Progress in Physics

Edited by
A. Jaffe, G. Parisi,
and D. Ruelle

Critical Phenomena

1983 Braşov School Conference

Valentin Ceauşescu,
Gabriel Costache, and
Vladimir Georgescu,
editors

Springer Science+Business Media, LLC

Progress in Physics
Vol. 11

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STAR-TRIANGLE AND INVERSION RELATIONS IN STATISTICAL MECHANICS

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Introduction

The study of the two-dimensional models in lattice statistics and the related exactly integrable 1+1 dimensional field theory has developed intensively over the past ten years and this evolution, this developement does not seem to decrease. There are several reasons why one can be interested in such models : the need to leave the framework of perturbation theory with the hope to describe particle spectrum and scattering, the fact that some 1+1 dimensional exactly solved models have physical applications for some important solid state theories (Solitons in magnets [61,11], Peierls-Frohlich model, Anderson localization, Kondo effects [19,37,1]),the importance of exactly solved two-dimensional models of lattice statistics to understand the critical phenomena (universality...).

The recent step forward in such a field corresponds mainly to the recognition that the several ways used to solve models in different domains of mathematical physics are often connected, and lead to an essentially unique mathematical structure. One must recall the emergence of the quantum inverse scattering method [18] which realizes the synthesis of the ideas developed in statistical mechanics by Bethe [10], Onsager [49], Yang [56], Lieb [42] and Baxter [2] and the twenty years old inverse scattering method introduced by Kruskal and coworkers [13,23] and also developed by Lax [40], Zakharov [57] ...

This quantum inverse scattering method shows some aspects which are paradoxically simpler than the corresponding onesin the classical inverse scattering method. The new algebraic Bethe ansatz of the QISM is more efficient and powerful than the old coordinate Bethe ansatz. There exists a lot of good reviews which describe the results and technical aspects of exactly solvable models and expiain the relations

between the different concepts and ideas one can encounter on these models (Faddeev [17], Kulish and Sklyanin [38], Olshanetsky and Perelomov [48], Baxter [3], Lieb [43], Kasteleyn [34], Thacker [54], Zamolodchikov [58] ...).

For our purpose we will emphasize the following point: there exists a very simple property which is the keystone of all the methods and technicalities used to solve these models whatever the domain of mathematical physics we deal with . For particle physics in 1+1 dimensions this property is called the S matrix factorization and one must mention Zamolodchikov's work [59], for one-dimensional Bose gas with δ function interaction it is called the Yang relations [56] for two dimensional statistical mechanics on lattices it is called the star triangle relation [3][7] (it is for instance the key point in the solution of the Baxter model [2] and also of the 2-d Ising model [7] . There is a funny anecdote about that 2-d Ising model : in the seventies Onsager revealed [50] that not only did he know, as far back as 1942, the existence of the star-triangle relation but he used intensively this very property to get the solution of the 2-d Ising model ; "unfortunately", as we know, he preferred to give a completely algebraic solution (Clifford algebra...) of this model).

This property is clearly also the keypoint of the QISM and the reason why this method is so elegant and simple.

This property also occurs in Lie group theory, where, as Semenov-Tian-Shansky [39] pointed out, the factorization property of the intertwinning operators coincides with the Yang-Baxter relation. Thus, beyond the particular problem studied , it seems that all these exactly solvable systems share this very simple mathematical structure, which can be thought as some homological condition that describes the symplectic structure of these systems. We will try in the following to explain, in a very pedestrian way, why a so simple property is so constraintfull.

The second point we would like to emphasize is that there is a complete formal identification between a 1+1 dimensional factorized S-matrix theory and a two dimensional vertex model with a star-triangle property in statistical mechanics : as we said, the S-matrix factorization is nothing more than the star triangle property; besides, the well known crossing and unitarity property [33] of the S matrix have two correspondents in statistical mechanis, for instance, the rotation of $\pi/2$

of the lattice (or some other symmetry) and what is called the inversion relation [52][62]. Because of these relations we will from now on restrict ourselves without much loss of generality to lattice models in statistical mechanics.

Let us now outline the plan of the paper :

- we will give a definition of the star -triangle relation (S.T.R.)

- we will also define another very simple relation which occurs simultaneously with the S.T.R. for the two-dimensional (2-d) exact models : the inversion relation (I.R.)

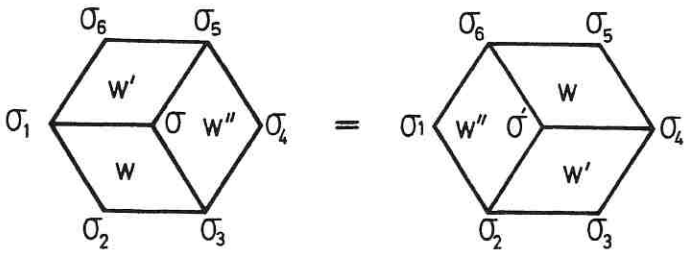
- we will study the connection between the S.T.R. and the I.R. : we will see that the S.T.R. is deeply connected to the I.R., but, on the contrary, we will see that the I.R. can exist even when no S.T.R. exists, as we will show for the 2-d anisotropic Potts model by exhibiting an inverse functional equation satisfied by the partition function.

- having recognized the I.R. as an interesting concept, we will use it by looking at the analytical consequences of this I.R, and, at last, we will come back to the S.T.R., examining some consequences of the I.R. on the S.T.R.

The star triangle relation

As we said, the star triangle relation is the keystone to the exact solution of most lattice models ; the reason for this is essentially due to the fact that the transfer matrices of the corresponding models commute provided the star-triangle relation is satisfied.

