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Canonical decomposition of irreducible linear differential operators with symplectic or orthogonal differential Galois groups

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Abstract

We first revisit an order-six linear differential operator, already introduced in a previous paper, having a solution which is a diagonal of a rational function of three variables. This linear differential operator is such that its exterior square has a rational solution, indicating that it has a selected differential Galois group, and is actually homomorphic to its adjoint. We obtain the two corresponding intertwiners giving this homomorphism to the adjoint. We show that these intertwiners are also homomorphic to their adjoint and have a simple decomposition, already underlined in a previous paper, in terms of order-two self-adjoint operators. From these results, we deduce a new form of decomposition of operators for this selected order-six linear differential operator in terms of three order-two self-adjoint operators. We generalize this decomposition to decomposition in terms of three self-adjoint operators of arbitrary orders, provided the three orders have the same parity. We then generalize the previous decomposition to decompositions in terms of an arbitrary number of self-adjoint operators of the same parity order. This yields an infinite family of linear differential operators homomorphic to their adjoint, and, thus, with a selected differential Galois group. We show that the equivalence of such operators, with selected differential Galois groups, is compatible with these canonical decompositions. The rational solutions of the symmetric, or exterior, squares of these selected operators are, noticeably, seen to depend only on the rightmost self-adjoint operator in the decomposition. These results, and tools, are applied on operators of large orders. For instance, it is seen that a large set

of (quite massive) operators, associated with reflexive 4-polytopes defining Calabi–Yau three-folds, obtained recently by Lairez, correspond to a particular form of the decomposition detailed in this paper. All the results of this paper can be seen as providing an algebraic characterization of linear differential operators with selected symplectic or orthogonal differential Galois groups.

Keywords: differential Galois groups, self-adjoint operators, Calabi–Yau ODEs, Ising model operators, homomorphism or equivalence of differential operators, euclidean division of operators, diagonals of rational functions

1. Introduction

The n -fold integrals occurring in theoretical physics (lattice statistical mechanics, enumerative combinatorics, ...) are, quite systematically⁵, seen to be highly selected. For instance, the corresponding series expansions are *globally bounded* [1], the linear differential operators, which annihilate them, are not only *Fuchsian*, but *globally nilpotent* [2]. This is sometimes encapsulated in the wording ‘modularity’, well-defined in a Calabi–Yau framework [3–5], but a work-in-progress concept⁶ outside this framework. It corresponds to two different kinds of ‘special properties’: firstly, properties of algebraic geometry, or of arithmetic character [3, 4] (occurrence of miscellaneous series with *integer coefficients* like the nome, or the Yukawa couplings, emergence of *modular forms* [6], algebraic varieties of Kodeira dimension zero [7], ...), and, secondly, properties of differential geometry character (the associated linear differential operators have *selected* (or *special* [8]) *differential Galois groups* [9–11]), and this can be rephrased as *differential algebra properties* [12, 13]: these operators are *homomorphic to their adjoint*, the symmetric, or exterior, powers of these operators, or of equivalent operators, have *rational solutions*. We have addressed these two different kinds of ‘special properties’ in two recent sets of papers. In a first set of papers [3, 4], we have shown that the n -fold integrals $\chi^{(n)}$, associated with the n -particle contribution to the magnetic susceptibility of the Ising model [14], as well as various other n -fold integrals of the ‘Ising class’ [15, 16], or n -fold integrals from enumerative combinatorics [17], like lattice Green functions, are actually *diagonals of rational functions*⁷. As a consequence, they are solutions of linear differential equations ‘*Derived From Geometry*’, and their power series expansions are *globally bounded* [1], which means that, after just one rescaling of the expansion variable, they can be cast into series expansions with *integer coefficients*. In a second set of papers [12, 13], we revisited miscellaneous linear differential operators, mostly associated with lattice Green functions in arbitrary dimensions [17, 18], but also *Calabi–Yau operators* [19, 20], and order-seven operators corresponding to *exceptional differential Galois groups* [21, 22]. We have shown that the fact that these irreducible operators have special differential Galois groups⁸ can be simply understood, in a differential algebra viewpoint, from the fact that they are *homomorphic to their* (formal) *adjoints*⁹, and this can also

⁵ In an ‘experimental mathematics’ approach.

⁶ The mix between analytic, arithmetic, algebraic-geometry, differential geometry, differential algebra, ... properties being often a source of confusion in the literature.

⁷ Diagonals of rational functions can be seen as the simplest generalization of algebraic functions to transcendental (holonomic) functions [3, 4].

⁸ In the regular case the differential Galois group forms the Zariski closure of the monodromy group (Schlesinger [23]).

⁹ The adjoint of a linear differential operator is the (formal) adjoint defined as in [12] (see equations (3) and (4) in [12]). As far as formal calculations in Maple (DEtools) are concerned, there is a command ‘adjoint’ which can be used, see also [24] the command ‘Homomorphisms’.

be seen in the fact that the *symmetric squares*, or the *exterior squares*, of these operators, or of equivalent operators, have a *rational solution*. Furthermore, in the examples displayed in [12, 13], we saw that this property of homomorphism to the adjoint always corresponded to a decomposition [12, 13] of the order- $2p$ linear differential operator $M_{2p}^{(n,2p-n)}$, as (see equations (60), (83), (90), (91) in [13])

$$M_{2p}^{(n,2p-n)} = L_{2p-n} \cdot a(x) \cdot L_n + \frac{\lambda}{a(x)}, \tag{1}$$

or, introducing¹⁰ $\tilde{M}_{2p}^{(n,2p-n)} = a(x) \cdot M_{2p}^{(n,2p-n)}$, as

$$\tilde{M}_{2p}^{(n,2p-n)} = a(x) \cdot L_{2p-n} \cdot a(x) \cdot L_n + \lambda, \tag{2}$$

where the L_m 's are *self-adjoint operators* of order m . Such operators are, naturally, homomorphic to their adjoint, with intertwiners corresponding to these decompositions (1) and (2):

$$\begin{aligned} L_n \cdot a(x) \cdot M_{2p}^{(n,2p-n)} &= \text{adjoint}\left(M_{2p}^{(n,2p-n)}\right) \cdot a(x) \cdot L_n, \\ M_{2p}^{(n,2p-n)} \cdot a(x) \cdot L_{2p-n} &= L_{2p-n} \cdot a(x) \cdot \text{adjoint}\left(M_{2p}^{(n,2p-n)}\right). \end{aligned} \tag{3}$$

In other words, these decompositions (1), or (2), are closely related to the left, or right, intertwiners of the operator with its adjoint. These decompositions have been seen in all the quite large number of non-trivial lattice statistical physics examples, or enumerative combinatorics examples in [12, 13]. Note that such decompositions enable us to understand why certain differential Galois groups, appearing in lattice Green, are included in *orthogonal groups* $O(n, \mathbb{C})$ or *symplectic groups* $Sp(n, \mathbb{C})$.

On all the examples of (minimal order) linear differential operators we have encountered in lattice statistical physics, and beyond, in enumerative combinatorics (see for instance [2–4, 6, 15, 25–31]), we have verified¹¹ that they were actually homomorphic to their adjoint.

Since many *derived from geometry* n -fold integrals ('Periods' [32]) occurring in physics are seen to be *diagonals of rational functions* [3, 4], we also addressed in [12, 13] several examples of (minimal order) operators annihilating diagonals of rational functions (not necessarily emerging from physics), and remarked, again, that their irreducible factors¹² were, systematically, *homomorphic to their adjoint*. This leads to envisaging the conjecture¹³ that all the *irreducible factors* of the *minimal order* linear differential operator annihilating a *diagonal of a rational function should be homomorphic to their adjoint* (possibly on an algebraic extension).

Again a decomposition like (1), or (2), has been seen in all these examples of diagonals of rational functions in [12, 13], except an order-six operator¹⁴ \mathcal{L}_6 that was too large (see section 2 below) to quickly check whether it is *homomorphic to its adjoint*. This order-six linear differential operator annihilates the diagonal of the (three variables) rational function

¹⁰ Do note that the $\tilde{M}_{2p}^{(n,2p-n)}$ operators (2) are such that the functions, annihilated by L_n , are automatically *eigenfunctions* of $\tilde{M}_{2p}^{(n,2p-n)}$ with *eigenvalue* λ .

¹¹ Except on the order twelve and order-21 operators occurring [25, 27] with $\chi^{(5)}$ and $\chi^{(6)}$, because of their sizes.

¹² The associated Hodge mixed structure explains, to some extent, why the linear differential operators annihilating diagonals of rational functions (like the $\chi^{(n)}$'s) have a large number of factors.

¹³ This conjecture will be ruled out below in section 7.

¹⁴ This operator is obtained from a creative telescopic code (we thank A Bostan for this calculation).

$$R(x, y, z) = \frac{1}{1 - 3x - 5y - 7z + xy + 2yz^2 + 3x^2z^2}, \quad (4)$$

whose series expansion reads¹⁵:

$$\text{Diag}(R(x, y, z)) = 1 + 616x + 947175x^2 + 1812651820x^3 + \dots \quad (5)$$

This expansion of the rational function (4) can also be obtained from an expansion using *multinomial coefficients* (see appendix A).

All these results are a strong incentive to accumulate other examples of minimal order operators annihilating *diagonals of rational functions*, and analyze all their irreducible factors to confirm, or discard, the previous conjecture that these factors are necessarily homomorphic to their adjoints, and see whether this homomorphism to the adjoint property is always associated to decompositions like (1) or (2).

Let us try to address this conjecture revisiting the order-six operator \mathcal{L}_6 in [12, 13], in order to see if \mathcal{L}_6 is also of the form (1) or (2).

2. Revisiting the order-six operator \mathcal{L}_6

A first sketchy analysis of this operator \mathcal{L}_6 was performed in [12, 13], which we recall now. We saw, for instance, that this operator is *not* MUM¹⁶: it has four solution-series analytic at the origin $x = 0$, one, among them, being not globally bounded [1], and two being log-dependent formal series solutions.

Even though the order-six operator \mathcal{L}_6 , which annihilates the diagonal of the rational function (4), was quite large, we were able to check that its *exterior square* is of generic order 15. Switching to the associated differential *theta-system*, we have been able to see that \mathcal{L}_6 (seen as a differential system) is actually *homomorphic to its adjoint*. Furthermore, one actually finds that the *exterior square* of the *associated differential system* has a *rational solution* (but not its symmetric square). The differential Galois group thus corresponds to a *symplectic structure*.

Since this order-six operator \mathcal{L}_6 has this symplectic structure, one can expect that its order-15 *exterior square* has a *rational solution*. Actually, after some formal calculations work, we have first been able to find this rational solution $R(x)$ which can be written as $R(x) = p_{10}/p_{12}/x$, where p_{10} and p_{12} are two polynomials¹⁷ of degree ten and twelve, with integer coefficients given in appendix A.3.

We can also consider the order-six linear differential operator \mathcal{L}_6 in [12, 13], seen as a linear differential operator with *polynomial coefficients*. The head polynomial h_6 of the order-six operator \mathcal{L}_6 , such that $\mathcal{L}_6 = h_6 \cdot D_x^6 + \dots$, reads $h_6 = x^2 \cdot p_{12} \cdot p_{43}$, where p_{12} is the previous degree-twelve polynomial, and where p_{43} is a polynomial with integer coefficients of degree 43 in x , given in A.3. Roots of polynomial p_{43} correspond to *apparent singularities* of the order-six operator \mathcal{L}_6 , whereas the roots of the degree twelve polynomial p_{12} correspond to *true singularities* of the order-six operator \mathcal{L}_6 .

¹⁵ Use the maple command `mtaylor(F, [x,y,z], terms)`, to get the series in three variables, then take the diagonal. Another method, in mathematica is to install the risc package `RISCErgoSum` (`RISCErgoSum` is a collection of packages created at the Research Institute for Symbolic Computation (RISC), Linz, Austria <http://risc.jku.at/research/combinat/software/ergosum/>), and in `HolonomicFunctions` use the command `FindCreativeTelescoping`.

¹⁶ MUM means maximally unipotent monodromy [6, 17, 33].

¹⁷ Everywhere in this paper p_n will denote a polynomial of degree n in x , with integer coefficients.

2.1. Homomorphisms of \mathcal{L}_6 with its adjoint

Let us now focus on the fundamental relation, underlined in [12, 13], between a linear differential operator and its adjoint, seeking for a homomorphism between \mathcal{L}_6 and its adjoint, and the associated intertwiners. In a second step we will also consider the homomorphism of the previous intertwiners with their adjoints, and so on. We will see that finding this ‘tower’ of intertwiners eventually yields a simple decomposition of the order-six operator \mathcal{L}_6 .

2.1.1. Homomorphism of \mathcal{L}_6 with its adjoint: the \mathcal{L}_4 intertwiner. After some large formal calculations, performed using the DEtools Maple command ‘Homomorphisms(\mathcal{L}_6 , adjoint(\mathcal{L}_6))’, we obtained an intertwiner, which we will denote \mathcal{L}_4 , such that

$$\text{adjoint}(\mathcal{L}_4) \cdot \mathcal{L}_6 = \text{adjoint}(\mathcal{L}_6) \cdot \mathcal{L}_4. \quad (6)$$

The intertwiner \mathcal{L}_4 is a quite large *order-four* linear differential operator. The coefficients of D_x^n , appearing in the operator $p_{43}^2 \cdot \mathcal{L}_4$, are (quite large¹⁸) polynomials with integer coefficients. This intertwiner \mathcal{L}_4 is *not conjugated to its adjoint*, which *excludes decompositions of \mathcal{L}_6 of the form (1) or (2)*.

Remarkably the order-four intertwiner \mathcal{L}_4 is such that its *exterior square* has the *same rational function solution* $R(x) = p_{10}/p_{12}/x$ as \mathcal{L}_6 . We explain this result later on in the paper (see remark 1 in section 3.2). Since \mathcal{L}_4 has this symplectic structure, it is natural to seek for a decomposition of \mathcal{L}_4 of the form (1), or (2), by looking at the homomorphisms of \mathcal{L}_4 with its adjoint¹⁹. Performing these calculations, we, indeed, obtained a decomposition of this form for \mathcal{L}_4 , namely

$$\mathcal{L}_4 = (N \cdot P + 1) \cdot r(x), \quad (7)$$

where N and P are two order-two *self-adjoint operators*, and where $r(x)$ is a rational function. The operators N and P , and the rational function $r(x)$, are given in appendix A.1.

2.1.2. Decomposition of \mathcal{L}_6 . Let us now perform the *euclidean right division* of \mathcal{L}_6 by \mathcal{L}_4 :

$$\mathcal{L}_6 = M \cdot \mathcal{L}_4 + \mathcal{L}_2. \quad (8)$$

The two operators M and \mathcal{L}_2 are two order-two operators. One remarks, from direct calculations, that the order-two operator M is *exactly self-adjoint*. The exact expression of the order-two operator M is given in appendix A.1 (see equation (A.10)).

One also remarks that the order-two operator \mathcal{L}_2 is *exactly equal to the product*

$$\mathcal{L}_2 = P \cdot r(x), \quad (9)$$

where P is the self-adjoint operator introduced in (7). Using (6) and (7), and the fact that M is self-adjoint, we note that \mathcal{L}_2 can be seen as an intertwiner of the homomorphism of \mathcal{L}_4 with its adjoint:

$$\text{adjoint}(\mathcal{L}_2) \cdot \mathcal{L}_4 = \text{adjoint}(\mathcal{L}_4) \cdot \mathcal{L}_2. \quad (10)$$

The fact that P is self-adjoint, and that $\mathcal{L}_2 = P \cdot r(x)$, corresponds to a last intertwining relation of \mathcal{L}_2 with its adjoint:

¹⁸ The polynomial coefficient of D_x^n is of degree $38 + n$, the head polynomial being, up to an integer factor, the product of x^2 and a polynomial p_{28} of degree 28.

¹⁹ Namely performing the Maple DEtools command ‘Homomorphisms (\mathcal{L}_4 , adjoint(\mathcal{L}_4))’ and then ‘Homomorphisms(adjoint(\mathcal{L}_4), \mathcal{L}_4)’.

$$\text{adjoint}(\mathcal{L}_0) \cdot \mathcal{L}_2 = \text{adjoint}(\mathcal{L}_2) \cdot \mathcal{L}_0, \quad \mathcal{L}_0 = r(x). \tag{11}$$

Decomposition of \mathcal{L}_6 : From (7) and (8) one immediately deduces a *very simple decomposition* for \mathcal{L}_6 , generalizing the decompositions (1) or (2) of [12, 13]:

$$\begin{aligned} \mathcal{L}_6 &= M \cdot (N \cdot P + 1) \cdot r(x) + P \cdot r(x) \\ &= (M \cdot N \cdot P + M + P) \cdot r(x). \end{aligned} \tag{12}$$

The other intertwining relation between \mathcal{L}_6 and its adjoint reads

$$\mathcal{L}_6 \cdot \mathcal{M}_4 = \text{adjoint}(\mathcal{M}_4) \cdot \text{adjoint}(\mathcal{L}_6), \tag{13}$$

where the *order-four* intertwiner \mathcal{M}_4 can be simply expressed in terms of the two previous self-adjoint order-two operators M and N :

$$\mathcal{M}_4 = \frac{1}{r(x)} \cdot (N \cdot M + 1). \tag{14}$$

2.2. Similar decompositions

This order-six operator, \mathcal{L}_6 , associated with the diagonal of a rational function [3, 4], shows that there exist operators, with selected differential Galois groups, with decompositions that *do not reduce* to the decompositions of [12, 13], namely (1) or (2). Let us now show two other examples also generalizing decompositions (1) and (2).

