Hyperbolic Coxeter Groups, Symmetry Group Invariants for Lattice Models in Statistical Mechanics, and the Tutte-Beraha Numbers

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Abstract—The symmetry groups, generated by the inversion relations of lattice models of statistical mechanics, are analysed for vertex models and for the standard scalar Potts model with two and three site interactions on triangular lattices. These groups are generated by three inversion relations and are noticeably generically very large ones: hyperbolic groups. Various situations for which the representations of these groups degenerate into smaller ones, hopefully compatible with integrability, are considered. For instance, the group becomes smaller for q-state Potts models for particular values of q, the so-called Tutte-Beraha numbers. For this model, algebraic varieties, including the known ferromagnetic critical variety, happen to be invariant under such large groups of symmetries. This analysis provides nice birational representations of hyperbolic Coxeter groups. Remarkable varieties breaking the symmetry of the lattice are seen to occur specifically for the Tutte-Beraha numbers. A detailed analysis of these Potts models is performed for q = 3. In particular, the algebraic varieties corresponding to conditions for the symmetry group to be finite order are carefully examined. Finally, specifically for the Tutte-Beraha numbers, the introduction of algebraic group invariants is discussed in detail for q = 3 in order to get closed expressions for the spontaneous magnetization of the edge Potts models.

Keywords—Tutte-Beraha numbers, Standard scalar Potts model, Yang-Baxter equations, Baxterisation, Birational representations of hyperbolic Coxeter groups.

1. INTRODUCTION

In previous papers [1,2] it has been shown that there exist nontrivial, nonlinear discrete symmetries acting on the parameter space of lattice models of statistical mechanics, are analysed for vertex models and for the standard scalar Potts model with two and three site interactions on triangular lattices. These groups are generated by three inversion relations and are noticeably generically very large ones: hyperbolic groups. Various situations for which the representations of these groups degenerate into smaller ones, hopefully compatible with integrability, are considered. For instance, the group becomes smaller for q-state Potts models for particular values of q, the so-called Tutte-Beraha numbers. For this model, algebraic varieties, including the known ferromagnetic critical variety, happen to be invariant under such large groups of symmetries. This analysis provides nice birational representations of hyperbolic Coxeter groups. Remarkable varieties breaking the symmetry of the lattice are seen to occur specifically for the Tutte-Beraha numbers. A detailed analysis of these Potts models is performed for q = 3. In particular, the algebraic varieties corresponding to conditions for the symmetry group to be finite order are carefully examined. Finally, specifically for the Tutte-Beraha numbers, the introduction of algebraic group invariants is discussed in detail for q = 3 in order to get closed expressions for the spontaneous magnetization of the edge Potts models.

It is important to note that these groups exist as (discrete) symmetry groups of lattice models even when one is no longer restricted to an integrable framework [9-11].

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From this point of view, the straight but tedious, analysis of a three-dimensional model through transfer matrix formalism, or any other classical method of lattice statistical mechanics, is replaced by an analysis of the transformations corresponding to the symmetries. These symmetries act in the parameter space, and therefore, at first sight, are less sensitive to the dimension of the lattice.

However, in the cases, integrable or not, known in the literature, a drastic difference seems to appear between two-dimensional and three-dimensional models, suggesting an explanation of the "obstruction" for three-dimensional integrability associated with generic three-dimensional symmetry groups, and also suggesting an algebraic definition of the notion of dimension of the model. In this framework the dimension of the lattice re-emerges through the "size" of the (infinite discrete) symmetry group. As far as two-dimensional models on square lattices are concerned, the discrete symmetry groups known in the literature were either finite groups [12,13] or groups isomorphic to products of $\mathbb{Z}$, up to a semidirect product by a finite group [9,13,14]. On the other hand, for lattice models of dimension three, these symmetry groups are much larger: they are generically free groups with (at least) three generators. With such symmetry groups, the very existence of solutions of the tetrahedron equations\footnote{Generalizations of the Yang-Baxter equations in dimension three [9,15-17].} having a "generic three-dimensional symmetry" seems problematic [14]: the only possibility for solutions of the tetrahedron equations are probably cases where the representations of such "large" groups degenerate into products of $\mathbb{Z}$ or even into finite groups [18,19]. Actually, a recent solution of Korepanov [20] of the tetrahedron equations confirms this point of view: these solutions actually correspond to a case for which the discrete symmetry group degenerates into a finite order group of order 32.

It will be shown here, that the analysis of the symmetry group of models on triangular lattices "weaken" this opposition between dimension two and dimension three. We will analyse vertex model on the triangular lattice, as well as the standard scalar $q$-state Potts model with two- and three-spin interaction also on the triangular lattice [21,22]. Generically, their symmetry groups are free groups with two generators. One recovers a situation similar to the one encountered in dimension three: these models on triangular lattices thus provide examples giving hints for the analysis of such large symmetry groups in dimension three.

However, these hyperbolic Coxeter groups of symmetries can actually degenerate into more "reasonable" groups, leaving room for integrability in the case of Tutte-Beraha numbers\footnote{The Tutte-Beraha numbers are particular numbers occurring in the analysis of chromatic polynomials [23].}

We will consider the consequences of these symmetries, with a special emphasis on criticality conditions. We will pay particular attention to a (self-dual) critical variety given by Wu [21,24], on the two- and three-site interaction Potts model on the triangular lattice, which we will revisit here.

Particular attention is devoted to the three-state Potts model. In this respect it will be shown that symmetry group invariants occurring specifically for the Tutte-Beraha numbers, seem to be useful to "decipher the complexity" of the (resummed low-temperature) expansions of various physical quantities and in particular, the spontaneous magnetization.

We hope such analysis will open a new class of lattice models for which a quite large set of exact calculations can be performed without having any "Yang-Baxter integrability"\footnote{Such models do exist: for instance, disorder solutions [25,26] provide some examples of "computable" models that are not Yang-Baxter-integrable. However, such disorder solutions correspond to dimensional reductions of the model. We are seeking here for two-dimensional (or higher-dimensional) models with a genuine two-dimensional complexity.}

2. RECALLS ON SYMMETRIES OF LATTICE MODELS

Let us recall the symmetry group generated by the inversion relations [3,5,6] for lattices of coordination number six, first, on the cubic three-dimensional vertex model [18,19] and then on the triangular lattice.
2.1. Inversion Relations and the Group $\Gamma_{3D}$

Let us consider vertex model on a three-dimensional cubic lattice of size $M \times M$. With each bond, is associated a variable with $q$ possible states. A Boltzmann weight $w(i, j, k, l, m, n)$ is assigned to each vertex configuration [27], and can be represented pictorially by:

\[
\begin{array}{c}
\text{i} \\
\text{j} \\
\text{k} \\
\text{l} \\
\text{R} \\
\text{m} \\
\text{n}
\end{array}
\]

The $q^6$ homogeneous weights $w(i, j, k, l, m, n)$ are first arranged in a $q^3 \times q^3$ matrix $R$ of entries:

\[
R_{imn}^{ijk} = w(i, j, k, l, m, n).
\tag{2.1}
\]

One may [19] introduce an involution $I$ which transforms $R$ into $IR$ according to:

\[
\sum_{\alpha_1, \alpha_2, \alpha_3} (IR)_{\alpha_1\alpha_2\alpha_3}^{i_1i_2i_3} \cdot R_{\beta_1\beta_2\beta_3}^{i_1i_2i_3} = \lambda \cdot \delta_{i_1j_1} \delta_{i_2j_2} \delta_{i_3j_3},
\tag{2.2}
\]

where $\lambda$ is an arbitrary multiplicative factor. This relation can be represented pictorially:

\[
\begin{array}{c}
i_d \\
i_m \\
i_s \\
I(R) \\
j_m \\
j_d \\
R
\end{array}
\]

The inversion transformation $I$ amounts to taking the inverse of the $q^3 \times q^3$ matrix $R$. One also introduces the partial transpositions $t_1, t_2, t_3$ with:

\[
(t_1R)_{i_1i_2i_3}^{j_1j_2j_3} = R_{i_1i_2i_3}^{j_1j_2j_3},
\tag{2.3}
\]

and similar definitions for $t_2$ and $t_3$.

For three-dimensional vertex models, one has four such involutions acting as symmetries of the $R$-matrix [19]:

\[
I_2 = I, \quad I_3 = t_1t_2t_3, \quad I_4 = t_2t_3t_1, \quad I_1 = t_3t_1t_2.
\tag{2.4}
\]

These four involutions generate an infinite discrete group $\Gamma_{3D}$ [19]. Let us note that the full transposition is nothing but the product $t = t_1 \cdot t_2 \cdot t_3$.

Considering the parameter space as a projective space (the entries of the $R$-matrix are homogeneous parameters), the elements of the group $\Gamma_{3D}$ have a nonlinear representation in terms of birational transformations. This group of symmetry of the parameter space of the model is very large. This is, in fact, a hyperbolic Coxeter group [28–33].

Remark. Coming back to integrability, it has been shown that the tetrahedron equations (generalization in three dimensions of the Yang-Baxter equations [10,15,16,34]) do have an infinite
group of symmetry generated by four involutions $K_1, K_2, K_3, K_4$ [19]. They satisfy various relations, for instance $(K_1 K_2 K_3 K_4)^2 = I_d$, where $I_d$ denotes the identity transformation. This group of symmetry of the tetrahedron equation is quite "large", since the number of elements of length smaller than $l$ is of exponential growth with respect to $l$, unlike the symmetry group of the Yang-Baxter equations which identifies with the affine Coxeter group $A^{(1)}_2$ [18,19,29].

In contrast the infinite discrete symmetry group of the square lattice is generated by two involutions (inversion relations) and therefore, is isomorphic to the infinite dihedral group. Let us introduce $I$ and $J$, the two inverse transformations on the square lattice vertex model [19]. A Boltzmann weight $w(i,j,k,l)$ is assigned to each square vertex configuration [27]:

$$
\begin{array}{c}
i \\
R \\
k \\
l
\end{array}
$$

The $q^4$ homogeneous weights $w(i,j,k,l)$ are first arranged in a $q^2 \times q^2$ matrix $R$:

$$R_{kl}^{ij} = w(i,j,k,l).$$

We introduce (see [1,18,19]) the inverse $I$ by:

$$\sum_{\alpha,\beta} R_{\alpha\beta}^{ij} \cdot (IR)_{\alpha\beta} = \lambda \cdot \delta_i^\alpha \delta_j^\beta = \sum_{\alpha,\beta} (IR)_{\alpha\beta} \cdot R_{\alpha\beta}^{ij},$$

and the other inverse $J$ by:

$$\sum_{\alpha,\beta} R_{\alpha\beta}^{ij} \cdot (JR)_{\alpha\beta} = \mu \cdot \delta_i^\alpha \delta_j^\beta = \sum_{\alpha,\beta} (JR)_{\alpha\beta} \cdot R_{\alpha\beta}^{ij}.$$ 

Similarly to the situation occurring for the cubic lattice, $I$ and $J$ are two involutions related by a partial transposition (denoted $t_1$ in [35]) of the indices: $J = t_1 I t_1$. Namely, $t_1$ reads: $(t_1 R_{kl}^{ij}) = R_{kl}^{ij}$. 

2.2. Inversion Relations of Triangular Vertex Models

For the triangular lattice the vertex Boltzmann weight [21] also reads $w(i,j,k,l,m,n)$, and can be represented by:

$$
\begin{array}{c}
i \\
R \\
k \\
l
\end{array}
$$

Similarly to the cubic model [18,19], the weights may be arranged in an $q^3 \times q^3$ matrix. However, for the triangular model there are only three inversion transformations, $I_1, I_2, I_3$, which actually coincide with three among the four of the cubic lattice (2.4). The fourth transformation $I_4$ corresponds to a nonplanar picture, which is meaningless for the triangular lattice. Let us denote $\Gamma_{\text{triang}}$ as the symmetry group generated by $I_1, I_2, I_3$. As will be shown in the following, using the equivalence between vertex and spin representation for this model [21], this group also generically has an exponential growth.

Let us recall the results obtained by Baxter, Temperley and Ashley on the triangular vertex and spin models [21]. They noticed that the integrable case discovered by Kelland for a triangular vertex model (a 20-vertex model [36]), actually corresponds to the following situation: the vertex
Boltzmann weight can alternatively be seen as either a left-hand side or a right-hand side of a Yang-Baxter equation (more generally, this refers to the $Z$-invariance concept [37]).

In the framework of this very model, they brought out the correspondence such a vertex model and the standard scalar $q$-state Potts model for anisotropic triangular lattices with two- and three-site interaction (only on up-pointing triangles) through the Lieb-Temperley algebra [21,38]. In terms of the two and three-site interaction spin model, these integrability conditions correspond to have no three spin interaction and also to be at the transition temperature [21].

There clearly exists here a drastic symmetry difference between the square and the cubic lattice, as far as the analysis of the group symmetries generated by the inversion relations is concerned. This difference stems from the fact that the number of involutions generating this very group is larger than two for the cubic lattice, which yields hyperbolic groups rather than the infinite dihedral group. The analysis of the symmetries on triangular models can be seen as a testing ground to study such hyperbolic groups, since the number of involutions generating this very group is larger than two. In the following sections, we will concentrate on spin models on triangular lattices and, more specifically, standard scalar Potts models [10].

### 3. TRIANGULAR SPIN MODEL

#### 3.1. Notations for the Spin Model

Let us now consider the standard scalar $q$-state Potts model on a triangular lattice with nearest neighbor interaction and three-spin interaction only on the up-pointing triangles:

\[
\begin{array}{c}
\text{3} \\
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\text{2} \\
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\]

The partition function of the models reads:

\[
Z = \sum_{\{\nu_i\}} \prod_{(i,j)} e^{K_{ij}\delta_{\nu_i,\nu_j}} \prod_{(j,k)} e^{K_{jk}\delta_{\nu_j,\nu_k}} \prod_{(k,l)} e^{K_{kl}\delta_{\nu_k,\nu_l}}
\]

(3.1)

The first three products denote the product over the edge two-side interaction Boltzmann weights along the three directions of the triangular model, and the last product denotes the product of all up-pointing triangles of the three-site interaction Boltzmann weights. The sum is taken over all spin configurations.

In this framework one can now introduce the following notations:

\[
y = xx_1x_2x_3 - (x_1 + x_2 + x_3) + 2, \quad \text{where} \quad x = e^K \quad \text{and} \quad x_i = e^{K_i}, \; i = 1, 2, 3.
\]

(3.2)

Of course for $q = 2$, the model degenerates into the nearest neighbor interaction triangular Ising model since the three-site interaction becomes irrelevant. Therefore one will not consider this $q = 2$ case in following (even if most of the results one will get are also valid in this very case).
3.2. Duality Transformation

Let us recall that a duality transformation does exist on this model [21,24]. With notations (3.2) this duality, denoted $D$, reads

$$D: \begin{cases} x_i \rightarrow x_i^* = 1 + q \frac{x_i - 1}{y}, \\ x \rightarrow x^* = \frac{x_1^2 + x_2^2 + x_3^2 - 2 + q^2/y}{x_1^2 x_2^2 x_3^2}, \\ y \rightarrow y^* = \frac{y^2}{y}. \end{cases} \quad (3.3)$$

This duality is associated with a rotation of 180° of the corresponding vertex model on a triangular lattice through the correspondence detailed in [21]. It should not be confused with the Kramers-Wannier for Potts models [39-41]: Kramers-Wannier duality maps the triangular lattice onto the honeycomb lattice. In fact duality (3.3) can be seen as the product of the Kramers-Wannier duality together with a star-triangle relation. $D$ is an involution.

In the isotropic limit $x_1 = x_2 = x_3 = u$, it gives:

$$\frac{(u - 3)u + u^3 x - q + 2}{u^3 x - 3u + 2}, \quad \frac{(u^3 x - 3u + 2 + 3qu - 3q + q^2)}{(u^3 x - 3u + 2 + qu - q)^3}.$$

For $q = 3$ it gives physical points only if $u \leq 1$ (antiferromagnetic edge coupling constants) and if the ferromagnetic condition $u^3 x - 1 \geq 0$ is satisfied or if $u \geq 1$ (ferromagnetic edge coupling constants) and if condition $u^3 x - 1 > 3 \cdot (u - 1)$ is satisfied.

Introducing well suited homogeneous variables, duality transformation (3.3) can be represented as a linear transformation $D_h$ (see Section 4.5), which satisfies relation: $D_h^2 = q^2 I_d$, where $I_d$ denotes the identity transformation. The hyperplanes stable by $D_h$ correspond to eigenforms associated with eigenvalues $\pm q$. The two self-dual varieties symmetric under permutations of 1, 2, and 3 can be written, respectively, as follows [21,24]:

$$y = -q \quad \text{and} \quad y = q.$$

Actually $y = q$ can be seen as the eigenform associated with eigenvalue $-q$ and reads:

$$xx_1 x_2 x_3 - (x_1 + x_2 + x_3) + 2 - q = 0. \quad (3.4)$$

Hyperplane (3.4) is a critical variety in some ferromagnetic region [24], whereas $y = -q$ has no such property. Let us notice that hyperplane (3.4) is the only variety that is stable point by point by duality $D$.

Note, that the well-known case of no three-site interaction, ($x = 1$), is not stable under $D$. Namely, variety $x = 1$ becomes:

$$(x_1 x_2 + x_2 x_3 + x_3 x_1 - x_1 + x_2 + x_3 - xx_1 x_2 x_3 + 1) \cdot y + q \cdot (x_1 - 1)(x_2 - 1)(x_3 - 1) = 0. \quad (3.5)$$

3.3. Disorder Solutions and Their Dual

Disorder varieties are algebraic varieties for which dimensional reductions occur for vertex or spin models, thus enabling the exact calculation of physical quantities such as the partition function per site, an infinite number of correlation functions ... [25,42]. A straightforward calculation, using a “disorder criterion” explained in [26], yields the following disorder conditions:

$$xx_1 x_2 x_3 - (x_1 + x_2 + x_3) + 2 - q \cdot x_i = 0, \quad i = 1, 2, 3. \quad (3.6)$$
When there is no three-sites interaction (\( z = 1 \)) one recovers the known disorder conditions of the two side nearest neighbor triangular Potts model [26,43].

These disorder conditions are "high-temperature" varieties. It is tempting to use duality (4.14) in order to single out some "low" temperature varieties. Unfortunately, when one transforms these disorder solutions by duality transformation (3.3), one gets:

\[
x_1 x_2 x_3 - (x_1 + x_2 + x_3) + 2 - q + q \cdot x_1 - \frac{q^2 \cdot x_i}{x_2 x_3 x_1} = 0.
\]

(3.7)

This is related to the fact that these disorder conditions are nothing but he vanishing\(^4\) conditions of the \( x_i^*\)'s.

For \( z = 1 \) the Kramers-Wannier dual [39] of these disorder varieties are algebraic varieties, on which the low-temperature expansions of the partition function per site (and many other quantities \( \ldots \)) simplify drastically to become the expansions of rational expressions [13]. Let us call these last varieties "order varieties". These "order varieties" are singled out: they do provide formal constraints\(^5\) on the low-temperature expansions of the model. Let us introduce low-temperature variables \( A, B, \) and \( C \):

\[
A = \frac{1}{x_1}, \quad B = \frac{1}{x_2}, \quad C = \frac{1}{x_3}.
\]

(3.8)

For instance, on the anisotropic triangular edge Potts model, relation:

\[
A + BC + (q - 2) \cdot ABC = 0,
\]

(3.9)

is a condition on which the low-temperature expansion of the partition function reduces to the (low-temperature) expansion of the partition function of an elementary triangular cell [13]. It is remarkable that these "order conditions" can actually be generalized to the (edge) checkerboard Potts model in a magnetic field\(^6\), thus providing nontrivial (formal) constraints on the (low-temperature resummed) expansion of the model [13]. Actually introducing the "order condition" (see [13]):

\[
D + ABC \cdot z + (q - 2) \cdot ABCD \cdot z = 0,
\]

(3.10)

one can show that the partition function per site is equal to a very simple expression when restricted to (3.10) and that the spontaneous magnetization restricted to (3.10) is actually equal to 1.

### 3.4. Inversion Relations

The inversion relations [7,25] for the two- and three-site interaction spin model can be represented pictorially as follows:

\[\alpha \quad \beta \quad \gamma \quad = \quad \alpha = \alpha' \quad \gamma\]

As it should [25], these three disorder varieties have no intersection with the ferromagnetic critical variety (3.4).

Most of the time these "order" conditions are not in the physical domain.

The fugacity is denoted \( z \).
which analytically means:

\[
\sum_{\beta} w(\alpha, \beta, \gamma) \cdot I(w)(\beta, \alpha', \gamma) = \lambda \cdot \delta_{\alpha, \alpha'}.
\]

(3.11)

The Boltzmann weight \(w(\alpha, \beta, \gamma)\) of model (3.1) is invariant under a common shift of each spin \(\alpha, \beta,\) and \(\gamma\). Therefore, \(\gamma\) can be fixed in a particular color, namely zero. Thus, the Boltzmann weight can be represented by a \(q \times q\) matrix (\(\alpha\) being the column index, and \(\beta\) the row one), with entries \(w(\alpha, \beta, 0)\). Equation (3.11) thus becomes the following matricial relation:

\[
\mathcal{W} \cdot I(\mathcal{W}) = \lambda \cdot I_{d, q},
\]

(3.12)

where \(I_{d, q}\) denotes the \(q \times q\) identity matrix, and the \(q \times q\) matrix Boltzmann weight \(\mathcal{W}\) reads:

\[
\mathcal{W} = \begin{pmatrix}
\tau_1 \tau_2 \tau_3 & \tau_2 & \cdots & \tau_2 \\
\tau_3 & 1 & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\tau_3 & 1 & 1 & \cdots & 1 \\
\end{pmatrix}
\]

Using a "\(Z_q\) Fourier transformation" [40,41], this \(q \times q\) matrix can be block-diagonalized into one \(2 \times 2\) block and a \((q - 2) \times (q - 2)\) matrix proportional to the identity matrix, \((x_1 - 1) \times I_{d, q-2}\). One can easily obtain the matrix inverse \(I(\mathcal{W})\). Note that \(I(\mathcal{W})\) is of the same form as \(\mathcal{W}\), \(x, \tau_1, \tau_2, \tau_3\), being changed to the following birational transformation \(I\):

\[
I(x) = \begin{cases}
(x_1 - 1)^2(x_1 + q - 2) / (xx_1^2 + xx_1(q - 3) - q + 2)(x_1 - 1), \\
x_1 \rightarrow xx_1^2 + xx_1(q - 3) - q + 2 / xx_1 - 1 = 2 - q - x_1 + x_1(x - 1) / x_1(x - 1), \\
x_2 \rightarrow x_1 - 1 / x_2(x_1 - 1), \\
x_3 \rightarrow x_1 - 1 / x_2(x_1 - 1).
\end{cases}
\]

(3.13)

\(I(x)\) can also be written:

\[
I(x) = x_1^2 \cdot (x_1 + q - 2) \cdot (x - 1)^2 + 2x_1 \cdot (x_1 - 1) \cdot (x_1 + q - 2) \cdot (x - 1) + (x_1 - 1)^2(x_1 + q - 2)
\]

\[
x_1(x_1 + q - 3)(x_1 - 1) \cdot (x - 1) + (x_1 - 1)^2(x_1 + q - 2)
\]

Obviously permutations of indices 1, 2, and 3 are also symmetries of the model. Introducing \(p_{23}\), the permutation of \(x_2\) and \(x_3\), and similarly \(p_{31}\) and \(p_{12}\), one can define the three following transformations:

\[
I_1 = p_{23}I = Ip_{23}, \quad I_2 = p_{31}p_{12}Ip_{12} = p_{12}Ip_{12}p_{31}, \quad I_3 = p_{12}p_{31}Ip_{31} = p_{31}Ip_{31}p_{12},
\]

corresponding to the three inversion transformations of the model [44].

\footnote{This existence of three involutions singles out the triangular lattice among the bidimensional models, from the symmetry group analysis point of view.}
4. THE SYMMETRY GROUP

Inversion $I$, permutations of $x_1, x_2, x_3$, and duality relation $D$ (defined by (3.3)) generate a symmetry group of the parameter space of the model, denoted $\Gamma_{\text{upt}}$ in the following.

At this point it is worth noticing that duality transformation $D$, does actually commute with $I$, and also with $S_3$, the group of permutations of $x_1, x_2, x_3$, and therefore, the whole group generated by $I$ and $S_3$. This commutation property enables us to see $\Gamma_{\text{upt}}$ as a hyperbolic Coxeter group generated by two infinite order transformations, up to the semidirect product by a finite group. These generically infinite order transformations read:

$$J_1 = I_3 I_2, \quad J_2 = I_1 I_3, \quad J_3 = I_2 I_1.$$  \hspace{1cm} (4.1)

By definition the $J_i$'s satisfy relation:

$$J_3 J_2 J_1 = \text{Identity}.$$ \hspace{1cm} (4.2)

Two of these $J_i$'s generate $\Gamma_{\text{upt}}$, up to the semidirect product by a finite group.

Let us recall that, for generic values of $q$ when $x = 1$, $\Gamma_{\text{upt}}$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$, up to a semidirect product by a finite group and degenerates into a finite group for Tutte-Beraha numbers [23] ($q = 2 - 2 \cos(k\pi/N)$). In fact, for $x = 1$, the $J_i$'s do commute and the elements of group $\Gamma_{\text{upt}}$ read:

$$y = J_1^{n_1} J_3^{n_3}.$$ \hspace{1cm} (4.3)

Generically, $n_1$ and $n_2$ are relative integers. For $q$, a Tutte-Beraha number associated with $N$, $n_1$ and $n_2$ run into $\{0, \ldots, N - 1\}$, the group $\Gamma_{\text{upt}}$ being therefore isomorphic to the product $\mathbb{Z}_N \times \mathbb{Z}_N \times \mathbb{Z}_2$.

