Pergamon

Mathl. Comput. Modelling Vol. 26, No. 8–10, pp. 169–225, 1997

0895-7177/97 \$17.00 + 0.00

PII: S0895-7177(97)00205-7

Hyperbolic Coxeter Groups, Symmetry Group Invariants for Lattice Models in Statistical Mechanics, and the Tutte-Beraha Numbers

J.-M. MAILLARD
Laboratorie de Physique Théorique et des Hautes Energies
Unité associée au C.N.R.S. (UA 280)
Université de Paris 6- Paris 7, Tour 16, 1[er] étage, boite 126
4 Place Jussieu, F-75252 Paris Cedex 05, France
maillard@lpthe.jussieu.fr

Abstract—The symmetry groups, generated by the inversion relations of lattice models of statistical mechanics, are analysed for vertex models and for the standard scalar Potts model with two and three site interactions on triangular lattices. These groups are generated by three inversion relations and are noticeably generically very large ones: hyperbolic groups. Various situations for which the representations of these groups degenerate into smaller ones, hopefully compatible with integrability, are considered. For instance, the group becomes smaller for q-state Potts models for particular values of q, the so-called Tutte-Beraha numbers. For this model, algebraic varieties, including the known ferromagnetic critical variety, happen to be invariant under such large groups of symmetries. This analysis provides nice birational representations of hyperbolic Coxeter groups. Remarkable varieties breaking the symmetry of the lattice are seen to occur specifically for the Tutte-Beraha numbers. A detailed analysis of these Potts models is performed for $q = 3$. In particular, the algebraic varieties corresponding to conditions for the symmetry group to be finite order are carefully examined. Finally, specifically for the Tutte-Beraha numbers, the introduction of algebraic group invariants is discussed in detail for $q = 3$ in order to get closed expressions for the spontaneous magnetization of the edge Potts models.

Keywords—Tutte-Beraha numbers, Standard scalar Potts model, Yang-Baxter equations, Baxterisation, Birational representations of hyperbolic Coxeter groups.

1. INTRODUCTION

In previous papers [1,2] it has been shown that there exist nontrivial, *nonlinear* discrete symmetries acting on the parameter space of *lattice models* of statistical mechanics generated by the so-called *inversion relations* [3–6]. These nonlinear groups of symmetries appeared as powerful tools to study integrable models in lattice statistical mechanics, as well as to find the *critical varieties* of their phase diagrams [7]. These symmetry groups can also be seen as *symmetries* of the Yang-Baxter equations (or star-triangle equations, when dealing with spin edge models) and their higher dimensional generalizations. This provides a solution for the so-called "*Baxterisation*" problem [8].

It is important to note that these groups exist as (*discrete) symmetry groups* of lattice models *even when one is no longer restricted to an integrable framework* [9–11].

Work supported by CNRS.
We thank J.-C. Anglès d'Auriac, A. J. Guttmann, H. Meyer, C. M. Viallet, and F. Y. Wu for many discussions and encouragements. We thank J.-C. Anglès d'Auriac and H. Meyer for communicating several Monte-Carlo results prior to publications.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

From this point of view, the straight but tedious, analysis of a three-dimensional model through transfer matrix formalism, or any other classical method of lattice statistical mechanics, is replaced by an analysis of the transformations corresponding to the *symmetries*. These symmetries act *in the parameter space*, and therefore, at first sight, are less sensitive to the *dimension* of the lattice.

However, in the cases, integrable or not, known in the literature, a drastic difference *seems* to appear between two-dimensional and three-dimensional models, suggesting an explanation of the "obstruction" for three-dimensional integrability associated with generic three-dimensional symmetry groups, and also suggesting an *algebraic definition* of the notion of *dimension* of the model. In this framework the *dimension* of the lattice re-emerges through the *"size" of the (infinite discrete) symmetry group*. As far as two-dimensional models on square lattices are concerned, the discrete symmetry groups known in the literature were either *finite groups* [12,13] or groups isomorphic to products of $\mathbb{Z}$, up to a semidirect product by a finite group [9,13,14]. On the other hand, for lattice models of dimension three, these symmetry groups are much larger: they are generically free groups with (at least) *three* generators. With such symmetry groups, the *very existence of solutions of the tetrahedron equations*[1] having a "*generic* three-dimensional symmetry" *seems problematic* [14]: the only possibility for solutions of the tetrahedron equations are probably cases where the representations of such "large" groups *degenerate into products* of $\mathbb{Z}$ or even into *finite groups* [18,19]. Actually, a recent solution of Korepanov [20] of the tetrahedron equations confirms this point of view: these solutions actually correspond to a case for which the discrete symmetry group *degenerates into a finite order* group of order 32.

It will be shown here, that the analysis of the symmetry group of models on *triangular* lattices "weakens" this opposition between dimension two and dimension three. We will analyse vertex model on the triangular lattice, as well as the standard scalar q-state Potts model with two- and *three-spin* interaction also on the triangular lattice [21,22]. Generically, their symmetry groups are free groups with two generators. One recovers a situation similar to the one encountered in dimension three: these models on *triangular lattices thus provide examples giving hints for the analysis of such large symmetry groups in dimension three.*

However, these *hyperbolic* Coxeter groups of symmetries can actually degenerate into more "reasonable" groups, leaving room for integrability in the case of *Tutte-Beraha numbers*[2].

We will consider the consequences of these symmetries, with a special emphasis on *criticality conditions*. We will pay particular attention to a (self-dual) critical variety given by Wu [21,24], on the two- and *three-site* interaction Potts model on the triangular lattice, which we will revisit here.

Particular attention is devoted to the three-state Potts model. In this respect it will be shown that symmetry group invariants occurring *specifically* for the *Tutte-Beraha numbers*, seem to be useful to "decipher the complexity" of the (resummed low-temperature) expansions of various physical quantities and in particular, the *spontaneous magnetization*.

We hope such analysis will open a new class of lattice models for which a quite large set of exact calculations can be performed without having any "Yang-Baxter integrability"[3].

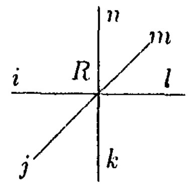
2. RECALLS ON SYMMETRIES OF LATTICE MODELS

Let us recall the symmetry group generated by the *inversion relations* [3,5,6] for lattices of coordination number six, first, on the cubic three-dimensional vertex model [18,19] and then on the triangular lattice.

[1]Generalizations of the Yang-Baxter equations in dimension three [9,15–17].
[2]The Tutte-Beraha numbers are particular numbers occurring in the analysis of *chromatic polynomials* [23].
[3]Such models do exist: for instance, *disorder solutions* [25,26] provide some examples of "computable" models that are not Yang-Baxter-integrable. However, such disorder solutions correspond to *dimensional reductions* of the model. We are seeking here for two-dimensional (or higher-dimensional) models with a *genuine* two-dimensional complexity.

2.1. Inversion Relations and the Group Γ_{3D}

Let us consider vertex model on a three-dimensional cubic lattice of size $M \times M$. With each bond, is associated a variable with q possible states. A Boltzmann weight $w(i,j,k,l,m,n)$ is assigned to each vertex configuration [27], and can be represented pictorially by:



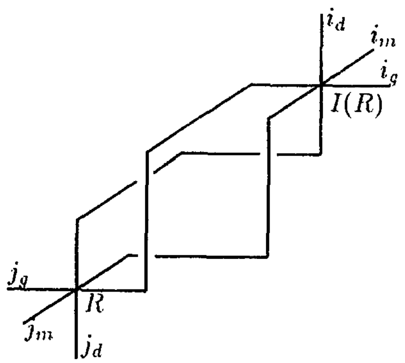
The q^6 homogeneous weights $w(i,j,k,l,m,n)$ are first arranged in a $q^3 \times q^3$ matrix R of entries:

$$R^{ijk}_{lmn} = w(i,j,k,l,m,n). \tag{2.1}$$

One may [19] introduce an involution I which transforms R into IR according to:

$$\sum_{\alpha_1,\alpha_2,\alpha_3} (IR)^{i_1 i_2 i_3}_{\alpha_1 \alpha_2 \alpha_3} \cdot R^{\alpha_1 \alpha_2 \alpha_3}_{j_1 j_2 j_3} = \lambda \cdot \delta^{i_1}_{j_1} \delta^{i_2}_{j_2} \delta^{i_3}_{j_3}, \tag{2.2}$$

where λ is an arbitrary multiplicative factor. This relation can be represented pictorially:



$$= \lambda \cdot \delta^{i_1}_{j_1}\, \delta^{i_2}_{j_2}\, \delta^{i_3}_{j_3}.$$

The inversion transformation I amounts to taking the inverse of the $q^3 \times q^3$ matrix R. One also introduces the partial transpositions t_1, t_2, and t_3 with:

$$(t_1 R)^{i_1 i_2 i_3}_{j_1 j_2 j_3} = R^{j_1 i_2 i_3}_{i_1 j_2 j_3}, \tag{2.3}$$

and similar definitions for t_2 and t_3.

For three-dimensional vertex models, one has *four* such involutions acting as symmetries of the R-matrix [19]:

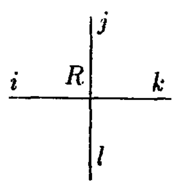
$$I_2 = I, \qquad I_3 = t_1 I t_2 t_3, \qquad I_4 = t_2 I t_3 t_1, \qquad I_1 = t_3 I t_1 t_2. \tag{2.4}$$

These four involutions generate an infinite discrete group Γ_{3D} [19]. Let us note that the full transposition is nothing but the product $t = t_1 \cdot t_2 \cdot t_3$.

Considering the parameter space as a projective space (the entries of the R-matrix are *homogeneous* parameters), the elements of the group Γ_{3D} have a *nonlinear* representation in terms of *birational transformations*. This group of symmetry of the parameter space of the model is very large. This is, in fact, a *hyperbolic* Coxeter group [28–33].

REMARK. Coming back to integrability, it has been shown that the tetrahedron equations (generalization in three dimensions of the Yang-Baxter equations [10,15,16,34]) do have an infinite group of symmetry generated by *four* involutions K_1, K_2, K_3, K_4 [19]. They satisfy various relations, for instance $(K_1K_2K_3K_4)^2 = \mathcal{I}d$, where $\mathcal{I}d$ denotes the identity transformation. This group of symmetry of the tetrahedron equation is quite "large", since the number of elements of length smaller than l is of exponential growth with respect to l, unlike the symmetry group of the Yang-Baxter equations which identifies with the affine Coxeter group $A_2^{(1)}$ [18,19,29].

In contrast the infinite discrete symmetry group of the square lattice is generated by *two* involutions (inversion relations) and therefore, is isomorphic to the infinite dihedral group. Let us introduce **I** and **J**, the two inverse transformations on the square lattice vertex model [19]. A Boltzmann weight $w(i,j,k,l)$ is assigned to each square vertex configuration [27]:



(2.5)

The q^4 homogeneous weights $w(i,j,k,l)$ are first arranged in a $q^2 \times q^2$ matrix R:

$$R^{ij}_{kl} = w(i,j,k,l).$$

We introduce (see [1,18,19]) the inverse I by:

$$\sum_{\alpha,\beta} R^{ij}_{\alpha\beta} \cdot (\mathbf{I}R)^{\alpha\beta}_{uv} = \lambda \cdot \delta^i_u \delta^j_v = \sum_{\alpha,\beta} (\mathbf{I}R)^{ij}_{\alpha\beta} \cdot R^{\alpha\beta}_{uv},$$

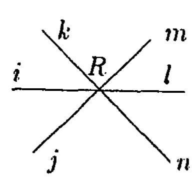
and the other inverse J by:

$$\sum_{\alpha,\beta} R^{\alpha u}_{v\beta} \cdot (\mathbf{J}R)^{\alpha i}_{j\beta} = \mu \cdot \delta^i_u \delta^j_v = \sum_{\alpha,\beta} (\mathbf{J}R)^{i\beta}_{\alpha j} \cdot R^{u\beta}_{\alpha v}.$$

Similarly to the situation occurring for the cubic lattice, **I** and **J** are two involutions related by a partial transposition (denoted t_1 in [35]) of the indices: $\mathbf{J} = t_1 \mathbf{I} t_1$. Namely, t_1 reads: $(t_1 R^{ij}_{kl}) = R^{kj}_{il}$.

2.2. Inversion Relations of Triangular Vertex Models

For the triangular lattice the vertex Boltzmann weight [21] also reads $w(i,j,k,l,m,n)$, and can be represented by:



Similarly to the cubic model [18,19], the weights may be arranged in an $q^3 \times q^3$ matrix. However, for the triangular model *there are only three inversion transformations*, I_1, I_2, I_3, which actually coincide with three among the four of the cubic lattice (2.4). The fourth transformation I_4 corresponds to a nonplanar picture, which is meaningless for the triangular lattice. Let us denote Γ_{triang} as the symmetry group generated by I_1, I_2, I_3. As will be shown in the following, using the equivalence between vertex and spin representation for this model [21], *this group also generically has an exponential growth.*

Let us recall the results obtained by Baxter, Temperley and Ashley on the triangular vertex and spin models [21]. They noticed that the integrable case discovered by Kelland for a triangular vertex model (a 20-vertex model [36]), actually corresponds to the following situation: the vertex

Boltzmann weight can alternatively be seen as either a left-hand side *or* a right-hand side of a Yang-Baxter equation (more generally, this refers to the $\mathcal{Z}$-invariance concept [37]).

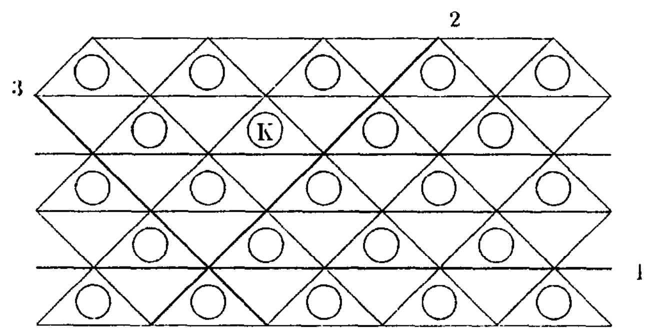
In the framework of this very model, they brought out the correspondence such a vertex model and the *standard scalar q-state Potts model for anisotropic triangular lattices with two- and three-site interaction* (only on up-pointing triangles) *through the Lieb-Temperley algebra* [21,38]. In terms of the two and three-site interaction spin model, these integrability conditions correspond to have *no three spin interaction* and also to be *at the transition temperature* [21].

There clearly exists here a drastic symmetry difference between the square and the cubic lattice, as far as the analysis of the group symmetries generated by the inversion relations is concerned. This difference stems from the fact that the *number of involutions* generating this very group is *larger than two* for the cubic lattice, which yields *hyperbolic groups* rather than the *infinite dihedral group*. The analysis of the symmetries on triangular models can be seen as a *testing ground* to study such hyperbolic groups, since the number of involutions generating this very group is *larger than two*. In the following sections, we will concentrate on spin models on *triangular* lattices and, more specifically, standard scalar Potts models [10].

3. TRIANGULAR SPIN MODEL

3.1. Notations for the Spin Model

Let us now consider the standard scalar q-state Potts model on a triangular lattice with nearest neighbor interaction and three-spin interaction only on the up-pointing triangles:



The partition function of the models reads:

$$\mathcal{Z} = \sum_{\{\sigma_i\}} \prod_{\langle i,j \rangle} e^{K_1 \delta_{\sigma_i,\sigma_j}} \prod_{\langle j,k \rangle} e^{K_2 \delta_{\sigma_j,\sigma_k}} \prod_{\langle k,l \rangle} e^{K_3 \delta_{\sigma_k,\sigma_l}} \prod_{\langle u.p.t \rangle} e^{K \delta_{\sigma_i,\sigma_j} \delta_{\sigma_j,\sigma_k}}. \tag{3.1}$$

The first three products denote the product over the edge two-side interaction Boltzmann weights along the *three* directions of the triangular model, and the last product denotes the product of all up-pointing triangles of the three-site interaction Boltzmann weights. The sum is taken over all spin configurations.

In this framework one can now introduce the following notations:

$$y = x x_1 x_2 x_3 - (x_1 + x_2 + x_3) + 2, \qquad \text{where } x = e^K \text{ and } x_i = e^{K_i},\ i = 1, 2, 3. \tag{3.2}$$

Of course for $q = 2$, the model degenerates into the nearest neighbor interaction triangular Ising model since the three-site interaction becomes irrelevant. Therefore one will not consider this $q = 2$ case in following (even if most of the results one will get are also valid in this very case).

3.2. Duality Transformation

Let us recall that a *duality transformation* does exist on this model [21,24]. With notations (3.2) this duality, denoted D, reads

$$D: \begin{cases} x_i \to x_i^* = 1 + q\dfrac{x_i - 1}{y}, \\ x \to x^* = \dfrac{x_1^* + x_2^* + x_3^* - 2 + q^2/y}{x_1^* x_2^* x_3^*}, \\ y \to y^* = \dfrac{q^2}{y}. \end{cases} \tag{3.3}$$

This duality is associated with a rotation of 180° of the corresponding vertex model on a triangular lattice through the correspondence detailed in [21]. It should not be confused with the Kramers-Wannier for Potts models [39–41]: Kramers-Wannier duality maps the triangular lattice onto the honeycomb lattice. In fact duality (3.3) can be seen as the product of the Kramers-Wannier duality together with a *star-triangle relation.* D is an involution.

In the isotropic limit $x_1 = x_2 = x_3 = u$, it gives:

$$(u,x) \to \left(\frac{(q-3)u + u^3x - q + 2}{u^3x - 3u + 2}, \frac{\left(u^3x - 3u + 2\right)^2 \left(u^3x - 3u + 2 + 3qu - 3q + q^2\right)}{\left(u^3x - 3u + 2 + qu - q\right)^3} \right).$$

For $q = 3$ it gives physical points only if $u \le 1$ (antiferromagnetic edge coupling constants) and if the ferromagnetic condition $u^3x - 1 \ge 0$ is satisfied or if $u \ge 1$ (ferromagnetic edge coupling constants) and if condition $u^3x - 1 > 3 \cdot (u-1)$ is satisfied.

Introducing well suited homogeneous variables, duality transformation (3.3) can be represented as a linear transformation D_h (see Section 4.5), which satisfies relation: $D_h^2 = q^2 \mathcal{I}d$, where $\mathcal{I}d$ denotes the identity transformation. The hyperplanes stable by D_h correspond to eigenforms associated with eigenvalues $\pm q$. The two self-dual varieties symmetric under permutations of 1, 2, and 3 can be written, respectively, as follows [21,24]:

$$y = -q \quad \text{and} \quad y = q.$$

Actually $y = q$ can be seen as the eigenform associated with eigenvalue $-q$ and reads:

$$x x_1 x_2 x_3 - (x_1 + x_2 + x_3) + 2 - q = 0. \tag{3.4}$$

Hyperplane (3.4) is a critical variety in some *ferromagnetic* region [24], whereas $y = -q$ has no such property. Let us notice that hyperplane (3.4) is *the only variety that is stable point by point by duality D.*

Note, that *the well-known case of no three-site interaction*, ($x = 1$), *is not stable under D.* Namely, variety $x = 1$ becomes:

$$(x_1x_2 + x_2x_3 + x_3x_1 - x_1 - x_2 - x_3 - x x_1 x_2 x_3 + 1) \cdot y + q \cdot (x_1 - 1)(x_2 - 1)(x_3 - 1) = 0. \tag{3.5}$$

3.3. Disorder Solutions and Their Dual

Disorder varieties are algebraic varieties for which *dimensional reductions* occur for vertex or spin models, thus enabling the exact calculation of physical quantities such as the partition function per site, an infinite number of correlation functions ... [25,42]. A straightforward calculation, using a "disorder criterion" explained in [26], yields the following disorder conditions:

$$x x_1 x_2 x_3 - (x_1 + x_2 + x_3) + 2 - q + q \cdot x_i = 0, \qquad i = 1,2,3. \tag{3.6}$$

When there is no three-sites interaction ($x = 1$) one recovers the known disorder conditions of the two side nearest neighbor triangular Potts model [26,43].

These disorder conditions are "high-temperature" varieties. It is tempting to use duality (4.14) in order to single out some "low" temperature varieties. Unfortunately, when one transforms these disorder solutions by duality transformation (3.3), one gets:

$$x_1x_2x_3 - (x_1 + x_2 + x_3) + 2 - q + q \cdot x_i \rightarrow \frac{q^2 \cdot x_i}{x_2x_3x_1x - (x_1 + x_2 + x_3) + 2}. \tag{3.7}$$

This is related to the fact that these disorder conditions are nothing but he vanishing[4] conditions of the x_i^*'s.

For $x = 1$ the Kramers-Wannier dual [39] of these disorder varieties are algebraic varieties, on which the *low-temperature* expansions of the partition function per site (and many other quantities ...) simplify drastically to become the expansions of rational expressions [13]. Let us call these last varieties "order varieties". These "order varieties" are singled out: they do provide *formal constraints*[5] on the low-temperature expansions of the model. Let us introduce low-temperature variables A, B, and C:

$$A = \frac{1}{x_1}, \qquad B = \frac{1}{x_2}, \qquad C = \frac{1}{x_3}. \tag{3.8}$$

For instance, on the anisotropic triangular edge Potts model, relation:

$$A + BC + (q-2) \cdot ABC = 0, \tag{3.9}$$

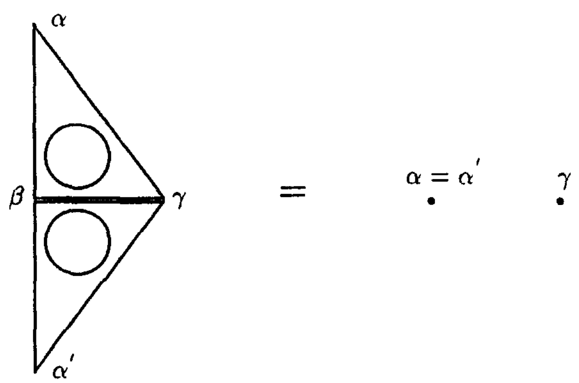
is a condition on which the low-temperature expansion of the partition function reduces to the (low-temperature) expansion of the partition function of an elementary triangular cell [13]. It is remarkable that these "order conditions" can actually be generalized to the (edge) checkerboard Potts model *in a magnetic field*[6], thus providing nontrivial (*formal*) constraints on the (low-temperature resummed) expansion of the model [13]. Actually introducing the "order condition" (see [13]):

$$D + ABC \cdot z + (q-2) \cdot ABCD \cdot z = 0, \tag{3.10}$$

one can show that the partition function per site is equal to a very simple expression when restricted to (3.10) and that the spontaneous magnetization restricted to (3.10) is actually equal to 1.

3.4. Inversion Relations

The inversion relations [7,25] for the two- and three-site interaction spin model can be represented pictorially as follows:



[4]As it should [25], these three disorder varieties have no intersection with the ferromagnetic critical variety (3.4).
[5]Most of the time these "order" conditions are not in the physical domain.
[6]The fugacity is denoted z.

which analytically means:

$$\sum_{\beta} w(\alpha,\beta,\gamma) \cdot I(w)(\beta,\alpha',\gamma) = \lambda \cdot \delta_{\alpha,\alpha'}. \tag{3.11}$$

The Boltzmann weight $w(\alpha,\beta,\gamma)$ of model (3.1) is invariant under a common shift of each spin α, β, and γ. Therefore, γ can be fixed in a particular color, namely zero. Thus, the Boltzmann weight can be represented by a $q \times q$ matrix (α being the column index, and β the row one), with entries $w(\alpha,\beta,0)$. Equation (3.11) thus becomes the following matricial relation:

$$\mathcal{W} \cdot I(\mathcal{W}) = \lambda \cdot \mathcal{I}d_q, \tag{3.12}$$

where $\mathcal{I}d_q$ denotes the $q \times q$ identity matrix, and the $q \times q$ matrix Boltzmann weight $\mathcal{W}$ reads:

$$\mathcal{W} = \begin{pmatrix} xx_1x_2x_3 & x_2 & x_2 & \dots & \dots & x_2 \\ x_3 & x_1 & 1 & 1 & \dots & 1 \\ x_3 & 1 & x_1 & 1 & \dots & 1 \\ \vdots & 1 & 1 & x_1 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 1 \\ x_3 & 1 & 1 & \dots & 1 & x_1 \end{pmatrix}.$$

Using a "$\mathbb{Z}_{q-1}$ Fourier transformation" [40,41], this $q \times q$ matrix can be block-diagonalized into one 2×2 block and a $(q-2) \times (q-2)$ matrix proportional to the identity matrix, $(x_1 - 1) \times \mathcal{I}d_{q-2}$. One can easily obtain the matrix inverse $I(\mathcal{W})$. Note that $I(\mathcal{W})$ is of the same form as $\mathcal{W}$, x, x_1, x_2, x_3, being changed to the following *birational* transformation I:

$$I : \begin{cases} x \rightarrow I(x) = \dfrac{(xx_1-1)^2(x_1+q-2)}{(xx_1^2 + xx_1(q-3) - q + 2)(x_1 - 1)}, \\ x_1 \rightarrow -\dfrac{xx_1^2 + xx_1(q-3) - q + 2}{xx_1 - 1} = 2 - q - x_1 + \dfrac{x_1(x-1)}{x_1x - 1}, \\ x_2 \rightarrow \dfrac{x_1 - 1}{x_3(xx_1 - 1)}, \\ x_3 \rightarrow \dfrac{x_1 - 1}{x_2(xx_1 - 1)}. \end{cases} \tag{3.13}$$

$I(x)$ can also be written:

$$I(x) = \frac{x_1^2 \cdot (x_1 + q - 2) \cdot (x-1)^2 + 2x_1 \cdot (x_1 - 1) \cdot (x_1 + q - 2) \cdot (x-1) + (x_1 - 1)^2(x_1 + q - 2)}{x_1(x_1 + q - 3)(x_1 - 1) \cdot (x-1) + (x_1 - 1)^2(x_1 + q - 2)}.$$

Obviously permutations of indices 1, 2, and 3 are also symmetries of the model. Introducing p_{23}, the permutation of x_2 and x_3, and similarly p_{31} and p_{12}, one can define the three following transformations:

$$I_1 = p_{23}I = Ip_{23}, \qquad I_2 = p_{31}p_{12}Ip_{12} = p_{12}Ip_{12}p_{31}, \qquad I_3 = p_{12}p_{31}Ip_{31} = p_{31}Ip_{31}p_{12},$$

corresponding to the three[7] inversion transformations of the model [44].

[7] This existence of *three* involutions singles out the *triangular lattice* among the bidimensional models, from the symmetry group analysis point of view.

4. THE SYMMETRY GROUP

Inversion I, permutations of x_1, x_2, x_3, and duality relation D (defined by(3.3)) generate a symmetry group of the parameter space of the model, denoted $\Gamma_{\rm upt}$ in the following.

At this point it is worth noticing that duality transformation D, *does actually commute* with I, and also with $\mathcal{S}_3$, the group of permutations of x_1, x_2, x_3, and therefore, the *whole* group generated by I and $\mathcal{S}_3$. This commutation property enables us to see $\Gamma_{\rm upt}$ as a *hyperbolic Coxeter group* generated by two infinite order transformations, up to the semidirect product by a finite group. These generically *infinite order* transformations read:

$$J_1 = I_3 I_2, \qquad J_2 = I_1 I_3, \qquad J_3 = I_2 I_1. \tag{4.1}$$

By definition the J_i's satisfy relation:

$$J_3 J_2 J_1 = \mathcal{I}\text{dentity}. \tag{4.2}$$

Two of these J_i's generate $\Gamma_{\rm upt}$, up to the semidirect product by a finite group.

Let us recall that, for generic values of q when $x = 1$, $\Gamma_{\rm upt}$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$, up to a semidirect product by a finite group and degenerates into a *finite group* for *Tutte-Beraha numbers* [23] ($q = 2 - 2\cos(k\pi/N)$). In fact, for $x = 1$, the J_i's *do commute* and the elements of group $\Gamma_{\rm upt}$ read:

$$\gamma = J_1^{n_1} J_2^{n_2} I_1^{\alpha_1}, \qquad \text{where } \alpha_1 = 0, 1. \tag{4.3}$$

Generically, n_1 and n_2 are relative integers. For q, a Tutte-Beraha number associated with N, n_1 and n_2 run into $\{0, \ldots, N-1\}$, the group $\Gamma_{\rm upt}$ being therefore isomorphic to the product $\mathbb{Z}_N \times \mathbb{Z}_N \times \mathbb{Z}_2$.

In order to analyse the general case ($x \neq 1$), let us introduce the 3-cycle $c = p_{31} p_{12}$, and let us write the J_i's in terms of c and of a single one (generically) infinite order transformation, namely $(cI)^2$:

$$J_1 = c(cI)^2 c^2, \qquad J_2 = c^2 (cI)^2 c, \qquad J_3 = (cI)^2. \tag{4.4}$$

4.1. Transformation $(cI)^2$

For the sake of simplicity, let us consider transformation $(cI)^2$ as a *homogeneous* transformation, introducing $x_0 = x x_1 x_2 x_3$ and a fifth homogenization variable t. One can then define a *homogeneous* inverse I_h (corresponding to (3.13)) and cI_h, which written as a *homogeneous* transformation, reads:

$$cI_h : (x_0, x_1, x_2, x_3, t) \to \left(-x_1 - (q-2)t, x_3, -(q-2)\frac{x_0 t - x_2 x_3}{x_1 - t} - x_0, x_2, \frac{x_0 t - x_2 x_3}{x_1 - t} \right).$$

One notices that $u_3 = x_1 + x_2 + (q-2)t$ and $v_3 = x_3 - x_0$ are just permuted by transformation cI_h: $u_3 \leftrightarrow v_3$. With these new variables, one also has:

$$cI_h : (x_1, x_3, t) \to (x_3, u_3 - x_1 - (q-2)t, F_0 t), \qquad \text{where } F_0 = \frac{x_0 t - x_2 x_3}{t(x_1 - t)}.$$

Transformation $(cI_h)^2$ then reads:

$$(cI_h)^2 : (u_3, v_3, x_1, x_3, t) \to (u_3, v_3, u_3 - x_1 - (q-2)t, v_3 - x_3 - (q-2)F_0 t, F_0 F_1 t),$$

where $F_1 = F_0(cI_h)$ is the same expression as F_0, where the x_i's have been replaced by their images by cI_h.

