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On the complexity of some birational transformations

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Abstract

Using three different approaches, we analyse the complexity of various birational maps constructed from simple operations (inversions) on square matrices of arbitrary size. The first approach comprises the study of the images of lines, and relies mainly on univariate polynomial algebra, the second approach is a singularity analysis and the third method is more numerical, using integer arithmetics. These three methods have their own domain of application, but they give corroborating results, and lead us to a conjecture on the algebraic entropy of a class of maps constructed from matrix inversions.

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1. Presentation

We investigate some properties of birational realizations of Coxeter groups on projective spaces of various dimensions. There are many motivation to examine these realizations. They appear naturally in the study of 'spin models' of statistical mechanics, in relation with the symmetries of the Yang–Baxter (alias star-triangle equations) [1, 2] as well as through their connection with association schemes and knot polynomials [3–8]. Among the basic ingredients in all these applications are various inversions of matrices (matrix inverse, block inverses, element by element inverse).

The typical group we consider is a Coxeter group with two involutive generators I and J, and no extra relations between I and J. They generate a discrete group Γ, the infinite dihedral group, isomorphic to the semi-direct product $\mathbb{Z} \ltimes \mathbb{Z}_2$. The action we analyse is the one of the infinite parts of Γ, i.e. to say the iterates of $K = I \circ J$.

Studying the iterates of K is the same thing as studying the rational (in fact bi-rational) discrete dynamical system. Such an investigation is a part of the algebraic dynamics, and we focus on the analysis of the complexity of the iterates [9–22].

We perform this analysis for a definite class of transformations, defined from elementary operations on matrices of size $q \times q$, the entries of the matrices being the natural coordinates of complex projective spaces $\mathbb{CP}_n$. Depending on the specific form of the matrices, the dimension n will take different values ($n \leqslant q^2 - 1$).

Another motivation for our choice is that arbitrary algebraic transformations are not invertible. Being by construction rational and invertible, matrix inversions constitute a factory of almost everywhere invertible transformations of any degree, and with any number of variables. They provide us with a variety of explicit birational dynamics. The specific choice we made here for the form of the matrices (see also [23, 24]) is motivated by their use in lattice statistical mechanics, and the richness of the structures of the systems we construct.

We explain, exemplify and confront three different approaches to the problem. We also present a conjecture for the value of the algebraic entropy for a family of transformations of interest to statistical mechanics.

The paper is organized as follows. We state in section 2 the problem of calculating the complexity of a birational transformation acting on a projective space, and define the basic objects of interest, in particular the algebraic entropy or equivalently the rate of growth of the degrees of the iterates of a map. We introduce four families of maps, which will be used for explicit calculations. In section 3 we indicate how to surmise the generating function of the sequence of degree of iterates of a map from its first terms. This provides a first method of calculation of the complexity. In section 4, we calculate exactly the sequence of degrees by an analysis of the singularity structure for one of the families of maps. In section 5, we describe an arithmetic approach, where we examine the action of iterates on rational points (integer homogeneous coordinates), and simply measure the growth of the size of the coordinates. This yields approximate values of the complexity. We conclude with a conjecture.

2. The problem

Let K be a birational transformation of complex projective space $\mathbb{CP}_n$. If we write K in terms of homogeneous coordinates, it appears as a polynomial transformation given by $n + 1$ homogeneous polynomials of the same degree d. With the rule that we should factorize out any common factor, d is well defined in a given system of coordinates. Of course it is *not* invariant by changes of coordinates. We may construct the sequence $\{d_n\}$ of the degrees of the iterates K^n of K.

We will use [15], as a measure of the complexity of K, the growth of the sequence d_n: in the absence of factorizations of the polynomials the sequence would just be

$$d_n = d_1^n = d^n. \tag{1}$$

What happens is that if some factorizations appear, they induce a drop of the degree, so that we only have an upper bound

$$d_n \leqslant d^n. \tag{2}$$

The drop may even be so important that the growth of d_n becomes polynomial or is bounded. A measure of the growth is the algebraic entropy [15][4]

$$\epsilon = \lim_{n\to\infty} \frac{1}{n} \log d_n, \tag{3}$$

or the rate of growth

$$\lambda = \exp(\epsilon). \tag{4}$$

[4] The existence of the limit is a consequence of the elementary relation $d_{n+m} \leqslant d_n \cdot d_m$.

Both the entropy ϵ and the rate of growth λ are invariant by any birational change of coordinates. They are canonically associated with the map K. Our aim is to calculate them for definite classes of maps, which we now describe.