2.2.1. A simple order-three operator. In fact, a much simpler example, corresponding to decomposition (12), can easily be found. Let us consider an order-two operator ($W(x)$ denotes its Wronskian)

$$\mathcal{L}_2 = a_2(x) \cdot \left(D_x^2 - \frac{1}{W(x)} \cdot \frac{dW(x)}{dx} \cdot D_x \right) + a_0(x), \tag{15}$$

and let us consider an order-three linear differential operator $\tilde{\mathcal{L}}_3$, equivalent²⁰ to the symmetric square of operator \mathcal{L}_2 given by²¹ (15):

$$I_1 \cdot \text{Sym}^2(\mathcal{L}_2) = \tilde{\mathcal{L}}_3 \cdot D_x \tag{16}$$

where I_1 denotes an order-one intertwiner. It is clear that this order-three operator $\tilde{\mathcal{L}}_3$ has, by construction, a selected differential Galois group, since it must reduce to the differential Galois group of the ‘underlying’ order-two operator \mathcal{L}_2 , namely $SL(2, \mathbb{C})$, which is known to be, up to a 2-to-1 homomorphism, isomorphic to the *orthogonal group* $SO(3, \mathbb{C})$. One easily finds that the symmetric square of this order-three operator $\tilde{\mathcal{L}}_3$ has a rational solution, which is nothing but $W(x)^2$, the square of the Wronskian of $\tilde{\mathcal{L}}_2$. Let us introduce the order-two intertwiner $\tilde{\mathcal{L}}_2$ corresponding to the homomorphism of $\tilde{\mathcal{L}}_3$ with its adjoint:

$$\text{adjoint}(\tilde{\mathcal{L}}_2) \cdot \tilde{\mathcal{L}}_3 = \text{adjoint}(\tilde{\mathcal{L}}_3) \cdot \tilde{\mathcal{L}}_2. \tag{17}$$

Let us perform the *euclidean right division* of $\tilde{\mathcal{L}}_3$ by $\tilde{\mathcal{L}}_2$:

$$\tilde{\mathcal{L}}_3 = M \cdot \tilde{\mathcal{L}}_2 + \tilde{\mathcal{L}}_1. \tag{18}$$

The order-one operator M is found to be *self-adjoint*. Let us perform, again, the euclidean right division of $\tilde{\mathcal{L}}_2$ by $\tilde{\mathcal{L}}_1$ (namely the rest of the previous euclidean right division (18)):

²⁰ In the sense of the equivalence of linear differential operators [11].

²¹ Just perform the right division by D_x of the LCLM of $\text{Sym}^2(\mathcal{L}_2)$ and D_x .

$$\tilde{\mathcal{L}}_2 = N \cdot \tilde{\mathcal{L}}_1 + \tilde{\mathcal{L}}_0. \tag{19}$$

The order-one operator N is found to be *self-adjoint*. One also finds that $\tilde{\mathcal{L}}_1 = P \cdot \tilde{\mathcal{L}}_0$, where $\tilde{\mathcal{L}}_0$ is a function $r(x)$, and where P is found to be *self-adjoint*. One thus deduces a decomposition of $\tilde{\mathcal{L}}_3$ also of the form (12)

$$\tilde{\mathcal{L}}_3 = (M \cdot N \cdot P + M + P) \cdot r(x), \tag{20}$$

but where the *self-adjoint* operators M , N and P are, this time, of *order one*.

2.2.2. Similar decompositions for simple order- n operators. In a similar way, one considers, for $n \geq 5$ odd ($n = 5, 7, \dots$) an order- n linear differential operator $\tilde{\mathcal{L}}_n$, equivalent to the symmetric $(n - 1)$ -th power of operator \mathcal{L}_2 , given by (15):

$$\mathcal{I}_1^{(n-1)} \cdot \text{Sym}^{n-1}(\mathcal{L}_2) = \tilde{\mathcal{L}}_n \cdot D_x, \tag{21}$$

where $\mathcal{I}_1^{(n-1)}$ denotes an order-one intertwiner. Again, one expects the differential Galois group of $\tilde{\mathcal{L}}_n$ to correspond to the differential Galois group of the underlying order-two operator \mathcal{L}_2 , namely $SL(2, \mathbb{C})$ or $PSL(2, \mathbb{C}) \simeq SO(3, \mathbb{C})$. Performing the same calculations as in the previous section 2.2.1, one thus deduces a decomposition of $\tilde{\mathcal{L}}_n$, also of the form (12)

$$\tilde{\mathcal{L}}_n = (M \cdot N \cdot P + M + P) \cdot r(x), \tag{22}$$

where the *self-adjoint* operators M , N are of *order one*, but where the *self-adjoint* operator P is of *odd order* $n - 2$. The symmetric square of $\tilde{\mathcal{L}}_n$ does not have a rational solution, but has a *drop of order*: its order is less than the order $n \cdot (n + 1)/2$ one expects generically for an order- n operator. In contrast the symmetric square of the *adjoint* of operator $\tilde{\mathcal{L}}_n$ has a rational solution which is the same as the *rational solution* of the symmetric square of operator M , namely the inverse of the head coefficient of the self-adjoint operator M .

Remark. For n even the order- n linear differential operator $\tilde{\mathcal{L}}_n$, equivalent to the symmetric $(n - 1)$ -th power of operator \mathcal{L}_2 (see (21)), gives decompositions of the form $\tilde{\mathcal{L}}_n = (M \cdot N + 1) \cdot r(x)$, corresponding to *symplectic Galois groups*, where M is of order two and N are of *even order* $n - 2$. This corresponds to the fact that the differential Galois group of the order-two operator \mathcal{L}_2 , namely $SL(2, \mathbb{C})$, is *also*²² a *symplectic group* $SL(2, \mathbb{C}) \simeq Sp(2, \mathbb{C})$. The *exterior square* of $\tilde{\mathcal{L}}_n$ has, for $n = 4$, a solution which is $W(x)^3$, but for n even, $n > 4$, this exterior square has no rational solution, it has a *drop of order*: its order is less than the order $n \cdot (n - 1)/2$ one expects generically for an order- n operator. In contrast the exterior square of the *adjoint* of operator $\tilde{\mathcal{L}}_n$ has a rational solution which is the same as the rational solution of the exterior square of operator M , namely the inverse of the head coefficient of the self-adjoint operator M . To be symplectic or orthogonal is a property of the representation. It is not an intrinsic property of the group.

2.3. Terminology: to be or not to be a selected differential Galois group

A simple generalization of section 2.2.1 amounts to introducing an order-three operator ($W(x)$ denotes its Wronskian)

²² $SL(2, \mathbb{C})$ is isomorphic to $Sp(2, \mathbb{C})$, to $Spin(3, \mathbb{C})$, and isomorphic, up to a 2-to-1 homomorphism, to $SO(3, \mathbb{C}) \simeq PSL(2, \mathbb{C})$.

$$\mathcal{L}_3 = a_3(x) \cdot \left(D_x^3 - \frac{1}{W(x)} \cdot \frac{dW(x)}{dx} \cdot D_x^2 \right) + a_1(x) \cdot D_x + a_0(x), \quad (23)$$

and considering, for instance, an order-six linear differential operator, equivalent to the symmetric square of operator \mathcal{L}_3 , given by (15):

$$\mathcal{I}_1 \cdot \text{Sym}^2(\mathcal{L}_3) = \tilde{\mathcal{L}}_6 \cdot D_x, \quad (24)$$

where \mathcal{I}_1 is an order-one intertwiner. This order-six operator $\tilde{\mathcal{L}}_6$ has, by construction, a ‘special’ differential Galois group, since it must reduce to the differential Galois group of the ‘underlying’ order-three operator \mathcal{L}_3 , namely $SL(3, C)$. However, the symmetric square, or exterior square, of this order-six operator *does not have a rational solution*, or even a *hyperexponential* [34] *solution*²³. This operator is *not homomorphic to its adjoint* (even in some algebraic extension).

We will *not* say that such an operator corresponds to ‘special geometry’ [8], even if it is clearly extremely ‘special’. By ‘special geometry’ we mean (*only*) that the operator is *homomorphic to its adjoint* [12, 13].

2.4. A first set of generalizations of this result

In this section we will consider *self-adjoint* linear differential operators, denoted M, N, P, Q, \dots not necessarily of the same order, but such that their orders have the *same parity* (all the operators are even order, or all the operators are odd order). Recalling the result that two operators A and B , such that their orders have the *same parity*, are such that²⁴ $\text{adjoint}(A + B) = \text{adjoint}(A) + \text{adjoint}(B)$, one immediately deduces relations like

$$\text{adjoint}(N \cdot P + 1) = P \cdot N + 1, \quad (25)$$

$$\text{adjoint}(M \cdot N \cdot P + M + P) = P \cdot N \cdot M + M + P, \quad (26)$$

enabling one to deduce a decomposition for the adjoint of operators like (7) or (12) without any new calculations.

The intertwining relations (6), (10), (11) form a ‘tower of intertwiners’. Once the decompositions of $\mathcal{L}_6, \mathcal{L}_4, \mathcal{L}_2$ in terms of self-adjoint linear differential operators, and of the function $r(x)$, are known (see (7), (9), (12)), the ‘Russian-doll’ structure of this ‘tower of intertwiners’ becomes obvious, corresponding, in fact, to *simple operator identities*. Actually the intertwining relation (6) is, because of (12), (26), (25), nothing but the identity:

$$\begin{aligned} (r(x) \cdot (P \cdot N + 1)) \cdot ((M \cdot N \cdot P + M + P) \cdot r(x)) \\ = (r(x) \cdot (P \cdot N \cdot M + M + P)) \cdot ((N \cdot P + 1) \cdot r(x)). \end{aligned} \quad (27)$$

Obviously, we also have the identity

$$\begin{aligned} ((M \cdot N \cdot P + M + P) \cdot r(x)) \cdot \frac{1}{r(x)} \cdot (N \cdot M + 1) \\ = \left((M \cdot N + 1) \cdot \frac{1}{r(x)} \right) \cdot (r(x) \cdot (P \cdot N \cdot M + M + P)), \end{aligned} \quad (28)$$

which actually corresponds to the other intertwining relation (13) between an operator, like \mathcal{L}_6 in (12), and its adjoint, the exact expression (14) of the intertwiner \mathcal{M}_4 being deduced, without any further calculations, from identity (28).

²³ This can be seen, more clearly, switching to the symmetric square of companion system of \mathcal{L}_3 .

²⁴ See also footnote 19 in [12].

Remark 1. As noticed²⁵ in a previous paper [12], the two intertwiners \mathcal{L}_4 and \mathcal{M}_4 are inverse operators modulo \mathcal{L}_6 . This, in fact, corresponds to the following identity:

$$\begin{aligned} & \left(\frac{1}{r(x)} \cdot (N \cdot M + 1) \right) \cdot ((N \cdot P + 1) \cdot r(x)) \\ &= 1 + \left(\frac{1}{r(x)} \cdot N \right) \cdot ((M \cdot N \cdot P + M + P) \cdot r(x)), \end{aligned} \quad (29)$$

which means that $\mathcal{M}_4 \cdot \mathcal{L}_4 = 1 \pmod{\mathcal{L}_6}$. Of course, we also have a ‘dual’ inverse identity for the adjoint of the operator (namely $\mathcal{L}_4 \cdot \mathcal{M}_4 = 1 \pmod{\text{adjoint}(\mathcal{L}_6)}$):

$$\begin{aligned} & ((N \cdot P + 1) \cdot r(x)) \cdot \left(\frac{1}{r(x)} \cdot (N \cdot M + 1) \right) \\ &= 1 + \left(N \cdot \frac{1}{r(x)} \right) \cdot (r(x) \cdot (P \cdot N \cdot M + M + P)). \end{aligned} \quad (30)$$

Remark 2. It is easy to generalize identities (27), (28) with more operators (we remove, here, the ‘dressing’ by the function $r(x)$):

$$\begin{aligned} & (Q \cdot P \cdot N + Q + N) \cdot (M \cdot N \cdot P \cdot Q + M \cdot Q + P \cdot Q + M \cdot N + 1) \\ &= (Q \cdot P \cdot N \cdot M + Q \cdot M + Q \cdot P + N \cdot M + 1) (N \cdot P \cdot Q + Q + N), \end{aligned} \quad (31)$$

and:

$$\begin{aligned} & (M \cdot N \cdot P \cdot Q + M \cdot Q + P \cdot Q + M \cdot N + 1) \cdot (P \cdot N \cdot M + M + P) \\ &= (M \cdot N \cdot P + M + P) (Q \cdot P \cdot N \cdot M + Q \cdot M + Q \cdot P + N \cdot M + 1). \end{aligned} \quad (32)$$

If one assumes that the four operators M, N, P and Q are *self-adjoint* operators of the *same parity order*, these identities can be interpreted as intertwining relations between an operator and its adjoint, the operator having the new decomposition:

$$L = (M \cdot N \cdot P \cdot Q + M \cdot Q + P \cdot Q + M \cdot N + 1) \cdot r(x). \quad (33)$$

Since these intertwining relations *do not require* that the *self-adjoint* operators are of the *same order*²⁶, we thus discover, with decompositions (12) or (33), *extremely large families* of linear differential operators for which we are sure that their *differential Galois groups will be special*. In the next section we generalize the decompositions (7), (12), (33), with, respectively, two, three, four *self-adjoint* operators, to an *arbitrary number* of *self-adjoint* operators.

Remark 3. The smaller factors, in the last two identities (31) and (32), can, thus, be seen as intertwiners. Again, one has two of these intertwiners which are *inverse operators modulo the operator*. This corresponds to the following identity generalizing (29):

$$\begin{aligned} & \left(\frac{1}{r(x)} \cdot (P \cdot N \cdot M + M + P) \right) \cdot ((N \cdot P \cdot Q + N + Q) \cdot r(x)) = -1 \\ &+ \left(\frac{1}{r(x)} \cdot (P \cdot N + 1) \right) \cdot ((M \cdot N \cdot P \cdot Q + M \cdot Q + P \cdot Q + M \cdot N + 1) \cdot r(x)), \end{aligned} \quad (34)$$

which actually amounts to saying that two intertwiners are *inverse operators modulo the operator* L given by (33). Of course, we also have the ‘dual’ inverse relation *modulo the*

²⁵ See the sentence after equation (8) in [12].

²⁶ But are of the same parity order.

adjoint of the operator L given by (33):

$$\begin{aligned} & ((N \cdot P \cdot Q + N + Q) \cdot r(x)) \cdot \left(\frac{1}{r(x)} \cdot (P \cdot N \cdot M + M + P) \right) \\ &= -1 + \left((N \cdot P + 1) \cdot \frac{1}{r(x)} \right) \\ & \quad \times (r(x) \cdot (Q \cdot P \cdot N \cdot M + Q \cdot M + Q \cdot P + N \cdot M + 1)). \end{aligned} \quad (35)$$

3. Tower of intertwiners and canonical decomposition of linear differential operators

With the previous identities (27), (28), one sees that the selected linear differential operator, and its successive intertwiners (between operators and their adjoints), have decompositions of a similar form. It is thus tempting to try to find, systematically, the decompositions of these selected operators from successive intertwiners of operators with their adjoints, ideally in an algorithmic recursion process.

In the next section (and in appendix B and appendix C) the operators will be denoted $L_{[N]}$, where N will not denote the order of the operators, as we always do [3, 4, 8, 12, 13], but an integer associated with the number of successive intertwiners. Similarly the integer n of the operators denoted U_n (or V_n in appendix B.2) does not correspond to the order of these operators.

3.1. The tower of intertwiners from a simple euclidean right division

Let us consider an order- q linear differential operator $L_{[N]}$, homomorphic to its adjoint. We have shown, in previous papers [12, 13], that this means that there exists an intertwiner, we will denote $L_{[N-1]}$, such that

$$\text{adjoint}(L_{[N]}) \cdot L_{[N-1]} = \text{adjoint}(L_{[N-1]}) \cdot L_{[N]}. \quad (36)$$

In Maple, `Homomorphisms(L[N], adjoint(L[N]))` is the command one should use to obtain this intertwiner $L_{[N-1]}$. From the previous intertwining relation (36), it is natural to compare the original operator $L_{[N]}$ and this new intertwiner $L_{[N-1]}$, performing a *euclidean right division*:

$$L_{[N]} = U_N \cdot L_{[N-1]} + L_{[N-2]}, \quad (37)$$

where U_N is the quotient of the euclidean right division, and $L_{[N-2]}$ is the remainder of the euclidean right division.

Reinjecting the euclidean right division decomposition (37) in the intertwining relation (36), one gets

$$\text{adjoint}(L_{[N]}) \cdot L_{[N-1]} = \text{adjoint}(L_{[N-1]}) \cdot (U_N \cdot L_{[N-1]} + L_{[N-2]}), \quad (38)$$

or, equivalently:

$$\left(\text{adjoint}(L_{[N]}) - \text{adjoint}(L_{[N-1]}) \cdot U_N \right) \cdot L_{[N-1]} = \text{adjoint}(L_{[N-1]}) \cdot L_{[N-2]}. \quad (39)$$

Since it was shown in [12, 13] that the intertwining relation between an *irreducible* operator $L_{[N-1]}$ and its adjoint is necessarily of the form (36), one deduces the equality

$$\text{adjoint}(L_{[N]}) - \text{adjoint}(L_{[N-1]}) \cdot U_N = \text{adjoint}(L_{[N-2]}), \quad (40)$$

the previous relation (39), rewriting as:

$$\text{adjoint}(L_{[N-2]}) \cdot L_{[N-1]} = \text{adjoint}(L_{[N-1]}) \cdot L_{[N-2]}, \quad (41)$$

which is a new intertwining relation exactly of the *same form* as the first intertwining relation (36).