Introducing $q_\pm$, the roots of the second order equation $z^2 + (q-2)z + 1 = 0$ and introducing the successive iterates of F_0 by transformation cI_h (namely: $F_{n+1} = F_n(cI_h)$), one can write down the general expression of transformation $(cI_h)^{2N}$ (see [45]).

Let us note that for q corresponding to a *Tutte-Beraha number*, the $q_\pm^2$ are *N^{th}-root of unity* [7].

4.2. Tutte-Beraha Numbers

Let us recall that, when there is no three site interaction (that is $x = 1$), there do exist particular values of q, the so-called *Tutte-Beraha numbers* [23,46], for which transformations J_i's, or equivalently transformation $(cI_h)^2$, become *finite order ones* [7,44].

Amazingly, this situation still holds for the generic case (with $x \neq 1$).

One can establish [45] for $q = 2 - 2\cos(k\pi/N)$, (a *Tutte-Beraha number*), that transformation $(cI_h)^{2N}$ *reduces an identity*, that is equivalently:

$$J_i^N = \mathcal{I}d, \qquad \text{with } i = 1, 2, 3. \tag{4.5}$$

REMARK. Such *Coxeter groups* can be seen as the *fundamental group* of a surface of genus g minus k points [33]. Here there is a genus zero Riemann surface minus three points. At this step, the Coxeter group, one has to deal with, is reminiscent of the Schwarz's triangular groups[8]. Considering a geodesic triangle of angles $\pi/n_1, \pi/n_2, \pi/n_3$, and considering S_1, S_2, S_3 the symmetries with respect to the edges of the triangle, and defining the "rotations":

$$R_1 = S_2S_3, \qquad R_2 = S_3S_1, \qquad R_3 = S_1S_2, \tag{4.6}$$

the R_i's verify:

$$R_i^{n_i} = \mathcal{I}d, \qquad \text{with } i = 1, 2, 3, \text{ and } R_1R_2R_3 = \mathcal{I}d. \tag{4.7}$$

In the study of these *triangular* groups, three different cases have to be distinguished: depending on $1/n_1 + 1/n_2 + 1/n_3$ greater, lower or equal to 1.

Because of the ternary symmetry of our triangular Potts model, one has here $n_1 = n_2 = n_3 = N$. *The only Euclidean case is* $N = 3$, while the other values of N yield hyperbolic triangles and hyperbolic geometries, $N = 2$ corresponds to $q = 2$, which is the Ising subspace of the model. In this case the three-site interaction becomes irrelevant. Thus, the *first interesting case is* $N = 3$, that is, $q = 3$ (or $q = 1$).

4.3. The "Euclidean Case": $q = 3$ or $q = 1$

In this section we will restrict ourselves to $N = 3$, that is $q = 3$ or $q = 1$. In this case $J_i^3 = \mathcal{I}d$. A "straight" analysis of this group, on the J_i's, is performed in Appendix A. In fact, in this specific $N = 3$ case, it is better suited to introduce the transformations:

$$G_1 = p_{12}J_1p_{31}, \qquad G_2 = p_{23}J_2p_{12}, \qquad G_3 = p_{31}J_3p_{23}. \tag{4.8}$$

By introducing these transformations, it is easier to show that, for $N = 3$, Γ_{upt} is no longer a group *with an "exponential growth", but reduces down to* $\mathbb{Z} \times \mathbb{Z}$ up to a semidirect product by a finite group (like the affine Coxeter group $A_2^{(1)}$ [18]).

First, one notices that the G_i's do satisfy a relation similar to relation (4.2):

$$G_3G_2G_1 = \mathcal{I}d. \tag{4.9}$$

Let us first study the group $\mathcal{G}$, generated by G_1, G_2, and G_3. The G_i's can be written in terms of transformation I and of the three-cycle c:

$$G_1 = c^2Ic^2Ic^2, \qquad G_2 = Ic^2Ic, \qquad G_3 = cIc^2I. \tag{4.10}$$

Using $(cI)^6 = \mathcal{I}\text{dentity}$, G_1G_2 reads:

$$\begin{aligned} G_1G_2 &= c^2Ic^2Ic^2Ic^2Ic = c^2(cI)^{-4}c^2 = c^2(cI)^2c^2 = IcIc^2 \\ &= G_2G_1. \end{aligned} \tag{4.11}$$

[8]Such groups have been obtained from the analysis of the ratios of solutions of second-order differential equations ramified in three points.

Thus, the G_i's *actually commute*. From relations (4.11) and (4.9), it is clear that a generic element of $\mathcal{G}$ reads:

$$g = G_1^{n_1} G_2^{n_2}, \tag{4.12}$$

where n_1 and n_2 are *relative integers*, which explicitly means that $\mathcal{G}$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$.

Γ_{upt} can be seen to be generated by I and c, up to some semidirect product by a finite group. From relation (4.10), one gets at once:

$$\begin{array}{llll} cIcI = G_1^{-1}c^2, & cIc^2I = G_3, & c^2IcI = G_2^{-1}, & c^2Ic^2I = G_1c, \\ IcIc = G_3^{-1}c^2, & IcIc^2 = G_3^{-1}, & Ic^2IC = G_2, & Ic^2Ic^2 = G_2c. \end{array}$$

Thus, Γ_{upt} is *isomorphic to* $\mathbb{Z} \to \mathbb{Z}$ up to a semidirect product by a finite group.

4.4. Numerical Analysis

These infinite order transformations, represented as *birational* transformations, act as (*symmetry*) transformations in the parameter space of the model. In the $x = 1$ subcase, it has already been noticed [47] that, for $0 < q < 4$, the infinite set of points of the orbits of the discrete group of birational transformations is *dense in an algebraic curve*, while in the other case, they accumulate to fixed points. One has a similar situation for $x \neq 1$. Therefore, *in this section we restrict here our study to* $0 < q < 4$.

To complete the analysis of the (infinite discrete) symmetry group, one has to study its (generically infinite order) generators (the J_i's). We draw here their orbits in the four dimensional parameter space ($\mathbb{C}P_4$) of the model. From relation (4.4), it is clear that the iterations of the J_i's amount to performing the iteration of transformation $(cI)^2$. For generic values of q (of course different from Tutte-Beraha numbers, see Section 4.2), the iteration of $(cI)^2$ *yields curves*. Figure 1 shows such a curve obtained for $q = 3.5$ (which is not a Tutte-Beraha number).

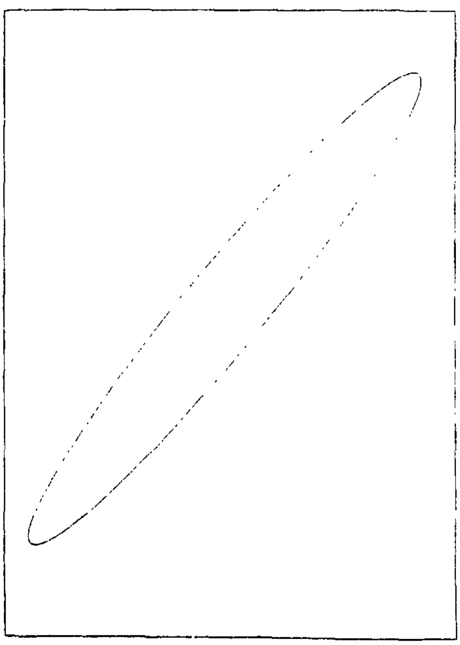
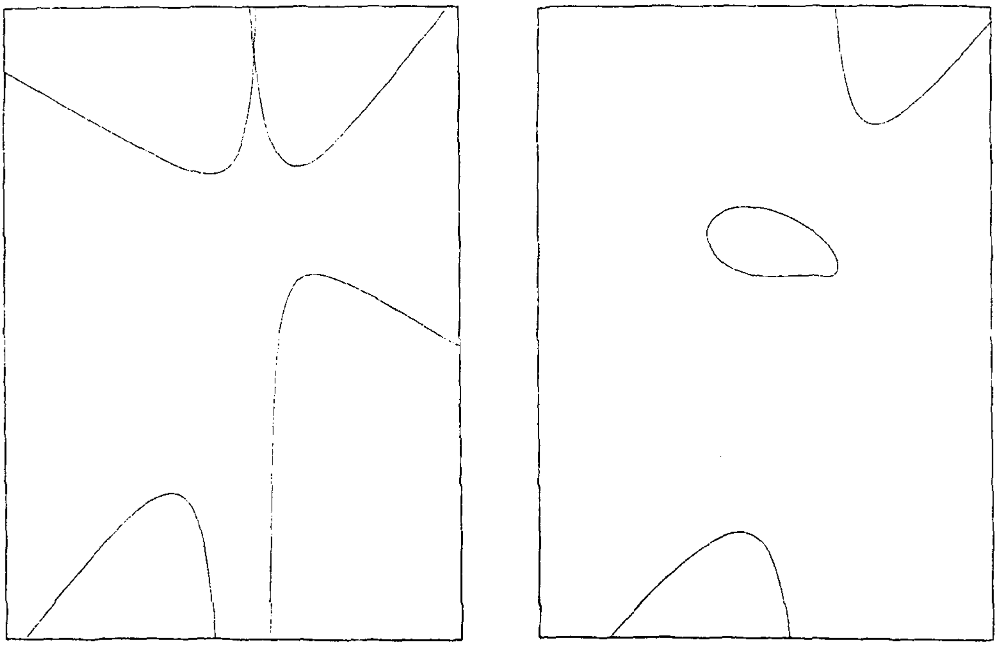


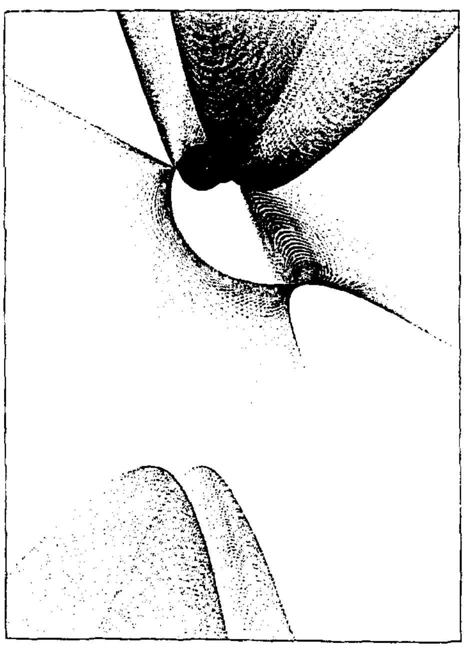
Figure 1. Two-dimensional projection of an orbit of transformation $(cI)^2$, for $q = 3.5$.

For *Tutte-Beraha numbers*, since the J_i's are *finite-order transformations*, one has to consider other elements of the group. As far as the Euclidean case is concerned ($q = 3$ or $q = 1$), let us recall that the G_i's are the *commuting* generators of the symmetry group isomorphic to $\mathbb{Z} \times \mathbb{Z}$.



(a) (b)

Two-dimensional projections of two orbits corresponding, respectively, to the iterations of transformations G_2 and G_3, for $q = 3$.



(c) Orbit generated by transformations G_2 and G_3, for $q = 3$: this surface is the product of the two curves of Figures 2a and 2b.

Figure 2.

Figure 2a illustrates the iteration of G_2 for $q = 3$. *Remarkably, curves are obtained once again.* Of course, iterating G_3 for $q = 3$ also yields curves, as can be seen of Figure 2b. Considering one orbit of the symmetry group generated by the G_i's one gets, as it should be, a *surface* which can clearly be seen on Figure 2c as the *product of curves* like Figures 2a and 2b. This last figure gives a nice illustration of the $\mathbb{Z} \times \mathbb{Z}$ structure of the group.

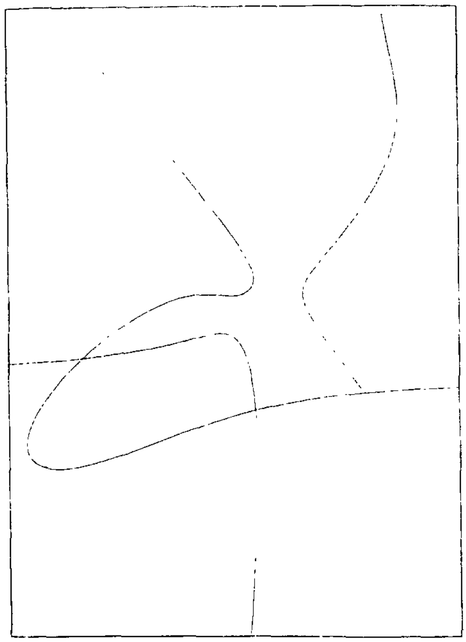


Figure 3. Two-dimensional projection of an orbit of transformation G_2 for $q = 0.5$.

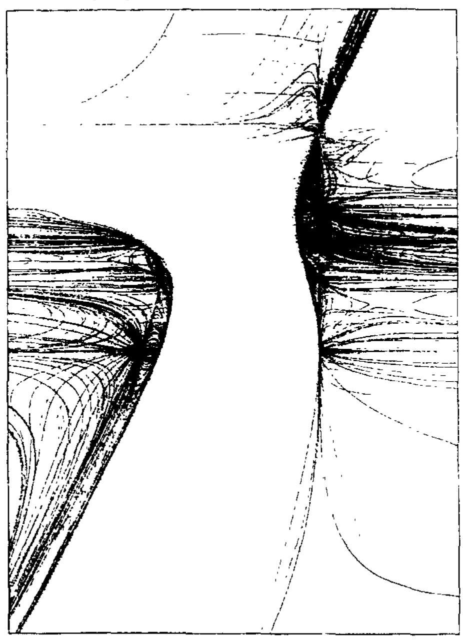


Figure 4. Two-dimensional projections of two orbits corresponding, respectively, to the iterations of transformation $J_1 J_2^2$ for a Tutte-Beraha number ($q = 2 + \sqrt{3}$).

Amazingly, the G_i's which *no longer commute* when q is no longer equal to 1 or 3, *do yield curves*, as can be seen on Figure 3 which represents the iteration of G_2 for $q = 0.5$ (not a Tutte-Beraha number).

All these results are remarkable: if one considers the iteration of more involved elements of the group, one generically gets quite "chaotic" figures (except of course for $q = 3$ or $q = 1$). Figure 4 shows such a quite "chaotic" orbit corresponding to the iteration of $J_1 J_2^2$ for a Tutte-Beraha number ($q = 2 + \sqrt{3}$).

These last figures and the study of many other orbits not given here, give a good hint of the "complexity" of these (infinite) hyperbolic Coxeter groups. They are generically of "exponential growth", *even when additional relations occur* (see relation (4.5)). This numerical study indicates that for generic values of q, the generators of the symmetry group (the J_i's) seem integrable since their iterations *yield curves* apparently in the whole parameter space. Moreover, the G_i's seem to satisfy the same property *for any value of* q, though they emerged from the analysis of the Euclidean case ($q = 3$ or $q = 1$).

A way to verify this assumption, is to give the algebraic equations of these curves. For this purpose, in the next section, we will seek *algebraic varieties invariant* under the J_i's and the G_i's.

4.5. Group Invariants and Well-Suited Variables

Let us first take into account that there exist three (homogeneous) polynomials, of degree, respectively, 1, 2, and 3, *invariant under permutation of* x_1, x_2, and x_3, and *covariant* under transformation I (see (3.13)). These three polynomials read:

$$\begin{aligned}
D_1 &= x_1 + x_2 + x_3 - x_0 + (q-2)t, \\
D_2 &= t(x_1 + x_2 + x_3 + x_0 - t) - x_1x_2 - x_2x_3 - x_3x_1, \\
D_3 &= t^2x_0 - x_1x_2x_3.
\end{aligned}$$

Let us consider the cofactors (under the action of I) of D_1, D_2, and D_3:

$$C_1 = \frac{I(D_1)}{D_1} = \frac{x_1 - 1}{(xx_1 - 1)\cdot x_2x_3}, \qquad C_2 = \frac{I(D_2)}{D_2} = -\frac{xx_1^2 + qxx_1 - 2xx_1 + 1 - q}{(xx_1-1)^2\cdot x_2x_3}, \quad \text{and}$$
$$C_3 = \frac{I(D_3)}{D_3} = -\frac{(x_1-1)(xx_1^2 + (q-2)xx_1 + 1 - q)}{x_2^2x_3^2\cdot(xx_1-1)^3}.$$

One notes that the cofactor of D_3 is the product of the respective cofactors of D_1 and D_2. As a consequence, one directly gets an *invariant* under the *whole group* generated by I and the permutations of x_1, x_2 and x_3:

$$\Delta = \frac{D_1\cdot D_2}{D_3}. \tag{4.13}$$

This provides, for arbitrary q, a *canonical foliation of the parameter space* ($\mathbb{C}P_4$) by codimension one algebraic varieties (namely *cubics*).

Let us recall that *duality* transformation D, defined in Section 3.2, is also a *symmetry* of the model, which *commutes*[9] with transformation I and with permutations of 1, 2, and 3. Let us notice that duality D can actually be represented as a *linear* transformation when written in terms of homogeneous variables:

$$D_h\colon\begin{cases}
x_0 \to x_0 + (q-1)\cdot(x_1 + x_2 + x_3 + (q-2)t), \\
x_1 \to (q-1)\cdot x_1 + x_0 - x_2 - x_3 - (q-2)t, \\
x_2 \to (q-1)\cdot x_2 + x_0 - x_1 - x_3 - (q-2)t, \\
x_3 \to (q-1)\cdot x_3 + x_0 - x_2 - x_1 - (q-2)t, \\
t \to x_0 - x_1 - x_2 - x_3 + 2t.
\end{cases} \tag{4.14}$$

Considering the previous covariant polynomials, one sees that D_1 and D_2 simply transform under the duality transformation:

$$(D_1, D_2) \longrightarrow (-q\cdot D_1, q^2\cdot D_2). \tag{4.15}$$

The duality acts in a slightly more involved way on D_3:

$$D_3 \longrightarrow D_3^* = q^2\cdot(q\cdot D_3 - D_1\cdot D_2). \tag{4.16}$$

[9]This is related to the fact that duality (4.14) corresponds to a *weak-graph* transformation [48]: therefore, it has a *linear* representation and *commutes* with the inversion relations.

Therefore, as far as the other covariant polynomials are concerned, one can barter D_3 for a homogeneous polynomial, namely:

$$D_{3d} = 2q \cdot D_3 - D_1 D_2. \tag{4.17}$$

On this new "self-dual" covariant the duality gives:

$$D_{3d} \longrightarrow q^3 \cdot D_{3d}. \tag{4.18}$$

Algebraic varieties D_1, D_2, and D_{3d} *do have covariance properties with respect to the whole group* $\Gamma_{\rm upt}$ (*including* duality (4.14)), which is (generically) a *hyperbolic group*. From the point of view of effective algebraic geometry, this provides examples of algebraic varieties *with very large (discrete) groups of (birational) automorphisms*.

4.5.1. Curves generated by the J_i's or the G_i's

It has been seen in Section 4.4 that the iterations of the J_i's yield, *for arbitrary q, curves in the whole parameter space*. In order to prove that these *curves* are *actually algebraic*, one has to exhibit two other algebraic invariants for these very transformations. From relations (4.14), it is clear that the study can be limited to transformation $(cI)^2$. One can show that the two polynomials.

$$E_1 = x_1 + x_2 - x_3 + x_0 + (q-2)t, \qquad E_2 = t(x_1 + x_2 - x_3 - x_0 - t) - x_1x_2 + x_2x_3 + x_3x_1,$$

are actually covariant under the action of $(cI)^2$. These expressions happen to have, respectively, the same cofactors (under transformation $(cI)^2$) as D_1 and D_2. This immediately provides two additional *algebraic invariants* under $(cI)^2$:

$$\Delta_1 = \frac{D_1}{E_1}, \qquad \Delta_2 = \frac{D_2}{E_2}. \tag{4.19}$$

Curves like Figure 1 are thus given as intersections of *cubics, quadrics, and hyperplanes*, namely:

$$\Delta = \delta, \qquad \Delta_1 = \delta_1, \qquad \Delta_2 = \delta_2, \tag{4.20}$$

where the δ's denote arbitrary constants.

Considering the previous covariant polynomials, one notices that five of them are "eigen-polynomials" of the duality transformation (4.14). In addition to the previous two covariants D_1 and D_2 (see (4.13)), one gets:

$$(E_1, E_2) \longrightarrow (qE_1, q^2E_2). \tag{4.21}$$

Algebraic curves, with an infinite number of (*birational*) automorphisms are either *elliptic* (or rational) curves [49]. Amazingly, by eliminating x_0 and x_3 from relations (4.20) one gets (as expressed in inhomogeneous variables):

$$(\delta_1 + 1) \cdot (\delta_2 + 1) \cdot (x_1x_2 - 1) = (4\delta_1\delta_2\delta \cdot (x_1 + x_2 - 2) + (\delta_1 - 1) \cdot (\delta_2 - 1)) \cdot (x_1 + x_2 + q - 2),$$

which proves that these curves are actually *rational curves*.

Let us now consider the G_i's (or equivalently the $(cI_h)^{2N}$'s). The previous numerical analysis indicated remarkable *occurrence of curves*, when iterating the G_i's for any value of q. Let us, for instance, consider G_3. One notices that polynomials:

$$F_1 = x_3, \qquad F_4 = (x_1x_3 + x_2x_3 - x_3t - x_0t) \cdot (x_1x_2 + (q-3)x_0t - (q-2)x_0x_3),$$

are actually covariant under the action of G_3. The values of the cofactors of these F_i's yield two G_3-invariants:

$$\Delta_1' = \frac{D_1}{F_1}, \qquad \Delta_4' = \frac{D_1 D_3}{F_4}. \tag{4.22}$$

Figures like Figures 2a, 2b, or 3 are thus *algebraic (elliptic) curves* given by intersections of *cubics, hyperplanes, and quartics.*

Duality transformation (4.14) also acts on polynomials F_i's. One can barter them for new "self-dual" homogeneous polynomials, namely:

$$F_{1d} = 2q \cdot x_3 - D_1, \qquad F_{4d} = 2q \cdot F_4 - \left(q^2 - 3q + 1\right) \cdot (D_2 - x_3 D_2) \cdot D_1.$$

4.5.2. The "Euclidean cases": $q = 3$ and $q = 1$

Let us recall that for $q = 3$ (or $q = 1$), these G_i's *do commute* and that: $G_3 G_2 G_1 = \mathcal{I}d$. It has been seen that each of the G_i's generates *algebraic elliptic curves*. Therefore, for $q = 3$ (and $q = 1$), the orbits of the group generated by the G_i's yield *algebraic surfaces* which are *products of two elliptic curves*, as clearly seen on Figure 2c. Since this surface is stable under the group $\mathcal{S}_3$ of permutations of x_1, x_2, and x_3, it is natural to give its equation without referring to two of the G_i's, that is, without having any direction singled out.

Actually, for $q = 3$, there exists an additional polynomial:

$$D_5 = -x_1 x_2 x_3 \left(x_0^2 + x_1^2 + x_2^2 + x_3^2 - t^2\right) + x_0 \left(x_1^2 x_2^2 + x_2^2 x_3^2 + x_3^2 x_1^2\right), \tag{4.23}$$

symmetric under permutations of x_1, x_2, and x_3 and covariant under I, from which one deduces, taking into account its cofactor, the ($\mathcal{S}_3$-symmetric) *invariant*:

$$\Delta_5' = \frac{D_1^3 D_2}{D_5}. \tag{4.24}$$

Invariant (4.24), together with invariant (4.13), thus give $\mathcal{S}_3$-symmetric equations of these algebraic surfaces. For $q = 3$ one thus has a *foliation of the four-dimensional parameter space in algebraic surfaces* given by (4.13) and (4.24).

Similarly, for $q = 1$, a $\mathcal{S}_3$-symmetric polynomial covariant under the action of I, namely $D_3' = x_0 \cdot D_2$, yields the following $\mathcal{S}_3$-symmetric invariant:

$$\Delta_3' = \frac{D_1^3}{D_3'}, \tag{4.25}$$

and the duality gives, on this last covariant: $D_3' \rightarrow q^2 D_3'$.

For $q = 3$, the duality acts in the following way on D_5:

$$D_5 \longrightarrow D_5^* = 9 \cdot \left(27 \cdot D_5 - D_1^3 \cdot D_2\right), \qquad \text{yielding: } D_5 \rightarrow (D_5^*)^* = 3^{10} \cdot D_5.$$

For $q = 3$, one can thus substitute D_5 for a self-dual covariant:

$$D_{5d} = 54 D_5 - D_1^3 \cdot D_2.$$

This new self-dual covariant duality (4.14) yields:

$$D_{5d} \longrightarrow 3^5 \cdot D_{5d}. \tag{4.26}$$

4.5.3. Self-dual well-suited variables

One would like to trade the foliation given by (4.13) for an equivalent, yet *self-dual* one. In terms of Δ, given by (4.13), the duality transformation (4.14) reads:

$$\Delta \longrightarrow -q \cdot \frac{\Delta}{q - \Delta}, \tag{4.27}$$

and for $q = 3$ only, invariant Δ_5' becomes:

$$\Delta_5' \longrightarrow -27 \cdot \frac{\Delta_5'}{27 - \Delta_5'}. \tag{4.28}$$

From this one immediately gets a "pseudo" self-dual invariant, namely $\Delta_{\rm dual}$ given by:

$$\Delta_{\rm dual} = \frac{\Delta}{2 \cdot q - \Delta} = \frac{D_1 \cdot D_2}{2 \cdot q \cdot D_3 - D_1 \cdot D_2}. \tag{4.29}$$

Actually duality transformation (4.14) acts as follows on $\Delta_{\rm dual}$:

$$\Delta_{\rm dual} \longrightarrow -\Delta_{\rm dual}. \tag{4.30}$$

The *critical ferromagnetic variety* $D_1 = 0$ reads $\Delta_{\rm dual} = 0$.

Similarly, for $q = 3$ only, one can introduce invariant $\Delta_5^{\rm dual}$ such that duality relation (4.14) reads:

$$\Delta_5^{\rm dual} \longrightarrow -\Delta_5^{\rm dual}. \tag{4.31}$$

Invariant $\Delta_5^{\rm dual}$ reads:

$$\Delta_5^{\rm dual} = \frac{\Delta_5'}{54 - \Delta_5'} = \frac{D_1^3 \cdot D_2}{54 D_5 - D_1^3 \cdot D_2}. \tag{4.32}$$

For $q = 3$ one would also like to exchange the foliation of the parameter space in *algebraic surfaces* given by (4.13) and (4.24), for a foliation corresponding to two *explicitly self-dual algebraic conditions*.

Here, we introduce invariants denoted X and Y:

$$X = \frac{1}{\Delta} = \frac{D_3}{D_1 \cdot D_3} \quad \text{and} \quad Y = \frac{1}{\Delta_5'} = \frac{D_5}{D_1^3 \cdot D_2}, \tag{4.33}$$

and also the following "pseudo-self-dual" invariants:

$$\tilde{X} = \frac{1}{6} - X = \frac{-1}{6 \cdot \Delta_{\rm dual}} \quad \text{and} \quad \tilde{Y} = \frac{1}{54} - Y = \frac{1}{54 \cdot \Delta_5^{\rm dual}}, \tag{4.34}$$

which transform, under the duality (4.14), as follows: $(\tilde{X}, \tilde{Y}) \to (-\tilde{X}, -\tilde{Y})$. From the two "pseudo" self-dual invariants $\Delta_5^{\rm dual}$ and $\Delta_{\rm dual}$ (or $(\tilde{X}, \tilde{Y})$), one can easily get two *explicitly self-dual* invariants.

4.6. Two Remarkable Varieties: $x = 1$ and Its Dual

Let us come back to the vanishing of the three-spin interaction that is $x = 1$, or $D_3 = 0$. One notes that this variety is not self-dual. Variety $x = 1$ is well known [25] and plays a special role: the symmetry group $\Gamma_{\rm upt}$ is *isomorphic to* $\mathbb{Z} \times \mathbb{Z}$ (up to some semidirect product by a finite group) [10].

4.6.1. A remarkable variety: $x = 1$

In this $x = 1$-subcase ($D_3 = 0$), there exists a *rational parametrization* which amounts to introducing the well-suited variables [10]:

$$x_i = \frac{1 - t^3 \cdot u_i}{t \cdot u_i - t^2}, \qquad i = 1,2,3, \qquad \text{where } q = t^2 + \frac{1}{t^2} + 2. \tag{4.35}$$

In the $x = 1$ and $D_1 = 0$ subcase (critical condition for the edge Potts model) one can actually *calculate exactly* and very quickly the partition function per site, using the *inversion trick* and these well-suited variables [7].