Suppose M is a $q \times q$ matrix (the 'q' is reminiscent of the q-state Potts model of statistical mechanics), and consider the two simple rational involutions I and J: the involution I is the matrix inverse up to a factor (i.e. when written polynomially it amounts to replacing each entry by its cofactor). The involution J is the element by element inverse (also called a Hadamard inverse, which replaces each entry M_{ij} by its inverse $1/M_{ij}$). The two involutions I and J do not commute, and their composition $K = I \circ J$ is generically of infinite order.

The map K acts naturally on $\mathbb{CP}_{q^2-1}$. It is however possible to define various reductions to smaller projective spaces in the following way [25]. For a given size of square matrices, we define a pattern as a set of equalities between entries of the matrix. The set of all patterns is the set of all partitions of the entries of the matrix. An example of a pattern is 'all diagonal entries equal, all off-diagonal entries equal'. This corresponds to the partition of the entries in two parts (diagonal + off-diagonal). Clearly any pattern is preserved by the action of J. We call admissible a pattern which is also stable by I (or equivalently K).

All admissible patterns have been classified for $q = 4$ and some of them for $q = 5$ in [23–26]. It has been also shown that λ can vary considerably from one admissible pattern to another. For example for 5×5 cyclic *and* symmetric matrices one has $\lambda = 1$ (polynomial growth), whereas with the cyclic matrices one gets $\lambda = (7 + 3\sqrt{5})/2$.

We will focus on four fundamental admissible patterns, which exist whatever the size q of the matrices is. The first one is the pattern (S) of symmetric matrices. The second one (C) is the pattern of the cyclic matrices defined by $M_{i,j} = M_{i+1,j+1}$ (with indices taken modulo q). The third one is the pattern of matrices which are at the same time cyclic and symmetric (CS). The last one is the general pattern (G), without equality conditions between the entries. From the results obtained on these different patterns, we conjecture that, contrary to intuition and although their number of variables differ enormously, λ is the same for cyclic (S), symmetric (S) and general (G) patterns.

3. A first approach: generating functions

From the sequence of degrees $\{d_n\}$, it is possible to construct a generating function

$$f(u) = \sum_{n=0}^{\infty} d_n u^n. \tag{5}$$

Since the degrees are bounded by (2), the series (5) always has a non-zero radius of convergence ρ. Actually

$$\rho = \frac{1}{\lambda}, \tag{6}$$

with λ being the rate of growth defined above.

The calculation method is the following: calculate explicitly the first terms of the series, and try to guess the values of the generating function. The method is sensible if the generating function is rational.

The striking fact is that indeed the generating function $f(u)$ happens to be a rational fraction with integer coefficients in most cases. The consequence is that a finite number of terms of the series determine it completely. For reversible maps (i.e. when there exists a similarity relation between the map and its inverse), we have not found any counterexample to this rule. There are however non-reversible maps for which the generating function is not

Table 1. Generating functions for the cyclic symmetric (CS) patterns. The formulae for values of q tagged with a $(\star)$ can be proved. $n_{\max}$ is the maximum number of iteration performed, m refers to the Pad approximation, λ is the rate of growth and λ_{num} is the numerical complexity calculated in section 5.

q	$f_q(u)$	$n_{\max}$	m	λ	λ_{num}
$4(\star)$	$\frac{(1+u)^2}{(1-u)^2}$	∞	4	1	
$5(\star)$	$\frac{(1+u+2u^2)^2}{(1-u)^3(1+u+u^2)}$	14	9	1	1.00
6	$\frac{(1+2u)^2}{(1-u)(1-4u)}$	15	4	4	4.00
$7(\star)$	$\frac{(1+u+3u^2)^2}{(1-u)(1+u+u^2)(1-7u+u^2)}$	12	9	6.854 10	6.85
8	$\frac{(1+u)(1+2u-u^2)}{(1-u)(1-11u+7u^2-u^3)}$	11	7	10.331 85	10.33
$9(\star)$	$\frac{(1+u+3u^2-3u^3)^2}{(1-u)(1-13u+2u^2+u^3+12u^4-8u^5+u^6)}$	11	13	12.832 69	12.83
10	$\frac{(1+3u)^2}{(1-u)(1-18u+u^2)}$	9	5	17.944 27	17.94
$11(\star)$	$\frac{(1+u+5u^2)^2}{(1-u)(1+u+u^2)(1-23u+u^2)}$	7	9	22.956 44	22.95
12	$\frac{(1+4u-3u^2)(1+2u-u^2)}{(1-u)(1-27u+31u^2-9u^3)}$	8	8	25.812 54	25.81
$13(\star)$	$\frac{(1+u+6u^2)^2}{(1-u)(1+u+u^2)(1-34u+u^2)}$		9	33.970 56	33.97

rational [22]. Another consequence of the rationality of f is that λ is an algebraic integer, and we have no counterexample yet to that.