Let us give a definition and a graphical representation of the star-triangle relation (S.T.R.) . If W, W', W" are Boltzmann weights associated with the elementary cells (square) of three different square lattices (W depends on the configuration of the four spins at the four corners of the square) the star triangle relation means that the partition function of the two graphs below are equal :



σ_1, σ_2,...σ_6 are fixed spins located at the four corners of the three different squares. They may belong to Z_2 for Ising models, or Z_q for Potts model for instance. One sumsover all the configurations of the central spins σ and σ'. This relation analytically means :

$$\sum_{\sigma} W(\sigma_2,\sigma_3,\sigma,\sigma_1) \cdot W'(\sigma_1,\sigma,\sigma_5,\sigma_6) \cdot W''(\sigma,\sigma_3,\sigma_4,\sigma_5)$$

$$= \sum_{\sigma'} W(\sigma',\sigma_4,\sigma_5,\sigma_6) \cdot W'(\sigma_2,\sigma_3,\sigma_4,\sigma') \cdot W''(\sigma_1,\sigma_2,\sigma',\sigma_6).$$

To fix the ideas, let us consider the well-known Baxter model [2] : in this case the Boltzmann factors (W,W'and W") have certain symmetries (for example $W(\alpha,\beta,\gamma,\delta) = W(-\alpha,-\beta,-\gamma,-\delta)$ the spin reverse symmetry) and there are only four different values, usually called a, b, c, d corresponding to the different spin configurations α, β, γ, δ. If one introduces a', b',... and a", b"... for W' and W" the STR corresponds to a set of six equations ; let us give for information two of them :

$$aca" + da'd" = bc'b" + ca'c"$$

$$ab'c" + dd'b" = ba'c" + cc'b" \quad .$$

These equations are homogeneous and linear in a",b",c",d" (or a,b,c,d or a',...), and therefore in order to have a nontrivial solution some determinants must vanish. These vanishing conditions are algebraic relations between a, b, c, d and a', b', c', d' which should, at first sight, mix indissolubly these two sets of parameters. The "miracle" that occurs with exact models is a factorization of these relations : these relations are equivalent to saying that some algebraic expression $\varphi_i(a,b,c,d)$ is equal to the same one for a', b', c', d'

$$\varphi_i(a,b,c,d) = \varphi_i(a',b',c',d')$$

For the Baxter model [2] one has two such expressions

$$\varphi_1(a,b,c,d) = \frac{a^2+b^2-c^2-d^2}{ab} \quad , \quad \varphi_2 = \frac{ab}{cd} \quad .$$

One should say that, in general, the STR only admits trivial and uninteresting solutions. An obvious one is W=W', for $W''(g,c,d,e) = \delta_{g,d}$ (which, as we will see later, corresponds to the fact that the transfer matrix commutes with itself). There are also a great number of solutions such that the STR is satisfied whatever W, W', W'' are. Most of the time they correspond to models for which the partition function is the same as the one of a one-dimensional, or even zero-dimensional lattice model.

One getsquickly convinced for particular models that the set of trilinear homogeneous equations corresponding to the STR overdetermine the few non trivial solutions that must exist. Thus the temptation is strong to try to classify exhaustively all the non trivial solutions of the STR. In fact this is not an easy task at all : the number of equations, the number of parameters and even the number of terms in each equation is too important, and if one tries to examine one of the Boltzmann weights,one obtains extremely complicated algebraic relations (vanishing of some determinants). Moreover the beloved perturbation approach of the physicist does not work very well : if one tries to generalize a non trivial solution of the S.T.R. by examining the neighbourhood of this solution, one will, in general, get nothing. There are too many perturbation parameters : perturbation theory is great when one has one, or two, perturbation parameters but when one has ten or even more... Another difficulty among many others is due to the fact that the restriction of a model that satisfies a S.T.R. may no longer satisfy a S.T.R. (the *anisotropic* 2-d Ising model satisfies a S.T.R. but its restriction,the *isotropic* 2-d Ising model does not satisfy a S.T.R. relation except the trivial one W = W'). To see that a model satisfies a S.T.R., one has to merge it into a larger model that satisfies a S.T.R. So it seems that we do have a problem with a straightforward approch of the S.T.R. : we will come back to this problem with more appropriate approach at the end of this paper.

From a STR it is simple to show that the transfer matrices with periodic boundary conditions associated with W and W' commute

$$[T(W), T(W')] = 0 \quad , \qquad (1)$$

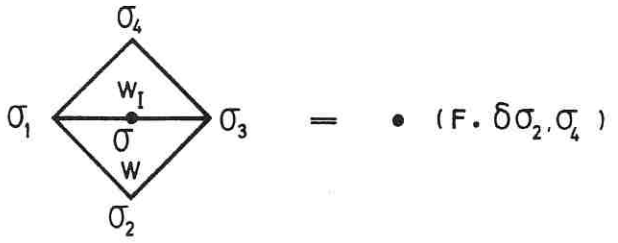
If one Taylor-expands T(W) and T(W') one sees that (1) will imply an infinite set of commutation relations $[T_i, T_j] = 0$; this is the expression for an infinite number of conservation laws for integrable models (infinite number of quantities in involution). We do not explain here the problem of simultaneous diagonalization by the Bethe ansatz of this family of commuting transfer matrices which depend on some continuous parameters. There are some nice and simple reviews (for instance [3,34]) that explain these ideas.

The inversion relation

The inversion relation (IR) was first introduced in statistical mechanics by Stroganov [52] and intensively used by R.J. Baxter [62]. It is formally identical with the unitarity relation of the two-body S-matrix people use in the framework of the Watson's equation (unitarity + crossing) to calculate the total S-matrix (Zamolodchikov [58]).

One interest in this relation is that, if one is not interested in sophisticated informations on the model (such as the spectrum, the eigenvectors of the transfer matrix...), the IR can be used as a short cut to calculate extremely simple and quickly the partition function.

Let us define the inverse relation [62] ; the inversion relation means that the partition functions of the two graphs below are equal :



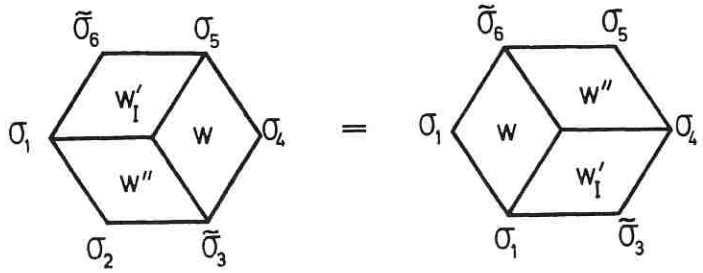
$\sigma_1, \sigma_2, \sigma_3$ and σ_4 are fixed spins and one sums over the configurations of spin σ. This relation analytically means that, whatever σ_1 and σ_3 are, one has :

$$\sum_{\sigma} W(\sigma_1,\sigma_2,\sigma_3,\sigma) \cdot W_I(\sigma_1,\sigma,\sigma_3,\sigma_4) = F \cdot \delta_{\sigma_2,\sigma_4}$$

(F is some known function).