By definition of the euclidean right division, the two terms $L_{[N]}$ and $U_N \cdot L_{[N-1]}$ in (37) are of the same order. Recalling, for two operators A and B of the same order (or even orders of the same parity), the result²⁷ that $\text{adjoint}(A - B) = \text{adjoint}(A) - \text{adjoint}(B)$, one gets from (37)

$$\begin{aligned} \text{adjoint}(L_{[N-2]}) &= \text{adjoint}(L_{[N]} - U_N \cdot L_{[N-1]}) \\ &= \text{adjoint}(L_{[N]}) - \text{adjoint}(U_N \cdot L_{[N-1]}) \\ &= \text{adjoint}(L_{[N]}) - \text{adjoint}(L_{[N-1]}) \cdot \text{adjoint}(U_N). \end{aligned} \quad (42)$$

Comparing (42) with (40), one deduces the result that the operator U_N , in the euclidean division (37), is, necessarily, *exactly self-adjoint*:

$$U_N = \text{adjoint}(U_N). \quad (43)$$

It is straightforward to see that one can go on recursively

$$\begin{aligned} L_{[N]} &= U_N \cdot L_{[N-1]} + L_{[N-2]}, & L_{[N-1]} &= U_{N-1} \cdot L_{[N-2]} + L_{[N-3]}, \\ L_{[N-2]} &= U_{N-2} \cdot L_{[N-3]} + L_{[N-4]}, & \dots & \\ L_{[N-p]} &= U_{N-p} \cdot L_{[N-p-1]} + L_{[N-p-2]}, & \dots & \end{aligned} \quad (44)$$

and

$$\begin{aligned} \text{adjoint}(L_{[N-1]}) \cdot L_{[N]} &= \text{adjoint}(L_{[N]}) \cdot L_{[N-1]}, \\ \text{adjoint}(L_{[N-2]}) \cdot L_{[N-1]} &= \text{adjoint}(L_{[N-1]}) \cdot L_{[N-2]}, \\ \text{adjoint}(L_{[N-3]}) \cdot L_{[N-2]} &= \text{adjoint}(L_{[N-2]}) \cdot L_{[N-3]}, \quad \dots \\ \text{adjoint}(L_{[N-p-1]}) \cdot L_{[N-p]} &= \text{adjoint}(L_{[N-p]}) \cdot L_{[N-p-1]}, \quad \dots \end{aligned} \quad (45)$$

thus building a ‘tower of intertwiners’. Let us see how this recursion stops. In the sequence of intertwining relation (45), the orders of the intertwiners decrease, and finally reach the moment where $L_{[N-p-1]}$ is just a function $r(x)$, which means that the operator $L_{[N-p]}$ is *simply conjugated* to its adjoint:

$$r(x) \cdot L_{[N-p]} = \text{adjoint}(L_{[N-p]}) \cdot r(x), \text{ or:} \quad (46)$$

$$\begin{aligned} U_{N-p} &= L_{[N-p]} \cdot \frac{1}{r(x)} = \frac{1}{r(x)} \cdot \text{adjoint}(L_{[N-p]}) \\ &= \text{adjoint}\left(L_{[N-p]} \cdot \frac{1}{r(x)}\right), \end{aligned} \quad (47)$$

which means that the operator U_{N-p} is *exactly self-adjoint*.

²⁷ Obvious from the definition of the adjoint of an operator, see [12, 13].

This is in agreement with the last euclidean division in (44), i.e. $L_{[1]} = U_1 \cdot L_{[0]} + L_{[-1]}$, with $L_{[0]} = r(x)$ and $L_{[-1]} = 0$, and where U_1 is exactly self-adjoint, namely:

$$L_{[1]} = U_1 \cdot r(x). \quad (48)$$

This operator U_1 appears, in the decomposition of $L_{[N]}$, as the *rightmost*²⁸ operator of this decomposition (see (50), ... (55), below). This rightmost self-adjoint operator U_1 plays a *selected role in the decomposition*. The rational solutions of the exterior, or symmetric, squares of the operators with special differential Galois groups depend only on $U_1 \cdot r(x)$, and *not* on the other U_n 's in the decomposition. It will be seen in section 3.3 that the order of U_1 is just constrained to have the *same parity* as the orders of the other U_n 's, the *even* order corresponding to *symplectic* differential Galois groups, and the *odd* order corresponding to *orthogonal* differential Galois groups.

3.2. Canonical decomposition

From these sequences of euclidean right-divisions on the successive intertwiners (44), together with the initial operators $L_{[0]} = r(x)$ and $L_{[-1]} = 0$, one immediately deduces canonical decompositions of the operator $L_{[N]}$. Let us show the first decompositions

$$L_{[0]} = r(x), \quad L_{[1]} = U_1 \cdot r(x), \quad (49)$$

$$L_{[2]} = U_2 \cdot L_{[1]} + L_{[0]} = (U_2 \cdot U_1 + 1) \cdot r(x), \quad (50)$$

$$L_{[3]} = U_3 \cdot L_{[2]} + L_{[1]} = (U_3 \cdot U_2 \cdot U_1 + U_1 + U_3) \cdot r(x), \quad (51)$$

$$\begin{aligned} L_{[4]} &= U_4 \cdot L_{[3]} + L_{[2]} = U_4 \cdot (U_3 \cdot L_{[2]} + L_{[1]}) + L_{[2]} \\ &= (U_4 \cdot U_3 \cdot U_2 \cdot U_1 + U_4 \cdot U_1 + U_2 \cdot U_1 + U_4 \cdot U_3 + 1) \cdot r(x), \end{aligned} \quad (52)$$

$$\begin{aligned} L_{[5]} &= U_5 \cdot L_{[4]} + L_{[3]} = U_5 \cdot (U_4 \cdot L_{[3]} + L_{[2]}) + L_{[3]} = \dots \\ &= (U_5 \cdot U_4 \cdot U_3 \cdot U_2 \cdot U_1 + U_5 \cdot U_4 \cdot U_1 \\ &\quad + U_5 \cdot U_2 \cdot U_1 + U_5 \cdot U_4 \cdot U_3 + U_3 \cdot U_2 \cdot U_1 + U_1 + U_3 + U_5) \cdot r(x), \end{aligned} \quad (53)$$

$$\begin{aligned} L_{[6]} &= U_6 \cdot L_{[5]} + L_{[4]} = U_6 \cdot (U_5 \cdot L_{[4]} + L_{[3]}) + L_{[4]} = \dots \\ &= (U_6 \cdot U_5 \cdot U_4 \cdot U_3 \cdot U_2 \cdot U_1 + U_6 \cdot U_5 \cdot U_4 \cdot U_1 + U_6 \cdot U_5 \cdot U_2 \cdot U_1 \\ &\quad + U_6 \cdot U_5 \cdot U_4 \cdot U_3 + U_6 \cdot U_3 \cdot U_2 \cdot U_1 + U_4 \cdot U_3 \cdot U_2 \cdot U_1 \\ &\quad + U_6 \cdot U_1 + U_6 \cdot U_3 + U_6 \cdot U_5 + U_4 \cdot U_1 + U_2 \cdot U_1 \\ &\quad + U_4 \cdot U_3 + 1) \cdot r(x), \end{aligned} \quad (54)$$

²⁸ Throughout the paper we call 'rightmost operator' in the decomposition of $L_{[N]}$, the rightmost operator appearing in the first and largest term of the decomposition: see the examples (50), ... (55) below.

$$\begin{aligned}
 L_{[7]} &= U_7 \cdot L_{[6]} + L_{[5]} = U_7 \cdot (U_6 \cdot L_{[5]} + L_{[4]}) + L_{[5]} = \dots \\
 &= (U_7 \cdot U_6 \cdot U_5 \cdot U_4 \cdot U_3 \cdot U_2 \cdot U_1 + U_7 \cdot U_6 \cdot U_5 \cdot U_4 \cdot U_1 \\
 &\quad + U_7 \cdot U_6 \cdot U_3 \cdot U_2 \cdot U_1 + U_7 \cdot U_4 \cdot U_3 \cdot U_2 \cdot U_1 + U_7 \cdot U_6 \cdot U_5 \cdot U_2 \cdot U_1 \\
 &\quad + U_7 \cdot U_6 \cdot U_5 \cdot U_4 \cdot U_3 + U_5 \cdot U_4 \cdot U_3 \cdot U_2 \cdot U_1 + U_7 \cdot U_6 \cdot U_1 \\
 &\quad + U_7 \cdot U_6 \cdot U_3 + U_7 \cdot U_6 \cdot U_5 + U_7 \cdot U_4 \cdot U_1 + U_7 \cdot U_2 \cdot U_1 \\
 &\quad + U_7 \cdot U_4 \cdot U_3 + U_5 \cdot U_4 \cdot U_1 + U_5 \cdot U_2 \cdot U_1 + U_5 \cdot U_4 \cdot U_3 \\
 &\quad + U_3 \cdot U_2 \cdot U_1 + U_1 + U_3 + U_5 + U_7) \cdot r(x), \tag{55}
 \end{aligned}$$

and so on. The number of terms in the $L_{[N]}$'s, namely 1, 2, 3, 5, 8, 13, 21, ... corresponds to the Fibonacci sequence, as a simple consequence of recursion (44).

Remark 1. In previous papers [12, 13] we reported on the equivalence of two properties, the homomorphism of an irreducible operator with its adjoint, and the occurrence of a rational (possibly hyperexponential²⁹) solution for the exterior, or symmetric, square of that operator. If we assume that the exterior (respectively symmetric) square of the operator $L_{[N]}$ has a rational solution $r(x)$, one immediately deduces, from the intertwining relation (36), that $r(x)$ is *also a solution* of the exterior (respectively symmetric) square of $L_{[N-1]}$. Using the tower of intertwining relations (45), one deduces that all the exterior (respectively symmetric) squares of all the $L_{[N-p]}$ intertwiners *have the same rational solution* $r(x)$, especially the last one, namely $L_{[1]} = U_1 \cdot r(x)$. It will be seen, in forthcoming sections, that the existence of a rational solution requires the self-adjoint operator U_1 to be of *order-one* for symmetric squares, and *order-two* for exterior squares. If the (irreducible) operator U_1 is of higher order, one *does not* have a rational solution but a *drop of the order* of the symmetric, or exterior, square.

Remark 2. Generically the $L_{[N]}$'s, in the tower of intertwiners described in (44) and (45) in section 3.1, are irreducible. One should note that we *have not used this irreducibility assumption*³⁰ in section 3.1.

3.3. Parity constraint on the order of the U_n 's

Let us now prove that these U_n 's have orders of the *same parity*. From (45) one easily deduces

$$(\text{adjoint}(L_{[N]}) - \text{adjoint}(L_{[N-2]})) \cdot L_{[N-1]} = \text{adjoint}(L_{[N-1]}) \cdot (L_{[N]} - L_{[N-2]}), \tag{56}$$

or, recalling (44), namely $L_{[N]} = U_N \cdot L_{[N-1]} + L_{[N-2]}$:

$$(\text{adjoint}(L_{[N]}) - \text{adjoint}(L_{[N-2]})) \cdot L_{[N-1]} = \text{adjoint}(L_{[N-1]}) \cdot U_N \cdot L_{[N-1]}. \tag{57}$$

From (57) and (44) one sees that $\text{adjoint}(L_{[N]})$ can be written alternatively:

$$\begin{aligned}
 \text{adjoint}(L_{[N]}) &= \text{adjoint}(L_{[N-2]}) + \text{adjoint}(L_{[N-1]}) \cdot U_N \\
 &= \text{adjoint}(L_{[N-2]}) + \text{adjoint}(U_N \cdot L_{[N-1]}) \\
 &= \text{adjoint}(L_{[N-2]} + U_N \cdot L_{[N-1]}). \tag{58}
 \end{aligned}$$

²⁹ Hyperexponential solutions [34] are obtained with the command 'expols' in DEtools in Maple.

³⁰ The euclidean right division can, of course, still be performed with reducible operators.

The equality (of the form $\text{adjoint}(A + B) = \text{adjoint}(A) + \text{adjoint}(B)$) between the two last terms in (58) can be fulfilled *only if* the parity of the order of $L_{[N-2]}$ is equal to the parity of the order of $U_N \cdot L_{[N-1]}$ (the order of $U_N \cdot L_{[N-1]}$ is, of course, the same as the order of $L_{[N]}$, see (37)).

Using, in the euclidean right divisions (44), relation $L_{[N-1]} = U_{N-1} \cdot L_{[N-2]} + L_{[N-3]}$, one finds that the parity of the order of $L_{[N-1]}$ is equal to the parity of the order of $U_{N-1} \cdot L_{[N-2]}$. Since the parity of the order of $L_{[N-2]}$ is equal to the parity of the order of $U_N \cdot L_{[N-1]}$, one straightforwardly deduces that the *parity of the order of U_{N-1} and U_N are the same*.

As a result, one finds, by recursion, that *all the U_n 's have orders of the same parity*.

Remark. The U_n 's, in these decompositions, have no reason to have the same order, they just need to have orders of the *same parity*. However, when one considers the successive intertwining relations (45), one knows that, *generically*, the intertwiner between two operators of the same order q (like $L_{[M]}$ and $\text{adjoint}(L_{[M]})$) is of order $q - 1$. If one assumes this 'generic' situation for all the intertwiners $L_{[M]}$ in the 'tower of intertwiners' (44), one finds that *all the U_n 's in the decomposition are of order one*. We will see in appendix E that this corresponds to a differential Galois group $SO(q, \mathbb{C})$. The other case corresponds to the intertwiner between the even order- q operators $L_{[M]}$ and $\text{adjoint}(L_{[M]})$ being of order $q - 2$. Again if one assumes this 'maximal even order' situation for all the intertwiners $L_{[M]}$, one finds that *all the U_n 's are of order two*. We will see, in appendix E, that this corresponds to a differential Galois group $Sp(q, \mathbb{C})$.

Definition. We will call 'generic' the decompositions where *all the U_n 's are of order one* (orthogonal differential Galois groups), or the decompositions where *all the U_n 's are of order two* (symplectic differential Galois groups).

It turns out that the examples from physics are, most of the time, *not generic* in the above mathematical sense³¹ (see the Calabi–Yau three-folds examples in section 6.1 below, and the order-six or eight lattice Green examples in [12, 13]), but most operators equivalent with these have such a 'generic' decomposition.

3.4. Canonical decomposition for the adjoint operator

Since these structures, and decompositions, rely on the homomorphisms between an operator $L_{[N]}$ and its adjoint, one can also consider the obvious viewpoint which amounts to seeing $\text{adjoint}(L_{[N]})$, the adjoint of an operator $L_{[N]}$, exactly on the *same footing* as $L_{[N]}$.

Switching to the adjoint of the operator one can get the decomposition of this adjoint in two ways. These two decompositions are detailed in appendix B. One decomposition, described in appendix B.1, amounts to performing *euclidean left divisions* on the adjoints of the tower of intertwiners described in section 3.1. However, it is known that the euclidean left division of a differential operator is more involved than the euclidean right division. As a consequence this decomposition is less efficient than the euclidean right divisions described in (3.1).

The other decomposition, described in appendix B.2, amounts to performing the euclidean right division described in (3.1), but, this time, *on the adjoint of the operator*. This adjoint operator is an operator of the same order, we can call $M_{[N]}$, for which the same euclidean right division calculations of section 3.1 can be performed, the first step corresponding to finding the intertwiner $M_{[N-1]}$ in the intertwining relation:

³¹ Operator \mathcal{L}_6 , the first order-six example of the paper, which has a 'generic' decomposition, is a mathematical example, it does not emerge from physics.

$$\text{adjoint}(M_{[N]}) \cdot M_{[N-1]} = \text{adjoint}(M_{[N-1]}) \cdot M_{[N]}. \quad (59)$$

The command `Homomorphisms($M_{[N]}$, $\text{adjoint}(M_{[N]})$)` gives this first intertwiner $M_{[N-1]}$ in the ‘tower’ of intertwiners (see section 3.1). Since we have in mind that $M_{[N]}$ is $\text{adjoint}(L_{[N]})$, the intertwining relation (59) is in fact

$$L_{[N]} \cdot M_{[N-1]} = \text{adjoint}(M_{[N-1]}) \cdot \text{adjoint}(L_{[N]}), \quad (60)$$

which is different from the intertwining relation (36), the DEtools Maple command, giving this first intertwiner $M_{[N-1]}$, being, now, `Homomorphisms($\text{adjoint}(L_{[N]})$, $L_{[N]}$)`. Performing the same calculations as in section 3.1 (see (50), (51), (52), ...), we will have another decomposition for these adjoints, deduced from successive right divisions. The relation between this decomposition for $\text{adjoint}(L_{[N]})$ and the decomposition for $L_{[N]}$ described in 3.1 is detailed in appendix B.2.

The fact that the (euclidean right division) decomposition of $\text{adjoint}(L_{[N]})$ will be more, or less, efficient than the (euclidean right division) decomposition of $L_{[N]}$, described in section (3.1), will depend on the very nature of $L_{[N]}$.

3.5. Getting the decomposition for very large differential operators

Before performing simple right (or left, see appendix B.1) euclidean divisions, all these various decompositions require one to find a first intertwiner, for instance from the DEtools Maple command `Homomorphisms($L_{[N]}$, $\text{adjoint}(L_{[N]})$)` or from the command `Homomorphisms($\text{adjoint}(L_{[N]})$, $L_{[N]}$)`.

Despite the fact that $L_{[N]}$ and $\text{adjoint}(L_{[N]})$ should be on the same footing, we have seen, experimentally, in a vast majority of physical examples, that, curiously, the command `Homomorphisms($L_{[N]}$, $\text{adjoint}(L_{[N]})$)` requires *much more* computer time and resources to be performed than the ‘reverse’ command `Homomorphisms($\text{adjoint}(L_{[N]})$, $L_{[N]}$)`. This corresponds to the fact that $L_{[N]}$ and $\text{adjoint}(L_{[N]})$ are, most of the time, *not on the same footing for most of the $L_{[N]}$ ’s emerging in physics*. This is illustrated in section 6.1, where the rightmost operators (U_1 in (50), (51), (52), ..., (55)) are of order *four*, when the leftmost operators (U_N in the $L_{[N]}$ ’s of (50), (51), (52), ..., (55)) are of order *two*: consequently, for operators $L_{[N]}$ of order q , the intertwiner obtained from `Homomorphisms($\text{adjoint}(L_{[N]})$, $L_{[N]}$)` is much simpler than the one obtained from `Homomorphisms($L_{[N]}$, $\text{adjoint}(L_{[N]})$)`, since they are respectively of order $q - 4$ and $q - 2$.