For practical purposes, it is necessary to push the calculation of the degree of the iterates as far as possible. Instead of evaluating the full iterate, it is sufficient to consider the image of a generic line l with a running point

$$l(t) = [a_0 + b_0 t, a_1 + b_1 t, \ldots, a_n + b_n t], \tag{7}$$

where a_i, b_i are arbitrary coefficients, and to evaluate the images of $l(t)$ by K^n. The degree d_n is read off from this image. The calculation may furthermore be improved by using integer coefficients in (7) and calculating (formal calculation software are quite efficient at that) over polynomial with coefficients in $\mathbb{Z}/\mathbb{Z}_p$ with p being a sufficiently large prime integer. Taking different values of p and of the coefficients a_i, b_i helps eliminate the accidental simplifications which may occur.

Suppose we have the degree d_n for the first values of n, say $n = 1 \ldots n_{\max}$. We may fit the series with a Padé approximant F, with numerator (respectively denominator) of degree N (respectively M), such that

$$N + M = n_{\max} - 1 \tag{8}$$

N running from 0 to $n_{\max} - 1$. Our experience is that, if $n_{\max}$ is large enough, the rational fraction F we find simplifies drastically, and stabilizes for some central values of N (i.e. the numerator and denominator are respectively of degree smaller than M and N). This usually means that the exact generating function has been reached.

Note that the expansion of the non-optimal $[N, M]$ Padé approximants yields non-integer, or negative coefficients in the expansion of F, in contradiction with these coefficients being a degree. Table 1 displays the 'exact' expression we have inferred for the generating function for various values of q for the (CS) pattern, as well as the value of $m = N + M$ and the value of $n_{\max}$.

When $n_{\max}$ is larger than m, we have a prediction on the next values of the degree, and this gives confidence that the result is exact.

In table 1, we also give the inverse of the modulus of the smallest zero of the denominator, as well as a numerical value computed as explained in section 5.

4. A second approach: singularity analysis

In this section we prove that the rate of growth of the patterns (CS) for prime q is a quadratic integer, by showing that the sequence of degrees verifies a linear recurrence relation of length 2 with integer coefficients. This implies that the generating function of the degrees is a rational fraction and corroborates a part of the results given in table 1.

4.1. Some notations

Let M be a cyclic symmetric matrix of size $q \times q$. The matrix M may be written in terms of the basic cycle of order q:

$$\sigma = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$M_{\mathrm{even}} = x_0 + x_1(\sigma + \sigma^{q-1}) + \cdots + x_{p-1}\sigma^{q/2}, \qquad p = \frac{q}{2} + 1$$

$$M_{\mathrm{odd}} = x_0 + x_1(\sigma + \sigma^{q-1}) + \cdots + x_{p-1}(\sigma^{(q-1)/2} + \sigma^{(q+1)/2}), \qquad p = \frac{q+1}{2}$$

when q is even and odd, respectively.

The parameter space is a projective space $\mathbb{CP}_{p-1}$ of dimension $p-1$, with $p = q/2 + 1$ if q is even and $p = (q+1)/2$ if q is odd. We use homogeneous coordinates $[x_0, \ldots, x_{p-1}]$.

We will study the two elementary transformations I and J acting on M. Both are rational involutions (and are thus birational transformations).

The Hadamard inverse J may be written polynomially in terms of the homogeneous coordinates

$$J : [x_0, \ldots, x_{p-1}] \longrightarrow \left[\prod_{k\neq 0} x_k, \prod_{k\neq 1} x_k, \ldots, \prod_{k\neq p-1} x_k\right]. \tag{9}$$

The matrix inverse I, up to a factor, transforms cyclic matrices into cyclic matrices, and symmetric matrices into symmetric matrices. It thus acts on cyclic symmetric matrices.

For cyclic symmetric matrices, the matrix inverse I is related to the Hadamard inverse J, by a similarity transformation:

$$I = C^{-1} \circ J \circ C. \tag{10}$$

The transformation C acts linearly on the p homogeneous coordinates. Denoting ω the qth root of unity, C is given by the $p \times p$ matrix with entries:

$$C_{r,0} = 1, \qquad C_{r,s} = \omega^{rs} + \frac{1}{\omega^{rs}} \qquad r = 0, \ldots, p-1, \quad s = 1, \ldots, p-1 \tag{11}$$

for q odd and

$$C_{r,0} = 1, \qquad C_{r,s} = \omega^{rs} + \frac{1}{\omega^{rs}} \qquad r = 0, \ldots, p-1, \quad s = 1, \ldots, p-2$$
$$C_{r,p-1} = (-1)^r \tag{12}$$

for q even.