In order to analyse the general case ($x \neq 1$), let us introduce the 3 cycle $c = p_{12} p_{13}$, and let us write the $J_i$'s in terms of $c$ and of a single one (generically) infinite order transformation, namely $(cI)^2$:

$$J_1 = (cI)^2 c^2, \quad J_2 = (cI)^2 c^3, \quad J_3 = (cI)^2.$$ \hspace{1cm} (4.4)

4.1. Transformation $(cI)^2$

For the sake of simplicity, let us consider transformation $(cI)^2$ as a homogeneous transformation, introducing $x_0 = x_1 x_2 x_3$ and a fifth homogenization variable $t$. One can then define a homogeneous inverse $I_h$ (corresponding to (3.13)) and $c I_h$, which written as a homogeneous transformation, reads:

\[
(cI)_h : (x_0, x_1, x_2, x_3, t) \rightarrow \left( -x_1 - (q - 2)t, x_3, -(q - 2) x_0 t - x_2 x_3 \frac{x_0 t - x_2 x_3}{x_1 - t} - x_0, x_2, \frac{x_0 t - x_2 x_3}{x_1 - t} \right).
\]

One notices that $u_3 = x_1 + x_2 + (q - 2)t$ and $v_3 = x_3 - x_0$ are just permuted by transformation $c I_h$: $u_3 \leftrightarrow v_3$. With these new variables, one also has:

\[
(cI)_h : (x_1, x_3, t) \rightarrow (x_2, x_3 - x_1 - (q - 2)t, F_0 t), \quad \text{where} \quad F_0 = \frac{x_0 t - x_2 x_3}{t(x_1 - t)}.
\]

Transformation $(cI)_h^2$ then reads:

\[
(cI)_h^2 : (u_3, v_3, x_1, x_3, t) \rightarrow (u_3, v_3, u_3 - x_1 - (q - 2)t, v_3 - x_3 - (q - 2) F_0 t, F_0 F_1 t),
\]

where $F_1 = F_0 (cI_h)$ is the same expression as $F_0$, where the $x_i$'s have been replaced by their images by $c I_h$.

Introducing $q_{\pm}$, the roots of the second order equation $z^2 + (q - 2)z + 1 = 0$ and introducing the successive iterates of $F_0$ by transformation $c I_h$ (namely: $F_{n+1} = F_n (cI_h)$), one can write down the general expression of transformation $(cI_h)^{2N}$ (see [45]).

Let us note that for $q$ corresponding to a Tutte-Beraha number, the $q_{\pm}$ are $N$th-root of unity [7].
4.2. Tutte-Beraha Numbers

Let us recall that, when there is no three site interaction (that is \( x = 1 \)), there do exist particular values of \( q \), the so-called Tutte-Beraha numbers \([23,46]\), for which transformations \( J_i \)'s, or equivalently transformation \((cI_h)^2\), become finite order ones \([7,44]\).

Amazingly, this situation still holds for the generic case (with \( x \neq 1 \)).

One can establish \([45]\) for \( q = 2 - 2 \cos(k\pi/N) \), (a Tutte-Beraha number), that transformation \((cI_h)^{2N}\) reduces an identity, that is equivalently:

\[
J_i^N = \text{Id}, \quad \text{with } i = 1, 2, 3. \tag{4.5}
\]

REMARK. Such Coxeter groups can be seen as the fundamental group of a surface of genus \( g \) minus \( k \) points \([33]\). Here there is a genus zero Riemann surface minus three points. At this step, the Coxeter group, one has to deal with, is reminiscent of the Schwarz's triangular groups \([8]\). Considering a geodesic triangle of angles \( \pi/n_1, \pi/n_2, \pi/n_3 \), and considering \( S_1, S_2, S_3 \) the symmetries with respect to the edges of the triangle, and defining the “rotations”:

\[
R_1 = S_2S_3, \quad R_2 = S_3S_1, \quad R_3 = S_1S_2, \tag{4.6}
\]

the \( R_i \)'s verify:

\[
R_i^{n_i} = \text{Id}, \quad \text{with } i = 1, 2, 3, \text{ and } R_1R_2R_3 = \text{Id}. \tag{4.7}
\]

In the study of these triangular groups, three different cases have to be distinguished: depending on \( 1/n_1 + 1/n_2 + 1/n_3 \) greater, lower or equal to 1.

Because of the ternary symmetry of our triangular Potts model, one has here \( n_1 = n_2 = n_3 = N \).

The only Euclidean case is \( N = 3 \), while the other values of \( N \) yield hyperbolic triangles and hyperbolic geometries, \( N = 2 \) corresponds to \( q = 2 \), which is the Ising subspace of the model.

In this case the three-site interaction becomes irrelevant. Thus, the first interesting case is \( N = 3 \), that is, \( q = 3 \) (or \( q = 1 \)).

4.3. The “Euclidean Case”: \( q = 3 \) or \( q = 1 \)

In this section we will restrict ourselves to \( N = 3 \), that is \( q = 3 \) or \( q = 1 \). In this case \( J^3 = \text{Id} \).

A “straight” analysis of this group, on the \( J_i \)'s, is performed in Appendix A. In fact, in this specific \( N = 3 \) case, it is better suited to introduce the transformations:

\[
G_1 = p_{12}J_1p_{31}, \quad G_2 = p_{23}J_2p_{12}, \quad G_3 = p_{31}J_3p_{23}. \tag{4.8}
\]

By introducing these transformations, it is easier to show that, for \( N = 3 \), \( \Gamma_{upt} \) is no longer a group with an “exponential growth”, but reduces down to \( \mathbb{Z} \times \mathbb{Z} \) up to a semidirect product by a finite group (like the affine Coxeter group \( A_2^{(1)} \) \([18]\)).

First, one notices that the \( G_i \)'s do satisfy a relation similar to relation (4.2):

\[
G_3G_2G_1 = \text{Id}. \tag{4.9}
\]

Let us first study the group \( G \), generated by \( G_1, G_2, \) and \( G_3 \). The \( G_i \)'s can be written in terms of transformation \( I \) and of the three-cycle \( c \):

\[
G_1 = c^2Ic^2Ic^2, \quad G_2 = Ic^2Ic, \quad G_3 = cIc^2I. \tag{4.10}
\]

Using \((cI)^6 = \text{Identity}, G_1G_2 \) reads:

\[
G_1G_2 = c^4Ic^2Ic^2Ic = c^2(cI)^{-2}c^2 = c^2(cI)^2c^2 = IcIc^2 = G_2G_1. \tag{4.11}
\]

*Such groups have been obtained from the analysis of the ratios of solutions of second-order differential equations ramified in three points.
Thus, the $G_i$'s actually commute. From relations (4.11) and (4.9), it is clear that a generic element of $\mathcal{G}$ reads:

$$g = G_1^{n_1}G_2^{n_2}, \quad (4.12)$$

where $n_1$ and $n_2$ are relative integers, which explicitly means that $\mathcal{G}$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$.

$\Gamma_{\text{opt}}$ can be seen to be generated by $I$ and $c$, up to some semidirect product by a finite group. From relation (4.10), one gets at once:

$$c I c = G_1^{-1}c^2, \quad c I c^2 = G_3, \quad c^2 I c = G_2^{-1}, \quad c^2 I c^2 = G_1 c,$$

$$I c I c = G_2^{-1}c^2, \quad I c I c^2 = G_3, \quad I^2 c I c = G_2, \quad I^2 c I c^2 = G_2 c.$$ 

Thus, $\Gamma_{\text{opt}}$ is isomorphic to $\mathbb{Z} \rightarrow \mathbb{Z}$ up to a semidirect product by a finite group.

4.4. Numerical Analysis

These infinite order transformations, represented as birational transformations, act as (symmetry) transformations in the parameter space of the model. In the $x = 1$ subcase, it has already been noticed [47] that, for $0 < q < 4$, the infinite set of points of the orbits of the discrete group of birational transformations is dense in an algebraic curve, while in the other case, they accumulate to fixed points. One has a similar situation for $x \neq 1$. Therefore, in this section we restrict here our study to $0 < q < 4$.

To complete the analysis of the (infinite discrete) symmetry group, one has to study its (generically infinite order) generators (the $J_i$'s). We draw here their orbits in the four dimensional parameter space ($\mathbb{C}P_4$) of the model. From relation (4.4), it is clear that the iterations of the $J_i$'s amount to performing the iteration of transformation $(cI)^2$. For generic values of $q$ (of course different from Tutte-Beraha numbers, see Section 4.2), the iteration of $(cI)^2$ yields curves. Figure 1 shows such a curve obtained for $q = 3.5$ (which is not a Tutte-Beraha number).

![Figure 1. Two-dimensional projection of an orbit of transformation $(cI)^2$, for $q = 3.5.$](image)

For Tutte-Beraha numbers, since the $J_i$'s are finite-order transformations, one has to consider other elements of the group. As far as the Euclidean case is concerned ($q = 3$ or $q = 1$), let us recall that the $G_i$'s are the commuting generators of the symmetry group isomorphic to $\mathbb{Z} \times \mathbb{Z}$. 
Two-dimensional projections of two orbits corresponding, respectively, to the iterations of transformations $G_2$ and $G_3$, for $q = 3$.

(c) Orbit generated by transformations $G_2$ and $G_3$, for $q = 3$: this surface is the product of the two curves of Figures 2a and 2b.

Figure 2.

Figure 2a illustrates the iteration of $G_2$ for $q = 3$. Remarkably, curves are obtained once again. Of course, iterating $G_2$ for $q = 3$ also yields curves, as can be seen of Figure 2b. Considering one orbit of the symmetry group generated by the $G_i$'s one gets, as it should be, a surface which can clearly be seen on Figure 2c as the product of curves like Figures 2a and 2b. This last figure gives a nice illustration of the $Z \times Z$ structure of the group.
Amazingly, the $G_i$'s which no longer commute when $q$ is no longer equal to 1 or 3, do yield curves, as can be seen on Figure 3 which represents the iteration of $G_2$ for $q = 0.5$ (not a Tutte-Beraha number).

All these results are remarkable: if one considers the iteration of more involved elements of the group, one generically gets quite "chaotic" figures (except of course for $q = 3$ or $q = 1$). Figure 4 shows such a quite "chaotic" orbit corresponding to the iteration of $J_1 J_2^2$ for a Tutte-Beraha number ($q = 2 + \sqrt{3}$).

These last figures and the study of many other orbits not given here, give a good hint of the "complexity" of these (infinite) hyperbolic Coxeter groups. They are generically of "exponential
growth*, even when additional relations occur (see relation (4.5)). This numerical study indicates that for generic values of \( q \), the generators of the symmetry group (the \( J_i \)'s) seem integrable since their iterations yield curves apparently in the whole parameter space. Moreover, the \( G_i \)'s seem to satisfy the same property for any value of \( q \), though they emerged from the analysis of the Euclidean case (\( q = 3 \) or \( q = 1 \)).

A way to verify this assumption, is to give the algebraic equations of these curves. For this purpose, in the next section, we will seek algebraic varieties invariant under the \( J_i \)'s and the \( G_i \)'s.

4.5. Group Invariants and Well-Suited Variables

Let us first take into account that there exist three (homogeneous) polynomials, of degree, respectively, 1, 2, and 3, invariant under permutation of \( x_1 \), \( x_2 \), and \( x_3 \), and covariant under transformation \( I \) (see (3.13)). These three polynomials read:

\[
D_1 = x_1 + x_2 + x_3 - x_0 + (q - 2)t, \\
D_2 = t(x_1 + x_2 + x_3 - t) - x_1x_2 - x_2x_3 - x_3x_1, \\
D_3 = t^2x_0 - x_1x_2x_3. 
\]

Let us consider the cofactors (under the action of \( I \)) of \( D_1 \), \( D_2 \), and \( D_3 \):

\[
C_1 = \frac{I(D_1)}{D_1} = \frac{x_1 - 1}{(xx_1 - 1) \cdot x_2x_3}, \quad C_2 = \frac{I(D_2)}{D_2} = -\frac{xx_1^2 + qxx_1 - 2xx_1 + 1 - q}{(xx_1 - 1)^2 \cdot x_2x_3}, \quad \text{and} \\
C_3 = \frac{I(D_3)}{D_3} = -\frac{(x_1 - 1)(xx_1^2 + (q - 2)xx_1 + 1 - q)}{x_2x_3^2 \cdot (xx_1 - 1)^3}. 
\]

One notes that the cofactor of \( D_3 \) is the product of the respective cofactors of \( D_1 \) and \( D_2 \). As a consequence, one directly gets an invariant under the whole group generated by \( I \) and the permutations of \( x_1 \), \( x_2 \), and \( x_3 \):

\[
\Delta = \frac{D_1 \cdot D_2}{D_3}. \quad (4.13)
\]

This provides, for arbitrary \( q \), a canonical foliation of the parameter space (\( CP_3 \)) by codimension one algebraic varieties (namely cubics).

Let us recall that duality transformation \( D \), defined in Section 3.2, is also a symmetry of the model, which commutes\(^9\) with transformation \( I \) and with permutations of 1, 2, and 3. Let us notice that duality \( D \) can actually be represented as a linear transformation when written in terms of homogeneous variables:

\[
D_h:\begin{cases}
  x_0 \rightarrow x_0 + (q - 1) \cdot (x_1 + x_2 + x_3 + (q - 2)t), \\
  x_1 \rightarrow (q - 1) \cdot x_1 + x_2 - x_3 + (q - 2)t, \\
  x_2 \rightarrow (q - 1) \cdot x_2 + x_0 - x_1 - (q - 2)t, \\
  x_3 \rightarrow (q - 1) \cdot x_3 + x_0 - x_2 - x_1 + (q - 2)t, \\
  t \rightarrow x_0 - x_1 - x_2 - x_3 + 2t.
\end{cases} \quad (4.14)
\]

Considering the previous covariant polynomials, one sees that \( D_1 \) and \( D_2 \) simply transform under the duality transformation:

\[
(D_1, D_2) \rightarrow (-q \cdot D_1, q^2 \cdot D_2). \quad (4.15)
\]

The duality acts in a slightly more involved way on \( D_3 \):

\[
D_3 \rightarrow D_3^* = q^2 \cdot (q \cdot D_3 - D_1 \cdot D_2). \quad (4.16)
\]

\(^9\)This is related to the fact that duality (4.14) corresponds to a weak-graph transformation [48]: therefore, it has a linear representation and commutes with the inversion relations.
Therefore, as far as the other covariant polynomials are concerned, one can barter $D_3$ for a homogeneous polynomial, namely:

$$D_{3d} = 2q \cdot D_3 - D_1 D_2.$$  \hfill (4.17)

On this new "self-dual" covariant the duality gives:

$$D_{3d} \rightarrow q^3 \cdot D_{3d}. \hfill (4.18)$$

Algebraic varieties $D_1$, $D_2$, and $D_{3d}$ do have covariance properties with respect to the whole group $\Gamma_{upt}$ (including duality (4.14)), which is (generically) a hyperbolic group. From the point of view of effective algebraic geometry, this provides examples of algebraic varieties with very large (discrete) groups of (birational) automorphisms.

4.5.1. Curves generated by the $J_i$'s or the $G_i$'s

It has been seen in Section 4.4 that the iterations of the $J_i$'s yield, for arbitrary $q$, curves in the whole parameter space. In order to prove that these curves are actually algebraic, one has to exhibit two other algebraic invariants for these very transformations. From relations (4.14), it is clear that the study can be limited to transformation $(\text{cl})^2$. One can show that the two polynomials:

$$E_1 = X_1 + X_2 - X_3 + X_0 + (q - 2)t, \quad E_2 = - (x_1 + x_2 - x_3 - x_0 - t) - x_1 x_2 + x_2 x_3 + x_3 x_1,$$

are actually covariant under the action of $(\text{cl})^2$. These expressions happen to have, respectively, the same cofactors (under transformation $(\text{cl})^2$) as $D_1$ and $D_2$. This immediately provides two additional algebraic invariants under $(\text{cl})^2$:

$$\Delta_1 = \frac{D_1}{E_1}, \quad \Delta_2 = \frac{D_2}{E_2}. \hfill (4.19)$$

Curves like Figure 1 are thus given as intersections of cubics, quadrics, and hyperplanes, namely:

$$\Delta = \delta, \quad \Delta_1 = \delta_1, \quad \Delta_2 = \delta_2, \hfill (4.20)$$

where the $\delta$'s denote arbitrary constants.

Considering the previous covariant polynomials, one notices that five of them are "eigen-polynomials" of the duality transformation (4.14). In addition to the previous two covariants $D_1$ and $D_2$ (see (4.13)), one gets:

$$(E_1, E_2) \rightarrow (q E_1, q^2 E_2). \hfill (4.21)$$

Algebraic curves, with an infinite number of (birational) automorphisms are either elliptic (or rational) curves \cite{49}. Amazingly, by eliminating $x_0$ and $x_3$ from relations (4.20) one gets (as expressed in inhomogeneous variables):

$$(\delta_1 + 1) \cdot (\delta_2 + 1) \cdot (x_1 x_2 - 1) = (4 \delta_1 \delta_2 \delta \cdot (x_1 + x_2 - 2) + (\delta_1 - 1) \cdot (\delta_2 - 1)) \cdot (x_1 + x_2 + q - 2),$$

which proves that these curves are actually rational curves.

Let us now consider the $G_i$'s (or equivalently the $(\text{cl}_h)^{2N_1}$'s). The previous numerical analysis indicated remarkable occurrence of curves, when iterating the $G_i$'s for any value of $q$. Let us, for instance, consider $G_3$. One notices that polynomials:

$$F_1 = x_3, \quad F_4 = (x_1 x_3 + x_2 x_3 - x_3 t - x_0 t) \cdot (x_1 x_2 + (q - 3) x_0 t - (q - 2) x_0 x_3),$$
are actually covariant under the action of $G_3$. The values of the cofactors of these $F_i$'s yield two $G_3$-invariants:
\[ \Delta_1' = \frac{D_1}{F_1}, \quad \Delta_4' = \frac{D_1 D_3}{F_4}. \] (4.22)

Figures like Figures 2a, 2b, or 3 are thus algebraic (elliptic) curves given by intersections of cubics, hyperplanes, and quartics.

Duality transformation (4.14) also acts on polynomials $F_i$'s. One can barter them for new "self-dual" homogeneous polynomials, namely:
\[ F_{1d} = 2q \cdot x_3 - D_1, \quad F_{4d} = 2q \cdot F_4 - (q^2 - 3q + 1) \cdot (D_2 - x_3 D_2) \cdot D_1. \]

4.5.2. The "Euclidean cases": $q = 3$ and $q = 1$

Let us recall that for $q = 3$ (or $q = 1$), these $G_i$'s do commute and that: $G_3 G_2 G_1 = \text{Id}$. It has been seen that each of the $G_i$'s generates algebraic elliptic curves. Therefore, for $q = 3$ (and $q = 1$), the orbits of the group generated by the $G_i$'s yield algebraic surfaces which are products of two elliptic curves, as clearly seen on Figure 2c. Since this surface is stable under the group $S_3$ of permutations of $x_1$, $x_2$, and $x_3$, it is natural to give its equation without referring to two of the $G_i$'s, that is, without having any direction singled out.

Actually, for $q = 3$, there exists an additional polynomial:
\[ D_5 = -x_1 x_2 x_3 (x_0^2 + x_1^2 + x_2^2 + x_3^2 - t^2) + x_0 (x_1^2 x_2 + x_2^2 x_3 + x_3^2 x_1), \] (4.23)
symmetric under permutations of $x_1$, $x_2$, and $x_3$ and covariant under $I$, from which one deduces, taking into account its cofactor, the ($S_3$-symmetric) invariant:
\[ \Delta_5' = \frac{D_3^3 D_2}{D_5}. \] (4.24)

Invariant (4.24), together with invariant (4.13), thus give $S_3$-symmetric equations of these algebraic surfaces. For $q = 3$ one thus has a foliation of the four-dimensional parameter space in algebraic surfaces given by (4.13) and (4.24).

Similarly, for $q = 1$, a $S_3$-symmetric polynomial covariant under the action of $I$, namely $D_5 = x_0 \cdot D_2$, yields the following $S_3$-symmetric invariant:
\[ \Delta_5' = \frac{D_3^3}{D_3}, \] (4.25)
and the duality gives, on this last covariant: $D_5' \rightarrow q^2 D_5'$.

For $q = 3$, the duality acts in the following way on $D_5$:
\[ D_5 \rightarrow D_5^* = 9 \cdot (27 \cdot D_5 - D_1^2 \cdot D_2), \quad \text{yielding:} \quad D_5 \rightarrow (D_5^*)^* = 3^{10} \cdot D_5. \]

For $q = 3$, one can thus substitute $D_5$ for a self-dual covariant:
\[ D_{5d} = 54 D_5 - D_3^3 \cdot D_2. \]

This new self-dual covariant duality (4.14) yields:
\[ D_{5d} \rightarrow 3^5 \cdot D_{5d}. \] (4.26)
4.5.3. Self-dual well-suited variables

One would like to trade the foliation given by (4.13) for an equivalent, yet self-dual one. In terms of $\Lambda$, given by (4.13), the duality transformation (4.14) reads:

$$\Lambda \rightarrow -q \cdot \frac{\Lambda}{q - \Delta^s}.$$  

(4.27)

and for $q = 3$ only, invariant $\Delta^s$ becomes:

$$\Delta^s \rightarrow -27 \cdot \frac{\Delta^s}{27 - \Delta^s}.$$  

(4.28)

From this one immediately gets a “pseudo” self-dual invariant, namely $\Delta_{\text{dual}}$ given by:

$$\Delta_{\text{dual}} = \frac{\Delta}{2 \cdot q - \Delta} = \frac{D_1 \cdot D_2}{2 \cdot q \cdot D_3 - D_1 \cdot D_2^*}.$$  

(4.29)

Actually duality transformation (4.14) acts as follows on $\Delta_{\text{dual}}$:

$$\Delta_{\text{dual}} \rightarrow -\Delta_{\text{dual}}.$$  

(4.30)

The critical ferromagnetic variety $D_1 = 0$ reads $\Delta_{\text{dual}} = 0$.

Similarly, for $q = 3$ only, one can introduce invariant $\Delta^s_{\text{dual}}$ such that duality relation (4.14) reads:

$$\Delta^s_{\text{dual}} \rightarrow -\Delta^s_{\text{dual}}.$$  

(4.31)

Invariant $\Delta^s_{\text{dual}}$ reads:

$$\Delta^s_{\text{dual}} = \frac{\Delta^s}{54 - \Delta^s} = \frac{D_1^3 \cdot D_2}{54D_5 - D_1^3 \cdot D_2}.$$  

(4.32)

For $q = 3$ one would also like to exchange the foliation of the parameter space in algebraic surfaces given by (4.13) and (4.24), for a foliation corresponding to two explicitly self-dual algebraic conditions.

Here, we introduce invariants denoted $X$ and $Y$:

$$X = \frac{1}{\Delta} = \frac{D_3}{D_1 \cdot D_3} \quad \text{and} \quad Y = \frac{1}{\Delta^s} = \frac{D_5}{D_1^3 \cdot D_2},$$  

(4.33)

and also the following “pseudo-self-dual” invariants:

$$\tilde{X} = \frac{1}{6} - X = \frac{-1}{6 \cdot \Delta_{\text{dual}}} \quad \text{and} \quad \tilde{Y} = \frac{1}{54} - Y = \frac{1}{54 \cdot \Delta^s_{\text{dual}}},$$  

(4.34)

which transform, under the duality (4.14), as follows: $(\tilde{X}, \tilde{Y}) \rightarrow (-\tilde{X}, -\tilde{Y})$. From the two “pseudo” self-dual invariants $\Delta^s_{\text{dual}}$ and $\Delta_{\text{dual}}$ (or $(\tilde{X}, \tilde{Y})$), one can easily get two explicitly self-dual invariants.

4.6. Two Remarkable Varieties: $z = 1$ and Its Dual

Let us come back to the vanishing of the three-spin interaction that is $z = 1$, or $D_3 = 0$. One notes that this variety is not self-dual. Variety $z = 1$ is well known [25] and plays a special role: the symmetry group $\Gamma_{\text{upt}}$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$ (up to some semidirect product by a finite group) [10].
4.6.1. A remarkable variety: $x = 1$

In this $x = 1$-subcase ($D_3 = 0$), there exists a rational parametrization which amounts to introducing the well-suited variables [10]:

$$x_i = \frac{1 - t^3 \cdot u_i}{t \cdot u_i^2 - t^2}, \quad i = 1, 2, 3, \quad \text{where } q = t^2 + \frac{1}{t^2} + 2. \quad (4.35)$$

In the $x = 1$ and $D_1 = 0$ subcase (critical condition for the edge Potts model) one can actually calculate exactly and very quickly the partition function per site, using the inversion trick and these well-suited variables [7].