In the simple algorithm described in section 3.1, performing *euclidean right divisions* of differential operators is *almost instantaneous*, even for very large operators, *once the first intertwiner* `Homomorphisms($L_{[N]}$, $\text{adjoint}(L_{[N]})$)` *has been obtained*. Unfortunately, in practice, for the very large differential operators emerging in physics, this first intertwiner `Homomorphisms($L_{[N]}$, $\text{adjoint}(L_{[N]})$)` corresponds to calculations that require³² much larger computer resources than the resources required for the other intertwiner, `Homomorphisms($\text{adjoint}(L_{[N]})$, $L_{[N]}$)`. Fortunately, the intertwiner `Homomorphisms($L_{[N]}$, $\text{adjoint}(L_{[N]})$)` can actually be simply obtained from the simpler intertwiner, `Homomorphisms($\text{adjoint}(L_{[N]})$, $L_{[N]}$)`. This is a consequence of the fact that they are *inverse of each other, modulo the operator $L_{[N]}$* .

A first set of such *inversion relations* between these two intertwiners was already noticed in section 2.4 (see (29), (30) (34), (35)). These inversion relations were also noticed in section 2.1 of [12].

³² From the DEtools Maple command, or through a different algorithm which amounts to switching to linear differential theta systems.

Appendix C shows explicitly how these two intertwiners of the operator with its adjoint are *actually inverse of each other, modulo the operator and the adjoint operator*. Recalling the intertwining relation (36) and the intertwining relation (59) (or (60)), one actually has the two inversion relations:

$$M_{[N-1]} \cdot L_{[N-1]} = \Omega_{ML} \cdot L_{[N]} + C_{ML}, \quad (61)$$

$$L_{[N-1]} \cdot M_{[N-1]} = \Omega_{LM} \cdot \text{adjoint}(L_{[N]}) + C_{LM}, \quad (62)$$

where the constants C_{ML} and C_{LM} are equal, and where Ω_{ML} and Ω_{LM} are two operators adjoint of each other: $\Omega_{LM} = \text{adjoint}(\Omega_{ML})$.

Since getting the inverse of a given operator, modulo a given operator, is just a *linear* problem, we will use these inversion relations, when studying quite massive linear differential operators, to get the (hard to get) intertwiner we need for the (euclidean right division) decomposition, namely Homomorphisms($L_{[N]}$, $\text{adjoint}(L_{[N]})$), from the (easier to get) intertwiner Homomorphisms($\text{adjoint}(L_{[N]})$, $L_{[N]}$).

For very large linear differential operators with selected differential Galois groups, using these inversion relations is, in practice, the simplest way to get the intertwiner corresponding to Homomorphisms($L_{[N]}$, $\text{adjoint}(L_{[N]})$). In a following section (6.1), these inversion relations will be systematically³³ used on quite ‘massive’ linear differential operators, recently obtained by Lairez [34, 35], annihilating periods arising from *mirror symmetries* associated with reflexive 4-polytopes defining various selected Calabi–Yau three-folds, in order to obtain Homomorphisms($L_{[N]}$, $\text{adjoint}(L_{[N]})$), the intertwiner required to get the (euclidean right division) decomposition of these operators.

4. Compatibility of the decompositions with the equivalence of operators and generic decompositions

Operators with the previously described decompositions have necessarily *selected differential Galois groups* [12, 13]. Two equivalent operators necessarily have the same differential Galois groups [11]. Let us see what happens to these decompositions when the operator is changed into an *equivalent one* [11]. We follow, here, an *experimental mathematics approach*. We built a large number of linear differential operators corresponding to the previous decompositions (since building self-adjoint linear differential operators of arbitrary order is quite easy), and performed, systematically, the algorithm described in section 3.1, to get the new decomposition.

4.1. Equivalence of operators for generic decompositions

In section 3.2 we have defined two kinds of ‘generic decompositions’, namely the decompositions where *all* the self-adjoint operators U_n ’s are of degree *one* (which will be seen to correspond to differential Galois groups in $SO(q, \mathbb{C})$), and the decompositions where *all* the self-adjoint operators U_M ’s are of degree *two* (which will be seen to correspond to differential Galois groups in $Sp(q, \mathbb{C})$).

We have obtained the following experimental result: the form of a ‘generic decomposition’ is (generically) stable by operator equivalence [11]. For instance, if one considers

³³ The intertwiner corresponding to Homomorphisms($L_{[N]}$, $\text{adjoint}(L_{[N]})$) can be obtained with the ‘standard’ Maple DEtools command, only for the simplest order-six operators considered below in section 6.1, but this requires a lot of time and computer memory.

an operator L_5 given by (53), where the five U_n 's are *all order-one* (respectively *all order-two*) self-adjoint operators, the equivalent operator \tilde{L}_5 defined by³⁴ (\mathcal{I}_n is an order- n intertwiner)

$$\mathcal{I}_n \cdot L_5 = \tilde{L}_5 \cdot D_x^n, \tag{63}$$

has (generically) also a decomposition of the *same form* as (53):

$$\begin{aligned} \tilde{L}_5 = & (\tilde{U}_5 \cdot \tilde{U}_4 \cdot \tilde{U}_3 \cdot \tilde{U}_2 \cdot \tilde{U}_1 + \tilde{U}_5 \cdot \tilde{U}_4 \cdot \tilde{U}_1 + \tilde{U}_5 \cdot \tilde{U}_2 \cdot \tilde{U}_1 \\ & + \tilde{U}_5 \cdot \tilde{U}_4 \cdot \tilde{U}_3 + \tilde{U}_3 \cdot \tilde{U}_2 \cdot \tilde{U}_1 + \tilde{U}_1 + \tilde{U}_3 + \tilde{U}_5) \cdot \rho(x), \end{aligned} \tag{64}$$

where all the \tilde{U}_n 's are *all order-one* (respectively *all order-two*) self-adjoint operators, and $\rho(x)$ is a function.

4.2. Equivalence of operators for non-generic decompositions

Let us now consider, also from an experimental mathematics viewpoint, a few ‘non-generic’ decompositions, where the order of *all* the U_M 's does not reduce to order one or order two.

Let us first consider an *order-five* linear differential operator \mathcal{L}_5 of the form (51)

$$\mathcal{L}_5 = M_3 \cdot N_1 \cdot P_1 + M_3 + P_1, \tag{65}$$

where N_1 and P_1 are self-adjoint operators of *order one*, but where the first self-adjoint operator M_3 is of *order three*. If one changes \mathcal{L}_5 into an equivalent operator $\tilde{\mathcal{L}}_5$ (\mathcal{I}_n is an order- n intertwiner)

$$\mathcal{I}_n \cdot \mathcal{L}_5 = \tilde{\mathcal{L}}_5 \cdot D_x^n, \quad n \geq 2, \tag{66}$$

one finds that the equivalent operator $\tilde{\mathcal{L}}_5$ has the *generic* decomposition (53), or (64) where the \tilde{U}_M are *all order-one* self-adjoint operators, and $\rho(x)$ is a function.

Let us now consider an *order-twelve* linear differential operator \mathcal{L}_{12} of the form (51)

$$\mathcal{L}_{12} = M_4 \cdot N_4 \cdot P_4 + M_4 + P_4, \tag{67}$$

where M_4 , N_4 and P_4 are self-adjoint operators of *order four*. If one changes \mathcal{L}_{12} into an equivalent operator $\tilde{\mathcal{L}}_{12}$ (\mathcal{I}_n is an order- n intertwiner)

$$\mathcal{I}_n \cdot \mathcal{L}_{12} = \tilde{\mathcal{L}}_{12} \cdot D_x^n, \quad n \geq 3, \tag{68}$$

one finds that the equivalent operator $\tilde{\mathcal{L}}_{12}$ has the *generic* decomposition (54), where the *six* U_M are *all order-two* self-adjoint operators, and $r(x)$ is a function.

Let us give another simple illustration of these results with the decomposition of an operator equivalent to an order-three self-adjoint operator. Let us consider an order-three self-adjoint operator

$$\mathcal{L}_3 = a_3(x) \cdot D_x^3 + \frac{3}{2} \cdot \frac{da_3(x)}{dx} \cdot D_x^2 + a_1(x) \cdot D_x + \frac{1}{2} \cdot \frac{da_1(x)}{dx} - \frac{1}{4} \cdot \frac{d^3a_3(x)}{dx^3}, \tag{69}$$

and its equivalent operator³⁵

$$\mathcal{I}_3 \cdot \mathcal{L}_3 = \tilde{\mathcal{L}}_3 \cdot D_x^3, \tag{70}$$

³⁴ Of course D_x^n can be replaced by more involved operators.

³⁵ The D_x^3 intertwiner in (70), the equivalent between \mathcal{L}_3 and $\tilde{\mathcal{L}}_3$ can be replaced by the order-two operator corresponding to $D_x^3 \bmod \mathcal{L}_3$.

where \mathcal{I}_3 is an order-three intertwiner. Introducing the (order-two) intertwiner $\tilde{\mathcal{L}}_2$, obtained from the Maple DEtools command Homomorphisms($\tilde{\mathcal{L}}_3$, adjoint($\tilde{\mathcal{L}}_3$)), such that

$$\text{adjoint}(\tilde{\mathcal{L}}_2) \cdot \tilde{\mathcal{L}}_3 = \text{adjoint}(\tilde{\mathcal{L}}_3) \cdot \tilde{\mathcal{L}}_2, \tag{71}$$

and performing the successive euclidean right-divisions:

$$\begin{aligned} \tilde{\mathcal{L}}_3 &= U_3 \cdot \tilde{\mathcal{L}}_2 + \tilde{\mathcal{L}}_1, & \tilde{\mathcal{L}}_2 &= U_2 \cdot \tilde{\mathcal{L}}_1 + \tilde{\mathcal{L}}_0, \\ \tilde{\mathcal{L}}_1 &= U_1 \cdot r(x), & \tilde{\mathcal{L}}_0 &= r(x), \end{aligned} \tag{72}$$

one finds that the U_n 's are actually self-adjoint *order-one* operators, $r(x)$ being a function. Thus, one gets that the order-three operator $\tilde{\mathcal{L}}_3$, equivalent to a self-adjoint order-three operator, has a decomposition (51):

$$\tilde{\mathcal{L}}_3 = (U_3 \cdot U_2 \cdot U_1 + U_1 + U_3) \cdot r(x), \tag{73}$$

where the U_n 's are *order-one* self-adjoint operators.

More generally, the simplest example of non-generic decomposition corresponds to (49), namely an operator $L_{[1]} = U_1 \cdot r(x)$, where the *unique* operator U_1 is a *self-adjoint* operator of order $q \geq 3$. This operator is such that its differential Galois group is $SO(q, \mathbb{C})$, for q *odd*, where one finds a drop of order of the *symmetric square* of U_1 , and $Sp(q, \mathbb{C})$, for q *even*, where one finds a drop of order of the *exterior square* of U_1 . Again, an equivalent operator (like (63), (66)), will, for n large enough³⁶, change $L_{[1]}$ into an equivalent operator with a *generic decomposition*, namely a decomposition where all the U_n 's are of order one, for q *odd*, and a decomposition where all the U_n 's are of *order two*, for q *even*.

More involved examples of equivalence of operators with *exceptional differential Galois groups*, already sketched in [12], are detailed in appendix D. Again, on these highly non-trivial examples, it is shown that the equivalents of these operators have *quite involved* (generic or non-generic) decompositions, like (55) (see also (D.7) below).

To sum-up: For *involved enough equivalence* of operators, the non-generic decomposition of an order- q operator turns into a *generic decomposition* for its equivalent operator. The order of *all* the self-adjoint operators U_n 's, in the decomposition of the equivalent operator, is *one* (when the self-adjoint operators in the non-generic decomposition of the initial operator are of *odd* orders, which will be seen, in appendix E.1, to correspond to differential Galois groups in $SO(q, \mathbb{C})$). The order of all the self-adjoint operators U_n 's, in the decomposition of the equivalent operator, is *two* when the self-adjoint operators in the non-generic decomposition of the initial operator are of *even* orders (corresponding to differential Galois groups in $Sp(q, \mathbb{C})$, see appendix E.2).

4.3. Towards the generic situation for selected differential Galois groups: reduced form for differential systems

In the previous sections we saw that the existence of a homomorphism between an operator and its adjoint (characteristic of selected differential Galois group [12, 13]) is the key ingredient to get the canonical decomposition of the operator. Another way to see that operators with selected differential Galois groups *necessarily have* the decomposition,

³⁶ For instance, for $n = 1$, and $q = 5$, an equivalent operator of U_1 has a (non-generic) decomposition $(M_1 N_1 P_3 + M_1 + P_3) \cdot r(x)$. However $n = 2$ and $q = 5$ yields a generic decomposition in five order-one self-adjoint operators (see (53)). For $n = 1$, and $q = 6$, an equivalent operator of U_1 has a (non-generic) decomposition $(M_2 N_4 + 1) \cdot r(x)$, when for $n = 2$, and $q = 6$, one has the generic decomposition $(M_2 N_2 P_2 + M_2 + P_2) \cdot r(x)$.

described in section 3.1, is sketched in appendix E, analyzing, separately, in appendix E.1 the operators with orthogonal differential Galois groups, and in appendix E.2 the operators with symplectic differential Galois groups. This approach amounts to introducing the concept of *reduced form* [37, 38] for *linear differential systems* associated with these selected differential operators. More specifically, appendix E shows that the *most general operators with selected differential Galois group correspond to the ‘generic decompositions’*.

To sum-up: Our different ‘experimental mathematics’ approaches show that *all the linear differential operators, with selected differential Galois groups, correspond to the decompositions described in section 3.1*. In other words these decompositions can be seen as an *algebraic description* of the operators with selected differential Galois groups.

5. Rational solutions of the exterior or symmetric squares of the operators versus decompositions

We have seen, in previous papers [12, 13], that the existence of rational solutions for the exterior, or symmetric, square of an operator (or drop of the order of these squares) was equivalent to the existence of a homomorphism between the operator and its adjoint. Since the existence of a homomorphism between the operator and its adjoint is the key ingredient to get the canonical decomposition of operators described in this paper, let us see what is the relation between these decompositions and the rational solutions of the exterior, or symmetric, square of the operator.

Following, again, our experimental mathematics approach, we have considered for all our examples of operators with their decompositions (see section 3.2, namely (50)–(54), ...), their symmetric and exterior squares, seeking for rational solutions of these squares. For simplicity, and without any loss of generality, we restrict the decompositions (50)–(54), ... to $r(x) = 1$ (finding the symmetric, or exterior, square of an operator $L \cdot r(x)$, from the symmetric, or exterior, square of an operator L , is straightforward).

For decompositions such that the self-adjoint operators U_n ’s are all of *odd order*, one gets, for the symmetric square of these order- q operators, either a rational solution of the symmetric square or a drop of the order of the symmetric square, the order being less than $q(q + 1)/2$ (the two situations corresponding to differential Galois groups in $SO(q, \mathbb{C})$). For decompositions such that the self-adjoint operators U_n ’s are all of *even order*, one gets, for the exterior square of these order- q operators, either a rational solution of the exterior square or a drop of the order of the exterior square, the order being less than $q(q - 1)/2$ (the two situations corresponding to differential Galois groups in $Sp(q, \mathbb{C})$). In both cases (odd or even order), one finds that the rational solution, or the drop of the order of the symmetric, or exterior, square of the operator, *depends only on* U_1 , the rightmost self-adjoint operator in the larger product in these decompositions. If the order of U_1 is higher than one or two, one has a drop of the order of the symmetric, or exterior, square of the operator. One has a rational solution for the symmetric, or exterior, square of the operator *only* when the self-adjoint operator is of *order one*

$$U_1 = a_1(x) \cdot D_x + \frac{1}{2} \cdot \frac{da_1(x)}{dx}, \quad (74)$$

and the rational solution of the *symmetric square* of the operator is $1/a_1(x)$, or when the self-adjoint operator is of *order two*

$$U_1 = a_2(x) \cdot D_x^2 + \frac{da_2(x)}{dx} \cdot D_x + a_0(x), \tag{75}$$

and the rational solution of the *exterior square* of the operator is $1/a_2(x)$.

5.1. Drop of the order of the exterior or symmetric squares of the operators

We have considered a large number of examples corresponding to the drop of the order of the squares of the operators, when the order q of the self-adjoint operator U_1 is higher than one or two:

$$\begin{aligned} U_1 = & a_q(x) \cdot D_x^q + \frac{q}{2} \cdot \frac{da_q(x)}{dx} \cdot D_x^{q-1} + a_{q-2}(x) \cdot D_x^{q-2} \\ & + \frac{q-2}{2} \cdot \left(\frac{da_{q-2}(x)}{dx} - \frac{q \cdot (q-1)}{12} \cdot \frac{d^3a_q(x)}{dx^3} \right) \cdot D_x^{q-3} + a_{q-4}(x) \cdot D_x^{q-4} \\ & + \frac{q-4}{2} \cdot \left(\frac{da_{q-4}(x)}{dx} - \frac{(q-2) \cdot (q-3)}{12} \cdot \frac{d^3a_{q-2}(x)}{dx^3} \right. \\ & \left. + \frac{q \cdot (q-1) \cdot (q-2) \cdot (q-3)}{120} \cdot \frac{d^5a_q(x)}{dx^5} \right) \cdot D_x^{q-5} + a_{q-6}(x) \cdot D_x^{q-6} + \dots \end{aligned} \tag{76}$$

We found the following result. If one changes the order- q operator (76) for an equivalent one (\mathcal{I}_n is an order- n intertwiner)

$$\mathcal{I}_n \cdot U_1 = \tilde{L}_q \cdot D_x^n, \tag{77}$$

one finds that the exterior, or symmetric, squares of the equivalent operator \tilde{L}_q have, again, a rational solution³⁷ for large enough n .

For instance, for odd orders m of the self-adjoint operator (76), one recovers a rational solution for the symmetric square of \tilde{L}_q , for $q = 3$ with $n = 1$, for $q = 5$ with $n = 2$, for $q = 7$ with $n = 3$, and more generally, with $n = (q - 1)/2$. The rational solution of the symmetric square of that equivalent operator \tilde{L}_m is found to be $1/a_q(x)$, the inverse of the head coefficient of operator (76).