The matrix C verifies $C^2 = \mathbf{1}$.

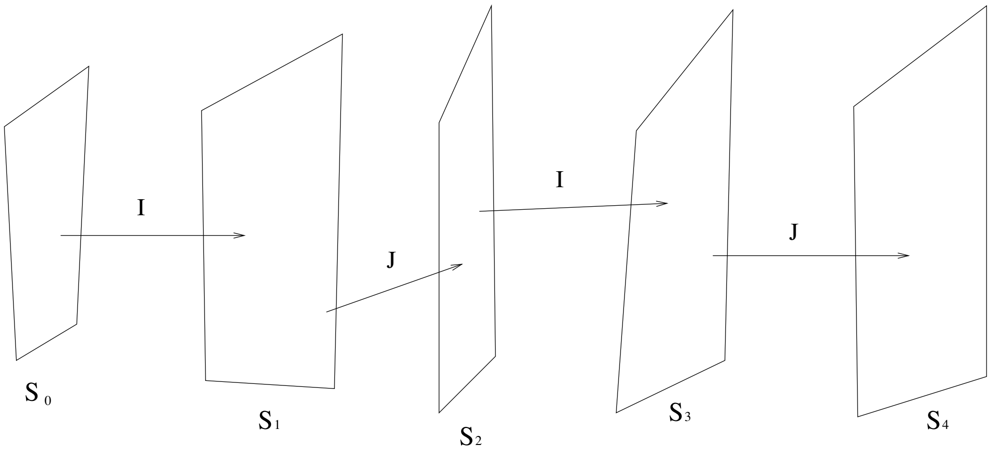

Figure 1. Successive images.

4.2. Sequences of surfaces and degrees

Consider now a sequence of hypersurfaces in $\mathbb{CP}_{p-1}$, obtained by applying successively I, J, then I and so on, starting with a generic hyperplane S_0 (see figure 1). Each surface S_n has a polynomial equation, of degree d_n, which we also denote as S_n. Since for non-singular points,

$$x \in S_{2n} \quad \leftrightarrow \quad J(x) \in S_{2n-1}, \tag{13}$$

S_{2n} can be obtained from S_{2n-1} by substituting the coordinates of x with the homogeneous polynomial expressions of the coordinates of $J(x)$ in $S_{2n-1}(x)$. Note that, since J is an involution, S_{2n-1} may be obtained from S_{2n} in the same manner.

What happens at the level of the equations is that $S_{2n-1}(J(x))$ may factorize. One of the factors is $S_{2n}(x)$. The only other possible factors are powers of the coordinates of $x = (x_0, \ldots, x_{p-1})$ as explained in the lemma below. The relation

$$S_{2n-1}(J(x)) = S_{2n}(x) \cdot \prod_{k=0}^{p-1} x_k^{\alpha_{2n-1}^{(k)}} \tag{14}$$

defines the exponents $\alpha_{2n-1}^{(k)}$.

4.3. A lemma

The previous relation is crucial. Its proof is elementary and goes as follows.

Suppose B is a birational involution. When written in terms of the homogeneous coordinates, B^2 appears as the multiplication by some common polynomial factor of all the coordinates, that is to say the identity transformation in projective space.

$$B(B(x)) = [\kappa_B(x) \cdot x_0, \kappa_B(x) \cdot x_2, \ldots, \kappa_B(x) \cdot x_{p-1}] \tag{15}$$

with $\kappa_B(x) =$ some polynomial.

We then have, if two algebraic hypersurfaces S and S' are the proper images of each other by involution B:

$$S(B(x)) = S'(x) \cdot R(x) \tag{16}$$
$$S'(B(x)) = S(x) \cdot T(x) \tag{17}$$

with R and T being some polynomial expressions of the coordinates. We then have, using (16) and (17):

$$\kappa_B(x)^{\deg(S)} \cdot S(x) = S(x) \cdot R(B(x)) \cdot T(x), \tag{18}$$

i.e. to say

$$\kappa_B(x)^{\deg(S)} = R(B(x)) \cdot T(x). \tag{19}$$

Equation (19) shows that the only factors in the right-hand side of equation (17) are the equations of S, and polynomial expressions $T(x)$ which divide $\kappa_B(x)$, possibly raised to some power.

In the specific example $B = J$, and $S = S_{2n-1}$, using

$$\kappa_J(x) = \prod_{i=0}^{p-1} x_i^{p-2} \tag{20}$$

we obtain

$$S_{2n-1}(J(x)) = S_{2n}(x) \cdot \prod_{i=0}^{p-1} x_i^{\rho_i}, \tag{21}$$

with $x_i(x)$ being the ith coordinate of t and ρ_i some integer power.