In almost all exactly solved two-dimensional lattice models one

finds out simultaneously a STR and an IR for these models (with some exceptions like the gaussian model). The STR has a very nice stability property with respect to the IR : for instance, acting simultaneously on the $(\sigma_1,\sigma_6,\sigma_5)$ and $(\sigma_2,\sigma_3,\sigma_4)$ side of the two hexagons of the STR with the inverse of W' (W_I') and using the very definition of the IR one gets immediately a new STR



(of course W' does not play a special role and one gets also a large number of other STR by making use of W_I or W_I''). This new relation leads in the same way, to the commutation of the transfer matrices associated to W and W_I' :

$$[T(W),\ T(W_I')] = 0 \quad .$$

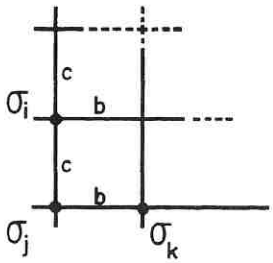
We will see later that this profound connection between the STR and the IR can help us to study the S.T.R.

It is possible to interpret this connection between the STR and the IR in the case of the vertex model, by saying that these two relations generate a non-trivial representation of the group of permutations [24].

So S.T.R. and I.R. are deeply connected : nonetheless the question remains of whether the inverse relation can be used independently of the star-triangle relation (I.R. without S.T.R.). We will reply positively to this question : for this purpose we will exhibit an inverse relation and an associated functional relation for the 2-d anisotropic Potts model at all temperatures (and not only at the critical temperature where it is known that a S.T.R. occurs). We make the choice of the Potts model because of its importance in solid state physics and statistical mechanics, but the ideas we will develop here are not restricted to this model.

Inverse relation for the Potts model

First sum up the q-state, scalar, two-dimensional, *anisotropic* Potts model for a square lattice [55]. If σ_i and σ_j, belonging to Z_q, which are spins on a square lattice are in the same state, the statistical weight associated with the corresponding vertical bond will be c, if not it will be +1 ; if σ_j and σ_k (horizontal bonds) are in the same state it will be b if not +1.

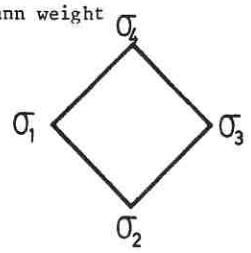


The partition function is therefore

$$Z = \sum_{\{\sigma\}} \prod_{<ij>} c^{\delta_{\sigma_i\sigma_j}} \prod_{<jk>} b^{\delta_{\sigma_j\sigma_k}}$$

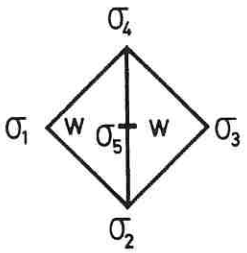
$\sum_{\{\sigma\}}$ denotes the sum over all the spin configurations on the lattice , $\prod_{<>}$ the products over respectively horizontal and vertical bonds.

There are two different ways of introducing the inverse relation for the Potts model. We can for example introduce the following Boltzmann weight



$= b^{\delta_{\sigma_1,\sigma_3}} \cdot c^{\delta_{\sigma_2,\sigma_4}}$; $\sigma_i \in Z_q$.

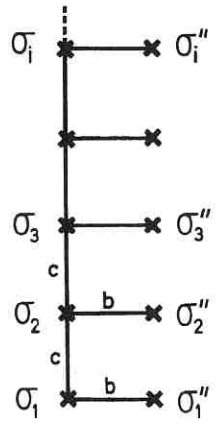
It is easy to see that we consider thus simultaneously two shifted copies of the same Potts model and that one has, with the definition of the IR, a corresponding Boltzmann weight W_I for $b_I = 2-q-b$ $c_I = \frac{1}{c}$ and $F = (b-1)(1-q-b)$



$$= \sum_{\sigma_5} b^{\delta_{\sigma_1,\sigma_5}} (2-q-b)^{\delta_{\sigma_3,\sigma_5}} c^{\delta_{\sigma_2,\sigma_4}} \left(\frac{1}{c}\right)^{\delta_{\sigma_2,\sigma_4}}$$

$$= (b-1)(1-q-b) \cdot \delta_{\sigma_1,\sigma_3}$$

The second way, and it is the most natural from our point of view, is to introduce the well-known transfer matrix formalism for that model. An immediate generalization of the Ising case leads to write the transfer matrix $T_{\sigma,\sigma''}$ as a matrix product $T_{1\ \sigma,\sigma'} \cdot T_{2\ \sigma',\sigma''}$ with



$$T_1 = \left(\prod_i c^{\delta_{\sigma_i,\sigma_{i+1}}}\right) \cdot \bigotimes_i \delta_{\sigma_i,\sigma_i'}$$

and $$T_2 = \bigotimes_i b^{\delta_{\sigma_i',\sigma_i''}}$$

T_1 is merely a diagonal $q^N \times q^N$ matrix and one gets easily

$$T_1(c) \cdot T_1\left(\frac{1}{c}\right) = 1,$$

where 1 is the $q^N \times q^N$ identity matrix. T_2 is the tensoral product of N times the same q×q matrix whose inverse (in the usual sense of the inverse of a matrix) is obtained by replacing b by 2-q-b (up to a known (b-1)(1-q-b) multiplicative factor). Therefore one has :

$$T_2(b) \cdot T_2(2-q-b) = (b-1)^N (1-q-b)^N \cdot 1 \quad . \qquad (1)$$

Instead of T, introduce $\tilde{T} = T_2^{1/2}\ T_1\ T_2^{1/2}$ (it does not change the partition function). $\tilde{T}$ satisfies :

$$\tilde{T}(b,c) \cdot \tilde{T}\left(2-q-b, \frac{1}{c}\right) = (b-1)^N (1-q-b)^N\ 1 \quad .$$

