables in order to integrate the previous recurrences (when integrable). We will see that the variables q_n's take into account a *three-parameter group* of symmetry of the recurrences.

Let us first remark that the variables f_n always occur through the product $f_n \cdot f_{n+1}$. As a consequence, it is tempting to consider the new variables $g_n = f_n \cdot f_{n+1}$. In terms of these new homogeneous polynomials one of the simplest recurrence on the f_n's already introduced (for instance recurrence (3.24)) reads

$$\frac{g_n\, g_{n+2}^3 - g_{n+3}\, g_{n+1}^3}{g_{n-1}\, g_{n+1}\, g_{n+3}^2 - g_n^2\, g_{n+2}\, g_{n+4}} \cdot \frac{g_n\, g_{n+3}}{g_{n+1}\, g_{n+2}} = \text{same expression with } n \to n+1\ . \tag{4.50}$$

Remarkably, this new equation has a *three-parameter dependent symmetry group*, namely

$$g_n \longrightarrow a^{n^2} \cdot b^n \cdot c \cdot g_{n+1} \ . \qquad (4.51)$$

It is thus interesting to introduce a new variable, invariant by the two parameters b and c of group (4.51):

$$r_n = \frac{g_n \, g_{n+2}}{g_{n+1}^2} = \frac{f_n \, f_{n+3}}{f_{n+1} \, f_{n+2}} \ . \qquad (4.52)$$

In terms of this new variable, recurrence (4.50) reads

$$\frac{r_n - r_{n+1}}{r_{n-1} - r_{n+2}} \cdot \frac{1}{r_n \, r_{n+1}} = \text{same expression with } n \longrightarrow n+1 \ . \qquad (4.53)$$

Let us remark that variable r_n actually identifies with q_{n+2} introduced in Sec. 4.6. The homogeneous variable q_n transforms, under the three-parameter group (4.51), according to

$$q_n \longrightarrow a^2 \cdot q_{n+1} \ . \qquad (4.54)$$

In contrast, the variable $x_n = q_{n+1}/q_n$ (see Sec. 4.6) is actually *invariant under the whole three-parameter symmetry group* (4.51).

4.7.1. *Comment on the integration of the recurrences on the x_n's*

One should note that one can also integrate our recurrences in terms of the x_n's, using the integration performed with the well-suited variables q_n's (see Sec. 4.6). Let us, for instance, consider the integration of one of our recurrences in terms of two biquadratics (see for instance, denoted $B_1(q_n, q_{n+1})$ and $B_2(q_{n+1}, q_{n+2})$ (*in our previous examples one remarks that one has* $B_2(q_{n+1}, q_{n+2}) = B_1(q_{n+2}, q_{n+1})$). Using the very relation between the x_n's and the q_n's, the system of these two biquadratic relations reads

$$B_1(q_n, q_n \cdot x_n) = 0 \ , \quad B_2(q_n \cdot x_n, q_n \cdot x_n \cdot x_{n+1}) = 0 \ . \qquad (4.55)$$

Eliminating the homogeneous variable q_n, one immediately gets a relation between x_n and x_{n+1}. Let us consider, for instance, the simplest example of integrable recurrence, that is recurrence (4.4) (or equivalently (4.5)). For this example the two q_n-biquadratics, B_1 and B_2, identify (see (4.27)) and the resultant between them yields a *bicubic* relation.

The other recurrences can similarly be integrated in terms of the variables x_n's, and one also gets *bicubic relations.*

Recalling the occurrence of general (nonsymmetric) biquadratic relations (see for instance the "pre-Bethe Ansatz" for the sixteen vertex model[9]), it is interesting to see what kind of relations can be deduced from the previous procedure. The result is the following: the elimination of q_n between $B(q_n, q_n \cdot x_n)$ and $B(q_n \cdot x_n \cdot x_{n+1}, q_n \cdot x_n)$[b] for a general biquadratic B with its nine coefficients, yields a ***biquartic*** (of a particular form).

Let us however note that the elimination[c] of q_n between $B(q_n, q_n \cdot x_n)$ and $B(q_n \cdot x_n, q_n \cdot x_n \cdot x_{n+1})$, for a general biquadratic B, yields much higher degree relations.

4.8. *Biquadratic relations versus Weierstrass's canonical form*

The elliptic curve (4.27) can be rewritten, after several transformations, in the *canonical Weierstrass's form*[9,17]:

$$y^2 = 4\,x^3 - g_2\,x - g_3 \tag{4.56}$$

where

$$12\,g_2 = 16\,\lambda^2 + 8\,\lambda\,\rho^2 - 24\,\rho\,\mu + \rho^4 \tag{4.57}$$

$$216\,g_3 = -\,48\,\lambda^2\rho^2 - 64\,\lambda^3 + 144\,\lambda\,\rho\,\mu - 216\,\mu^2 - 12\,\lambda\,\rho^4 + 36\,\rho^3\,\mu - \rho^6 \tag{4.58}$$

and the discriminant reads

$$\Delta = g_2^3 - 27\,g_3^2 = -\,\mu^2\,(16\,\lambda^3 - \rho^3\,\mu + 8\,\lambda^2\,\rho^2 + \lambda\,\rho^4 - 36\,\lambda\,\rho\,\mu + 27\,\mu^2)\,. \tag{4.59}$$

The relation between integrable recurrences on the various variables we introduced (f_n, g_n, k_n, l_n, x_n, q_n, see (3.24), (4.50), (3.28), (4.4), (3.35), (4.53)) and biquadratic relations, such as (4.27), is reminiscent of the relation between the integrable birational mappings associated with the *sixteen-vertex model* and their associated biquadratic relations corresponding to the (pre-) Bethe Ansatz on the same model.[9] In fact, all these integrable mappings and recurrences can be seen to be associated with *different representations of the same elliptic curve* represented in the simplest way by a canonical Weierstrass form like (4.56).[9,17] Conversely, one can get Eq. (4.53) from the biquadratic equation (4.27). Let us consider the biquadratic relation (4.27), written in a homogeneous way, and the same equations where n is

[b]Remark the permutation of the arguments for the second biquadratic: $B(q_n \cdot x_n \cdot x_{n+1}, q_n \cdot x_n)$ instead of $B(q_n \cdot x_n, q_n \cdot x_n \cdot x_{n+1})$.

[c]This elimination procedure gives quite different results from a direct elimination of q_{n+1} between $B(q_n, q_{n+1})$ and $B(q_{n+2}, q_{n+1})$ which remarkably gives a *symmetric biquadratic* relation between q_n and q_{n+2}, while the elimination of q_{n+1} between $B(q_n, q_{n+1})$ and $B(q_{n+1}, q_{n+2})$ gives a *biquartic* relation between q_n and q_{n+2}.

shifted by one each time, yielding the system of equations

$$\begin{aligned}
\lambda_0 + \lambda_1\,(q_n + q_{n+1}) + \lambda_2\, q_n\, q_{n+1} + \lambda_3\, q_n\, q_{n+1}\,(q_n + q_{n+1}) &= 0\\
\lambda_0 + \lambda_1\,(q_{n+1} + q_{n+2}) + \lambda_2\, q_{n+1}\, q_{n+2} + \lambda_3\, q_{n+1}\, q_{n+2}\,(q_{n+1} + q_{n+2}) &= 0\\
\lambda_0 + \lambda_1\,(q_{n+2} + q_{n+3}) + \lambda_2\, q_{n+2}\, q_{n+3} + \lambda_3\, q_{n+2}\, q_{n+3}\,(q_{n+2} + q_{n+3}) &= 0\\
\lambda_0 + \lambda_1\,(q_{n+3} + q_{n+4}) + \lambda_2\, q_{n+3}\, q_{n+4} + \lambda_3\, q_{n+3}\, q_{n+4}\,(q_{n+3} + q_{n+4}) &= 0\,.
\end{aligned} \tag{4.60}$$