In this $x = 1$-subcase an algebraic *antiferromagnetic* variety has been proposed by Martin and Maillard [50] on the basis of the analysis of the discrete group generated by the inversion relations that is:

$$\begin{aligned} M_q(x_1, x_2x_3) = (q-2) \cdot x_1x_2x_3 + 2 \cdot (x_1x_2 + x_2x_3 + x_3x_1) \\ + (q-2) \cdot (x_1 + x_2 + x_3) + (q-2)^2 - 2 = 0. \end{aligned} \tag{4.36}$$

With this rational parametrization, the antiferromagnetic relation (4.36) and the *critical* condition $D_1 = 0$ read, respectively:

$$\frac{(t^4-1)^3 \cdot (tu_1u_2u_3 + 1)}{t^5(t-u_1)(t-u_2)(t-u_3)} = 0 \quad \text{and} \quad -\frac{(t^4-1)^2 \cdot (tu_1u_2u_3 - 1)}{t^3(t-u_1)(t-u_2)(t-u_3)} = 0,$$

and the ratio:

$$\frac{M_q(x_1,x_2,x_3)}{D_1} = \frac{1-t^4}{t^2} \cdot \frac{(tu_1u_2u_3+1)}{(tu_1u_2u_3-1)}. \tag{4.37}$$

Algebraic expression D_2 reads in the $x = 1$ limit:

$$D_2 = (x_3 - 1) \cdot (x_2 - 1) \cdot (x_1 - 1) \cdot \frac{(t^2+1)^3 (tu_1 - 1) \cdot (tu_2 - 1) \cdot (tu_3 - 1)}{t^3(t-u_1) \cdot (t-u_2) \cdot (t-u_3)}.$$

For $q = 3$ (that is $t^6 = -1$), D_5 also reduces to a very simple expression in the $x = 1$ limit, namely:

$$\begin{aligned} D_5 &= -x_1x_2x_3(x_3-1) \cdot (x_3+1) \cdot (x_2-1) \cdot (x_2+1) \cdot (x_1-1) \cdot (x_1+1) \\ &= -\frac{(t^4-1)^3 (t^3u_1-1)(t^3u_2-1)(t^3u_3-1)(tu_1-1)(tu_1+1)(tu_2-1)(tu_2+1)(tu_3-1)(tu_3+1)}{t^9 \cdot (t-u_1)^3(t-u_2)^3(t-u_3)^3}. \end{aligned}$$

The ratio (4.37) is *actually an invariant of the group* Γ_{triang} [7]. In fact, this invariant can be replaced by other ones using the following remarkable identity:

$$\begin{aligned} &M_q^2 - q \cdot (q-4) \cdot D_1^2 \\ &\quad = 4 \cdot \left(1 + (q-2) \cdot x_1 + x_2^2\right) \cdot \left(1 + (q-2) \cdot x_2 + x_2^2\right) \cdot \left(1 + (q-2) \cdot x_3 + x_3^2\right). \end{aligned} \tag{4.38}$$

One can also, instead of the ratio (4.37), introduce the following algebraic invariant [7]:

$$\mathcal{I} = \frac{(q-2) \cdot x_1x_2x_3 + (x_1x_2 + x_2x_3 + x_3x_1) - 1}{(x_1x_2 + x_2x_3 + x_3x_1) + (q-2) \cdot (x_1 + x_2 + x_3) + (q-1) \cdot (q-3)}. \tag{4.39}$$

Equivalently one can introduce invariant:

$$\mathcal{I}_2 = \frac{x_1x_2x_3 - (x_1 + x_2 + x_3) - (q-2)}{(q-2) \cdot x_1x_2x_3 + (x_1x_2 + x_2x_3 + x_3x_1) - 1} = \frac{x_1x_2x_3 - (x_1 + x_2 + x_3) - (q-2)}{-x_1x_2x_3 \cdot W(x_1,x_2,x_3)},$$

where

$$W(x_1,x_2,x_3) = \frac{1}{x_1}\frac{1}{x_2}\frac{1}{x_3} - \left(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}\right) - (q-2). \tag{4.40}$$

In the numerator of (4.40) one recognizes D_1 for $x = 1$ and the denominator is nothing but the numerator where the x_i's are changed into the $1/x_i$'s.

4.6.2. Another remarkable variety: The dual of $x = 1$

Since duality (4.14) commutes with $\Gamma_{\rm upt}$, the *dual variety of* $x = 1$ *also corresponds to the degeneracy of* $\Gamma_{\rm upt}$ *into a group isomorphic to* $\mathbb{Z} \times \mathbb{Z}$ (up to some semidirect product by a finite group). This remarkable variety (3.5) also reads:

$$D_3^* = q^2(qD_3 - D_1D_2) = 0, \tag{4.41}$$

or explicitly,

$$\begin{aligned}&\left(x_1^2x_2^2x_3^2x^2 + \left(x_1^2x_2 + x_1^2x_3 + x_2^2x_3 + x_2x_3^2 + x_1x_2^2 + x_1x_3^2\right) - \left(x_1^2 + x_3^2 + x_2^2\right) + 1\right)\\ &\quad + (q-3)\cdot\left((x_1x_2 + x_1x_3 + x_2x_3) - (x_1 + x_2 + x_3) + x_1x_2x_3 + 1\right) = 0.\end{aligned} \tag{4.42}$$

In the isotropic limit $x_1 = x_2 = x_3 = u$ and, for $q = 3$, this variety reads

$$u^6 \cdot x^2 + u^3 \cdot \left(1 - 3u^2\right) \cdot x + \left(1 - 6u^2 + 6u^3\right) = 0. \tag{4.43}$$

There is also a rational parametrization of (4.42) as follows:

$$x_i = \frac{n_i}{d}, \qquad i = 1,2,3 \text{ and } x = \frac{n_{1x}^2 \cdot n_{2x}}{t \cdot d_{1x} \cdot d_{2x} \cdot d_{3x}}, \tag{4.44}$$

with

$$\begin{aligned}
n_1 &= (u_1u_2u_3t - 1)\cdot\left(t^4 - t^2 + 1\right) - t^4u_1u_3 - u_2u_3 - t^4u_1u_2 + t^5u_1 + tu_3 + tu_2,\\
n_2 &= (u_1u_2u_3t - 1)\cdot\left(t^4 - t^2 + 1\right) - t^4u_2u_3 - u_1u_3 - t^4u_1u_2 + t^5u_2 + tu_3 + tu_1,\\
n_3 &= (u_1u_2u_3t - 1)\cdot\left(t^4 - t^2 + 1\right) - t^4u_2u_3 - u_1u_2 - t^4u_1u_3 + t^5u_3 + tu_2 + tu_1,\\
n_{1x} &= t^3 \cdot u_1u_2u_3 \cdot \left(t^2 - 2\right) + t^2(u_1u_2 + u_1u_3 + u_2u_3) - t^3(u_1 + u_2 + u_3) + 2t^2 - 1,\\
n_{2x} &= u_1u_2u_3\left(t^2 + t^6 - 1\right) - t^5(u_1u_2 + u_1u_3 + u_2u_3) + t^2(u_1 + u_2 + u_3) + t\cdot\left(t^6 - t^4 - 1\right),\\
d_{1x} &= (u_1u_2u_3t - 1)\cdot\left(t^4 - t^2 + 1\right) - t^4u_1u_3 - u_2u_3 - t^4u_1u_2 + t\cdot\left(t^4u_1 + u_2 + u_3\right),\\
d_{2x} &= (u_1u_2u_3t - 1)\cdot\left(t^4 - t^2 + 1\right) - t^4u_2u_3 - u_1u_3 - t^4u_1u_2 + t\cdot\left(t^4u_2 + u_3 + u_1\right),\\
d_{3x} &= (u_1u_2u_3t - 1)\cdot\left(t^4 - t^2 + 1\right) - t^4u_2u_3 - u_1u_2 - t^4u_1u_3 + t\cdot\left(t^4u_3 + u_2 + u_1\right).
\end{aligned}$$

With this rational parametrization (4.44) inversion relation (4.13) takes a very simple form:

$$(u_1, u_2, u_3) \longrightarrow \left(\frac{t^2}{u_1}, \frac{1}{t^2 \cdot u_3}, \frac{1}{t^2 \cdot u_2}\right), \tag{4.45}$$

and the polynomial expressions D_1, D_2, D_3 read, respectively,

$$\begin{aligned}
D_1 &= \frac{\left(t^4 - 1\right)^2 \cdot (t \cdot u_1u_2u_3 - 1)}{t^2 \cdot d},\\
D_2 &= \frac{-\left(t^2 + 1\right)^3(tu_1 - 1)(u_1 - t)(tu_2 - 1)(u_2 - t)(tu_3 - 1)(u_3 - t)}{t \cdot d^2}, \quad \text{and}\\
D_3 &= \frac{-\left(t^4 - 1\right)^2\left(t^2 + 1\right)(tu_3 - 1)(u_3 - t)(tu_2 - 1)(u_2 - t)(tu_1 - 1)(u_1 - t)(tu_1u_2u_2 - 1)}{t \cdot d^3},
\end{aligned} \tag{4.46}$$

with

$$d = u_1u_2u_3 \cdot t^5 - (2u_1u_2u_3 + u_2 + u_3 + u_1)t^3 + (u_2u_3 + u_1u_2 + u_1u_3 + 2)t^2 - 1.$$

4.7. Additional Algebraic Invariant for $x = 1$. Additional Algebraic Invariants for Finite Group Subcases

At first glance, there seems to exist an incompatibility between invariants (4.13), (4.24), and invariant (4.39) (or (4.40)) of the *rational parametrization* [7,10] introduced for $x = 1$ (no three-spin interactions).

The previous occurrence of an additional invariant *specific* of $x = 1$ (see (4.39)) and of another one specific of $q = 3$ (see (4.24)), deserves some comments. Invariant (4.39) exists *for arbitrary* q when $x = 1$ and cannot, at first glance, be simply extended to $x \neq 1$. Similarly, invariant (4.24) only exists when $q = 3$. When *restricted to* $q = 3$, it may seem that there is an incompatibility between these various invariants *when one considers the* $x = 1$ limit. In fact, the answer to this paradox is the following: when $q = 3$ *and* $x = 1$ the discrete group degenerates into a *finite group*. The orbits of the group are no longer surfaces, or curves, but *a finite set of points* and *one can have as many (independent) "algebraic" invariants as the dimension of the parameter space*. In $x = 1$, $q = 3$ subcase, besides the $x = 1$ ($D_3 = 0$) condition, one has the two invariants Δ_5' and $\mathcal{I}$ ((4.13) and (4.39)) and (at least) a *third* invariant (see Appendices D and E.4 for more details). Actually, in this $x = 1$, $q = 3$ subcase, one has an *additional covariant* namely:

$$\begin{aligned} p_9 = {} & (2 + 2x_3 + 3x_2x_3 + 2x_2 - x_1 + x_1x_2x_3) \cdot (3x_1x_3 + x_1x_2x_3 + 2x_3 + 2x_1 + 2 - x_2) \\ & \times (3x_1x_3 + x_1 + 2x_1x_2x_3 + x_3 + 1 + x_2) \cdot (x_1x_2x_3 + 3x_1x_2 + 2x_2 + 2x_1 + 2 - x_3) \\ & \times (x_1 + 3x_1x_2 + 2x_1x_2x_3 + x_3 + 1 + x_2) \cdot (x_1 + 1 + 2x_1x_2x_3 + 3x_2x_3 + x_3 + x_2) \\ & \times (2x_3 - x_2 + x_1x_2x_3 - x_1 - 1) \cdot (2x_2 - x_3 - x_1 - 1 + x_1x_2x_3) \\ & \times (x_1x_2x_3 + 2x_1 - x_2 - x_3 - 1). \end{aligned} \tag{4.47}$$

Note, that p_9 *obtained, from the disorder condition,* (3.6) taking the product of the images of this disorder condition under the group Γ_{triang}, which *happens to be finite* for $q = 3$. Note, that p_9 transforms as follows under the inversion relation:

$$p_9 \longrightarrow \frac{p_9}{x_2^9 \cdot x_3^9}. \tag{4.48}$$

One has, therefore, an *additional invariant* in this $q = 3$ ($x = 1$) subcase:

$$I_9 = -\frac{p_9}{D_1^9}. \tag{4.49}$$

Invariant (4.49) is basically built as the product, over the discrete group, of the disorder conditions. Appendix C.1 give calculations corresponding to a straight generalization of such an invariant in the $x = 1$ and $q = 3$ subcase, but *for the checkerboard Potts model.* One can also build another invariant from the product, over the discrete group, of the "*order conditions*" described in Section 3.3. These two invariants (product over the group of *disorder conditions*, product over the group of "*order*" conditions) can both be used. However, the invariant product over the group of "order" conditions is better suited for the analysis of *low-temperature expansions* and the analysis of quantities of a "low-temperature" nature like the *spontaneous magnetization* (see Appendix D).

From this new invariant, which does not exist[10] for arbitrary q, one gets a new foliation of the parameter space in terms of the *algebraic surfaces*, I_9 = constant. Invariants $\mathcal{I}$ and I_9 can be shown to be algebraically independent. One can verify that the intersection of $x = 1$ (i.e., $D_3 = 0$), Δ_5 = *constant*, $\mathcal{I}$ = *constant*, and I_9 = *constant* gives *a finite set of points* corresponding to the orbits of the discrete symmetry group.

[10]In the $x = 1$ subcase the group *becomes finite for the Tutte-Beraha numbers*. When the group is finite, and *only in this case*, one can get additional invariants, similar to (4.9), built as product over the group of disorder (or "order") varieties.

REMARK. If one can imagine, for $q = 3$ and $x = 1$, that the critical manifolds correspond to quite involved relations between these various invariants [51], at first glance, one expects the phase diagram of the model for $q = 3$ and $x \neq 1$, to *depend only on invariant* (4.13) and the phase diagram of the model for $x = 1$ and $q \neq 3$, to *depend only on invariant* (4.39). Therefore, a question naturally pops out, namely the "continuity" of the phase diagram [51] for $q \simeq 3$ and $x \simeq 1$. We will address this question in parallel publications.

5. REMARKABLE ALGEBRAIC VARIETIES AND MONTE-CARLO SIMULATIONS: TOWARDS CRITICAL MANIFOLDS

Critical manifolds have to be compatible with *all the symmetries* of the lattice model: the *discrete* symmetries generated by the inversion relations, as well as the (*continuous*) weak-graph ("gauge") transformations [48]. The self-dual variety (3.4), that is $D_1 = 0$, *is already known as a critical variety of this very Potts model, for arbitrary* q, in some ferromagnetic region [24]. This variety[11] is, as it should be, *stable under the whole hyperbolic Coxeter group* [44]. In a rather general framework, one can write a critical manifold as follows:

$$F\left(\Delta_{\text{dual}}, \Delta_5^{\text{dual}}\right) = 0 \quad \text{or} \quad \mathcal{F}\left(\tilde{X}, \tilde{Y}\right) = 0, \tag{5.1}$$

where F (or $\mathcal{F}$) is any (transcendental) function such that:

$$F\left(\Delta_{\text{dual}}, \Delta_5^{\text{dual}}\right) = F\left(-\Delta_{\text{dual}}, -\Delta_5^{\text{dual}}\right) \quad \text{or} \quad \mathcal{F}\left(\tilde{X}, \tilde{Y}\right) = \mathcal{F}\left(-\tilde{X}, -\tilde{Y}\right). \tag{5.2}$$

In order to get some hint on the *critical manifolds*, one also has to take into account other exact results. For instance, there should not be any intersection between the disorder conditions[12] and the critical manifolds or, if any, this should localize very precisely a tricritical manifold [52].

Several varieties pop out as natural conditions for criticality since they have been shown to be stable under the whole infinite Coxeter group, namely (besides $D_1 = 0$) the self-dual varieties $D_2 = 0$, $D_{3d} = 0$ (that is, $6 \cdot D_3 - D_1 \cdot D_2 = 0$), and for $q = 3$ only, $D_{5d} = 0$ (that is, $54 \cdot D_5 - D_1^3 \cdot D_2 = 0$). Unfortunately, these "candidates" for criticality are ruled out by Monte-Carlo simulations [52].

5.1. $D_1 = 0$ and a Tricritical Manifold

Monte-Carlo simulations have been performed in the $q = 3$ *isotropic* limit, on $3 \times L^2$ sites *triangular* lattices with both $2\pi/3$-rotation invariance and translation invariance, for various values of L [52]. The Monte-Carlo calculations [52] show that $D_1 = 0$ is actually a *critical* variety, and that, on the algebraic variety $D_1 = 0$, *both first-order and second-order transitions occur*. Therefore, there must exist a *codimension-two tricritical submanifold* which corresponds to the "frontier" between these two types of transitions on the codimension-one variety $D_1 = 0$. Our objective is to get the equation of such a variety.

It is clear, from the two invariants (4.13) and (4.24), that, when restricted to $D_1 = 0$, the only expression invariant under the group which does not trivialize because of $D_1 = 0$, is the ratio:

$$\mathcal{I}_{\text{new}} = \frac{\Delta_5'}{\Delta^3} = \frac{D_3^3}{d_5 \cdot D_2^3}. \tag{5.3}$$

The $(x, u) = (4, 1)$ point (critical pure three-spin point) seems to be, in the case of the isomorphic mode, a tricritical point [52]: this yields the following value $-1/2$ for the new invariant $\mathcal{I}_{\text{new}}$, which singles out the algebraic variety:

$$2 \cdot D_3^3 + D_5 \cdot D_2^2 = 0. \tag{5.4}$$

[11] The *globally* self-dual variety $y = -q$ cannot be critical one since it is not stable under Γ_{upt}.

[12] The analysis of the action of the hyperbolic group on these varieties also has to be performed, in order to generalize the analysis already achieved in the $x = 1$ limit [26].

Monte-Carlo calculations confirm that $D_1 = 0$ is actually a critical variety even for quite anisotropic points. However, the interpretation of the possible *tricritical behaviour* of the intersection of $D_1 = 0$ and (5.4) is a more delicate question that will be addressed elsewhere.

6. FINITE ORDER CONDITIONS

6.1. $G_1^4 = \mathcal{I}$dentity for $q = 3$

In a previous Section 5.1, it was indicated that the intersection of $D_1 = 0$ together with variety (5.4) was a possible candidate for the equation of a critical manifold. We see here that this very intersection has a remarkable interpretation in terms of the discrete symmetry group.

Let us recall (the generically infinite order) transformation $G_1 = c^2 \cdot I \cdot c \cdot I$ (see (4.10)) which reads for the three-state model:

$$\begin{aligned} G_1(x_1, x_2, x_3, x) &= (x_{1G}, x_{2G}, x_{3G}, x_G), \qquad \text{with} \\ x_{1G} &= \frac{x_2 \cdot (x_1^2 x - 1) \cdot (1 - x_1 + x_3 x x_1 - x_3)}{(x_3 x_1^2 x - x_1^2 x - x x_1 + x_1 + 1 - x_3) \cdot (x_1 - 1)}, \\ x_{2G} &= -\frac{x_3^2 x_1^2 x - x_1^2 + 1 - x_3^2}{(1 + x_3 x_1^2 x - x_1^2 x - x x_1 + x_1 - x_3) \cdot x_3}, \\ x_{3G} &= -\frac{1 - x_1 + x_3 x x_1 - x_3}{1 + x_3 x_1^2 x - x_1^2 x - x x_1 + x_1 - x_3}, \qquad \text{and} \\ x_G &= \frac{(x_3 x_1^2 x - x_1^2 x - x x_1 + x_1 + 1 - x_3)^2 \cdot (x_1 - 1 + x_3 x x_1 - x_3)}{(1 - x_1^2 - x_3^2 + x_3^2 x_1^2 x) \cdot (1 - x_1 + x_3 x x_1 - x_3) \cdot (x_1^2 x - 1)}. \end{aligned} \tag{6.1}$$

In fact, by writing condition $G_1^4 = \mathcal{I}$dentity, one gets *a dimension-one (self-dual) algebraic variety*:

$$2 \cdot D_3^3 + D_5 \cdot D_2^2 - D_3^2 \cdot D_1 \cdot D_2 = 0. \tag{6.2}$$

Under the duality transformations relation (6.2) gives:

$$2 \cdot D_3^3 + D_5 \cdot D_2^2 - D_3^2 \cdot D_1 \cdot D_2 \longrightarrow 3^9 \cdot \left(2 \cdot D_3^3 + D_5 \cdot D_2^2 - D_3^2 \cdot D_1 \cdot D_2\right). \tag{6.3}$$

In the *isotropic limit* relation (6.2) reads $u = 1$ together with:

$$\begin{aligned} &x^4 u^{11} - 6u^{10}x^3 + u^{10}x^4 - 3u^9x^3 + 9u^9x^2 + u^9x^4 + 8u^8x^2 - 3u^8x^3 + 2u^7x^2 \\ &-24xu^7 + 12u^6x + 2x^2u^6 + 20u^5 - u^5x - 5u^5x^2 - 28u^4 + u^4x^2 + 5u^4x \\ &-4u^3x + x + 8u^3 + u^3x^2 + 7u^2 - 5u + 1 = 0, \end{aligned} \tag{6.4}$$

which in the $x = 1$ limit, gives

$$(u + 1)^3 \cdot (u - 1)^8 = 0.$$

6.2. $G_1^N = \mathcal{I}$dentity Varieties for $q = 3$

For $q = 3$, Section 4.3 shows that the *whole discrete group* is finite if *transformation* G_1 is actually of *finite order*.

Let us study systematically these *finite-order conditions*.

It should be noted that the conditions corresponding to $G_1 = \mathcal{I}$dentity and $G_1^2 = \mathcal{I}$dentity do not yield codimension-one algebraic variety symmetric under the permutations of x_1, x_2, x_3. Therefore, the analysis of the algebraic varieties $G_1^N = \mathcal{I}$dentity starts with $N \geq 3$.

All the (no three-spin) points $x = 1$ (that is $D_3 = 0$) yield $G_1^3 = \mathcal{I}$dentity. Conversely, $G_1^3 = \mathcal{I}$dentity corresponds to $D_3 = 0$ and its dual, $D_1 \cdot D_2 - 3 \cdot D_3 = 0$, or equivalently:

$$X \cdot (3 \cdot X - 1) = 0 \quad \text{or} \quad \left(6 \cdot \tilde{X} + 1\right) \cdot \left(6 \cdot \tilde{X} - 1\right) = 0. \tag{6.5}$$

This last condition, $D_1 \cdot D_2 - 3 \cdot D_3 = 0$, reads in terms of the x_i's:

$$\begin{aligned} & 1 - (x_1x_2 + x_2x_3 + x_3x_1) - (x_1^2 + x_2^2 + x_3^2) + x_1x_2x_3x + x_1^2x_2^2x_3^2x^2 \\ & + x_3x_1^2 + x_2x_1^2 + x_3^2x_1 + x_2^2x_3 + x_2x_3^2 + x_1x_3^2 - x_2^2x_3^2x_1x - x_2x_3^2x_1^2x - x_2^2x_3x_1^2x = 0, \end{aligned} \tag{6.6}$$

which, in the *isotropic* case $x_1 = x_2 = x_3 = u$, gives

$$1 - 6u^2 + u^3x + x^2u^6 - 3xu^5 + 6u^3 = 0. \tag{6.7}$$

The $G_1^4 = \mathcal{I}$dentity variety has been detailed in the previous section. Next is the $G_1^5 = \mathcal{I}$dentity variety which reads a *self-dual codimension-one* variety namely:

$$\begin{aligned} & -D_2^4 \cdot D_5^2 + \left(D_1^2 \cdot D_2^4 \cdot D_3 - 7D_1 \cdot D_2^3 \cdot D_3^2 + 14 \cdot D_2^2 \cdot D_3^3\right) \cdot D_5 \\ & + D_3^4 \cdot D_1^2 \cdot D_2^2 - 5 \cdot D_1 \cdot D_2 \cdot D_3^5 + 5 \cdot D_3^6 = 0. \end{aligned} \tag{6.8}$$

After a few calculations one can also find that condition $G_1^6 = \mathcal{I}$dentity, written as $G_1^3 = G_1^{(-3)}$, yields the previous $G_1^3 = \mathcal{I}$dentity conditions, together with *another algebraic variety*:

$$\mathcal{C}_6 = 2 \cdot D_5^2 D_2^4 - D_2^2 D_3 \cdot \left(10D_3^2 - 5D_3D_2D_1 + D_2^2D_1^2\right) \cdot D_5 + D_3^5 \cdot (D_2D_1 - D_3) = 0,$$

which can also be written in terms of the (pseudo-self) dual invariants $\tilde{X}$ and $\tilde{Y}$ (4.34):

$$\tilde{\mathcal{C}}_6 = 3456 \cdot \tilde{Y}^2 - 288 \cdot \tilde{X} \cdot \left(1 + 60\tilde{X}^2\right) \cdot \tilde{Y} - \left(12\tilde{X}^2 + 1\right) \cdot \left(144\tilde{X}^4 - 72\tilde{X}^2 + 1\right) = 0. \tag{6.9}$$

One verifies immediately by looking at (6.9), that condition $G_1^6 = \mathcal{I}$dentity is *self-dual*. In the isotropic limit it reads an involved expression given in Appendix B.

Large formal calculations enable us to write down explicitly the algebraic varieties corresponding to the *finite-order conditions* $G_1^N = \mathcal{I}$dentity for larger values of N. One should note that, remarkably, all these conditions $G_1^N = \mathcal{I}$dentity are *codimension-one self-dual varieties*, which we denote $\mathcal{C}_N = 0$.

The $G_1^N = \mathcal{I}$dentity varieties seem, for arbitrary N ($N \geq 3$), to be *codimension-one algebraic varieties* of the form:

$$0 = P_0(X) \cdot Y^N + P_1(X) \cdot Y^{N-1} + P_2(X) \cdot Y^{N-2} + P_3(X) \cdot Y^{N-3} + \cdots, \tag{6.10}$$

with $P_0(X) = P_0(0)$ and $P_i(X) = 0$ for $i = 1, \ldots, N$. For example, by introducing $Y = 1/\Delta_5'$ and $X = 1/\Delta$, condition $G_1^N = \mathcal{I}$dentity reads, respectively, for $N = 3, 4, 6$:

$$\begin{aligned} & \mathcal{C}_3 = X \cdot (3 \cdot X - 1) = 0, \qquad \mathcal{C}_4 = Y + 2X^3 - X^2 = 0, \quad \text{and} \\ & \mathcal{C}_5 = Y^2 - X \cdot \left(14X^2 - 7X + 1\right) \cdot Y - X^4 \cdot \left(1 - 5X + 5X^2\right) = 0, \\ & \mathcal{C}_6 = 2Y^2 - X \cdot \left(10X^2 - 5X + 1\right) \cdot Y + X^5 \cdot (1 - X) = 0, \end{aligned} \tag{6.11}$$

or in terms of $\tilde{X} = 1/6 - X$ and $\tilde{Y} = 1/54 - Y$ (see (4.34)):

$$\begin{aligned} & \tilde{\mathcal{C}}_3 = \left(6 \cdot \tilde{X} - 1\right) \cdot \left(6 \cdot \tilde{X} + 1\right), \qquad \tilde{\mathcal{C}}_4 = -6 \cdot \tilde{Y} - \tilde{X} \cdot \left(12\tilde{X}^2 - 1\right), \\ & \tilde{\mathcal{C}}_5 = 1728 \cdot \tilde{Y}^2 - 288 \cdot \tilde{X}\left(84\tilde{X}^2 - 1\right) \cdot \tilde{Y} - 1 + 12\tilde{X}^2 + 1872\tilde{X}^4 - 8640\tilde{X}^6, \\ & \tilde{\mathcal{C}}_6 = 3456 \cdot \tilde{Y}^2 - 288 \cdot \tilde{X}\left(1 + 60\tilde{X}^2\right) \cdot \tilde{Y} - \left(12\tilde{X}^2 + 1\right) \cdot \left(144\tilde{X}^4 - 72\tilde{X}^2 + 1\right). \end{aligned}$$

Let us give some additional examples. The codimension-one variety corresponding to $G_1^7 = \mathcal{I}$dentity is also self-dual and reads in terms of the two invariants X and Y:

$$\begin{aligned} \mathcal{C}_7 = 0 = {} & Y^4 + \left(5X - 1 - 10X^2\right) \cdot X \cdot Y^3 + \left(111X^2 - 36X + 4 - 111X^3\right) \cdot X^3 \cdot Y^2 \\ & + \left(13X - 291X^4 + 194X^5 + 192X^3 - 69X^2 - 1\right) \cdot X^4 \cdot Y + \left(7X - 1 + 7X^3 - 14X^2\right) \cdot X^9, \end{aligned}$$

or equivalently,

$$\begin{aligned}\tilde{C}_7 = 0 = {} & 2985984 \cdot \tilde{Y}^4 - 497664 \cdot \tilde{X} \cdot \left(1 + 60 \cdot \tilde{X}^2\right) \cdot \tilde{Y}^3 \\ & + \left(1728 - 435456 \cdot \tilde{X}^2 + 30606336 \cdot \tilde{X}^4 - 331444224 \cdot \tilde{X}^6\right) \cdot \tilde{Y}^2 \\ & + 288\tilde{X} \cdot \left(2011392\tilde{X}^8 - 20736\tilde{X}^6 - 6048\tilde{X}^4 + 48\tilde{X}^2 + 1\right) \cdot \tilde{Y} \\ & + \left(20901888\tilde{X}^{12} - 17418240\tilde{X}^{10} - 2301696\tilde{X}^8 - 7920\tilde{X}^4 + 269568\tilde{X}^6 + 120\tilde{X}^2 - 1\right).\end{aligned}$$

The codimension-one variety corresponding to $G_1^* = \mathcal{I}\text{dentity}$ is self-dual and reads:

$$\begin{aligned}\mathcal{C}_8 &= -Y^4 + \left(2 - 14X + 28X^2\right) \cdot X \cdot Y^3 + \left(240X^3 - 93X^2 - 240X^4 + 16X - 1\right) \cdot X^2 \cdot Y^2 \\ &= \left(12X - 195X^4 - 58X^2 + 130X^5 + 144X^3 - 1\right) \cdot X^4 \cdot Y + \left(2X^2 - 4X + 1\right) \cdot X^{10} = 0,\end{aligned}$$

or equivalently,

$$\begin{aligned}\tilde{C}_8 = 0 = {} & -4478976 \cdot \tilde{Y}^4 + 1492992 \cdot \tilde{X} \cdot \left(84\tilde{X}^2 - 1\right) \cdot \tilde{Y}^3 \\ & + \left(10368 - 1074954240\tilde{X}^6 - 746496 \cdot \tilde{X}^2 + 31352832\tilde{X}^4\right) \cdot \tilde{Y}^2 \\ & + 432 \cdot \tilde{X} \cdot \left(12\tilde{X}^2 + 1\right) \cdot \left(112320\tilde{X}^6 + 2736\tilde{X}^4 - 156\tilde{X}^2 + 1\right) \cdot \tilde{Y} \\ & + 3 \cdot \left(144\tilde{X}^4 + 48\tilde{X}^2 - 1\right) \cdot \left(20736\tilde{X}^8 - 34560\tilde{X}^6 + 864\tilde{X}^4 - 48\tilde{X}^2 + 1\right).\end{aligned}$$

The following (codimension-one) self-dual varieties $\mathcal{C}_N = 0$ are given in Appendix B for $N = 9, \dots, 15$. One can easily write down these conditions in terms of the "pseudo-self-dual" invariants $\tilde{X}, \tilde{Y}$ in order to make explicit the self-dual character of conditions $\mathcal{C}_N = 0$.