In this $x = 1$-subcase an algebraic antiferromagnetic variety has been proposed by Martin and Maillard [50] on the basis of the analysis of the discrete group generated by the inversion relations that is:

$$M_q(x_1, x_2, x_3) = (q - 2) \cdot x_1 x_2 x_3 + 2 \cdot (x_1 x_2 + x_2 x_3 + x_3 x_1) + (q - 2) \cdot (x_1 + x_2 + x_3) + (q - 2)^2 - 2 = 0. \quad (4.36)$$

With this rational parametrization, the antiferromagnetic relation (4.36) and the critical condition $D_1 = 0$ read, respectively:

$$\left(\frac{t^4 - 1}{t^5 (t - u_1)(t - u_2)(t - u_3)}\right)^3 = 0 \quad \text{and} \quad - \left(\frac{t^4 - 1}{t^5 (t - u_1)(t - u_2)(t - u_3)}\right)^2 = 0,$$

and the ratio:

$$\frac{M_q(x_1, x_2, x_3)}{D_1} = \frac{1 - t^4}{t^2} \cdot \frac{(tu_1 u_2 u_3 + 1)}{(tu_1 u_2 u_3 - 1)}. \quad (4.37)$$

Algebraic expression $D_2$ reads in the $x = 1$ limit:

$$D_2 = (x_3 - 1) \cdot (x_2 - 1) \cdot (x_1 - 1) \cdot \left(\frac{(t^2 + 1)^3 (t^2 u_1 - 1) \cdot (t^2 u_2 - 1) \cdot (t^2 u_3 - 1)}{t^3 (t - u_1)(t - u_2)(t - u_3)}\right).$$

For $q = 3$ (that is $t^6 = -1$), $D_5$ also reduces to a very simple expression in the $x = 1$ limit, namely:

$$D_5 = -x_1 x_2 x_3 (x_3 - 1) \cdot (x_3 + 1) \cdot (x_2 - 1) \cdot (x_2 + 1) \cdot (x_1 - 1) \cdot (x_1 + 1)$$

$$= -\left(\frac{t^4 - 1}{t^5 (t - u_1)}\right)^3 \left(\frac{t^3 u_1 - 1}{(t^3 u_2 - 1)(t^3 u_3 - 1)(t^3 u_1 - 1)}\right).$$

The ratio (4.37) is actually an invariant of the group $\Gamma_{\text{triang}}$ [7]. In fact, this invariant can be replaced by other ones using the following remarkable identity:

$$M_q^2 - q \cdot (q - 4) \cdot D_1^2$$

$$= 4 \cdot (1 + (q - 2) \cdot x_1 + x_2^2) \cdot (1 + (q - 2) \cdot x_2 + x_3^2) \cdot (1 + (q - 2) \cdot x_3 + x_1^2). \quad (4.38)$$

One can also, instead of the ratio (4.37), introduce the following algebraic invariant [7]:

$$I = \frac{(q - 2) \cdot x_1 x_2 x_3 + (x_1 x_2 + x_2 x_3 + x_3 x_1) - 1}{(x_1 x_2 + x_2 x_3 + x_3 x_1) + (q - 2) \cdot (x_1 + x_2 + x_3) + (q - 1) \cdot (q - 3)}. \quad (4.39)$$

Equivalently one can introduce invariant:

$$I_2 = \frac{x_1 x_2 x_3 - (x_1 + x_2 + x_3) - (q - 2)}{(q - 2) \cdot x_1 x_2 x_3 + (x_1 x_2 + x_2 x_3 + x_3 x_1) - 1} = \frac{x_1 x_2 x_3 - (x_1 + x_2 + x_3) - (q - 2)}{-x_1 x_2 x_3 \cdot W(x_1, x_2, x_3)},$$

where

$$W(x_1, x_2, x_3) = \frac{1}{x_1} \cdot \frac{1}{x_2} \cdot \frac{1}{x_3} - \left(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}\right) - (q - 2). \quad (4.40)$$

In the numerator of (4.40) one recognizes $D_1$ for $x = 1$ and the denominator is nothing but the numerator where the $x_i$'s are changed into the $1/x_i$'s.
4.6.2. Another remarkable variety: The dual of \( x = 1 \)

Since duality (4.14) commutes with \( \Gamma_{\text{up}} \), the dual variety of \( x = 1 \) also corresponds to the degeneracy of \( \Gamma_{\text{up}} \) into a group isomorphic to \( \mathbb{Z} \times \mathbb{Z} \) (up to some semidirect product by a finite group). This remarkable variety (3.5) also reads:

\[
D_3' = q^2(q D_3 - D_1 D_2) = 0, \tag{4.41}
\]

or explicitly,

\[
(x_1^2 x_2^2 x_3^2 + (x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_2 x_3^2 + x_1 x_2 x_3^2) - (x_1^2 + x_2^2 + x_3^2) + 1) \\
+ (q - 3) \cdot ((x_1 x_2 + x_1 x_3 + x_2 x_3) - (x_1 + x_2 + x_3) + x_1 x_2 x_3 + 1) = 0. \tag{4.42}
\]

In the isotropic limit \( x_1 = x_2 = x_3 = u \) and, for \( q = 3 \), this variety reads

\[
u^6 \cdot x^2 + u^3 \cdot (1 - 3u^2) \cdot x + (1 - 6u^2 + 6u^3) = 0. \tag{4.43}
\]

There is also a rational parametrization of (4.42) as follows:

\[
x_i = \frac{n_i}{d}, \quad i = 1, 2, 3 \text{ and } x = \frac{n_1^2 \cdot n_2 x \cdot n_3 x}{t \cdot d_{1x} \cdot d_{2x} \cdot d_{3x}}, \tag{4.44}
\]

with

\[
\begin{align*}
n_1 &= (u_1 u_2 u_3 t - 1) \cdot (t^4 - t^2 + 1) - t^4 u_1 u_3 - u_2 u_3 - t^4 u_1 u_2 + t^5 u_1 + t u_3 + t u_2, \\
n_2 &= (u_1 u_2 u_3 t - 1) \cdot (t^4 - t^2 + 1) - t^4 u_2 u_0 - u_1 u_3 - t^4 u_1 u_2 + t^5 u_2 + t u_3 + t u_1, \\
n_3 &= (u_1 u_2 u_3 t - 1) \cdot (t^4 - t^2 + 1) - t^4 u_2 u_3 - u_1 u_2 - t^4 u_1 u_3 + t^5 u_3 + t u_2 + t u_1, \\
n_{1x} &= t^3 \cdot u_1 u_2 u_3 \cdot (x^2 - 2) + t^2 (u_1 u_2 + u_1 u_3 + u_2 u_3) - t^3 (u_1 + u_2 + u_3) + 2t^2 - 1, \\
n_{2x} &= u_1 u_2 u_3 \left( t^6 + 1 \right) - t^5 (u_1 u_2 + u_1 u_3 + u_2 u_3) + t^5 (u_1 + u_2 + u_3) + t \cdot (t^6 - t^4 - 1), \\
d_{1x} &= (u_1 u_2 u_3 t - 1) \cdot (t^4 - t^2 + 1) - t^4 u_1 u_3 - u_2 u_3 - t^4 u_1 u_2 + t \cdot (t^4 u_1 + u_2 + u_3), \\
d_{2x} &= (u_1 u_2 u_3 t - 1) \cdot (t^4 - t^2 + 1) - t^4 u_2 u_3 - u_1 u_3 - t^4 u_1 u_2 + t \cdot (t^4 u_2 + u_3 + u_1), \\
d_{3x} &= (u_1 u_2 u_3 t - 1) \cdot (t^4 - t^2 + 1) - t^4 u_2 u_3 - u_1 u_2 - t^4 u_1 u_3 + t \cdot (t^4 u_3 + u_2 + u_1).
\end{align*}
\]

With this rational parametrization (4.44) inversion relation (4.19) takes a very simple form:

\[
(u_1, u_2, u_3) \longrightarrow \left( \frac{t^2}{u_1}, \frac{1}{t^2}, \frac{1}{u_3}, \frac{1}{t^2 \cdot u_2} \right), \tag{4.45}
\]

and the polynomial expressions \( D_1, D_2, D_3 \) read, respectively,

\[
D_1 = \frac{(t^4 - 1)^2 \cdot (t \cdot u_1 u_2 u_3 - 1)}{t^2 \cdot d},
\]

\[
D_2 = -\frac{(t^2 + 1)^2 \cdot (tu_1 - 1)(tu_2 - 1)(tu_3 - 1)(u_3 - t)}{t \cdot d^2}, \quad \text{and} \tag{4.46}
\]

\[
D_3 = -\frac{(t^4 - 1)^2 \cdot (t^2 + 1)(tu_3 - 1)(u_3 - t)(tu_2 - 1)(u_2 - t)(tu_1 - 1)(u_1 - t)(tu_1 u_2 u_3 - 1)}{t \cdot d^3},
\]

with

\[
d = u_1 u_2 u_3 \cdot t^5 - (2u_1 u_2 u_3 + u_2 + u_3 + u_1)t^3 + (u_2 u_3 + u_1 u_2 + u_1 u_3 + 2)t^2 - 1.
\]
4.7. Additional Algebraic Invariant for $z = 1$

Additional Algebraic Invariants for Finite Group Subcases

At first glance, there seems to exist an incompatibility between invariants (4.13), (4.24), and invariant (4.39) (or (4.40)) of the rational parametrization [7,10] introduced for $z = 1$ (no three-spin interactions).

The previous occurrence of an additional invariant specific of $z = 1$ (see (4.39)) and of another one specific of $q = 3$ (see (4.24)), deserves some comments. Invariant (4.39) exists for arbitrary $q$ when $z = 1$ and cannot, at first glance, be simply extended to $z \neq 1$. Similarly, invariant (4.24) only exists when $q = 3$. When restricted to $q = 3$, it may seem that there is an incompatibility between these various invariants when one considers the $x = 1$ limit. In fact, the answer to this paradox is the following: when $q = 3$ and $x = 1$ the discrete group degenerates into a finite group. The orbits of the group are no longer surfaces, or curves, but a finite set of points and one can have as many (independent) “algebraic” invariants as the dimension of the parameter space. In $x = 1, q = 3$ subcase, besides the $x = 1 (D_3 = 0)$ condition, one has the two invariants $\Delta_5'$ and $I$ ((4.13) and (4.39)) and (at least) a third invariant (see Appendices D and E.4 for more details). Actually, in this $x = 1, q = 3$ subcase, one has an additional covariant namely:

$$p_9 = (2 + 2x_3 + 3x_2x_3 + 2x_2 - x_1 + x_1x_2x_3) \cdot (3x_1x_2 + x_1x_2x_3 + 2x_3 + 2x_1 + 2 - x_2)$$
$$\times (3x_1x_3 + x_1 + 2x_1x_2x_3 + x_3 + 1 + x_2) \cdot (x_1x_2 + 3x_1x_2 + 2x_2 + 2x_1 + 2 - x_3)$$
$$\times (x_1 + 3x_1x_2 + 2x_1x_2x_3 + x_3 + 1 + x_2) \cdot (x_1 + 1 + 2x_1x_2x_3 + 3x_2x_3 + x_3 + x_2)$$
$$\times (2x_3 - x_2 + x_1x_2x_3 - x_1 - 1) \cdot (2x_2 - x_3 - x_1 - 1 + x_1x_2x_3)$$
$$\times (x_1x_2x_3 + 2x_1 - x_2 - x_3 - 1).$$

(4.47)

Note, that $p_9$, obtained, from the disorder condition, (3.6) taking the product of the images of this disorder condition under the group $\Gamma_{\text{triang}}$, which happens to be finite for $q = 3$. Note, that $p_9$ transforms as follows under the inversion relation:

$$p_9 \rightarrow \frac{p_9}{x_2^9 \cdot x_3^9}.$$  (4.48)

One has, therefore, an additional invariant in this $q = 3$ ($x = 1$) subcase:

$$I_9 = \frac{p_9}{D_1^9}.$$  (4.49)

Invariant (4.49) is basically built as the product, over the discrete group, of the disorder conditions. Appendix C.1 give calculations corresponding to a straight generalization of such an invariant in the $x = 1$ and $q = 3$ subcase, but for the checkerboard Potts model. One can also build another invariant from the product, over the discrete group, of the “order conditions” described in Section 3.3. These two invariants (product over the group of disorder conditions, product over the group of “order” conditions) can both be used. However, the invariant product over the group of “order” conditions is better suited for the analysis of low-temperature expansions and the analysis of quantities of a “low-temperature” nature like the spontaneous magnetization (see Appendix D).

From this new invariant, which does not exist\(^\text{10}\) for arbitrary $q$, one gets a new foliation of the parameter space in terms of the algebraic surfaces, $I_9 = \text{constant}$. Invariants $I$ and $I_9$ can be shown to be algebraically independent. One can verify that the intersection of $x = 1$ (i.e., $D_3 = 0$), $\Delta_5 = \text{constant}$, $I = \text{constant}$, and $I_9 = \text{constant}$ gives a finite set of points corresponding to the orbits of the discrete symmetry group.

\(^{10}\)In the $x = 1$ subcase the group becomes finite for the Tutte-Berge numbers. When the group is finite, and only in this case, one can get additional invariants, similar to (4.9), built as product over the group of disorder (or “order”) varieties.
Remark. If one can imagine, for \( q = 3 \) and \( x = 1 \), that the critical manifolds correspond to quite involved relations between these various invariants [51], at first glance, one expects the phase diagram of the model for \( q = 3 \) and \( x \neq 1 \), to depend only on invariant (4.13) and the phase diagram of the model for \( x = 1 \) and \( q \neq 3 \), to depend only on invariant (4.39). Therefore, a question naturally pops out, namely the "continuity" of the phase diagram [51] for \( q \approx 3 \) and \( x \approx 1 \). We will address this question in parallel publications.

5. Remarkable Algebraic Varieties and Monte-Carlo Simulations: Towards Critical Manifolds

Critical manifolds have to be compatible with all the symmetries of the lattice model: the discrete symmetries generated by the inversion relations, as well as the (continuous) weak-graph ("gauge") transformations [48]. The self-dual variety (3.4), that is \( D_1 = 0 \), is already known as a critical variety of this very Potts model, for arbitrary \( q \), in some ferromagnetic region [24]. This variety\(^{11}\) is, as it should be, stable under the whole hyperbolic Coxeter group [44]. In a rather general framework, one can write a critical manifold as follows:

\[
\mathcal{F}(\Delta_{\text{dual}}, \Delta_5^{\text{dual}}) = 0 \quad \text{or} \quad \mathcal{F}(\hat{X}, \hat{Y}) = 0,
\]

where \( \mathcal{F} \) (or \( \mathcal{J} \)) is any (transcendental) function such that:

\[
\mathcal{F}(\Delta_{\text{dual}}, \Delta_5^{\text{dual}}) = \mathcal{F}(\Delta_{\text{dual}}, \Delta_5^{\text{dual}}) \quad \text{or} \quad \mathcal{F}(\hat{X}, \hat{Y}) = \mathcal{F}(\hat{X}, \hat{Y}).
\]

In order to get some hint on the critical manifolds, one also has to take into account other exact results. For instance, there should not be any intersection between the disorder conditions\(^{12}\) and the critical manifolds or, if any, this should localize very precisely a tricritical manifold [52]. Several varieties pop out as natural conditions for criticality since they have been shown to be stable under the whole infinite Coxeter group, namely (besides \( D_1 = 0 \)) the self-dual varieties \( D_2 = 0, D_{3d} = 0 \) (that is, \( 6 \cdot D_3 = D_1 \cdot D_2 = 0 \)), and for \( q = 3 \) only, \( D_{5d} = 0 \) (that is, \( 54 \cdot D_5 - D_{3d}^2 \cdot D_2 = 0 \)). Unfortunately, these "candidates" for criticality are ruled out by Monte-Carlo simulations [52].

5.1. \( D_1 = 0 \) and a Tricritical Manifold

Monte-Carlo simulations have been performed in the \( q = 3 \) isotropic limit, on \( 3 \times L^2 \) sites triangular lattices with both \( 2\pi/3 \)-rotation invariance and translation invariance, for various values of \( L \) [52]. The Monte-Carlo calculations [52] show that \( D_1 = 0 \) is actually a critical variety, and that, on the algebraic variety \( D_1 = 0 \), both first-order and second-order transitions occur. Therefore, there must exist a codimension-two tricritical submanifold which corresponds to the "frontier" between these two types of transitions on the codimension-one variety \( D_1 = 0 \). Our objective is to get the equation of such a variety.

It is clear, from the two invariants (4.13) and (4.24), that, when restricted to \( D_1 = 0 \), the only expression invariant under the group which does not trivialize because of \( D_1 = 0 \), is the ratio:

\[
\mathcal{I}_{\text{new}} = \frac{\Delta_6}{\Delta_5^3} = \frac{D_3^3}{d_5 \cdot D_5^2}.
\]

The \( (x, u) = (4, 1) \) point (critical pure three-spin point) seems to be, in the case of the isomorphic mode, a tricritical point [52]; this yelds the following value \(-1/2\) for the new invariant \( \mathcal{I}_{\text{new}} \), which singles out the algebraic variety:

\[
2 \cdot D_3^3 + D_5 \cdot D_2^2 = 0.
\]

\(^{11}\)The globally self-dual variety \( y = -q \) cannot be critical one since it is not stable under \( \Gamma_{\text{upt}} \).

\(^{12}\)The analysis of the action of the hyperbolic group on these varieties also has to be performed, in order to generalize the analysis already achieved in the \( x = 1 \) limit [26].
Monte-Carlo calculations confirm that $D_1 = 0$ is actually a critical variety even for quite anisotropic points. However, the interpretation of the possible tricritical behaviour of the intersection of $D_1 = 0$ and (5.4) is a more delicate question that will be addressed elsewhere.

6. finite order conditions

6.1. $G^4_1 = \text{Identity for } q = 3$

In a previous Section 5.1, it was indicated that the intersection of $D_1 = 0$ together with variety (5.4) was a possible candidate for the equation of a critical manifold. We see here that this very intersection has a remarkable interpretation in terms of the discrete symmetry group.

Let us recall (the generically infinite order) transformation $G_1 = c^2 \cdot I \cdot c \cdot I$ (see (4.10)) which reads for the three-state model:

$$G_1(x_1, x_2, x_3, x) = (x_{1G}, x_{2G}, x_{3G}, x_G),$$

with

$$x_{1G} = x_2 \cdot (x_1^2 x - 1) \cdot (1 - x_1 + x_3 x x_1 - x_3),$$

$$x_{2G} = \frac{x_2^2 x_2 x - x_1^2 + 1 - x_3^2}{(1 + x_3 x_2 x_2 - x_1 x_1 + x_1 - x_3)} \cdot x_3,$$

$$x_{3G} = \frac{1 - x_1 + x_3 x x_1 - x_3}{1 + x_3 x_2 x_2 - x_1 x_1 + x_1 - x_3},$$

and

$$x_G = \frac{(x_3 x_2^2 x - x_1^2 - x_1 + x_2)^2 \cdot (x_1 - 1 + x_3 x x_1 - x_3)}{(1 - x_1^2 + x_3^2 x_2^2 x_2) \cdot (1 - x_1 + x_3 x x_1 - x_3) \cdot (x_1^2 x - 1)}.$$  

(6.1)

In fact, by writing condition $G^4_1 = \text{Identity}, one gets a dimension-one (self-dual) algebraic variety:

$$2 \cdot D_3^3 + D_5 \cdot D_2^2 - D_3^2 \cdot D_1 \cdot D_2 = 0.$$  

(6.2)

Under the duality transformations relation (6.2) gives:

$$2 \cdot D_3^3 + D_5 \cdot D_2^2 - D_3^2 \cdot D_1 \cdot D_2 \longrightarrow 3^9 \cdot (2 \cdot D_3^3 + D_5 \cdot D_3^2 - D_3^2 \cdot D_1 \cdot D_2).$$  

(6.3)

In the isotropic limit relation (6.2) reads $u = 1$ together with:

$$x^4 u^{11} - 6u^{10} x^3 + u^{10} x^4 - 3u^9 x^3 + 9u^9 x^2 + u^9 x^4 + 8u^8 x^2 - 3u^8 x^3 + 2u^7 x^2$$

$$-24u x^7 + 12u x^2 + 2x u^5 + 20u^5 - u^5 x - 5u^5 x^2 - 28u^4 + u^4 x^2 + 5u^4 x$$

$$-4u^3 x + x + 8u^3 + u^3 x^2 + 7u^2 - 5u + 1 - 0,$$

which in the $x = 1$ limit, gives

$$(u + 1)^3 \cdot (u - 1)^8 = 0.$$  

(6.4)

6.2. $G^N_1 = \text{Identity Varieties for } q = 3$

For $q = 3$, Section 4.3 shows that the whole discrete group is finite if transformation $G_1$ is actually of finite order.

Let us study systematically these finite-order conditions.

It should be noted that the conditions corresponding to $G_1 = \text{Identity}$ and $G^4_1 = \text{Identity}$ do not yield codimension-one algebraic variety symmetric under the permutations of $x_1, x_2, x_3$. Therefore, the analysis of the algebraic varieties $G^N_1 = \text{Identity}$ starts with $N \geq 3$.

All the (no three-spin) points $x = 1$ (that is $D_3 = 0$) yield $G^3_1 = \text{Identity}$. Conversely, $G^4_1 = \text{Identity}$ corresponds to $D_3 = 0$ and its dual, $D_1 \cdot D_2 - 3 \cdot D_3 = 0$, or equivalently:

$$X \cdot (3 \cdot X - 1) = 0 \quad \text{or} \quad (6 \cdot \bar{X} + 1) \cdot (6 \cdot \bar{X} - 1) = 0.$$  

(6.5)
This last condition, $D_1 \cdot D_2 - 3 \cdot D_3 = 0$, reads in terms of the $x_i$'s:

$$\begin{align*}
1 - (x_1 x_2 + x_2 x_3 + x_3 x_1) - (x_1^2 + x_2^2 + x_3^2) + x_1 x_2 x_3 x + x_1^2 x_2^2 x_3^2 x^2 \\
+x_3 x_1^2 + x_2 x_1 + x_2^2 x_3 + x_2 x_3 + x_1 x_3 - x_2^2 x_3 x_1 x - x_2 x_3 x_1 x^2 - x_2^2 x_3 x_1^2 x = 0,
\end{align*}$$

which, in the isotropic case $x_1 = x_2 = x_3 = u$, gives

$$1 - 6u^2 + u^3 + x^2 u^5 - 3xu^5 + 6u^3 = 0.$$

The $G_1^I = I$ identity variety has been detailed in the previous section. Next is the $G_1^I = I$ identity variety which reads a self-dual codimension-one variety namely:

$$\begin{align*}
-D_1^2 \cdot D_2^2 + (D_1^2 \cdot D_2^2 \cdot D_3 - 7D_1 \cdot D_2 \cdot D_3^2 + 14 \cdot D_1^2 \cdot D_3^2) \cdot D_6 \\
+D_4^2 \cdot D_2^2 - 5 \cdot D_2 \cdot D_2 \cdot D_2 + 5 \cdot D_5^2 = 0.
\end{align*}$$

After a few calculations one can also find that condition $G_2^I = I$ identity, written as $G_1 = G_1^{(-3)}$, yields the previous $G_1^I = I$ identity conditions, together with another algebraic variety:

$$C_6 = 2 \cdot D_2^2 D_2^2 - D_2^3 D_3 \cdot (10D_3^2 - 5D_3 D_2 D_3 + D_2 D_3) \cdot D_6 + D_4^2 \cdot D_4 \cdot (D_2 D_2 - D_3) = 0,$$

which can also be written in terms of the (pseudo-self) dual invariants $X$ and $Y$ (4.34):

$$\tilde{C}_6 = 3456 \cdot Y^2 - 288 \cdot X \cdot \left(1 + 60X^2 \right) \cdot \tilde{Y} - \left(12X^2 + 1 \right) \cdot \left(144X^4 - 72X^2 + 1 \right) = 0.$$

One verifies immediately by looking at (6.9), that condition $G_0^I = I$ identity is self-dual. In the isotropic limit it reads an involved expression given in Appendix B.

Large formal calculations enable us to write down explicitly the algebraic varieties corresponding to the finite-order conditions $G_1^I = I$ identity for larger values of $N$. One should note that, remarkably, all these conditions $G_1^N = I$ identity are codimension-one self-dual varieties, which we denote $C_N = 0$.

The $G_N^I = I$ identity varieties seem, for arbitrary $N (N \geq 3)$, to be codimension-one algebraic varieties of the form:

$$0 = P_0(X) \cdot Y^N + P_1(X) \cdot Y^{N-1} + P_2(X) \cdot Y^{N-2} + P_3(X) \cdot Y^{N-3} + \cdots,$$

with $P_0(X) = P_0(0)$ and $P_0(X) = 0$ for $i = 1, \ldots, N$. For example, by introducing $Y = 1/\Delta$ and $X = 1/\Delta$, condition $G_1^N = I$ identity reads, respectively, for $N = 3, 4, 6$:

$$C_3 = X \cdot (3 \cdot X - 1) = 0, \quad C_4 = Y + 2X^3 - X^2 = 0, \quad \text{and} \quad C_5 = Y^2 - X \cdot \left(14X^2 - 7X + 1 \right) \cdot Y - X^4 \cdot \left(1 - 5X + 5X^2 \right) = 0,$$

or in terms of $\tilde{X} = 1/6 - X$ and $\tilde{Y} = 1/54 - Y$ (see (4.34)):

$$\begin{align*}
\tilde{C}_3 &= \left(6 \cdot \tilde{X} - 1 \right) \cdot \left(6 \cdot \tilde{X} + 1 \right), \quad \tilde{C}_4 = -6 \cdot \tilde{Y} - \tilde{X} \cdot \left(12 \tilde{X}^2 - 1 \right), \\
\tilde{C}_5 &= 1728 \cdot \tilde{Y}^2 - 288 \cdot \tilde{X} \left(84\tilde{X}^2 - 1 \right) \cdot \tilde{Y} - 1 + 12\tilde{X}^2 + 1872\tilde{X}^4 - 8640\tilde{X}^6, \\
\tilde{C}_6 &= 3456 \cdot \tilde{Y}^2 - 288 \cdot \tilde{X} \left(1 + 60\tilde{X}^2 \right) \cdot \tilde{Y} - \left(12\tilde{X}^2 + 1 \right) \cdot \left(144\tilde{X}^4 - 72\tilde{X}^2 + 1 \right).
\end{align*}$$

Let us give some additional examples. The codimension-one variety corresponding to $G_1^I = I$ identity is also self-dual and reads in terms of the two invariants $X$ and $Y$:

$$\begin{align*}
C_7 &= 0 = Y^4 + \left(5X - 1 - 10X^2 \right) \cdot X \cdot Y^3 + \left(111X^2 - 36X + 4 - 111X^3 \right) \cdot X^3 \cdot Y^2 \\
&\quad + \left(13X - 291X^4 + 194X^5 + 192X^3 - 69X^2 - 1 \right) \cdot X^4 \cdot Y + \left(7X - 1 + 7X^3 - 14X^2 \right) \cdot X^9,
\end{align*}$$
or equivalently,
\[
\tilde{C}_7 = 0 = 2985984 \cdot Y^4 - 497664 \cdot \tilde{X} \cdot \left(1 + 60 \cdot \tilde{X}^2\right) \cdot \tilde{Y}^3 \\
+ \left(1728 - 435456 \cdot \tilde{X}^2 + 30606336 \cdot \tilde{X}^4 - 331444224 \cdot \tilde{X}^6\right) \cdot \tilde{Y}^2 \\
+ 288 \cdot \tilde{X} \cdot \left(2011392 \tilde{X}^8 - 20736 \tilde{X}^6 - 6048 \tilde{X}^4 + 48 \tilde{X}^2 + 1\right) \cdot \tilde{Y} \\
+ \left(20901888 \tilde{X}^{12} - 17418240 \tilde{X}^{10} - 2301696 \tilde{X}^8 - 7920 \tilde{X}^4 + 269568 \tilde{X}^6 + 120 \tilde{X}^2 - 1\right).
\]
The codimension-one variety corresponding to \(G_1 = \text{Identity}\) is self-dual and reads:
\[
\tilde{C}_8 = -Y^4 + (2 - 14X + 28X^2) \cdot X \cdot Y^3 + (240X^3 - 93X^2 - 240X^4 + 16X - 1) \cdot X^2 \cdot Y^2 \\
= (12X - 195X^4 - 58X^2 + 130X^5 + 144X^3 + 1) \cdot X^4 \cdot Y + (2X^2 - 4X + 1) \cdot X^6 \cdot 10 = 0,
\]
or equivalently,
\[
\tilde{C}_8 = 0 = -4478976 \cdot Y^4 + 1492992 \cdot \tilde{X} \cdot \left(84 \tilde{X}^2 - 1\right) \cdot \tilde{Y}^3 \\
+ \left(10368 - 1074954240 \tilde{X}^6 - 746496 \cdot \tilde{X}^2 + 31352832 \tilde{X}^4\right) \cdot \tilde{Y}^2 \\
+ 432 \cdot \tilde{X} \cdot \left(12 \tilde{X}^2 + 1\right) \cdot \left(112320 \tilde{X}^4 + 2736 \tilde{X}^4 - 156 \tilde{X}^2 + 1\right) \cdot \tilde{Y} \\
+ 3 \cdot \left(144 \tilde{X}^4 + 48 \tilde{X}^2 - 1\right) \cdot \left(20736 \tilde{X}^8 - 34560 \tilde{X}^6 + 864 \tilde{X}^4 - 48 \tilde{X}^2 + 1\right).
\]
The following (codimension-one) self-dual varieties \(C_N = 0\) are given in Appendix B for \(N = 9, \ldots, 15\). One can easily write down these conditions in terms of the “pseudo-self-dual” invariants \(\tilde{X}, \tilde{Y}\) in order to make explicit the self-dual character of conditions \(C_N = 0\).