The elimination of the variables λ_0, λ_1, λ_2, λ_3 yields a determinantal compatibility condition *which actually identifies with recurrence* (4.53):

$$\begin{vmatrix}
1 & (q_n + q_{n+1}) & q_n\, q_{n+1} & q_n\, q_{n+1}\,(q_n + q_{n+1})\\
1 & (q_{n+1} + q_{n+2}) & q_{n+1}\, q_{n+2} & q_{n+1}\, q_{n+2}\,(q_{n+1} + q_{n+2})\\
1 & (q_{n+2} + q_{n+3}) & q_{n+2}\, q_{n+3} & q_{n+2}\, q_{n+3}\,(q_{n+2} + q_{n+3})\\
1 & (q_{n+3} + q_{n+4}) & q_{n+3}\, q_{n+4} & q_{n+3}\, q_{n+4}\,(q_{n+3} + q_{n+4})
\end{vmatrix} = 0 \tag{4.61}$$

4.9. *Finite orbits*

The search for solution of the so-called *tetrahedron relations* (and their higher dimensional generalization)[6,18] is clearly an important challenge in lattice statistical mechanics.[19–22] It has been argued that, generically, the symmetry groups of these (over-determined) relations is too large to allow solutions to survive to such strong constraints.[5,23] Solutions should correspond to cases where such large symmetry group degenerate into symmetry groups similar to the one of the two-dimensional models, or, more probably, *the best "bet" for finding solutions should amount to seeking for models (or conditions on the models) for which these groups degenerate into finite groups.*[5,23] *This situation amounts to say that birational transformations like K are of finite order.*[5,23,d]

Unfortunately, finding the algebraic varieties corresponding to finite orbits for birational mappings in $\mathbb{C}P_{q^2-1}$ yields too large formal calculations. Let us seize the opportunity we have here to *actually associate to these birational mappings* in $\mathbb{C}P_{q^2-1}$ *simpler recurrences on a single variable*, in order to get these *finite order conditions.* As a consequence of this correspondence, the algebraic varieties corresponding to finite orbits for birational mappings in $\mathbb{C}P_{q^2-1}$ *are included* in the (algebraic varieties) conditions corresponding to write down the conditions for these recurrences to be periodic.

[d] One can also recall some examples coming from lattice statistical mechanics with a $\mathbb{C}P_2$ parameter space, such that the iteration of the corresponding birational mappings has been shown to yield elliptic curves foliating the whole parameter space $\mathbb{C}P_2$.[24,25] The set of points for which these birational transformations are of finite order have been shown to be particular curves of this linear pencil of elliptic curves.[24,25] The knowledge of these finite orbit curves gives a precious hint to get the algebraic invariants enabling to write down this foliation.

Recurrence (4.4) can be seen to have finite order orbits, which can be written alternatively as algebraic conditions bearing on the x_n's, or the q_n's, or *even on the three parameters* λ, μ and ρ (see (4.27) and (4.53)), thus giving as many examples of "special" Weierstrass's elliptic curves (4.56).

For instance the orbits of order four ($x_n = x_{n+4}$, $q_n = q_{n+4}$) correspond to the following algebraic conditions:

$$x_n\, x_{n+2} - 1 = 0\ , \quad \mu = \lambda\, \rho \quad \text{or} \quad q_{n+1}\, q_{n+3} - q_n\, q_{n+2} = 0\ . \tag{4.62}$$

These conditions can also be seen to correspond to the following relations mixing λ and the q_n's:

$$q_n\, q_{n+2} = \lambda\ . \tag{4.63}$$

The orbits of order three are also simple. They read

$$x_n\, x_{n+1}\, x_{n+2} - 1 = 0\ . \tag{4.64}$$

The orbits of order five in variables q_n, x_n or λ, μ and ρ respectively read

$$(x_n\, x_{n+1}^2\, x_{n+2} - 1)\,(x_n\, x_{n+2} - 1) = x_{n+2}\,(x_n\, x_{n+1}\, x_{n+2} - 1)\,(x_{n+1} - 1) \tag{4.65}$$

or

$$(q_{n+2}\, q_{n+3} - q_n\, q_{n+1})\,(q_{n+1}\, q_{n+3} - q_n\, q_{n+2}) = q_n\, q_{n+3}\,(q_{n+3} - q_n)\,(q_{n+2} - q_{n+1}) \tag{4.66}$$

or

$$\mu^2 - \mu\, \lambda\, \rho + \lambda^3 = 0\ . \tag{4.67}$$

Amazingly, recalling the bicubic relation between x_n and x_{n+1}, it is worth noticing that the vanishing of three of the coefficients of the bicubic *are nothing but finite order conditions*, respectively third order, fourth order and fifth order conditions.

Clearly, and, similarly to what happened in Refs. 24, 25, the finite order conditions become quickly involved. One "handable" way to describe these finite order conditions seems to write them down in terms of the three parameters λ, μ and ρ.

One should also note that these conditions yield remarkably simple conditions on the *products* of the x_n's (involving larger number of consecutive x_n's compared to conditions like (4.65)). This can be understood as follows: all the previous recurrences can be written in a short way, using symbolic notations similar to the one of (4.2), as

$$E_{n+s} = x_n^{(k)} \cdot E_n \tag{4.68}$$

where E_n denotes some expression in the variables x_n. A condition of finite order N on the x_n's, of course implies a condition of the same finite order N on the E_n's ($E_{n+N} = E_n$). One thus directly gets a condition bearing on the factors $x_n^{(k)}$:

$$x_{n+sN}^{(k)} \cdot x_{n+(s-1)N}^{(k)} \cdots x_n^{(k)} = 1 \ . \tag{4.69}$$

It is important to note that such conditions on the q_n's are also, to some extent, finite order conditions for all the recurrences of the hierarchy detailed in Sec. 4.6.3 (see (4.49)). More precisely, for the *nonintegrable recurrences* ((4.19), (4.18) ...) one can always look for such finite order conditions among the finite order conditions occurring for recurrences closer to the basic recurrence (4.4) at the top of the "hierarchical diagram" previously described (see Sec. 4.6.3).

5. Relation with the Sixteen-Vertex Model

In the case of 4×4 matrices, a particular permutation of the entries of the matrix, t_1, has been introduced in the framework of the symmetries of the sixteen-vertex model.[9] This permutation corresponds to

$$m_{12} \leftrightarrow m_{21} \ , \quad m_{32} \leftrightarrow m_{41} \ , \quad m_{23} \leftrightarrow m_{14} \ , \quad m_{43} \leftrightarrow m_{34} \ . \tag{5.1}$$

Remarkably, the symmetry group generated by the matrix inverse $\widehat{I}$ and transformation t_1, or by the infinite generator $K_{t_1} = t_1 \cdot \widehat{I}$, *has been shown to yield algebraic elliptic curves* given by intersection of quadrics.[9] We analyze here this particular permutation of the entries, since, as it will be seen in the following, it surprisingly yields closely related results.

5.1. *Factorization properties for the partial transposition t_1*

Let us consider a 4×4 matrix M_0 and the successive matrices obtained by iteration of transformation $K_{t_1} = t_1 \cdot I$, where t_1 is the permutation of the coefficients of the 4×4 matrix previously described associated with the sixteen-vertex model.[5,9] The first matrices and determinants read

$$M_1 = K_{t_1}(M_0) \ , \quad M_2 = K_{t_1}(M_1) \ , \quad F_1 = \det(M_0) \ , \quad F_2 = \det(M_1) \ .$$

From the third action of K, factorizations appear, they read for arbitrary n:

$$M_{n+2} = \frac{K_{t_1}(M_{n+1})}{F_n^2} \ , \quad F_{n+2} = \frac{\det(M_{n+1})}{F_n^3}$$
$$\text{and} \quad \frac{K_{t_1}(M_{n+2})}{\det(M_{n+2})} = \frac{M_{n+3}}{F_{n+1}F_{n+3}} \ . \tag{5.2}$$

From these factorizations, one can easily get linear recurrences on the exponents α_n, β_n, μ_n and ν_n, and the following expressions for their generating functions:

$$\alpha(x) = \frac{4(1+3x^2)}{(1-x)^3} , \quad \beta(x) = \frac{4x}{(1-x)^3} , \quad \mu(x) = \frac{x^2(3-x)}{(1-x)^3} , \quad \nu(x) = \frac{2x^2}{(1-x)^3} .$$

The expressions of the exponents α_n, β_n, μ_n and ν_n read respectively:

$$\alpha_n = 4\,(2\,n^2+1) , \quad \beta_n = 2\,n\,(n+1) , \quad \mu_n = n^2 - 1 , \quad \nu_n = n\,(n-1) .$$

It is worth noticing that these factorizations *are also satisfied for* K (corresponding to transposition $m_{12} \leftrightarrow m_{21}$) instead of K_{t_1}, however the associated polynomials F_n's are no longer the optimal factorizations; namely one recovers from (5.2) the factorization relations (3.3) and (3.4) for $q = 4$, changing K_{t_1} into K and F_n into $f_{n-1}\,f_n$.