The coefficients quickly become quite large, however one can get simpler expressions and coefficients by introducing:

$$\tilde{X} \longrightarrow \hat{X} = \frac{\tilde{X}}{\sqrt{12}}, \qquad \tilde{Y} \longrightarrow \hat{Y} = \frac{\tilde{Y}}{\sqrt{12^3}}. \tag{6.12}$$

With these last invariants, conditions $\tilde{C}_7$ and $\tilde{C}_8$ are replaced by

$$\begin{aligned}\hat{C}_7 = {} & \hat{Y}^4 - 2\hat{X} \cdot \left(5\hat{X}^2 + 1\right) \cdot \hat{Y}^3 + \left(1 - 21\hat{X}^2 + 123\hat{X}^4 - 111\hat{X}^6\right) \cdot \hat{Y}^2 \\ & + 2\hat{X} \cdot \left(4\hat{X}^2 + 97\hat{X}^8 - 42\hat{X}^4 + 1 - 12\hat{X}^6\right) \cdot \hat{Y} \\ & - 1 + 10\hat{X}^2 - 55\hat{X}^4 + 156\hat{X}^6 - 111\hat{X}^8 - 70\hat{X}^{10} + 7\hat{X}^{12}, \\ \hat{C}_8 = {} & 12\hat{X} \cdot \left(7\hat{X}^2 - 1\right) \cdot \hat{Y}^3 - 3\hat{Y}^4 + 12\left(1 - 60\hat{X}^6 - 6\hat{X}^2 + 21\hat{X}^4\right) \cdot \hat{Y}^2 \\ & + 6 \cdot \hat{X} \cdot \left(65\hat{X}^6 + 19\hat{X}^4 - 13\hat{X}^2 + 1\right) \cdot \left(\hat{X}^2 + 1\right) \cdot \hat{Y} \\ & + 6 \cdot \left(\hat{X}^4 + 4\hat{X}^2 - 1\right) \cdot \left(\hat{X}^4 + 4\hat{X}^3 - 2\hat{X}^2 + 1\right) \cdot \left(\hat{X}^4 - 4\hat{X}^3 - 2\hat{X}^2 + 1\right).\end{aligned}$$

In the $x = 1$-limit, which corresponds to $D_3 = 0$, i.e., $X = 0$, the previous *finite-order conditions* $\mathcal{C}_N = 0$ $(N = 3, \dots, 15)$ read $Y^M = 0$. *All these equations* $G_1^N = \mathcal{I}\text{dentity}$ therefore *degenerate into*:

$$u \cdot (1 + u) = 0, \tag{6.13}$$

in the isotropic limit when $x = 1$. *Therefore, it seems likely that no* $x = 1$ *isotropic point* (except $u = 0$ and $u = -1$) *can belong to a finite-order condition* $G_1^N = \mathcal{I}\text{dentity}$ for $N \neq 3$.

REMARK. All these relations are $\mathcal{S}_3$-symmetric, therefore they identify with conditions $G_2^N = \mathcal{I}\text{dentity}$ and $G_3^N = \mathcal{I}\text{dentity}$, and are therefore *sufficient conditions* for having a *finite group*.

6.3. Genus of the Finite-Order Conditions

It is remarkable that all these finite order conditions are *codimension-one varieties*. They can all be written[13] in terms of the two group invariants X and Y. These relations between X and Y ($\hat{X}$ or $\hat{Y}$) cannot be obviously deduced from any further analysis of the discrete group generated by the inversion relations. Thus, several questions pop out. Where do the explicit equations of these curves come from? What is the nature of these curves? *For instance, are these algebraic curves $P(X,Y)=0$ elliptic curves?*

At least for $G_1^N = \mathcal{I}$dentity for $N=7$ (or 8), one can get some hint concerning this last question since the polynomial is of degree four in Y:

$$A_0 \cdot Y^4 + 4 \cdot A_1 Y^3 + 6A_2 \cdot Y^2 + 4A_3 \cdot Y + A_4 = 0, \tag{6.14}$$

where the A_i's are polynomial in X. Introducing:

$$g_2 = A_0A_4 - 4A_1A_3 + 3A_2^2, \qquad g_3 = A_0A_2A_4 + 2A_1A_2A_3 - A_0A_3^2 - A_4A_1^2 - A_2^3,$$

and the discriminant:

$$\Delta = g_2^3 - 27g_3^2,$$

discriminant Δ reads for $N=7$:

$$\begin{aligned} 4^4 \cdot \Delta(N=7) = X^{15} \cdot \big(29376X^6 - 29376X^5 + 16200X^4 - 5360X^3 \\ + 1003X^2 - 97X + 4\big) \cdot (3X-1)^{15}. \end{aligned}$$

These curves are not rational curves. Recognized in this discriminant are the two following rational cases $X = 1/3$ and $X = 0$, namely $D_3 = 0$ and its dual variety. In fact, using the *Macaulay algebraic geometry computing system* [54], one gets that condition $\mathcal{C}_4 = 0$ is a *genus one* curve, that $\mathcal{C}_5 = 0$ is a *genus ten* curve and that $\mathcal{C}_9 = 0$ is a genus 78 curve.

In the (X,Y)-plane most of these infinite sets of *finite-order conditions* are algebraic curves of *genus greater than one*, amazingly associated with a foliation of P_4 in algebraic (elliptic) surfaces.

6.4. Generalization for Arbitrary q

These calculations can be generalized for arbitrary q. The (generically infinite order) birational formation G_1 reads:

$$\begin{aligned}
X_{1G} &= \frac{\big(x_1^2x + x_1qx - 3x_1x - q + 2\big)\cdot x_2 \cdot (1 - x_1 + x_1x_3x - x_3)}{(x_1 - x_1^2x - (q-2)\cdot x_1x + (q-2) + x_1^2x_3x + (q-3)\cdot x_1x_3x - (q-2)\cdot x_3)\cdot(x_1-1)}, \\
x_{2G} &= \frac{N_{2G}}{D_{2G}}, \qquad \text{where} \\
N_{2G} &= \big(1 - 3x_3x - x_3^2qx + 2x_3^2x + x_3qx\big)\cdot x_1^2 \\
&\quad + \big(-(q-3)\cdot x_3 + (q-3) - \big(q^2 - 5q + 6\big)\cdot x_3^2x + \big(q^2-5q+6\big)\cdot x_3x\big)\cdot x_1 \\
&\quad + \big(q^2 - 4q + 4\big)\cdot x_3^2 - \big(q^2 - 5q + 6\big)\cdot x_3 - (q-2), \qquad\qquad (6.15)\\
D_{2G} &= \big(x_1 - x_1^2x - x_1qx + q + 2x_1x - 2 + x_1^2x_3x + x_1x_3qx - 3x_1x_3x - x_3q + 2x_3\big)\,x_3, \\
x_{3G} &= -\frac{1 - x_1 + x_1x_3x - x_3}{x_1 - x_1^2x - x_1qx + q + 2x_1x - 2 + x_1^2x_3x + x_1x_3qx - 3x_1x_3x - x_3q + 2x_3}, \\
x_G &= \frac{N_{1g}\cdot N_{2g}}{D_{1g}\cdot D_{2g}}, \qquad \text{where} \\
N_{1g} &= \big(x_1 - x_1^2x - x_1qx + q + 2x_1x - 2 + x_1^2x_3x + x_1x_3qx - 3x_1x_3x - x_3q + 2x_3\big)^2,
\end{aligned}$$

[13]This is not surprising since one can show [53] that conditions $G_i^N = \mathcal{I}$dentity are automatically invariant by the discrete group generated by the inversion relations.

$$
\begin{aligned}
N_{2g} &= x_1 - 1 + (q-2)\cdot x_1x_3x - (q-2)\cdot x_3, \\
D_{1g} &= -2x_1^2x_3^2x - qx_1^2x_3x + 2x_1^2x_3x + qx_1^2x_3^2x - 5x_1x_3^2qx \\
&\quad + 6x_1x_3^2x - 6x_1x_3x + x_1x_3^2q^2x \qquad (6.15)(\text{cont.}) \\
&\quad + 5x_1x_3qx - x_1x_3q^2x - x_1^2 - 3x_1x_3 + x_1x_3q + 3x_1 - x_1q - (q-2)^3\cdot x_3^2 \\
&\quad + \left(q^2 - 5q + 6\right)\cdot x_3 + (q-2), \\
D_{2g} &= (1 - x_1 + x_1x_3x - x_3)\cdot\left(x_1^2x + x_1qx - 3x_1x - q + 2\right).
\end{aligned}
$$

Writing down the $G_1^N = \mathcal{I}$dentity relations yields calculations that are too large. However, it is possible to get the solution for $N = 3$. Writing down the relations: $G_1 = G_1^{-2}$ on the four coordinates again yields the two relations:

$$
q\cdot D_3 - D_1\cdot D_2 = 0 \quad \text{and} \quad D_3 = 0. \tag{6.16}
$$

Conversely, one can easily see that G_1 reduces, for $x = 1$, to an *order-three transformation*:

$$
(x_1, x_2, x_3, 1) \longrightarrow \left(x_2, -\frac{1 + q\cdot x_3 - 2\cdot x_3}{x_3}, \frac{-1}{x_1 - 2 + q}, 1\right). \tag{6.17}
$$

Of course by duality, this is also the case for $q\cdot D_3 - D_1\cdot D_2 = 0$.

REMARK. *For the Tutte-Beraha numbers the group is finite for* $D_3 = 0$ (that is $x = 1$) *and therefore restricted to its dual variety*: $q\cdot D_3 - D_1\cdot D_2 = 0$. Do other algebraic varieties exist, such that the group becomes finite for *arbitrary values of q*? Do other codimension-one (or codimension-two) algebraic varieties also exist such that the group becomes finite for the *Tutte-Beraha numbers*? The calculations become unfortunately quickly very large.

REMARK. For $x = 1$, but for arbitrary q, it is known [10] that the group is isomorphic (up to semidirect product with finite groups) to $\mathbb{Z}\times\mathbb{Z}$. Therefore, to some extent, the $x = 1$ subcase can be compared to the $q = 3$ subcase: in both cases, the group is (up to a semidirect product) isomorphic to $\mathbb{Z}\times\mathbb{Z}$. In the $x = 1$ subcase, one has to introduce the J_i's in order to see it, these J_i's being order three for $q = 3$ (and arbitrary x), while in the $q = 3$ subcase, it is necessary to introduce the G_i's in order to see that the group isomorphic to $\mathbb{Z}\times\mathbb{Z}$ (see Section 4.3), these G_i's being of order three when $x = 1$ (and arbitrary q).

7. SYMMETRY GROUP INVARIANT APPROXIMANTS FOR SPONTANEOUS MAGNETIZATION

The previous algebraic group invariants are certainly well-suited variables to analyze the "analytical complexity" of the various physical quantity one can encounter. However, most of the physical quantities depend, in a quite nontrivial way, of various "spectral" parameters [47]. In this respect, some "one-point functions" like, for instance, the *spontaneous magnetization* can be seen as *remarkable group invariant expression* which should not depend "too much" of various "spectral" parameters (whatever they are [49] ...). Of course we do not expect the spontaneous magnetization of the edge Potts model to be a closed *algebraic* expression (like for the Ising model: see 7.1 in the following). It can be seen as a "transcendental invariant" for the group. Taking advantage of the previous analysis which singles out a "canonical" invariant (namely Δ_5') corresponding to $x \neq 1$ deformations of the edge Potts mode, one may ask the following question: is it possible, to write down the spontaneous magnetization as a (more or less involved) function of invariants like the group-invariants Δ_5', $\mathcal{I}$, and I_9, (respectively, defined by (4.24), (4.39), (4.49))? Is it possible to write a closed expression of these invariants which could be a good approximation for the spontaneous magnetization?

Let us consider the $q = 3$, $x = 1$ subcase of this model. This subcase can be seen as a *testing ground* for providing an example of *exact calculation* using the *symmetries* (and various analyticity assumptions), but *no Yang-Baxter structure*. We use here the specificity of the Tutte-Beraha numbers and of the $x = 1$ (no three-spin interaction) case. The calculations are sketched in Appendices C, D, and E.

Let us first recall the exact expression of *modulus of the elliptic functions* occurring in the *elliptic parametrization of the checkerboard Ising model* [49,55]:

$$k^2 = \prod_{i=1} \frac{t_i \cdot (1-t_i) \cdot (1+t_i) \cdot (t_i^* + t_j^* \cdot t_k^* \cdot t_l^*)}{t_i^* \cdot (1-t_i^*) \cdot (1+t_i^*) \cdot (t_i + t_j \cdot t_k \cdot t_l)}, \tag{7.1}$$

where the t_i's denote the usual high-temperature variable $t_i = th(K_i)$, and the t_i^*'s denote their dual $t_i^* = (1-t_i)/(1+t_i)$. It is clear on (7.1) that this valuable expression, which is a key ingredient to foliating the parameter space, parameterizing the model, finding the critical variety (namely $k^2 = 1$) and to actually solve the model, could have been "guessed" from the various degeneracies of the model, namely $K_i = 0$, $K_i = \infty$ and the *disorder solutions, their* (Kramers-Wannier) *dual* and the action of the discrete group generated by the inversion relations. Let us also recall the exact expression of the spontaneous magnetization of the *checkerboard Ising model* [55]:

$$M = (1-k^2)^{1/8}. \tag{7.2}$$

The spontaneous magnetization has a remarkably simple expression in terms of the algebraic invariant k^2. When restricted to the critical condition, $k^2 = 1$, it vanishes. Furthermore, when restricted to the "order" solutions (dual of the disorder solutions), namely $t_i^* + t_j^* \cdot t_k^* \cdot t_l^* = 0$, the spontaneous magnetization M *becomes equal* to 1. This can be checked formally on low temperature resummed expansions of the spontaneous magnetization [56,57]. Both quantities M and k^2 do have *the same symmetries* (symmetry of the square, inversion relation symmetries, ...).

It is tempting to try to generalize this result to q-state edge Potts models, and, in particular, to the *three-state* standard scalar edge *checkerboard* Potts model [58], in order to get simple closed expressions (or approximations), for instance, for the spontaneous magnetization. The calculations are sketched in Appendix C.1 for the *checkerboard* lattice because this very lattice provides a "nice" representation of the Kramers-Wannier duality, namely $k \to 1/k$. Unfortunately these calculations are too naive and yield an algebraic invariant which does not suite well with the resummed expansions (see Appendices C.2 and C.3).

Actually we will see that the *honeycomb* lattice (dual of the triangular one) is better suited to address this question (see Appendix D).

7.1. Algebraic Invariant for the Honeycomb Lattice

Let us recall the low-temperature variables (3.8) and the well-known (ferromagnetic) critical variety of the three-state honeycomb Potts model [39]:

$$\mathcal{C}_0^{\text{honey}} = -1 + (A+B+C) + 2 \cdot (BA + CA + BC) + ABC = 0. \tag{7.3}$$

The results are the following. Let us introduce the following (see Appendix D.2 for more details) group invariant for the *honeycomb* lattice:

$$\begin{aligned} k_{\text{Potts}} &= \frac{\mathcal{N}_{\text{Potts}}}{(\mathcal{D}_{\text{Potts}})^3}, \qquad \text{where} \\ \mathcal{N}_{\text{Potts}} &= 27(ABC + AA + BC)(BCA + B + CA)(ABC + C + BA)(ABC + A + B + 2BA + 1) \\ &\quad \times (ABC + C + B + 2BC + 1)(ABC + C + A + 2CA + 1)(A - C + B + BA + 1) \\ &\quad \times (C + A - B + CA + 1)(C - A + B + BC + 1), \qquad \text{and} \\ \mathcal{D}_{\text{Potts}} &= 1 + (A+B+C) + 2 \cdot (BC + CA + BA) - (A^2 + B^2 + C^2) \end{aligned} \tag{7.4}$$

$$
\begin{aligned}
&- \left(A^3+B^3+C^3\right) - \left(CA^2+B^2A+C^2A+BA^2+B^2C+BC^2\right) + 6ABC \\
&- 2\left(A^3B+B^3A+A^3C+B^3C+C^3A+C^3B\right) + 2\cdot\left(B^2A^2+C^2A^2+B^2A^2\right) \\
&- 3\left(BC^2A+3B^2CA+3BCA^2\right) \\
&- ABC\cdot\left(3CA+3BA+3BC+5A^2+5B^2+5C^2\right) \qquad (7.4)(\text{cont.}) \\
&- 2ABC\left(CA^2+B^2C+C^2A+BC^2+B^2A+BA^2+6ABC\right) \\
&- 5A^2B^2C^2(C+B+A) - 2\cdot A^2B^2C^2\cdot(AC+BC+AB) - A^3B^3C^3.
\end{aligned}
$$

Let us introduce a first *approximation* for the derivative of the (low temperature normalized) partition function, $z\cdot\frac{d}{dz}\ln(\Lambda)$, namely:

$$
D_{\mathrm{er}} = \left(1-k_{\mathrm{Potts}}\right)^{1/9} - 1. \tag{7.5}
$$

The (resummed low temperature [10]) expansion of $z\cdot\frac{d}{dz}\ln(\Lambda)$ can be written as follows:

$$
\begin{aligned}
z\cdot\frac{d}{dz}\ln(\Lambda) &= 2\cdot\frac{A(C+BA)(B+CA)}{(1-A^2)^2} \\
&+ 2\cdot\frac{A^2\cdot(A-1)\cdot\left(A^2-4A-1\right)}{(1-A^2)^3}\cdot(B+C)\cdot BC + 4\cdot\frac{A^4}{(1-A^2)^3}\cdot(B+C)^3 \\
&+ 2\cdot\frac{(B+2CA)(C+BA)\left(A^3-5A^2+A-1\right)(A-1)^3}{(1-A^2)^4}\cdot BC \\
&+ 4\cdot\frac{(B+CA)\cdot(C+BA)\cdot A\cdot\left(1+5A^2+2A^4\right)}{(1-A^2)^4}\cdot(B+C)^2 \\
&+ 2\cdot\frac{\left(3A^5+4A^4+3A^3-A^2-1\right)\cdot(A-1)^3\cdot A\cdot(B+CA)\cdot(C+BA)}{(1-A^2)^3\cdot(1-A^3)^2}\cdot BC \\
&+ 4\cdot\frac{A^2\left(2A^5+4A^4+5A^3+5A^2+2A+1+A^6\right)(1-A)(B+CA)(C+BA)}{(1-A^2)^3\cdot(1-A^3)^2}(B+C)^2 \\
&+ \frac{6A^6}{(1-A^2)^4}\cdot(B+C)^4 + 2\cdot\frac{(A-1)A^4\cdot\left(A^2-9A-4\right)}{(1-A^2)^4}\cdot BC\cdot(B+C)^2 \\
&- 2\cdot\frac{(A-1)^2(A-3)(A+1)A^3}{(1-A^2)^4}\cdot B^2C^2 \\
&+ 4\cdot\frac{A^2(C+BA)(B+CA)(4A-5)(1-A)^2}{(1-A^2)^4}\cdot BC \\
&- 4\cdot\frac{A^2(C+BA)(B+CA)\left(2A^3+7A\right)}{(1-A^2)^4}\cdot(B+C)^2.
\end{aligned} \tag{7.6}
$$

A straightforward calculation enables us to write (7.6) in terms of D_{er} as follows:

$$
\frac{-3}{2}\cdot z\cdot\frac{d}{dz}\ln(\Lambda) = D_{\mathrm{er}}\cdot\left(1+\lambda\cdot D_{\mathrm{er}}+\mathcal{R}_{\mathrm{est}}+\cdots\right), \tag{7.7}
$$

where $\mathcal{R}_{\mathrm{est}}$ has a remarkably simple expression for $\lambda=-3/2$, namely:

$$
\begin{aligned}
\mathcal{R}_{\mathrm{est}} &= \frac{A\cdot\left(1+A-4A^2+A^3+A^4\right)}{(1-A^3)^2}\cdot BC + \frac{2A^3}{(1-A^3)^2}\cdot(B+C)^2 \\
&= \frac{A\cdot\left(1-A^5\right)}{(1-A)\cdot(1-A^3)^2}\cdot BC - \frac{5\cdot A^3}{(1-A^3)^2}\cdot BC + \frac{2A^3}{(1-A^3)^2}\cdot(B+C)^2.
\end{aligned} \tag{7.8}
$$

One should also note that invariant $k_{\rm Potts}$ is built in such a way that it *does vanish* when one is restricted to the "*order conditions*" of the three-state edge Potts model on a *honeycomb* lattice, $(A + BC + ABC = 0, \ldots)$. Furthermore, when one restricts oneself to the *critical condition* of the three-state edge Potts model on a *honeycomb* lattice, $k_{\rm Potts}$ *actually becomes equal to* 1. Therefore, the closed expression (7.5) *leads to the recovery of the critical (magnetic) exponent* of the three-state Potts model [39], $\beta = 1/9$, on the critical variety $C_0^{\rm honey} = 0$.

Conversely, condition $k_{\rm Potts} - 1 = 0$ *factorizes into this very critical condition* together with three other algebraic varieties, namely:

$$\begin{aligned} k_{\rm Potts} - 1 = 0 \Longleftrightarrow C_0^{\rm honey} \cdot \left(C_1^{\rm honey} \cdot C_2^{\rm honey} \cdot C_3^{\rm honey}\right)^2 &= 0, \qquad \text{with} \\ C_0^{\rm honey} &= -1 + (A+B+C) + 2 \cdot (BA + CA + BC) + ABC, \\ C_1^{\rm honey} &= ABC^2 + 2ABC + A + 2CA + B + 2BC - C^2 + 1, \\ C_2^{\rm honey} &= ACB^2 + 2BA + A + 2ABC + 1 - B^2 + C + 2BC + 1, \\ C_3^{\rm honey} &= BCA^2 - A^2 + 2BA + B + 2CA + C + 2ABC + 1. \end{aligned} \tag{7.9}$$

This factorization of condition $k_{\rm Potts} - 1 = 0$ is reminiscent of the factorization for the checkerboard Ising model (see equation (C.16) in Appendix C.1). If formula (7.5) is taken for granted, one should expect a $\beta = 2/9$-magnetic critical exponent [39] on these three new varieties (7.9).

By performing a Kramers-Wannier duality [40,41], one deduces three remarkable varieties for the *triangular* lattice from the last three algebraic varieties. In terms of the variable x_i's one of these reads:

$$\begin{gathered} x_2 x_3 x_1^2 + 2x_2 x_3 x_1 - x_1^2 + 2x_2 x_1 + 2x_1 x_3 + 1 + x_3 + x_2 = 0, \qquad \text{or} \\ (x_1 + 2) \cdot (x_1 x_2 x_3 - (x_1 + x_2 + x_3) - 1) + 3 \cdot (x_1 + 1) \cdot (1 + x_2 + x_3) = 0, \end{gathered}$$

which have a remarkably simple form in terms of the rational parameterization of the $x = 1$ subcase of the Potts model (see (4.35)):

$$u_2 \cdot u_3 + t^2 \cdot u_1^2 = 0, \qquad \text{with: } t^6 = -1. \tag{7.10}$$

Such varieties, remarkably simple in terms of the well-suited variables u_i's *are specific of the Tutte-Beraha numbers* (see Appendix E.2).

7.2. Comments on Invariant $k_{\rm Potts}$

Since expression (7.4) seems to be particularly well-suited to "decipher" the *resummed* low-temperature expansions of the spontaneous magnetization of the three-state edge Potts model on the *honeycomb* lattice (see (7.8) and Appendix D.3), it is tempting to compare (7.5) to other expansions available in the literature, and in particular to the expansions on which the most extensive studies have been performed, namely *isotropic low-temperature expansions* [59,60]. Unfortunately for the three-state edge Potts model, only *high-field* expansions are specifically dedicated to the *honeycomb* lattice [59]. From this expansion, the first coefficient of a low-temperature expansion can be deduced. The agreement between (7.5) and this deduced low-temperature expansion for the spontaneous magnetization is quite good (see Appendix D.4). The largest low-temperature expansions for the spontaneous magnetization have been obtained recently for the *square lattice* up to order A^{47} in [60]. By setting the limits $C = 0$ and $B = A$, one can easily get from (7.5), the equivalent of (7.5) for the *isotropic three-state edge Potts model* on the *square lattice*. For the square lattice, $k_{\rm Potts}$ reads:

$$k_{\rm Potts}^{\rm square} = \frac{27 \cdot A^4 \cdot (2A^2 + 2A + 1) \cdot (A+1)^4}{(1 + 2A - 4A^3 - 2A^4)^3}. \tag{7.11}$$

The *low-temperature* expansion of the spontaneous magnetization of the three-state edge Potts model square lattice reads [60]:

$$\begin{aligned} M_{\text{square}} = 1 - \frac{3}{2} \cdot z \cdot \frac{d}{dz} \ln(\Lambda) = 1 &- 3A^4 - 12A^6 - 12A^7 - 36A^8 - 108A^9 \\ &- 210A^{10} - 480A^{11} - 1746A^{12} - 2340A^{13} - 10566A^{14} - 19500A^{15} - 53976A^{16} \\ &- 152604A^{17} - 329424A^{18} - 971304A^{19} - 2403291A^{20} - 5955576A^{21} - 16858584A^{22} \\ &- 40337376A^{23} - 110301321A^{24} - 287061696A^{25} - 730223208A^{26} - 1985703720A^{27} \\ &- 5070001716A^{28} - 13446444720A^{29} - 35650214232A^{30} - 92442918828A^{31} \\ &- 247542929499A^{32} - 648347258796A^{33} - 1713912378552A^{34} - 4559593914288A^{35} \\ &- 11991311519034A^{36} - 31943103715128A^{37} - 84599939924118A^{38} \\ &- 224265087762144A^{39} - 597511883594619A^{40} - 1584231404110704A^{41} \\ &- 4220295103426356A^{42} - 11234571367790256A^{43} - 29892611571334848A^{44} \\ &- 79763126301078204A^{45} - 212500082474434470A^{46} - 567062477783225940A^{47}, \end{aligned}$$

and the expansion of (7.5) (but for (7.11)) yields:

$$\begin{aligned} (1 - k_{\text{Potts}}^{\text{square}})^{1/9} - 1 = &-3A^4 - 12A^6 - 12A^7 - 45A^8 - 96A^9 - 234A^{10} - 576A^{11} - 1446A^{12} \\ &- 3468A^{13} - 9108A^{14} - 22032A^{15} - 57774A^{16} - 144000A^{17} - 373800A^{18} - 952128A^{19} \\ &- 2466738A^{20} - 6353328A^{21} - 16509432A^{22} - 42797448A^{23} - 111617214A^{24} - 290794368A^{25} \\ &- 760795740A^{26} - 1990352736A^{27} - 5221859292A^{28} - 13707847944A^{29} - 36056085192A^{30} \\ &- 94917607680A^{31} - 250248685599A^{32} - 660357590880A^{33} - 1744686518304A^{34} \\ &- 4613473905696A^{35} - 12211923142131A^{36} - 32351256927936A^{37} - 85779066068604A^{38} \\ &- 227613073998444A^{39} - 604434359093817A^{40} - 1606203980794368A^{41} \\ &- 4271216035201722A^{42} - 11365197818350656A^{43} - 30260209228058898A^{44} \\ &- 80615490837540204A^{45} - 214886746688288580A^{46} - 573101698152234528A^{47}. \end{aligned}$$

The difference between these two expressions reads:

$$\begin{aligned} &9A^8 - 12A^9 + 24A^{10} + 96A^{11} - 300A^{12} + 1128A^{13} - 1458A^{14} + 2532A^{15} + 3798A^{16} \\ &\quad - 8604A^{17} + 44376A^{18} - 19176A^{19} + 63447A^{20} + 397752A^{21} - 349152A^{22} + 2460072A^{23} \\ &\quad + 1315893A^{24} + 3732672A^{25} + \cdots . \end{aligned}$$

The agreement between these two expansion is remarkable[14]. For instance, if one compares the coefficient of A^{22}, A^{23}, A^{24}, A^{25}, A^{39}, A^{46}, and A^{47}, the coefficients in these previous two expansions are actually equal up to 0.0207, 0.0609, 0.0119, 0.0130, 0.0149, 0.0112, and 0.0105, respectively.

The agreement between these two expansion is so good that it is tempting to imagine that the Kramers-Wannier dual of (7.4) is also well-suited to express the spontaneous magnetization of the three-state edge Potts model on the *triangular* lattice and that there may exist a *single* expression generalizing the previous two for the three-state edge Potts model on the *checkerboard* lattice. Unfortunately, the *Kramers-Wannier dual* of (7.4) is not a well-suited expression for the (low-temperature) resummed expansion of the spontaneous magnetization of the three-state edge Potts model on the *triangular* lattice (see Appendix E.4). The *resummed* expansion of the spontaneous magnetization of the three-state Potts model on the anisotropic triangular lattice corresponds to:

$$z \cdot \frac{d}{dz} \ln(\Lambda) = \frac{2A^2C^2}{(1 - A^2C^2)^2} \cdot B^2 + \frac{4A^4C^4}{(1 - A^2C^2)^3} \cdot B^3 + 2\frac{M_4 \cdot A^2C^2}{(1 - A^3C^3)^2 (1 - A^2C^2)^4} \cdot B^4 + \cdots ,$$

[14]In particular when one recalls the problems encountered in the Pade analysis of the three-state Potts models.

where

$$\begin{aligned}M_4 &= 2A^2 + 2C^2 + 6A^3C^3 - 8A^2C^2 + A^4C^4 + 2C^4A^2 - 4A^5C^3 + 2A^8C^6 - 4A^7C^5 \\ &+ 2A^{10}C^8 - 4C^5A^3 + 2C^8A^6 - 4C^7A^5 + 2C^{10}A^8 - 2A^7C^6 - 2A^6C^7 - 2A^6C^6 + 8A^5C^5 \\ &- 8A^8C^8 + 3A^{10}C^{10} - 2A^4C^3 - 2A^3C^4 + 2CA^2 + 2C^8A^9 + 2C^2A + 2C^9A^8 + 2A^4C^2.\end{aligned}$$

Another invariant seems to dominate the expansion of the triangular lattice (see Appendix E.4). It should be quite close to the following invariant:

$$Y_{\text{triang}} = -\frac{ABC \cdot (1+A) \cdot (1+B) \cdot (1+C)}{(ABC + (BC + CA + BA) - 1)^3}. \tag{7.12}$$

In the case of the triangular three-state edge Potts mode, Monte-Carlo simulations have been performed[15] in order to see if the new varieties (7.10) could not be critical varieties. Monte-Carlo calculations show that these varieties *are not critical* varieties with a magnetic critical exponent given by formula (7.5) (namely $\beta = 2/9$), but it may be possible that these varieties could be "special" in some way (see Appendix E.3).