For even orders q of the self-adjoint operator (76) one recovers a rational solution for the exterior square of \tilde{L}_q , for $q = 4$ with $n = 1$, for $q = 6$ with $n = 2$, for $q = 8$ with $n = 3$, and more generally, with $n = (q - 2)/2$. The rational solution of the exterior square of that equivalent operator \tilde{L}_q is also found to be $1/a_q(x)$, the inverse of the head coefficient of operator (76).

Remark. Of course, if one considers the *adjoint* of an operator having one of the decompositions described in section 3, $L_{[N]} = (U_N \cdots U_1 + \cdots) \cdot r(x)$, the rational solution of the symmetric, or exterior, squares of this adjoint depends only on U_N (instead of $U_1 \cdot r(x)$).

6. Decomposition of large operators with selected differential Galois groups

The euclidean right division of an operator with a particular intertwiner provides a simple well-defined algorithm to get, very quickly, a canonical decomposition of operators with selected differential Galois groups, as well as a ‘tower’ of intertwiners. However, most of the

³⁷ This is in agreement with the previous section: for large enough n , non-generic decompositions reduce to generic decompositions (which have rational solutions for their exterior or symmetric squares).

time in physics, or in the challenging problems of mathematical physics, the operators with selected differential Galois groups are quite ‘massive’ operators (several Mega-octets, ...) of quite large order (12, 21, ... see [8]), and one remarks, experimentally, that the intertwiner one needs to calculate in the first step of the algorithm, namely $\text{Homomorphisms}(\text{Oper}, \text{adjoint}(\text{Oper}))$, requires massive computer resources compared to $\text{Homomorphisms}(\text{adjoint}(\text{Oper}), \text{Oper})$. Furthermore the ‘Homomorphisms’ command, implemented in DEtools in Maple, is not efficient enough for such ‘massive’ operators.

Let us sketch, or fully perform, in the next section, the study of some quite ‘massive’ operators that are important in physics, or mathematical physics. This study is done using the ideas of section 3.5, together with a brand new algorithm that requires to work on the linear *theta-system* associated with the operators.

6.1. Operators annihilating periods arising from mirror symmetries

Lairez obtained³⁸ recently, in a systematic analysis, a set of 210 explicit linear differential operators annihilating periods arising from *mirror symmetries*³⁹ (associated with reflexive 4-polytopes defining 68 topologically different Calabi–Yau three-folds, see [35, 36, 39]). These periods are also *diagonals of rational functions* [1, 3, 4].

Among these 210 operators many correspond to the ‘standard’ Calabi–Yau ODEs that have already been analyzed in various papers [6]. They are *order-four* irreducible operators satisfying the ‘*Calabi–Yau condition*’⁴⁰ [6] corresponding to saying that the exterior square of these order-four operators is of *order five*. However, remarkably, the other operators are *higher order* operators of even orders $N = 6, 8, 10, 12, 14, 16, \dots, 24$.

The study of such ‘massive’ operators relies on the ideas of section 3.5. The intertwiner $\text{Homomorphisms}(\text{Oper}, \text{adjoint}(\text{Oper}))$, which one needs to calculate in the first step of the euclidean right division algorithm described in section 3.1, is, in fact, obtained from the (much easier to get) intertwiner $\text{Homomorphisms}(\text{adjoint}(\text{Oper}), \text{Oper})$, since these two intertwiners are *inverse of each other modulo the operator* one considers: getting one intertwiner from the other one is essentially a *linear* problem. The complexity of the problem is reduced to calculating the (much easier to get) intertwiner $\text{Homomorphisms}(\text{adjoint}(\text{Oper}), \text{Oper})$. However, for most of the ‘massive’ operators of this list [36] of 210 operators, even obtaining these (simpler to get) intertwiners remains beyond our computer resources, using the ‘Homomorphisms’ command of DEtools in Maple. In order to achieve this first step, we have developed a brand new algorithm that requires to work on the linear *theta-system* associated with the operators. The details of these calculations are slightly technical and will be explained elsewhere. Let us just give the result of these (still massive⁴¹) calculations.

Performing these calculations, we found⁴² the following decompositions for operators up to order sixteen:

³⁸ We thank Lairez for generously sending us these explicit examples of selected operators before public access on the web [36].

³⁹ Using a criterion of Namikawa, Batyrev and Kreuzer found [39] 30241 reflexive 4-polytopes such that the corresponding Calabi–Yau hypersurfaces are smoothable by a flat deformation. In particular, they found 210 reflexive 4-polytopes defining 68 topologically different Calabi–Yau three-folds with $h_{11} = 1$.

⁴⁰ They are, up to a conjugation by a function, irreducible *order-four self-adjoint* operators [12].

⁴¹ Switching from the operator approach to a linear theta-system approach yields drastic reduction of the computing time as well as the memory required to perform the calculations.

⁴² These calculations do not use the ‘Homomorphisms’ command available in Maple in DEtools. The calculations are performed on the associated differential theta-systems, the Homomorphism of operator corresponding to ‘gauge transformations’ on the system. This will be explained elsewhere.

$$\mathcal{L}_6 = (M_2 \cdot N_4 + 1) \cdot r(x), \quad (78)$$

$$\mathcal{L}_8 = (M_2 \cdot N_2 \cdot P_4 + M_2 + P_4) \cdot r(x), \quad (79)$$

$$\mathcal{L}_{10} = (M_2 \cdot N_2 \cdot P_2 \cdot Q_4 + M_2 \cdot Q_4 + P_2 \cdot Q_4 + M_2 \cdot N_2 + 1) \cdot r(x),$$

$$\begin{aligned} \mathcal{L}_{12} = & (M_2 \cdot N_2 \cdot P_2 \cdot Q_2 \cdot R_4 + M_2 \cdot N_2 \cdot R_4 + M_2 \cdot Q_2 \cdot R_4 \\ & + M_2 \cdot N_2 \cdot P_2 + P_2 \cdot Q_2 \cdot R_4 + M_2 + P_2 + R_4) \cdot r(x), \end{aligned} \quad (80)$$

$$\begin{aligned} \mathcal{L}_{14} = & (M_2 \cdot N_2 \cdot P_2 \cdot Q_2 \cdot R_2 \cdot S_4 + M_2 \cdot N_2 \cdot R_2 \cdot S_4 + M_2 \cdot N_2 \cdot P_2 \cdot S_4 \\ & + M_2 \cdot N_2 \cdot P_2 \cdot Q_2 + M_2 \cdot Q_2 \cdot R_2 \cdot S_4 + P_2 \cdot Q_2 \cdot R_2 \cdot S_4 \\ & + M_2 \cdot S_4 + M_2 \cdot Q_2 + M_2 \cdot N_2 + P_2 \cdot S_4 + R_2 \cdot S_4 \\ & + P_2 \cdot Q_2 + 1) \cdot r(x), \end{aligned} \quad (81)$$

$$\begin{aligned} \mathcal{L}_{16} = & (M_2 \cdot N_2 \cdot P_2 \cdot Q_2 \cdot R_2 \cdot S_2 \cdot T_4 + M_2 \cdot N_2 \cdot P_2 \cdot Q_2 \cdot T_4 \\ & + M_2 \cdot N_2 \cdot R_2 \cdot S_2 \cdot T_4 + M_2 \cdot Q_2 \cdot R_2 \cdot S_2 \cdot T_4 + M_2 \cdot N_2 \cdot P_2 \cdot S_2 \cdot T_4 \\ & + M_2 \cdot N_2 \cdot P_2 \cdot Q_2 \cdot R_2 + P_2 \cdot Q_2 \cdot R_2 \cdot S_2 \cdot T_4 + M_2 \cdot N_2 \cdot T_4 \\ & + M_2 \cdot N_2 \cdot R_2 + M_2 \cdot N_2 \cdot P_2 + M_2 \cdot Q_2 \cdot T_4 + M_2 \cdot S_2 \cdot T_4 \\ & + M_2 \cdot Q_2 \cdot R_2 + P_2 \cdot Q_2 \cdot T_4 + P_2 \cdot S_2 \cdot T_4 + P_2 \cdot Q_2 \cdot R_2 \\ & + R_2 \cdot S_2 \cdot T_4 + T_4 + R_2 + P_2 + M_2) \cdot r(x), \end{aligned} \quad (82)$$

where $r(x)$ is a rational function, and where the $M_n, N_n, P_n, Q_n, R_n, S_n$ and T_n operators are *self-adjoint* operators of order n . One notes that the ‘rightmost’ self-adjoint operator (in the first and largest term of the decomposition) is always of order *four*.

For instance, for the order-twelve operator \mathcal{L}_{12} , the intertwiners \mathcal{L}_{10} and $\tilde{\mathcal{L}}_8$

$$\mathcal{L}_{12} \cdot \tilde{\mathcal{L}}_8 = \text{adjoint}(\tilde{\mathcal{L}}_8) \cdot \text{adjoint}(\mathcal{L}_{12}), \quad (83)$$

$$\text{adjoint}(\mathcal{L}_{12}) \cdot \mathcal{L}_{10} = \text{adjoint}(\mathcal{L}_{10}) \cdot \mathcal{L}_{12}, \quad (84)$$

read respectively

$$\mathcal{L}_{10} = (N_2 \cdot P_2 \cdot Q_2 \cdot R_4 + N_2 \cdot R_4 + Q_2 \cdot R_4 + N_2 \cdot P_2 + 1) \cdot r(x), \quad (85)$$

and

$$\tilde{\mathcal{L}}_8 = \frac{1}{r(x)} \cdot (Q_2 \cdot P_2 \cdot N_2 \cdot M_2 + Q_2 \cdot M_2 + Q_2 \cdot P_2 + N_2 \cdot M_2 + 1). \quad (86)$$

In this order-twelve operator \mathcal{L}_{12} example, we first obtain (from a theta-system approach) the intertwiner $\tilde{\mathcal{L}}_8$. We then obtain the \mathcal{L}_{10} intertwiner using the fact that $\tilde{\mathcal{L}}_8$ and \mathcal{L}_{10} are inverse of each other modulo \mathcal{L}_{12} (see section 3.5).

Remark 1. The form (80) of the decomposition of, for instance, \mathcal{L}_{12} with the rightmost self-adjoint *order-four* operator R_4 yields that the intertwiner $\tilde{\mathcal{L}}_8$ is of *order eight*, when the other intertwiner \mathcal{L}_{10} is of *order ten*. This explains why, in all this set of 210 operators (and, apparently, many others in physics), the intertwiner corresponding to Homomorphisms(adjoint(Oper), Oper) is much easier to obtain than the intertwiner corresponding to Homomorphisms(Oper, adjoint(Oper)).

Remark 2. From the fact that the ‘rightmost’ self-adjoint operator U_1 in these decompositions (78), ..., (80), (81), (82) is actually an *order-four* operator, one immediately deduces (see section 5.1) that its exterior square has a drop of order (no rational solution), but that the

exterior square of its *adjoint* has a rational solution (corresponding to the exterior square of the left-most order-two operator M_2).

The results: Switching not only to differential systems, but differential *theta systems*, to obtain Homomorphisms(*adjoint*(Oper), Oper), and, then, using the inversion relation to obtain the intertwiner corresponding to Homomorphisms(Oper, *adjoint*(Oper)), were two crucial steps to get these results for such very large operators. Even so, the calculations still remain quite ‘massive’. For instance, among the last *order-fourteen* operators, for which we have been able to perform these calculations, for the order-fourteen (degree 185) operator, denoted v.23.592 in the list [36], the first theta system step required 381 CPU hours, when the inversion relation step, which is essentially a linear calculation, took only 3 CPU hours. In contrast the last euclidean right-division step takes less than two minutes.

Among the three *order sixteen* operators, for which we have been able to perform these calculations, one is denoted v.23.696 in the list [36], another one is denoted v.19.5882 in [36], and the last one is denoted v.21.120 in [36]. The first operator v.23.696 is the same as operator v22.1476 in the list [36], associated with topology 13 (see [35]). It is of degree 190. This operator annihilates a diagonal of a rational function having the series expansion:

$$1 + 18 \cdot t^2 + 138 \cdot t^3 + 2094 \cdot t^4 + 29\,520 \cdot t^5 + 465\,210 \cdot t^6 + 7\,569\,240 \cdot t^7 \\ + 128\,131\,710 \cdot t^8 + 2\,225\,959\,680 \cdot t^9 + 39\,546\,740\,268 \cdot t^{10} + \dots \quad (87)$$

For this order sixteen operator the first theta system step required 683 CPU hours (i.e. one month) and 39 Gbytes of memory, when the inversion relation step, which is essentially a linear calculation, took only 6 CPU hours. In contrast the last euclidean right-division step takes less than ten minutes.

The second order sixteen operator v.19.5882 in [36] is in fact the same as operator v.21.845, associated with topology 22 (see [35]). It is of degree 206. This operator annihilates a diagonal of a rational function having the series expansion:

$$1 + 12 \cdot t^2 + 96 \cdot t^3 + 1068 \cdot t^4 + 13\,500 \cdot t^5 + 176\,520 \cdot t^6 + 2\,441\,040 \cdot t^7 \\ + 34\,844\,460 \cdot t^8 + 511\,438\,200 \cdot t^9 + 7\,670\,635\,812 \cdot t^{10} + \dots \quad (88)$$

For this order sixteen operator the first theta system step required 1280 CPU hours (i.e. 53 days) and 25 Gbytes of memory, when the inversion relation step took only 10 CPU hours. Again, the last euclidean right-division step takes less than ten minutes.

The third order sixteen operator v.21.120 in [36] is in fact the same operator as v21.2347 and v22.1519, associated with topology 10 (see [35]). It is of degree 221. This operator annihilates a diagonal of a rational function having the series expansion:

$$1 + 16 \cdot t^2 + 114 \cdot t^3 + 1584 \cdot t^4 + 20\,940 \cdot t^5 + 303\,190 \cdot t^6 + 4\,580\,520 \cdot t^7 \\ + 71\,654\,800 \cdot t^8 + 1\,152\,128\,040 \cdot t^9 + 18\,933\,486\,516 \cdot t^{10} + \dots \quad (89)$$

For this order sixteen operator the first theta system step required 1389 CPU hours (i.e. 57.89 days), when the inversion relation step took only 9 CPU hours. The last euclidean right-division step takes less than ten minutes.

To have results for all the order sixteen operators, one still needs to perform these calculations on the order sixteen operator v22.316 in [36] which is in fact the same operator as operators v22.357 and v23.42 associated with topology 14. It is a degree 236. This operator annihilates a diagonal of a rational function having the series expansion:

$$1 + 16 \cdot t^2 + 132 \cdot t^3 + 1776 \cdot t^4 + 25\,080 \cdot t^5 + 374\,380 \cdot t^6 + 5896\,800 \cdot t^7 \\ + 95\,873\,680 \cdot t^8 + 1603\,427\,280 \cdot t^9 + 27\,402\,671\,016 \cdot t^{10} + \dots \quad (90)$$

To sum-up we found that all the thirty-two *order-six* operators were of the form (50), all the seven *order-eight* operators were of the form (51), all the sixteen *order-ten* operators were of the form (52), all the fifteen *order-twelve* operators were of the form (53), all the *order-fourteen* operators⁴³ were of the form (54), and, finally, that three (among four) *order-sixteen* operators, were of the form (55). For all these operators all the U_n 's are *order-two* self-adjoint operators, except the rightmost operator U_1 which is an *order-four* self-adjoint operator.

Conjecture: We conjecture that *all the other*⁴⁴ operators of higher orders (16, 18, 20, 22, 24) also have decompositions generalizing the form (54), namely the rightmost self-adjoint operator being of *order four*, all the other self-adjoint operators being of *order two*.

By-product: As a by-product these decompositions show that the differential Galois groups of all these operators of even order q are included in the symplectic groups $Sp(q, \mathbb{C})$. This is coherent with previous results of Bogner [21] on Calabi–Yau operators.

Remark 3. Considering the successive euclidean right divisions described in section 3.1, starting with the first euclidean right division of the operators by the intertwiner corresponding to Homomorphisms(Adjoint(Oper), Oper), yields a set of operators $L_{[n]}$ ($n = 1, \dots, N - 1$, see (44)), and a set of self-adjoint operators U_N . One finds that the Wronskians of all these operators are rational functions. Furthermore, the critical exponents of all the singularities of these operators (the $L_{[n]}$'s in the tower of intertwiners (44) and the U_n 's) are *rational numbers* which are half integers, and, in fact, *most of the time integers*. However, one actually finds that these underlying operators are *not globally nilpotent*⁴⁵. These results are in agreement with the results in [12, 13] where the decompositions of an order-six and an order-eight operator yield self-adjoint operators that are, also, *not globally nilpotent* (see sections 3.6 and 3.7 in [12]). Along a similar line, note that the series-solutions, analytic at $x = 0$, of these ‘underlying’ operators are, also, *not globally bounded* [1, 3, 4] in contrast with the 210 operators [36] (which correspond to diagonal of rational functions). This excludes the possibility that these underlying operators could annihilate diagonals of rational functions. One finds, however, that the rightmost self-adjoint order-four operator U_1 corresponds to a maximal unipotent monodromy (MUM) structure⁴⁶ (see section 6.2 in [6]).

In the decomposition of these operators [36], corresponding to Calabi–Yau manifolds, the rightmost self-adjoint operators U_1 satisfy a large number of the properties defining the ‘standard’ Calabi–Yau order-four ODEs [12, 20]: they are self-adjoint, they satisfy the ‘Calabi–Yau condition’, see section 4 in [12], they have rational Wronskians, the critical exponents of all their singularities are rational numbers, however they are *not globally nilpotent*, and the series-solutions, analytic at $x = 0$, of these operators are *not globally bounded*⁴⁷, and, probably, these series-solutions cannot be represented as n -fold integrals. Therefore, these order-four self-adjoint operators U_1 are *not ‘standard’ Calabi–Yau order-*

⁴³ With the notations of [36], the order-fourteen operators are v.26.354, v.23.469, v.23.473 and v.23.592, v.23.375 and v.23.585. The operators v.23.473 and v.23.592 are in fact the same operator. The operators v.23.375 and v.23.585 are also the same operator.