This ends the proof of formula (14).

4.4. Recurrence relation

Similar to equation (14), we have

$$S_{2n}(J(x)) = S_{2n-1}(x) \cdot \prod_{i=0}^{p-1} x_i^{\alpha_{2n}^{(i)}}, \tag{22}$$

with the constraint

$$\kappa_J(x)^{d_{2n}} = \prod_{i=0}^{p-1} x_i^{\alpha_{2n-1}^{(i)}}(x) \cdot \prod_{j=0}^{p-1} x_j^{\alpha_{2n}^{(j)}}(J(x)). \tag{23}$$

We also have the corresponding equations for the action of I.

$$S_{2n}(I(x)) = S_{2n+1}(x) \cdot \prod_{k=0}^{p-1} X_k^{\beta_{2n}^{(k)}} \tag{24}$$

$$S_{2n+1}(I(x)) = S_{2n}(x) \cdot \prod_{i=0}^{p-1} X_i^{\beta_{2n+1}^{(i)}}, \tag{25}$$

where X_i is the ith coordinate of Cx.

To make relations more uniform, we introduce a slight change of notation: define the sequences $\{u_n^i\}$ and $\{v_n^i\}$ with the convention that

$$\alpha_{2k+1}^i = u_{2k+1}^i, \qquad \alpha_{2k}^i = v_{2k}^i, \tag{26}$$

$$\beta_{2k+1}^i = v_{2k+1}^i, \qquad \beta_{2k}^i = u_{2k}^i. \tag{27}$$

At step n we have $2p + 1$ variables (d_n, u_n^i and v_n^i).

A first equation simply expresses the factorization:

$$d_n = (p-1)d_{n-1} - \sum_{i=0}^{p-1} u_{n-1}^i. \tag{28}$$

Another set of equations is obtained by expressing that both I and J are involutions:

$$(p-2)d_n = v_{n-1}^i + \sum_{j\neq i} u_n^i, \qquad i = 0 \ldots p-1. \tag{29}$$

It is easy to obtain from equations (28) and 29:

$$v_n^i = (p-2)d_{n-1} + u_{n-1}^i - \sum_{j=0}^{p-1} u_{n-1}^j, \qquad i = 0 \ldots p-1. \tag{30}$$

4.5. Singularity structure

We need p additional equations to complete the previous system. They will be given, under some constraints, by the analysis of the singularity structure. The basic idea is that the numbers α_n^i, β_n^i (or equivalently u_n^i, v_n^i) have a geometrical meaning: they are the multiplicity of some specific points on the surface S_n.

The singularity structure of J is very simple. A singular point is a point whose image is undetermined: this happens when all polynomial expressions giving the image (9) vanish simultaneously. Any point with more than two vanishing coordinates is singular for J.

We will look at the action of the pair I, J on the hypersurfaces composing the factor κ_J of equation (23). Those are just the n hyperplanes Π_k, $k = 0 \ldots p-1$ of equation

$$\Pi_k : \qquad \{x_k = 0\}. \tag{31}$$

All intersections of these hyperplanes are made out of singular points of J. Some points are in a sense *maximally* singular. They are the intersections of all but one of the planes Π_r, i.e. all but one of their coordinates vanish. There are p such points

$$P_k = [0, \ldots, 0, 1, 0, \ldots, 0], \qquad k = 0 \ldots p-1 \tag{32}$$

with 1 in $(k+1)$ th position.

The singularity structure of I is the same as that of J, up to the linear change of coordinates C. There are in particular p distinguished singular points Q_k, $k = 1 \ldots p$ of I:

$$Q_k = C^{-1} P_k, \qquad k = 0 \ldots p-1. \tag{33}$$

To complete the set of equations (28), (29), (30), we need to explore in some more details the singularity structure of the maps. What matters is the interplay between I and J.

The map J sends the hyperplane Π_k (31) onto the point P_k (32). The subsequent images depend on what q is.

The situation is tractable when q is a prime number, in which case the subsequent images of Π_k *always go back to the point P_k after a finite number of steps*, actually one or three steps. There, we meet a singularity, *and the equation of Π_k factorizes*. We will examine the case where q is a prime number, $q = 2p-1$.

The coordinate x_0 plays a special role and the point P_0 behaves differently from the other points P_s, $s = 1 \ldots p-1$.