Furthermore, expression (7.11) for the square lattice, inherited from the one for the honeycomb lattice (see (7.4)) does not yield a simple representation of the Kramers-Wannier duality. The Kramers-Wannier dual of (7.11) reads:

$$k_{\text{KW}}^{\text{square}} = \frac{(1+2A)^2(2+A)^4\,(5+2A+2A^2)\,(A-1)^4}{(10A - 1 + 12A^2 + 4A^3 + 2A^4)^3}. \tag{7.13}$$

Eliminating variable A between (7.11) and (7.13) yields a quite complicated (involutive) algebraic relation between $k_{\text{Potts}}^{\text{square}}$ and $k_{\text{KW}}^{\text{square}}$ (see Appendix D.3).

These two facts suggest, for the checkerboard and square lattices, that one should introduce, instead of a *single one*, (at least) *two* invariants, one being dominant for the *honeycomb* limit and another for the *triangular* limit. A "nice" representation of Kramers-Wannier duality probably requires considering *at least* these two invariants. This could be consistent with the fact that varieties (7.10) are not $\beta = 2/9$-critical varieties. This also suggests that expressions like (7.4) are *just approximations* for the *dominant singular part* of the spontaneous magnetization[16] and have to be improved. Finally, this suggests that, even for the honeycomb three-state Potts mode, one should be able to improve the previous results (see (7.4) and (7.7)) and get "improved" symmetry-invariant approximations for the spontaneous magnetization.

7.3. Towards "Improved" Algebraic Invariants

In order to "improve"the invariants let us remark that the resummed expansions of the various group invariants provide either simple $1 - A^2$, (respectively, $1 - AC$) singularities, or on the contrary, quite involved singularities *but not the N^{th} root singularities $1 - A^N$ known to occur in the resummed expansions of the spontaneous magnetization* of the three-state (or q-state) edge Potts models [10]. Actually, though (7.5) enables us to retrieve the exact critical exponent $\beta = 1/9$, its (B, C small) expansion *yields only* $1 - A^2$ singularities for the honeycomb lattice. This could suggest that the $1 - A^N$ singularities in the *resummed* expansions, *are not related to the dominant singular part* [13] *of the spontaneous magnetization* but to sub-dominant singularities.

In order to understand the occurrence of these N^{th} root of unity, it is interesting to consider the resummed expansion of the partition function per site of the three-state Potts model on

[15]We thank J.-C. Anglès d'Auriac and H. Meyer for communicating these results prior to publications.

[16]This situation is reminiscent of the susceptibility of the Ising model where the closed expression of Syozi and Naya [61,62] gives the *dominant singular part* of the susceptibility (see [13]).

rubber bands [6,63]. Such an example is given in Appendix F for the square lattice. The (low-temperature normalized) partition function per site, denoted Λ, is one of the three solutions of a polynomial equation:

$$P(\Lambda) = \Lambda^3 + C_2 \cdot \Lambda^2 + C_1 \cdot \Lambda + C_0 = 0, \qquad \text{where}$$
$$C_n = C_{n0} + C_{n1} + C_{n2} + \cdots, \qquad n = 0, 1, 2,$$

where the C_{nj}'s are of order j in B. Let us introduce the (resummed low-temperature) expansion of Λ:

$$\Lambda = 1 + 3 \cdot (q-1) \cdot \frac{W_2}{1-A^2} \cdot B^2 + (q-1) \cdot \sum_{n=3}^{\infty} \frac{W_n}{(1-A^2)^{n-1}} \cdot B^n. \tag{7.14}$$

One can actually understand, on this very example, that the occurrence of the singularity $1 - A^3$, is related to the following relation:

$$3 + C_{10} + 2 \cdot C_{20} = \left(1 - A^2\right) \cdot \left(1 - A^3\right) = \frac{d}{d\Lambda} P(\Lambda), \qquad (B = 0). \tag{7.15}$$

Appendix F also enables to understand that singularity $1 - A^3$ does not occur for W_2. Since we know that the $1 - A^N$ singularities occur in the resummed expansion of the spontaneous magnetization for *all the values of the integer* N [10], this seems to suggest that the "polynomial" needed for a closed algebraic formula for the spontaneous magnetization is of "infinite" degree.

8. CONCLUSION

The *discrete symmetry group* generated by *inversion relations* has been analysed for the standard scalar Potts model with two- and three-sites interactions on the *triangular lattice* [45]. The group generated by three involutions is seen to be generically a very large one (like a free group), namely *hyperbolic groups*.

In this analysis a remarkable situation pops out for q-state Potts models for particular values of q, the so-called *Tutte-Beraha numbers* [23,46]. For these values of q, some of the (generically infinite order) generators are of *finite order*. However, even with such additional relations on the generators, one still gets groups with an *"exponential growth"*, *except* for $q = 3$ (or $q = 1$). Additional relations on the generators can also occur on particular algebraic varieties, yielding a degeneracy of the group into products of $\mathbb{Z}$. We have seen that $x = 1$ and its dual variety (4.41) are such varieties. It would be interesting to systematically seek more examples of such varieties.

A detailed analysis of the $q = 3$ case has been performed. For $q = 3$ the *finite-order* conditions for which the group degenerates into a *finite-order* group are found to be *codimension one* varieties. In this $q = 3$ subcase, a rather systematic study of well-suited *group symmetry invariants* has been performed. It could be of some help to analyze the *analytical structure* of certain physical quantities like, for instance, the *spontaneous magnetization*. In particular, we have got a closed algebraic expression which is a quite *good approximation of the spontaneous magnetization for the honeycomb lattice.*

As a byproduct, this analysis provides nice *birational representations of hyperbolic Coxeter groups* as well as providing *algebraic varieties having such large groups of (birational) automorphisms*. It is clear to see that many calculations, performed on the hyperbolic Coxeter groups of symmetries of *triangular* Potts models, can *simply be generalized to three- (or higher-) dimensional vertex models mutatis mutandis*. This first analysis of *hyperbolic Coxeter symmetry groups for lattice models*, including degeneracy subcases, *should help a better understanding of the symmetries of three-dimensional models* and provide tools to perform *exact calculations* based on the *symmetry* analysis of these higher-dimensional models.

APPENDIX A
MORE RELATIONS ON THE GENERATORS OF THE HYPERBOLIC COXETER GROUPS

The analysis of $\Gamma_{\rm triang}$ can be performed directly on the J_i's. Let us now consider the *Tutte-Beraha* subcases. One has the following relations between the J_i's:

$$J_i^N = \mathcal{I}d, \qquad i = 1, 2, 3. \tag{A.1}$$

One can easily deduce many other relations for the Tutte-Beraha numbers, for instance:

$$J_3 = J_2^{N-1} J_1^{N-1}. \tag{A.2}$$

Since J_3 can be rewritten in terms of J_1 and J_2, the Coxeter group can be seen as generated by J_1 and J_2 [28]. Then, using relations (A.1) with $i = 1$ or $i = 2$, one obtains the form of a general element of the group:

$$J_1^{n_1} J_2^{n_2} J_1^{n_3} J_2^{n_4} \cdots J_1^{n_k} J_2^{n_{k+1}}, \tag{A.3}$$

where $n_1, n_{k+1} = 0, 1, \dots, N-1$; $n_\alpha = 1, 2, \dots, N-1$, $\alpha = 2, 3, \dots, k$.

Generically, there is no further relation between the J_i's thus Γ is a free group generated by two infinite generators (let us say, for example, J_1 and J_2).

Introducing the well-suited transformations:

$$\tilde{G}_1 = J_1 J_3 J_2, \qquad \tilde{G}_2 = J_2 J_1 J_3, \qquad \tilde{G}_3 = J_3 J_2 J_1, \tag{A.4}$$

one can show that $N = 3$ is *singled out*, Γ reducing to $\mathbb{Z} \times \mathbb{Z}$ up to a semidirect product by a finite group.

At first, let us study the group, G, generated by $\tilde{G}_1, \tilde{G}_2$ and $\tilde{G}_3$. Relation (A.2) can also be written in the following way by using relation (A.2):

$$J_1 J_2 = J_3^{N-1}, \qquad J_2 J_3 = J_1^{N-1}, \qquad J_3 J_1 = J_2^{N-1}, \tag{A.5}$$

then $\tilde{G}_2 \tilde{G}_3$ reads:

$$\tilde{G}_2 \tilde{G}_3 = (J_2 J_1 J_3)(J_3 J_2 J_1) = J_2 J_1 J_3^2 J_2 J_1.$$

Notice that for $N = 3$, one can use relation (A.5) and obtain:

$$\tilde{G}_2 \tilde{G}_3 = J_2 J_1^2 J_2^2 J_1 = J_2^2 J_3^2 J_1^2 = J_3 J_1^2 J_2^2 J_3 = J_3 J_2 J_3^2 J_1 J_3 = J_3 J_2 J_1 J_2 J_1 J_3 = \tilde{G}_3 \tilde{G}_2. \tag{A.6}$$

Thus, the $\tilde{G}_i$'s *actually commute if and only if* $N = 3$. Furthermore, they do satisfy a relation of the same structure as (4.2):

$$\tilde{G}_1 \tilde{G}_2 \tilde{G}_3 = \mathcal{I}d. \tag{A.7}$$

Let us now suppose that $N = 3$. A generic element of G reads:

$$g = \tilde{G}_1^{n_1} \tilde{G}_2^{n_2}, \tag{A.8}$$

where n_1 and n_2 are relative integers, which explicitly means that G is isomorphic to $\mathbb{Z} \times \mathbb{Z}$.

Let us now demonstrate that Γ is isomorphic to G, up to a semidirect product by a finite group. If γ denotes a generic element of Γ, it can be written as follows:

$$\gamma = \left(\prod T_{i,j,k}\right) J_1^{\alpha_1} J_2^{\alpha_2}, \tag{A.9}$$

where $T_{i,j,k} = J_i J_j J_k$ and $(\alpha_1, \alpha_2) = (0, 1, 2)$.

One can immediately replace, $\mathbf{T}_{1,2,3}$, $\mathbf{T}_{3,1,2}$, and $\mathbf{T}_{2,3,1}$ by the identity transformation, and $\mathbf{T}_{1,3,2} = \tilde{G}_1$, $\mathbf{T}_{2,1,3} = \tilde{G}_2$, $\mathbf{T}_{3,2,1} = \tilde{G}_3$. Besides, one has some kind of "pseudo"-commutation relations between the $\tilde{G}_i$'s and J_i's:

$$\begin{array}{lll} J_1\tilde{G}_1 = \tilde{G}_2 J_1, & J_2\tilde{G}_1 = \tilde{G}_2 J_2, & J_3\tilde{G}_1 = \tilde{G}_2 J_3, \\ J_1\tilde{G}_2 = \tilde{G}_3 J_1, & J_2\tilde{G}_2 = \tilde{G}_3 J_2, & J_3\tilde{G}_2 = \tilde{G}_3 J_3, \\ J_1\tilde{G}_3 = \tilde{G}_1 J_1, & J_2\tilde{G}_3 = \tilde{G}_1 J_2, & J_3\tilde{G}_3 = \tilde{G}_1 J_3, \end{array} \tag{A.10}$$

then γ reads:

$$\gamma = \tilde{G}_1^{n_1}\tilde{G}_2^{n_2}\tilde{G}_3^{n_3}\left(\prod T_{i,j,k}\right) J_1^{\beta_1} J_2^{\beta_2}. \tag{A.11}$$

In fact, all the $\mathbf{T}_{i,j,k}$'s, where $\{i,j,k\} = \{1,2,3\}$, have already been replaced by "words" in terms of the $\tilde{G}_i$'s. Thus, using relation $J_3 = J_2^2 J_1^2$, the only $\mathbf{T}_{i,j,k}$'s, appearing in (8.11) are:

$$\mathbf{T}_{1,1,2} = \mathbf{T}_1, \quad \mathbf{T}_{1,2,1} = \mathbf{T}_2, \quad \mathbf{T}_{2,1,1} = \mathbf{T}_3, \quad \mathbf{T}_{2,2,1} = \mathbf{T}_4, \quad \mathbf{T}_{2,1,2} = \mathbf{T}_5, \quad \mathbf{T}_{1,2,2} = \mathbf{T}_6.$$

However, it is easy to remark that these T_i's satisfy the following relations:

$$\mathbf{T}_2 = \tilde{G}_3\mathbf{T}_1, \quad \mathbf{T}_3 = \tilde{G}_2\tilde{G}_3\mathbf{T}_1, \quad \mathbf{T}_6 = \tilde{G}_1\mathbf{T}_4, \quad \mathbf{T}_5 = \tilde{G}_2\tilde{G}_1\mathbf{T}_4,$$

and besides

$$\mathbf{T}_1^2 = \tilde{G}_1^2\tilde{G}_2\mathbf{T}_4, \qquad \mathbf{T}_4^2 = \tilde{G}_3^2\tilde{G}_2\mathbf{T}_1, \qquad \mathbf{T}_1\mathbf{T}_4 = \mathbf{T}_4\mathbf{T}_1 = \mathcal{I}d.$$

Thus, Γ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$ up to a semidirect product by a finite group.

APPENDIX B

FINITE-ORDER CONDITIONS $G_1^N = \mathcal{I}$DENTITY FOR $Q = 3$

Let us give here a list of the *codimension-one self-dual varieties corresponding to* $G_1^N = \mathcal{I}$dentity. They read:

$$\mathcal{C}_n = \sum_i c_i \cdot Y^i = 0, \tag{B.12}$$

where the c_i's are polynomial expressions in X *with (relative) integer coefficients.*

The codimension-one variety, corresponding to $G_1^9 = \mathcal{I}$dentity, is self-dual and reads:

$$\begin{aligned} \mathcal{C}_9 = {} & 3 \cdot Y^6 - \left(6 - 36X + 72X^2\right) \cdot X \cdot Y^5 + \left(4 - 46X + 219X^2 - 504X^3 + 504X^4\right) \cdot X^2 \cdot Y^4 \\ & + \left(679X^3 - 1842X^6 - 147X^2 - 1842X^4 - 1 + 2763X^5 + 18X\right) \cdot X^3 \cdot Y^3 \\ & + \left(2394X^4 - 1197X^5 + 669X^2 - 1803X^3 - 123X + 9\right) \cdot X^7 \cdot Y^2 \\ & + \left(401X^3 - 896X^4 + 16X + 414X^7 - 1035X^6 + 1239X^5 - 1 - 108X^2\right) \cdot X^8 \cdot Y \\ & + \left(3X^3 - 9X^2 + 6X - 1\right) \cdot X^{15} = 0, \end{aligned}$$

or equivalently:

$$\begin{aligned} \tilde{\mathcal{C}}_9 = {} & 15479341056 \cdot \tilde{Y}^6 - 371504185344 \cdot \tilde{X}^3 \cdot \tilde{Y}^5 \\ & + \left(2600529297408\tilde{X}^6 - 11943936 + 46438023168\tilde{X}^4\right) \cdot \tilde{Y}^4 \\ & - 497664 \cdot \tilde{X} \cdot \left(5 - 1248\tilde{X}^2 + 54432\tilde{X}^4 + 19097856\tilde{X}^8\right) \cdot \tilde{Y}^3 \\ & + \lambda_2\left(\tilde{X}\right) \cdot \tilde{Y}^2 + 864\tilde{X} \cdot \lambda_1\left(\tilde{X}\right) \cdot \tilde{Y} + \lambda_0\left(\tilde{X}\right) = 0, \end{aligned}$$

with

$$\begin{aligned}\lambda_2\left(\tilde{X}\right) &= 5184 - 124416\tilde{X}^2 - 48522240\tilde{X}^4 - 8707129344\tilde{X}^8\\ &\quad + 2020054007808\tilde{X}^{10} + 1755758592\tilde{X}^6 - 6176257081344\tilde{X}^{12},\\ \lambda_1\left(\tilde{X}\right) &= -112320\tilde{X}^6 - 87340032\tilde{X}^{10} + 188116992\tilde{X}^{12} + 11088\tilde{X}^4 - 188\tilde{X}^2 + 1\\ &\quad + 2472394752\tilde{X}^{14} - 3366144\tilde{X}^8,\\ \lambda_0\left(\tilde{X}\right) &= -1 + 36\tilde{X}^2 - 194586624\tilde{X}^{10} + 20777472\tilde{X}^8 + 15479341056\tilde{X}^{18} + 9792\tilde{X}^4\\ &\quad + 3929554944\tilde{X}^{12} - 34828517376\tilde{X}^{16} - 822528\tilde{X}^6 - 19349176320\tilde{X}^{14}.\end{aligned}$$

Introducing invariants (6.12), these last expressions become simpler namely:

$$\begin{aligned}\hat{C}_9 &= 3\cdot\hat{Y}^6 - 72\hat{Y}^5\hat{X}^3 + 4\cdot\left(126\hat{X}^6 + 27\hat{X}^4 - 1\right)\cdot\hat{Y}^4\\ &\quad - 2\hat{X}\cdot\left(378\hat{X}^4 + 5 - 104\hat{X}^2 + 921\hat{X}^8\right)\cdot\hat{Y}^3\\ &\quad + 3\left(196\hat{X}^6 - 65\hat{X}^4 - 399\hat{X}^{12} + 1566\hat{X}^{10} - 2\hat{X}^2 + 1 - 81\hat{X}^8\right)\cdot\hat{Y}^2\\ &\quad + 2\hat{X}\cdot\left(-47\hat{X}^2 - 487\hat{X}^8 + 231\hat{X}^4 + 3 - 195\hat{X}^6 + 207\hat{X}^{14} - 1053\hat{X}^{10} + 189\hat{X}^{12}\right)\cdot\hat{Y}\\ &\quad + 3\hat{X}^2 - 1 - 476\hat{X}^6 - 782\hat{X}^{10} + 1002\hat{X}^8 + 1316\hat{X}^{12} + 3\hat{X}^{18} + 68\hat{X}^4 - 540\hat{X}^{14} - 81\hat{X}^{16}.\end{aligned}$$

The codimension-one variety corresponding to $G_1^{10} = \mathcal{I}$dentity is self-dual and reads:

$$\begin{aligned}\mathcal{C}_{10} &= Y^6 + \left(3X - 1 - 6X^2\right)\cdot X\cdot Y^5 + \left(255X^2 - 85X - 255X^3 + 10\right)\cdot X^3\cdot Y^4\\ &\quad + \left(-1185X^4 - 300X^2 + 800X^3 + 60X + 790X^5 - 5\right)\cdot X^4\cdot Y^3\\ &\quad + \left(4506X^6 - 17X - 4013X^5 + 2055X^4 + 1 - 2253X^7 + 136X^2 - 660X^3\right)\cdot X^5\cdot Y^2\\ &\quad + \left(868X^5 - 95X^2 - 685X^4 + 264X^7 + 330X^3 - 660X^6 + 15X - 1\right)\cdot X^8\cdot Y\\ &\quad + \left(1 - 3X + X^2\right)\cdot X^{16} = 0,\end{aligned}$$

or equivalently

$$\begin{aligned}\tilde{C}_{10} &= 5159780352\cdot\tilde{Y}^6 - 2579890176\cdot\tilde{X}\cdot\left(12\tilde{X}^2 + 1\right)\cdot\tilde{Y}^5\\ &\quad + \left(14929920 - 1612431360\tilde{X}^2 - 1315743989760\tilde{X}^6 + 109645332480\tilde{X}^4\right)\cdot\tilde{Y}^4\\ &\quad + 2488320\tilde{X}\cdot\left(12\tilde{X}^2 + 1\right)\cdot\left(136512\tilde{X}^6 - 9648\tilde{X}^4 + 156\tilde{X}^2 - 1\right)\cdot\tilde{Y}^3\\ &\quad - 1728\cdot\left(6727421952\tilde{X}^{12} + 5 - 792\tilde{X}^2 - 350853120\tilde{X}^{10} - 2592000\tilde{X}^6\right.\\ &\quad \left.+ 62640\tilde{X}^4 + 36391680\tilde{X}^8\right)\cdot\tilde{Y}^2 + 1152\tilde{X}\cdot\lambda_1\left(\tilde{X}\right)\cdot\left(12\tilde{X}^2 + 1\right)\cdot\tilde{Y}\\ &\quad + \left(1728\tilde{X}^6 + 1584\tilde{X}^4 - 60\tilde{X}^2 + 1\right)\cdot\lambda_2\left(\tilde{X}\right) = 0,\end{aligned}$$

with

$$\begin{aligned}\lambda_1\left(\tilde{X}\right) &= 98537472\tilde{X}^{12} + 28366848\tilde{X}^{10} + 62208\tilde{X}^8 - 152064\tilde{X}^6 + 6192\tilde{X}^4 - 120\tilde{X}^2 + 1,\\ \lambda_2\left(\tilde{X}\right) &= 2985984\tilde{X}^{12} - 12441600\tilde{X}^{10} + 311040\tilde{X}^8 - 158976\tilde{X}^6 + 11376\tilde{X}^4 - 216\tilde{X}^2 + 1.\end{aligned}$$

Again in terms of invariants (6.12) this reads:

$$\begin{aligned}
\hat{C}_{10} = {} & \hat{Y}^6 - 6\hat{X}\left(\hat{X}^2+1\right)\cdot\hat{Y}^5 + 5\left(1-51\hat{X}^6-9\hat{X}^2+51\hat{X}^4\right)\cdot\hat{Y}^4 \\
& + 10\hat{X}\cdot\left(\hat{X}^2+1\right)\cdot\left(79\hat{X}^6-67\hat{X}^4+13\hat{X}^2-1\right)\cdot\hat{Y}^3 \\
& + \left(-2253\hat{X}^{12}-1755\hat{X}^8-5+66\hat{X}^2+1410\hat{X}^{10}+1500\hat{X}^6-435\hat{X}^4\right)\cdot\hat{Y}^2 \\
& + 8\hat{X}\cdot\left(33\hat{X}^{12}+114\hat{X}^{10}+3\hat{X}^8-88\hat{X}^6+43\hat{X}^4-10\hat{X}^2+1\right)\cdot\left(\hat{X}^2+1\right)\cdot\hat{Y} \\
& + \left(\hat{X}^6+11\hat{X}^4-5\hat{X}^2+1\right)\cdot\left(\hat{X}^6-6\hat{X}^5-7\hat{X}^4-4\hat{X}^3+7\hat{X}^2+2\hat{X}-1\right) \\
& \times\left(\hat{X}^6+6\hat{X}^5-7\hat{X}^4+4\hat{X}^3+7\hat{X}^2-2\hat{X}-1\right).
\end{aligned}$$

The codimension-one variety corresponding to $G_1^{11} = \mathcal{I}$dentity is self-dual and reads (B.12) with:

$$\begin{aligned}
c_{10} &= -1, \\
c_9 &= 2n - 26X^2 + 52X^3, \\
c_8 &= 1098X^5 - 6X^2 - 441X^4 - 1098X^6 + 82X^3, \\
c_7 &= 9462X^9 + 9432X^7 - 14193X^8 + 4X^3 - 3482X^6 + 742X^5 - 85X^4, \\
c_6 &= 29X^5 - 5800X^8 - 10785X^{10} + 10729X^9 - 317X^6 - 2955X^{12} + 5910X^{11} + 1796x^7 - X^4, \\
c_5 &= 450007X^{12} - 36X^7 - 8300X^9 - 183756X^{11} - 701604X^{13} - 254088X^{15} + 635220X^{14} \\
&\quad + 819X^8 + 48942X^{10}, \\
c_4 &= 96392X^{12} - 905355X^{17} - 1181576X^{15} + 9X^8 - 238X^9 - 315601X^{13} \\
&\quad + 301785X^{18} - 20390X^{11} + 1306971X^{16} + 2852X^{10} + 728509X^{14}, \\
c_3 &= 107428X^{14} - 416364X^{21} + 2469751X^{18} - 359872X^{15} + 1457274X^{20} - 24116X^{13} \\
&\quad + 3951X^{12} - 1756987X^{17} + 31X^{10} + 915917X^{16} - 446X^{11} - X^9 - 2406174X^{19}, \\
c_2 &= 3921X^{17} - 23425X^{18} - 376065X^{22} + 221796X^{23} - 221400X^{20} - 55449X^{24} \\
&\quad + 362881X^{21} - 378X^{16} + 88794X^{19} + 16X^{15}, \\
c_1 &= 1834X^{19} - 23859X^{26} + 71812X^{23} + 52821X^{25} - 279X^{18} - X^{16} \\
&\quad + 25X^{17} - 48670X^{22} - 74438X^{24} + 5302X^{27} + 23358X^{21} - 7891X^{20}, \\
c_0 &= -X^{25} + 11X^{26} - 44X^{27} + 77X^{28} - 55X^{29} + 11X^{30}.
\end{aligned}$$

The codimension-one variety corresponding to $G_1^{12} = \mathcal{I}$dentity (B.12) with:

$$\begin{aligned}
c_8 &= -2, \\
c_7 &= 6X - 38X^2 + 76X^3, \\
c_6 &= 1088X^5 - 449X^4 - 1088X^6 + 88X^3 - 7X^2, \\
c_5 &= 9310X^9 - 79X^4 + 9120X^7 - 13965X^8 + 684X^5 - 3278X^6 + 4X^3, \\
c_4 &= 30X^5 - 93939X^{10} - 406X^6 - 15416X^8 - 52838X^{12} + 48070X^9 + 3164X^7 \\
&\quad - X^4 + 105676X^{11}, \\
c_3 &= 3076X^9 - 81979X^{12} + 106594X^{13} - 13977X^{10} - 421X^8 + 41512X^{11} - 86080X^{14} - X^6 \\
&\quad + 32X^7 + 34432X^{15}, \\
c_2 &= 1692X^{11} + 88472X^{15} + 24X^9 - 53463X^{14} - 7428X^{12} + 70530X^{17} \\
&\quad - 100506X^{16} - 260X^{10} + 23314X^{13} - X^8 - 23510X^{18}, \\
c_1 &= 20X^{13} - 2849X^{20} + 814X^{21} + 5144X^{17} + 848X^{15} - 173X^{14} - 2594X^{16} - X^{12} \\
&\quad - 6661X^{18} + 5544X^{19}, \\
c_0 &= X^{22} + X^{24} - 4X^{23}.
\end{aligned}$$

The codimension-one variety corresponding to $G_1^{13} = \mathcal{I}\text{dentity}$ (B.12) with:

$$
\begin{aligned}
c_{14} &= 1, \\
c_{13} &= 22X^2 - 4X - 44X^3, \\
c_{12} &= 6X^2 - 46X^3 + 131X^4 + 166X^6 - 166X^5, \\
c_{11} &= 19166X^9 + 16920X^7 - 28749X^8 - 9X^4 - 4870X^6 + 630X^5 - 4X^3, \\
c_{10} &= -376735X^{12} - 2061X^6 - 102535X^8 + 753470X^{11} + 332081X^9 - 662045X^{10} + 89X^5 \\
&\quad + X^4 + 19408X^7, \\
c_9 &= 179240X^9 - 5677671X^{12} - 84X^6 - 7084940X^{14} + 2172X^7 - 828356X^{10} + 2618872X^{11} \\
&\quad + 2833976X^{15} - 25539X^8 + 8163172X^{13}, \\
c_8 &= 36X^7 - 25841583X^{14} - 3595995X^{12} + 42698208X^{15} - 1116X^8 - 48502152X^{16} \\
&\quad - 140616X^{10} + 11259027X^{13} + 840420X^{11} + 34213977X^{17} - 11404659X^{18} + 16016X^9, \\
c_7 &= 62366X^{11} - 255092649X^{18} - 474376X^{12} + 2669814X^{13} + 43023480X^{21} - 9X^8 \\
&\quad - 150582180X^{20} - 95144007X^{16} - 5741X^{10} + 331X^9 + 37645872X^{15} + 248402658X^{19} \\
&\quad - 11431638X^{14} + 181864551X^{17}, \\
c_6 &= 468445155X^{21} - 375453015X^{22} + 107275X^{13} - 48164520X^{24} + X^9 + 884X^{11} \\
&\quad - 16596807X^{16} + 54132501X^{17} + 273496776X^{19} - 414599493X^{20} + 192658080X^{23} \\
&\quad - 745487X^{14} - 138078891X^{18} - 11557X^{12} - 43X^{10} + 3984560X^{15}, \\
c_5 &= -4216X^{15} - 23246946X^{19} + 912409755X^{24} - 685515X^{17} - 716767191X^{25} \\
&\quad - 76780498X^{27} - 797360400X^{23} + 120X^{14} - 239693730X^{21} + 68625X^{16} \\
&\quad + 505814490X^{22} + 85879152X^{20} + 4691515X^{18} + 345512241X^{26}, \\
c_4 &= 4790014X^{20} - 160266120X^{23} - 19969205X^{21} + 64161465X^{22} \\
&\quad + 311586450X^{24} - 10627X^{17} - 463526472X^{27} - 468914811X^{25} + 22887455X^{230} - 16X^{15} \\
&\quad + 285221689X^{28} - 114437275X^{29} - 869740X^{19} + 115599X^{18} + 539368809X^{26} + 604X^{16}, \\
c_3 &= X^{16} - 9546X^{19} + 30033130X^{24} + 70613026X^{32} - 187554800X^{31} + 791X^{18} \\
&\quad - 12838732X^{33} \\
&\quad - 76069635X^{25} + 157330983X^{26} - 41X^{17} + 360982362X^{28} \\
&\quad - 388448166X^{29} + 318744046X^{30} - 265247394X^{27} - 9639880X^{23} + 80811X^{20} \\
&\quad + 2490950X^{22} - 510070X^{21}, \\
c_2 &= 12740149X^{33} - 4697117X^{30} - 527066X^{36} + 84186X^{27} - 12970962X^{32} - 456030X^{28} \\
&\quad + 9220752X^{31} + 3162396X^{35} - 10326X^{26} - 8271495X^{34} + 1728297X^{29} \\
&\quad - 25X^{24} + 756X^{25}, \\
c_1 &= 17524X^{39} + 989925X^{33} - 3735X^{28} + X^{25} - 31X^{26} + 438X^{27} + 263504X^{31} + 21456X^{29} \\
&\quad - 87768X^{30} - 113906X^{38} + 1155490X^{35} - 780578X^{36} + 370214X^{37} - 1242823X^{34} \\
&\quad - 589817X^{32}, \\
c_0 &= +182X^{40} + 13X^{42} - 91X^{41} - 156X^{39} + 65X^{38} - 13X^{37} + X^{36}.
\end{aligned}
$$