The coefficients quickly become quite large, however one can get simpler expressions and coefficients by introducing:
\[
\tilde{X} = \frac{X}{\sqrt{12}}, \quad \tilde{Y} = \frac{Y}{\sqrt{12^3}}.
\]
With these last invariants, conditions \(\tilde{C}_7\) and \(\tilde{C}_8\) are replaced by
\[
\tilde{C}_7 = Y^4 - 2X \cdot (5X^2 + 1) \cdot Y^3 + (1 - 21X^2 + 123X^4 - 111X^6) \cdot Y^2 \\
+ 2X \cdot (4X^2 + 97X^8 - 42X^4 + 1 - 12X^6) \cdot Y \\
- 1 + 10X^2 - 55X^4 + 156X^6 - 111X^8 - 70X^{10} + 7X^{12},
\]
\[
\tilde{C}_8 = 12X \cdot (7X^2 - 1) \cdot Y^3 - 3Y^4 + 12 \left(1 - 60X^6 - 6X^2 + 21X^4\right) \cdot Y^2 \\
+ 6 \cdot \tilde{X} \cdot (65X^6 + 19X^4 - 13X^2 + 1) \cdot \left(\tilde{X}^2 + 1\right) \cdot \tilde{Y} \\
+ 6 \cdot \left(X^4 + 4X^2 - 1\right) \cdot \left(X^4 + 4X^3 - 2X^2 + 1\right) \cdot \left(X^4 - 4X^3 - 2X^2 + 1\right).
\]
In the \(x = 1\)-limit, which corresponds to \(D_3 = 0\), i.e., \(X = 0\), the previous finite-order conditions \(C_N = 0\) (\(N = 3, \ldots, 15\)) read \(Y^M = 0\). All these equations \(G_N^N = \text{Identity}\) therefore degenerate into:
\[
u \cdot (1 + u) = 0,
\]
in the isotropic limit when \(x = 1\). Therefore, it seems likely that no \(x = 1\) isotropic point (except \(u = 0\) and \(u = -1\)) can belong to a finite-order condition \(G_1^N = \text{Identity}\) for \(N \neq 3\).

Remark. All these relations are \(S_3\)-symmetric, therefore they identify with conditions \(G_3^N = \text{Identity}\) and \(G_3^N = \text{Identity}\), and are therefore sufficient conditions for having a finite group.
6.3. Genus of the Finite-Order Conditions

It is remarkable that all these finite order conditions are codimension-one varieties. They can all be written\(^{13}\) in terms of the two group invariants \(X\) and \(Y\). These relations between \(X\) and \(Y\) (\(X\) or \(Y\)) cannot be obviously deduced from any further analysis of the discrete group generated by the inversion relations. Thus, several questions pop out. Where do the explicit equations of these curves come from? What is the nature of these curves? For instance, are these algebraic curves \(P(X, Y) = 0\) elliptic curves?

At least for \(G_1^N = \text{Identity}\) for \(N = 7\) (or 8), one can get some hint concerning this last question since the polynomial is of degree four in \(Y\):

\[
A_0 \cdot Y^4 + 4 \cdot A_1 Y^3 + 6 A_2 \cdot Y^2 + 4 A_3 \cdot Y + A_4 = 0,
\]

where the \(A_i\)'s are polynomial in \(X\). Introducing:

\[
g_2 = A_0 A_4 - 4 A_1 A_3 + 3 A_2^2, \quad g_3 = A_0 A_2 A_4 + 2 A_1 A_2 A_3 - A_0 A_2^2 - A_4 A_3^2 - A_3^2,
\]

and the discriminant:

\[
\Delta = g_2^3 - 27 g_3^2,
\]

discriminant \(\Delta\) reads for \(N = 7\):

\[
4^4 \cdot \Delta(N = 7) = X^{15} \cdot (29376X_6 - 29376X_5 + 16200X_4 - 5360X^3 + 1003X^2 - 97X + 4) \cdot (3X - 1)^{15}.
\]

These curves are not rational curves. Recognized in this discriminant are the two following rational cases \(X = 1/3\) and \(X = 0\), namely \(C_4 = 0\) and its dual variety. In fact, using the Macaulay algebraic geometry computing system \([54]\), one gets that condition \(C_4 = 0\) is a genus one curve, that \(C_5 = 0\) is a genus ten curve and that \(C_6 = 0\) is a genus 78 curve.

In the \((X, Y)\)-plane most of these infinite sets of finite-order conditions are algebraic curves of genus greater than one, amazingly associated with a foliation of \(P_4\) in algebraic (elliptic) surfaces.

6.4. Generalization for Arbitrary \(q\)

These calculations can be generalized for arbitrary \(q\). The (generically infinite order) birational formation \(G_1\) reads:

\[
X_{1G} = \frac{(x_1^2 x + x_1 q x - 3x_1 x - q + 2) \cdot x_2 \cdot (1 - x_1 + x_1 x_3 x - x_3)}{(x_1 - x_1 x_3 x + x_1 x_3 x - x_3)},
\]

\[
x_{2G} = \frac{N_{2G}}{D_{2G}}, \quad \text{where}
\]

\[
N_{2G} = (1 - 3x_3 x - x_3 x q + 2x_3 x + x_3 q x) \cdot x_1^2
\]

\[
\quad + (- (q - 3) \cdot x_3 + (q - 3) - (q^2 - 5q + 6) \cdot x_1 x + (q^2 - 5q + 6) \cdot x_3) \cdot x_1
\]

\[
\quad + (q^2 - 4q + 4) \cdot x_3^2 - (q^2 - 5q + 6) \cdot x_3 - (q - 2),
\]

\[
D_{2G} = (x_1 - x_1 x q + q + 2x_1 x - 2 + x_1 x q x + x_1 x_3 x q - 3x_1 x_3 x - x_3 q + 2x_3) x_3,
\]

\[
x_{3G} = \frac{-x_1 - x_1^2 x - x_1 x q + q + 2x_1 x - 2 + x_1^2 x_3 x + x_1 x q x - 3x_1 x_3 x - x_3 q + 2x_3}{1 - x_1 + x_1 x_3 x - x_3},
\]

\[
x_{G} = \frac{N_{1G} \cdot N_{2G}}{D_{1G} \cdot D_{2G}}, \quad \text{where}
\]

\[
N_{1G} = (x_1 - x_1 x q + q + 2x_1 x - 2 + x_1^2 x_3 x + x_1 x q x - 3x_1 x_3 x - x_3 q + 2x_3)^2,
\]

\(^{13}\)This is not surprising since one can show \([53]\) that conditions \(G_1^N = \text{Identity}\) are automatically invariant by the discrete group generated by the inversion relations.
\[ N_{2q} = x_1 - 1 + (q - 2) \cdot x_1 x_3 x - (q - 2) \cdot x_3, \]
\[ D_{1q} = -2x_1^2 x_3^2 - qx_1^2 x_3 x + 2x_1^2 x_3 x + qx_1^2 x_3^2 x - 5x_1 x_3^2 q x \]
\[ + 6x_1 x_3^2 - 6x_1 x_3 x + x_1 x_3^2 q^2 x + 5x_1 x_3 x q - 3x_1 x_3 q + 3x_1 - x_1 q - (q - 2)^3 \cdot x_3^3 \]
\[ + (q^2 - 5q + 6) \cdot x_3 + (q - 2). \]
\[ D_{2q} = (1 - x_1 + x_1 x_3 x - x_3) \cdot (x_1^2 x + x_1 x q - 3x_1 x - q + 2). \]

Writing down the \( G_i^N = \text{Identity relations} \) yields calculations that are too large. However, it is possible to get the solution for \( N = 3 \). Writing down the relations: \( G_i = G_i^{-2} \) on the four coordinates again yields the two relations:
\[ q \cdot D_3 - D_1 \cdot D_2 = 0 \quad \text{and} \quad D_3 = 0. \quad (6.16) \]

Conversely, one can easily see that \( G_1 \) reduces, for \( x = 1 \), to an \textit{order-three transformation}:
\[ (x_1, x_2, x_3, 1) \rightarrow \left( x_2, \frac{1 + q \cdot x_3 - 2 \cdot x_3}{x_3}, \frac{-1}{x_1 - 2 + q}, 1 \right). \quad (6.17) \]

Of course by duality, this is also the case for \( q \cdot D_3 - D_1 \cdot D_2 = 0 \).

**Remark.** For the Tutte-Beraha numbers the group is finite for \( D_3 = 0 \) (that is \( x = 1 \)) and therefore restricted to its dual variety: \( q \cdot D_3 - D_1 \cdot D_2 = 0 \). Do other algebraic varieties exist, such that the group becomes finite for arbitrary values of \( q \)? Do other codimension-one (or codimension-two) algebraic varieties also exist such that the group becomes finite for the Tutte-Beraha numbers? The calculations become unfortunately quickly very large.

**Remark.** For \( x = 1 \), but for arbitrary \( q \), it is known [10] that the group is isomorphic (up to semidirect product with finite groups) to \( \mathbb{Z} \times \mathbb{Z} \). Therefore, to some extent, the \( x = 1 \) subcase can be compared to the \( q = 3 \) subcase: in both cases, the group is (up to a semidirect product) isomorphic to \( \mathbb{Z} \times \mathbb{Z} \). In the \( x = 1 \) subcase, one has to introduce the \( J_i \)'s in order to see it, these \( J_i \)'s being order three for \( q = 3 \) (and arbitrary \( x \)), while in the \( q = 3 \) subcase, it is necessary to introduce the \( G_i \)'s in order to see that the group isomorphic to \( \mathbb{Z} \times \mathbb{Z} \) (see Section 4.3), these \( G_i \)'s being of order three when \( x = 1 \) (and arbitrary \( q \)).

### 7. Symmetry Group Invariant Approximants for Spontaneous Magnetization

The previous algebraic group invariants are certainly well-suited variables to analyze the “analytical complexity” of the various physical quantity one can encounter. However, most of the physical quantities depend, in a quite nontrivial way, of various “spectral” parameters [47]. In this respect, some “one-point functions” like, for instance, the spontaneous magnetization can be seen as \textit{remarkable group invariant expression} which should not depend “too much” of various “spectral” parameters (whatever they are [49]. . . ). Of course we do not expect the spontaneous magnetization of the edge Potts model to be a closed \textit{algebraic expression} (like for the Ising model: see 7.1 in the following). It can be seen as a “transcendental invariant” for the group.

Taking advantage of the previous analysis which singles out a “canonical” invariant (namely \( \Delta^e_1 \)) corresponding to \( x \neq 1 \) deformations of the edge Potts mode, one may ask the following question: is it possible, to write down the spontaneous magnetization as a (more or less involved) function of invariants like the group-invariants \( \Delta^e_i, I, \) and \( J_0 \), (respectively, defined by (4.24), (4.39), (4.49))? Is it possible to write a closed expression of these invariants which could be a good approximation for the spontaneous magnetization?

Let us consider the \( q = 3, \ x = 1 \) subcase of this model. This subcase can be seen as a \textit{testing ground} for providing an example of \textit{exact calculation} using the \textit{symmetries} (and various
analyticity assumptions), but no Yang-Baxter structure. We use here the specificity of the Tutte-Beraha numbers and of the $x = 1$ (no three-spin interaction) case. The calculations are sketched in Appendices C, D, and E.

Let us first recall the exact expression of \textit{modulus of the elliptic functions} occurring in the \textit{elliptic parametrization} of the checkerboard Ising model [49,55]:

\begin{equation}
\begin{aligned}
k^2 &= \prod_{i=1}^{l} \frac{t_i \cdot (1 - t_i) \cdot (1 + t_i) \cdot (t_i^* + t_j^* \cdot t_k^* \cdot t_l^*)}{t_i^* \cdot (1 - t_i^*) \cdot (1 + t_i^*) \cdot (t_i + t_j \cdot t_k \cdot t_l)},
\end{aligned}
\end{equation}

where the $t_i$'s denote the usual high-temperature variable $t_i = \t(K_i)$, and the $t_i^*$'s denote their dual $t_i^* = (1 - t_i)/(1 + t_i)$. It is clear on (7.1) that this valuable expression, which is a key ingredient to foliating the parameter space, parameterizing the model, finding the critical variety (namely $k^2 = 1$) and to actually solve the model, could have been “guessed” from the various degeneracies of the model, namely $K_i = 0$, $K_i = \infty$ and the disorder solutions, their (Kramers-Wannier) dual and the action of the discrete group generated by the inversion relations. Let us also recall the exact expression of the spontaneous magnetization of the checkerboard Ising model [55]:

\begin{equation}
\begin{aligned}
M &= (1 - k^2)^{1/4}.
\end{aligned}
\end{equation}

The spontaneous magnetization has a remarkably simple expression in terms of the algebraic invariant $k^2$. When restricted to the critical condition, $k^2 = 1$, it vanishes. Furthermore, when restricted to the “order” solutions (dual of the disorder solutions), namely $t^*_i + t^*_j t^*_k t^*_l = 0$, the spontaneous magnetization $M$ becomes equal to 1. This can be checked formally on low temperature resummed expansions of the spontaneous magnetization [56,57]. Both quantities $M$ and $k^2$ do have the same symmetries (symmetry of the square, inversion relation symmetries, ...).

It is tempting to try to generalize this result to $q$-state edge Potts models, and, in particular, to the three-state standard scalar edge checkerboard Potts model [58], in order to get simple closed expressions (or approximations), for instance, for the spontaneous magnetization. The calculations are sketched in Appendix C.1 for the checkerboard lattice because this very lattice provides a “nice” representation of the Kramers-Wannier duality, namely $k \rightarrow 1/k$. Unfortunately these calculations are too naive and yield an algebraic invariant which does not suite well with the resummed expansions (see Appendices C.2 and C.3).

Actually we will see that the honeycomb lattice (dual of the triangular one) is better suited to address this question (see Appendix D).

### 7.1. Algebraic Invariant for the Honeycomb Lattice

Let us recall the low-temperature variables (3.8) and the well-known (ferromagnetic) critical variety of the three-state honeycomb Potts model [39]:

\begin{equation}
\begin{aligned}
C^\text{honey}_0 &= -1 + (A + B + C) + 2 \cdot (BA + CA + BC) + ABC = 0.
\end{aligned}
\end{equation}

The results are the following. Let us introduce the following (see Appendix D.2 for more details) group invariant for the honeycomb lattice:

\begin{equation}
\begin{aligned}
k^\text{Potts} &= \frac{N^\text{Potts}}{(D^\text{Potts})^3}, \quad \text{where} \\
N^\text{Potts} &= 27(ABC + AA + BC)(BCA + B + CA)(ABC + C + BA)(ABC + A + B + 2BA + 1) \\
&\quad \times (ABC + C + B + 2BC + 1)(ABC + C + A + 2CA + 1)(A - C + B + BA + 1) \\
&\quad \times (C + A - B + CA + 1)(C - A + B + BC + 1), \quad \text{and} \\
D^\text{Potts} &= 1 + (A + B + C) + 2 \cdot (BC + CA + BA) - (A^2 + B^2 + C^2).
\end{aligned}
\end{equation}
\[- \frac{1}{2} (A^3 + B^3 + C^3) - (CA^2 + B^2A + C^2A + RA^2 + R^2C + RC^2) + 6ABC \]
\[- 2(A^3B + B^3A + A^3C + B^3C + C^3A + C^3B) + 2 \cdot (B^2A^2 + C^2A^2 + B^2A^2) \]
\[- 3 (BC^2A + 3B^2CA + 3BCA^2) \]
\[- ABC \cdot (3CA + 3BA + 3BC + 5A^2 + 5B^2 + 5C^2) \]
\[- 2ABC (CA^2 + B^2C + C^2A + BA^2 + B^2A + BA^2 + 6ABC) \]
\[- 5A^2B^2C^2(C + B + A) - 2 \cdot A^2B^2C^2 \cdot (AC + BC + AB) - A^3B^3C^3. \]

Let us introduce a first approximation for the derivative of the (low temperature normalized) partition function, \( z \cdot \frac{d}{dz} \ln(\Lambda) \), namely:

\[ D_{er} = (1 - k_{Potts})^{1/9} - 1. \]  

(7.5)

The (resummed low temperature [10]) expansion of \( z \cdot \frac{d}{dz} \ln(\Lambda) \) can be written as follows:

\[
z \cdot \frac{d}{dz} \ln(\Lambda) = 2 \cdot \frac{A(C + BA)(B + CA)}{(1 - A^2)^2} + 2 \cdot \frac{A^2 \cdot (A - 1) \cdot (A^2 - 4A - 1)}{(1 - A^3)^3} \cdot (B + C) \cdot BC + 4 \cdot \frac{A^4}{(1 - A^2)^3} \cdot (B + C)^3 \]
\[
+ 2 \cdot \frac{(B + 2CA)(C + BA)}{(1 - A^2)^4} \cdot (A^3 - 5A^2 + A - 1) \cdot (A - 1)^3 \cdot BC \]
\[
+ 4 \cdot \frac{(B + CA) \cdot (C + BA) \cdot A \cdot (1 + 5A^2 + 2A^4)}{(1 - A^2)^4} \cdot (B + C)^2 \]
\[
+ 2 \cdot \frac{(3A^5 + 4A^4 + 3A^3 - A^2 - 1) \cdot (A - 1)^3 \cdot A \cdot (B + CA) \cdot (C + BA) \cdot BC}{(1 - A^2)^3 \cdot (1 - A^3)^2} \]
\[
+ 4 \cdot \frac{A^2 (2A^5 + 4A^4 + 5A^3 + 5A^2 + 2A + 1 + A^6)}{(1 - A^2)^4} \cdot \frac{(1 - A)(B + CA)(C + BA)}{(B + C)^2} \]
\[
+ \frac{6A^6}{(1 - A^2)^4} \cdot (B + C)^4 + 2 \cdot \frac{(A - 1)A^4 \cdot (A^2 - 9A - 4)}{(1 - A^2)^4} \cdot BC \cdot (B + C)^2 \]
\[
- 2 \cdot \frac{(A - 1)^2(A - 3)(A + 1)A^3}{(1 - A^2)^4} \cdot B^2C^2 \]
\[
+ 4 \cdot \frac{A^2(C + BA)(B + CA)(4A - 5)(1 - A)^2}{(1 - A^2)^4} \cdot BC \]
\[
- 4 \cdot \frac{A^2(C + BA)(B + CA)(2A^3 + 7A)}{(1 - A^2)^4} \cdot (B + C)^2. \]

(7.6)

A straightforward calculation enables us to write (7.6) in terms of \( D_{er} \) as follows:

\[
\frac{-3}{2} \cdot z \cdot \frac{d}{dz} \ln(\Lambda) = D_{er} \cdot (1 + \lambda \cdot D_{er} + R_{est} + \cdots), \]  

(7.7)

where \( R_{est} \) has a remarkably simple expression for \( \lambda = -3/2 \), namely:

\[
R_{est} = \frac{A \cdot (1 + A - 4A^2 + A^3 + A^4)}{(1 - A^3)^2} \cdot BC + \frac{2A^3}{(1 - A^3)^2} \cdot (B + C)^2 \]
\[
= \frac{A \cdot (1 - A^5)}{(1 - A) \cdot (1 - A^3)^2} \cdot BC - \frac{5 \cdot A^3}{(1 - A^3)^2} \cdot BC + \frac{2A^3}{(1 - A^3)^2} \cdot (B + C)^2. \]  

(7.8)
One should also note that invariant $k_{\text{Potts}}$ is built in such a way that it does vanish when one is restricted to the "order conditions" of the three-state edge Potts model on a honeycomb lattice, $(A + BC + ABC = 0, \ldots)$. Furthermore, when one restricts oneself to the critical condition of the three-state edge Potts model on a honeycomb lattice, $k_{\text{Potts}}$ actually becomes equal to 1. Therefore, the closed expression (7.5) leads to the recovery of the critical (magnetic) exponent of the three-state Potts model $[39]$, $\beta = 1/9$, on the critical variety $C_0^{\text{honey}} = 0$.

Conversely, condition $k_{\text{Potts}} - 1 = 0$ factorizes into this very critical condition together with three other algebraic varieties, namely:

$$
k_{\text{Potts}} - 1 = 0 \iff C_0^{\text{honey}} \cdot \left(C_1^{\text{honey}} \cdot C_2^{\text{honey}} \cdot C_3^{\text{honey}}\right)^2 = 0,
\text{ with}
$$

$$
C_0^{\text{honey}} = -1 + (A + B + C) + 2 \cdot (BA + CA + BC) + ABC,
C_1^{\text{honey}} = ABC^2 + 2ABC + A + 2CA + B + 2BC - C^2 + 1,
C_2^{\text{honey}} = A CB^2 + 2BA + A + 2ABC + 1 - B^2 + C + 2BC + 1,
C_3^{\text{honey}} = BCA^2 - A^2 + 2BA + B + 2CA + C + 2ABC + 1.
$$

This factorization of condition $k_{\text{Potts}} - 1 = 0$ is reminiscent of the factorization for the checkerboard Ising model (see equation (C.16) in Appendix C.1). If formula (7.5) is taken for granted, one should expect a $\beta = 2/9$-magnetic critical exponent $[39]$ on these three new varieties (7.9).

By performing a Kramers-Wannier duality $[40,41]$, one deduces three remarkable varieties for the triangular lattice from the last three algebraic varieties. In terms of the variable $x_i$'s one of these reads:

$$
x_3x_4x_1^2 + 2x_3x_1x_2 - x_3^2 + 2x_3x_1 + 2x_3x_2 + 1 + x_2 + x_3 = 0, \quad \text{or}
$$

$$(x_1 + 2) \cdot (x_1x_2x_3 - (x_1 + x_2 + x_3) - 1) + 3 \cdot (x_1 + 1) \cdot (1 + x_2 + x_3) = 0,$$

which have a remarkably simple form in terms of the rational parameterization of the $x = 1$ subcase of the Potts model (see (4.35)):

$$
u_2 \cdot u_3 + t^2 \cdot u_3^2 = 0, \quad \text{with: } t^6 = -1.
$$

Such varieties, remarkably simple in terms of the well-suited variables $u_i$'s are specific of the Tutte-Beraha numbers (see Appendix E.2).

### 7.2. Comments on Invariant $k_{\text{Potts}}$

Since expression (7.4) seems to be particularly well-suited to "decipher" the resummed low-temperature expansions of the spontaneous magnetization of the three-state edge Potts model on the honeycomb lattice (see (7.8) and Appendix D.3), it is tempting to compare (7.5) to other expansions available in the literature, and in particular to the expansions on which the most extensive studies have been performed, namely isotropic low-temperature expansions $[59,60]$. Unfortunately for the three-state edge Potts model, only high-field expansions are specifically dedicated to the honeycomb lattice $[59]$. From this expansion, the first coefficient of a low-temperature expansion can be deduced. The agreement between (7.5) and this deduced low-temperature expansion for the spontaneous magnetization is quite good (see Appendix D.4).

The largest low-temperature expansions for the spontaneous magnetization have been obtained recently for the square lattice up to order $A^{47}$ in $[60]$. By setting the limits $C = 0$ and $B = A$, one can easily get from (7.5), the equivalent of (7.5) for the isotropic three-state edge Potts model on the square lattice. For the square lattice, $k_{\text{Potts}}$ reads:

$$
k_{\text{Potts}}^{\text{square}} = \frac{27 \cdot A^4 \cdot (2A^2 + 2A + 1) \cdot (A + 1)^4}{(1 + 2A - 4A^3 - 2A^4)^3}.
$$

(7.11)
The low-temperature expansion of the spontaneous magnetization of the three-state edge Potts model square lattice reads [60]:

\[ M_{\text{square}} = 1 - \frac{3}{2} \cdot z \cdot \frac{d}{dz} \ln(A) = 1 - 3A^4 - 12A^6 - 12A^7 - 36A^8 - 108A^9 \]

- 210A^{10} - 480A^{11} - 1746A^{12} - 2340A^{13} - 10566A^{14} - 19500A^{15} - 53976A^{16} - 152604A^{17} - 329424A^{18} - 971304A^{19} - 2403291A^{20} - 595576A^{21} - 1685884A^{22} - 4033736A^{23} - 110301321A^{24} - 287061696A^{25} - 730223028A^{26} - 1985703720A^{27} - 507001716A^{28} - 13446444720A^{29} - 3560214232A^{30} - 9242918828A^{31} - 247542929499A^{32} - 684347258796A^{33} - 1713912378552A^{34} - 455959391288A^{35} - 1199131590344A^{36} - 3194310371528A^{37} - 8459993992118A^{38} - 224265087762144A^{39} - 59751183594619A^{40} - 158423140110704A^{41} - 422029510342656A^{42} - 11234571367790256A^{43} - 29892165171334848A^{44} - 79763126301078204A^{45} - 21250008247443470A^{46} - 56706277783225940A^{47},

and the expansion of (7.5) (but for (7.11)) yields:

\[(1 - k_t \zeta) \ln(A) = -3A^4 - 12A^6 - 12A^7 - 45A^8 - 96A^9 - 234A^{10} - 576A^{11} - 1464A^{12} - 3468A^{13} - 9108A^{14} - 22032A^{15} - 57774A^{16} - 144000A^{17} - 373800A^{18} - 952128A^{19} - 2466738A^{20} - 6353328A^{21} - 16509432A^{22} - 42797448A^{23} - 11161712A^{24} - 290794368A^{25} - 760795740A^{26} - 1990352736A^{27} - 522185922A^{28} - 13707847944A^{29} - 3656058192A^{30} - 94917607680A^{31} - 250248655992A^{32} - 660357590880A^{33} - 1744608618304A^{34} - 4613473905696A^{35} - 1221192342131436 - 32351256927936A^{37} - 8577906608604A^{38} - 22761307398444A^{39} - 60443435903817A^{40} - 160620398079436A^{41} - 427121603520722A^{42} - 11365107818350656A^{43} - 302602902858808A^{44} - 80615490837540204A^{45} - 214886746688288580A^{46} - 57310169815224528A^{47}.