Whatever q, the transformation of the hyperplane Π_0 reads

$$\Pi_0 \overset{J}{\rightarrowtail} P_0 \overset{I}{\longrightarrow} P_0 \overset{J}{\rightsquigarrow} \Pi_0. \tag{34}$$

We use the following convention concerning the arrows: when a variety is sent by the map onto a variety of same codimension we use the plain arrow $\longrightarrow$. When the codimension of the image is lower (blow-down) we use the symbol $\rightarrowtail$, and when it is larger (blow-up) we use the squiggly arrow $\rightsquigarrow$. A blow-up for the birational mapping K corresponds to a blow-down for its inverse K^{-1}.

The action of I and J on the hyperplane Π_s reads

$$\Pi_s \overset{J}{\rightarrowtail} P_s \overset{I}{\longrightarrow} R_s \overset{J}{\longrightarrow} R_s \overset{I}{\longrightarrow} P_s \overset{J}{\rightsquigarrow} \Pi_s. \tag{35}$$

The points R_s have coordinates $[\pm 1, \pm 1, \ldots, \pm 1]$. For example for $q = 5$, $R_2 = [+1, +1, -1]$ and $R_3 = [+1, -1, +1]$, while, for $q = 7$, $R_2 = [+1, -1, -1, +1]$, $R_3 = [-1, -1, +1, +1]$ and $R_4 = [-1, +1, -1, +1]$.

The pattern is similar for the points Q_k. It is obtained from the previous one by the linear change of coordinates defined by C. The planes Π_k are replaced by the planes $\Pi'_k = C^{-1}\Pi_k$ and the points R_s are replaced by the points $R'_s = C^{-1}R_s$.

When q is not a prime number, the pattern is different: the successive actions of I and J lead to singular points other than the P_k's and Q_k's. In appendix A the case $q = 9$ is studied as an example.

Relations (34), (35) allow us to relate the multiplicities of the singular points P_k on different surfaces S_n. Since $P_0 \rightarrow P_0$ in (equation 34) we have

$$u_n^0 = v_{n-1}^0 \tag{36}$$

and since $P_s \rightarrow P_s$ in (equation 35) we obtain

$$u_n^s = v_{n-3}^s, \qquad s = 1 \ldots p-1. \tag{37}$$

4.6. *End of the proof*

The previous analysis shows that when q is prime, the factors x_i (respectively X_i) $1 \leqslant i < p$ appear with the same exponent. In other words, for q a prime number, the points $P_1, P_2, \ldots P_{p-1}$ play an equivalent role; they will have the *same multiplicities* on each S_n; and we will use u_n^1 to denote their common value.

Using (36), (37) together with (28) and (30) we get

$$\begin{aligned} d_n &= (p-1)d_{n-1} - u_{n-1}^0 - (p-1)u_{n-1}^1, \\ u_n^0 &= (p-2)d_{n-2} - (p-1)u_{n-2}^1, \\ u_n^1 &= (p-2)d_{n-4} - u_{n-4}^0 - (p-2)u_{n-4}^1. \end{aligned} \tag{38}$$

The rate of growth of the d_n's is the inverse of the modulus of the smallest eigenvalues of the 12×12 matrix given by the above linear system. The outcome is that the rate of growth of K is the inverse of the smaller root of

$$x^2 + (2 - (p-1)^2)x + 1 = 0. \tag{39}$$

To obtain the full expression of the generating functions, we need to specify the initial values of d_n, u_n^0 and u_n^1. They can easily be calculated with the help of formal calculation software. The results are summarized in table 2.

Note that when q is not prime, we may still write a set of recursions similar to (38). The system is not complete, and cannot be obtained from the analysis presented in section 4 (see appendix A).

Table 2. Initial values of d_n, u_n^0 and u_n^1 for $0 \leqslant n < 4$.

n	d_n	u_n^0	u_n^1
0	1	0	0
1	$p-1$	0	0
2	$(p-1)^2$	$p-2$	0
3	p^3-3p^2+2p+1	$(p-1)(p-2)$	0
4	$(p-1)(p^3-3p^2+p+3)$	$(p-1)^2(p-2)$	$p-2$

5. Arithmetical approach: complexity through number of digits

The third approach consists in calculating the image of integer points, and evaluating the growth of the size of the coordinates, through the number of digits. It means that we do not try to calculate the iterates formally. This method was already experimented in [13].