5.2. *Recurrences for the partial transposition* t_1

Amazingly, the F_n's corresponding to K_{t_1} do satisfy *exactly the same recurrences as* (3.24) *where the* f_n*'s are replaced by the* F_n*'s*:

$$\frac{F_n F_{n+3}^2 - F_{n+4}F_{n+1}^2}{F_{n-1}F_{n+3}F_{n+4} - F_n F_{n+1}F_{n+5}} = \frac{F_{n-1}F_{n+2}^2 - F_{n+3}F_n^2}{F_{n-2}F_{n+2}F_{n+3} - F_{n-1}F_n F_{n+4}} \tag{5.3}$$

which in terms of the l_n's reads

$$\frac{l_{n+4}l_{n+3}^2 l_{n+2} - 1}{l_{n+5}l_{n+4}^3 l_{n+3}^4 l_{n+2}^3 l_{n+1} - 1} = \frac{l_{n+3}l_{n+2}^2 l_{n+1} - 1}{l_{n+4}l_{n+3}^3 l_{n+2}^4 l_{n+1}^3 l_n - 1} \cdot l_n l_{n+1}^2 l_{n+2}^2 l_{n+3}^2 l_{n+4} \tag{5.4}$$

or, for instance,

$$\frac{F_n^2 F_{n+2}F_{n+4} - F_{n-1}F_{n+1}F_{n+3}^2}{F_{n+1}^3 F_{n+3} - F_n F_{n+2}^3} = \frac{F_{n+1}^2 F_{n+3}F_{n+5} - F_n F_{n+2}F_{n+4}^2}{F_{n+2}^3 F_{n+4} - F_{n+1}F_{n+3}^3} \tag{5.5}$$

which in terms of the l_n's reads

$$\frac{l_{n+2}\,l_{n+3} - 1}{l_{n+1}\,l_{n+2}^2\,l_{n+3}^2\,l_{n+4} - 1} = \frac{l_{n+1}\,l_{n+2} - 1}{l_n\,l_{n+1}^2\,l_{n+2}^2\,l_{n+3} - 1} \cdot l_n\,l_{n+1}^2\,l_{n+2}^3\,l_{n+3}^2 \tag{5.6}$$

and in terms of the x_n's reads

$$\frac{x_{n+2} - 1}{x_{n+1}\,x_{n+2}\,x_{n+3} - 1} = \frac{x_{n+1} - 1}{x_n\,x_{n+1}\,x_{n+2} - 1}\,x_n\,x_{n+1}\,x_{n+2}^2 . \tag{5.7}$$

Among the various recurrences verified by the l_n's, one also has

$$\frac{l_{n+5}l_{n+4}^3l_{n+3}^3l_{n+2}-1}{l_{n+6}l_{n+5}^4l_{n+4}^7l_{n+3}^7l_{n+2}^4l_{n+1}-1} = \frac{l_{n+4}l_{n+3}^3l_{n+2}^3l_{n+1}-1}{l_{n+5}l_{n+4}^4l_{n+3}^7l_{n+2}^7l_{n+1}^4l_n-1} \cdot l_n l_{n+1}^2 l_{n+2}^2 l_{n+3}^2 l_{n+4} \,. \tag{5.8}$$

On the x_n's, one has the following recurrences (coded with the symbolic sequences previously described):

$$\begin{aligned}
\mathcal{S}_{1\,1}^{(1)} &= ((1),\,(1),\,(1,\,1,\,1),\,(1,\,1,\,2)) \ , \\
\mathcal{S}_{1\,2}^{(1)} &= ((1),\,(1),\,(1,\,2,\,1),\,(1,\,1,\,1))\cdots \\
\mathcal{S}_{1\,1}^{(2)} &= ((2),\,(1),\,(1,\,1,\,1),\,(1,\,2,\,3,\,2)) \ , \\
\mathcal{S}_{1\,2}^{(2)} &= ((2),\,(1),\,(1,\,2,\,1),\,(1,\,2,\,2,\,1))\cdots \\
\mathcal{S}_{2\,1}^{(1)} &= ((1),\,(1,\,1),\,(1,\,1,\,1,\,1),\,(1,\,1,\,2,\,2)) \ , \\
\mathcal{S}_{2\,2}^{(1)} &= ((1),\,(1,\,1),\,(1,\,2,\,2,\,1),\,(1,\,1,\,1,\,1)) \\
\mathcal{S}_{2\,3}^{(1)} &= ((1),\,(1,\,1),\,(1,\,3,\,3,\,1),\,(1,\,1,\,1,\,0))\,, \ \cdots \,.
\end{aligned} \tag{5.9}$$

Recurrences (5.5) or (5.6) are nothing but recurrence $S_{1\,2}^{(1)}$ on the x_n's.

Let us give some examples of such relations. $\mathcal{S}_{2\,2}^{(1)}$ reads

$$\frac{x_{n+2}\,x_{n+3}-1}{x_{n+1}\,x_{n+2}^2\,x_{n+3}^2\,x_{n+4}-1} = \frac{x_{n+1}\,x_{n+2}-1}{x_n\,x_{n+1}^2\,x_{n+2}^2\,x_{n+3}-1}\,x_n\,x_{n+1}\,x_{n+2}\,x_{n+3} \tag{5.10}$$

(this equation is nothing but (5.3) or (5.4)) and $\mathcal{S}_{3\,8}^{(1)}$ (see Appendix E) reads

$$\begin{aligned}
&\frac{x_{n+2}\,x_{n+3}^2\,x_{n+4}-1}{x_{n+1}\,x_{n+2}^3\,x_{n+3}^4\,x_{n+4}^3\,x_{n+5}-1} \\
&\quad = \frac{x_{n+1}\,x_{n+2}^2\,x_{n+3}-1}{x_n\,x_{n+1}^3\,x_{n+2}^4\,x_{n+3}^3\,x_{n+4}-1}\,x_n\,x_{n+1}\,x_{n+2}\,x_{n+3}
\end{aligned} \tag{5.11}$$

(this equation is nothing but (5.8)). *Again one has a whole hierarchy of recurrences.* The analysis performed in Sec. 4.6.3) for transposition $m_{[1,\,2]} - m_{[2,\,1]}$ is still valid.

Amazingly a recurrence in the variables l_n for transposition $m_{12} \leftrightarrow m_{21}$ is also a recurrence in the variables x_n for t_1.

For instance, recurrence (5.10) (resp. (5.11)) in the variables x_n, for t_1, identifies with recurrence (3.31) (resp. (3.32)) on the variables l_n for transposition $m_{12} \leftrightarrow m_{21}$.