The difference between these two expressions reads:

\[9A^8 - 12A^9 + 24A^{10} + 96A^{11} - 300A^{12} + 1128A^{13} - 1458A^{14} + 2532A^{15} + 3798A^{16} - 8604A^{17} + 44376A^{18} - 19176A^{19} + 63447A^{20} + 397752A^{21} - 349152A^{22} + 2460072A^{23} + 1315893A^{24} + 3732672A^{25} + \ldots.\]

The agreement between these two expansion is remarkable. For instance, if one compares the coefficient of \(\Lambda^{22}, \Lambda^{23}, \Lambda^{24}, \Lambda^{25}, \Lambda^{36}, \Lambda^{46}\), and \(\Lambda^{47}\), the coefficients in these previous two expansions are actually equal up to 0.0207, 0.0609, 0.0119, 0.0130, 0.0149, 0.0112, and 0.0105, respectively.

The agreement between these two expansion is so good that it is tempting to imagine that the Kramers-Wannier dual of (7.4) is also well-suited to express the spontaneous magnetization of the three-state edge Potts model on the triangular lattice and that there may exist a single expression generalizing the previous two for the three-state edge Potts model on the checkerboard lattice. Unfortunately, the Kramers-Wannier dual of (7.4) is not a well-suited expression for the (low-temperature) resummed expansion of the spontaneous magnetization of the three-state edge Potts model on the triangular lattice (see Appendix E.4). The resummed expansion of the spontaneous magnetization of the three-state Potts model on the anisotropic triangular lattice corresponds to:

\[z \cdot \frac{d}{dz} \ln(A) = \frac{2A^2C^2}{(1 - A^2C^2)^2} \cdot B^2 + \frac{4A^4C^4}{(1 - A^2C^2)^3} \cdot B^3 + 2 \cdot M_4 \cdot \frac{A^2C^2}{(1 - A^4C^3)^2(1 - A^2C^4)} \cdot B^4 + \ldots.\]

14In particular when one recasts the problems encountered in the Padé analysis of the three-state Potts models.
where

\[
M_4 = 2A^2 + 2C^2 + 6A^3C^3 - 8A^2C^2 + A^4C^4 + 2C^4A^2 - 4A^5C^3 + 2A^6C^6 - 4A^7C^5
+ 2A^{10}C^8 - 4C^5A^3 + 2C^6A^6 - 4C^7A^5 + 2C^{10}A^8 - 2A^7C^6 - 2A^6C^7 - 2A^6C^6 + 8A^5C^5
- 8A^8C^8 + 3A^{10}C^{10} - 2A^4C^3 - 2A^3C^4 + 2CA^2 + 2C^8A^6 + 2C^2A + 2C^9A^5 + 2A^4C^2.
\]

Another invariant seems to dominate the expansion of the triangular lattice (see Appendix E.4). It should be quite close to the following invariant:

\[
Y_{\text{triang}} = \frac{ABC \cdot (1 + A) \cdot (1 + B) \cdot (1 + C)}{(ABC + (BC + CA + BA) - 1)^3}.
\] (7.12)

In the case of the triangular three-state edge Potts mode, Monte-Carlo simulations have been performed\(^{15}\) in order to see if the new varieties (7.10) could not be critical varieties. Monte-Carlo calculations show that these varieties are not critical varieties with a magnetic critical exponent given by formula (7.5) (namely \(\beta = 2/9\)), but it may be possible that these varieties could be "special" in some way (see Appendix E.3).

Furthermore, expression (7.11) for the square lattice, inherited from the one for the honeycomb lattice (see (7.4)) does not yield a simple representation of the Kramers-Wannier duality. The Kramers-Wannier dual of (7.11) reads:

\[
k_{\text{square}}^{KW} = \frac{(1 + 2A)^2(2 + A)^4(5 + 2A + 2A^2)(A - 1)^4}{(10A - 1 + 12A^2 + 4A^3 + 2A^4)^3}.
\] (7.13)

Eliminating variable \(A\) between (7.11) and (7.13) yields a quite complicated (involutive) algebraic relation between \(k_{\text{Potts}}^{\text{square}}\) and \(k_{\text{KW}}^{\text{square}}\) (see Appendix D.3).

These two facts suggest, for the checkerboard and square lattices, that one should introduce, instead of a single one, (at least) two invariants, one being dominant for the honeycomb limit and another for the triangular limit. A “nice” representation of Kramers-Wannier duality probably requires considering at least these two invariants. This could be consistent with the fact that varieties (7.10) are not \(\beta = 2/9\)-critical varieties. This also suggests that expressions like (7.4) are just approximations for the dominant singular part of the spontaneous magnetization\(^{16}\) and have to be improved. Finally, this suggests that, even for the honeycomb three-state Potts mode, one should be able to improve the previous results (see (7.4) and (7.7)) and get “improved” symmetry-invariant approximations for the spontaneous magnetization.

### 7.3. Towards “Improved” Algebraic Invariants

In order to “improve” the invariants let us remark that the resummed expansions of the various group invariants provide either simple \(1 - A^2\), (respectively, \(1 - AC\)) singularities, or on the contrary, quite involved singularities but not the \(N^\text{th}\) root singularities \(1 - A^N\) known to occur in the resummed expansions of the spontaneous magnetization of the three-state (or q-state) edge Potts models \(^{10}\). Actually, though (7.5) enables us to retrieve the exact critical exponent \(\beta = 1/9\), its \((B, C\) small\) expansion yields only \(1 - A^2\) singularities for the honeycomb lattice. This could suggest that the \(1 - A^N\) singularities in the resummed expansions, are not related to the dominant singular part \(^{13}\) of the spontaneous magnetization but to sub-dominant singularities.

In order to understand the occurrence of these \(N^\text{th}\) root of unity, it is interesting to consider the resummed expansion of the partition function per site of the three state Potts model on

\(^{15}\)We thank J.-C. Anglès d'Auriac and H. Meyer for communicating these results prior to publications.

\(^{16}\)This situation is reminiscent of the susceptibility of the Ising model where the closed expression of Syzgi and Naya \(^{61,62}\) gives the dominant singular part of the susceptibility (see \(^{13}\)).
rubber bands [6,63]. Such an example is given in Appendix F for the square lattice. The (low-
temperature normalized) partition function per site, denoted \( \Lambda \), is one of the three solutions of a
polynomial equation:

\[
P(\Lambda) = \Lambda^3 + C_2 \cdot \Lambda^2 + C_1 \cdot \Lambda + C_0 = 0,
\]

where

\[
C_n = C_{n0} + C_{n1} + C_{n2} + \cdots, \quad n = 0, 1, 2,
\]

where the \( C_m \)'s are of order \( j \) in \( B \). Let us introduce the (resummed low-temperature) expansion
of \( \Lambda \):

\[
\Lambda = 1 + 3 \cdot (q - 1) \cdot \frac{W_2}{1 - A^2} \cdot B^2 + (q - 1) \cdot \sum_{n=3}^{\infty} \frac{W_n}{(1 - A^2)^{n-1}} \cdot B^n.
\]

(7.14)

One can actually understand, on this very example, that the occurrence of the singularity \( 1 - A^3 \),
is related to the following relation:

\[
3 + C_{10} + 2 \cdot C_{20} = (1 - A^2) \cdot (1 - A^3) = \frac{d}{d\Lambda} P(\Lambda), \quad (B = 0).
\]

(7.15)

Appendix F also enables to understand that singularity \( 1 - A^3 \) does not occur for \( W_2 \). Since
we know that the \( 1 - A^N \) singularities occur in the resummed expansion of the spontaneous
magnetization for all the values of the integer \( N \) [10], this seems to suggest that the “polynomial”
needed for a closed algebraic formula for the spontaneous magnetization is of “infinite” degree.

8. CONCLUSION

The discrete symmetry group generated by inversion relations has been analysed for the stand-
ard scalar Potts model with two- and three-sites interactions on the triangular
lattice [45]. The group generated by three involutions is seen to be generically a very large one (like a free group),
namely hyperbolic groups.

In this analysis a remarkable situation pops out for \( q \)-state Potts models for particular values
of \( q \), the so-called Tutte-Beraha numbers [23,46]. For these values of \( q \), some of the (generically
infinite order) generators are of finite order. However, even with such additional relations on
the generators, one still gets groups with an “exponential growth”, except for \( q = 3 \) (or \( q = 1 \)).
Additional relations on the generators can also occur on particular algebraic varieties, yielding a
degeneracy of the group into products of \( \mathbb{Z} \). We have seen that \( x = 1 \) and its dual variety (4.41)
are such varieties. It would be interesting to systematically seek more examples of such varieties.

A detailed analysis of the \( q = 3 \) case has been performed. For \( q = 3 \) the finite-order
conditions for which the group degenerates into a finite-order group are found to be codimension
one varieties. In this \( q = 3 \) subcase, a rather systematic study of well-suited group symmetry
invariants has been performed. It could be of some help to analyze the analytical structure
of certain physical quantities like, for instance, the spontaneous magnetization. In particular, we
have got a closed algebraic expression which is a quite good approximation of the spontaneous
magnetization for the honeycomb lattice.

As a byproduct, this analysis provides nice birational representations of hyperbolic Coxeter
groups as well as providing algebraic varieties having such large groups of (birational) automor-
phisms. It is clear to see that many calculations, performed on the hyperbolic Coxeter groups of
symmetries of triangular Potts models, can simply be generalized to three- (or higher-) dimen-
sional vertex models mutatis mutandis. This first analysis of hyperbolic Coxeter symmetry groups
for lattice models, including degeneracy subcases, should help a better understanding of the sym-
metries of three-dimensional models and provide tools to perform exact calculations based on the
symmetry analysis of these higher-dimensional models.
APPENDIX A

MORE RELATIONS ON THE GENERATORS
OF THE HYPERBOLIC COXETER GROUPS

The analysis of $\Gamma_{\text{triang}}$ can be performed directly on the $J_i$'s. Let us now consider the Tutte-Beraha subcases. One has the following relations between the $J_i$'s:

$$J_i^N = I_d, \quad i = 1, 2, 3. \quad (A.1)$$

One can easily deduce many other relations for the Tutte-Beraha numbers, for instance:

$$J_3 = J_2^{N-1} J_1^{N-1}. \quad (A.2)$$

Since $J_3$ can be rewritten in terms of $J_1$ and $J_2$, the Coxeter group can be seen as generated by $J_1$ and $J_2$ [28]. Then, using relations (A.1) with $i = 1$ or $i = 2$, one obtains the form of a general element of the group:

$$J_1^{n_1} J_2^{n_2} J_1^{n_3} J_2^{n_4} \cdots J_1^{n_k} J_2^{n_{k+1}}, \quad (A.3)$$

where $n_1, n_{k+1} = 0, 1, \ldots, N - 1$; $n_2 = 1, 2, \ldots, N - 1$; $\alpha = 2, 3, \ldots, k$.

Generically, there is no further relation between the $J_i$'s thus $\Gamma$ is a free group generated by two infinite generators (let us say, for example, $J_1$ and $J_2$).

Introducing the well-suited transformations:

$$G_1 = J_1 J_3 J_2, \quad G_2 = J_2 J_1 J_3, \quad G_3 = J_3 J_2 J_1, \quad (A.4)$$

one can show that $N = 3$ is singled out, $\Gamma$ reducing to $\mathbb{Z} \times \mathbb{Z}$ up to a semidirect product by a finite group.

At first, let us study the group, $G$, generated by $G_1, G_2$ and $G_3$. Relation (A.2) can also be written in the following way by using relation (A.2):

$$J_1 J_2 = J_3^{N-1}, \quad J_2 J_3 = J_1^{N-1}, \quad J_3 J_1 = J_2^{N-1}, \quad (A.5)$$

then $G_2 G_3$ reads:

$$G_2 G_3 = (J_2 J_1 J_3) (J_3 J_2 J_1) = J_2 J_1 J_3^2 J_2 J_1. \quad (A.6)$$

Notice that for $N = 3$, one can use relation (A.5) and obtain:

$$G_2 G_3 = J_2 J_1^2 J_2 J_1 = J_2 J_2^2 J_1 = J_3 J_1^2 J_3 J_2 = J_3 J_2 J_3 J_1 J_2 J_3 = J_3 J_2 J_1 J_2 J_1 J_3 = G_3 G_2. \quad (A.7)$$

Thus, the $G_i$'s actually commute if and only if $N = 3$. Furthermore, they do satisfy a relation of the same structure as (4.2):

$$G_1 G_2 G_3 = I_d. \quad (A.8)$$

Let us now suppose that $N = 3$. A generic element of $G$ reads:

$$g = G_1^{n_1} G_2^{n_2}, \quad (A.9)$$

where $n_1$ and $n_2$ are relative integers, which explicitly means that $G$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$.

Let us now demonstrate that $\Gamma$ is isomorphic to $G$, up to a semidirect product by a finite group. If $\gamma$ denotes a generic element of $\Gamma$, it can be written as follows:

$$\gamma = \left( \prod T_{i,j,k} \right) J_1^{\alpha_1} J_2^{\alpha_2}, \quad (A.9)$$

where $T_{i,j,k} = J_i J_j J_k$ and $(\alpha_1, \alpha_2) = (0, 1, 2)$. 

One can immediately replace, $T_{1,2,3}$, $T_{3,1,2}$, and $T_{2,3,1}$ by the identity transformation, and $T_{1,3,2} = G_1$, $T_{2,1,3} = G_2$, $T_{3,2,1} = G_3$. Besides, one has some kind of "pseudo"-commutation relations between the $G_i$'s and $J_i$'s:
\[
\begin{align*}
J_1 G_1 & = G_2 J_1, \\
J_2 G_1 & = G_3 J_2, \\
J_3 G_1 & = G_2 J_3, \\
J_1 G_2 & = G_3 J_1, \\
J_2 G_2 & = G_3 J_2, \\
J_3 G_2 & = G_3 J_3, \\
J_1 G_3 & = G_1 J_1, \\
J_2 G_3 & = G_1 J_2, \\
J_3 G_3 & = G_1 J_3,
\end{align*}
\]
(A.10)

then $\gamma$ reads:
\[
\gamma = \tilde{C}_1^{n_1} \tilde{C}_2^{n_2} \tilde{C}_3^{n_3} \left( \prod_{1 \leq i,j,k \leq 3} T_{i,j,k} \right) J_1^{n_1} J_2^{n_2} J_3^{n_3}.
\]
(A.11)

In fact, all the $T_{i,j,k}$'s, where $(i,j,k) = \{1,2,3\}$, have already been replaced by "words" in terms of the $G_i$'s. Thus, using relation $J_3 = J_2^2 J_1^3$, the only $T_{i,j,k}$'s, appearing in (8.11) are:
\[
\begin{align*}
T_{1,1,2} = T_1, & \quad T_{1,2,1} = T_2, & \quad T_{2,1,1} = T_3, & \quad T_{2,2,1} = T_4, & \quad T_{2,1,2} = T_5, & \quad T_{1,2,2} = T_6.
\end{align*}
\]

However, it is easy to remark that these $T_i$'s satisfy the following relations:
\[
\begin{align*}
T_2 = G_3 T_1, & \quad T_3 = G_2 G_3 T_1, & \quad T_6 = G_1 T_4, & \quad T_5 = G_2 G_1 T_4, \\
T_1^2 = G_1^2 G_2 T_4, & \quad T_4^2 = G_2^2 G_3 T_1, & \quad T_1 T_4 = T_4 T_1 = Id.
\end{align*}
\]

Thus, $\Gamma$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$ up to a semidirect product by a finite group.

\section*{APPENDIX B}

\textbf{FINITE-ORDER CONDITIONS $G_1^Q = \text{IDENTITY FOR } Q = 3$}

Let us give here a list of the codimension-one self-dual varieties corresponding to $G_1^Q = \text{Identity}$. They read:
\[
C_n = \sum_i c_i \cdot Y^i = 0,
\]
(B.12)

where the $c_i$'s are polynomial expressions in $X$ with (relative) integer coefficients.

The codimension-one variety, corresponding to $G_1^Q = \text{Identity}$, is self-dual and reads:
\[
\begin{align*}
C_9 = 3 \cdot Y^6 - (6 - 36 X + 72 X^2) \cdot X \cdot Y^5 + (4 - 46 X + 219 X^2 - 504 X^3 + 504 X^4) \cdot X^2 \cdot Y^4 \\
+ (670 X^3 - 1842 X^6 - 147 X^2 - 1842 X^4 - 1 + 2763 X^5 + 18 X) \cdot X^3 \cdot Y^3 \\
+ (2394 X^4 - 1197 X^5 + 669 X^2 - 1803 X^3 - 123 X + 9) \cdot X^7 \cdot Y^2 \\
+ (401 X^3 \cdot 806 X^4 + 16 X + 414 X^7 - 1035 X^6 + 1239 X^5 - 1 - 108 X^2) \cdot X^8 \cdot Y \\
+ (3 X^3 - 9 X^2 + 6 X - 1) \cdot X^{15} = 0,
\end{align*}
\]
or equivalently:
\[
\begin{align*}
\tilde{C}_9 = 15479341056 \cdot Y^6 - 371504185344 \cdot X^3 \cdot Y^5 \\
+ \left( 2600529297408 \tilde{X}^8 - 11943936 + 46438023168 \tilde{X}^4 \right) \cdot \tilde{Y}^4 \\
- 497664 \cdot \tilde{X} \left( 5 - 1248 \tilde{X}^2 + 54432 \tilde{X}^4 + 19097856 \tilde{X}^8 \right) \cdot \tilde{Y}^3 \\
+ \lambda_2 \left( \tilde{X} \right) \cdot \tilde{Y}^2 + 864 \tilde{X} \cdot \lambda_1 \left( \tilde{X} \right) \cdot \tilde{Y} + \lambda_0 \left( \tilde{X} \right) = 0,
\end{align*}
\]
with

$$\lambda_2 \left( \tilde{X} \right) = 5184 - 124416 \tilde{X}^2 - 48522240 \tilde{X}^4 - 8707129344 \tilde{X}^8$$
$$+ 2020054007808 \tilde{X}^{10} + 1755758592 \tilde{X}^{12} - 6176257081344 \tilde{X}^{14},$$

$$\lambda_1 \left( \tilde{X} \right) = -112320 \tilde{X}^6 - 87340032 \tilde{X}^{10} + 188116992 \tilde{X}^{12} + 11088 \tilde{X}^4 - 188 \tilde{X}^2 + 1$$
$$+ 2472394752 \tilde{X}^{14} - 3366144 \tilde{X}^8,$$

$$\lambda_0 \left( \tilde{X} \right) = -1 + 36 \tilde{X}^2 - 194586624 \tilde{X}^{10} + 20777472 \tilde{X}^8 + 15479341056 \tilde{X}^{16} + 9792 \tilde{X}^4$$
$$+ 392954944 \tilde{X}^{12} - 34826517376 \tilde{X}^{16} - 822528 \tilde{X}^6 - 19349176320 \tilde{X}^{14}.$$

Introducing invariants (6.12), these last expressions become simpler namely:

$$\tilde{C}_9 = 3 \tilde{Y}^6 - 72 \tilde{X}^5 \tilde{Y} + 4 \left( 126 \tilde{X}^6 + 27 \tilde{X}^4 - 1 \right) \tilde{Y}^4$$
$$- 2 \tilde{X} \cdot \left( 378 \tilde{X}^4 + 5 - 104 \tilde{X}^2 + 921 \tilde{X}^8 \right) \tilde{Y}^3$$
$$+ 3 \left( 196 \tilde{X}^6 - 65 \tilde{X}^4 - 399 \tilde{X}^{12} + 1566 \tilde{X}^{10} - 2 \tilde{X}^2 + 1 - 81 \tilde{X}^8 \right) \tilde{Y}^2$$
$$+ 2 \tilde{X} \cdot \left( -47 \tilde{X}^2 - 487 \tilde{X}^6 + 231 \tilde{X}^4 + 3 - 195 \tilde{X}^6 + 207 \tilde{X}^{14} - 1053 \tilde{X}^{10} + 189 \tilde{X}^{12} \right) \tilde{Y}$$
$$+ 3 \tilde{X}^2 - 1 - 476 \tilde{X}^6 - 782 \tilde{X}^{10} + 1002 \tilde{X}^8 + 1316 \tilde{X}^{12} + 3 \tilde{X}^{16} + 68 \tilde{X}^4 - 540 \tilde{X}^{14} - 81 \tilde{X}^{16}.$$

The codimension-one variety corresponding to $G_1^{10} = \text{Identity}$ is self-dual and reads:

$$\tilde{C}_{10} = \tilde{Y}^5 + (3 \tilde{X} - 1 - 6 \tilde{X}^2) \cdot \tilde{X} \cdot \tilde{Y}^5 + (255 \tilde{X}^2 - 85 \tilde{X} - 255 \tilde{X}^3 + 10) \cdot \tilde{X}^3 \cdot \tilde{Y}^4$$
$$+ (1185 \tilde{X}^4 - 300 \tilde{X}^2 + 800 \tilde{X}^3 + 60 \tilde{X} + 790 \tilde{X}^5 - 5) \cdot \tilde{X}^4 \cdot \tilde{Y}^3$$
$$+ (450 \tilde{X}^6 - 17 \tilde{X} - 4013 \tilde{X}^5 + 2055 \tilde{X}^4 + 1 - 2253 \tilde{X}^7 + 136 \tilde{X}^2 - 660 \tilde{X}^3) \cdot \tilde{X}^5 \cdot \tilde{Y}^2$$
$$+ (868 \tilde{X}^5 - 95 \tilde{X}^2 + 685 \tilde{X}^4 + 264 \tilde{X}^7 + 330 \tilde{X}^3 - 660 \tilde{X}^6 + 15 \tilde{X} - 1) \cdot \tilde{X}^8 \cdot \tilde{Y}$$
$$+ (1 - 3 \tilde{X} + \tilde{X}^2) \cdot \tilde{X}^{16} = 0,$$

or equivalently

$$\tilde{C}_{10} = 51597080352 \tilde{Y}^6 + 2579890176 \tilde{X} \cdot \left( 12 \tilde{X}^2 + 1 \right) \tilde{Y}^5$$
$$+ \left( 14929920 - 1612431360 \tilde{X}^2 - 1315743089760 \tilde{X}^6 + 10964532480 \tilde{X}^4 \right) \cdot \tilde{Y}^4$$
$$+ 2488329 \tilde{X} \cdot \left( 12 \tilde{X}^2 + 1 \right) \cdot \left( 136512 \tilde{X}^6 - 9648 \tilde{X}^4 + 156 \tilde{X}^2 - 1 \right) \tilde{Y}^3$$
$$- 1728 \cdot \left( 6727421952 \tilde{X}^{12} + 5 - 792 \tilde{X}^2 - 350653120 \tilde{X}^{10} - 2592000 \tilde{X}^8 \right.$$
$$+ 62640 \tilde{X}^4 + 36391680 \tilde{X}^8 \right) \tilde{Y}^2 + 1152 \tilde{X} \cdot \lambda_1 \left( \tilde{X} \right) \cdot \left( 12 \tilde{X}^2 + 1 \right) \tilde{Y}$$
$$+ \left( 1728 \tilde{X}^6 + 1584 \tilde{X}^4 - 60 \tilde{X}^2 + 1 \right) \cdot \lambda_2 \left( \tilde{X} \right) = 0,$$

with

$$\lambda_1 \left( \tilde{X} \right) = 98537472 \tilde{X}^{12} + 28366848 \tilde{X}^{10} + 62208 \tilde{X}^8 - 152064 \tilde{X}^6 + 6192 \tilde{X}^4 - 120 \tilde{X}^2 + 1,$$

$$\lambda_2 \left( \tilde{X} \right) = 2985984 \tilde{X}^{12} - 12441600 \tilde{X}^{10} + 311040 \tilde{X}^8 - 158976 \tilde{X}^6 + 11376 \tilde{X}^4 - 216 \tilde{X}^2 + 1.$$
Again in terms of invariants (6.12) this reads:

\[
\begin{align*}
\hat{C}_{10} &= \hat{Y}^6 - 6\hat{X} \left( \hat{X}^2 + 1 \right) \cdot \hat{Y}^5 + 5 \left( 1 - 51\hat{X}^6 - 9\hat{X}^2 + 51\hat{X}^4 \right) \cdot \hat{Y}^4 \\
&\quad + 10\hat{X} \cdot \left( \hat{X}^2 + 1 \right) \cdot \left( 79\hat{X}^6 - 67\hat{X}^4 + 13\hat{X}^2 - 1 \right) \cdot \hat{Y}^3 \\
&\quad + \left( -2253\hat{X}^{12} - 1755\hat{X}^8 - 5 + 66\hat{X}^2 + 1410\hat{X}^{10} + 1500\hat{X}^6 - 435\hat{X}^4 \right) \cdot \hat{Y}^2 \\
&\quad + 8\hat{X} \cdot \left( 33\hat{X}^{12} \cdot 114\hat{X}^{10} / 3\hat{X}^8 - 88\hat{X}^6 + 43\hat{X}^4 - 10\hat{X}^2 + 1 \right) \cdot \left( \hat{X}^2 + 1 \right) \cdot \hat{Y} \\
&\quad + \left( \hat{X}^6 + 11\hat{X}^4 - 5\hat{X}^2 + 1 \right) \cdot \left( \hat{X}^6 - 6\hat{X}^5 - 7\hat{X}^4 - 4\hat{X}^3 + 7\hat{X}^2 + 2\hat{X} - 1 \right) \\
&\quad \times \left( \hat{X}^6 + 6\hat{X}^5 - 7\hat{X}^4 + 4\hat{X}^3 + 7\hat{X}^2 - 2\hat{X} - 1 \right).
\end{align*}
\]