⁴⁴ There are eight (different) operators of order 18, two operators of order 20, three of order 22 and one operator of order 24.

⁴⁵ We thank A Bostan for providing, here, several p -curvature calculations, which confirmed our more limited p -curvature calculations.

⁴⁶ All the other $L_{[n]}$'s in the ‘tower of intertwiner’, of order higher than four, are *not* MUM.

⁴⁷ And this is also the case for the corresponding nome or Yukawa couplings [12, 13].

four ODEs [12, 20]: they correspond to some interesting generalization of Calabi–Yau order-four ODEs.

7. Speculations on diagonals of rational functions and selected differential Galois groups

All the small factors of the minimal order operators annihilating the $\chi^{(n)}$'s components of the magnetic susceptibility of the Ising model we have obtained and studied [8] correspond to *globally bounded* [1, 3, 4] series solutions, in fact hypergeometric series with *integer coefficients*, having an elliptic function, or *modular function*, interpretation. These solutions are *diagonals of rational functions*. It was shown in [3, 4] that the $\chi^{(n)}$'s are actually *diagonals of rational functions*, but we have seen that *all the factors* of the operators annihilating the $\chi^{(n)}$'s, themselves, seem to have solutions that are *diagonals of rational functions*. Is this property verified for any diagonal of a rational function, or is it a consequence of our 'physical framework'?

Furthermore, we have seen [3, 4] that all these factors correspond to *selected differential Galois groups, these operators being homomorphic to their adjoints*. This raises the question to see whether the factors of the (minimal order) operators, *annihilating diagonals of rational functions, are, quite systematically, homomorphic to their adjoint*, thus corresponding to *selected differential Galois groups*, or if this 'duality property' is, on the contrary, a *consequence of our 'physical framework'*. A first experimental examination of hundreds of (simple enough) diagonals of three variables seems to *systematically yield* such a *duality* (homomorphism of the factor to its adjoint), but one must be careful before generalizing too quickly these results. In fact, the (minimal order) operators, annihilating diagonals of a rational function, are *not systematically* homomorphic to their adjoint. Let us consider the series expansion of the simple hypergeometric function⁴⁸ ${}_3F_2([1/3, 1/3, 1/3], [1, 1], 3^6x)$. It is a series with *integer coefficients*. One verifies easily that this hypergeometric series is the *Hadamard cube* of the series (with integer coefficients) of the algebraic function $(1 - 9x)^{-1/3}$, and is, thus, the diagonal of a rational function [3, 4]. On the other side, one can show that the order-three operator annihilating this hypergeometric series has a differential Galois group which is $SL(3, \mathbb{C})$, and thus it *cannot*⁴⁹ *be homomorphic to its adjoint* (even in some involved algebraic extension).

Accumulating more examples of operators annihilating diagonals of rational functions, in order to find if they are homomorphic to their adjoints or not, should help to clarify the *relation between diagonals of rational functions and selected differential Galois groups* (and associated decompositions), in order to understand why the diagonals of rational functions *emerging in physics* seem to have, systematically, this 'duality property'.

8. Conclusion

Trying to understand why an order-six operator \mathcal{L}_6 was homomorphic to its adjoint, thus having a selected differential Galois group, we have discovered that this was, in fact, a consequence of a decomposition (12) of this order-six operator into three order-two *self-*

⁴⁸ We thank A Bostan for providing this simple hypergeometric example.

⁴⁹ Using the fact that the symmetric square of that irreducible order-three operator has no rational solution and has logarithms in its series-solutions [40], or using the algorithm in [41] and showing that there is no invariant of degree 2, 3, 4, 6, 8, 9, 12.

adjoint operators. This provides a first example of a *selected differential Galois group* that does not emerge from a decomposition like (1) or (2), but actually emerges from a more general *new type of decomposition* namely (12). A first set of *selected differential Galois groups* actually emerges from a decomposition (12), where *one just needs to impose that the orders of the three self-adjoint operators have the same parity* (not necessarily even). We, then, discovered in section 3, a recursion, *based on euclidean right-division of operators*, on a sequence of linear differential operators where the *intertwiner in the homomorphism with the adjoint of an operator $L_{[N]}$ is actually the next operator $L_{[N-1]}$ in the sequence*: we do have a ‘*tower of intertwiners*’. These canonical decompositions in an *arbitrary number* of self-adjoint operators provide an *infinite number* of linear differential operators which have, automatically, *selected differential Galois groups*. We defined the ‘generic’ decompositions as the ones where the self-adjoint operators U_n , in the decompositions, are *all of order one*, or *all of order two*. It was found⁵⁰ ‘experimentally’ that the *most general operators with selected differential Galois groups do correspond to what we have called the ‘generic decompositions’*. To some extent, this provides an *algebraic approach of differential Galois groups*: the existence of such simple algebraic decompositions of the operators is the ‘*deus ex machina*’ for selected differential Galois groups.

According to the *parity* order of the underlying self-adjoint operators required to build them, the *exterior, or symmetric*, squares of the selected differential operators, or equivalent operators, described in this paper, have *rational solutions*. We have seen that the ‘generic’ character of the decomposition is (generally) *preserved by the operator equivalence*. In contrast, operators equivalent to operators with ‘non-generic’ decomposition eventually have a ‘generic’ decomposition for involved enough operator equivalence. Non-genericity⁵¹ is, thus, not preserved by the operator equivalence. We have also found the remarkable result that the rational solutions (or drop of order) of the symmetric, or exterior, squares of the operators with these decompositions *depend only on $U_1 \cdot r(x)$* , where U_1 is the *rightmost self-adjoint operator*. Rational solutions always emerge for equivalent operators, for an involved enough equivalence.

Since we have a well-defined algorithm to get these canonical decompositions of operators with selected differential Galois groups, we used it to obtain the decompositions of various remarkable (and quite massive) operators. For instance, in a quite systematic analysis of a set, obtained recently by Lairez [35, 36], of 210 explicit linear differential operators associated with reflexive 4-polytopes defining 68 topologically different Calabi–Yau three-folds, we found, with quite large calculations, that all the order-six, order-eight, order-ten, order-twelve, order-fourteen operators, and a first *order-sixteen* operator, had the decompositions detailed in this paper, all the U_n ’s being *order-two* self-adjoint operators except the rightmost operator U_1 which is an *order-four* self-adjoint operator. We conjecture that all these operators of higher order (14, 16, 18, 20, 22, 24) also have this kind of particular decomposition (U_N of order-two and the rightmost operator U_1 of *order-four*).

All the structures, decompositions, discovered in this paper, can be seen as a simple *algebraic description* of the linear differential operators with selected differential Galois groups. We have seen that the various linear differential operators emerging in the Ising model, in a large number of integrable models of lattice statistical mechanics, or enumerative combinatorics (lattice Green functions [3, 4, 12, 13, 18, 44, 45]), correspond to selected linear

⁵⁰ Using Kolchin’s result [42, 43] that the most general operators with selected differential Galois groups can be reduced, up to operator equivalence, to operators associated with some simple ‘reduced form’ [35, 36] for the associated linear differential systems.

⁵¹ An apparently quite frequent situation in physics, or enumerative combinatorics [12, 13].

differential operators. They are globally nilpotent [2] operators, or operators associated with reflexive polytopes (and Calabi–Yau three-folds), all associated with *diagonals of rational functions* [3, 4]), which also correspond to selected differential Galois groups. This paper provides simple, and computationally efficient, *algebraic tools* to study, and describe, these selected operators and their selected differential Galois groups.

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Appendix A. Analysis of the order-six operator \mathcal{L}_6

This order-six linear differential operator \mathcal{L}_6 in [12, 13], analyzed in section 2 annihilates the diagonal of the (three variables) rational function

$$R(x, y, z) = \frac{1}{1 - P(x, y, z)} = \frac{1}{1 - 3x - 5y - 7z + xy + 2yz^2 + 3x^2z^2}. \quad (\text{A.1})$$

The expansion (5) of the rational function (A.1) can also be obtained from an expansion using *multinomial coefficients*:

$$\begin{aligned} R(x, y, z) &= \sum_{n=0}^{\infty} P(x, y, z)^n \\ &= \sum_{m_1 \cdots m_6} \binom{m_1 + \cdots + m_6}{m_1, m_2, \dots, m_6} \cdot A_{m_i}(x, y, z), \quad \text{where:} \\ A_{m_i}(x, y, z) &= (3x)^{m_1} (5y)^{m_2} (7z)^{m_3} (-xy)^{m_4} (-2yz^2)^{m_5} (-3x^2z^2)^{m_6}. \end{aligned}$$

Then the expansion (5) of the diagonal of this rational function (A.1) reads

$$\text{Diag}(R(x, y, z)) = \sum_{M=0}^{\infty} \sum_{N, m_1, m_5} A_{N, m_1, m_5} \cdot x^M, \quad (\text{A.2})$$

where:
$$A_{N, m_1, m_5} = \binom{N}{m_1, m_5, p_1, p_2, p_3, p_4} \cdot 2^{m_5} \cdot 3^{q_1} \cdot 5^{q_2} \cdot 7^{q_3} \cdot (-1)^{q_4},$$

with

$$\begin{aligned} p_1 &= 3m_1 - 2N - 4M - 5m_5, & p_2 &= 2N - 2m_1 + 2m_5 - 3M, \\ p_3 &= 2N - 3m_1 - 3M + 4m_5, & p_4 &= 2M - N + m_1 - 2m_5, \end{aligned} \quad (\text{A.3})$$

and:

$$\begin{aligned} q_1 &= 2M - N + 2m_1 - 2m_5, & q_2 &= 3m_1 - 5m_5 + 4M - 2N, \\ q_3 &= 2N - 2m_1 + 2m_5 - 3M, & q_4 &= N - 2m_1 - M + 3m_5, \end{aligned} \quad (\text{A.4})$$

where the summation in (A.2) is taken over all the integers m_1, m_5 and N , provided the p_i 's in (A.3) are not negative.

A.1. Decomposition of the order-four operator \mathcal{L}_4

The order-four linear differential operator \mathcal{L}_4 , of section 2, reads

$$\mathcal{L}_4 = (N \cdot P + 1) \cdot r(x), \quad (\text{A.5})$$

where N and P are two order-two *self-adjoint operators*, and where $r(x)$ is the rational function

$$r(x) = c_r \cdot x \cdot \left(\frac{P_{28}}{P_{43} \cdot P_{10}} \right)^2,$$

with: $c_r = -175\,746\,210\,353\,375\,850\,313\,251\,934\,961\,664\,000\,000$, (A.6)

where p_{43} is an *apparent polynomial* of degree 43 given below in (A.16), where p_{10} is given below in (A.12), and where p_{28} is a polynomial of degree 28 given in (A.13). The two self-adjoint operators N and P are quite large order-two operators. The order-two self-adjoint operator N reads:

$$\begin{aligned} \frac{1}{c_N} \cdot N &= \frac{x^2 \cdot P_{28}^3}{P_{10} \cdot P_{43}^4} \cdot D_x^2 \\ &+ \frac{x P_{28}^2 P_{81}}{P_{10}^2 P_{43}^5} \cdot D_x + \frac{x P_{28}^2 P_{123}}{P_{10}^2 P_{43}^6}, \end{aligned}$$

with: $c_N = -738\,664\,786\,498\,877\,076\,270\,961\,404\,149\,760\,000\,000$, (A.7)

where p_{81} is a polynomial of degree 81 in x , with integer coefficients, and where p_{123} is a polynomial of degree 123, also with integer coefficients.

The order-two self-adjoint operator P reads:

$$c_P \cdot P = \frac{P_{10}^3 \cdot P_{12} \cdot P_{43}^4}{x \cdot P_{28}^4} \cdot D_x^2 + \frac{P_{10}^2 \cdot P_{93} \cdot P_{43}^3}{x^2 \cdot P_{28}^5} \cdot D_x + \frac{P_{10}^2 \cdot P_{164} \cdot P_{43}^2}{x^3 \cdot P_{28}^6},$$

with: $c_P = 305\,114\,948\,530\,166\,406\,793\,840\,164\,864\,000\,000$, (A.8)

where p_{93} and p_{164} are polynomials with integer coefficients of degree 93 and 164 respectively. The order-two operator P can be redefined in a simpler way introducing another self-adjoint operator $r(x) \cdot P \cdot r(x)$:

$$\frac{1}{c} \cdot r(x) \cdot P \cdot r(x) = \frac{P_{12}}{P_{10}} \cdot x \cdot D_x^2 + \frac{P_{22}}{P_{10}^2} \cdot D_x + 2 \frac{P_{21}}{P_{10}^2},$$

with: $c = 101\,229\,817\,163\,544\,489\,780\,433\,114\,537\,918\,464\,000\,000$, (A.9)

where p_{22} and p_{21} are degree 22 and 21 polynomials with integer coefficients (see (A.14), (A.15) given below).

A.2. Euclidean right division of \mathcal{L}_6 by \mathcal{L}_4

Recalling the euclidean right division (8) of the order-six operator \mathcal{L}_6 by the order-four operator \mathcal{L}_4 , namely $\mathcal{L}_6 = M \cdot \mathcal{L}_4 + \mathcal{L}_2$, the order-two self-adjoint operator M reads:

$$c_M \cdot M = \frac{p_{43}^3}{p_{28}} \cdot D_x^2 + \frac{p_{70} \cdot p_{43}^2}{p_{28}^2} \cdot D_x + \frac{q_{70} \cdot p_{43}^2}{x \cdot p_{28}^2},$$

with: $c_M = 697\,405\,596\,640\,380\,358\,385\,920\,376\,832\,000\,000$, (A.10)

where p_{70} and q_{70} are two polynomials of degree 70 in x , with integer coefficients,

A.3. The polynomials occurring in the analysis of \mathcal{L}_6

The two polynomials p_{10} and p_{12} occurring in section 2, in the rational function of the exterior square of \mathcal{L}_6 , are two polynomials of degree ten and twelve with integer coefficients, which read respectively:

$$\begin{aligned} p_{10} = & 145\,212\,480\,x^{10} - 1851\,804\,864\,x^9 - 471\,355\,865\,712\,x^8 - 2127\,618\,407\,544\,x^7 \\ & - 2504\,188\,513\,576\,x^6 - 4687\,345\,201\,826\,x^5 - 9012\,222\,844\,732\,x^4 \\ & - 18\,285\,528\,253\,377\,x^3 - 22\,232\,025\,838\,680\,x^2 - 10\,741\,640\,355\,390\,x \\ & - 1453\,000\,612\,770, \end{aligned} \quad (\text{A.11})$$

and

$$\begin{aligned} p_{12} = & 474\,360\,768\,x^{12} - 21\,346\,234\,560\,x^{11} - 9830\,736\,566\,352\,x^{10} \\ & - 49\,730\,104\,754\,400\,x^9 + 332\,161\,583\,716\,500\,x^8 + 2281\,890\,913\,038\,548\,x^7 \\ & + 4259\,219\,378\,255\,537\,x^6 + 2726\,995\,508\,245\,092\,x^5 + 266\,148\,400\,806\,530\,x^4 \\ & + 1339\,537\,706\,508\,092\,x^3 + 2659\,961\,825\,724\,861\,x^2 + 1339\,804\,492\,447\,785\,x \\ & - 484\,333\,537\,590. \end{aligned} \quad (\text{A.12})$$

The degree-28 polynomial p_{28} , occurring in the rational function (A.6) emerging in the decomposition of the order-four intertwiner (7) and also in the decomposition of \mathcal{L}_6 (see (12)), reads:

$$\begin{aligned}
p_{28} = & 222\,000\,402\,253\,116\,211\,200\,x^{28} + 16\,671\,177\,334\,581\,067\,210\,752\,000\,x^{27} \\
& + 997\,686\,473\,094\,955\,341\,487\,964\,160\,x^{26} \\
& - 424\,558\,055\,468\,758\,430\,724\,178\,378\,752\,x^{25} \\
& - 5352\,263\,005\,372\,177\,429\,702\,409\,969\,664\,x^{24} \\
& + 28\,532\,549\,906\,147\,363\,004\,230\,907\,949\,056\,x^{23} \\
& - 817\,011\,419\,352\,436\,691\,241\,775\,865\,192\,448\,x^{22} \\
& - 16\,274\,396\,213\,266\,656\,792\,629\,292\,697\,691\,136\,x^{21} \\
& - 90\,232\,371\,132\,900\,789\,295\,067\,378\,226\,024\,960\,x^{20} \\
& - 500\,637\,681\,004\,648\,369\,297\,528\,874\,000\,878\,080\,x^{19} \\
& - 4993\,606\,665\,623\,240\,703\,449\,658\,338\,613\,645\,312\,x^{18} \\
& - 30\,245\,133\,133\,694\,295\,061\,972\,050\,348\,437\,464\,512\,x^{17} \\
& - 185\,743\,134\,083\,879\,510\,307\,905\,273\,106\,717\,264\,704\,x^{16} \\
& - 976\,930\,467\,381\,467\,189\,083\,133\,231\,381\,070\,485\,120\,x^{15} \\
& - 3367\,039\,413\,961\,560\,591\,653\,749\,004\,449\,857\,274\,584\,x^{14} \\
& - 7294\,585\,923\,067\,316\,376\,931\,832\,664\,546\,378\,582\,832\,x^{13} \\
& - 9938\,986\,858\,255\,476\,817\,819\,640\,602\,774\,955\,842\,078\,x^{12} \\
& - 8869\,602\,400\,042\,451\,664\,718\,223\,608\,577\,965\,685\,292\,x^{11} \\
& - 791\,573\,427\,792\,097\,707\,512\,450\,306\,736\,489\,196\,2671\,x^{10} \\
& - 13\,323\,343\,876\,410\,843\,499\,695\,772\,267\,774\,078\,620\,818\,x^9 \\
& - 23\,035\,143\,510\,457\,266\,975\,680\,110\,399\,567\,225\,817\,550\,x^8 \\
& - 27\,995\,703\,147\,600\,523\,231\,627\,072\,322\,667\,435\,627\,270\,x^7 \\
& - 23\,240\,989\,943\,800\,867\,264\,930\,403\,245\,305\,575\,019\,984\,x^6 \\
& - 12\,663\,707\,457\,789\,877\,548\,806\,711\,493\,284\,323\,118\,250\,x^5 \\
& - 4207\,201\,352\,793\,289\,661\,461\,971\,125\,476\,855\,152\,780\,x^4 \\
& - 802\,705\,633\,170\,266\,051\,006\,530\,606\,597\,781\,060\,580\,x^3 \\
& - 154\,800\,451\,744\,689\,329\,012\,558\,083\,417\,852\,734\,900\,x^2 \\
& - 66\,136\,165\,509\,758\,890\,646\,569\,575\,954\,920\,184\,000\,x \\
& - 17\,095\,610\,140\,484\,667\,152\,552\,076\,876\,139\,296\,000.
\end{aligned} \tag{A.13}$$