Obviously the integer coordinates become extremely large, as large as 10^{6000} and we used the GMP library to implement the program [27]. At each step of the calculation we factor out the greatest common divisor of the components. We assume that the existence of a common factor between all the coordinates is due to a factorization of the underlying polynomial. This assumption is valid, at least after the first step where an accidental factorization could occur. The degree of the polynomial is estimated as the number of bits used to store a typical entry (i.e., $\log_2(M_{ij})$). The algorithm proceeds as follows: (i) construct a random matrix of integers respecting the equalities of the pattern under consideration, (ii) replace each term by its cofactor, (iii) divide every term by the greatest common factor of all of them, (iv) replace each term by the product of all others, (v) divide every term by the greatest common factor of all of them, (vi) record the number of digits used to store the matrix elements. Note that one can exchange (ii) and (iv) without altering the results. The procedure is iterated for as many steps as possible, and possibly several runs with different initial matrices are performed. Note that for pattern involving only very few variables, it can be efficient to write directly the recursion relation over the variables.

The results are summarized in table 3, giving the value of the complexity for various values of q and for the four patterns introduced above. For cyclic matrices and general q it has been shown in [15] that the rate of growth for $K = I \circ J$ is a quadratic integer which is the inverse of the smaller root of

$$x^2 + (2 - (q-2)^2)x + 1. \tag{40}$$

In table 3 an empty cell means that we have not been able to compute the corresponding λ. This is due to the fast growth of the coordinates, preventing us to perform a sufficient number of numerical iterations. The number of digits displayed is just an indication of the estimated accuracy of our numerical result.

6. Conclusion

The three different approaches we have used give corroborating results. The first two methods are indeed measuring the same object, i.e. the rate of growth of the degree of iterates. The third method is evaluating something which may *a priori* be different, but appears to coincide with the previous one. We see by comparing the last two columns of table 3 that λ_G happens to be extremely close to λ_S, as well as to λ_C. This allows us to state the two conjectures:

Table 3. Complexities of $K = I \circ J$ for various values of q, for patterns (CS), (C), (S) and (G). The numerical and analytical results are displayed. The number in brackets is the number of iterations of K we have been able to calculate.

q		Cyclic symmetric λ_{CS}	Cyclic λ_C	Symmetric λ_S	General λ_G
5	Numerical	1.000	6.85 [7]	6.86 [7]	6.86 [6]
	Analytical	1	6.854		
6	Numerical	4.00 [10]	13.93 [5]	13.88 [5]	13.9 [4]
	Analytical	4	13.928		
7	Numerical	6.854 [7]	22.96 [4]	22.97 [4]	22.97 [4]
	Analytical	6.854	22.956		
8	Numerical	10.33 [6]	33.97 [4]	33.97 [3]	34.1 [3]
	Analytical		33.970		
9	Numerical	12.83 [5]	47.0 [3]	47.0 [3]	47.0 [3]
	Analytical		46.978		
10	Numerical	17.9 [4]	62.0 [3]	62.0[3]	62.0 [2]
	Analytical		61.984		
11	Numerical	22.96 [4]	79.0 [3]	79.1 [2]	80.7 [2]
	Analytical	22.956	78.987		
12	Numerical	25.8 [4]	98.0 [3]	99.1 [2]	100.3 [2]
	Analytical		97.990		
13	Numerical	33.97 [3]	130.3 [3]	121.6 [2]	121.5 [1]
	Analytical	33.970	118.992		
14	Numerical	39.1 [2]	142.8 [2])	144.5 [2]	144.2 [1]
	Analytical		141.993		
15	Numerical	42.19 [2]	167. [2]	170. [2]	
	Analytical		166.99		
16	Numerical	49.10 [2]	194. [2]		
	Analytical		193.995		
17	Numerical	61.6 [2]	224. [2]		
	Analytical	61.984	222.995		

Proposition 1. *Evaluating the rate of growth of the 'size' of iterates on rational points yields the same value as the rate of growth of the degree of the iterates.*

Proposition 2. *The rate of growth of the degree of the iterates of the transformation $K = I \circ J$ for the general matrices (pattern (G)), for symmetric matrices (pattern (S)), and for cyclic matrices (pattern (C)) are the same. Their common value is the inverse of the modulus of the smaller root of $x^2 - (q^2 - 4q + 2)x + 1 = 0$.*

Such a statement means that although the number of parameters of patterns (G) and (S) is much bigger than the one of pattern (C), the latter captures the entirety of the complexity of the product of inversions $K = I \circ J$. This might be related to the structure of bialgebra of the set of square matrices equipped with ordinary matrix product and Hadamard product. Phrased differently, the skeleton formed by the cyclic matrices encodes the structure of the whole bialgebra. This deserves further investigations which are beyond the scope of this paper.

Appendix. The cyclic symmetric case for $q = 9$

We consider in this appendix the case $q = 9$. Since q is not a prime number, our result of section 4 does not apply.