The codimension-one variety corresponding to \(G_{11} \neq \text{Identity} \) is self-dual and reads (B.12) with:

\[
\begin{align*}
c_{10} &= -1, \\
c_9 &= 2n - 26X^2 + 52X^3, \\
c_8 &= \frac{1098X^3 - 6X^2 - 441X^4 - 1098X^6 + 82X^3}{6}, \\
c_7 &= \frac{9462X^9 + 9432X^7 - 14193X^8 + 4X^3 - 3482X^6 + 742X^5 - 85X^4}{6}, \\
c_6 &= \frac{29X^9 - 5800X^8 - 10785X^{10} + 10729X^9 - 317X^6 - 2955X^{12} + 5910X^{11} + 1796X^7 - X^4}{6}, \\
c_5 &= \frac{450007X^{12} - 36X^7 - 8300X^9 - 183756X^{11} - 701604X^{13} - 254088X^{15} + 635220X^{14}}{6}, \\
&\quad + \frac{819X^8 + 48942X^{10}}{6}, \\
c_4 &= \frac{96392X^{12} - 905355X^{17} - 1181576X^{15} + 9X^8 - 238X^9 - 315601X^{13}}{6}, \\
&\quad + \frac{301785X^{18} - 238200X^{11} + 1306971X^{16} + 2852X^{10} + 728509X^{14}}{6}, \\
c_3 &= \frac{107428X^{14} - 416364X^{21} + 2469751X^{18} - 359872X^{15} + 1457274X^{20} - 24116X^{13}}{6}, \\
&\quad + \frac{3951X^{12} - 175698X^{17} + 31X^{10} + 91917X^{16} - 446X^{11} - X^9 - 240617X^{19}}{6}, \\
c_2 &= \frac{3921X^{17} - 23425X^{18} - 376065X^{22} + 221796X^{23} - 221400X^{20} + 55449X^{24}}{6}, \\
&\quad + \frac{362881X^{21} - 378X^{16} + 88794X^{19} + 16X^{15}}{6}, \\
c_1 &= \frac{1834X^{19} - 23850X^{26} + 71812X^{23} + 52821X^{25} - 279X^{18} - X^{16}}{6}, \\
&\quad + \frac{25X^{17} - 48670X^{22} - 74438X^{24} + 5302X^{27} + 2358X^{21} - 7891X^{20}}{6}, \\
c_0 &= \frac{-X^{25} + 11X^{26} - 44X^{27} + 77X^{28} + 55X^{29} + 11X^{30}}{6}.
\end{align*}
\]

The codimension-one variety corresponding to \(G_{12} \neq \text{Identity} \) (B.12) with:

\[
\begin{align*}
c_8 &= -2, \\
c_7 &= 0X - 38X^2 + 76X^3, \\
c_6 &= \frac{1088X^5 - 449X^4 - 1088X^6 + 88X^3 - 7X^2}{6}, \\
c_5 &= \frac{931X^9 - 9X^4 + 9120X^7 - 13995X^6 + 684X^5 - 3278X^9 + 4X^3}{6}, \\
c_4 &= \frac{30X^5 - 93939X^{10} - 406X^6 - 15416X^8 - 52838X^{12} + 48070X^9 + 3164X^7}{6}, \\
&\quad - X^4 + 105676X^{11}, \\
c_3 &= \frac{3076X^9 - 81979X^{12} + 106594X^{13} - 13977X^{10} - 421X^8 + 41512X^{11} - 86080X^{14} - X^6}{6}, \\
&\quad + 32X^7 + 34432X^{15}, \\
c_2 &= \frac{1692X^{11} + 88472X^{15} + 24X^9 - 53463X^{14} - 7428X^{12} + 70530X^{17}}{6}, \\
&\quad - 100506X^{16} - 260X^{10} + 23314X^{13} - X^8 - 23510X^{18}, \\
c_1 &= \frac{20X^{13} - 2849X^{20} + 814X^{21} + 5144X^{17} + 848X^{15} - 173X^{14} - 2594X^{16} - X^{12}}{6}, \\
&\quad - 6661X^{18} + 5544X^{19}, \\
c_0 &= X^{22} + X^{24} - 4X^{23}.
\end{align*}
\]
The codimension-one variety corresponding to $G_{14}^{13} = \text{Identity}$ (B.12) with:

\[
n_{14} = 1, \\
_{13} = 22X^2 - 4X - 44X^3, \\
_{12} = 6X^2 - 46X^3 + 131X^4 + 166X^5 - 166X^6, \\
_{11} = 19166X^9 + 16920X^7 - 28749X^8 - 9X^4 - 4870X^6 + 630X^3 - 4X^3, \\
_{10} = -376735X^{12} - 2061X^6 - 102535X^8 + 753470X^{11} + 332081X^9 - 662045X^{10} + 89X^5 + X^4 + 19408X^7, \\
_{9} = 179240X^9 - 5677671X^{12} - 84X^6 - 7084940X^{14} + 2172X^7 - 828356X^{10} + 2618872X^{11} + 2833976X^{15} - 25539X^8 + 8163172X^{13}, \\
_{8} = 36X^7 - 2581583X^{14} - 3595995X^{12} + 42698208X^{15} - 1116X^8 - 48502152X^{16} - 140616X^{10} + 11259027X^{13} + 840420X^{11} + 34213977X^{17} - 11406659X^{18} + 16016X^9, \\
_{7} = 62366X^{11} - 255902649X^{18} - 474376X^{12} + 2669814X^{13} + 43023480X^{21} - 9X^8 - 150582180X^{20} - 95144007X^{16} - 5741X^{10} + 331X^9 + 3764572X^{15} + 248402658X^{19} - 11431638X^{14} + 181864551X^{17}, \\
_{6} = 46844515X^{21} - 375453015X^{22} + 107275X^{13} - 48164520X^{24} + X^9 + 884X^{11} - 16596807X^{16} + 54132501X^{17} + 273496776X^{19} - 414599493X^{20} + 192658080X^{23} - 745487X^{14} - 138078991X^{18} - 11557X^{12} - 43X^9 + 3984560X^{15}, \\
_{5} = -4216X^{15} - 23246046X^{19} + 912409755X^{24} - 685515X^{17} - 71676719X^{25} - 76780498X^{27} - 797360400X^{23} + 120X^{14} - 239693730X^{21} + 68625X^{16} + 50581440X^{22} + 85870152X^{20} + 4601515X^{18} + 34551224X^{26}, \\
_{4} = 4790014X^{20} - 160266120X^{23} - 19969205X^{21} + 64161465X^{22} + 311586450X^{24} - 10627X^{17} - 463526472X^{27} - 40991481X^{25} + 2288745X^{230} - 16X^{15} + 28522169X^{26} - 114437275X^{29} - 869740X^{16} + 115599X^{18} + 539368809X^{26} + 604X^{16}, \\
_{3} = X^{16} - 9546X^{19} + 3003310X^{24} + 70613026X^{32} - 187554800X^{31} + 791X^{18} - 12838732X^{33} - 76069635X^{35} + 15330983X^{26} - 41X^{17} + 36098236X^{28} - 388448166X^{29} + 318744046X^{30} - 265247394X^{27} - 9639880X^{23} + 80811X^{20} + 2490905X^{22} - 510070X^{21}, \\
_{2} = 12740149X^{33} - 489711X^{30} - 527066X^{36} + 84186X^{27} - 12970962X^{32} - 456030X^{28} + 9220752X^{31} + 3162396X^{35} - 10326X^{29} - 8271495X^{34} + 1728297X^{29} - 25X^{24} + 756X^{26}, \\
_{1} = 17524X^{39} + 989925X^{33} - 3735X^{26} + X^{25} - 31X^{26} + 438X^{27} + 263504X^{31} + 21456X^{29} - 87768X^{30} - 113906X^{38} + 1155490X^{35} - 780578X^{36} + 370214X^{37} - 1242823X^{34} - 589817X^{32}, \\
_{0} = +182X^{40} + 13X^{42} - 91X^{41} - 156X^{39} + 65X^{38} - 13X^{37} + X^{36}. \\

The codimension-one variety corresponding to $G_{14}^{14} = \text{Identity}$ reads:

\[
c_{12} = 1, \\
_{11} = 33X^2 - 5X - 66X^3, \\
_{10} = 2091X^6 + 10X^2 + 820X^4 - 2091X^5 - 146X^3, \\
_{9} = 51360X^8 - 2295X^6 - 10X^3 - 33080X^7 + 11575X^6 - 34240X^9 + 240X^4, \\
_{8} = 88200X^8 - 548064X^{11} + 274032X^{12} + 495978X^{10} + 5X^4 + 2513X^6. \\
\]
\[-181X^5 - 262710X^9 - 18084X^7,\]
\[c_7 = -76615X^9 - 280584X^{13} + 981498X^{11} + 11998X^8 + 61X^6 - X^5 - 1176X^7 - 953352X^{15} + 329343X^{10} + 2027148X^{12} + 2383380X^{14},\]
\[c_6 = 533918X^{12} + 4675440X^{15} + 3080X^9 - 325383X^{17} + 3128625X^{14} + 24990X^{10} - 8X^7 - 137046X^{11} - 1512021X^{13} + 231X^8 + 1084401X^{18} + 4872273X^{16},\]
\[c_5 = 9069711X^{16} - 17768520X^{17} + 248101X^{15} + X^8 + 44870X^{12} - 6076X^{11} + 15511356X^{20} - 25335870X^{19} + 581X^{10} + 1058345X^{14} + 25532766X^{18} - 3518896X^{15} - 35X^9 - 4431816X^{21},\]
\[c_4 = 8031600X^{20} - 9501450X^{20} - 1330735X^{15} + 4278750X^{19} - 120X^{14} + 282775X^{17} + 14352285X^{21} + 3240X^{15} - 39345X^{16} + 14064945X^{22} - 2007900X^{24},\]
\[c_3 = 2264885X^{21} + 16X^{10} + 378374X^{27} + 239790X^{19} + 5816880X^{23} - 5670600X^{24} + 3856071X^{25} - 1702683X^{26} - 870000X^{20} - 46285X^{18} - 458X^{16} - 4268665X^{22} + 5949X^{17},\]
\[c_2 = 31X^{17} - X^{16} - 1656115X^{24} + 329065X^{22} + 2594400X^{25} - 23132X^{20} - 159531X^{30} - 3167946X^{26} + 2921346X^{27} - 443X^{18} + 100465X^{21} + 3870X^{19} + 797655X^{29} - 1915618X^{26} + 833240X^{23},\]
\[c_1 = 280X^{23} - 1855X^{24} + 65535X^{30} + 52220X^{27} + 34496X^{31} + X^{21} - 24395X^{26} + 87155X^{29} - 11517X^{32} - 79950X^{28} - 25X^{22} + 9094X^{33} + 8085X^{25},\]
\[c_0 = \pm X^{28} - X^{33} - 6X^{35} + 5X^{34}.\]

The codimension-one variety corresponding to $G_1^5 = \text{Identity}$ reads:
\[c_{16} = -5,\]
\[c_{15} = 290X^3 - 145X^2 + 25X,\]
\[c_{14} = 7125X^5 - 3045X^4 + 630X^3 - 55X^2 - 7125X^6,\]
\[c_{13} = 9443X^5 - 41741X^6 - 1211X^4 + 108766X^9 + 109872X^7 - 163149X^8 + 70X^3,\]
\[c_{12} = 97320X^7 + 2448680X^{11} + 1323X^5 - 56X^4 + 12017988X^9 - 14635X^6 - 421235X^8 - 1224340X^{12} + 2236919X^{10},\]
\[c_{11} = 678039X^9 + 8680056X^{15} + 12452X^7 - 111989X^8 - 17091363X^{12} + 21700140X^{14} + 8592370X^{11} + 25307166X^{13} - 853X^6 + 28X^5 - 2870244X^{10},\]
\[c_{10} = 80817464X^{15} + 469562X^{10} + 61854X^9 + 301X^7 - 95397318X^{16} + 2520133X^{11} - 9815259X^{12} - 5496X^8 - 21548015X^{18} + 64644045X^{17} + 28080170X^{13} - 59010131X^{14} - 8X^6,\]
\[c_9 = 350896976X^{18} - 12832X^{10} + 1643388X^{13} + 992X^9 - 13544232X^{15} + X^7 + 251206109X^{20} - 45X^8 + 13410264X^{16} - 71773174X^{21} - 541988X^{12} + 104413X^{11} - 38866798X^{19} - 201684145X^{17} + 1229013X^{14},\]
\[c_8 = 3563999094X^{20} - 10X^{11} + 86208330X^{16} - 212834044X^{19} - 10X^{10} - 754395375X^{18} + 4377647070X^{21} - X^{10} - 16979532X^{15} - 329227860X^{17} + 373740739X^{12} + 14823X^{12} + 497002248X^{24} + 2442285X^{14},\]
\[c_7 = 105X^{12} - 549761134X^{27} + 2473925103X^{26} - 4005X^{13} + 72131X^{14} - 813468X^{15} - 351027851X^{21} - 58891036X^{15} - 5443072605X^{25} + 771628769X^{24} + 168676936X^{18} + 596131603X^{22} - 7821106908X^{23} + 6422415X^{16} - 37590360X^{17} + 1619651049X^{20},\]
\[c_6 = 19904754327X^{26} - 15X^{13} - 846010922X^{21} + 992243662X^{30} + 11914238576X^{28} - 12843X^{15} - 55750129X^{19} + 240297445X^{20} + 164818X^{18} + 10350804X^{18}.\]
In terms of the "pseudo-self-dual" variables \( \tilde{X} = 1/6 - X \) and \( \tilde{Y} = 1/54 - Y \) (see (4.34)), these finite order conditions, respectively, become:

\[
\begin{align*}
\tilde{C}_{11} & \propto -3 \cdot \tilde{Y}^{10} + \tilde{X} \cdot \left(156 \tilde{X}^2 - 1\right) \cdot \tilde{Y}^9 + \ldots , \\
\tilde{C}_{12} & \propto -6 \cdot \tilde{Y}^8 + \tilde{X} \cdot \left(228 \tilde{X}^2 - 1\right) \cdot \tilde{Y}^7 + \ldots , \\
\tilde{C}_{13} & \propto 3 \cdot \tilde{Y}^{14} - \tilde{X} \cdot \left(132 \tilde{X}^2 + 1\right) \cdot \tilde{Y}^{13} + \ldots , \\
\tilde{C}_{14} & \propto 2 \cdot \tilde{Y}^{12} - \tilde{X} \cdot \left(132 \tilde{X}^2 - 1\right) \cdot \tilde{Y}^{11} + \ldots , \\
\tilde{C}_{15} & \propto -6 \cdot \tilde{Y}^{16} + \tilde{X} \cdot \left(1 + 348 \tilde{X}^2\right) \cdot \tilde{Y}^{15} + \ldots .
\end{align*}
\]

In the isotropic limit, these conditions yield fairly involved expressions. For instance, condition \( C_6^I = I_{\text{identity}} \) reads:

\[
\begin{align*}
u^{24} x^9 & + \left(u^{24} - u^{21} + 3u^{22} - 12u^{23}\right) x^8 \\
& + \left(60u^{22} - 9u^{20} - 12u^{21} + 3u^{19} - 12u^{21} + 4u^{18} - u^{15}\right) x^7 \\
& + \left(165u^{19} + 54u^{22} - 4u^{19} - 175u^{21} + 3u^{14} + 9u^{16} - 95u^{18} + u^{12} - 42u^{20}\right) x^6 \\
& + \left(6u^{16} + 13u^{15} - 620u^{16} + 198u^{20} + 9u^{11} - 15u^{14}\right) x^5 \\
& + \left(267u^{17} - 13u^{12} + 393u^{19} - 108u^{21} - u^9\right) x^4 \\
& + \left(744u^{17} - 1063u^{16} + 600u^{16} + 81u^{20} - 45u^{10} + 135u^{19} + 123u^{13} + 14u^9 - 916u^{15}\right) x^4 \\
& + \left(60u^{11} - 69u^{12} + 210u^{14}\right) x^3 \\
& + \left(807u^{16} + 789u^{17} + 360u^{10} - 35u^9 + 78u^{13} - 2730u^{15} - 63u^8 + 27u^7 - 279u^{12} - 387u^{11}\right) x^3 \\
& + \left(-378u^{18} + 1899u^{14} - 4u^6\right) x^2.
\end{align*}
\]
\[ + (633u^8 - 2618u^{15} + 999u^{13} + 2154u^{11} + u^3 + 720u^{15} + 48u^6 - 6u^5 + 2607u^{14} - 1242u^9) x^2 \\
+ (-3u^4 - 3722u^{12} - 186u^7 + 579u^{10}) x^2 \\
+ (1047u^7 + 306u^{10} - 1 + 2880u^{13} - 94u^3 + 6u + 3495u^{11} - 4982u^{12} + 309u^4 - 415u^8) x \\
+ (-1018u^9 - 294u^5 - 624ur^4 - 65u^2 + 1) z \\
- 78u^4 + 3 + 120u^2 + 1578u^8 + 996u^5 - 30u + 1781u^6 - 203u^3 + 2880u^{10} \\
+ 912u^7 + 224u^{12} - 1248u^{11} - 3374u^9 = 0. \]

which, within the \( x = 1 \) limit, gives:

\[ 2 \cdot (u + 1)^9 \cdot (u - 1)^{18} = 0. \]

Another example is condition \( G^3 = \text{Identity} \) which reads:

\[ u^5 \cdot (u + 1) \cdot x^2 - u^4 \cdot (1 + 3 \cdot u) \cdot x + 1 - 2u - u^2 + 4u^3 = 0, \]

together with a polynomial relation of degree 16 in \( x \), 42 in \( u \) which corresponds to the sum of 310 monomials and \( G^0 = \text{Identity} \) reads:

\[ x^3u^9 - 3u^8x^2 + (u^3 + 8u^5 + 3u^4 - u^2 + 3u^3) x + (1 - 3u + u^2 + 7u^3 - 9u^4 + 2u^5) = 0, \]

together with a polynomial relation of degree 24 in \( x \), 63 in \( u \) which corresponds to the sum of 695 monomials.

### APPENDIX C

**TOWARDS ALGEBRAIC INVARIANTS FOR THE THREE-STATE CHECKERBOARD POTTS MODEL**

#### C.1. CHECKERBOARD MODELS

Since, in the following, one considers low (or high) temperature expansions, let us introduce the low temperature variables:

\[ A = \frac{1}{x_1}, \quad B = \frac{1}{x_2}, \quad C = \frac{1}{x_3}, \quad D = \frac{1}{x_4}. \]  

(C.13)

The checkerboard Potts model (without magnetic field) is self-dual with respect to the Kramers-Wannier duality [40,41,55] which reads

\[ A \longrightarrow A^* = \frac{1 - A}{1 + (q - 1) \cdot A} = \frac{x_1 - 1}{x_1 + q - 1}. \]  

(C.14)

The modulus of the elliptic function of the checkerboard *Ising* model:

\[ k^2 = \prod_{i=1}^{4} \left( \frac{1}{t_i} \cdot (1 - t_i) \cdot (1 + t_i) \cdot (t_i^* + t_i^2) \cdot t_i^4 \right), \]  

(C.15)

reads (after simplifications) in terms of the low-temperature variables:

\[ k^2 = \frac{N_k}{D_k}, \quad \text{where} \]

\[ N_k = 16 \cdot (A + BCD) \cdot (B + ACD) \cdot (C + BAD) \cdot (D + BCA), \quad \text{and} \]

\[ D_k = (-1 - BC - AC - BA + CD + BD + AD + BCAD) \times (BC - 1 + AC - BA + CD - BD - AD + BCAD) \times (BC - 1 - AC + BA - CD + BD - AD + ABCD) \times (-1 - BC + AC + BA - CD - BD + AD + ABCD). \]
Remarkably condition \( k^2 - 1 = 0 \) factorizes the well-known critical condition of the model \( (F_{0}^{\text{acto}} = 0) \) together with three other varieties breaking the \( C_{4v} \) symmetry of the square of the lattice:

\[
k^2 - 1 = \frac{F_{1}^{\text{acto}} \cdot (F_{1}^{\text{acto}} \cdot F_{2}^{\text{acto}} \cdot F_{3}^{\text{acto}})}{D_k},
\]

where

\[
F_0^{\text{acto}} = ABCD - (AD + BD + CD + AC + BC + BA) + 1,
\]

\[
F_1^{\text{acto}} = ABCD + (AD + BD + CD + AC + BC + BA) + 1 - 2 \cdot (AD + BC),
\]

\[
F_2^{\text{acto}} = ABCD + (AD + BD + CD + AC + BC + BA) + 1 - 2 \cdot (AB + CD),
\]

\[
F_3^{\text{acto}} = ABCD + (AD + BD + CD + AC + BC + BA) + 1 - 2 \cdot (AC + BD).
\]

It is important to note that the set of these three conditions \( F_{i}^{\text{acto}} = 0, (i = 1, 2, 3) \) is stable by the two inversion relations of the checkerboard Ising model, and that these three additional varieties are also critical.

It is tempting to try to generalize a pattern like (C.15) to \( q \)-state Potts models. However, the analysis of the successive images of the disorder solutions under the (infinite) discrete group of birational transformations, generated by the inversion relations for the \( q \)-state edge checkerboard Potts model [25], shows that one gets an infinite number of such image varieties. In order to avoid infinite product expressions for a tentative substitute of \( k^2 \) (see (C.15)), it is necessary to restrict oneself to Tutte-Beraha numbers. Taking into account the remarkable properties of the Euclidean subcase \( q = 3 \) (see Section 4.3), the three-state standard scalar edge checkerboard Potts model, pops out as a good candidate for building the equivalent of the algebraic expression \( k^2 \), and hopefully, get exact expressions for the spontaneous magnetization.

**Checkerboard Potts Model for \( q = 3 \)**

Let us restrict to \( q = 3 \). Let us first of all, consider a polynomial \( Q \) which is the product of the four disorder conditions:

\[
Q = (ABCD + BCD + A) \cdot (ABCD + ACD + B) \cdot (ABCD + BAD + C) \cdot (ABCD + BCA + D).
\]

Denoting \( Q_{1}, Q_{2}, Q_{12}, Q_{21}, \) and \( Q_{121} \) the algebraic expressions corresponding to the action of:

\[
I_1, I_2, I_1 \cdot I_2, I_2 \cdot I_1, I_1 \cdot I_2 \cdot I_1
\]

with:

\[
I_1: (A,B,C,D) \rightarrow \left( \frac{1}{A} - \frac{B}{(1+B)}, \frac{1}{C}, 1, \frac{D}{(1+D)} \right),
\]

\[
I_2: (A,B,C,D) \rightarrow \left( \frac{1}{1+A}, \frac{1}{B}, \frac{1}{(1+C)}, \frac{1}{D} \right).
\]

One can see that the product of the action of the whole group (generated by the two inversion relations of the checkerboard model) is actually equal to the product \( Q_{\text{prod}} = Q \cdot Q_{1} \cdot Q_{2} \cdot Q_{12} \cdot Q_{21} \cdot Q_{121} \). This new product happens to be a perfect square. If one introduces the square root of \( Q_{\text{prod}} \), it reads \( G_{\text{check}} = N_{\text{check}}/D_{\text{check}} \) where \( N_{\text{check}} \) and \( D_{\text{check}} \) read:

\[
N_{\text{check}} = (ABCD + BCD + A) \cdot (ABCD + ACD + B) \cdot (ABCD + BAD + C) \cdot (BD + BCD + A + AD + AB + B) \cdot (AC + BCA + D + CD + AD + AC) \cdot (AC + ACD + B + BC + AB + BCA) \cdot (BD - 1 + CD + AD + ACD + BCD + BAD + ABCD) \cdot (BD - 1 + BC + AB + BCA + BCD + BAD + ABCD) \cdot (AC - 1 + CD + ACD + BC + BCA + BCD + ABCD) \cdot (AC - 1 + AD + ACD + AB + BCA + BAD + ABCD),
\]

\[17\] With the "awkward position" that the partition function is a multivalued function with infinite valuation [26].
\[ D_{\text{check}} = A^4 B^4 C^4 D^4 \cdot (1 + A)^4 \cdot (1 + B)^4 \cdot (1 + C)^4 \cdot (1 + D)^4. \]  
(C.21)

Let us denote \( G^* \) the (Kramers-Wannier) dual expression of \( G_{\text{check}} \). In order to have a nice representation of the Kramers-Wannier duality [40,41,55] (like for the checkerboard Ising model: \( k^2 \to 1/k^2 \)), one can introduce the following ratio:

\[ G = \frac{G_{\text{check}}}{G^*}, \quad \text{with} \quad G^* = \frac{N_{G^*}}{D_{G^*}}, \]  
(C.22)

where the denominator \( D_{G^*} \) and the numerator \( N_{G^*} \), respectively, read:

\[ D_{G^*} = (2 + A)^4 \cdot (2 + B)^4 \cdot (2 + C)^4 \cdot (2 + D)^4 \cdot (A - 1)^4 \]
\[ \times (B - 1)^4 \cdot (C - 1)^4 \cdot (D - 1)^4 \cdot (1 + 2A)^4 \cdot (1 + 2B)^4 \cdot (1 + 2C)^4 \cdot (1 + 2D)^4, \]

and

\[ N_{G^*} = 3^{12} \cdot N_1 \cdot N_2 \cdot N_3 \cdot N_4 \cdot N_5 \cdot N_6 \cdot N_7 \cdot N_8 \cdot N_9 \cdot N_{10} \cdot N_{11} \cdot N_{12}, \]

with, for instance,

\[ N_1 = -1 - 2BCD + ACD + BAD + BCA \]
\[ - 2CD - 2BD - 2BC + AD + AB + AC + 3ABCD. \]

Algebraic expression \( G \) is invariant by the group generated by the two inversions (C.18) and (C.19) of the checkerboard model. Note, that \( G \) is not symmetric under the whole group \( S_4 \) of permutation of \( \{A, B, C, D\} \) but only (as it should) under the symmetry group of the square \( C_4 \).