Clearly, performing the ratios of the left-hand sides and the right-hand sides of Eqs. (5.10) and (5.11), one can get an invariant I_n for the action of K_{t_1}:

$$I_n = \frac{x_{n+1}\, x_{n+2}^2\, x_{n+3} - 1}{x_n\, x_{n+1}^3\, x_{n+2}^4\, x_{n+3}^3\, x_{n+4} - 1} \cdot \frac{x_n\, x_{n+1}^2\, x_{n+2}^2\, x_{n+3} - 1}{x_{n+1}\, x_{n+2} - 1}$$
$$= \text{same expression with } n \to n+1 \ . \qquad (5.12)$$

Of course many more invariants can be obtained from similar ratios of recurrences for K_{t_1} and also for K (for transposition $m_{12} \leftrightarrow m_{21}$). The relation between these invariants and some quadratic covariants under the action of K_{t_1} or K (see Ref. 9) has been detailed elsewhere.[8]

The relation between the various variables k_n, l_n, x_n and F_n read as follows: one has the same relations as (3.36), (3.37), (3.38) and (3.39) but of course for $q = 4$, and one also has the following expression for k_n in term of the F_n's:

$$k_n = (F_1^{3^{n-3}} \cdot F_2^{3^{n-4}} \cdot F_3^{3^{n-5}} \cdots F_{n-4}^{3^2} \cdot F_{n-3}^3 \cdot F_{n-2})^8 \cdot F_{n-1}^3 \cdot F_{n+1} \qquad (5.13)$$

One recovers Eq. (3.40), for $q = 4$, again changing F_n into $f_{n-1}\, f_n$. Combining (5.13) and (3.39), one gets x_n in terms of F_n's:

$$x_n = \frac{k_{n+1}}{k_n^3} = \frac{(F_1^{3^{n-2}} \cdot F_2^{3^{n-3}} \cdot F_3^{3^{n-4}} \cdots F_{n-3}^{3^2} \cdot F_{n-2}^3 \cdot F_{n-1})^8 \cdot F_n^3 \cdot F_{n+2}}{[(F_1^{3^{n-3}} \cdot F_2^{3^{n-4}} \cdot F_3^{3^{n-5}} \cdots F_{n-4}^{3^2} \cdot F_{n-3}^3 \cdot F_{n-2})^8 \cdot F_{n-1}^3 \cdot F_{n+1}]^3}$$
$$= \frac{F_n^3\, F_{n+2}}{F_{n+1}^3\, F_{n-1}} = \frac{f_{n-1}^2\, f_{n+2}}{f_{n+1}^2\, f_{n-2}} \ . \qquad (5.14)$$

Similarly to what has been done for transposition $m_{12} \leftrightarrow m_{21}$, one can extend t_1 to $q \times q$ matrices, defining again t_1 as

$$m_{12} \leftrightarrow m_{21}\ , \quad m_{32} \leftrightarrow m_{41}\ , \quad m_{23} \leftrightarrow m_{14}\ , \quad m_{43} \leftrightarrow m_{34}\ , \qquad (5.15)$$

the other entries m_{ij} being unchanged.

Unfortunately, the factorizations do not yield a polynomial growth of the complexity, and we have not been able to find any simple recurrence. Many other generalizations of the partial transposition t_1 to $q \times q$ matrices can also be seen to fail yielding recurrences on some f_n's.[3]

6. Demonstration of the Factorizations and of the Recurrences

We will here establish the previous factorizations and recurrences. We will consider the $q \times q$ matrix R_q (2.1).

As the transposition t acts symmetrically on only two entries of matrix R_q, it can be seen as a "deformation" of the matrix in a fixed direction led by matrix P:

$$t(R_q) = R_q + \Delta_0 \cdot P$$

where $\Delta_0 = [R_q]_{21} - [R_q]_{12} = m_{21} - m_{12}$, and P reads

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots \\ -1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} .$$

The inhomogeneous transformation $\widehat{K}$, can thus be seen as a "deformation" of the matricial inverse $\widehat{I}$:

$$\widehat{K}(R_q) = \widehat{I}(R_q) - \Delta_1 \cdot P \tag{6.1}$$

where

$$\Delta_1 = [\widehat{K}(R_q)]_{21} - [\widehat{K}(R_q)]_{12} = [\widehat{I}(R_q)]_{12} - [\widehat{I}(R_q)]_{21} . \tag{6.2}$$

Let us introduce matrix $U = R_q \cdot \widehat{K}(R_q)$, which is, by construction, very close from the identity matrix ($U = \mathcal{I}d_q - \Delta_1 \, R_q \cdot P$)

$$U = \begin{bmatrix} 1 + \Delta_1 m_{12} & -\Delta_1 m_{11} & 0 & 0 & \dots \\ \Delta_1 m_{22} & 1 - \Delta_1 m_{21} & 0 & 0 & \dots \\ \Delta_1 m_{32} & -\Delta_1 m_{31} & 1 & 0 & \dots \\ \vdots & \vdots & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} .$$

This expression of U gives at once the determinant

$$\begin{aligned} \det(U) &= \det(R_q) \cdot \det(\widehat{K}(R_q)) = l_0 \, l_1 = x_0 \\ &= (1 + \Delta_1 m_{12})(1 - \Delta_1 m_{21}) + \Delta_1^2 m_{11} m_{22} \\ &= 1 + (m_{12} - m_{21}) \Delta_1 + (m_{11} m_{22} - m_{12} m_{21}) \Delta_1^2 \\ &= 1 - \Delta_0 \Delta_1 + N_0 \Delta_1^2 \end{aligned} \tag{6.3}$$

where $N_0 = (m_{11} m_{22} - m_{12} m_{21})$ (that is the 2×2 minor corresponding to rows and columns 1 and 2 of the R_q matrix). Let us notice that this provides an easy way to calculate the determinant of $\widehat{K}(R_q)$:

$$\det(\widehat{K}(R_q)) = x_0 / \det(R_q) . \tag{6.4}$$

One is now able to calculate easily the second step of the iteration

$$\widehat{K}^2(R_q) = t(\widehat{I}(\widehat{K}(R_q))) = t(\widehat{I}(U).R_q) \tag{6.5}$$

where $\widehat{I}(U)$ reads

$$\widehat{I}(U) = \begin{bmatrix} (1-\Delta_1 m_{21})/x_0 & \Delta_1 m_{11}/x_0 & 0 & 0 & \dots \\ -\Delta_1 m_{22}/x_0 & (1+\Delta_1 m_{12})/x_0 & 0 & 0 & \dots \\ T_1 & T_2 & 1 & 0 & \dots \\ \vdots & \vdots & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{6.6}$$

with the notations

$$T_1 = -\frac{\Delta_1 (m_{32} + \Delta_1 (m_{31} m_{22} - m_{32} m_{21}))}{x_0}$$

$$T_2 = -\frac{\Delta_1 (m_{31} + \Delta_1 (m_{31} m_{12} - m_{32} m_{11}))}{x_0} .$$

Thus the explicit form of $\widehat{K}^2(R_q)$ reads

$$\widehat{K}^2(R_q) = \begin{bmatrix} m_{11}/x_0 & (m_{21} - \Delta_1 N_0)/x_0 & (m_{13} + \Delta_1 (m_{11} m_{23} - m_{21} m_{13}))/x_0 & \dots \\ (m_{12} + \Delta_1 N_0)/x_0 & m_{22}/x_0 & (m_{23} + \Delta_1 (m_{12} m_{23} - m_{22} m_{13}))/x_0 & \dots \\ T_2/\Delta_1 & -T_1/\Delta_1 & T_1 m_{13} + T_2 m_{23} + m_{33} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} . \tag{6.7}$$

Since these results have been calculated for a generic matrix R_q, they can be applied successively on each matrix $\widehat{K}^n(R_q)$. Thus, all the equations given above in this section are actually recurrence relations. Furthermore, the expressions of $\widehat{K}(R_q)$ and $\widehat{K}^2(R_q)$ will permit us to demonstrate, first the recurrence on x_n's and, in a second step, the factorizations of the matrices $K^n(R_q)$ and of their determinants.