Let $|\Omega\rangle$ be the eigenvector corresponding to the largest eigenvalue $\lambda(b,c)$ of $\tilde{T}(b,c)$; equation (1) leads to

$$\tilde{T}(b,c)\ \tilde{T}(2-q-b,\ \tfrac{1}{c})\ |\Omega\rangle = (b-1)^N(1-q-b)^N\ |\Omega\rangle$$

$$= \lambda(b,c)\ \tilde{T}(2-q-b,\ \tfrac{1}{c})\ |\Omega\rangle \quad ,$$

empirically equal to $\lambda(b,c)\lambda(2-q-b,\frac{1}{c})|\Omega\rangle$ and in consequence, if one denotes by Z the partition function per site of the model in the thermodynamic limit (which is of course $\lim_{N\to\infty} \lambda^{1/N}(b,c)$), one gets [28]

$$Z(b,c)\ .\ Z(2-q-b,\ \tfrac{1}{c}) = (b-1)\ (1-q-b)\ . \qquad (2)$$

This relation takes place between the function Z(b,c) and the same function at $(2-q-b,\ \frac{1}{c})$. What is meant by saying the same function ? This does not mean the partition function with the new values of the parameters : for instance in the (q=2) Ising case, one can show easily that

$$\mathrm{Tr}(\tilde{T}(b,c))^M = \mathrm{Tr}(\tilde{T}(-b,\ \tfrac{1}{c}))^M,$$

that is to say

$$Z(-b,\ \tfrac{1}{c}) = Z(b,c),$$

therefore Z^2 would be a known algebraic function. This is not the case of course. In fact the relation takes place between the function Z(b,c) and an *analytical continuation* of this function. One can get convinced of this by looking at one dimensional or quasi-one dimensional (rubber bands) problems. For more delicate cases, Stroganov [52] and also Baxter [62] seem to justify this fact by considering the complete integrability of the models which, because of the family of commuting transfer matrices, provides an analytical path leading from the eigenvalue at the point λ, $\lambda(\theta)$, to its analytical continuation at the inverse point $-\theta$.

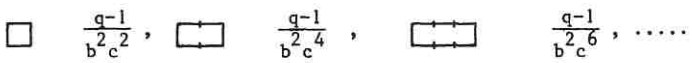
So the question is : is equation (2) correct for the Potts model which, precisely, does not satisfy a S.T.R. (except at the critical point)? If we reply positively, we will get that the inverse relation and its consequence, the inverse functional relation on the partition function, can be used outside the framework of complete integrability. In fact, this problem truly involves two complex variables and any justification of (2) seems to be quite complicated. The best way to get convinced of (2) is to show it directly, using empirical methods, for

instance diagrammatic expansions for the Potts model. A diagrammatic expansion exists for the Potts model [25] ; it is a generalization of the well-known expansions of the Ising model : for example the low temperature expansion $\frac{1}{b} \to 0$, $\frac{1}{c} \to 0$ gives

$$Z(b,c) = bc \left[1 + \frac{(q-1)}{b^2 c^2} + \ldots \right] = bc\, \Lambda(b,c)$$

(Λ is a normalized partition function which tends to 1 in the low temperature limit). The first term $\frac{q-1}{b^2 c^2}$ corresponds to the diagram  , the following term $\frac{q-1}{b^4 c^2}$ to  , the term $\frac{q-1}{b^2 c^4}$ to  .

In order to verify equation (2), it is necessary to have an expansion stable by the inverse transformation (if $\frac{1}{b} \to 0$, $\frac{1}{c} \to 0$ then $\frac{1}{2-q-b} \to 0$ but unfortunately $1/(1/c) \to \infty$). The expansion one needs is, for instance, one with small $\frac{1}{b}$ values but arbitrary values for c. Fortunately, it is possible to obtain this new type of diagrammatic expansion. Let us look at all the terms in $\frac{1}{b^2}$:



It is easy to sum over all these diagrams, which compose a geometrical series,to get $\frac{q-1}{b^2} \frac{1}{c^2-1}$. Relation (2) leads in terms of $\Lambda(b,c)$ to :

$$\ell n\, \Lambda(b,c) + \ell n\, \Lambda(2-q-b, \tfrac{1}{c}) = \ell n \left[\frac{(b-1)(1-q-b)}{b(2-q-b)} \right] . \qquad (3)$$

Up to $\frac{1}{b^2}$, relation (3) is actually satisfied :

$$\frac{(q-1)}{b^2 (c^2-1)} + \frac{(q-1)}{b^2} \left(\frac{-c^2}{c^2-1} \right) = \frac{1-q}{b^2} . \qquad (4)$$

We see that relation (3) imposes strong restrictions the $\frac{1}{c^2-1}$ singularity not only must vanish, but the sum of the two terms on the left hand side of relation (4) must be independent of c because the right hand side of (3) does not depend on c.

In fact it is possible to show, up to $\frac{1}{c^5}$, that relation (3) is satisfied [28].