Table A1. The initial values of d_n, u_n^0, u_n^1 and u_n^2 for $0 \leqslant n \leqslant 4$.

n	d_n	u_n^0	u_n^1	u_n^2
0	1	0	0	0
1	4	0	0	0
2	16	3	0	2
3	59	12	0	8
4	216	46	3	32

The number of homogeneous variables is $p = (q+1)/2 = 5$. We use the same notation as in the text for the hyperplane Π_k and the point P_i. In addition we define the three points $Q_1 = (1, 1, -1, -1, 1)$, $Q_2 = (1, 1, 1, -1, -1)$ and $Q_4 = (1, -1, 1, -1, 1)$. We also introduce the codimension-two variety $\Pi_{0,3}$ defined by the two equations $x_0 = 0$ and $x_3 = 0$. The singularity structure is

$$\begin{aligned}
&\Pi_0 \xrightarrow{J} P_0 \xrightarrow{I} P_0 \overset{J}{\rightsquigarrow} \Pi_0\\
&\Pi_1 \xrightarrow{J} P_1 \xrightarrow{I} Q_1 \xrightarrow{J} Q_1 \xrightarrow{I} P_1 \overset{J}{\rightsquigarrow} \Pi_1\\
&\Pi_2 \xrightarrow{J} P_2 \xrightarrow{I} Q_2 \xrightarrow{J} Q_2 \xrightarrow{I} P_2 \overset{J}{\rightsquigarrow} \Pi_2\\
&\Pi_3 \xrightarrow{J} P_3 \xrightarrow{I} Q_3 \overset{J}{\rightsquigarrow} \Pi_{0,3}\\
&\Pi_4 \xrightarrow{J} P_4 \xrightarrow{I} Q_4 \xrightarrow{J} Q_4 \xrightarrow{I} P_4 \overset{J}{\rightsquigarrow} \Pi_4
\end{aligned}$$

the subsequent iterates of $\Pi_{0,3}$ are non-singular. We see that there will be six sets of exponents: u_n^0 and v_n^0 related to x_0; u_n^1 and v_n^1 related to x_1, x_2 and x_4; and finally u_n^2 and v_n^2 related to x_3. The equations expressing the degree drop due to the factorization, and the fact that I and J are involutions, are

$$\begin{aligned}
d_{n+1} &= 4d_n - u_n^0 - 3u_n^1 - u_n^2,\\
v_{n+1}^0 &= 3d_n - 3u_n^1 - u_n^2,\\
v_{n+1}^1 &= 3d_n - u_n^0 - 2u_n^1 - u_n^2,\\
v_{n+1}^2 &= 3d_n - u_n^0 - 3u_n^1.
\end{aligned}$$

Moreover, the singularity structure shown above yields

$$u_{n+1}^0 = v_n^0 \qquad u_{n+1}^1 = v_{n-2}^1.$$

It is clear that an equation is missing to close the system:

$$\begin{aligned}
d_{n+1} &= 4d_n - u_n^0 - 3u_n^1 - u_n^2\\
u_{n+1}^0 &= 3d_{n-1} - 3u_{n-1}^1 - u_{n-1}^2\\
u_{n+1}^1 &= 3d_{n-3} - u_{n-3}^0 - 2u_{n-3}^1 - u_{n-3}^2.
\end{aligned}$$

If we suppose that there exists a recursion relation of the form

$$u_{n+1}^2 = ad_{n-q} + bu_{n-q}^0 + cu_{n-q}^1 + du_{n-q}^2 + e,$$

where the a, b, c, d, e, as well as the shift q are integer constants. The hypothesis $q = 1$ yields

$$u_{n+1}^2 = 2d_{n-1} - 3u_{n-1}^1. \tag{A.1}$$

Introducing, with obvious notations, the generating functions

$$d(s) = \sum d_n s^n, \qquad u_i(s) = \sum u_n^i s^n, \qquad i = 1, 2, 3,$$

one easily finds (see table A1)

$$d(s) = 1 + \frac{(4 - s^2 - s^6)s}{P(s)}, \qquad u_0(s) = \frac{(2s^2 - 3)(1 + s^2)s^4}{P(s)},$$

$$u_1(s) = \frac{(2s^2 - 3)s^4}{P(s)}, \qquad u_2(s) = \frac{(3s^4 - 2s^2 - 2)s^2}{P(s)},$$

with

$$P(s) = (1 - s) \cdot (1 - 3s - 2s^2 - s^3 + 2s^4 + 2s^5 - s^6)$$

from which

$$f_9(u) = \frac{(1 + u + 3u^2 - 3u^3)^2}{(1 - u)(1 - 13u + 2u^2 + u^3 + 12u^4 - 8u^5 + u^6)}.$$

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