It is worth noticing that the orbits of $\widehat{K}^2$ move inside a five dimensional affine space (see also Appendix A) which depends on the initial matrix R_q (or equivalently any of its iterates $\widehat{K}^{2n}(R_q)$).

$$\widehat{K}^{2n}(R_q) = \frac{1}{x_0 \, x_2 \cdots x_{2n-2}} \cdot \Big(R_q + a_n \cdot P + b_n \cdot \widehat{K}^2(R_q) + c_n \cdot \widehat{K}^4(R_q) + d_n \cdot \widehat{K}^6(R_q) + e_n \cdot \widehat{K}^8(R_q)\Big) . \tag{6.8}$$

6.1. *Demonstration of the recurrences*

Straightforward calculations on the expression of $\widehat{K}^{n+2}(R_q)$ as a function of $\widehat{K}^n(R_q)$, allow us to write the following relations:

$$N_{n+2} = \frac{N_n}{x_n} \tag{6.9}$$

$$\Delta_{n+2} = \frac{-\Delta_n + 2\,\Delta_{n+1}\,N_n}{x_n}\,. \tag{6.10}$$

Using the expression of x_n, namely (6.3):

$$x_n = 1 - \Delta_n\,\Delta_{n+1} + N_n\,\Delta_{n+1}^2 \tag{6.11}$$

it is easy to eliminate N_n to obtain two equations on x_n and Δ_n:

$$(x_n\,\Delta_{n+2} + \Delta_n)\,\Delta_{n+1} = 2\,\Delta_{n+1}^2\,N_n = 2\,(x_n - 1 + \Delta_n\,\Delta_{n+1})$$

$$N_{n+2} = \frac{x_{n+2} - 1 + \Delta_{n+2}\,\Delta_{n+3}}{\Delta_{n+3}^2} = \frac{N_n}{x_n} = \frac{x_n - 1 + \Delta_n\,\Delta_{n+1}}{\Delta_{n+1}^2\,x_n}\,. \tag{6.12}$$

Introducing the well-suited variables $p_n = \Delta_n\,\Delta_{n+1} - 2$, Eqs. (6.12) become

$$x_n\,p_{n+1} = p_n \tag{6.13}$$

$$\frac{x_{n+2} + 1 + p_{n+2}}{(p_{n+2} + 2)^2} = \frac{x_n + 1 + p_n}{(p_{n+1} + 2)^2\,x_n}\,. \tag{6.14}$$

The last equations straightforwardly give

$$\left(x_{n+2} + 1 + \frac{p_n}{x_n\,x_{n+1}}\right)\left(\frac{p_n}{x_n} + 2\right)^2 x_n = (x_n + 1 + p_n)\left(\frac{p_n}{x_n\,x_{n+1}} + 2\right)^2\,. \tag{6.15}$$

One obtains a second equation shifting n by one and replacing p_{n+1} by p_n/x_n

$$\left(x_{n+3} + 1 + \frac{p_n}{x_n\,x_{n+1}\,x_{n+2}}\right)\left(\frac{p_n}{x_n\,x_{n+1}} + 2\right)^2 x_{n+1}$$
$$= \left(x_{n+1} + 1 + \frac{p_n}{x_n}\right)\left(\frac{p_n}{x_n\,x_{n+1}\,x_{n+2}} + 2\right)^2\,. \tag{6.16}$$

The elimination of p_n amounts to calculating the resultant of (6.15) and (6.16) with respect to variable p_n. This resultant, up to the trivial solutions ($x_{n+1} = 1$

and $x_{n+2} = 1$), yields the recurrence

$$(x_{n+2}-1)(x_n\, x_{n+1}\, x_{n+2}-1)-(x_{n+1}-1)(x_{n+1}\, x_{n+2}\, x_{n+3}-1)\, x_n\, x_{n+2} = 0 \; . \quad (6.17)$$

One remarks that recurrence (6.17) is nothing but recurrence (4.4), which yields the whole hierarchy of recurrences detailed in Sec. 4.6.3.

Let us remark that q_n and p_n introduced in Secs. 4.6 and 6.1 (see for instance Eq. (6.13)) are simply related: $q_n = q_0\, p_0/p_n$.

One should however note that the q_n's *are homogeneous variables, which is not the case of the* p_n's.

One recovers Eq. (4.4). Similarly, one can deduce a recurrence bearing only on p_n using (6.13) in (6.17). One gets

$$\frac{1}{p_n\, p_{n+3}} \cdot \frac{p_n - p_{n+3}}{p_{n+1} - p_{n+2}} = \frac{1}{p_{n+1}\, p_{n+4}} \cdot \frac{p_{n+1} - p_{n+4}}{p_{n+2} - p_{n+3}} \; . \quad (6.18)$$

Amazingly, this recurrence between p_n, p_{n+1}, p_{n+2}, p_{n+3} and p_{n+4} is compatible with a shorter recurrence between only p_n, p_{n+1}, p_{n+2} and p_{n+3}:

$$p_{n+3} = \frac{p_n\, p_{n+2}\, (p_{n+1} + 2)^2}{-p_n\, p_{n+1}^2 - p_n\, p_{n+2}\, p_{n+1}^2 + p_n\, p_{n+2}^2 + p_{n+1}\, p_{n+2}^2 + 4 p_{n+2}\, p_{n+1} + 4 p_{n+1} + p_n\, p_{n+1}\, p_{n+2}^2} \; . \quad (6.19)$$

6.2. *Demonstration of the factorizations*

Factorization properties are obviously meaningless on matrices $\widehat{K}^n(R_q)$, which have no polynomial entries, but rational ones. Let us thus change the previous variables to the homogeneous ones:

$$K(R_q) = \det(R_q)\, \widehat{K}(R_q) \; ;$$

K being homogeneous of degree $q-1$ one obtains:

$$\begin{aligned} K^2(R_q) &= \det(R_q)^{q-1}\, K(\widehat{K}(R_q)) \\ &= \det(R_q)^{q-1}\, \det(\widehat{K}(R_q))\, \widehat{K}^2(R_q) \\ &= \det(R_q)^{q-2}\, x_0\, \widehat{K}^2(R_q) \; . \end{aligned} \quad (6.20)$$

Let us now consider the form of $x_0\, \widehat{K}^2(R_q)$ given by Eq. (6.7). One remarks that its entries are polynomials in the entries of the matrix R_q and *quadratic* in Δ_1. The definition of Δ_1 (relation (6.2)) straightforwardly shows that its denominator is $\det(R_q)$. The entries of the matrix $x_0\, \det(R_q)^2\, \widehat{K}^2(R_q)$ are thus polynomials, and

Eq. (6.20) proves the first step of the factorization:

$$K^2(R_q) = \det(R_q)^{q-4}\, M_2 \ . \tag{6.21}$$

The same kind of demonstration can be performed on $\det(K(R_q))$:

$$\begin{aligned}\det(K(R_q)) &= \det(R_q)^q \, \det(\widehat{K}(R_q)) \\ &= \det(R_q)^{q-1}\, x_0 \ .\end{aligned}$$

The expression of x_0, namely (6.3), is also quadratic in Δ_1. One thus has the factorization

$$\det(K(R_q)) = \det(R_q)^{q-3}\, f_2 \ . \tag{6.22}$$

Notice that (6.21) is valid for $q > 3$, and (6.22) for $q > 2$. Considering successively the explicit expressions of $K^n(R_q)$ and of their determinants, we noticed (see (3.3), (3.4)) that there are further factorizations: they can actually be obtained the same way. Let us assume these factorizations (they have however been strictly obtained by formal computer calculations), and show how these factorizations "propagate" for generic n. Let us assume the following recurrence hypothesis, up to order m, and demonstrate the next step:

$$(f_n)_K = f_1^{\mu_n}\, f_{n+1} \tag{6.23}$$

$$\det(M_n) = f_{n-2}^{q-1}\, f_{n-1}^3\, f_n^{q-3}\, f_{n+1} \tag{6.24}$$

$$(M_n)_K = f_1^{\nu_n}\, M_{n+1} \tag{6.25}$$

$$K(M_n) = f_{n-2}^{q-2}\, f_{n-1}^2\, f_n^{q-4}\, M_{n+1} \tag{6.26}$$

$$\text{for any} \quad n \leq m \ .$$

Let us first calculate $K((M_n)_K)$ in two different ways, using respectively the right and left action of K:

$$\begin{aligned}K((M_m)_K) &= f_1^{(q-1)\,\nu_m}\ K(M_{m+1}) \\ &= (f_{m-2})_K^{q-2}\ (f_{m-1})_K^2\ (f_m)_K^{q-4}\ (M_{m+1})_K \\ &= f_1^{(q-2)\,\mu_{m-2}+2\,\mu_{m-1}\,(q-4)\,\mu_m}\ f_{m-1}^{q-2}\, f_m^2\ f_{m+1}^{q-4}\ (M_{m+1})_K \ .\end{aligned} \tag{6.27}$$

One thus obtains the equation

$$\begin{aligned}&f_1^{(q-1)\,\nu_m-((q-2)\,\mu_{m-2}+2\,\mu_{m-1}\,(q-4)\,\mu_m)}\ K(M_{m+1}) \\ &\quad = f_{m-1}^{q-2}\, f_m^2\ f_{m+1}^{q-4}\ (M_{m+1})_K \ .\end{aligned} \tag{6.28}$$