The polynomials p_{22} and p_{21} , occurring as coefficients of the order-two self-adjoint operator $r(x) \cdot P \cdot r(x)$ (see (A.9),) read:

$$\begin{aligned}
p_{22} = & 206\,649\,310\,607\,953\,920\,x^{22} - 9713\,173\,628\,131\,319\,808\,x^{21} \\
& - 2426\,922\,106\,370\,025\,707\,520\,x^{20} \\
& + 70\,600\,350\,852\,546\,385\,296\,384\,x^{19} \\
& + 14\,163\,950\,463\,400\,033\,391\,748\,096\,x^{18} \\
& + 130\,185\,601\,423\,920\,337\,305\,765\,888\,x^{17} \\
& + 286\,975\,902\,003\,933\,399\,495\,977\,088\,x^{16} \\
& - 623\,168\,847\,910\,615\,654\,838\,222\,592\,x^{15} \\
& - 3538\,729\,092\,035\,340\,386\,109\,171\,216\,x^{14} \\
& - 10\,952\,980\,116\,620\,522\,700\,303\,057\,376\,x^{13}
\end{aligned}$$

$$\begin{aligned}
& - 43\,198\,517\,288\,449\,764\,379\,831\,927\,088\,x^{12} \\
& - 145\,043\,720\,261\,824\,787\,740\,194\,197\,788\,x^{11} \\
& - 367\,790\,107\,127\,544\,232\,638\,909\,982\,720\,x^{10} \\
& - 659\,752\,751\,521\,470\,013\,936\,611\,019\,364\,x^9 \\
& - 760\,856\,686\,133\,164\,797\,192\,096\,477\,412\,x^8 \\
& - 514\,919\,966\,483\,376\,918\,743\,532\,932\,868\,x^7 \\
& - 189\,221\,289\,620\,448\,844\,441\,499\,010\,432\,x^6 \\
& - 70\,630\,559\,928\,767\,816\,259\,500\,053\,080\,x^5 \\
& - 79\,750\,467\,708\,757\,827\,084\,054\,819\,315\,x^4 \\
& - 64\,947\,815\,604\,716\,490\,637\,198\,723\,800\,x^3 \\
& - 25\,997\,244\,208\,301\,276\,956\,743\,517\,260\,x^2 \\
& - 3893\,473\,497\,037\,260\,884\,458\,428\,900\,x \\
& + 703\,736\,926\,903\,331\,829\,024\,300,
\end{aligned} \tag{A.14}$$

and

$$\begin{aligned}
p_{21} = & 33\,738\,662\,956\,400\,640\,x^{21} - 1065\,211\,258\,519\,296\,000\,x^{20} \\
& + 137\,751\,133\,024\,444\,133\,376\,x^{19} + 13\,971\,600\,975\,964\,488\,256\,512\,x^{18} \\
& + 2072\,616\,787\,347\,233\,631\,183\,360\,x^{17} + 19\,143\,502\,759\,621\,498\,372\,119\,936\,x^{16} \\
& + 111\,056\,760\,563\,865\,193\,378\,425\,600\,x^{15} + 609\,147\,860\,679\,604\,401\,066\,131\,808\,x^{14} \\
& + 2284\,440\,127\,728\,770\,464\,121\,151\,168\,x^{13} + 3816\,824\,280\,036\,865\,582\,719\,277\,232\,x^{12} \\
& - 916\,281\,024\,394\,298\,349\,389\,193\,376\,x^{11} - 17\,325\,258\,089\,488\,435\,492\,938\,995\,946\,x^{10} \\
& - 47\,923\,688\,835\,171\,521\,983\,291\,837\,594\,x^9 - 85\,220\,533\,413\,397\,361\,142\,814\,764\,127\,x^8 \\
& - 95\,547\,159\,126\,942\,042\,724\,808\,925\,384\,x^7 - 57\,905\,642\,741\,824\,134\,044\,096\,538\,894\,x^6 \\
& - 6830\,662\,192\,589\,519\,324\,624\,321\,892\,x^5 + 16\,304\,405\,412\,896\,867\,488\,602\,623\,040\,x^4 \\
& + 12\,975\,784\,425\,095\,916\,109\,623\,440\,580\,x^3 + 3377\,129\,346\,163\,911\,842\,811\,657\,630\,x^2 \\
& - 415\,617\,758\,773\,209\,736\,393\,116\,600\,x - 216\,750\,973\,486\,226\,203\,339\,484\,400.
\end{aligned} \tag{A.15}$$

The polynomials p_{43} , occurring in the head coefficient of \mathcal{L}_6 , reads:

$$\begin{aligned}
p_{43} = & 697\,115\,132\,002\,046\,172\,480\,720\,076\,800\,000\,x^{43} \\
& + 46\,966\,788\,589\,019\,958\,554\,494\,032\,194\,568\,192\,000\,x^{42} \\
& - 21\,099\,793\,949\,237\,885\,955\,113\,953\,175\,859\,206\,553\,600\,x^{41} \\
& - 12\,218\,070\,507\,034\,966\,285\,297\,837\,769\,765\,669\,044\,224\,000\,x^{40} \\
& - 14\,194\,373\,444\,333\,574\,120\,149\,253\,605\,673\,357\,064\,273\,920\,x^{39} \\
& + 95\,681\,633\,992\,983\,148\,850\,231\,449\,432\,522\,337\,810\,723\,635\,200\,x^{38} \\
& + 5520\,522\,767\,992\,453\,240\,508\,293\,629\,010\,426\,228\,437\,862\,318\,080\,x^{37} \\
& + 114\,296\,137\,092\,097\,802\,339\,644\,622\,301\,518\,644\,478\,404\,747\,591\,680\,x^{36} \\
& + 417\,910\,689\,150\,890\,521\,997\,273\,968\,958\,589\,501\,995\,213\,946\,224\,640\,x^{35} \\
& - 31\,508\,493\,936\,457\,860\,589\,092\,556\,909\,074\,176\,639\,521\,503\,684\,624\,384\,x^{34}
\end{aligned}$$

$$\begin{aligned}
& - 903\,936\,764\,481\,753\,141\,424\,148\,779\,858\,885\,101\,259\,602\,207\,109\,447\,680\,x^{33} \\
& - 14\,332\,265\,098\,518\,509\,709\,934\,412\,539\,054\,897\,619\,963\,416\,778\,500\,300\,800\,x^{32} \\
& - 149\,481\,107\,350\,715\,305\,817\,082\,579\,283\,208\,875\,899\,765\,288\,515\,401\,691\,136\,x^{31} \\
& - 1033\,538\,497\,267\,096\,011\,790\,975\,433\,433\,897\,490\,651\,064\,449\,893\,529\,853\,952\,x^{30} \\
& - 4821\,442\,010\,547\,424\,238\,037\,959\,567\,906\,394\,473\,464\,364\,860\,695\,632\,913\,408\,x^{29} \\
& - 19\,978\,165\,382\,552\,239\,749\,293\,130\,530\,244\,544\,553\,933\,980\,930\,279\,985\,998\,848\,x^{28} \\
& - 140\,867\,377\,223\,273\,230\,659\,981\,893\,431\,123\,146\,189\,730\,880\,776\,359\,488\,689\,664\,x^{27} \\
& - 1159\,867\,839\,188\,205\,256\,138\,512\,414\,882\,994\,988\,836\,222\,652\,247\,822\,650\,381\,184\,x^{26} \\
& - 6351\,896\,417\,634\,861\,870\,867\,164\,399\,785\,963\,779\,858\,852\,813\,259\,236\,265\,509\,632\,x^{25} \\
& - 19\,971\,763\,593\,409\,793\,575\,488\,427\,179\,831\,444\,424\,654\,145\,339\,690\,894\,395\,028\,704\,x^{24} \\
& - 29\,276\,418\,982\,909\,322\,276\,170\,630\,931\,225\,213\,840\,998\,278\,949\,306\,320\,170\,659\,232\,x^{23} \\
& + 9654\,630\,434\,461\,854\,336\,187\,543\,726\,596\,136\,785\,120\,347\,219\,785\,702\,648\,099\,920\,x^{22} \\
& + 85\,853\,277\,293\,687\,339\,742\,074\,678\,932\,453\,004\,464\,244\,583\,743\,701\,538\,472\,162\,264\,x^{21} \\
& - 160\,054\,706\,451\,279\,429\,494\,171\,232\,085\,742\,639\,976\,407\,036\,877\,062\,557\,712\,330\,264\,x^{20} \\
& - 2258\,895\,857\,748\,459\,378\,897\,561\,858\,150\,946\,902\,130\,393\,238\,862\,424\,109\,243\,558\,684\,x^{19} \\
& - 10\,587\,987\,305\,051\,901\,240\,209\,836\,269\,848\,805\,589\,728\,925\,726\,684\,541\,835\,278\,857\,721\,x^{18} \\
& - 31\,197\,739\,833\,767\,694\,054\,120\,129\,704\,437\,013\,088\,350\,311\,917\,897\,807\,218\,379\,021\,982\,x^{17} \\
& - 61\,702\,779\,900\,612\,867\,731\,214\,872\,614\,165\,179\,314\,709\,755\,885\,273\,509\,788\,298\,565\,787\,x^{16} \\
& - 84\,666\,583\,576\,312\,950\,514\,740\,894\,147\,953\,872\,788\,530\,635\,983\,760\,466\,811\,973\,453\,234\,x^{15} \\
& - 84\,581\,321\,256\,078\,449\,348\,114\,201\,458\,605\,757\,961\,462\,588\,640\,328\,430\,113\,579\,371\,469\,x^{14} \\
& - 66\,579\,501\,385\,417\,743\,645\,896\,831\,995\,666\,623\,321\,786\,452\,789\,989\,601\,988\,868\,930\,220\,x^{13} \\
& - 46\,422\,144\,734\,306\,012\,640\,614\,115\,287\,296\,842\,756\,324\,723\,788\,172\,904\,199\,227\,424\,928\,x^{12} \\
& - 34\,109\,048\,367\,984\,268\,267\,671\,082\,613\,328\,375\,922\,384\,379\,783\,373\,944\,543\,955\,842\,292\,x^{11} \\
& - 31\,579\,571\,071\,937\,009\,738\,017\,989\,004\,416\,659\,378\,464\,666\,295\,456\,745\,499\,499\,451\,123\,x^{10} \\
& - 31\,383\,477\,409\,339\,726\,943\,053\,162\,665\,868\,124\,737\,503\,098\,706\,615\,744\,587\,744\,129\,055\,x^9 \\
& - 25\,192\,805\,778\,861\,042\,021\,357\,196\,910\,841\,906\,492\,061\,035\,149\,750\,125\,700\,349\,112\,500\,x^8 \\
& - 15\,484\,947\,468\,128\,477\,413\,170\,938\,540\,582\,845\,959\,272\,279\,364\,669\,181\,133\,069\,873\,400\,x^7 \\
& - 8825\,647\,465\,199\,759\,499\,489\,079\,986\,639\,230\,436\,941\,230\,964\,864\,953\,414\,638\,740\,550\,x^6 \\
& - 5592\,482\,889\,305\,405\,022\,888\,968\,560\,846\,164\,620\,653\,256\,382\,698\,605\,479\,016\,168\,500\,x^5 \\
& - 3175\,821\,728\,174\,664\,086\,144\,593\,028\,022\,649\,626\,932\,243\,816\,098\,004\,497\,271\,215\,500\,x^4 \\
& - 1197\,312\,010\,623\,323\,136\,288\,515\,293\,659\,836\,786\,239\,749\,091\,267\,282\,912\,166\,901\,000\,x^3 \\
& - 261\,689\,835\,566\,995\,733\,872\,747\,571\,036\,555\,312\,536\,936\,557\,291\,652\,102\,921\,551\,000\,x^2 \\
& - 27\,051\,165\,864\,672\,155\,225\,865\,153\,613\,729\,609\,826\,904\,997\,195\,126\,025\,314\,800\,000\,x \\
& - 3137\,634\,679\,643\,321\,707\,303\,313\,577\,336\,229\,275\,827\,023\,029\,827\,551\,200\,000. \tag{A.16}
\end{aligned}$$

Appendix B. Canonical decomposition for the adjoint

Switching to the adjoint of the operator one can get the decomposition of this adjoint in two ways. One amounts to performing *euclidean left division* on the adjoints of the tower of intertwiners described in (3.1). The other one corresponds to performing the euclidean right division described in (3.1) but, this time, on the adjoint of the operator.

B.1. Canonical decomposition for the adjoint: euclidean left division

If one takes the adjoint of the relations (44) of section 3.1, one has (since the U_n 's are self-adjoint):

$$\begin{aligned}
 \text{adjoint}(L_{[N]}) &= \text{adjoint}(L_{[N-1]}) \cdot U_N + \text{adjoint}(L_{[N-2]}), \\
 \text{adjoint}(L_{[N-1]}) &= \text{adjoint}(L_{[N-2]}) \cdot U_{N-1} + \text{adjoint}(L_{[N-3]}), \\
 \text{adjoint}(L_{[N-2]}) &= \text{adjoint}(L_{[N-3]}) \cdot U_{N-2} + \text{adjoint}(L_{[N-4]}), \quad \dots \\
 \text{adjoint}(L_{[N-p]}) &= \text{adjoint}(L_{[N-p-1]}) \cdot U_{N-p} + \text{adjoint}(L_{[N-p-2]}), \quad \dots \quad (\text{B.1})
 \end{aligned}$$

the intertwining relations (45) remaining the same (they are globally self-adjoint). Of course, $L_{[N]}$ and $\text{adjoint}(L_{[N]})$ are on the same footing. A similar ‘tower’ of intertwiners can be built on the $\text{adjoint}(L_{[N]})$ and on the adjoint of the successive intertwiners $\text{adjoint}(L_{[N-p]})$, the only difference being that one must consider the *euclidean left division* of these successive adjoints. The first decompositions of these adjoints read:

$$\begin{aligned}
 \text{adjoint}(L_{[0]}) &= r(x), & \text{adjoint}(L_{[1]}) &= r(x) \cdot U_1, \\
 \text{adjoint}(L_{[2]}) &= r(x) \cdot (U_1 \cdot U_2 + 1), \\
 \text{adjoint}(L_{[3]}) &= r(x) \cdot (U_1 \cdot U_2 \cdot U_3 + U_1 + U_3), \\
 \text{adjoint}(L_{[4]}) &= r(x) \cdot (U_1 \cdot U_2 \cdot U_3 \cdot U_4 + U_1 \cdot U_4 + U_3 \cdot U_4 + U_1 \cdot U_2 + 1). \quad (\text{B.2})
 \end{aligned}$$

B.2. Canonical decomposition for the adjoint: euclidean right division

The adjoint operator $\text{adjoint}(L_{[N]})$ is an operator, we can call $M_{[N]}$, for which the same euclidean right division calculations of section 3.1 can be performed, the first step corresponding to find the intertwiner $M_{[N-1]}$ in the intertwining relation:

$$\text{adjoint}(M_{[N]}) \cdot M_{[N-1]} = \text{adjoint}(M_{[N-1]}) \cdot M_{[N]}. \quad (\text{B.3})$$

The command Homomorphisms($M_{[N]}$, $\text{adjoint}(M_{[N]})$) gives this first intertwiner $M_{[N-1]}$ in the tower of intertwiners (see section 3.1). Since we have in mind that $M_{[N]}$ is $\text{adjoint}(L_{[N]})$, the intertwining relation (C.2) is in fact

$$L_{[N]} \cdot M_{[N-1]} = \text{adjoint}(M_{[N-1]}) \cdot \text{adjoint}(L_{[N]}), \quad (\text{B.4})$$

which is different from the intertwining relation (36), the DEtools Maple command giving this first intertwiner $M_{[N-1]}$ being now Homomorphisms($\text{adjoint}(L_{[N]})$, $L_{[N]}$). Performing the same calculations as in section 3.1 (see (49), (50), (51), (52), ...), we will have another decomposition for these adjoints, deduced from successive right-divisions:

$$\begin{aligned}
 M_{[0]} &= \rho(x), \quad M_{[1]} = V_1 \cdot \rho(x), \\
 M_{[2]} &= (V_2 \cdot V_1 + 1) \cdot \rho(x), \\
 M_{[3]} &= (V_3 \cdot V_2 \cdot V_1 + V_1 + V_3) \cdot \rho(x), \\
 M_{[4]} &= (V_4 \cdot V_3 \cdot V_2 \cdot V_1 + V_4 \cdot V_1 + V_2 \cdot V_1 + V_4 \cdot V_3 + 1) \cdot \rho(x), \quad \dots .
 \end{aligned} \quad (\text{B.5})$$