Let us now consider the two polynomials:

\[ S = (1 - A) \cdot (1 - B) \cdot (1 - C) \cdot (1 - D) \quad \text{and} \quad T = (1 + 2A) \cdot (1 + 2B) \cdot (1 + 2C) \cdot (1 + 2D), \]

as well as the following polynomials:

\[ U = (1 + A) \cdot (1 + B) \cdot (1 + C) \cdot (1 + D) \quad \text{and} \quad R = ABCD. \]

Let us denote \( ST \) their product. The product corresponding to the action of (C.18) and (C.19) on \( ST \), (namely the product \( P_H = H \cdot H_1 \cdot H_2 \cdot H_{12} \cdot H_{21} \cdot H_{121} \)) happens to be exactly equal to \( P_{ST} = S^4 T^4 W^4 / U^4 / R^4 \). Let us introduce, as a multiplicative correction term, \( C_G \) equal to \( P_{ST} / 3^{24} \). Expression \( C_G \) is, by construction, invariant under the group generated by the inversion relations. One now introduces, instead of \( G \), a new algebraic expression:

\[ \bar{G}_1 = \frac{G}{C_G} = 3^{24} \cdot \frac{\bar{G}}{P_{ST}}. \]  
(C.23)

We get the remarkable property that the critical condition of the \( q = 3 \) checkerboard Potts model, namely \([56,57]\):

\[ x_1 x_2 x_3 x_4 - (x_1 + x_2 + x_3 + x_4) - (x_1 x_2 + x_1 x_3 + x_1 x_4 + x_2 x_3 + x_2 x_4 + x_3 x_4) = 0, \]

or

\[ 1 - (ABC + BCD + ABD + ACD) - (BA + CA + AD + BC + BD + DC) = 0, \]  
(C.24)

reads in terms of invariant \( \bar{G}_1 \):

\[ \bar{G}_1 = 1. \]  
(C.25)

Note that \( \bar{G}_1 \), on the contrary, is not equal to 1 on the critical condition (C.24).

The algebraic expression \( \bar{G}_1 \) satisfies all the symmetries of the spontaneous magnetization (permutation symmetries of the three \( x_i \)'s, inversion relation, duality ...). It vanishes on the disorder conditions (and its inverse vanishes on the "order" conditions) and becomes equal to 1 on the critical variety. It can thus be seen, as first glance, as a generalization of the Ising modulus of elliptic functions (C.15).
C.2. RESUMMED EXPANSIONS FOR THE CHECKERBOARD POTTs MODEL

The resummed expansion of the partition function of the checkerboard q-state Potts model, in the presence of a magnetic field, reads [10,25,56,57] (denoting z the fugacity):

\[
\ln(\Lambda) = (q - 1) \cdot \frac{(BX + DY)}{2} + \frac{(q - 1)(q - 2)}{2} \cdot \frac{(BX^2 + DY^2)}{2} \\
+ \frac{(q - 1)}{2} \cdot \frac{A^2z^2 + C^2z^2 + 2A^2C^2z^4}{(1 - A^2C^2z^4)} \cdot (BD + BY + DX)^2 \\
+ \frac{(q - 1)(q - 2)}{2} \cdot \frac{ACz^2 (A + C + 2A^2C^2z^2)}{1 - A^4C^3z^4} \cdot (BD + BY + DX)^2 \\
+ \frac{(q - 1)(q - 2)}{2} \cdot \frac{ACz (2BDAX + ACz (B^2A^2 + D^2Y^2))}{(1 - A^2C^2z^2)} \\
+ \frac{(q - 1)^2}{2} \cdot \left( \frac{(A^2C^2z^2 - 7)}{(1 - A^2C^2z^2)^3} \right) \cdot (B^4 + D^4) \\
- \frac{(q - 1)^2}{2} \cdot \left( \frac{(8 + 4A^2C^2z^2)}{(1 - A^2C^2z^2)^3} \cdot (B^3D + BD^3) \right) \\
- \frac{(q - 1)^2}{2} \cdot \left( \frac{(5 + A^2C^2z^2)}{(1 - A^2C^2z^2)^3} \cdot A^2C^2z^2B^2D^2 \right) + \ldots .
\]

with

\[ X = \frac{ACz \cdot (D + ABCz)}{1 - A^2C^2z^2} \quad \text{and} \quad Y = \frac{ACz \cdot (B + ADCz)}{1 - A^2C^2z^2}, \]

and where the A, B, C, D denote the low-temperature variables (C.13). Performing the derivative of (C.26) with respect to z, one gets:

\[
z \cdot \frac{d}{dz} \ln(\Lambda) = (q - 1) \cdot ACz \cdot (ACz \cdot (B^2 + D^2) \cdot (1 + A^2C^2z^2) \cdot BD) \\
+ 2 \cdot (q - 1) \cdot (q - 2) \cdot \frac{A^4C^4z^4}{(1 - A^2C^2z^2)^3} \cdot (B^3 + D^3) \]

\[
+ (q - 1) \cdot (q - 2) \cdot \frac{A^2C^2z^2 \cdot (1 + A^2C^2z^2 + A^3C^3z^3 + 3ACz)}{(1 - A^2C^2z^2)} \cdot BD \cdot (B + D) \\
+ c_{44} \cdot (R^4 + D^4) + c_{13} \cdot (R^3D + RD^3) + c_{20} \cdot R^2D^2 + \ldots .
\]

The derivative of (C.28) is nothing (up to a multiplicative factor) but the magnetization minus one. Recalling the "order" conditions (see Section 3.3), let us introduce the product of the two "order conditions" (see [13]):

\[
P = (D + ABCz + (q - 2) \cdot ABCDz) (B + ADCz + (q - 2) \cdot ABCDz) \\
= (1 + A^2C^2z^2) \cdot BD + ACz \cdot (B^2 + D^2) \\
+ (q - 2) \cdot ACz \cdot (1 + ACz) \cdot (B^2D + BD^2) \]

\[
+ (q - 2)^2 \cdot A^2C^2z^2 \cdot B^2D^2 .
\]

The "order conditions" are such that, restricted to them, the (low-temperature expansion of the) magnetization becomes formally equal to 1 [13]. Actually, at this order, one can verify for (C.28):

\[
z \cdot \frac{d}{dz} \ln(\Lambda) = P \cdot H,
\]

\[
(C.30)
\]
where $H$, at this order, reads:

$$
H = \frac{(q-1)ACz}{(A^2C^2z^2 - 1)^3} + 2 \cdot \frac{(q-1) \cdot (q-2) \cdot A^3C^3z^3}{(1 - A^2C^2z^2)^3} \cdot (B + D) \\
+ \sum \cdots .
$$

(C.31)

At this order, relation (C.30) corresponds to the following relation between $c_{13}, c_{44}, c_{22}$:

$$
\left(1 + A^4C^4z^4\right) \cdot c_{44} - \left(1 + A^2C^2z^2\right)^2 \cdot c_{13} - \frac{(q-1) \cdot (q-2)^2 A^3C^3z^3 \cdot (2ACz - 1)}{(1 - A^2C^2z^2)^2} + c_{22} = 0.
$$

The expressions of $H_2$ and $H_3$ in terms of $c_{13}, c_{44}$, and $c_{22}$ read:

$$
H_2 = \frac{c_{44}}{ACz}, \quad H_3 = c_{34} - \frac{(1 + A^2C^2z^2)}{ACz} \cdot c_{44} - \frac{2(q-2)^2 A^4C^4z^4(1 + ACz)(q-1)}{(1 - A^2C^2z^2)^3} \cdot \frac{1}{ACz}.
$$

The expressions of $c_{13}, c_{44}$, and $c_{22}$ are quite involved, and therefore, will not be given here.

C.3. REMARK ON THE IMAGE OF THE ORDER CONDITION BY THE INVERSION

RELATIONS FOR CHECKERBOARD MODELS.

RESUMMED EXPANSIONS VERSUS

ALGEBRAIC INVARIANTS

In the numerator of $G_1$ (see (C.20)), the image, by the inversion relations, of the four "order" conditions, read, respectively,

$$
BD + BCD + A + AD + BA + BAD = 0 \quad \text{and} \quad BD + BAD + C + CD + BC + BCD = 0,
$$

together with six other algebraic varieties. Among these varieties the previous two are compatible with the (low temperature) resummed expansions of the checkerboard model (namely $B$ and $D$ small).

The expressions occurring in the resummed expansion of the spontaneous magnetization (see (C.28)) have $N$th roots of unity $1 - ANC^N$, while the rational expressions in the (low temperature) resummed expansion of $G_1$ have much more complicated denominators. Unfortunately, all the "correcting" terms by which one can multiply $G_1$, cannot easily change this situation, suppressing the "unpleasant" singularities in (C.22) and replacing them by "nice" $N$th root singularities of (C.28).

In fact, when one substitutes (C.32) in the resummed expansion (C.28), for $z = 1$ and $q = 3$, one does not get zero (as one could expect from a naive interpretation of the automorphy property of (C.28)). In fact, if the spontaneous magnetization can be as a automorphic function of several complex variables with respect to our discrete group of birational transformations, it is a multivalued function with a very complicated covering. Furthermore, if one considers the (anisotropic) triangular limit of invariant $G_1$, namely $G_1^{\text{triang}}$, this expression is not invariant under the permutations of $x_1, x_2, x_3$.

One also remarks that the ($B, D$ small) expansion of $G_1$ is paradoxically more involved than the resummed expansion of the spontaneous magnetization (see (C.28)). In particular, the rational expressions occurring in the resummed expansion of the spontaneous magnetization only have $N$th roots of unity $1 - ANC^N$, while the rational expressions in the ($B, D$ small) expansion of $G_1$ paradoxically have much more complicated denominators. Unfortunately, all the "correcting" terms by which one can multiply $G_1$, cannot easily change this situation, suppressing the "unpleasant" singularities in (C.22) and replacing them by "nice" $N$th root singularities of (C.28).

In fact, even an "improved" $G_1$ is probably not sufficient enough to describe the resummed expansion of the spontaneous magnetization ((C.28) for $z = 1$ and $q = 3$).

In the following we will try to clarify this point considering two limits of the checkerboard Potts model: the triangular model and, more particularly, namely the *honeycomb model* (and more precisely the algebraic invariant built from product, over the group, of "order" varieties), which seem to correspond to less analytically "subtle" situations.
APPENDIX D
TOWARDS ALGEBRAIC INVARIANTS
FOR THE HONEYCOMB POTTS MODEL

D.1. SPONTANEOUS MAGNETIZATION FOR Q-STATE
EDGE POTTS MODEL ON THE HONEYCOMB LATTICE

In the Ising case, one has for the spontaneous magnetization of the honeycomb lattice:

\[ M = (1 - k_A^2)^{1/8}, \quad \text{where} \quad k_A^2 = \frac{16 \cdot (1 + ABC) \cdot (A + BC) \cdot (B + AC) \cdot (C + AB)}{(1 - A^2)^3 \cdot (1 - B^2)^3 \cdot (1 - C^2)^2}. \]

For arbitrary \( q \), the relation between the spontaneous magnetization and the derivative with respect to \( z \) of \( \ln(\Lambda) \) reads:

\[ z \cdot \frac{d}{dz} \ln(\Lambda) = \frac{q - 1}{q} \cdot (1 - M) \quad \text{or} \quad M = 1 - \frac{q}{q - 1} \cdot z \cdot \frac{d}{dz} \ln(\Lambda). \quad (D.33) \]

Taking the \( C = 1 \) limit\(^{18}\) one gets the equivalent of expansion (C.28) for the three-state honeycomb Potts model:

\[
\begin{align*}
&z \cdot \frac{d}{dz} \ln(\Lambda) = (q - 1) \cdot \frac{A \cdot (C + BA)(B + CA)}{(1 - A^2)^2} + 2 \cdot (q - 1) \cdot (q - 2) \cdot \frac{A^4 \cdot (B + C)^3}{(1 - A^2)^3} \\
&\quad + (q - 1) \cdot (q - 2) \cdot \frac{A^2 \cdot (A - 1) \cdot (A^2 - 4A - 1) \cdot BC \cdot (B + C)}{(1 - A^2)^3} \\
&\quad + 2 \cdot (q - 1) \cdot \frac{A \cdot (2A^4 + 1 + 5A^2) \cdot (B + CA) \cdot (C + BA) \cdot (B + C)^2}{(1 - A^2)^4} \\
&\quad + (q - 1) \cdot \frac{(A^2 - 5A^2 + A - 1) \cdot (A - 1)^2 \cdot (B + CA) \cdot (C + BA) \cdot BC}{(1 - A^2)^4} \\
&\quad + 2 \cdot (q - 1) \cdot (q - 2) \cdot \frac{A^2 \cdot (2A^6 + 6A^4 + 15A^3 + 30A^2 + 30A + 1 + A^6) \cdot (B + CA)(C + BA)(B + C)^2}{(1 - A^2)^4 \cdot (1 - A^3)^2 \cdot (1 + A)} \\
&\quad + (q - 1) \cdot (q - 2) \cdot \frac{(1 + A^2 - 3A^2 - 4A^4 - 3A^6) \cdot (1 - A^2) \cdot A \cdot (B + CA) \cdot (C + BA) \cdot BC}{(1 - A^2)^2 \cdot (1 - A^3)^2 \cdot (1 + A)} \\
&\quad + 3 \cdot (q - 1) \cdot (q - 2) \cdot \frac{A^6 \cdot (B + C)^4}{(1 - A^2)^4} + (q - 1) \cdot (q - 2) \cdot \frac{A^3 \cdot (A - 1) \cdot (A^2 - 9A - 4) \cdot (B + C)^2 \cdot BC}{(1 - A^2)^3} \\
&\quad + (q - 1) \cdot (q - 2) \cdot \frac{A^4 \cdot (A - 1) \cdot (A^2 - 9A - 4) \cdot (B + C)^2 \cdot BC}{(1 - A^2)^4} \\
&\quad - (q - 1) \cdot 2 \cdot A^2 \cdot (2A^3 + 7A) \cdot (C + BA) \cdot (B + CA) \cdot (B + C)^2 \\
&\quad - (q - 1)^2 \cdot \frac{A^2 \cdot (5 - 4A) \cdot (A - 1)^2 \cdot (B + CA) \cdot (B + CA) \cdot BC}{(1 - A^2)^4}. \quad (D.34)
\end{align*}
\]

D.2. SEEKING FOR INVARIANTS
FOR THE HONEYCOMB LATTICE

Let us first recall the two invariants of the triangular lattice. Written in terms of the low-temperature variables (C.13) invariants \( Y \) in (4.33) and (4.37) read:

\[ Y_{\text{triang}} = - \frac{(1 + A)(1 + B)(1 + C)BCA}{(BC + CA + BA - 1 + BCA)^3}, \quad \text{and} \quad (D.36) \]

\[ M_{\text{triang}} = - \frac{-1 - 2 \cdot (A + B + C) - (BC + CA + BA) + BCA}{BC + CA + BA - 1 + BCA}. \quad (D.36) \]

\(^{18}\)And replacing \( D \) by \( C \).
Resummed expansion (D.34) has to be compared to the Kramers-Wannier dual of (D.35) and (D.36) together with an invariant originating from product of "order" varieties denoted $\mathcal{P}_{\text{rod}}$:

$$\mathcal{P}_{\text{rod}} = 2 \cdot \frac{\mathcal{P}_N}{\mathcal{P}_D}, \quad \text{with}$$

$$\mathcal{P}_D = (-1 - C - A - B + RCA)^6 \cdot (-1 + A + B + C + (BC + CA + BA) + ABC)^3,$$

and

$$\mathcal{P}_N = (BCA + A + BC) \cdot (BCA + B + CA) \cdot (BCA + C + BA) \times (BCA + A + D + 2BA + 1) \cdot (BCA + C + D + 2BC + 1) \times (BCA + C + A + 2CA + 1) \cdot (A - C + B + BA + 1) \cdot (C + A - B + CA + 1) \times (A - C - D - DC - 1).$$

Let us introduce the ferromagnetic critical variety of the honeycomb lattice:

$$c_0^{\text{honey}} = -1 + 2 \cdot (BC + CA + BA) + (A + B + C) + ABC = 0. \quad \text{(D.39)}$$

One notes that $\mathcal{P}_{\text{rod}}$ has a $(c_0^{\text{honey}})^{-3}$ singularity. One immediately verifies that $\mathcal{P}_{\text{rod}}$ is invariant under the honeycomb inversion relation:

$$\begin{align*}
(A, D, C) \longrightarrow & \left( \frac{-A}{1 + A}, \frac{1}{B'}, \frac{-C}{1 + C'} \right). \\
\text{(D.40)}
\end{align*}$$

One notes that this invariant cannot be obtained from the one on checkerboard Potts model (namely $\mathcal{G}_1$) since the honeycomb limit of this invariant gives an expression which is not $S_3$ symmetric (see Appendix C.3).

The Kramers-Wannier dual of (D.35) and (D.36) read, respectively,

$$Y^{\text{honey}} = \frac{(C + 2) \cdot (2 + B) \cdot (A + 2) \cdot (C - 1) \cdot (B - 1) \cdot (1 + 2C) \cdot (1 + 2B) \cdot (1 + 2A)}{27 \cdot (BCA - 1 + BC + CA + BA)^3},$$

$$M^{\text{honey}} = \frac{3 \cdot (BCA - 1 + A - D - C)}{BCA - 1 + 2(BC + CA + BA) + A + C + B}.$$

Expression $Y^{\text{honey}}$ is invariant under the honeycomb inversion relation (D.40), while $M^{\text{honey}}$ becomes $-M^{\text{honey}}$.

In the case of the (honeycomb) three-state Potts model, since the discrete group is finite, one can introduce many other invariants. For instance, by introducing:

$$K^{\text{honey}} = \frac{ABC \cdot (1 + A) \cdot (1 + B) \cdot (1 + C)}{(1 + 2A) \cdot (1 + 2B) \cdot (1 + 2C) \cdot (1 - A) \cdot (1 - B) \cdot (1 - C) \cdot (A + 2) \cdot (B + 2) \cdot (C + 2)},$$

one can easily verify that it also transforms like $K^{\text{honey}} \rightarrow -K^{\text{honey}}$ under the inversion relation of the honeycomb three-state Potts model (D.40). Of course one can also introduce many other invariants:

$$\frac{(M^{\text{honey}} + 3) \cdot (M^{\text{honey}} - 3)}{(M^{\text{honey}})^2} = -4 \cdot \frac{(A + B + C + BC + CA + BA)(BCA - 1 + BC + CA + BA)}{(-1 - A - B - C + BCA)^3},$$

$$- 3^6 \cdot \frac{K^{\text{honey}} \cdot Y^{\text{honey}}}{(M^{\text{honey}})^3} = - \frac{(1 + A)(1 + B)(1 + C)ABC}{(ABC - 1 - A - C - B)^3} = \frac{A}{(A + 1)^3} \cdot BC + \cdots.$$
For $B$ and $C$ small, these invariants expand as follows:

\[
y_{\text{honey}} = -\frac{4 \cdot (A + 2) \cdot (1 + 2A)}{27(A - 1)^2} + \frac{2 \cdot (1 + 2A) \cdot (A + 1) \cdot (B + C)}{3(A - 1)^3} + \cdots,
\]

\[
M_{\text{honey}} = \frac{3 \cdot \frac{1 + A}{1 - A} + 6 \cdot (1 + A + A^2) \cdot (B + C)}{(1 - A)^2} + \cdots,
\]

\[
K_{\text{honey}} = \frac{A \cdot (A + 1)}{4 \cdot (1 + 2A)(1 - A)(A + 2)} \cdot BC + \cdots.
\]

One then gets:

\[
z \cdot \frac{d}{dz} \ln(\Lambda) = P_{\text{rod}} \cdot \left(1 - 18 \cdot \frac{A \cdot (H + C)}{1 - A^2} + \frac{T_2}{(1 + A^2 \cdot (1 - A^2)^2} \cdots\right),
\]

where

\[
T_2 = -A^2 \cdot (18A^5 - 99A^4 - 218A^3 - 254A^2 - 135) \cdot (B + C)^2
+ (1 - A) \cdot (A + 1) \cdot (18A^6 + 17A^5 - 3A^4 - 54A^3 - 69A^2 - 53A - 18) \cdot BC.
\]

It is necessary to introduce other (resummed expansion well-suited) polynomials. Another additional invariant is particularly interesting to introduce because it yields a nice $(B, C)$ small expansion:

\[
Q_1 = \frac{\left((M_{\text{honey}})^2 + 27 \cdot y_{\text{honey}} - 1\right)}{3 \cdot (M_{\text{honey}})^2} = \frac{q_1}{(1 - A - B - C + BCA)^2(-1 + A + B + C + 2BC + 2CA + 2BA + ABC)}
\]

\[
- 2BA^2 - 2C^2A - 3BCA + 3(BC^2A + BCA^2 + B^2CA^2)
\]

$Q_1$ expands as follows:

\[
Q_1 = \frac{2 \cdot A}{1 - A^2} \cdot (C + B) \cdot \frac{(A - 1) \cdot (2A^3 + A^2 - 5A - 2)}{(1 - A^2)^2} \cdot BC + \frac{2A^2(A + 5)(C + B)^2}{(1 - A^2)^2} \cdots.
\]

An invariant which expands like (see the right-hand side of (D.42)):

\[
1 - 18 \cdot \frac{A \cdot (H + C)}{1 - A^2} + \cdots,
\]

and which can cancel the $(C_{\text{honey}})^{-3}$ singularity of $P_{\text{rod}}$, is for instance:

\[
Q_3 = (1 + 3 \cdot Q_1)^{-3} = \frac{(-1 - A - B - C + BCA)^6(-1 + A + B + C + 2(BC + CA + BA) + BCA)^3}{q_3^3}
\]

where $q_3$ reads:

\[
+ 5(A^3BC + B^3CA + C^3BA) + 2BC^2A^3
\]
The expansion of $Q_3$ is remarkably simple:

$$Q_3 = 1 + 18 \cdot \frac{A}{1 - A^2} \cdot (C + B) + \frac{9 (2A^3 + 22A^2) \cdot (C + B)^2}{(1 - A^2)^2} + \frac{9(A - 1) \cdot (2A^3 + A^2 - 5A - 2) \cdot BC}{(1 - A^2)^2} + \ldots .$$

One thus exchanges $P_{rod}$ for a new invariant, denoted $L_{ast}$:

$$L_{ast} = P_{rod} \cdot Q_3 = 2 \cdot \frac{P_N}{Q_3} \cdot \text{(D.44)}.$$

Amazingly, one verifies that $L_{ast}$ is such that, restricted to the critical variety of the honeycomb lattice, namely (D.39), becomes equal to $2/27$.

Invariant, $L_{ast}$ expands as follows:

$$L_{ast} = \frac{2A(B + CA)(C + BA)}{(1 - A^2)^2} + \ldots = \frac{2A^3(C + B)^2}{(1 - A^2)^2} + \frac{2A \cdot BC \cdot (1 - A^2)^2}{(1 - A^2)^2} + \ldots .$$

In term of this new invariant $L_{ast}$, relation (D.42) reads:

$$z \cdot \frac{d}{dz} \ln(A) = L_{ast} \cdot (1 + S_2(L_{ast}, J_1, \ldots) + \cdots),$$

where $J_i$ denotes various well-suited invariants, and $S_2$ is of order two in $B$ and $C$.

Recalling (D.33), one gets from (D.45):

$$M - 1 = \frac{3}{2} \cdot \frac{z \cdot \frac{d}{dz} \ln(A)}{k_{\text{Potts}}} = \left(1 - k_{\text{Potts}}\right)^{1/9} 1 + \ldots$$

$$(D.46)$$

The $L_{ast} = -2/27$ limit corresponds to $k_{\text{Potts}} = 1$. In order to get a simple expression for the spontaneous magnetization at criticality, one thus finally introduces:

$$k_{\text{Potts}}^{\text{honey}} = \frac{27}{2} \cdot L_{ast}, \quad \text{such that } C_0^{\text{honey}} = 0 \Rightarrow k_{\text{Potts}}^{\text{honey}} = 1.$$  \hspace{1cm} (D.47)

Conversely,

$$k_{\text{Potts}}^{\text{honey}} - 1 = \frac{C_0^{\text{honey}} \cdot \left(C_1^{\text{honey}} \cdot C_2^{\text{honey}} \cdot C_3^{\text{honey}}\right)^2}{Q_3^3}.$$  \hspace{1cm} (D.48)

with (see (7.9)):

$$C_0^{\text{honey}} = -1 + 2 \cdot (BC + CA + BA) + (A + B + C) + ABC,$$

$$C_1^{\text{honey}} = (BC - 1) \cdot A^2 + 2 \cdot (B + C + BC) \cdot A + (B + C) + 1, \ldots.$$  \hspace{1cm} (D.49)

One can actually verify that this set of three last additional varieties (D.49) (see also (7.9)) (which break the $S_3$ symmetry of permutation of $x_1$, $x_2$, and $x_3$) have simple covariance properties with respect to the inversion relation of the honeycomb model.

These three "cousin varieties" of the critical variety $C_0^{\text{honey}} = 0$ are reminiscent of the situation one has in the case of the Ising honeycomb model (see also (C.16) for the checkerboard model).