We can now be confident with this relation and imagine determining analytically the partition function from the inverse functional equation and from the obvious symmetry equation between b and c :

$$Z(b,c)\ Z(2-q-b, \frac{1}{c}) = (b-1)(1-q-b)$$

$$Z(b,c) = Z(c,b) \ .$$

These equations are very like the functional equations for automorphic functions. The partition function is some generalization to several complex variables of automorphic functions, with respect to a group G generated by the two involutions

$$I : (b,c) \to (2-q-b, \frac{1}{c}) \text{ and } S : (b,c) \to (c,b) \ .$$

If one introduces appropriate new variables $x = \frac{b-q_+}{b-q_-}$ and $y = \frac{c-q_+}{c-q_-}$ (with $q_\pm = 1 - \frac{q}{2} \pm \frac{1}{2}\sqrt{q(q-4)}$) the transformations I and S take a simple multiplicative form

$$I : (x,y) \to \left(\frac{1}{x}, \frac{q_+^2}{y}\right) \quad S : (x,y) \to (y,x) \ .$$

With these new variables it becomes quite clear that this group is an infinite discrete group which satisfies the exact sequence

$$0 \to Z \to G \to Z_2 \to 0 \ .$$

In the special case of the critical temperature $(b-1)(c-1) = q$, or $xy = -q_+$, it is known that one has a STR and that it is possible to calculate the partition function (using a mapping with a symmetric six vertex that can be solved by the Bethe ansatz method [8]). Is it possible to use our two functional equations to get more quickly the partition function ? The answer is yes. Let us write these two functional equations :

$$Z(x) \ . \ Z\left(\frac{1}{x}\right) = -qq_+ \frac{\left(1+\frac{x}{q_+}\right)\left(1+\frac{1}{q_+ x}\right)}{(1-x)\left(1-\frac{1}{x}\right)}$$

$$Z(x) = Z\left(\frac{-q_+}{x}\right) \ .$$

The function Z(x) has singularities at least for the orbit by the group G of x=1. If we look at the solution which has only this set of singularities (maximal analyticity assumption) it is unique, and it is clear that it is given by some kind of iteration between these two equations :

$$Z(x) = \sqrt{-qq_+}\ \frac{A(x)\ A\left(-\frac{q_+}{x}\right)}{A(-q_+x)\ A\left(\frac{q_+^2}{x}\right)}$$

where

$$A(x) = \frac{\prod_{n=0}^{\infty}(1+q_+^{2n-1}x)}{\prod_{n=0}^{\infty}(1-q_+^{2n}x)}$$

One can verify on the known solution for the critical Potts model that this eulerian product shape solution is correct [8] . This calculation is completely identical to the determination of the S matrix using the principles of analyticity, unitarity and crossing [33].

Coming back to the general case for all temperatures, it is possible to find the minimal solution of our two functional equations : it is of the form $Z(x,y) = f(x)\ f(y)$ where f has again some eulerian product shape.

This solution is physically shoking because it factorizes into an expression which depends only on the vertical coupling constant and another one which depends only on the horizontal one. In fact, this solution is ruled out, as can be seen on a large q expansion for instance [29]. This reminds the well known CDD ambiguities (Castilleja, Dalitz, Dyson 1956) phenomena ; but one sees also, on that large q expansion, that the usual examples of CDD ambiguities [14] one finds in the literature are not sufficient to find the physical solution of this Potts model : this expansion even excludes a Veneziano-type form like $\frac{f(x)\ f(y)}{g(\frac{x}{y})\ h(xy)}$ or even more elaborate forms.

Therefore the partition function does satisfy a very nice set of functional equations ; however the physical solution is a very complicated one (except at criticality). These ideas open a new class of models which satisfy nice functional equations without being exactly soluble. This class is a very large one [44] and contains important statistical models like, for instance, the three dimensional Ising model [27] or the three dimensional Potts model. In fact it is possible to generalize what has been said for the 2-d Potts model to the 3-d anisotropic cubic Potts model [30] and get that way the following functional equation :

$$Z(a,b,c)\ Z(2-q-a,b,c) = (a-1)(1-q-b)$$

(a,b,c correspond to the three different axes of a cubic lattice). The corresponding group G is a little bit more complicated : let's only say that it has a normal subgroup H isomorphic to Z×Z. It is also possible to verify this functional equation diagramatically. There is a particularly interesting subcase which is the (q=2) cubic Ising model. We will come back to this point later.

One should notice that the inversion relation is a much more universal concept than, for instance, the Kramers-Wannier (KW) duality [35]. One does not know any self-duality property for the 3-d Ising model; however an inversion relation exists for that model. This is not specific of that model : the inversion relation exists for a number of models much more important than the number of models with a self-dual property (non-planar lattice, models with magnetic field with three, four spins interactions...). This is encouraging because the IR is much more constraintfull than the self-dual property.

Before looking at the different analytical consequences of the IR, let us make a few remarks : the IR on the transfer matrix seems to imply a functional equation not only for the largest eigenvalue (the partition function) but for other eigenvalues of the matrix. For instance, for the symmetric six vertex model (which is equivalent from the point of view of the partition function to the critical Potts model), the ratio $\frac{\Lambda_o}{\Lambda_2}$ of the largest over the next largest eigenvalue has been evaluated exactly [32] :

$$\left(\frac{\Lambda_2}{\Lambda_o}\right)^{1/2}(x) = t^{1/2}(x^{1/2}+x^{-1/2}) \prod_{m=1}^{\infty} \frac{(1+t^{4m}x)\cdot\left(1+\frac{t^{4m}}{x}\right)}{(1+t^{4m-2}x)\cdot\left(1+\frac{t^{4m-2}}{x}\right)} \quad ;$$

one verifies easily that

$$\left(\frac{\Lambda_2}{\Lambda_o}\right)^{1/2}\left(\frac{1}{x}\right) = \left(\frac{\Lambda_2}{\Lambda_o}\right)^{1/2}(x) \qquad \text{(crossing symmetry)}$$

and

$$\left(\frac{\Lambda_2}{\Lambda_o}\right)^{1/2}(x)\cdot\left(\frac{\Lambda_2}{\Lambda_o}\right)^{1/2}\left(\frac{t^2}{x}\right) = 1 \qquad \text{(inversion relation)}.$$

This can be also checked on the symmetric eight vertex model, or the anisotropic 2-d Ising model (this ratio is linked to the interface energy which has a very simple expression for the 2-d Ising model). As the relation seems to be satisfied by other eigenvalues of the transfer

matrix, it is thus natural to look for an inverse relation on the correlation functions. Such relations exist [30] but we will just mention this fact without any detail.