As we assumed that, generically, f_1 *has no common factor with other* f_n's, $K(M_{m+1})$ actually factorizes $f_{m-1}^{q-2}\, f_m^2\, f_{m+1}^{q-4}$. *Assuming that, generically, there is no further factorization*, it then allows one to define the next matrix M_{m+2} by the relation

$$K(M_{m+1}) = f_{m-1}^{q-2}\; f_m^2\; f_{m+1}^{q-4}\; M_{m+2}\;. \tag{6.29}$$

One thus obtains

$$(M_{m+1})_K = f_1^{(q-1)\,\nu_m-((q-2)\mu_{m-2}+2\mu_{m-1}(q-4)\mu_m)} M_{m+2}\;. \tag{6.30}$$

These two equations have the right form to go on with the recurrence, if one defines ν_{m+1} as

$$\nu_{m+1} = (q-1)\,\nu_m - ((q-2)\,\mu_{m-2} + 2\,\mu_{m-1}\,(q-4)\,\mu_m)\;. \tag{6.31}$$

To obtain the two last equations, one just has to proceed the same way, calculating $\det((M_m)_K)$

$$\begin{aligned}\det((M_m)_K) &= f_1^{q\,\nu_m}\;\det(M_{m+1})\\ &= (f_{m-2})_K^{q-1}\,(f_{m-1})_K^3\,(f_m)_K^{q-3}\;(f_{m+1})_K\\ &= f_1^{(q-1)\,\mu_{m-2}+3\,\mu_{m-1}+(q-3)\,\mu_m}\;f_{m-1}^{q-1}\;f_m^3\;f_{m+1}^{q-3}\;(f_{m+1})_K\;;\end{aligned} \tag{6.32}$$

that is, the equation

$$\begin{aligned}&f_1^{q\,\nu_m-((q-1)\,\mu_{m-2}+3\,\mu_{m-1}+(q-3)\,\mu_m)}\det(M_{m+1})\\ &\quad= f_{m-1}^{q-1}\;f_m^3\;f_{m+1}^{q-3}\;(f_{m+1})_K\;.\end{aligned} \tag{6.33}$$

Similarly one gets the "minimal" factorization (that we assumed to be the exact one) of $\det(M_{m+1})$ and defines f_{m+2}:

$$\det(M_{m+1}) = f_{m-1}^{q-1} f_m^3\, f_{m+1}^{q-3}\; f_{m+2}\;. \tag{6.34}$$

Besides $(f_{m+1})_K$ reads

$$(f_{m+1})_K = f_1^{q\,\nu_m-((q-1)\,\mu_{m-2}+3\,\mu_{m-1}+(q-3)\,\mu_m)}\; f_{m+2}\;. \tag{6.35}$$

We have thus demonstrated the recurrence hypothesis at order $m+1$ and μ_{m+1} reads

$$\mu_{m+1} = q\nu_m - ((q-1)\,\mu_{m-2} + 3\,\mu_{m-1} + (q-3)\,\mu_m)\;. \tag{6.36}$$

From (6.31) and (6.36) one recovers (3.20) and (3.21).

Let us show that the factorization properties (6.26) of the successive matrices M_n in fact enable one to obtain the factorization properties (6.24) of their determinants.

Let us recall that $K(M_n) = \det(M_n) \cdot t(M_n^{-1})$, (6.26) can thus be rewritten in the following way:

$$\det(M_n) \cdot M_n^{-1} = f_{n-2}^{q-2} f_{n-1}^2 f_n^{q-4} \cdot t(M_{n+1}) \,. \tag{6.37}$$

Taking the determinant of the previous relation one directly gets

$$\det(M_n)^{q-1} = (f_{n-2}^{q-2} f_{n-1}^2 f_n^{q-4})^q \cdot \det(t(M_{n+1})) \,. \tag{6.38}$$

One can get easily convinced that this yields the factorization of $\det(M_n)$ by $f_{n-2}^{q-1} f_{n-1}^3 f_n^{q-3}$ and one recovers (6.24).

Let us now show that the right and left factorizations are also related, and are to some extent equivalent.

At first, let us assume, for any n, up to order m, some left factorization properties:

$$K(M_n) = M_{n+1} \cdot f_1^{u_n} \cdot f_2^{u_{n-1}} \cdot f_3^{u_{n-2}} \cdots f_{n-1}^{u_2} \cdot f_n^{u_1} \tag{6.39}$$

and

$$\det(M_n) = f_1^{v_n} \cdot f_2^{v_{n-1}} \cdot f_3^{v_{n-2}} \cdots f_{n-1}^{v_2} \cdot f_n^{v_1} \cdot f_{n+1} \,. \tag{6.40}$$

Assuming the right factorization properties (6.23) and (6.25), we will show that (6.39) and (6.40) are still valid for $n = m + 1$, thus establishing the left factorization properties of K.

Combining relation (6.39) and relation (6.24), one can write

$$\begin{aligned} K(M_{m+1}) &= \frac{K((M_m)_K)}{f_1^{(q-1)\,\nu_m}} \\ &= M_{m+2} \cdot \frac{f_1^{\mu_1 u_m + \mu_2 u_{m-1} + \cdots + \mu_m u_1} \cdot f_2^{u_m} \cdot f_3^{u_{m-1}} \cdots f_{m-1}^{u_3} \cdot f_m^{u_2} \cdot f_1^{\nu_{m+1}}}{f_1^{(q-1)\,\nu_m}} \\ &\quad \cdot f_{m+1}^{u_1} \\ &= M_{m+2} \cdot f_1^{\mu_1 u_m + \mu_2 u_{m-1} + \cdots + \mu_m u_1 + \nu_{m+1} - (q-1)\,\nu_m} \\ &\quad \cdot f_2^{u_m} \cdot f_3^{u_{m-1}} \cdots f_{m-1}^{u_3} \cdot f_m^{u_2} \cdot f_{m+1}^{u_1} \,. \end{aligned} \tag{6.41}$$

This result is nothing but (6.39) with u_{m+1} reading

$$u_{m+1} = \mu_1 u_m + \mu_2 u_{m-1} + \cdots + \mu_m u_1 + \nu_{m+1} - (q-1)\,\nu_m \,. \tag{6.42}$$

Similarly combining relations (6.40) and (6.23) one gets

$$\det(M_{n+1}) = \frac{\det((M_n)_K)}{f_1^{q\,\nu_1}}$$
$$= \frac{f_1^{v_n\,\mu_1+v_{n-1}\,\mu_2+\cdots+\mu_n\,v_1+\mu_{n+1}} \cdot f_2^{v_n} \cdot f_3^{v_{n-1}} \cdots f_{n-1}^{v_3} \cdot f_n^{v_2} \cdot f_{n+1}^{v_1} \cdot f_{n+2}}{f_1^{q\,\nu_1}} \,. \tag{6.43}$$

This result is nothing but (6.40) with v_{m+1} reading

$$v_{m+1} = v_m\,\mu_1 + v_{m-1}\,\mu_2 + \cdots + \mu_m\,v_1 + \mu_{m+1} - q\,\nu_m \,. \tag{6.44}$$

In order to show the reciprocal statement, that is, that the left factorizations (6.39) and (6.40) yield the right factorizations (6.23) and (6.25), let us assume (6.39) and (6.40) for every n, and assume (6.23) and (6.25) for $n = 1, 2, \ldots\ m$. We will demonstrate (6.23) and (6.25) for $n = m + 1$.

$$(f_{m+1})_K = \frac{\det((M_m)_K)}{f_1^{v_m\,\mu_1+v_{m-1}\,\mu_2+\cdots+\mu_m\,v_1} \cdot f_2^{v_m} \cdot f_3^{v_{m-1}} \cdots f_{m-1}^{v_3} \cdot f_m^{v_2} \cdot f_{m+1}^{v_1}}$$
$$= \frac{f_1^{q\,\nu_m} \cdot \det(M_{m+1})}{f_1^{v_m\,\mu_1+v_{m-1}\,\mu_2+\cdots+\mu_m\,v_1} \cdot f_2^{v_m} \cdot f_3^{v_{m-1}} \cdots f_{m-1}^{v_3} \cdot f_m^{v_2} \cdot f_{m+1}^{v_1}}$$
$$= f_1^{v_{m+1}+q\,\nu_m-(v_m\,\mu_1+v_{m-1}\,\mu_2+\cdots+\mu_m\,v_1)} \cdot f_{m+2} \,. \tag{6.45}$$