In the Ising case, condition $k_{\text{h}}^2 = 1$ reads (using (D.33)):

$$k_{\text{h}}^2 - 1 = -\frac{F_0^{\text{acto}} \cdot F_1^{\text{acto}} \cdot F_2^{\text{acto}} \cdot F_3^{\text{acto}}}{T_{\text{iv}}^2}, \quad \text{with}$$

$$T_{\text{iv}} = (A - 1)(1 + A)(B - 1)(B + 1)(C - 1)(1 + C), \quad F_0^{\text{acto}} = ABC - (BC + AB + AC) - (A + B + C) + 1,$$

$$F_1^{\text{acto}} = (AB - 1 + A + B) \cdot C + 1 + (A + B) - BA, \quad F_2^{\text{acto}} = (BC - 1 + B + C) \cdot A + 1 + (B + C) - BC,$$

$$F_3^{\text{acto}} = (AC - 1 + A + C) \cdot B + 1 + (A + C) - AC.$$  \hspace{1cm} (D.50)
D.3. INVARIANT OF THE HONEYCOMB LATTICE VERSUS RESUMMED EXPANSIONS. FINE TUNING

Let us recall $D_{\text{er}}$:

$$D_{\text{er}} = (1 - k_{\text{Potts}})^{1/9} - 1.$$  \hfill (D.51)

The expansion of this very expression, for $B$ and $C$ small, reads:

$$D_{\text{er}} - \frac{2A(C + BA)(B + CA)}{(1 - A^2)^2} + \cdots = \frac{2A \cdot (1 - A)^2}{(1 - A^2)^2} \cdot BC + \frac{2A^2}{(1 - A^2)^2} \cdot (B + C)^2 + \cdots.$$  

On the other side, $z \cdot \frac{d}{dz} \ln(\Lambda)$ has the following resummed low-temperature expansion for the three-state Potts model on the honeycomb lattice (see (D.34)):

$$z \cdot \frac{d}{dz} \ln(\Lambda) = \frac{2A(C + BA)(B + CA)}{(1 - A^2)^2} + \frac{2A^2(A - 1) \left(A^2 - 4A - 1\right)}{(1 - A^2)^3} \cdot (B + C) \cdot BC$$

$$+ \frac{4A^2}{(1 - A^2)^3} \cdot (C + B)^3 + \cdots.$$  \hfill (D.52)

One verifies that:

$$z \cdot \frac{d}{dz} \ln(\Lambda) - D_{\text{er}} = \left(\frac{2A \cdot (1 - A)^2}{(1 - A^2)^2} \cdot BC\right) \cdot \frac{-\alpha_2 \cdot A}{(1 - A^2)^2 \cdot (1 + A)^2},$$  \hfill (D.53)

where $\alpha_2$ reads:

$$\alpha_2 = A \cdot \left(3A^4 + 4A^3 + 5A^2 + 4A + 3\right) \cdot (B^2 + C^2)$$

$$+ \left(2A^6 + 3A + 9A^2 + 10A^3 + 9A^4 + 3A^5 + 2\right) \cdot BC,$$  \hfill (D.54)

or equivalently at this order of the expansions (namely order two in $B$ and $C$ see (D.45)):

$$z \cdot \frac{d}{dz} \ln(\Lambda) = D_{\text{er}} \cdot \left(1 - \frac{A \cdot W_2}{(1 + A)^2 \cdot (1 - A^3)^2}\right) = D_{\text{er}} \cdot (1 + \lambda \cdot D_{\text{er}} + R_{\text{est}} + \cdots),$$  \hfill (D.55)

with

$$W_2 = A \left(3A^4 + 4A^3 + 5A^2 + 4A + 3\right) \cdot (B + C)^2 + (A^2 - A + 1) \cdot (2A^2 + 3A + 2) \cdot (A - 1)^2 \cdot BC.$$  

At this step, without additional information, there is some ambiguity in the determining of $\lambda$. If one converts the two following quantities (where $P$ and $S$ denote, respectively, $BC$ and $B + C$), to partial fraction form:

$$- \frac{A \cdot W_2}{(1 + A)^2 \cdot (1 - A^3)^2} = \frac{3S^2 - 12P}{4(A + 1)} - \frac{3S^2 - 12P}{4(A + 1)^2} - \frac{19S^2}{36(A - 1)} - \frac{19S^2}{36(A - 1)^2}$$

$$- \frac{6S^2 - 18P + 2AS^2 - 9PA}{9 \cdot (A^2 + A + 1)} + \frac{2S^2 - 6P + 2AS^2 - 6PA}{3 \cdot (A^2 + A + 1)^2},$$

and

$$2 \cdot A \cdot \frac{(C + BA) \cdot (B + CA)}{(1 - A^2)^3} = \frac{S^2}{2(A - 1)} + \frac{S^2}{2(A + 1)} \cdot (S^2 - 4P) + \frac{S^2 - 4P}{2(A + 1)^2},$$

two values of $\lambda$ pop out: $\lambda = -3/2$ and $\lambda = -10/18$. For the first value of $\lambda$ one gets a particularly simple expression for $R_{\text{est}}$, namely (7.8).
D.4. GROUP-INARIANT VERSUS LOW-TEMPERATURE EXPANSIONS FOR THE HONEYCOMB LATTICE

It is tempting to compare the expansion of (7.5) with the low-temperature expansion of the spontaneous magnetization in the case of the honeycomb lattice. The isotropic limit of (7.4) reads:

\[ k_{\text{Potts}}^{\text{honey}} = \frac{27 \cdot (A + 1)^3 \cdot A^3}{(1 - 3A^2 - A^3)^3} \]

\[ (1 - k_{\text{Potts}}^{\text{honey}})^{1/9} - 1 = 2A^3 + 6A^4 + 24A^5 + 86A^6 + 324A^7 + 1224A^8 + 4722A^9 + 18432A^{10} + \cdots. \]  

\[(D.56)\]

One cannot find directly low-temperature expansion for the spontaneous magnetization of the three-state honeycomb lattice in the literature but rather high field expansions \([59]\). However, from these high field expansions, one can get (by derivation with respect to the magnetic field) an expansion for the spontaneous magnetization, namely:

\[-\frac{2}{3} (M_{\text{honey}} - 1) = z \cdot \frac{d}{dz} \ln(A) = 2A^3 + 6A^4 + 24A^5 + 82A^6 + 300A^7 + 1176A^8 + 4434A^9 + 15720A^{10} + \cdots.\]

\[(D.57)\]

Taking into account the fact that (D.57) is basically a high field expansion \([59]\) and not a low temperature one (only the first terms are correct), one remarks, however, a quite good agreement between these two expansions.

D.5. COMMENTS ON \(k_{\text{Potts}}^{\text{honey}}\) FOR THE THREE-STATE HONEYCOMB POTTs MODEL

The isotropic limit \(C = B = A\), of the invariant of the honeycomb lattice (7.4), is simple\(^\text{19}\):

\[ k_{\text{Potts}}^{\text{honey}} = \left( \frac{3A(A + 1)}{A^3 - 3A^2 + 1} \right)^3 = (k_{\text{iso}})^3. \]  

\[(D.58)\]

Condition \(k_{\text{Potts}} = 1\) reads two conditions:

\[ k_{\text{iso}} - 1 = 1 - 3A - 6A^2 - A^3 \quad \text{and} \quad (k_{\text{iso}})^2 + k_{\text{iso}} + 1 = (1 + A + A^2)^3. \]  

\[(D.59)\]

The first condition \(k_{\text{iso}} - 1 = 0\) is the well-known critical condition of the isotropic honeycomb lattice in terms of the low-temperature variables.

The Kramers-Wannier dual of (D.58) reads:

\[ k_{\text{KW}}^{\text{honey}} = \left( \frac{3A^* (A^* + 1)}{-(A^*)^3 - 3(A^*)^2 + 1} \right)^3 = \left( \frac{(1 + 2A)(2 + A)(A - 1)}{3A - 1 + 6A^2 + A^3} \right)^3. \]  

\[(D.60)\]

Let us recall the rational parameterization (4.35). Invariant \(k_{\text{Potts}}^{\text{honey}}\) has a rather involved form when written in terms of the \(u_1, u_2, u_3\) variables. The simplest part is the denominator, which reads \((D_{u_1u_2u_3})^3\) with:

\[ D_{u_1u_2u_3} = 9 \cdot (u_1^2 + u_2^2 + u_3^2 - 3u_1u_2u_3) \cdot t^2 + 9 \cdot (2u_1^2u_2^2 + 2u_1^2u_3^2 + 2u_2^2u_3^2 + 3u_2u_3u_1^2) \cdot t^3 + 9 \cdot (2u_1^3 + 2u_2^3 + 2u_3^3 + 3u_1u_2u_3) \cdot t + 9 \cdot (3u_2u_3u_1^2 - u_1u_2^3 - u_2u_1^3 - u_3u_2^3). \]

\[^{19}\text{This simple form for the isotropic limit of the model enables us to write down very simply the unit circle } |k_{\text{Potts}}| = 1, \text{ namely } |k_{\text{iso}}| = 1. \text{ However, one should not expect the unit circle to play a key role for the honeycomb Potts model [64]. This used to happen for the Ising model as a consequence of the elliptic parameterization that we do not expect here.}\]
In the square limit $C = 0$, invariant $k_{\text{Potz}}$ becomes:
\[
\]

It has been noted, in Section 7, that this expression, inherited from the one for the honeycomb lattice, does not yield a "nice" representation of the Kramers-Wannier duality (like for instance, $k \to k/(\alpha \cdot k - 1)$). In the isotropic case, (D.61) becomes (7.11) and the Kramers-Wannier dual of (7.11) is given by (7.13). One can eliminate $A$ between these two expressions to get a relation between (7.11) and (7.13). Actually, the resultant yields (the square of):
\[
R = 2^{39} \cdot k^6K^6 - 141 \cdot 2^{33} \cdot k^5K^5 \cdot (K + k)
+ 3 \cdot 2^{36}k^4K^4 \cdot (4429 (k^2 + K^2) + 9306kkK^2)
- 2^{30} \cdot k^3K^3 \cdot (k + k) \cdot (210748 \cdot (k^2 + K^2) + 213467kkK^2)
+ 3 \cdot 2^{10} \cdot k^2K^2 \cdot (48719 \cdot (k^4 + k^2) - 24169286 \cdot (k^3K + kK^3) - 47735079k^2K^2)
- 3 \cdot 2^7 \cdot kK \cdot (k + k) \cdot (5687 \cdot (k^4 + K^4))
- 805800479 \cdot (k^3K^3 + kK^3) - 2859101424 \cdot k^2K^2)
+ 2634567894 \cdot (kK^5 + k^5K) - 1398272669644k^3K^3
- 44823873571 \cdot (k^2K^2 + k^2K^2) + 1331 \cdot (K^6 + k^6)
+ 3 \cdot (k + k) \cdot (141113 \cdot (k^4 + K^4) - 42754046948 \cdot (k^3K + kK^3) + 1936209422 \cdot k^2K^2)
+ 9853471948k^2K^2 + 246865154424 \cdot (kK^3 + k^3K) + 50285730 \cdot (k^4 + K^4)
+ 7^7 \cdot (K + k) \cdot (30524825 \cdot (k^2 + K^2) - 3221730809kkK)
+ 3 \cdot 2^{10} \cdot (124093734 \cdot kK + 16041625 (k^2 + K^2)) - 2^{15} \cdot 5^4 \cdot 5037 \cdot (K + k) + 5^8 \cdot 2^{17}.
\]

**APPENDIX E**

**TOWARDS ALGEBRAIC INVARIANTS FOR THE TRIANGULAR LATTICE**

**E.1. THE TRIANGULAR LATTICE: SINGLED OUT ALGEBRAIC VARIETIES**

From the critical variety and the three additional varieties for the honeycomb lattice (see (7.9)) one can deduce (using the Kramers-Wannier duality) the critical variety and three singled out varieties for the triangular lattice:

\[
1 - ABC - (BC + AC + BA) = 0,
\]
\[
1 + 2A - BC + 2AC + 2BA + BCA^2 + BA^2 + CA^2 = 0,
\]
\[
1 + 2C + 2BC - AC + 2BA + B^2A + B^2CA + B^2C = 0,
\]

or, in terms of the $x_i$'s:

\[
x_1x_2x_3 - (x_1 + x_2 + x_3) = 1 = 0, \quad \text{and}
\]
\[
c_{1}^{\text{triang}} = x_2x_3x_4^2 + 2x_2x_3x_1 - x_1^2 + 2x_2x_1 + 2x_1x_3 + 1 + x_3 + x_2
+ (x_1 + 2) \cdot (x_2x_3x_1 - 1 - x_1 - x_2 + x_3) + 3(x_1 + 1)(1 + x_3 + x_4) = 0,
\]
\[
c_{2}^{\text{triang}} = (x_2 + 2) \cdot (x_2x_3x_1 - 1 - x_1 - x_2 - x_3) + 3 \cdot (x_2 + 1)(x_1 + x_3 + 1) = 0,
\]
\[
c_{3}^{\text{triang}} = (x_3 + 2) \cdot (x_2x_3x_1 - 1 - x_1 - x_2 + x_3) + 3 \cdot (x_3 + 1)(x_1 + x_2 + 1) = 0.
\]

In the isotropic limit $c_{1}^{\text{triang}} = 0$ becomes $(1 + u + u^2)^2 = 0$. In the anisotropic square limit (E.63) yields:

\[
c_{1}^{\text{triang}} = x_2x_3^2 + 4x_2x_1 - x_1^2 + 2x_1 + 2 + x_2,
\]
\[
c_{2}^{\text{triang}} = x_2x_1 + 4x_2x_1 - x_2^2 + 2x_2 + 2 + x_1,
\]
\[
c_{3}^{\text{triang}} = x_1x_2 + x_1 + x_2.
\]
E.2. TUTTE-BERAHANUMBERS

For the Ising model, the additional varieties for the triangular and honeycomb lattices read, respectively,

\[ AB - AC + BC + 1 = 0 \quad \text{or} \quad u_3 u_1 + u_2 \cdot t = 0, \quad \text{with} \]
\[ t^4 = -1 \quad \text{and} \quad u_2 u_3 - t^3 \cdot u_1 = 0, \quad \text{with} \quad t^4 = -1. \]

More generally, considering the inversion relation of the triangular lattice:

\[(u_1, u_2, u_3) \rightarrow \left(\frac{t^2}{u_1}, \frac{1}{t^2 \cdot u_2}, \frac{1}{t^2 \cdot u_3}\right), \quad (E.65)\]

and a variety of the form:

\[ u_1 \cdot u_2 \cdot u_3 \cdot \alpha \cdot u_1^M = 0, \quad (E.66)\]

it is straightforward to see that \((E.66)\) is invariant by the three inversion relations \((E.65)\) if:

\[ t^4 = 1 \quad \text{and} \quad \alpha^2 \cdot t^2 \cdot t^2 \cdot u_1 = 1, \quad \text{that is} \quad \alpha = \pm t^{M-1}. \quad (E.67)\]

Of course this can easily be generalized to the checkerboard lattice.

This makes clear that such varieties only occur for Tutte-Beraha numbers.

E.3. CRITICALITY OF THESE ADDITIONAL VARIETIES?

Since, in the case of the Ising model, three other varieties similar\(^{20}\) to \((E.63)\), occur in addition to the critical variety which also happen to be critical (see also Appendix C.1), it is natural to wonder if the additional varieties \((E.63)\) could not be also critical varieties. If one takes for granted expression \((7.5)\) to represent the dominant singular behaviour of the spontaneous magnetization, one expects the magnetic critical exponent \(\beta\), corresponding to these three new critical varieties \(C_i^{\text{honey}}\) (and also \(C_i^{\text{triang}}\)), to be \(\beta = 2/9\). If one assumes that the (well-known) relations for the critical (or tricritical) exponents of Potts models are still valid (see relations \((5.23)\) in [39]), in particular:

\[ 1 - S = \alpha m' \quad (E.68)\]

one gets a thermal exponent: \(\alpha = 5/3\). In order to examine the critical character of the additional varieties \((E.63)\), Monte-Carlo calculations have been performed on the anisotropic edge triangular Potts model. Unfortunately, Monte-Carlo calculations seem to indicate that the points of any of the three varieties \((E.63)\) are not \(\beta = 2/9\)-critical points. Note however, that \(C^\text{triang}_3 = 0\) in \((E.64)\) is nothing but the antiferromagnetic critical condition of Baxter [65] for the square lattice in the \(q = 3\) limit:

\[ (x_1 + 1) \cdot (x_2 + 1) = 4 - q. \quad (E.69)\]

It is thus possible, in view \((E.64)\) and \((E.70)\), to imagine that these additional \(S_3\)-symmetry breaking) varieties could however be critical varieties with other exponents.

E.4. ALGEBRAIC INVARIANTS FOR THE TRIANGULAR LATTICE

For the anisotropic triangular model we have (at least) four group-invariants:

\[ \mathcal{K} = \frac{ABC \cdot (1 + A)(1 + B)(1 + C)}{(1 + 2A)(1 + 2B)(1 + 2C)(1 - A)(1 - B)(1 - C)(A + 2)(B + 2)(C + 2)}, \quad (E.70)\]

which transforms into its opposite \(-\mathcal{K}\) under the inversion relation of the triangular lattice,

\[ M^\text{triang} = -\frac{-1 - 2(A + B + C) - (BC + CA + BA) + ABC}{BC + CA + BA - 1 + BCA}, \quad (E.71)\]

\(^{20}\)Breaking in particular spontaneously the symmetry of the lattice.
which transforms into its opposite $-M_{\text{triang}}$, under the inversion relation of the triangular lattice and:

$$Y_{\text{triang}} = -\frac{ABC \cdot (1 + A)(1 + B)(1 + C)}{(BC + CA + BA - 1 + BCA)^3}, \quad (E.72)$$

and also (form (4.49) and $M_{\text{triang}}$):

$$G_{\text{triang}} = -\frac{N_{\text{triang}}}{27(BC + CA + BA - 1 + BCA)^3(-1 - 2(A + B + C) - BC - CA - BA + ABC)^3},$$

where

$$N_{\text{triang}} = 2 \cdot (1 + 2BCA + 3A - BC + 2CA + 2BA)(1 + 2BCA + 3B + 2BC - CA + 2BA) \times (1 + 2BCA + 3C + 2BC + 2CA - BA) \cdot (2 + BCA + 3B + BC + CA + BA) \times (-1 + BC + CA - 2BA + ABC)(-1 + BC - 2CA + BA + ABC) \times (-1 - 2BC + CA + BA + BCA).$$

$Y_{\text{triang}}$ and $G_{\text{triang}}$ are invariant under the inversion relation of the triangular lattice. Curiously $K$ is also covariant for the inversion relation of the honeycomb lattice (see $K_{\text{honey}}$ in Appendix D.2); one can easily verify that it transforms like $K \rightarrow -K$ under the inversion relation of the honeycomb three-state Potts model.

The expansion of three of these (up to a sign) invariants yields:

$$K = \frac{AC \cdot (1 + A)(1 + C)}{2 \cdot (1 + 2A). (1 + 2C). (A - 1) \cdot (C - 1) \cdot (A + 2)(C + 2)} \cdot B + \ldots;$$

$$M_{\text{triang}} = \frac{AC + 2A + 2C + 1}{AC - 1} + \ldots; \quad (E.73)$$

$$Y_{\text{triang}} = \frac{AC \cdot (1 + A)(1 + C)}{(AC - 1)^3} \cdot B + \ldots.$$

One should note that these three invariants yield only $1 - A \cdot C$ singularities, or simple singularities like $1 - A, C + 2 \ldots$. One cannot get the $1 - A^N \cdot C^N$ singularities known to occur on resummed expansions [10,25] in this way. The last invariant $G_{\text{triang}}$ yields more involved singularities, for instance $CA + 2A + 2C + 1$ singularities.

The resummed expansion of the spontaneous magnetization of the edge three-state Potts model on the anisotropic triangular lattice corresponds to (see (7.12)):

$$z \cdot \frac{d}{dz} \ln(A) = \frac{2A^2C^2}{(1 - A^2C^2)^2} \cdot B^2 + \frac{4A^4C^4}{(1 - A^2C^2)^3} \cdot B^3 + \ldots.$$

More precisely, in the $z = 1$ triangular limit (namely $D = 0$), expression (C.28) becomes:

$$z \cdot \frac{d}{dz} \ln(A) = (q - 1) \left( \frac{A^2C^2}{1 - A^2C^2} + \frac{A^4C^4}{(1 - A^2C^2)^2} \right) B^2 + (q - 1) \cdot (q - 2) \cdot \left( \frac{2A^4C^4}{(1 - A^2C^2)^2} + \frac{2A^6C^6}{(1 - A^2C^2)^3} \right) \cdot B^3$$

$$+ \frac{2(q - 1)A^2C^2 \cdot (A^2 + C^2 + A^4C^2 + C^4A^2 + 3A^2C^2 + A^4C^4)}{(1 - A^2C^2)^4} \cdot B^4$$

$$+ \frac{2(q - 1) \cdot (q - 2) \cdot (3A^2C^2 - A^4C^4 - A^6C^6 - A^8C^8 - A^7C^7 + A + C - A^5C^5)A^3C^3}{(1 - A^2C^2)^3(1 - A^3C^3)^2} \cdot B^4$$

$$+ \frac{3(q - 1) \cdot (q - 2)^2 A^6C^6}{(1 - A^2C^2)^4} \cdot B^4 - \frac{(q - 1)^2 A^4C^4(2A^2C^2 + 7)}{(1 - A^2C^2)^4} \cdot B^4.$$
In contrast with the honeycomb lattice, there are many ways to get a resummed expansion like (E.74), in terms of group invariants of the triangular lattice. For the honeycomb lattice, invariant (7.4) was built from the modification of an invariant which corresponded to taking the product over the group of the "order varieties". It seems that the equivalent of (7.4), but for the triangular lattice, could rather be an invariant corresponding to a modification of $Y_{\text{triang}}$.

Thus, coming back to the checkerboard lattice, we could therefore, have (at least) two invariants, one which "dominates" in the honeycomb limit and the other in the triangular limit.

APPENDIX F

OCCURRENCE OF $N$th ROOTS OF UNITY ON ANISOTROPIC SQUARE LATTICE RUBBER BANDS

In order to understand the occurrence of $1 - A^N$ singularities in the resummed expansions of the edge Potts models, let us consider an anisotropic square lattice $q$-state Potts model on rubber bands, for instance the one for which the transfer matrices are represented as $3 \times 3$ matrices in [6,63].

After extracting the leading low-temperature terms, the largest eigenvalue $\Lambda$ can be seen to be solution of the following algebraic equation of third degree (characteristic polynomial of the $3 \times 3$ transfer matrix):

$$
\Lambda^3 + C_2 \cdot \Lambda^2 + C_1 \cdot \Lambda + C_0 = 0, \quad \text{with}
$$

$$
C_0 = A^5 (B - 1)^5 (Bq + 1 - R)^4, \\
C_1 = A^2 (B - 1)^2 (Bq + 1 - B) \cdot c_1, \\
C_2 = A^3 B^3 \cdot q^3 + 3A^2 B^2 (2BA - B - A) \cdot q^2 \\
+ (15A^3 B^2 - 7A^2 B^2 - 14B^3 A^3 + 11B^3 A^2 - B^3 - 3A^3 B - BA^2) \cdot q \\
+ (D - 1) \cdot (D^2 - 11A^2 D^2 + 13A^3 D^2 + BA^2 - 8A^3 B + B + A^3 + A^2 + 1),
$$

where

$$
c_1 = A^3 R^3 - q^3 - AR^2 (6R A^2 - 2B - 1 - 3A^2) \cdot q^2 \\
+ (11B^3 A^3 - AB^2 + B^3 - 7AB^3 + 3B^2 - 12A^3 B^2 + 3A^3 B + 2BA) \cdot q \\
- (B - 1) \cdot (7A^3 B^2 - 5AB^2 + B^2 - 5A^3 B + 4B - 5BA + A^2 + A + 1).
$$

The expansion of $\Lambda$, when $B$ is small, reads:

$$
\Lambda = 1 + \frac{3 \cdot (q - 1) \cdot W_2}{1 - A^2} \cdot B^2 + (q - 1) \cdot \sum_{n=3}^{\infty} \frac{W_n}{(1 - A^2)^{n-1}} \cdot B^n, \quad \text{with}
$$

$$
W_2 = A^2, \\
W_3 = (3q - 5) \cdot A^4 + 4A^2 + 1, \\
W_4 = 3q \cdot (q - 2) \cdot A^6 - 3 (A^4 - A^2 - 1) A^2 \\
+ 3A^2(q - 2) \cdot (7A^3 + 8A^3 - 2A^2 + 5A - 2A^6 + 2 + 3A^4) \\
+ 3A^3 + A + 1
$$

$$
W_5 = 3A^5(q - 2) \cdot q^2 - 3A^2 \cdot (A^6 + 5A^4 + 3A^2 + 1) \\
+ 3 \cdot (q - 2) \cdot \frac{3A^2 \cdot Q_{52}}{(A^2 + A + 1)} + 3 \cdot (q - 2) \cdot \frac{3A^2 \cdot Q_{53}}{(A^2 + A + 1)^2}, \quad \text{with}
$$

$$
Q_{52} = 1 - 4A - 15A^2 - 46A^3 - 102A^4 - 116A^5 - 107A^6 - 72A^7 - 19A^8 + 2A^9 + A^{10}, \\
Q_{53} = -q \cdot A \cdot (4A^8 + 8A^8 + A^7 - 2 - 28A^6 - 50A^5 - 64A^4 - 55A^3 - 30A^2 + 9A).
$$

Note, that $W_7$ also has a $(1 + A + A^2)^4$ singularity. In fact the characteristic equation reads:

$$
P(\Lambda) = \Lambda^3 + C_2 \cdot \Lambda^2 + C_1 \cdot \Lambda + C_0 = 0, \quad \text{where}
$$

$$
C_0 = C_{00} + C_{01} + \cdots + C_{09}, \\
C_1 = C_{10} + C_{11} + \cdots + C_{16}, \\
C_2 = C_{20} + C_{21} + C_{22} + C_{23},
$$

(F.76)
where the $C_{ij}$'s are of order $j$ in $B$, reads successive equations of the form:

$$R_{\text{ext}}(W_1, W_2, \ldots, W_{n-1}) + (3 + C_{10} + 2 \cdot C_{20}) \cdot W_n = 0.$$  \hspace{1cm} (F.77)

The occurrence of a new singularity is thus related to the following combination of the $C_{ij}$'s:

$$3 + C_{10} + 2 \cdot C_{20} = (1 - A^2) \cdot (1 - A^3).$$  \hspace{1cm} (F.78)

This key expression can be seen as the derivative of the characteristic polynomial (F.76) with respect to $A$, in the $B = 0$ limit.

$$3 + C_{10} + 2C_{20} = \frac{d}{dA} P(A)(B = 0).$$  \hspace{1cm} (F.79)

Let us note, however, that for the first coefficient, $W_2$, there is the following equation:

$$(A^2 - 1) \cdot (C_{22} + C_{02} + C_{12}) - 3B^2 \cdot (3 + C_{10} + 2C_{20}) \cdot (q - 1) \cdot W_2 = 0,$$

where

$$C_{22} + C_{02} + C_{12} = 3 \cdot (q - 1) \cdot A^2B^2 \cdot (A^3 - 1).$$  \hspace{1cm} (F.80)

Therefore, one sees that the singularity $1 - A^3$ cancels out for this first coefficient $W_2$. A mechanism where all the $N$th root of unity occur, clearly needs to consider "polynomial" relations of infinite degree.

**REFERENCES**