Analytical consequences of the inverse relation

Let us come back to the low temperature expansion in the case of the two dimensional Ising model. The inverse relation for the normalized partition function Λ becomes [62] :

$$\ell n\ \Lambda(b,c) + \ell n\ \Lambda(-b, \frac{1}{c}) = \ell n(1 - \frac{1}{b^2}) \cdot$$

In the 2-d Ising case, the resumming of the geometrical series leads to terms where only the $b^2=1$ singularity occurs, so that one can write

$$\ell n\ \Lambda(b,c) = \sum_{r=1}^{\infty} \frac{1}{b^{2r}} \cdot \frac{Pr(c^2)}{(c^2-1)^{2r-1}} \qquad (5)$$

where Pr is a polynomial of degree less than 2r-1. One has for Pr

$$\frac{Pr(c^2)}{(c^2-1)^{r-1}} + \left(\frac{-c^2}{c^2-1}\right)^{2r-1} \cdot Pr\left(\frac{1}{c^2}\right) = -\frac{1}{r} \quad .$$

This relation indicates that if the first r-1 coefficients of Pr are known, the polynomial Pr is determined completely. If one assumes in a recurrence that one knowns $P_1,\ldots,P_{r-1}$ then from the symmetry $\ell n\ \Lambda(b,c) = \ell n\ \Lambda(c,b)$ one can determine the preceding r-1 coefficients of Pr and thus Pr.

Therefore, *using only the inverse and symmetry relations, one can say that the partition function is determined in a unique way.* Of course the analyticity hypothesis (5) is very important : for the Potts model $c^n=1$ for singularities for every integer n occur [29] and it is no longer possible to determine the partition function.

In the anisotropic 3-d Ising model (cubic lattice) if one uses a high temperature expansion, and introducing the more convenient parameters

$$u = th\ K_1,\ v = th\ K_2,\ w = th\ K_3$$

(K_i are the three different coupling constants along the three axis of the cubic lattice) one has [27] :

$$\ell n\ \Lambda(u,v,w) + \ell n\ \Lambda(\frac{1}{u},-v,-w) = \ell n(1-v^2)(1-w^2)$$

and

$$\ell n\ \Lambda(u,v,w) = (v^2+w^2)\ .\ \frac{u^2}{1-u^2} + (v^4+w^4)\ \frac{u^2}{(1-u^2)^3}\left(1-\frac{u^2}{2}+\frac{u^4}{2}\right)$$

$$+\ v^2w^2\ .\ \frac{A+Bu^2+Cu^4+Du^6}{(1-u^2)^3} + \ldots$$

From the inverse and symmetry relations, and from a knowledge of the coefficient of the $u^2v^2w^2$ term (one sees easily that it is 16), one can immediately determine A, B, C, D which gives us the anisotropic high temperature expansion up to the 8^{th} order in th K_i. The simple isotropic expansion has had a chequered career : there has been an accumulation of mistakes (and misprints) on this expansion because of the difficulty to evaluate the disconnected terms and mainly the number of embeddings of self avoiding rings of large size [53]. On that anecdotic example one will appreciate the *possibility to have an exact and constraintfull functional equation to check this kind of expansion.* To higher orders it seems that, as in the two dimensional case, only $u^2=1$ singularities occur : if this is true,it is certainly a very important analytical property of the 3-d Ising model.

Determination of critical manifold by the group G [45]

It is well known that it is possible to localize the critical temperature, when it is unique, as the stable point of the KW duality [35]. It is natural to extend this line of argument (with or without self dual property available) : it can easily be seen that if (x,y) is a singularity for the partition function, then the automorphic properties associated with the group G imply that the orbits of this point under the group G are also singularity points. Therefore the critical manifold has to be stable under the action of the group G. In general manifolds which are stable under such a discrete infinite group are very complicated. Accordingly, we assume that the critical manifold is an algebraic variety $f(x,y) = 0$. Such an assumption is supported by almost all the exactly known critical manifolds. We make the following change of variables $u = xy$, $v = \frac{x}{y}$, $f(x,y) = g(u,v)$. A particular element of G, $(SI)^2$ is such that

$$(SI)^2 : (u,v) \to (u, q_+^4 v)$$

Hence g must be stable under the transformation $v \to q_+^4 v$; however this

property (periodicity) is in contradiction with the algebraic character of g, unless g does not depend on v. Therefore the critical variety is necessarily of the form u = C, where C is a constant depending only on q (the number of states of the Potts model). This constant can be determined using the invariance under : $(u,v) \to \left(\frac{q_+^2}{u}, \frac{1}{q_+^2 v}\right)$ which leads to $C^2 = q_+^2$, and thus to the two critical manifolds $xy = -q_+$ and $xy = q_+$. The first one gives the critical temperature of the *anisotropic* ferromagnetic Potts model and the second one gives exactly the equation recently obtained by Baxter [5] for the critical temperature of the *anisotropic antiferromagnetic* case. One can also get very quickly the partition function for this antiferromagnetic critical variety using the I.R. and symmetry relation and it coincides with the exact expression [5]. The same ideas generalize easily to the 3-d Potts model and one obtains that, if the critical manifold is an algebraic variety, it can only be one of the two varieties $xyz = q_+^2$ and $xyz = -q_+^2$.

However, these two varieties are excluded by precise numerical estimates for the simple cubic lattice (Ditzian and Kadanoff [16], Blöte, Swendsen [12]). This leads to the conclusion that, in this case, the critical manifold *is not an algebraic one*. This confirms that some profound differences occur for this model between d=2 and d=3. The same ideas can also be applied to many other models and for instance one gets that way the exact critical varieties for the Potts model on the triangular, honeycomb or chequerboard square lattice [45].

Lee-Yang singularities of the partition function

As analyticity properties seem to be deeply involved in the action of the automorphy group G, it is tempting to look at the relationship between the group structure and the Lee-Yang singularities of the partition function. Let us introduce G_o, the subgroup of G, whose elements are such that their decomposition into I and S contains an even number of I. The set of zeros of the partition function must obviously, because of the automorphy property of the partition function, be stable by the group G_o. In the case of the Potts model a generalization of the Lee-Yang theorem enablesto locate the Lee-Yang singularities [41,26,47]. They are located, using our variables x and y, at $\left|\frac{xy}{-q_+}\right| = 1$ for the square lattice. This manifold is obviously stable by G_o and even by G. This result stands only for q > 4. From the simple expressior for this manifold when expressed in the variables well adapted to the automorphy

group G, it is quite clear that the Lee Yang singularities are deeply connected to the group. It should be noticed that, in the isotropic case, this variety splits , for $q < 4$, into two circles $|b-1| = \sqrt{q}$ and $|b+1| = \sqrt{4-q}$ which reduces at $q=2$ to the equation of the well-known Fisher's circles [45].