This result is nothing but (6.23) with μ_{m+1} reading

$$\mu_{m+1} = v_{m+1} + q\,\nu_m - (v_m\,\mu_1 + v_{m-1}\,\mu_2 + \cdots + \mu_m\,v_1) \,. \tag{6.46}$$

Similarly

$$(M_{m+1})_K = \frac{K((M_m)_K)}{f_2^{u_m} \cdot f_3^{u_{m-1}} \cdots f_{m-1}^{u_3} \cdot f_m^{u_2} \cdot f_1^{\mu_1\,u_m+\mu_2\,u_{m-1}+\cdots+\mu_m\,u_1}}$$
$$= M_{m+2} \cdot f_1^{(q-1)\,\nu_m+u_{m+1}-(u_1\,\mu_m+u_2\,\mu_{m-1}+\cdots+u_m\,\mu_1)} \,. \tag{6.47}$$

This result is nothing but (6.25) with ν_{m+1} reading

$$\nu_{m+1} = (q-1)\,\nu_m + u_{m+1} - (u_1\,\mu_m + u_2\,\mu_{m-1} + \cdots + u_m\,\mu_1) \,. \tag{6.48}$$

7. Conclusion

In the framework of the analysis of groups of birational transformations generated by involutions,[4,5,23-25] an interesting heuristic example has emerged[8]; it is well suited to present the various properties related to such birational transformations.

More precisely, we have concentrated on the relations between various structures and properties, and in particular, the relation between *polynomial growth, the occurrence of recurrences, and the integrability.*

These particular birational transformations have been seen to be *integrable* and to yield *elliptic curves*, when iterated.[8] We have revisited here this particular example in more detail, looking systematically at the successive iterates of such a birational mapping. The iteration of this transformation, seen as a homogeneous polynomial transformation, has been shown to satisfy *remarkable factorization properties.* As a consequence, we have noticed a *polynomial growth* of the degree of the successive iterated transformations (instead of a $(q-1)^n$ generic exponential growth). This polynomial growth can be seen to be, to some extent, a consequence of the integrability (more precisely of the occurrence of elliptic curves in a $\mathbb{C}P_n$-projective space). More generally, examples of birational transformations, the iterations of which yield *abelian varieties, could be shown to also have a polynomial growth.*[3] Nevertheless the precise relation between polynomial growth and the occurrence of abelian varieties, for the iteration of birational mappings remains an open question.[3]

In a forthcoming publication,[2] it will be seen that factorization properties, in fact occur for any birational transformation generated by the two following simple algebraic involutions, namely taking the matrix inverse of $q \times q$-matrices together with permuting *two* entries of such a matrix. Even more generally, factorizations *do occur for quite general permutations of the entries* of the matrices.[3]

Factorization properties of transformation K have enabled us to define homogeneous polynomials (the f_n's). These polynomials are simply related to *determinants* of the successive iterates of the $q \times q$-matrix. Remarkably they also satisfy nonlinear recurrences in one variable.

We have been able to show that these recurrences are in fact organized in a whole *hierarchy of recurrences*, deduced from one basic recurrence. Without referring to the matrix framework of our birational transformations anymore, one can consider these *recurrences for themselves* as many discrete dynamical systems: the previously mentioned basic recurrence is actually an *integrable recurrence, as well as a subset of recurrences* of our hierarchy. All these integrable recurrences can be integrated to get elliptic curves represented as *biquadratic* equations in terms of some homogeneous variables, or particular *bicubic* relations in terms of inhomogeneous variables.

Surprisingly, though our hierarchy emerged from the analysis of an integrable mapping, all the recurrences *are not integrable*, *even if they all are reversible.*

All these analyses have also been performed for another birational transformation associated with the *sixteen vertex model*, yielding very similar results. In particular, we have noticed a remarkable correspondence (identification up to a simple change of variable!) between the two hierarchies of recurrences.

These transformations generated by two simple involutions, the matrix inverse and a particular transposition of two entries of a $q \times q$ matrix, can therefore be represented either as birational transformations on $q^2 - 1$ variables, polynomial transformations on q^2 homogeneous variables (the entries of the $q \times q$ matrix) yielding integrable mappings for arbitrary q (see Sec. 3.4), or nonlinear integrable recurrences on *only one variable*, for instance, the *determinant* of the image of these matrices under these transformations.

This function of the q^2 variables (the determinant) realizes a "canonical" embedding of the birational mappings in $q^2 - 1$ variables into mappings in very few variables. In fact, all these integrable mappings in $q^2 - 1$ variables and all these integrable recurrences yield *the same elliptic curve.*

The occurrence of *Plücker variables*[26] in the analysis of birational transformations in $\mathbb{C}P_{15}$,[8] and here, the natural occurrence of determinants for analyzing the same birational transformations should be interpreted in a close future as Grassmannian structures[26] associated with elliptic curves.

Finally, we would like to underline that all the results, structures and properties detailed here *are actually valid for* $q \times q$*-matrices for arbitrary* q.

In particular, these mappings provide *an example of integrable mapping in arbitrary dimension.*

Acknowledgment

We thank M. Bellon and C. M. Viallet for many discussions and much encouragement. We would like also to thank B. Grammaticos and A. Ramani for very fruitful conversations.

Appendix A

Let us revisit some of the relations and properties introduced and analyzed in Sec. 3. The matrices M_n defined in Sec. 3 (see relations (3.3) and (3.4)) satisfy many additional properties and relations. This list would be too long. Let us just give a couple of interesting properties. It should first be noticed that, for a given initial matrix M_0, the successive matrices M_n belong to two simple matrix affine spaces. For n even one has

$$M_n = a_n^{(0)} \cdot M_0 + a_n^{(1)} \cdot P + a_n^{(2)} \cdot M_2 + a_n^{(3)} \cdot M_4 + a_n^{(4)} \cdot M_6 + a_n^{(5)} \cdot M_8 \qquad (7.1)$$

where P denotes the matrix P previously defined (6.1).

The successive matrices M_n for n odd also belong to a five-dimensional affine subspace:

$$M_n = b_n^{(0)} \cdot M_1 + b_n^{(1)} \cdot P + b_n^{(2)} \cdot M_3 + b_n^{(3)} \cdot M_5 + b_n^{(4)} \cdot M_7 + b_n^{(5)} \cdot M_9 \,. \qquad (7.2)$$

It is worth noticing that matrix P belongs to both the "odd" and the "even" affine subspace.

Another interesting property corresponds to the particular form of the product of two successive matrices M_n and M_{n+1} (see Sec. 6), where the introduction of the matrix U and relation (6.3) are the key points to actually prove the recurrences of this paper).

Let us introduce the following $q \times q$ matrices:

$$\mathcal{U}_n = M_n \cdot M_{n+1} \, . \qquad (7.3)$$

One can verify that these matrices have the form

$$\mathcal{U}_n = f_{n-1} \cdot f_n \cdot \begin{bmatrix} f_{n-1} \cdot u_{11} & f_{n-1} \cdot u_{12} & 0 & 0 & \cdots \\ f_{n-1} \cdot u_{21} & f_{n-1} \cdot u_{22} & 0 & 0 & \cdots \\ v_{31} & v_{32} & f_{n-2} \cdot f_{n+1} & 0 & \cdots \\ \vdots & \vdots & 0 & f_{n-2} \cdot f_{n+1} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \qquad (7.4)$$

where the determinant of the 2×2 block up left reads

$$u_{11}\, u_{22} - u_{12}\, u_{21} = f_{n-2} \cdot f_{n+2} \, . \qquad (7.5)$$

On the form (7.4) for the matrix $\mathcal{U}_n = M_n \cdot M_{n+1}$, one sees that additional factorization properties occur.