The codimension-one variety corresponding to $G_1^{14} = \mathcal{I}\text{dentity}$ reads:

$$
\begin{aligned}
c_{12} &= 1, \\
c_{11} &= 33X^2 - 5X - 66X^3, \\
c_{10} &= 2091X^6 + 10X^2 + 820X^4 - 2091X^5 - 146X^3, \\
c_9 &= 51360X^8 - 2295X^5 - 10X^3 - 33080X^7 + 11575X^6 - 34240X^9 + 240X^4, \\
c_8 &= 88200X^8 - 548064X^{11} + 274032X^{12} + 495978X^{10} + 5X^4 + 2513X^6
\end{aligned}
$$

$$
\begin{aligned}
&\quad - 181X^5 - 262710X^9 - 18984X^7,\\
c_7 &= -76615X^9 - 2808084X^{13} - 981498X^{11} + 11998X^8 + 61X^6 - X^5 - 1176X^7 - 953352X^{15}\\
&\quad + 329343X^{10} + 2027148X^{12} + 2383380X^{14},\\
c_6 &= 533918X^{12} - 4675440X^{15} - 3080X^9 - 3253383X^{17} + 3128625X^{14} + 24990X^{10} - 8X^7\\
&\quad - 137046X^{11} - 1512021X^{13} + 231X^8 + 1084461X^{18} + 4872273X^{16},\\
c_5 &= 9069711X^{16} - 17768520X^{17} - 248101X^{13} + X^8 + 44870X^{12} - 6076X^{11} + 15511356X^{20}\\
&\quad - 25335870X^{19} + 581X^{10} + 1058345X^{14} + 25532766X^{18} - 3518896X^{15} - 35X^9\\
&\quad - 4431816X^{21},\\
c_4 &= 8031600X^{23} - 9501450X^{20} - 1330735X^{18} + 4278750X^{19} - 120X^{14} + 282775X^{17}\\
&\quad + 14352285X^{21} + 3240X^{15} - 39345X^{16} - 14064945X^{22} - 2007900X^{24},\\
c_3 &= 2264885X^{21} + 16X^{15} + 378374X^{27} + 239790X^{19} + 5816880X^{23} - 5670600X^{24}\\
&\quad + 3856071X^{25}\\
&\quad - 1702683X^{26} - 870000X^{20} - 46285X^{18} - 458X^{16} - 4268865X^{22} + 5949X^{17},\\
c_2 &= 31X^{17} - X^{16} - 1656115X^{24} - 329065X^{22} + 2594400X^{25} - 23132X^{20} - 159531X^{30}\\
&\quad - 3167946X^{26} + 2921346X^{27} - 443X^{18} + 100465X^{21} + 3870X^{19} + 797655X^{29}\\
&\quad - 1915618X^{28} + 833240X^{23},\\
c_1 &= 280X^{23} - 1855X^{24} - 66535X^{30} + 52220X^{27} + 34496X^{31} + X^{21} - 24395X^{26} + 87155X^{29}\\
&\quad - 11517X^{32} - 79950X^{28} - 25X^{22} + 2094X^{33} + 8085X^{25},\\
c_0 &= +X^{28} - X^{33} - 6X^{35} + 5X^{34}.
\end{aligned}
$$

The codimension-one variety corresponding to $G_1^{15} = \mathcal{I}\text{dentity}$ reads:

$$
\begin{aligned}
c_{16} &= -5,\\
c_{15} &= 290X^3 - 145X^2 + 25X,\\
c_{14} &= 7125X^5 - 3045X^4 + 630X^3 - 55X^2 - 7125X^6,\\
c_{13} &= 9443X^5 - 41741X^6 - 1211X^4 + 108766X^9 + 109872X^7 - 163149X^8 + 70X^3,\\
c_{12} &= 97320X^7 + 2448680X^{11} + 1323X^5 - 56X^4 + 1207988X^9 - 14635X^6 - 421235X^8\\
&\quad - 1224340X^{12} - 2236919X^{10},\\
c_{11} &= 678039X^9 + 8680056X^{15} + 12452X^7 - 111989X^8 - 17991363X^{12} - 21700140X^{14}\\
&\quad + 8592370X^{11} + 25307166X^{13} - 853X^6 + 28X^5 - 2870244X^{10},\\
c_{10} &= 89817464X^{15} - 469562X^{10} + 61854X^9 + 301X^7 - 95397318X^{16} + 2520133X^{11}\\
&\quad - 9815259X^{12} - 5496X^8 - 21548015X^{18} + 64644045X^{17} + 28080170X^{13}\\
&\quad - 59010131X^{14} - 8X^6,\\
c_9 &= 350898976X^{18} - 12832X^{10} + 1643388X^{13} + 992X^9 - 13544232X^{15} + X^7 + 251206109X^{20}\\
&\quad - 45X^8 + 73140264X^{16} - 71773174X^{21} - 541988X^{12} + 104413X^{11} - 388667984X^{19}\\
&\quad - 201684145X^{17} - 1229013X^{14},\\
c_8 &= 3563999094X^{20} - 412X^{11} + 86208330X^{16} - 2128340448X^{19} - 1988008992X^{23}\\
&\quad - 242475X^{13} + 958395375X^{18} - 4377674070X^{21} - X^{10} - 16979532X^{15} - 329227860X^{17}\\
&\quad + 3737407239X^{22} + 14823X^{12} + 497002248X^{24} + 2442285X^{14},\\
c_7 &= 105X^{12} - 549761134X^{27} + 2473925103X^{26} - 4005X^{13} + 72131X^{14} - 813468X^{15}\\
&\quad - 3510271851X^{21} - 588891036X^{19} - 5443027605X^{25} + 7716287679X^{24} + 168476936X^{18}\\
&\quad + 5961316803X^{22} - 7821106908X^{23} + 6422415X^{16} - 37590360X^{17} + 1619651049X^{20},\\
c_6 &= 19904754327X^{26} - 15X^{13} - 840610922X^{21} + 992243662X^{30} + 11914238576X^{28}\\
&\quad - 12843X^{15} - 55750129X^{19} + 240297445X^{20} + 164818X^{16} + 10350804X^{18}
\end{aligned}
$$

$$
\begin{aligned}
&+ 10743198954X^{24} - 1503457X^{17} - 18227502951X^{27} - 5631133635X^{23} \\
&- 16487353161X^{25} + 2404443576X^{22} - 4961218310X^{29} + 635X^{14}, \\
c_5 &= 8778582X^{20} - 27448399922X^{29} + 199877136X^{22} + 22417411968X^{30} + 2159640588X^{24} \\
&- 5388866436X^{25} + 1072X^{16} - 1358539X^{19} - 18778235985X^{27} - 13134426396X^{31} \\
&- 897364830X^{33} - 719793207X^{23} + 11118193815X^{26} + 4935506565X^{32} - 46125999X^{21} \\
&- 15754X^{17} + X^{14} - 47X^{15} + 25572366009X^{28} + 167097X^{18}, \\
c_4 &= 1077500X^{22} - 982X^{19} - 125600532X^{25} + 1870397532X^{34} - 1065255286X^{27} - 139693X^{21} \\
&+ 5684483477X^{30} - 6416078038X^{31} - 3818595163X^{33} + 13802X^{20} + 31826207X^{24} \\
&- 607816272X^{35} - 4017479622X^{29} - 6539598X^{23} + 2293042980X^{28} + 101302712X^{36} \\
&+ 5671486990X^{32} + 45X^{18} + 404446392X^{26} - X^{17}, \\
c_3 &= 47841314X^{28} - 43X^{21} + 859026789X^{32} + 3452857X^{26} + 93396082X^{38} - 664178X^{25} \\
&+ 296245849X^{30} - 14287286X^{27} - 11090X^{23} - 554173054X^{31} - 969339604X^{35} \\
&+ 872X^{22} + 632198614X^{36} - 131110277X^{29} - 1099890449X^{33} - 300905050X^{37} \\
&- 14368628X^{39} + 99226X^{24} + X^{20} + 1152423789X^{34}, \\
c_2 &= 1728951X^{41} + 31X^{29} + 4761X^{31} + 8897265X^{37} + 9214176X^{39} - 246993X^{42} \\
&- 5301420X^{36} - 5200554X^{40} - 10811551X^{38} - 763250X^{34} - 34588X^{32} + 187480X^{33} \\
&- 474X^{30} - X^{28} + 2329805X^{35}, \\
c_1 &= X^{31} - 58804X^{36} + 3230X^{45} - 3001X^{34} - 24225X^{44} - 234815X^{42} + 410594X^{41} \\
&+ 93258X^{43} - 514154X^{40} + 381X^{33} + 15807X^{35} + 470232X^{39} - 317671X^{38} + 159098X^{37} \\
&- 29X^{32}, \\
c_0 &= -8X^{47} + 14X^{46} - 7X^{45} + X^{44} + X^{48}.
\end{aligned}
$$

In terms of the "pseudo-self-dual" variables $\tilde{X} = 1/6 - X$ and $\tilde{Y} = 1/54 - Y$ (see (4.34)), these finite order conditions, respectively, become:

$$
\begin{aligned}
\tilde{C}_{11} &\propto -3 \cdot \tilde{Y}^{10} + \tilde{X} \cdot \left(156\tilde{X}^2 - 1\right) \cdot \tilde{Y}^9 + \cdots, \\
\tilde{C}_{12} &\propto -6 \cdot \tilde{Y}^{8} + \tilde{X} \cdot \left(228\tilde{X}^2 - 1\right) \cdot \tilde{Y}^7 + \cdots, \\
\tilde{C}_{13} &\propto 3 \cdot \tilde{Y}^{14} - \tilde{X} \cdot \left(132\tilde{X}^2 + 1\right) \cdot \tilde{Y}^{13} + \cdots, \\
\tilde{C}_{14} &\propto 2 \cdot \tilde{Y}^{12} - \tilde{X} \cdot \left(132\tilde{X}^2 - 1\right) \cdot \tilde{Y}^{11} + \cdots, \\
\tilde{C}_{15} &\propto -6 \cdot \tilde{Y}^{16} + \tilde{X} \cdot \left(1 + 348\tilde{X}^2\right) \cdot \tilde{Y}^{15} + \cdots.
\end{aligned}
$$

In the isotropic limit, these conditions yield fairly involved expressions. For instance, condition $G_{\mathrm{I}}^6 = \mathcal{I}\mathrm{dentity}$ reads:

$$
\begin{aligned}
&u^{24}x^9 + \left(u^{24} - u^{21} + 3u^{22} - 12u^{23}\right)x^8 \\
&\quad + \left(63u^{22} - 9u^{20} - 12u^{23} + 3u^{19} - 12u^{21} + 4u^{18} - u^{15}\right)x^7 \\
&\quad + \left(165u^{19} + 54u^{22} - 4u^{15} - 175u^{21} + 3u^{14} + 9u^{16} - 95u^{18} + u^{12} - 42u^{20}\right)x^6 \\
&\quad + \left(6u^{16} + 13u^{15} - 620u^{18} + 198u^{20} + 6u^{11} - 15u^{14}\right. \\
&\quad + \left.267u^{17} - 13u^{12} + 393u^{19} - 108u^{21} - u^{9}\right)x^5 \\
&\quad + \left(744u^{17} - 1063u^{18} + 600u^{16} + 81u^{20} - 45u^{10} + 135u^{19} + 123u^{13} + 14u^9 - 916u^{15}\right)x^4 \\
&\quad + \left(60u^{11} - 69u^{12} + 210u^{14}\right)x^4 \\
&\quad + \left(807u^{16} + 789u^{17} + 360u^{10} - 35u^9 + 78u^{13} - 2730u^{15} - 63u^8 + 27u^7 - 279u^{12} - 387u^{11}\right)x^3 \\
&\quad + \left(-378u^{18} + 1899u^{14} - 4u^6\right)x^3
\end{aligned}
$$

$$
\begin{aligned}
&+ \left(633u^8 - 2618u^{15} + 999u^{13} + 2154u^{11} + u^3 + 720u^{16} + 48u^6 - 6u^5 + 2607u^{14} - 1242u^9\right) x^2 \\
&+ \left(-3u^4 - 3722u^{12} - 186u^7 + 579u^{10}\right) x^2 \\
&+ \left(1047u^7 + 306u^{10} - 1 + 2880u^{13} - 94u^3 + 6u + 3495u^{11} - 4982u^{12} + 309u^4 - 415u^6\right) x \\
&+ \left(-1618u^9 - 294u^5 - 624u^{14} - 6u^8\right) x \\
&- 78u^4 + 3 + 120u^2 + 1578u^8 + 996u^5 - 30u - 1781u^6 - 203u^3 + 2880u^{10} \\
&+ 912u^7 + 224u^{12} - 1248u^{11} - 3374u^9 = 0,
\end{aligned}
$$

which, within the $x = 1$ limit, gives:

$$2 \cdot (u+1)^6 \cdot (u-1)^{18} = 0.$$

Another example is condition $G_1^8 = \mathcal{I}\text{dentity}$ which reads:

$$u^5 \cdot (u+1) \cdot x^2 - u^4 \cdot (1 + 3 \cdot u) \cdot x + 1 - 2u - u^2 + 4u^3 = 0,$$

together with a polynomial relation of degree 16 in x, 42 in u which corresponds to the sum of 310 monomials and $G_1^{10} = \mathcal{I}\text{dentity}$ reads:

$$x^3u^9 - 3u^8x^2 + \left(u^3 - 8u^5 + 3u^4 - u^2 + 8u^6\right) \cdot x + \left(1 - 3u + u^2 + 7u^3 - 9u^4 + 2u^5\right) = 0,$$

together with a polynomial relation of degree 24 in x, 63 in u which corresponds to the sum of 695 monomials.

APPENDIX C
TOWARDS ALGEBRAIC INVARIANTS FOR THE THREE-STATE CHECKERBOARD POTTS MODEL

C.1. CHECKERBOARD MODELS

Since, in the following, one considers low (or high) temperature expansions, let us introduce the *low temperature* variables:

$$A = \frac{1}{x_1}, \qquad B = \frac{1}{x_2}, \qquad C = \frac{1}{x_3}, \qquad D = \frac{1}{x_4}. \tag{C.13}$$

The checkerboard Potts model (*without magnetic field*) is self-dual with respect to the *Kramers-Wannier duality* [40,41,55] which reads

$$A \longrightarrow A^* = \frac{1 - A}{1 + (q-1) \cdot A} = \frac{x_1 - 1}{x_1 + q - 1}. \tag{C.14}$$

The modulus of the elliptic function of the checkerboard *Ising* model:

$$k^2 = \prod_{i=1}^{4} \frac{t_i \cdot (1 - t_i) \cdot (1 + t_i) \cdot \left(t_i^* + t_j^* \cdot t_k^* \cdot t_l^*\right)}{t_i^* \cdot (1 - t_i^*) \cdot (1 + t_i^*) \cdot (t_i + t_j \cdot t_k \cdot t_l)}, \tag{C.15}$$

reads (after simplifications) in terms of the low-temperature variables:

$$
\begin{aligned}
k^2 &= \frac{N_k}{D_k}, \qquad \text{where} \\
N_k &= 16 \cdot (A + BCD) \cdot (B + ACD) \cdot (C + BAD) \cdot (D + BCA), \qquad \text{and} \\
D_k &= (-1 - BC - AC - BA + CD + BD + AD + BCAD) \\
&\quad \times (BC - 1 + AC - BA + CD - BD - AD + BCAD) \\
&\quad \times (BC - 1 - AC + BA - CD + BD - AD + ABCD) \\
&\quad \times (-1 - BC + AC + BA - CD - BD + AD + ABCD).
\end{aligned}
$$

Remarkably condition $k^2 - 1 = 0$ factorizes the well-known critical condition of the model ($\mathrm{F}_0^{\mathrm{acto}} = 0$) *together with three other varieties* breaking the C_{4v} symmetry of the square of the lattice:

$$
\begin{aligned}
k^2 - 1 &= \frac{\mathrm{F}_1^{\mathrm{acto}} \cdot (\mathrm{F}_1^{\mathrm{acto}} \cdot \mathrm{F}_2^{\mathrm{acto}} \cdot \mathrm{F}_3^{\mathrm{acto}})}{D_k}, \qquad \text{where} \\
\mathrm{F}_0^{\mathrm{acto}} &= ABCD - (AD + BD + CD + AC + BC + BA) + 1, \\
\mathrm{F}_1^{\mathrm{acto}} &= ABCD + (AD + BD + CD + AC + BC + BA) + 1 - 2 \cdot (AD + BC), \\
\mathrm{F}_2^{\mathrm{acto}} &= ABCD + (AD + BD + CD + AC + BC + BA) + 1 - 2 \cdot (AB + CD), \\
\mathrm{F}_3^{\mathrm{acto}} &= ABCD + (AD + BD + CD + AC + BC + BA) + 1 - 2 \cdot (AC + BD).
\end{aligned} \tag{C.16}
$$

It is important to note that the set of these three conditions $\mathrm{F}_i^{\mathrm{acto}} = 0$, $(i = 1, 2, 3)$ is stable by the two inversion relations of the checkerboard Ising model, and that these three additional varieties *are also critical.*

It is tempting to try to generalize a pattern like (C.15) to q-state *Potts models*. However, the analysis of the successive images of the disorder solutions under the (infinite) discrete group of birational transformations, generated by the inversion relations for the q-state edge checkerboard Potts model [25], shows that one gets an *infinite number* of such image varieties[17]. In order to avoid infinite product expressions for a tentative substitute of k^2 (see (C.15)), *it is necessary to restrict oneself to Tutte-Beraha numbers.* Taking into account the remarkable properties of the Euclidean subcase $q = 3$ (see Section 4.3), the *three-state* standard scalar edge *checkerboard* Potts model, pops out as a good candidate for building the equivalent of the algebraic expression k^2, and hopefully, get exact expressions for the spontaneous magnetization.

Checkerboard Potts Model for $q = 3$

Let us restrict to $q = 3$. Let us first of all, consider a polynomial Q which is the product of the *four disorder conditions*:

$$
\begin{aligned}
Q &= (ABCD + BCD + A) \cdot (ABCD + ACD + B) \\
&\quad \times (ABCD + BAD + C) \cdot (ABCD + BCA + D).
\end{aligned} \tag{C.17}
$$

Denoting Q_1, Q_2, Q_{12}, Q_{21}, and Q_{121} the algebraic expressions corresponding to the action of: I_1, I_2, $I_1 \cdot I_2$, $I_2 \cdot I_1$, $I_1 \cdot I_2 \cdot I_1$ with:

$$
I_1\colon \quad (A, B, C, D) \longrightarrow \left(\frac{1}{A}, -\frac{B}{(1+B)}, \frac{1}{C}, -\frac{D}{(1+D)}\right), \tag{C.18}
$$

$$
I_2\colon \quad (A, B, C, D) \longrightarrow \left(-\frac{A}{(1+A)}, \frac{1}{B}, -\frac{C}{(1+C)}, \frac{1}{D}\right). \tag{C.19}
$$

One can see that the product of the action of the whole group (generated by the two inversion relations of the checkerboard model) is actually equal to the product $Q_{\mathrm{prod}} = Q \cdot Q_1 \cdot Q_2 \cdot Q_{12} \cdot Q_{21} \cdot Q_{121}$. This new product *happens to be a perfect square.* If one introduces the square root of Q_{prod}, it reads $G_{\mathrm{check}} = N_{\mathrm{check}}/D_{\mathrm{check}}$ where N_{check} and D_{check} read:

$$
\begin{aligned}
N_{\mathrm{check}} &= (ABCD + BCD + A) \cdot (ABCD + ACD + B) \cdot (ABCD + BAD + C) \\
&\quad \times (ABCD + BCA + D) \cdot (BD + BAD + C + CD + BC + BCD) \\
&\quad \times (BD + BCD + A + AD + AB + BAD) \\
&\quad \times (AC + BCA + D + CD + AD + ACD) \\
&\quad \times (AC + ACD + B + BC + AB + BCA) \\
&\quad \times (BD - 1 + CD + AD + ACD + BCD + BAD + ABCD) \\
&\quad \times (BD - 1 + BC + AB + BCA + BCD + BAD + ABCD) \\
&\quad \times (AC - 1 + CD + ACD + BC + BCA + BCD + ABCD) \\
&\quad \times (AC - 1 + AD + ACD + AB + BCA + BAD + ABCD),
\end{aligned} \tag{C.20}
$$

[17] With the "awkward position" that the partition function is a multivalued function with *infinite valuation* [26].

and

$$D_{\text{check}} = A^4B^4C^4D^4 \cdot (1+A)^4 \cdot (1+B)^4 \cdot (1+C)^4 \cdot (1+D)^4. \tag{C.21}$$

Let us denote G^* the (Kramers-Wannier) dual expression of G_{check}. In order to have a nice representation of the Kramers-Wannier duality [40,41,55] (like for the checkerboard Ising model: $k^2 \to 1/k^2$), one can introduce the following ratio:

$$\mathcal{G} = \frac{G_{\text{check}}}{G^*}, \qquad \text{with } G^* = \frac{N_{G^*}}{D_{G^*}}, \tag{C.22}$$

where the denominator D_{G^*} and the numerator N_{G^*}, respectively, read:

$$\begin{aligned} D_{G^*} &= (2+A)^4 \cdot (2+B)^4 \cdot (2+C)^4 \cdot (2+D)^4 \cdot (A-1)^4 \\ &\quad \times (B-1)^4 \cdot (C-1)^4 \cdot (D-1)^4 \cdot (1+2A)^4 \cdot (1+2B)^4 \cdot (1+2C)^4 \cdot (1+2D)^4, \qquad \text{and} \\ N_{G^*} &= 3^{12} \cdot \mathcal{N}_1 \cdot \mathcal{N}_2 \cdot \mathcal{N}_3 \cdot \mathcal{N}_4 \cdot \mathcal{N}_5 \cdot \mathcal{N}_6 \cdot \mathcal{N}_7 \cdot \mathcal{N}_8 \cdot \mathcal{N}_9 \cdot \mathcal{N}_{10} \cdot \mathcal{N}_{11} \cdot \mathcal{N}_{12}, \end{aligned}$$

with, for instance,

$$\begin{aligned} \mathcal{N}_1 &= -1 - 2BCD + ACD + BAD + BCA \\ &\quad - 2CD - 2BD - 2BC + AD + AB + AC + 3ABCD. \end{aligned}$$

Algebraic expression $\mathcal{G}$ *is invariant by the group generated by the two inversions* (C.18) and (C.19) of the checkerboard model. Note, that $\mathcal{G}$ *is not symmetric under the whole group* S_4 *of permutation of* (A, B, C, D) but only (as it should) *under the symmetry group of the square* C_{4v}.

Let us now consider the two polynomials:

$$S = (1-A) \cdot (1-B) \cdot (1-C) \cdot (1-D) \quad \text{and} \quad T = (1+2A) \cdot (1+2B) \cdot (1+2C) \cdot (1+2D),$$

as well as the following polynomials:

$$U = (1+A) \cdot (1+B) \cdot (1+C) \cdot (1+D) \quad \text{and} \quad R = ABCD.$$

Let us denote ST their product. The product corresponding to the action of (C.18) and (C.19) on ST, (namely the product $P_H = H \cdot H_1 \cdot H_2 \cdot H_{12} \cdot H_{21} \cdot H_{121}$) happens to be exactly equal to $P_{ST} = S^4T^4W^4/U^4/R^4$. Let us introduce, as a multiplicative correction term, C_G equal to $P_{ST}/3^{24}$. Expression C_G is, by construction, *invariant under the group generated by the inversion relations*. One now introduces, instead of $\mathcal{G}$, a new algebraic expression:

$$\mathcal{G}_1 = \frac{\mathcal{G}}{C_G} = 3^{24} \cdot \frac{\mathcal{G}}{P_{ST}}. \tag{C.23}$$

We get the remarkable property that *the critical condition of the* $q = 3$ *checkerboard Potts model*, namely [56,57]:

$$x_1x_2x_3x_4 - (x_1 + x_2 + x_3 + x_4) - (x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4) = 0,$$

or

$$1 - (ABC + BCD + ABD + ACD) - (BA + CA + AD + BC + BD + DC) = 0, \tag{C.24}$$

reads in terms of invariant $\mathcal{G}_1$:

$$\mathcal{G}_1 = 1. \tag{C.25}$$

Note that $\mathcal{G}$, on the contrary, is not equal to 1 on the critical condition (C.24).

The algebraic expression $\mathcal{G}_1$ *satisfies all the symmetries of the spontaneous magnetization* (permutation symmetries of the three x_i's, inversion relation, duality ...). *It vanishes on the disorder conditions* (and its inverse vanishes on the "order" conditions) and *becomes equal to* 1 *on the critical variety*. It can thus be seen, as first glance, as a generalization of the Ising *modulus of elliptic functions* (C.15).

C.2. RESUMMED EXPANSIONS FOR THE CHECKERBOARD POTTS MODEL

The *resummed expansion* of the partition function of the *checkerboard* q-state Potts model, *in the presence of a magnetic field*, reads [10,25,56,57] (denoting z the fugacity):

$$
\begin{aligned}
\ln(\Lambda) &= (q-1)\cdot\frac{(B\mathcal{X}+D\mathcal{Y})}{2} + (q-1)(q-2)\cdot\frac{(B\mathcal{X}^2+D\mathcal{Y}^2)}{2} \\
&+ \frac{(q-1)}{2}\cdot\frac{(A^2z^2+C^2z^2+2A^2C^2z^4)\,(BD+B\mathcal{Y}+D\mathcal{X})^2}{(1-A^2C^2z^4)} \\
&+ \frac{(q-1)(q-2)}{2}\cdot\frac{ACz^2\,(A+C+2A^2C^2z^2)\,(BD+B\mathcal{Y}+D\mathcal{X})^2}{1-A^3C^3z^4} \\
&+ \frac{(q-1)(q-2)^2}{2}\cdot\frac{ACz\,(2BD\mathcal{X}\mathcal{Y}+ACz\,(B^2\mathcal{X}^2+D^2\mathcal{Y}^2))}{(1-A^2C^2z^2)} \\
&+ \frac{(q-1)^2}{2}\cdot\left(\frac{(A^2C^2z^2-7)\cdot A^4C^4z^4}{(1-A^2C^2z^2)^3}\right)\cdot(B^4+D^4) \\
&- \frac{(q-1)^2}{2}\cdot\left(\frac{(8+4A^2C^2z^2)\,A^3C^3z^3\cdot(B^3D+BD^3)}{(1-A^2C^2z^2)^3}\right) \\
&- \frac{(q-1)^2}{2}\cdot\left(\frac{(5+A^2C^2z^2)\,(1+2A^2C^2z^2)\cdot A^2C^2z^2B^2D^2}{(1-A^2C^2z^2)^3}\right)+\cdots,
\end{aligned}
\tag{C.26}
$$

with

$$
\mathcal{X} = \frac{ACz\cdot(D+ABCz)}{1-A^2C^2z^2} \quad\text{and}\quad \mathcal{Y} = \frac{ACz\cdot(B+ADCz)}{1-A^2C^2z^2}, \tag{C.27}
$$

and where the A, B, C, D denote the *low-temperature variables* (C.13). Performing the derivative of (C.26) with respect to z, one gets:

$$
\begin{aligned}
z\cdot\frac{d}{dz}\ln(\Lambda) &= (q-1)\cdot ACz\frac{(ACz\cdot(B^2+D^2)\,(1+A^2C^2z^2)\cdot BD)}{(1-A^2C^2z^2)^2} \\
&+ 2\cdot(q-1)\cdot(q-2)\cdot\frac{A^4C^4z^4}{(1-A^2C^2z^2)^3}\cdot(B^3+D^3) \\
&+ (q-1)\cdot(q-2)\cdot\frac{A^2C^2z^2\cdot(1+A^2C^2z^2+A^3C^3z^3+3ACz)}{(1-A^2C^2z^2)^3}\cdot BD\cdot(B+D) \\
&+ c_{44}\cdot(B^4+D^4)+c_{13}\cdot(B^3D+BD^3)+c_{22}\cdot B^2D^2+\cdots.
\end{aligned}
\tag{C.28}
$$

The derivative of (C.28) is nothing (up to a multiplicative factor) but the magnetization minus one. Recalling the "order" conditions (see Section 3.3), let us introduce the product of the two "order conditions" (see [13]):

$$
\begin{aligned}
P &= (D+ABCz+(q-2)\cdot ABCDz)\,(B+ADCz+(q-2)\cdot ABCDz) \\
&= (1+A^2C^2z^2)\cdot BD + ACz\cdot(B^2+D^2) \\
&+ (q-2)\cdot ACz\cdot(1+ACz)\cdot(B^2D+BD^2) \\
&+ (q-2)^2\cdot A^2C^2z^2\cdot B^2D^2.
\end{aligned}
\tag{C.29}
$$

The "order conditions" are such that, restricted to them, the (*low-temperature* expansion of the) magnetization becomes *formally* equal to 1 [13]. *Actually, at this order, one can verify for* (C.28):

$$
z\cdot\frac{d}{dz}\ln(\Lambda) = P\cdot H, \tag{C.30}
$$

where H, at this order, reads:

$$H = \frac{(q-1)ACz}{(A^2C^2z^2-1)^2} + \frac{2\cdot(q-1)\cdot(q-2)\cdot A^3C^3z^3}{(1-A^2C^2z^2)^3}\cdot(B+D) + H_2\cdot\left(B^2+D^2\right) + H_3\cdot BD + \cdots . \tag{C.31}$$

At this order, relation (C.30) corresponds to the following relation between c_{13}, c_{44}, c_{22}:

$$\left(\frac{1+A^4C^4z^4}{A^2C^2z^2}\right)\cdot c_{44} - \frac{(1+A^2C^2z^2)}{ACz}\cdot c_{13} - \frac{(q-1)\cdot(q-2)^2A^3C^3z^3\cdot(2ACz-1)}{(1-A^2C^2z^2)^2} + c_{22} = 0.$$

The expressions of H_2 and H_3 in terms of c_{13}, c_{44}, and c_{22} read:

$$H_2 = \frac{c_{44}}{ACz}, \quad H_3 = c_{34} - \frac{(1+A^2C^2z^2)}{ACz}\cdot c_{44} - \frac{2(q-2)^2A^4C^4z^4(1+ACz)(q-1)}{(1-A^2C^2z^2)^3}\cdot\frac{1}{ACz}.$$

The expressions of c_{13}, c_{44}, and c_{22} are quite involved, and therefore, will not be given here.