The inversion relation and critical properties

The inversion relation implies strong constraints on the partition function. Does it tell us something on the critical property of the partition function that is to say on the singular part of the partition function ? The inversion relation can clearly be restricted to the singular part of the partition function. For instance in the symmetric eight vertex model, introducing with Baxter (appendix E of [2]) the variables $\mu = \pi\xi/K\lambda$ and $U = \frac{\pi v}{K\lambda}$ (spectral variable) which remain finite near the critical temperature, one can write

$$-\beta f_{sing}(U) = -\beta f_{sing}(-U) \qquad \text{(crossing symmetry)}$$

$$-\beta f_{sing}(U) + (-\beta f_{sing})(2\mu-U) = 0 \quad \text{(inversion relation)}.$$

If one assumes a Kadanoff-type form such as

$$-\beta f_{sing} = a(U,\mu) \cdot q^{2-\alpha(\mu)} + \ldots$$

(q vanishes as T tends to T_c) one gets

$$a(U,\mu) = a(-U,\mu) \tag{6}$$

$$a(U,\mu) + a(2\mu-U,\mu) = 0 \quad . \tag{7}$$

Paradoxically the inversion and symmetry relation contain the sophisticated information, the amplitude $a(U,\mu)$ that multiplies the scaling law $q^{2-\alpha}$ (weak universality), but does not give any information on the critical exponant $2-\alpha$. From the exact known solution of the symmetric eight-vertex one sees that

$$-\beta f_{sing} = 4 \cos \frac{\pi U}{2\mu} \cdot \cot g \frac{\pi^2}{2\pi} \cdot q^{\pi/\mu} + \ldots$$

The amplitude satisfies equations (6) (7) and is a periodic function as it should be.

On the contrary, we have seen that the IR, combined with the symmetry relation and some analyticity assumptions, completely determines the partition function and therefore the critical exponent α.

Thus the question remains open to know if a more sophisticated use of the inversion relation would constrain the critical exponents. For instance the critical exponents of the 2-d Potts model are known exactly (conjecture of den Nijs, Nienhuis et al [47]) and become rational numbers when the group G degenerate into a finite group. This relation between the critical exponents and the group G suggests that it might be possible to get these exact expressions using the I.R concept.

The inversion relation and the star-triangle relation [31]

Finally, we come back to the study of the STR and show that the IR can help us to find STR and, perhaps, to classify exhaustively some family of them. Usually, the STR is used to derive the commutation of large transfer matrices, i.e. in the thermodynamic limit, when their size N goes to infinity. Nonetheless the same commutation is quite deducible whatever the value of N is.

Hence, it is quite feasible for the smallest values of N, 1,2,3,4,... to examine the conditions implied by the commutation and thus to get *necessary* conditions for satisfying the triangle relation. The S.T.R. leads to a so constraintfull overdetermined system of equations that it seems natural to search for necessary conditions which would enable some systematic procedure to limit the research range. On the contrary, to find sufficient conditions, one needs an extraordinary feat of intuition. Let us consider the most general model with the following parameters corresponding to the 16 different configurations of Ising spins σ_1, σ_2, σ_3, σ_4 (I.R.F. model, interaction-round-a-face model).

(σ_3,σ_4) \ (σ_1,σ_2)	(++)	(+-)	(-+)	(--)
(++)	a	b	c	d
(+-)	e	f	g	h
(-+)	i	j	k	l
(--)	m	n	o	p

The commutation of the transfer matrices with periodic boundary conditions, for the size N=1 leads to the two simple equations

$$[T_N(W),\ T_N(W')] = 0 \quad N = 1 \ :$$

$$\frac{a-p}{d} = \frac{a'-p'}{d'}\ ,\quad \frac{d}{m} = \frac{d'}{m'}$$

(a',...,p' denote the parameters associated with the Boltzmann weight W'). The factorization into a form $\varphi_i(W) = \varphi_i(W')$ which is so important in exact models and enables their uniformization, occurs here very simply. For N=2 one gets (using also the N=1 equations)

$$\frac{bc}{b'c'} = \frac{ei}{e'i'} = \frac{no}{n'o'} = \frac{h\ell}{h'\ell'} = \frac{d^2}{d'^2} = \frac{m^2}{m'^2} = \frac{a^2-p^2}{a'^2-p'^2}$$

$$= \frac{fk+gj-a^2}{f'k'+g'j'-a'^2} \quad \text{if} \quad \frac{d^2}{m^2} \neq \frac{bc}{ei} \cdot \frac{no}{h\ell} \quad .$$

Therefore in order that the STR be satisfied one must have:

$$\varphi_1(W) = \frac{a-p}{d} = \varphi_1(W') = \text{some constant}$$

$$\varphi_2(W) = \frac{d}{m} = \text{cte} \qquad \varphi_3(W) = \frac{bc}{ei} = \text{constant}$$

etc... If one adds the N=3 algebraic expression φ_i and the N=4, and, due to the fact that one can gauge-away some variables (weak graph duality [22]), one finds that the number of constraints "$\varphi_i(W)$ = constant " is equal, or greater, than the number of relevant variables : the only solution is the trivial solution $W' = \lambda W$ (the transfer matrix commutes with itself). Therefore it seems necessary to restrict the model to the $\frac{d^2}{m^2} = \frac{bc}{ei} \cdot \frac{no}{h\ell}$ case.

Let us make use of the connection between the IR and the STR : we have indicated that the S.T.R. implies $[T_N(W), T_N(W')] = 0$ but also $[T_N(W), T_N(W_I')] = 0$. If one gets, for some finite N, some algebraic relation $\varphi_i(W) = \varphi_i(W')$, one can say that $\varphi_i(W') = \varphi_i(W) = \varphi_i(W_I')$.