Appendix B

A first biquadratic relation

Introducing the variables q_n from $x_n = q_{n+1}/q_n$, Eq. (4.15) reads

$$\frac{q_{n+3} - q_{n+1}}{(q_{n+4} - q_n)\, q_{n+1}\, q_{n+2}\, q_{n+3}} = \frac{q_{n+4} - q_{n+2}}{(q_{n+5} - q_{n+1})\, q_{n+2}\, q_{n+3}\, q_{n+4}} \, . \qquad (7.6)$$

Equation (7.6) can be "integrated": one first remarks that the right hand side and the left hand side of recurrence (7.6) identify up to a shift of 1 of n, which means that they have to be equal to one constant we denote λ:

$$\frac{1}{q_{n+1}} - \frac{1}{q_{n+3}} = \lambda \cdot (q_{n+4} - q_n) \cdot q_{n+2} \, . \qquad (7.7)$$

This now yields to introduce two new constants ρ_1 and ρ_2 namely

$$\lambda \cdot q_n \; q_{n+2} + \frac{1}{q_{n+1}} = \rho_n = \rho_{n+2} \, . \qquad (7.8)$$

From the two relations (7.8) one gets

$$\rho_n \cdot q_{n+1}\, q_{n+3} - q_{n+3} = \rho_{n+1} \cdot q_n\, q_{n+2} - q_n \tag{7.9}$$

which yields to introduce a new constant of integration μ:

$$\rho_{n+1} \cdot q_n\, q_{n+2} - q_n - q_{n+1} - q_{n+2} = \mu\ . \tag{7.10}$$

Eliminating q_{n+2} between (7.10) and (7.8) finally yields a biquadratic relation:

$$(\rho_n\, q_{n+1} - 1) \cdot (\rho_{n+1}\, q_n - 1) - \lambda \cdot q_n\, q_{n+1}\, (\mu + q_n + q_{n+1}) = 0\ . \tag{7.11}$$

A second biquadratic relation

Introducing the variables q_n from $x_n = q_{n+1}/q_n$, Eq. (4.21) reads

$$\frac{q_{n+5} - q_{n+3}}{(q_{n+5}\, q_{n+6} - q_{n+2}\, q_{n+3}) \cdot q_{n+4}} = \frac{q_{n+3} - q_{n+1}}{(q_{n+3}\, q_{n+4} - q_n\, q_{n+1}) \cdot q_{n+2}}\ . \tag{7.12}$$

In fact, Eq. (7.12) can be "integrated": one first remarks that the right hand side and the left hand side of recurrence (7.12) identify, up to a shift of 2 of n, which means that they have to be equal to two constants, λ_1 and λ_2, depending on whether n is odd or even:

$$(q_{n+3}\, q_{n+4} - q_n\, q_{n+1}) \cdot q_{n+2} = -\lambda_n \cdot (q_{n+3} - q_{n+1})\ . \tag{7.13}$$

with $\lambda_n = \lambda_{n+2}$, This means that $q_n\, q_{n+1}\, q_{n+2} + \lambda_n\, q_{n+1}$ is invariant when one shifts n of 2, yielding to introduce two new constants ρ_1 and ρ_2:

$$q_n\, q_{n+1}\, q_{n+2} + \lambda_n\, q_{n+1} = \rho_n \tag{7.14}$$

with $\rho_n = \rho_{n+2}$. Considering the ratio of $q_n\, q_{n+1}\, q_{n+2}$ and $q_{n+1}\, q_{n+2}\, q_{n+3}$ one easily gets

$$(\rho_n - \lambda_n\, q_{n+1}) \cdot q_{n+3} = (\rho_{n+1} - \lambda_{n+1}\, q_{n+2}) \cdot q_n \tag{7.15}$$

and also:

$$(\rho_{n+1} - \lambda_{n+1}\, q_{n+2}) \cdot q_{n+4} = (\rho_n - \lambda_n\, q_{n+3}) \cdot q_{n+1}\ . \tag{7.16}$$

From these two relations (7.15) and (7.16), it is clear that one can introduce a new constant μ:

$$(\rho_n - \lambda_n\, q_{n+1}) \cdot q_{n+3} + \rho_n \cdot q_{n+1} + \rho_{n+1} \cdot q_{n+2} = \mu \tag{7.17}$$

$$(\rho_{n+1} - \lambda_{n+1}\, q_n) \cdot q_{n+2} + \rho_{n+1} \cdot q_n + \rho_n \cdot q_{n+1} = \mu\ . \tag{7.18}$$

From relation (7.14) and (7.18) one easily eliminates q_{n+2} to get

$$(\rho_n - \lambda_n\, q_{n+1}) \cdot (\rho_{n+1} - \lambda_{n+1}\, q_n) = (\mu - (\rho_{n+1} \cdot q_n + \rho_n \cdot q_{n+1})) \cdot q_n\, q_{n+1}\ . \tag{7.19}$$

References

1. S. Boukraa, J-M. Maillard and G. Rollet, "A New Approach for Lattice Models in Statistical Mechanics" LPTHE preprint 94-27 (1994).
2. S. Boukraa, J-M. Maillard and G. Rollet, "Integrable Mappings and Polynomial Growth", *Physica A* (1994), to be published.
3. S. Boukraa, J-M. Maillard and G. Rollet, *Int. J. Mod. Phys.* **B8**, 137–174 (1994).
4. M. P. Bellon, J-M. Maillard and C-M. Viallet, *Phys. Lett.* **A157** (1991), pp. 343–353.
5. M. P. Bellon, J-M. Maillard and C-M. Viallet, *Phys. Lett.* **B260**, 87–100 (1991).
6. M. T. Jaekel and J. M. Maillard, *J. Phys.* **A15**, 2241 (1982).
7. C. M. Viallet and G. Falqui, *Comm. Math. Phys.* **154**, 111–125 (1993).
8. S. Boukraa, J-M. Maillard and G. Rollet, "Integrable Mappings and Factorization Properties", LPTHE preprint 93-27 (1993).
9. M. P. Bellon, J-M. Maillard and C-M. Viallet, *Phys. Lett.* **B281**, 315–319 (1992).
10. A. Ramani, B. Grammaticos and J. Hietarinta, *Phys. Rev. Lett.* **67**, 1829–1832 (1991).
11. G. R. W. Quispel, J. A. G. Roberts and C. J. Thompson, *Phys. Lett.* **A126**, 419 (1988).
12. G. R. W. Quispel, J. A. G. Roberts and C. J. Thompson, *Physica* **D34**, 183–192 (1989).
13. G. R. W. Quispel and F. W. Nijhoff, *Phys. Lett.* **A161**, 419 (1992).
14. B. Grammaticos, A. Ramani and V. Papageorgiou, *Phys. Rev. Lett.* **67**, 1825–1827 (1991).
15. A. Ramani, B. Grammaticos and K. M. Tamizhmani, *J. Phys.* **A25**, L883-L886 (1992).
16. V. G. Papageorgiou, F. W. Nijhoff and H. W. Capel, *Phys. Lett.* **A147**, 106–114 (1990).
17. S. Boukraa and J-M. Maillard, *Int. J. Mod. Phys.* (1992).
18. J. M. Maillard, *J. Math. Phys.* **27**, 2776 (1986).
19. A. B. Zamolodchikov, *Comm. Math. Phys.* **79**, 489 (1981).
20. R. J. Baxter, *Comm. Math. Phys.* **88**, 185–205 (1983).
21. R. J. Baxter, *Physica* **D18**, 321–347 (1986).
22. R. J. Baxter and Bazhanov, *Int. J. Mod. Phys.* **B7** (1993).
23. M. P. Bellon, J-M. Maillard and C-M. Viallet, *Phys. Rev. Lett.* **67**, 1373–1376 (1991).
24. M. P. Bellon, J-M. Maillard and C-M. Viallet, *Phys. Lett.* **A159**, 221–232 (1991).
25. M. P. Bellon, J-M. Maillard and C-M. Viallet, *Phys. Lett.* **A159**, 233–244 (1991).
26. P. Griffiths and J. Harris, *Principles of Algebraic Geometry* (John Wiley, 1978).