C.3. REMARK ON THE IMAGE OF THE ORDER CONDITION BY THE INVERSION RELATIONS FOR CHECKERBOARD MODELS. RESUMMED EXPANSIONS VERSUS ALGEBRAIC INVARIANTS

In the numerator of $\mathcal{G}_1$ (see (C.20)), the image, by the inversion relations, of the four "order" conditions, read, respectively,

$$BD + BCD + A + AD + BA + BAD = 0 \quad \text{and} \quad BD + BAD + C + CD + BC + BCD = 0,$$

together with six other algebraic varieties. Among these varieties the previous two are compatible with the (low-temperature) resummed expansions of the checkerboard model (namely B and D small).

$$A = -\frac{BD\cdot(1+C)}{(1+B)\cdot(1+D)} \quad \text{and} \quad C = -\frac{BD\cdot(1+A)}{(1+B)\cdot(1+D)}. \tag{C.32}$$

In fact, when one substitutes (C.32) in the resummed expansion (C.28), for $z = 1$ and $q = 3$, one *does not get* zero (as one could expect from a naive interpretation of the automorphy property of (C.28)). In fact, if the spontaneous magnetization can be as a *automorphic function of several complex variables* with respect to our discrete group of birational transformations [10], it is a *multivalued function* [25] with a *very complicated covering*. Furthermore, if one considers the (anisotropic) triangular limit of invariant $\mathcal{G}_1$, namely $\mathcal{G}_1^{\mathrm{triang}}$, this expression *is not invariant under the permutations of* x_1, x_2, x_3.

One also remarks that the (B, D small) expansion of $\mathcal{G}_1$ is paradoxically more involved that the resummed expansion of the spontaneous magnetization (see (C.28)). In particular, the rational expressions occurring in the resummed expansion of the spontaneous magnetization *only have N^{th} roots of unity* $1 - A^NC^N$, while the rational expressions in the (B, D small) expansion of $\mathcal{G}_1$ *paradoxically have much more complicated denominators*. Unfortunately, all the "correcting" terms by which one can multiply $\mathcal{G}_1$, cannot easily change this situation, suppressing the "unpleasant" singularities in (C.22) and replacing them by "nice" N^{th} root singularities of (C.28).

In fact, even an "improved" $\mathcal{G}_1$ is probably not sufficient enough to describe the resummed expansion of the spontaneous magnetization ((C.28) for $z = 1$ and $q = 3$).

In the following we will try to clarify this point considering two limits of the checkerboard Potts model: the triangular model and, more particularly, namely the *honeycomb model* (and more precisely the algebraic invariant built from product, over the group, of "order" varieties), which seem to correspond to less analytically "subtle" situations.

APPENDIX D
TOWARDS ALGEBRAIC INVARIANTS FOR THE HONEYCOMB POTTS MODEL

D.1. SPONTANEOUS MAGNETIZATION FOR Q-STATE EDGE POTTS MODEL ON THE HONEYCOMB LATTICE

In the Ising case, one has for the spontaneous magnetization of the *honeycomb* lattice:

$$M = \left(1 - k_h^2\right)^{1/8}, \qquad \text{where } k_h^2 = \frac{16 \cdot (1 + ABC) \cdot (A + BC) \cdot (B + AC) \cdot (C + AB)}{(1 - A^2)^2 \cdot (1 - B^2)^2 \cdot (1 - C^2)^2}.$$

For arbitrary q, the relation between the spontaneous magnetization and the derivative with respect to z of $\ln(\Lambda)$ reads:

$$z \cdot \frac{d}{dz} \ln(\Lambda) = \frac{q-1}{q} \cdot (1 - M) \quad \text{or} \quad M = 1 - \frac{q}{q-1} \cdot z \cdot \frac{d}{dz} \ln(\Lambda). \tag{D.33}$$

Taking the $C = 1$ limit[18] one gets the equivalent of expansion (C.28) for the *three-state honeycomb Potts model*:

$$\begin{aligned}
z \cdot \frac{d}{dz} \ln(\Lambda) &= (q-1) \cdot \frac{A \cdot (C + BA)(B + CA)}{(1 - A^2)^2} + 2 \cdot (q-1) \cdot (q-2) \cdot \frac{A^4 \cdot (B + C)^3}{(1 - A^2)^3} \\
&+ (q-1) \cdot (q-2) \cdot \frac{A^2 \cdot (A-1) \cdot (A^2 - 4A - 1) \cdot BC \cdot (B + C)}{(1 - A^2)^3} \\
&+ 2 \cdot (q-1) \cdot \frac{A \cdot (2A^4 + 1 + 5A^2) \cdot (B + CA) \cdot (C + BA) \cdot (B + C)^2}{(1 - A^2)^4} \\
&+ (q-1) \cdot \frac{(A^3 - 5A^2 + A - 1) \cdot (A-1)^3 \cdot (B + CA) \cdot (C + BA) \cdot BC}{(1 - A^2)^4} \\
&+ 2 \cdot (q-1) \cdot (q-2) \cdot \frac{A^2 \, (2A^5 + 4A^4 + 5A^3 + 5A^2 + 2A + 1 + A^6) \, (B + CA)(C + BA)(B + C)^2}{(1 - A^2)^2 \cdot (1 - A^3)^2 \cdot (1 + A)} \\
&+ (q-1) \cdot (q-2) \cdot \frac{(1 + A^2 - 3A^5 - 4A^4 - 3A^3) \cdot (1 - A^2) \cdot A \cdot (B + CA) \cdot (C + BA) \cdot BC}{(1 - A^2)^2 \cdot (1 - A^3)^2 \cdot (1 + A)} \\
&+ 3 \cdot (q-1) \cdot (q-2)^2 \cdot \frac{A^6 \cdot (B + C)^4}{(1 - A^2)^4} + (q-1) \cdot (q-2)^2 \cdot \frac{A^3 \cdot (A-1) \cdot (A-3) \cdot B^2 C^2}{(1 - A^2)^3} \\
&+ (q-1) \cdot (q-2)^2 \cdot \frac{A^4 \cdot (A-1) \cdot (A^2 - 9A - 4) \cdot (B + C)^2 \cdot BC}{(1 - A^2)^4} \\
&- (q-1)^2 \cdot \frac{A^2 \cdot (2A^3 + 7A) \cdot (C + BA) \cdot (B + CA) \cdot (B + C)^2}{(1 - A^2)^4} \\
&- (q-1)^2 \cdot \frac{A^2 \cdot (5 - 4A) \cdot (A-1)^2 \cdot (C + BA) \cdot (B + CA) \cdot BC}{(1 - A^2)^4}.
\end{aligned} \tag{D.34}$$

D.2. SEEKING FOR INVARIANTS FOR THE HONEYCOMB LATTICE

Let us first recall the two invariants of the triangular lattice. Written in terms of the *low-temperature variables* (C.13) invariants Y in (4.33) and (4.37) read:

$$Y^{\text{triang}} = -\frac{(1 + A)(1 + B)(1 + C)BCA}{(BC + CA + BA - 1 + BCA)^3}, \qquad \text{and} \tag{D.35}$$

$$M^{\text{triang}} = -\frac{-1 - 2 \cdot (A + B + C) - (BC + CA + BA) + BCA}{BC + CA + BA - 1 + BCA}. \tag{D.36}$$

[18] And replacing D by C.

Resummed expansion (D.34) has to be compared to the Kramers-Wannier dual of (D.35) and (D.36) together with *an invariant originating from product of "order" varieties* denoted $\mathcal{P}_{\rm rod}$:

$$\begin{aligned} \mathcal{P}_{\rm rod} &= 2 \cdot \frac{\mathcal{P}_N}{\mathcal{P}_D}, \qquad \text{with} \\ \mathcal{P}_D &= (-1 - C - A - B + BCA)^6 \cdot (-1 + A + B + C + 2(BC + CA + BA) + ABC)^3, \end{aligned} \tag{D.37}$$

and

$$\begin{aligned} \mathcal{P}_N = {} & (BCA + A + BC) \cdot (BCA + B + CA) \cdot (BCA + C + BA) \\ & \times (BCA + A + B + 2BA + 1) \cdot (BCA + C + B + 2BC + 1) \\ & \times (BCA + C + A + 2CA + 1) \cdot (A - C + B + BA + 1) \cdot (C + A - B + CA + 1) \\ & \times (A - C - B - BC - 1). \end{aligned} \tag{D.38}$$

Let us introduce the ferromagnetic critical variety of the honeycomb lattice:

$$\mathcal{C}_0^{\rm honey} = -1 + 2 \cdot (BC + CA + BA) + (A + B + C) + ABC = 0. \tag{D.39}$$

One notes that $\mathcal{P}_{\rm rod}$ has a $(\mathcal{C}_0^{\rm honey})^{-3}$ singularity. One immediately verifies that $\mathcal{P}_{\rm rod}$ is *invariant under the honeycomb inversion relation*:

$$(A, B, C) \longrightarrow \left(\frac{-A}{1+A}, \frac{1}{B}, \frac{-C}{1+C} \right). \tag{D.40}$$

One notes that this invariant *cannot* be obtained from the one on *checkerboard* Potts model (namely $\mathcal{G}_1$) since the honeycomb limit of this invariant gives an expression *which is not S_3 symmetric* (see Appendix C.3).

The Kramers-Wannier dual of (D.35) and (D.36) read, respectively,

$$\begin{aligned} \mathcal{Y}^{\rm honey} &= -\frac{(C+2)\cdot(2+B)\cdot(A+2)\cdot(A-1)\cdot(C-1)\cdot(B-1)\cdot(1+2C)\cdot(1+2B)\cdot(1+2A)}{27 \cdot (C - 1 + A + B + 2BC + 2CA + 2BA + BCA)^3}, \\ \mathcal{M}^{\rm honey} &= \frac{3 \cdot (BCA - 1 - A - B - C)}{BCA - 1 + 2(BC + CA + BA) + A + C + B}. \end{aligned}$$

Expression $\mathcal{Y}^{\rm honey}$ is *invariant under the honeycomb inversion relation* (D.40), *while* $\mathcal{M}^{\rm honey}$ becomes $-\mathcal{M}^{\rm honey}$.

In the case of the (honeycomb) *three-state Potts model*, since the discrete group is *finite*, one can introduce many other invariants. For instance, by introducing:

$$K_{\rm honey} = \frac{ABC \cdot (1+A) \cdot (1+B) \cdot (1+C)}{(1+2A) \cdot (1+2B) \cdot (1+2C) \cdot (1-A) \cdot (1-B) \cdot (1-C) \cdot (A+2) \cdot (B+2) \cdot (C+2)},$$

one can easily verify that it also transforms like $K_{\rm honey} \to -K_{\rm honey}$ under the inversion relation of the *honeycomb* three-state Potts model (D.40). Of course one can also introduce many other invariants:

$$\begin{aligned} & \frac{(\mathcal{M}^{\rm honey} + 3) \cdot (\mathcal{M}^{\rm honey} - 3)}{(\mathcal{M}^{\rm honey})^2} \\ & \qquad = -4 \cdot \frac{(A + B + C + BC + CA + BA)(BCA - 1 + BC + CA + BA)}{(-1 - A - B - C + BCA)^2}, \\ & - 3^6 \cdot \frac{K_{\rm honey} \cdot \mathcal{Y}^{\rm honey}}{(\mathcal{M}^{\rm honey})^3} = -\frac{(1+A)(1+B)(1+C)ABC}{(ABC - 1 - A - C - B)^3} = \frac{A}{(A+1)^2} \cdot BC + \cdots . \end{aligned}$$

For B and C small, these invariants expand as follows:

$$Y^{\text{honey}} = -\frac{4\cdot(A+2)\cdot(1+2A)}{27(A-1)^2} + \frac{2\cdot(1+2A)\cdot(A+2)\cdot(A+1)\cdot(B+C)}{3(A-1)^3} + \cdots,$$
$$M^{\text{honey}} = 3\cdot\frac{1+A}{1-A} + \frac{6\cdot(1+A+A^2)\cdot(B+C)}{(1-A)^2} + \cdots, \tag{D.41}$$
$$K_{\text{honey}} = \frac{A\cdot(A+1)}{4\cdot(1+2A)(1-A)(A+2)}\cdot BC + \cdots.$$

One then gets:

$$z\cdot\frac{d}{dz}\ln(\Lambda) = \mathcal{P}_{\text{rod}}\cdot\left(1 - 18\cdot\frac{A}{1-A^2}\cdot(B+C) + \frac{T_2}{(1+A)^2\cdot(1-A^3)^2}\cdots\right), \tag{D.42}$$

where

$$T_2 = -A^2\cdot\left(18A^5 - 99A^4 - 218A^3 - 373A^2 - 254A - 135\right)\cdot(B+C)^2$$
$$+ (1-A)\cdot(A+1)\cdot\left(18A^6 + 17A^5 - 3A^4 - 54A^3 - 69A^2 - 53A - 18\right)\cdot BC.$$

It is necessary to introduce other (resummed expansion well-suited) polynomials. Another additional invariant is particularly interesting to introduce because it yields a nice (B, C small) expansion:

$$Q_1 = \frac{\left(\left(M^{\text{honey}}\right)^2 + 27\cdot Y^{\text{honey}} - 1\right)}{3\cdot(M^{\text{honey}})^2}$$
$$= \frac{q_1}{(-1-A-B-C+BCA)^2(-1+A+B+C+2BC+2CA+2BA+ABC)}, \quad \text{with}$$
$$q_1 = -9ABC - 2(BC+CA+BA) - 2B^2C^2 - 2B^2C - 2BC^2 - 2C^2A^2 - 2CA^2 - 2B^2A^2$$
$$- 2BA^2 - 2C^2A - 3BCA^2 - 3BC^2A - 3B^2CA + 3(B^2C^2A + BC^2A^2 + B^2CA^2)$$
$$+ 2A^3BC + 2B^3CA + 2ABC^3 + 2BC^2A^3 + 2B^2CA^3 + 2B^2C^2A^3 + 2B^2C^3A$$
$$+ 2BC^3A^2 + 2A^2C^3B^2 + 2B^3C^2A^2 + 2B^3C^2A - 2B^2A + 2B^3CA^2 + 9B^2C^2A^2.$$

Q_1 expands as follows:

$$Q_1 = \frac{2\cdot A}{1-A^2}\cdot(C+B) + \frac{(A-1)\left(2A^3+A^2-5A-2\right)}{\left(1-A^2\right)^2}\cdot BC + \frac{2A^2(A+5)(C+B)^2}{\left(1-A^2\right)^2}\cdots.$$

An invariant which expands like (see the right-hand side of (D.42)):

$$1 - 18\cdot\frac{A}{1-A^2}\cdot(B+C) + \cdots,$$

and which can cancel the $(\mathcal{C}_0^{\text{honey}})^{-3}$ singularity of $\mathcal{P}_{\text{rod}}$, is for instance:

$$Q_3 = (1+3\cdot Q_1)^{-3}$$
$$= \frac{(-1-A-B-C+BCA)^6(-1+A+B+C+2(BC+CA+BA)+BCA)^3}{q_3^3},$$

where q_3 reads:

$$q_3 = -1 - 2(BC+CA+BA) - 2\left(B^2C^2 + C^2A^2 + B^2A^2\right) + BC^2 + B^2C$$
$$+ CA^2 + BA^2 + C^2A + B^2A - (A+B+C) + \left(A^2+B^2+C^2\right) + A^3 + B^3 + C^3$$
$$+ 3B^2C^2A + 3BC^2A^2 + 3B^2CA^2 + 12B^2C^2A^2$$
$$+ 5\left(A^3BC + B^3CA + C^3BA\right) + 2BC^2A^3 \tag{D.43}$$
$$+ 2B^2C^3A + 2BC^3A^2 + 5\left(B^2C^2A^3 + B^2C^3A^2 + B^3C^2A^2\right) + 2B^3C^2A + 2B^2CA^3$$
$$+ 2B^3CA^2 + 2B^3C + 2B^3A + 2C^3B + 2C^3A + 2A^3C + 2A^3B + 2B^3C^3A^2$$
$$+ 2B^2C^3A^3 + 2B^3C^2A^3 + B^3C^3A^3 - 6ABC + 3ABC\cdot(A+B+C).$$

The expansion of Q_3 is remarkably simple:

$$Q_3 = 1 + 18 \cdot \frac{A}{1-A^2} \cdot (C+B) + \frac{9\left(2A^3+22A^2\right)\cdot(C+B)^2}{\left(1-A^2\right)^2} + \frac{9(A-1)\cdot\left(2A^3+A^2-5A-2\right)\cdot BC}{\left(1-A^2\right)^2} + \cdots .$$

One thus exchanges $\mathcal{P}_{\rm rod}$ for a new invariant, denoted $L_{\rm ast}$:

$$L_{\rm ast} = \mathcal{P}_{\rm rod} \cdot Q_3 = 2 \cdot \frac{\mathcal{P}_N}{q_3^3}. \tag{D.44}$$

Amazingly, one verifies that $L_{\rm ast}$ is such that, *restricted to the critical variety* of the honeycomb lattice, namely (D.39), *becomes equal to* $2/27$.

Invariant, $L_{\rm ast}$ expands as follows:

$$L_{\rm ast} = \frac{2A(B+CA)(C+BA)}{\left(1-A^2\right)^2} + \cdots = \frac{2A^3(C+B)^2}{\left(1-A^2\right)^2} + \frac{2A\cdot BC\cdot(1-A)^2}{\left(1-A^2\right)^2} + \cdots .$$

In term of this new invariant $L_{\rm ast}$, relation (D.42) reads:

$$z \cdot \frac{d}{dz}\ln(\Lambda) = L_{\rm ast}\cdot\left(1 + S_2(L_{\rm ast}, \mathcal{J}_i, \dots) + \cdots\right), \tag{D.45}$$

where $\mathcal{J}_i$ denotes various well-suited invariants, and S_2 is of order two in B and C.

Recalling (D.33), one gets from (D.45):

$$\begin{aligned} M - 1 &= -\frac{3}{2}\cdot z\cdot\frac{d}{dz}\ln(\Lambda) = (1-k_{\rm Potts})^{1/9} - 1 + \cdots \\ &= -\frac{k_{\rm Potts}}{9} - \frac{4(k_{\rm Potts})^2}{81} + \cdots = -\frac{3}{2}\cdot L_{\rm ast} + \cdots . \end{aligned} \tag{D.46}$$

The $L_{\rm ast} = -2/27$ limit corresponds to $k_{\rm Potts} = 1$. In order to get a simple expression for the spontaneous magnetization at criticality, one thus finally introduces:

$$k_{\rm Potts}^{\rm honey} = \frac{27}{2}\cdot L_{\rm ast}, \qquad \text{such that } \mathcal{C}_0^{\rm honey} = 0 \Rightarrow k_{\rm Potts}^{\rm honey} = 1. \tag{D.47}$$

Conversely,

$$k_{\rm Potts}^{\rm honey} - 1 = \frac{\mathcal{C}_0^{\rm honey}\cdot\left(\mathcal{C}_1^{\rm honey}\cdot\mathcal{C}_2^{\rm honey}\cdot\mathcal{C}_3^{\rm honey}\right)^2}{q_3^3}, \tag{D.48}$$

with (see (7.9)):

$$\begin{aligned} \mathcal{C}_0^{\rm honey} &= -1 + 2\cdot(BC+CA+BA) + (A+B+C) + ABC, \\ \mathcal{C}_1^{\rm honey} &= (BC-1)\cdot A^2 + 2\cdot(B+C+BC)\cdot A + (B+C) + 1, \dots . \end{aligned} \tag{D.49}$$

One can actually verify that this set of three last additional varieties (D.49) (see also (7.9)) (which break the $\mathcal{S}_3$ symmetry of permutation of x_1, x_2, and x_3) have simple covariance properties with respect to the inversion relation of the *honeycomb* model.

These three "cousin varieties" of the critical variety $\mathcal{C}_0^{\rm honey} = 0$ are reminiscent of the situation one has in the case of the *Ising honeycomb* model (see also (C.16) for the checkerboard model). In the Ising case, condition $k_h^2 = 1$ reads (using (D.33)):

$$\begin{aligned} k_h^2 - 1 &= -\frac{\mathrm{F}_0^{\rm acto}\cdot\left(\mathrm{F}_1^{\rm acto}\cdot\mathrm{F}_2^{\rm acto}\cdot\mathrm{F}_3^{\rm acto}\right)}{T_{\rm riv}^2}, \qquad \text{with} \\ \mathrm{T}_{\rm riv} &= (A-1)(1+A)(B-1)(B+1)(C-1)(1+C), \\ \mathrm{F}_0^{\rm acto} &= ABC - (BC+AB+AC) - (A+B+C) + 1, \\ \mathrm{F}_1^{\rm acto} &= (AB-1+A+B)\cdot C + 1 + (A+B) - BA, \\ \mathrm{F}_2^{\rm acto} &= (BC-1+B+C)\cdot A + 1 + (B+C) - BC, \\ \mathrm{F}_3^{\rm acto} &= (AC-1+A+C)\cdot B + 1 + (A+C) - AC. \end{aligned} \tag{D.50}$$

D.3. INVARIANT OF THE HONEYCOMB LATTICE VERSUS RESUMMED EXPANSIONS. FINE TUNING

Let us recall D_{er}:

$$D_{\mathrm{er}} = (1 - k_{\mathrm{Potts}})^{1/9} - 1. \tag{D.51}$$

The expansion of this very expression, for B and C small, reads:

$$D_{\mathrm{er}} = \frac{2A(C+BA)(B+CA)}{(1-A^2)^2} + \cdots = \frac{2A\cdot(1-A)^2}{(1-A^2)^2}\cdot BC + \frac{2A^2}{(1-A^2)^2}\cdot(B+C)^2 + \cdots.$$

On the other side, $z\cdot\frac{d}{dz}\ln(\Lambda)$ has the following resummed low-temperature expansion for the three-state Potts model on the honeycomb lattice (see (D.34)):

$$\begin{aligned} z\cdot\frac{d}{dz}\ln(\Lambda) = {} & \frac{2A(C+BA)(B+CA)}{(1-A^2)^2} + \frac{2A^2(A-1)\left(A^2-4A-1\right)}{(1-A^2)^3}\cdot(B+C)\cdot BC \\ & + \frac{4A^2}{(1-A^2)^3}\cdot(C+B)^3 + \cdots. \end{aligned} \tag{D.52}$$

One verifies that:

$$z\cdot\frac{d}{dz}\ln(\Lambda) - D_{\mathrm{er}} = \left(\frac{2A\cdot(1-A)^2}{(1-A^2)^2}\cdot BC\right)\cdot\frac{-\alpha_2\cdot A}{(1-A^3)^2\cdot(1+A)^2}, \tag{D.53}$$

where α_2 reads:

$$\begin{aligned} \alpha_2 = {} & A\cdot\left(3A^4+4A^3+5A^2+4A+3\right)\cdot\left(B^2+C^2\right) \\ & + \left(2A^6+3A+9A^2+10A^3+9A^4+3A^5+2\right)\cdot BC, \end{aligned} \tag{D.54}$$

or equivalently at this order of the expansions (namely order two in B and C see (D.45)):

$$z\cdot\frac{d}{dz}\ln(\Lambda) = D_{\mathrm{er}}\cdot\left(1 - \frac{A\cdot\mathcal{W}_2}{(1+A)^2\cdot(1-A^3)^2}\right) = D_{\mathrm{er}}\cdot(1+\lambda\cdot D_{\mathrm{er}} + \mathcal{R}_{\mathrm{est}} + \cdots), \tag{D.55}$$

with

$$\mathcal{W}_2 = A\left(3A^4+4A^3+5A^2+4A+3\right)\cdot(B+C)^2 + \left(A^2-A+1\right)\cdot\left(2A^2+3A+2\right)(A-1)^2\cdot BC.$$

At this step, without additional information, there is some ambiguity in the determining of λ. If one converts the two following quantities (where P and S denote, respectively, BC and $B+C$), to partial fraction form:

$$\begin{aligned} -\frac{A\cdot\mathcal{W}_2}{(1+A)^2\cdot(1-A^3)^2} = {} & \frac{3S^2-12P}{4(A+1)} - \frac{3S^2-12P}{4(A+1)^2} - \frac{19S^2}{36(A-1)} - \frac{19S^2}{36(A-1)^2} \\ & - \frac{6S^2-18P+2AS^2-9PA}{9\cdot(A^2+A+1)} + \frac{2S^2-6P+2AS^2-6PA}{3\cdot(A^2+A+1)^2}, \quad \text{and} \end{aligned}$$

$$2\cdot A\cdot\frac{(C+BA)\cdot(B+CA)}{(1-A^2)^2} = \frac{S^2}{2(A-1)} + \frac{S^2}{2(A-1)^2} - \frac{S^2-4P}{2(A+1)} + \frac{S^2-4P}{2(A+1)^2},$$

two values of λ pop out: $\lambda = -3/2$ and $\lambda = -19/18$. For the first value of λ one gets a particularly simple expression for $\mathcal{R}_{\mathrm{est}}$, namely (7.8).

D.4. GROUP-INVARIANT VERSUS LOW-TEMPERATURE EXPANSIONS FOR THE HONEYCOMB LATTICE

It is tempting to compare the expansion of (7.5) with the *low-temperature* expansion of the spontaneous magnetization in the case of the *honeycomb lattice*. The isotropic limit of (7.4) reads:

$$
\begin{aligned}
k_{\text{Potts}}^{\text{honey}} &= \frac{27 \cdot (A+1)^3 \cdot A^3}{(1-3A^2-A^3)^3}, \qquad \text{and} \\
(1-k_{\text{Potts}}^{\text{honey}})^{1/9} - 1 &= 2A^3 + 6A^4 + 24A^5 + 86A^6 + 324A^7 + 1224A^8 \\
&\quad + 4722A^9 + 18432A^{10} + \cdots .
\end{aligned} \tag{D.56}
$$

One cannot find directly low-temperature expansion for the spontaneous magnetization of the three-state honeycomb lattice in the literature but rather *high field* expansions [59]. However, from these high field expansions, one can get (by derivation with respect to the magnetic field) an expansion for the spontaneous magnetization, namely:

$$
\begin{aligned}
-\frac{2}{3} \cdot (M_{\text{honey}} - 1) &= z \cdot \frac{d}{dz} \ln(\Lambda) \\
&= 2A^3 + 6A^4 + 24A^5 + 82A^6 + 300A^7 \\
&\quad + 1176A^8 + 4434A^9 + 15720A^{10} + \cdots .
\end{aligned} \tag{D.57}
$$

Taking into account the fact that (D.57) is basically a *high field* expansion [59] and not a low temperature one (only the first terms are correct), one remarks, however, a quite good agreement between these two expansions.

D.5. COMMENTS ON $K_{\text{Potts}}^{\text{honey}}$ FOR THE THREE-STATE HONEYCOMB POTTS MODEL

The *isotropic limit* $C = B = A$, of the invariant of the honeycomb lattice (7.4), is simple[19]:

$$
k_{\text{Potts}}^{\text{honey}} = \left(\frac{3A(A+1)}{-A^3 - 3A^2 + 1}\right)^3 = (k_{\text{iso}})^3. \tag{D.58}
$$

Condition $k_{\text{Potts}} = 1$ reads two conditions:

$$
k_{\text{iso}} - 1 = 1 - 3A - 6A^2 - A^3 \quad \text{and} \quad (k_{\text{iso}})^2 + k_{\text{iso}} + 1 = \left(1 + A + A^2\right)^3. \tag{D.59}
$$

The first condition $k_{\text{iso}} - 1 = 0$ is the well-known critical condition of the *isotropic honeycomb* lattice in terms of the low-temperature variables.

The Kramers-Wannier dual of (D.58) reads:

$$
k_{\text{KW}}^{\text{honey}} = \left(\frac{3A^*(A^*+1)}{-(A^*)^3 - 3(A^*)^2 + 1}\right)^3 = \left(-\frac{(1+2A)(2+A)(A-1)}{3A - 1 + 6A^2 + A^3}\right)^3. \tag{D.60}
$$

Let us recall the rational parameterization (4.35). Invariant $k_{\text{Potts}}^{\text{honey}}$ has a rather involved form when written in terms of the u_1, u_2, u_3 variables. The simplest part is the denominator, which reads $(D_{u_1,u_2,u_3})^3$ with:

$$
\begin{aligned}
D_{u_1u_2u_3} &= 9 \cdot \left(u_1^3 + u_2^3 + u_3^3 - 3u_2u_3u_1\right) \cdot t^3 + 9 \cdot \left(2u_2^3u_3^3 + 2u_3^3u_1^3 + 2u_2^3u_1^3 + 3u_2^2u_3^2u_1^2\right) \cdot t^2 \\
&\quad - 9 \cdot \left(2u_1^3 + 2u_3^3 + 2u_2^3 + 3u_1u_2u_3\right) \cdot t + 9 \cdot \left(3u_2^2u_3^2u_1^2 - u_3^3u_1^3 - u_2^3u_1^3 - u_2^3u_3^3\right).
\end{aligned}
$$

[19]This simple form for the isotropic limit of the model enables us to write down very simply the unit circle $|k_{\text{Potts}}| = 1$, namely $|k_{\text{iso}}| = 1$. However, one should not expect the unit circle to play a key role for the *honeycomb* Potts model [64]. This used to happen for the *Ising* model as a consequence of the *elliptic* parameterization that we *do not expect here*.