If one compares this decomposition for $M_{[4]} = \text{adjoint}(L_{[4]})$ with the one in appendix B.1, which is nothing but the adjoint of the decomposition in section 3.1, one finds that they are, actually, the same decompositions provided $\rho(x) = r(x)$ for N even, and $\rho(x) = 1/r(x)$ for N odd, with the following change of operators:

$$\begin{aligned}
V_N &= r(x) \cdot U_1 \cdot r(x), & V_{N-1} &= \frac{1}{r(x)} \cdot U_2 \cdot \frac{1}{r(x)}, & V_{N-2} &= r(x) \cdot U_3 \cdot r(x), \\
V_{N-3} &= \frac{1}{r(x)} \cdot U_4 \cdot \frac{1}{r(x)}, & V_{N-4} &= r(x) \cdot U_5 \cdot r(x), & \dots &.
\end{aligned} \tag{B.6}$$

Therefore the $M_{[N]} = \text{adjoint}(L_{[N]})$ operator, and its successive intertwiners, read

$$\begin{aligned}
M_{[N]}^{(N)} &= (V_N \cdot V_{N-1} \cdots V_2 \cdot V_1 + \cdots) \cdot \rho(x) \\
&= r(x) \cdot (U_1 \cdot U_2 \cdots U_{N-1} \cdots U_N + \cdots), \\
M_{[N-1]}^{(N)} &= (V_{N-1} \cdot V_{N-2} \cdots V_2 \cdot V_1 + \cdots) \cdot \rho(x) \\
&= \frac{1}{r(x)} \cdot (U_2 \cdot U_3 \cdots U_{N-1} \cdots U_N + \cdots), \\
M_{[N-2]}^{(N)} &= (V_{N-2} \cdot V_{N-3} \cdots V_2 \cdot V_1 + \cdots) \cdot \rho(x) \\
&= r(x) \cdot (U_3 \cdot U_4 \cdots U_{N-1} \cdots U_N + \cdots), \quad \dots .
\end{aligned} \tag{B.7}$$

Appendix C. The two intertwiners of an operator with its adjoint are inverse of each other modulo the operator

From the two intertwining relations (36)

$$\text{adjoint}(L_{[N]}) \cdot L_{[N-1]} = \text{adjoint}(L_{[N-1]}) \cdot L_{[N]}, \tag{C.1}$$

and (59), (60)

$$\text{adjoint}(M_{[N]}) \cdot M_{[N-1]} = \text{adjoint}(M_{[N-1]}) \cdot M_{[N]}, \tag{C.2}$$

with $M_{[N]} = \text{adjoint}(L_{[N]})$, namely

$$L_{[N]} \cdot M_{[N-1]} = \text{adjoint}(M_{[N-1]}) \cdot \text{adjoint}(L_{[N]}), \tag{C.3}$$

one gets

$$\begin{aligned}
L_{[N]} \cdot M_{[N-1]} \cdot L_{[N-1]} &= \text{adjoint}(M_{[N-1]}) \cdot \text{adjoint}(L_{[N-1]}) \cdot L_{[N]} \\
&= \text{adjoint}(L_{[N-1]} \cdot M_{[N-1]}) \cdot L_{[N]},
\end{aligned} \tag{C.4}$$

and

$$\begin{aligned}
\text{adjoint}(L_{[N]}) \cdot L_{[N-1]} \cdot M_{[N-1]} &= \text{adjoint}(L_{[N-1]}) \cdot \text{adjoint}(M_{[N-1]}) \cdot \text{adjoint}(L_{[N]}) \\
&= \text{adjoint}(M_{[N-1]} \cdot L_{[N-1]}) \cdot \text{adjoint}(L_{[N]}).
\end{aligned} \tag{C.5}$$

As $L_{[N]}$ is irreducible, relation (C.4) implies that the right division of $M_{[N-1]} \cdot L_{[N-1]}$ by $L_{[N]}$ is a constant. Relation (C.5) means that the right division of $L_{[N-1]} \cdot M_{[N-1]}$ by $\text{adjoint}(L_{[N]})$ is a constant (see section 2.1 of [12]). This yields:

$$M_{[N-1]} \cdot L_{[N-1]} = \Omega_{ML} \cdot L_{[N]} + C_{ML}, \tag{C.6}$$

$$L_{[N-1]} \cdot M_{[N-1]} = \Omega_{LM} \cdot \text{adjoint}(L_{[N]}) + C_{LM}, \tag{C.7}$$

where Ω_{ML} and Ω_{LM} are two operators of appropriate orders, and where the constants C_{ML} and C_{LM} can be shown to be equal: $C_{ML} = C_{LM}$. As $L_{[N]}$ is defined modulo a constant, we may choose $C_{ML} = 1$.

Using $\text{adjoint}(A + B) = \text{adjoint}(A) + \text{adjoint}(B)$, when A and B are of the same parity order, and the fact that $\Omega_{LM} \cdot \text{adjoint}(L_{[N]})$ is of the same order as $L_{[N-1]} \cdot M_{[N-1]}$, and is thus of even parity, one finds, reinjecting (C.6) and (C.7)

$$\begin{aligned} L_{[N]} \cdot (\Omega_{ML} \cdot L_{[N]} + 1) &= L_{[N]} \cdot \Omega_{ML} \cdot L_{[N]} + L_{[N]} \\ &= \text{adjoint}(\Omega_{LM} \cdot \text{adjoint}(L_{[N]} + 1)) \cdot L_{[N]} \\ &= \text{adjoint}(\Omega_{LM} \cdot \text{adjoint}(L_{[N]})) \cdot L_{[N]} + L_{[N]} \\ &= L_{[N]} \cdot \text{adjoint}(\Omega_{LM}) \cdot L_{[N]} + L_{[N]}, \end{aligned} \tag{C.8}$$

yielding

$$L_{[N]} \cdot \Omega_{ML} \cdot L_{[N]} = L_{[N]} \cdot \text{adjoint}(\Omega_{LM}) \cdot L_{[N]}, \tag{C.9}$$

and thus:

$$\Omega_{LM} = \text{adjoint}(\Omega_{ML}). \tag{C.10}$$

In fact the operator Ω_{ML} in these inversion relations modulo $L_{[N]}$ is exactly $M_{[N-2]}^{(N-1)}$ of (B.7). For instance for $L_{[6]}$, using the notation of (B.7), one gets the following inversion relations, modulo $L_{[6]}$, on $M_{[5]}^{(6)}$ and $L_{[5]}$:

$$M_{[5]}^{(6)} \cdot L_{[5]} = \Omega_{ML} \cdot L_{[6]} + 1 = \text{adjoint}(\Omega_{LM}) \cdot L_6 + 1, \tag{C.11}$$

$$L_{[5]} \cdot M_{[5]}^{(6)} = \Omega_{LM} \cdot \text{adjoint}(L_{[6]}) + 1, \tag{C.12}$$

where the operator Ω_{ML} reads

$$\Omega_{ML} = \frac{1}{r(x)} \cdot (U_2 \cdot U_3 \cdot U_4 \cdot U_5 + U_2 \cdot U_5 + U_2 \cdot U_3 + U_4 \cdot U_5 + 1), \tag{C.13}$$

which is nothing but the adjoint of $M_{[4]}^{(5)}$ (see (B.7) in appendix B.2). The two relations (C.11) read:

$$M_{[5]}^{(6)} \cdot L_{[5]} = M_{[4]}^{(5)} \cdot L_{[6]} + 1 = M_{[4]}^{(5)} \cdot L_{[6]} - 1, \tag{C.14}$$

$$L_{[5]} \cdot M_{[5]}^{(6)} = \text{adjoint}(M_{[4]}^{(5)}) \cdot \text{adjoint}(L_{[6]}) - 1. \tag{C.15}$$

The same calculations for $L_{[7]}$ give an operator Ω_{ML} which reads

$$\begin{aligned} \Omega_{ML} &= (U_2 \cdot U_3 \cdot U_4 \cdot U_5 \cdot U_6 + U_2 \cdot U_3 \cdot U_6 + U_2 \cdot U_5 \cdot U_6 \\ &\quad + U_2 \cdot U_3 \cdot U_4 + U_4 \cdot U_5 \cdot U_6 + U_2 + U_4 + U_6) \cdot r(x), \end{aligned} \tag{C.16}$$

which is nothing but $M_{[5]}^{(6)}$, the two inversion relations reading:

$$M_{[6]}^{(7)} \cdot L_{[6]} = M_{[5]}^{(6)} \cdot L_{[7]} + 1 = M_{[5]}^{(6)} \cdot L_{[7]} - 1, \tag{C.17}$$

$$L_{[6]} \cdot M_{[6]}^{(7)} = \text{adjoint}(M_{[5]}^{(6)}) \cdot \text{adjoint}(L_{[7]}) + 1. \tag{C.18}$$

More generally one has the two relations (with the notations of (B.7)):

$$M_{[N-1]}^{(N)} \cdot L_{[N-1]} = M_{[N-2]}^{(N-1)} \cdot L_{[N]} - (-1)^N, \tag{C.19}$$

$$L_{[N-1]} \cdot M_{[N-1]}^{(N)} = \text{adjoint}\left(M_{[N-2]}^{(N-1)}\right) \cdot \text{adjoint}\left(L_{[N]}\right) - (-1)^N. \tag{C.20}$$

Seeing (C.19) and (C.20) as identities on the (self-adjoint same parity-order) U_n 's, identity (C.20) is nothing but identity (C.19), if the U_n 's are changed according to the involution $U_n \leftrightarrow U_{N+1-n}$.

Appendix D. Decomposition for operators equivalent to operators with exceptional differential Galois groups

Let us recall, for instance⁵², one of the six order-seven linear differential operators (called E_2 in [12]), which has the *exceptional differential Galois group* G_2 :

$$\begin{aligned} \mathcal{L}_7 = & \theta^7 - 128 \cdot x \cdot \left(8\theta^4 + 16\theta^3 + 20\theta^2 + 12\theta + 3\right)(2\theta + 1)^3 \\ & + 1048576x^2 \cdot (2\theta + 1)^2 \cdot (2\theta + 3)^2 \cdot (\theta + 1)^3, \end{aligned} \tag{D.1}$$

where $\theta = x \cdot D_x$.

D.1. A first equivalent operator

We introduce an operator $\tilde{\mathcal{L}}_7^{(3)}$ equivalent to \mathcal{L}_7 :

$$\mathcal{I}_3 \cdot \mathcal{L}_7 = \tilde{\mathcal{L}}_7^{(3)} \cdot D_x^3, \tag{D.2}$$

where \mathcal{I}_3 is an order-three linear differential operator. The equivalent operator $\tilde{\mathcal{L}}_7^{(3)}$ is such that its *symmetric square* has a (quite simple [12]) rational solution, namely $\rho(x) = 1/(1 - 4096x)^2/x^6$, when symmetric square of \mathcal{L}_7 has a drop of order (order 27 instead of the generic order 28, see [12]).

Performing the DEtools Maple command ‘Homomorphisms($\tilde{\mathcal{L}}_7^{(3)}$, adjoint($\tilde{\mathcal{L}}_7^{(3)}$)’, we obtained an *order-six* operator that we will denote $\tilde{\mathcal{L}}_6^{(3)}$, such that

$$\text{adjoint}\left(\tilde{\mathcal{L}}_6^{(3)}\right) \cdot \tilde{\mathcal{L}}_7^{(3)} = \text{adjoint}\left(\tilde{\mathcal{L}}_7^{(3)}\right) \cdot \tilde{\mathcal{L}}_6^{(3)}. \tag{D.3}$$

The successive euclidean right-divisions described in section 3.1, enable us to deduce, for $\tilde{\mathcal{L}}_7^{(3)}$, the (generic⁵³) decomposition (55), where the seven operators U_n 's are *all order-one self-adjoint* operators, $r(x)$ being a quite involved rational function. The rational solution $\rho(x)$ is also the rational solution of the symmetric square of the order-one operator $U_1 \cdot r(x)$ in this decomposition (55).

Remark. The order-one operator U_1 is quite involved, as well as the rational solution $r(x)$. In contrast the *self-adjoint* operator $r(x) \cdot U_1 \cdot r(x)$ has a remarkably simple expression in terms of the rational solution $\rho(x)$:

⁵² There is nothing particular with this example, the results are similar for the other order-seven linear differential operators with an exceptional differential Galois group G_2 in [12].

⁵³ In contrast, the order-seven operator \mathcal{L}_7 has the most extreme non-generic decomposition, since it is of the form $U_1 \cdot \rho(x)$ where $\rho(x)$ is a function and U_1 is an order-seven self-adjoint operator.

$$r(x) \cdot U_1 \cdot r(x) = -\frac{99\,225}{122\,825\,998\,336} \cdot \frac{1}{\rho(x)} \cdot \left(D_x - \frac{1}{2} \cdot \frac{d \ln(\rho(x))}{dx} \right). \tag{D.4}$$

D.2. A second equivalent operator

We now introduce an operator $\tilde{\mathcal{L}}_7^{(2)}$ equivalent to \mathcal{L}_7 :

$$\mathcal{M}_2 \cdot \mathcal{L}_7 = \tilde{\mathcal{L}}_7^{(2)} \cdot D_x^2, \tag{D.5}$$

where \mathcal{M}_2 is an order-two linear differential operator. The equivalent operator $\tilde{\mathcal{L}}_7^{(2)}$ is such that its exterior square has a drop of order, but its order-35 *exterior cube* [12] has a (quite simple) *rational solution*, namely $1/(1 - 4096x^3)/x^9$. Again, we follow the algorithm described in section 3.1, to get the decomposition of this order-seven operator $\tilde{\mathcal{L}}_7^{(2)}$. Performing the DEtools Maple command ‘Homomorphisms($\tilde{\mathcal{L}}_7^{(2)}$, adjoint($\tilde{\mathcal{L}}_7^{(2)}$))’, we obtained an *order-six* operator that we will denote $\tilde{\mathcal{L}}_6^{(2)}$, such that

$$\text{adjoint}\left(\tilde{\mathcal{L}}_6^{(2)}\right) \cdot \tilde{\mathcal{L}}_7^{(2)} = \text{adjoint}\left(\tilde{\mathcal{L}}_7^{(2)}\right) \cdot \tilde{\mathcal{L}}_6^{(2)}. \tag{D.6}$$

Performing the successive euclidean right-divisions described in section 3.1, we obtain the following decomposition, corresponding to decomposition (53), *but in the non-generic case*

$$\begin{aligned} \tilde{\mathcal{L}}_7^{(2)} = & (U_5 \cdot U_4 \cdot U_3 \cdot U_2 \cdot U_1 + U_5 \cdot U_4 \cdot U_1 + U_5 \cdot U_2 \cdot U_1 \\ & + U_5 \cdot U_4 \cdot U_3 + U_3 \cdot U_2 \cdot U_1 + U_1 + U_3 + U_5) \cdot r(x), \end{aligned} \tag{D.7}$$

where $r(x)$ is a rational function, where U_5, U_4, U_3, U_2 are four *order-one self-adjoint* operators, but U_1 is an *order-three self-adjoint* operator.

From this non-generic decomposition (D.7), it is clear, from the results of the section 5, that the symmetric square of $\tilde{\mathcal{L}}_6^{(2)}$ has a drop of order (as a consequence of the order-three self-adjoint operator U_1), but that the symmetric square of the adjoint of $\tilde{\mathcal{L}}_6^{(2)}$ has a rational solution, associated with the order-one self-adjoint operator U_5 .

One notes that the rational solution of the exterior cube of $\tilde{\mathcal{L}}_7^{(2)}$, namely $1/(1 - 4096x^3)/x^9$, is, as it should be, *the same* as the rational solution of the exterior cube of the order-three operator $U_1 \cdot r(x)$, or of the self-adjoint operator $r(x) \cdot U_1 \cdot r(x)$ (see (D.4)).

Appendix E. Towards the generic situation for selected differential Galois groups: reduced form for differential systems

Let us show (experimentally) in an alternative way that a linear differential operator with a *selected differential Galois group necessarily has* a decomposition as described in section 3.2. In order to get some hint on this very general question we need to recall the concept of *reduced form* [37, 38] for *linear differential systems*.

E.1. Reduced form of differential systems for orthogonal groups

If one considers a linear differential system corresponding to an *antisymmetric* $q \times q$ matrix $A(x)$, whose entries are rational functions of x

$$Y' = A(x) \cdot Y, \tag{E.1}$$

one is sure that the differential Galois group of this system will correspond to the orthogonal group (this is a result by Kolchin [38, 42, 43, 46]). Less obvious is the result by Kovacic and

Kolchin [37, 47] that *any* linear differential system with an *orthogonal* group for its differential Galois group can be reduced to this canonical form (E.1). Studying linear differential systems like (E.1) is, thus, a way to get some hint of ‘generic’ differential operators with differential Galois groups which correspond to orthogonal groups.

Following our experimental mathematics approach, we have built a large number of examples of order- q (mostly $q = 4$, but also $q = 5, 6, 7, 8$) linear differential operators \tilde{L}_N associated with such a linear differential system (E.1). Using the algorithm described in section 3.1, we obtained the decomposition of \tilde{L}_N . We found that all our (numerous) examples have a *generic decomposition*, the order of *all* the self-adjoint operators U_n ’s being *one* (corresponding to $SO(q, \mathbb{C})$ differential Galois groups).

E.2. Reduced form of differential systems for symplectic groups

The (generic) symplectic case can be sketched, in a similar way, introducing linear differential systems like $Y' = J \cdot A(x) \cdot Y$, where $A(x)$ is a *symmetric* $2p \times 2p$ matrix with rational functions entries, and where J is the *symplectic* matrix

$$J = \begin{bmatrix} 0 & Id \\ -Id & 0 \end{bmatrix}, \quad (\text{E.2})$$

where Id denotes the $p \times p$ identity matrix, and 0 denotes the $p \times p$ null matrix. Again, we have built a large number of examples of order- q (mostly $q = 6$, but also $q = 8$) linear differential operators \tilde{L}_q associated with such a ‘symplectic’ linear differential system. Using the algorithm described in section 3.1, we have obtained the decomposition of these \tilde{L}_q . We have found that all our (numerous) examples actually have a *generic decomposition*, the order of *all* the self-adjoint operators U_n ’s being *two* (corresponding to $Sp(q, \mathbb{C})$ differential Galois groups).

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