Thus the algebraic expression φ_i *has to be stable by the inversion relation I.* In fact there are not one, but two inversion relations (conjugated by the $\frac{\pi}{2}$ rotations). In general these transformations do not commute and generate an infinite discrete group and an infinite number of constraints. All the constraints from finite N commutation relationsand their transformed by IR seem to restrict the only non trivial solution (without excluded configurations)to essentially the Baxter model and the free fermion Felderhof model [20] : details of this study will be published elsewhere [46]. For this model most of the φ_i become trivial $\frac{bc}{ei} = 1,...$ and for other non trivial φ_i, equations like $\varphi_i(W) = \varphi_i(W_I)$ become identities.

This systematic approach can be applied to a lot of models (Ashkin-Teller model, staggered vertex models...). Let us just mention two examples : the hard hexagon model [4] (a=b=c=d=e=i=m=g=j=0) for which one gets three algebraic expressions called Δ_1, Δ_2, Δ_3 (in Baxter's notation) at respectively the N=2,3 and 4^{th} order and for which one can verify algebraically that $(\Delta_2)_I = \Delta_3 - \Delta_1$ (one has $\Delta_2 = \Delta_3 - \Delta_1$ for this model so that $\Delta_2 = (\Delta_2)_I$). Let us mention also the 2-d *anisotropic* Potts model where one gets at the N=3 order (with our preceding notations), the equation

$$b'c'(c^2b^2+1-b^2-c^2) + (q-2)(c'+b')(cb+1) = bc(c'^2b'^2+1-b'^2-c'^2) + (q-2)(c+b)(c'b'+1) \quad . \tag{8}$$

This equation is not immediately in a factorized form but for q=2, one gets

$$\frac{bc}{(1-b^2)(1-c^2)} = \frac{b'c'}{(1-b'^2)(1-c'^2)} \quad ;$$

for $q \neq 2$ one verifies that bc = b+c+q-1 (ferromagnetic critical condition) and bc+b+c = q-3 (antiferromagnetic critical condition) both satisfy the equation (8).

But, of course, it is especially in three dimensions (due to the tremendous complexity of the tetrahedron relation [60][6] which is the multidimensional generalization of the S,T,R.)that such ideas could be useful . Since the tetrahedron relation is known to imply the commutation of transfer matrices of arbitrary size, in the same way as the S.T.R. does, the same approach applies without any major modification in three dimensions : one obtains also necessary conditions from the commutation relation of the transfer matrices with periodic conditions along two directions of respective size N and M :

$$[T_{N,M}(W), T_{N,M}(W')] = 0 \quad .$$

For N=1, one recovers the two-dimensional necessary conditions of the I.R.F. model. Thus, the study of the commutation relation of the two-dimensional model can be used again in three dimensions.

In the case of the model studied by Zamolodchikov which satisfies a tetrahedron relation [60] one can get, as necessary conditions for this tetrahedron relation, algebraic equations such as $\varphi_i(W) = \varphi_i(W')$.

For instance

$$\varphi(W) = \frac{(P_o-Q_o)^4-(P_o+Q_o)^4+(P_1-Q_1)^4-(P_1+Q_1)^4}{R_oR_1R_2R_3}$$

$$+ \frac{(P_2+Q_2)^4 - (P_2-Q_2)^4+ (P_3-Q_3)^4 - (P_3+Q_3)^4}{R_oR_1R_2R_3}$$

(with the Baxter's notations [6]). Of course it is more tedious to get the algebraic expansion φ_i. A straightforward generalization of the connection between S.T.R. and I.R. enables to say in three dimensions that $\varphi_i(W) = \varphi_i(W') \Rightarrow \varphi_i(W) = \varphi_i(W_I)$. May be these simple calculations can be a systematic approach for the tetrahedron relation.

Conclusion

One would like to continue to develop exact models and go further in two dimensions : there are very few models that have been solved in the presence of a field (spherical model, KDP ferroelectric model, free-fermion Felderhof model [20]). It seems that exact solubility has more or less something to do with the vanishing field conditions and it seems unlikely that some generalized S.T.R. can be used to solve the Ising or other models in the presence of a field. It seems also that exact solubility has something to do with criticality conditions of a model (see for instance the Potts model) : the Ashkin-Teller model can be solved on its critical self-dual line and it is known that this line splits into two critical lines ; is it possible to calculate exactly the partition function on these two lines ? The conformal invariance is a property which exists for a lot of exactly solved 2-d models (it is far from being understood on the 2-d Ising model in the scaling limit for instance) : what are the relations between this property and the two preceding ones? Another important question is to know if the exact solvability of a model is related to the rationality of the critical exponents of the model : if it could be possible to calculate exactly the partition function, for instance of the Potts model at all temperatures, for the values of q for which the exponents are rational (and for which the group G degenerates: $q = 2+2\cos 2\pi \frac{k}{n}$ with k and n integers ; Tutte-Beraha numbers) it would be a giant step forward. The problem of the relation between complete integrability and some other properties for two-dimensional theories, raises a very important question : is

exact integrability an inherently two-dimensional phenomenon or does similar behaviour occur in (more realistic) three or four dimensional theories ? The answer is far from clear, but, for instance, the (1+1) dimensional non linear σ model is often mentioned in the literature to have much in common with the (3+1)dimensional Yang-Mills theory [51].

However to answer these questions, it seems important to understand precisely what makes a model integrable. There are a lot of new ideas (Kac-Moody algebra [21]). But it seems to us that the Yang-Baxter equations and their multidimensional generalizations (tetrahedron equations) are a key point to understand exact models : a lot of particular solutions to Y-B-E have been found and many people devote their energies to make progress the general Y-B-E theory ([9],[15],[36] group theoretical approach). In this framework we have shown that the inversion relation plays an important role for the Y-B-E (S.T.R.). We have indicated that the inversion relation is an important concept even when no STR exists. The study of models which are not integrable, but satisfy exact functional relations is an open subject. Let us just recall the 2-d and 3-d Potts model and the 3-d Ising model which are "natural" models in statistical mechanics.

How to use at best these exact relations on these models ? We have just started to try to answer the question but much remains to be done.

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