In the *square limit* $C = 0$, invariant $k^{\text{honey}}_{\text{Potts}}$ becomes:

$$k^{\text{square}}_{\text{Potts}} = \frac{27A^2B^2(A+B+2BA+1)(A+1)^2(B+1)^2(B-A-1)(B-A+1)}{(-1-2BA+BA^2+B^2A-2B^2A^2+2B^3A+2A^3B-B-A+A^2+B^2+A^3+B^3)^3}. \tag{D.61}$$

It has been noted, in Section 7, that this expression, inherited from the one for the honeycomb lattice, does not yield a "nice" representation of the Kramers-Wannier duality (like for instance, $k \longrightarrow k/(\alpha \cdot k - 1)$). In the isotropic case, (D.61) becomes (7.11) and the Kramers-Wannier dual of (7.11) is given by (7.13). One can eliminate A between these two expressions to get a relation between (7.11) and (7.13). Actually, the resultant yields (the square of):

$$\begin{aligned}
R = {} & 2^{39} \cdot k^6K^6 - 141 \cdot 2^{33} \cdot k^5K^5 \cdot (K+k) \\
& + 3 \cdot 2^{26}k^4K^4 \cdot \left(4429\left(k^2+K^2\right) + 9366kK\right) \\
& - 2^{30} \cdot k^3K^3 \cdot (K+k) \cdot \left(210748 \cdot \left(k^2+K^2\right) + 213467kK\right) \\
& + 3 \cdot 2^{13} \cdot k^2K^2 \cdot \left(48719 \cdot \left(K^4+k^4\right) - 24169286 \cdot \left(k^3K+kK^3\right) - 47735079k^2K^2\right) \\
& - 3 \cdot 2^7 \cdot kK \cdot (K+k) \cdot \left(5687 \cdot \left(k^4+K^4\right)\right. \\
& \left. - 805800479 \cdot \left(k^3K+kK^3\right) - 2859101424 \cdot k^2K^2\right) \\
& + 2634567894 \cdot \left(kK^5+k^5K\right) - 1398272669644k^3K^3 \\
& - 448223873571 \cdot \left(k^4K^2+k^2K^4\right) + 1331 \cdot \left(K^6+k^6\right) \\
& + 3 \cdot (K+k) \cdot \left(141113 \cdot \left(k^4+K^4\right) - 42754046948 \cdot \left(k^3K+kK^3\right) + 193362094422 \cdot k^2K^2\right) \\
& + 98553471948k^2K^2 + 246865154424 \cdot \left(kK^3+k^3K\right) + 50285730 \cdot \left(k^4+K^4\right) \\
& + 2^7 \cdot (K+k) \cdot \left(20524825 \cdot \left(k^2+K^2\right) - 3221730802kK\right) \\
& + 3 \cdot 2^{10} \cdot \left(124093734 \cdot kK + 16041625\left(k^2+K^2\right)\right) - 2^{15} \cdot 5^4 \cdot 5037 \cdot (K+k) + 5^8 \cdot 2^{17}.
\end{aligned}$$

APPENDIX E
TOWARDS ALGEBRAIC INVARIANTS FOR THE TRIANGULAR LATTICE

E.1. THE TRIANGULAR LATTICE: SINGLED OUT ALGEBRAIC VARIETIES

From the critical variety and the three additional varieties for the honeycomb lattice (see (7.9)) one can deduce (using the Kramers-Wannier duality) the critical variety and three singled out varieties for the *triangular* lattice:

$$\begin{aligned}
1 - ABC - (BC + AC + BA) &= 0, \qquad \text{and} \\
1 + 2A - BC + 2AC + 2BA + BCA^2 + BA^2 + CA^2 &= 0, \\
1 + 2C + 2BC + 2AC - BA + C^2A + BC^2A + BC^2 &= 0, \\
1 + 2B + 2BC - AC + 2BA + B^2A + B^2CA + B^2C &= 0,
\end{aligned} \tag{E.62}$$

or, in terms of the x_i's:

$$\begin{aligned}
& x_1x_2x_3 - (x_1+x_2+x_3) - 1 = 0, \qquad \text{and} \\
& C_1^{\text{triang}} = x_2x_3x_1^2 + 2x_2x_3x_1 - x_1^2 + 2x_2x_1 + 2x_1x_3 + 1 + x_3 + x_2 \\
& \qquad + (x_1+2) \cdot (x_2x_3x_1 - 1 - x_1 - x_2 - x_3) + 3(x_1+1)(1+x_3+x_2) = 0, \\
& C_2^{\text{triang}} = (x_2+2) \cdot (x_2x_3x_1 - 1 - x_1 - x_2 - x_3) + 3 \cdot (x_2+1)(x_1+x_3+1) = 0, \\
& C_3^{\text{triang}} = (x_3+2) \cdot (x_2x_3x_1 - 1 - x_1 - x_2 - x_3) + 3 \cdot (x_3+1) \cdot (x_1+x_2+1) = 0.
\end{aligned} \tag{E.63}$$

In the isotropic limit $C_1^{\text{triang}} = 0$ becomes $(1+u+u^2)^2 = 0$. In the anisotropic square limit (E.63) yields:

$$\begin{aligned}
C_1^{\text{triang}} &= x_2x_1^2 + 4x_2x_1 - x_1^2 + 2x_1 + 2 + x_2, \\
C_2^{\text{triang}} &= x_2^2x_1 + 4x_2x_1 - x_2^2 + 2x_2 + 2 + x_1, \\
C_3^{\text{triang}} &= x_1x_2 + x_1 + x_2.
\end{aligned} \tag{E.64}$$

E.2. TUTTE-BERAHA NUMBERS

For the *Ising model*, the additional varieties for the triangular and honeycomb lattices read, respectively,

$$AB - AC + BC + 1 = 0 \quad \text{or} \quad u_3 u_1 + u_2 \cdot t = 0, \text{ with}$$
$$t^4 = -1 \quad \text{and} \quad u_2 u_3 - t^3 \cdot u_1 = 0, \text{ with } t^4 = -1.$$

More generally, considering the inversion relation of the *triangular* lattice:

$$(u_1, u_2, u_3) \longrightarrow \left(\frac{t^2}{u_1}, \frac{1}{t^2 \cdot u_2}, \frac{1}{t^2 \cdot u_3} \right), \tag{E.65}$$

and a variety of the form:

$$u_1 \cdot u_2 \cdot u_3 + \alpha \cdot u_1^M = 0, \tag{E.66}$$

it is straightforward to see that (E.66) is invariant by the three inversion relations (E.65) if:

$$t^{4 \cdot M} = 1 \quad \text{and} \quad \alpha^2 \cdot t^2 \cdot t^{2 \cdot M} = 1, \qquad \text{that is } \alpha = \pm t^{M-1}. \tag{E.67}$$

Of course this can easily be generalized to the checkerboard lattice.

This makes clear that such varieties only occur for Tutte-Beraha numbers.

E.3. CRITICALITY OF THESE ADDITIONAL VARIETIES?

Since, in the case of the Ising model, three other varieties similar[20] to (E.63), occur in addition to the critical variety which *also happen to be critical* (see also Appendix C.1), it is natural to wonder if the additional varieties (E.63) could not be also critical varieties. If one takes for granted expression (7.5) to represent the *dominant* singular behaviour of the spontaneous magnetization, one expects the magnetic critical exponent β, corresponding to these three new critical varieties $\mathcal{C}_i^{\text{honey}}$ (and also $\mathcal{C}_i^{\text{triang}}$), to be $\beta = 2/9$. If one assumes that the (well-known) relations for the critical (or tricritical) exponents of Potts models are still valid (see relations (5.23) in [39]), in particular:

$$\alpha = \frac{1 - 8 \cdot \beta}{1 - 6 \cdot \beta}, \tag{E.68}$$

one gets a thermal exponent: $\alpha = 5/3$. In order to examine the critical character of the additional varieties (E.63), Monte-Carlo calculations have been performed on the anisotropic edge *triangular* Potts model. Unfortunately, Monte-Carlo calculations seem to indicate that the points of any of the three varieties (E.63) *are not* $\beta = 2/9$-critical points. Note however, that $\mathcal{C}_3^{\text{triang}} = 0$ in (E.64) is nothing but the *antiferromagnetic critical condition* of Baxter [65] for the square lattice in the $q = 3$ limit:

$$(x_1 + 1) \cdot (x_2 + 1) = 4 - q. \tag{E.69}$$

It is thus possible, in view (E.64) and (E.70), to imagine that these additional ($\mathcal{S}_3$-symmetry breaking) varieties could however be critical varieties with other exponents.

E.4. ALGEBRAIC INVARIANTS FOR THE TRIANGULAR LATTICE

For the anisotropic *triangular* model we have (at least) four group-invariants:

$$\mathcal{K} = \frac{ABC \cdot (1 + A)(1 + B)(1 + C)}{(1 + 2A)(1 + 2B)(1 + 2C)(1 - A)(1 - B)(1 - C)(A + 2)(B + 2)(C + 2)}, \tag{E.70}$$

which transforms into its opposite $-\mathcal{K}$ under the inversion relation of the triangular lattice,

$$M_{\text{triang}} = -\frac{-1 - 2(A + B + C) - (BC + CA + BA) + ABC}{BC + CA + BA - 1 + BCA}, \tag{E.71}$$

[20] Breaking in particular spontaneously the symmetry of the lattice.

which transforms into its opposite $-M_{\mathrm{triang}}$, under the inversion relation of the triangular lattice and:

$$Y_{\mathrm{triang}} = -\frac{ABC\cdot(1+A)(1+B)(1+C)}{(BC+CA+BA-1+BCA)^3}, \tag{E.72}$$

and also (form (4.49) and M_{triang}):

$$\mathcal{G}_{\mathrm{triang}} = -\frac{N_{\mathrm{triang}}}{27(BC+CA+BA-1+BCA)^3(-1-2(A+B+C)-BC-CA-BA+ABC)^6},$$

where

$$\begin{aligned} N_{\mathrm{triang}} = {} & 2\cdot(1+2BCA+3A-BC+2CA+2BA)(1+2BCA+3B+2BC-CA+2BA)\\ & \times(1+2BCA+3C+2BC+2CA-BA)\cdot(2+BCA+3B+BC+CA+BA)\\ & \times(2+BCA+3A+BC+CA+BA)\cdot(2+BCA+3C+BC+CA+BA)\\ & \times(-1+BC+CA-2BA+ABC)(-1+BC-2CA+BA+ABC)\\ & \times(-1-2BC+CA+BA+BCA). \end{aligned}$$

Y_{triang} and $\mathcal{G}_{\mathrm{triang}}$ are invariant under the *inversion relation of the triangular lattice*. Curiously $\mathcal{K}$ is also covariant *for the inversion relation of the honeycomb lattice* (see K_{honey} in Appendix D.2): one can easily verify that it transforms like $\mathcal{K} \to -\mathcal{K}$ under the inversion relation of the *honeycomb three-state* Potts model.

The expansion of three of these (up to a sign) invariants yields:

$$\begin{aligned} \mathcal{K} &= \frac{AC\cdot(1+A)(1+C)}{2\cdot(1+2A)\cdot(1+2C)\cdot(A-1)\cdot(C-1)\cdot(A+2)(C+2)}\cdot B+\cdots,\\ M_{\mathrm{triang}} &= \frac{AC+2A+2C+1}{AC-1}+\cdots,\\ Y_{\mathrm{triang}} &= -\frac{AC\cdot(1+A)(1+C)}{(AC-1)^3}\cdot B+\cdots. \end{aligned} \tag{E.73}$$

One should note that these three invariants *yield only* $1-A\cdot C$ singularities, or simple singularities like $1-A$, $C+2\ldots$. One cannot get the $1-A^N\cdot C^N$ singularities known to occur on resummed expansions [10,25] in this way. The last invariant $\mathcal{G}_{\mathrm{triang}}$ yields more involved singularities, for instance $CA+2A+2C+1$ singularities.

The resummed expansion of the spontaneous magnetization of the edge three-state Potts model on the anisotropic triangular lattice corresponds to (see (7.12)):

$$z\cdot\frac{d}{dz}\ln(\Lambda) = \frac{2A^2C^2}{(1-A^2C^2)^2}\cdot B^2+\frac{4A^4C^4}{(1-A^2C^2)^3}\cdot B^3+\cdots.$$

More precisely, in the $z=1$ *triangular limit* (namely $D=0$), expression (C.28) becomes:

$$\begin{aligned} z\cdot\frac{d}{dz}\ln(\Lambda) = {} & (q-1)\left(\frac{A^2C^2}{1-A^2C^2}+\frac{A^4C^4}{(1-A^2C^2)^2}\right)B^2\\ & +(q-1)\cdot(q-2)\cdot\left(\frac{2A^4C^4}{(1-A^2C^2)^2}+\frac{2A^6C^6}{(1-A^2C^2)^3}\right)\cdot B^3\\ & +\frac{2(q-1)A^2C^2\cdot(A^2+C^2+A^4C^2+C^4A^2+3A^2C^2+A^4C^4)}{(1-A^2C^2)^4}\cdot B^4\\ & +\frac{2(q-1)\cdot(q-2)\left(3A^2C^2-A^4C^4-A^6C^5-A^5C^6-A^7C^7+A+C-A^5C^5\right)A^3C^3}{(1-A^2C^2)^3(1-A^3C^3)^2}\cdot B^4\\ & +\frac{3(q-1)\cdot(q-2)^2A^6C^6}{(1-A^2C^2)^4}\cdot B^4-\frac{(q-1)^2A^4C^4\left(2A^2C^2+7\right)}{(1-A^2C^2)^4}\cdot B^4. \end{aligned} \tag{E.74}$$

In contrast with the honeycomb lattice, there are many ways to get a resummed expansion like (E.74), in terms of group invariants of the triangular lattice. For the honeycomb lattice, invariant (7.4) was built from the modification of an invariant which corresponded to taking the *product over the group of the "order varieties"*. It seems that the equivalent of (7.4), but for the triangular lattice, could rather be an invariant corresponding to a modification of Y_{triang}.

Thus, coming back to the checkerboard lattice, we could therefore, have (at least) *two* invariants, one which "dominates" in the honeycomb limit and the other in the triangular limit.

APPENDIX F
OCCURRENCE OF N^{th} ROOTS OF UNITY ON ANISOTROPIC SQUARE LATTICE RUBBER BANDS

In order to understand the occurrence of $1 - A^N$ singularities in the resummed expansions of the edge Potts models, let us consider an anisotropic square lattice q-state Potts model on *rubber bands*, for instance the one for which the transfer matrices are represented as 3×3 matrices in [6,63].

After extracting the leading low-temperature terms, the largest eigenvalue Λ can be seen to be solution of the following algebraic equation of third degree (characteristic polynomial of the 3×3 transfer matrix):

$$\begin{aligned}
&\Lambda^3 + C_2 \cdot \Lambda^2 + C_1 \cdot \Lambda + C_0 = 0, \qquad \text{with}\\
&C_0 = A^5(B-1)^5(Bq+1-B)^4,\\
&C_1 = A^2(B-1)^2(Bq+1-B)\cdot c_1,\\
&C_2 = -A^3B^3\cdot q^3 + 3A^2B^2(2BA - B - A)\cdot q^2\\
&\quad + \left(15A^3B^2 - 7A^2B^2 - 14B^3A^3 + 11B^3A^2 - B^3 - 3A^3B - BA^2\right)\cdot q\\
&\quad + (B-1)\cdot\left(B^2 - 11A^2B^2 + 13A^3B^2 + BA^2 - 8A^3B + B + A^3 + A^2 + 1\right),
\end{aligned} \tag{F.75}$$

where

$$\begin{aligned}
c_1 &= A^3B^3\cdot q^3 - AB^2(6BA^2 - 2B - 1 - 3A^2)\cdot q^2\\
&\quad + \left(11B^3A^3 - AB^2 + B^3 - 7AB^3 + 3B^2 - 12A^3B^2 + 3A^3B + 2BA\right)\cdot q\\
&\quad - (B-1)\cdot\left(7A^3B^2 - 5AB^2 + B^2 - 5A^3B + 4B - 5BA + A^3 + A + 1\right).
\end{aligned}$$

The expansion of Λ, when B is small, reads:

$$\begin{aligned}
\Lambda &= 1 + \frac{3\cdot(q-1)\cdot W_2}{1-A^2}\cdot B^2 + (q-1)\cdot\sum_{n=3}^{\infty}\frac{W_n}{(1-A^2)^{n-1}}\cdot B^n, \qquad \text{with}\\
W_2 &= A^2,\\
W_3 &= (3q-5)\cdot A^4 + 4A^2 + 1,\\
W_4 &= 3q\cdot(q-2)\cdot A^6 - 3\left(A^4 - A^2 - 1\right)A^2\\
&\quad + \frac{3A^2(q-2)\cdot\left(7A^2 + 8A^3 - 2A^5 + 5A - 2A^6 + 2 + 3A^4\right)}{A^2+A+1},\\
W_5 &= 3A^8(q-2)\cdot q^2 - 3A^2\cdot\left(A^6 + 5A^4 + 3A^2 + 1\right)\\
&\quad + 3\cdot(q-2)\cdot\frac{3A^2\cdot Q_{52}}{(A^2+A+1)} + 3\cdot(q-2)\cdot\frac{3A^2\cdot Q_{53}}{(A^2+A+1)^2}, \qquad \text{with}\\
Q_{53} &= 1 - 4A - 15A^2 - 46A^3 - 102A^4 - 116A^5 - 107A^6 - 72A^7 - 19A^8 + 2A^9 + A^{10},\\
Q_{52} &= -q\cdot A\cdot\left(4A^9 + 8A^8 + A^7 - 2 - 28A^6 - 50A^5 - 64A^4 - 55A^3 - 30A^2 - 9A\right).
\end{aligned}$$

Note, that W_7 also has a $(1 + A + A^2)^4$ singularity. In fact the characteristic equation reads:

$$\begin{aligned}
&P(\Lambda) = \Lambda^3 + C_2\cdot\Lambda^2 + C_1\cdot\Lambda + C_0 = 0, \qquad \text{where}\\
&C_0 = C_{00} + C_{01} + \cdots + C_{09},\\
&C_1 = C_{10} + C_{11} + \cdots + C_{16},\\
&C_2 = C_{20} + C_{21} + C_{22} + C_{23},
\end{aligned} \tag{F.76}$$

where the C_{ij}'s are of order j in B, reads successive equations of the form:

$$R_{\text{est}}(W_1, W_2, \ldots, W_{n-1}) + (3 + C_{10} + 2 \cdot C_{20}) \cdot W_n = 0. \tag{F.77}$$

The occurrence of a new singularity is thus related to the following combination of the C_{ij}'s:

$$3 + C_{10} + 2 \cdot C_{20} = \left(1 - A^2\right) \cdot \left(1 - A^3\right). \tag{F.78}$$

This key expression can be seen as the derivative of the characteristic polynomial (F.76) with respect to Λ, in the $B = 0$ limit.

$$3 + C_{10} + 2C_{20} = \frac{d}{d\Lambda} P(\Lambda)(B = 0). \tag{F.79}$$

Let us note, however, that for the first coefficient, W_2, there is the following equation:

$$\left(A^2 - 1\right) \cdot (C_{22} + C_{02} + C_{12}) - 3B^2 \cdot (3 + C_{10} + 2C_{20}) \cdot (q - 1) \cdot W_2 = 0,$$

where

$$C_{22} + C_{02} + C_{12} = 3 \cdot (q - 1) \cdot A^2 B^2 \cdot (A^3 - 1). \tag{F.80}$$

Therefore, one sees that the singularity $1 - A^3$ *cancels out for this first coefficient* W_2. A mechanism where all the N^{th} root of unity occur, clearly needs to consider "polynomial" relations of infinite degree.

REFERENCES

1. M.P. Bellon, J.-M. Maillard and C.-M. Viallet, Integrable Coxeter groups, *Physics Letters* **A159**, 221–232 (1991).
2. M.P. Bellon, J.-M. Maillard and C.-M. Viallet, Higher dimensional mappings, *Physics Letters* **A159**, 233–244 (1991).
3. Y.G. Stroganov, A new calculation method for partition functions in some lattice models, *Phys. Lett.* **A74**, 116–118 (1979).
4. R.J. Baxter, Exactly solved models in statistical mechanics, In *Proc. of the 1980 Enschede Summer School: Fundamental Problems in Statistical Mechanics V*, North-Holland, Amsterdam, (1981).
5. R.J. Baxter, The inversion relation method for some two-dimensional exactly solved models in lattice statistics, *J. Stat. Phys.* **28**, 1–41 (1982).
6. J.-M. Maillard, The inversion relation: Some simple examples (I) and (II), *Journal de Physique* **46**, 329–335 (1984).
7. J.-M. Maillard and R. Rammal, Some analytical consequences of the inverse relation for the Potts model, *J. Phys.* **A16**, 353–367 (1983).
8. V.F.R. Jones, Baxterization, Proc. of 'Yang-Baxter equations, conformal invariance and integrability in statistical mechanics and field theory', Canberra, *Int. J. Mod. Phys.* **B4**, 701–713 (1990).
9. M.T. Jaekel and J.-M. Maillard, Symmetry relations in exactly soluble models, *J. Phys.* **A15**, 1309–1325 (1982).
10. M.T. Jaekel and J.-M. Maillard, Inverse functional relations on the Potts model, *J. Phys.* **A15**, 2241–2257 (1982).
11. C. Viallet, Baxterization, dynamical systems, and the symmetries of integrability, In *Integrable Models and Strings, Lecture Notes in Physics*, (Edited by A. Alekseev *et al.*), Volume 436, pp. 11–35, Springer-Verlag, (1993).
12. R.J. Baxter, J.H.H. Perk and H. Au-Yang, New solutions of the star-triangle relations for the chiral Potts model, *Phys. Lett.* **A128**, 138–148 (1988).
13. D. Hansel and J.-M. Maillard, Symmetries of models with genus > 1, *Phys. Lett.* **A133**, 11–14 (1988).
14. M.P. Bellon, J.-M. Maillard and C.-M. Viallet, Rational mappings, arborescent iterations, and the symmetries of integrability, *Physical Review Letters* **67**, 1373–1376 (1991).
15. R.J. Baxter, On Zamolodchikov's solution of the tetrahedron equations, *Comm. Math. Phys.* **88**, 185–205 (1983).
16. A.B. Zamolodchikov, Tetrahedron equations and the relativistic S-matrix of straight-strings in $2+1$ dimensions, *Comm. Math. Phys.* **79**, 489–505 (1981).
17. V.V. Bazhanov, Inversion and symmetry relations for a three-dimensional solvable model, In *Yang-Baxter Equations in Paris,* (Edited by J.-M. Maillard), pp. 29–43, *World Scientific* (1993).

18. M.P. Bellon, J.-M. Maillard and C.-M. Viallet, Infinite discrete symmetry group for the Yang-Baxter equations: Spin models, *Physics Letters* **A157**, 343–353 (1991).
19. M.P. Bellon, J.-M. Maillard and C.-M. Viallet, Infinite discrete symmetry group for the Yang-Baxter equations: Vertex models, *Phys. Lett. B* **260**, 87–100 (1991).
20. I.G. Korepanov, Vacuum curves, classical integrable systems in discrete space-time and statistical physics, Preprint, **HEP-TH/9312197**.
21. R.J. Baxter, H.N.V. Temperley and S.E. Ashley, Triangular Potts model at its transition temperature, and related models, *Proc. R. Soc. Lond.* **A358**, 535–559 (1978).
22. H. Meyer, J.-C. Anglès d'Auriac, J.-M. Maillard and G. Rollet, Phase diagram of a six-state chiral Potts model, *Physica* **A208**, 223–236 (1994).
23. W.T. Tutte, *Studies in Graph Theory, Part II*, (Edited by D.R. Fulkerson), The Mathematical Association of America, (1975).
24. F.Y. Wu and R.K.P. Zia, Critical point of a triangular Potts model with two- and three-site interactions, *J. Phys.* **A14**, 721–727 (1981).
25. M.T. Jaekel and J.-M. Maillard, Inversion relations and disorder solutions on Potts models, *J. Phys.* **A17**, 2079–2094 (1984).
26. M.T. Jaekel and J.-M. Maillard, A criterion for disorder solutions of spin models, *J. Phys.* **A18**, 1229–1238 (1985).
27. E.H. Lieb and F.Y. Wu, Two-dimensional ferroelectric models, In *Phase Transitions and Critical Phenomena,* Volume 1, (Edited by C. Domb and M.S. Green), pp. 331–490, Academic Press, New York (1972).
28. H.S.M. Coxeter and W.O.J. Moser, *Generators and Relations for Discrete Groups*, Second edition, Springer-Verlag, (1965).
29. J.E. Humphreys, *Reflection Groups and Coxeter Groups*, University Press, Cambridge, (1990).
30. P.E. Conner, J.P. Alexander and G.C. Hawrick, *Old Older Group Actions and Classification of Innerproduct*, Springer-Verlag, Berlin, (1977).
31. A.F. Beardon, *The Geometry of Discrete Groups*, Graduated Texts in Mathematics, Springer-Verlag, (1983).
32. W.P. Thurston, *The Geometry and Topology of Three Manifolds*, Notes based on a course at Princeton, Final parts of the notes written by B. Floyd and S. Kerchoff, Princeton, (1978).
33. M. Godement, *Séminaire Godement*, Chapitre III; revêtements ramifiés et groupes discontinus; exposés de J.P. Serre, Paris, (1959).
34. R.J. Baxter, The Yang-Baxter equations and the Zamolodchikov model, *Physica* **D18**, 321–347 (1986).
35. M.P. Bellon, J.-M. Maillard and C.-M. Viallet, Quasi integrability of the sixteen-vertex model, *Phys. Lett.* **B281**, 315–319 (1992).
36. S.B. Kelland, Twenty-vertex model on a triangular lattice, *Aust. J. Phys.* **27**, 813–829 (1974).
37. R.J. Baxter, Solvable eight-vertex model on an arbitrary planar lattice, *Phil. Trans. R. Soc. London* **289**, 315–346 (1978).
38. H.N.V. Temperley and E.H. Lieb, Relations between the "percolation" and "colouring" problem and other graph-theoretical problems associated with regular planar lattices: Some exact results for the "percolation" problem, *Proc. R. Soc. Lond.* **A323**, 251–280 (1971).
39. F.Y. Wu, The Potts model, *Rev. Mod. Phys.* **54**, 235–268 (1982).
40. N.L. Biggs, On the duality of interaction models, *Math. Proc. Cambr. Philos. Soc.* **80**, 429–436 (1976).
41. N.L. Biggs, Interaction models, In *Lecture Notes Series 30*, Cambridge University Press, (1977).
42. P. Le Doussal, A. Georges, D. Hansel and J.-M. Maillard, Dimensional reduction and correlation functions on 3D lattice models, *J. Phys.* **A20**, 5299–5309 (1987).
43. P. Rujan, Cellular automata and statistical mechanical models, *J. Stat. Phys.* **49**, 139–222 (1984).
44. J.-M. Maillard, G. Rollet and F.Y. Wu, Inversion relations and symmetry groups for Potts models on the triangular lattice, *J. Phys.* **A27**, 3373–3379 (1994).
45. J.-M. Maillard and G. Rollet, Hyperbolic Coxeter groups for triangular Potts models, *J. Phys.* **A27**, 6963–6986 (1994).
46. S. Beraha, J. Kahane and N.J. Weiss, Limits of chromatic zeros of some falilies of maps, *J. Comb. Theory* **B28**, 52–65 (1980).
47. R.J. Baxter, *Exactly Solved Models in Statistical Mechanics*, London Acad. Press, (1981).
48. A. Gaaf and J. Hijmans, Symmetry relations in the sixteen-vertex model, *Physica* **A80**, 149–171 (1975).
49. J.-M. Maillard, Automorphism of algebraic varieties and Yang-Baxter equations, *Journ. Math. Phys.* **27**, 2776-2781 (1986).
50. P. Martin and J.-M. Maillard, Zeros of the partition function of the three-state Potts model, *J. Phys.* **A19L**, 547–551 (1986).
51. H. Parke, Three-state Potts model on a triangular lattice, *Phys. Rev.* **B49**, 12881–12887 (1994).
52. H. Meyer, J.-C. Anglès d'Auriac, J.-M. Maillard, G. Rollet and F.Y. Wu, Tricritical behavior of the three-state triangular Potts model, *Phys. Lett.* **A201**, 252–256 (1995).
53. S. Boukraa, J.-M. Maillard and G. Rollet, Determinantal identities on integrable mappings, *Int. J. Mod. Phys.* **B8**, 2157–2201 (1994).
54. D. Bayer and M. Stillman, *Macaulay User Manual, (1993).*
55. I. Syozi, Transformations of Ising models, In *Phase Transitions and Critical Phenomena*, Volume 1, (Edited by C. Domb and M.S. Green), pp. 270–329, Academic Press, New York, (1972).

56. D. Hansel and J.-M. Maillard, Formal constraints on series analysis on the Potts model, *Mod. Phys. Lett.* **B14**, 145–153 (1988).
57. D. Hansel and J.-M. Maillard, Series analysis of the q-state checkerboard Potts model, *Int. J. Mod. Phys.* **B2** (6), 1447-1462 (1989).
58. J.-M. Maillard and R. Rammal, S_4 symmetry on the checkerboard Potts models, *J. Phys.* **A18**, 833–846 (1985).
59. I.G. Enting, Series expansions for the Potts model II. Partial generating functions in two-dimensions, *J. Phys.* **A7**, 2181–2197 (1974).
60. K.M. Briggs, I.G. Enting and A.J. Guttmann, Series studies of the Potts model: II. Bulk series for the square lattice, *J. Phys.* **A27**, 1503–1523 (1994).
61. I. Syozi and S. Naya, Spontaneous magnetization, a generalized square lattice, *Prog. Theor. Phys.* **23**, 374–376 (1993).
62. I. Syozi and S. Naya, Symmetrical properties of two-dimensional Ising lattices, *Prog. Theor. Phys.* **24**, 829–839 (1994).
63. B. Derrida and J. Vannimenus, Transfer-matrix approach to percolation and phenomenological renormalization, *Journal de Phisique Lettres* **41**, 473–476 (1980).
64. G. Rollet, J.-C. Anglès d'Auriac, J.-M. Maillard and F.Y. Wu, Zeroes of the triangular Potts model partition function: A conjectured distribution, *Physica* **A206**, 441–453 (1994).
65. R.J. Baxter, Critical antiferromagnetic square-lattice Potts model, *Proc. R. Soc. London Ser.* **A383**, 43–54 (1982).