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The lambda extensions of the Ising correlation functions $C(M,N)$

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Abstract

We revisit, with a pedagogical heuristic motivation, the lambda extension of the low-temperature row correlation functions $C(M,N)$ of the two-dimensional Ising model. In particular, using these one-parameter series to understand the deformation theory around selected values of λ, namely $\lambda = \cos(\pi m/n)$ with m and n integers, we show that these series yield perturbation coefficients, generalizing form factors, that are D-finite functions. As a by-product these exact results provide an infinite number of highly non-trivial identities on the complete elliptic integrals of the first and second kind. These results underline the fundamental role of Jacobi theta functions and Jacobi forms, the previous D-finite functions being (relatively simple) rational functions of Jacobi theta functions, when rewritten in terms of the nome of elliptic functions.

Keywords: Ising correlation functions, form factors, lambda extension of correlation functions, sigma form of Painlevé VI, D-finite functions, globally bounded series, Jacobi forms

1. Introduction

We revisit, with a pedagogical heuristic motivation, the lambda extension [1] of the two-point correlation functions $C(M,N)$ of the two-dimensional Ising model. For simplicity we will examine in detail the lambda extension of a particular low-temperature diagonal correlation function, namely $C(1,1)$, in order to make crystal clear some structures and subtleties. However similar structures and results can be obtained on the two-point correlation functions

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$C(M,N)$ for the special case $\nu = -k$ studied in [2] where Okamoto sigma-forms of Painlevé VI equations also emerge.

In 1976 Wu *et al* [3] discovered, in the scaling limit $T \to T_c$ with $N \cdot (T - T_c)$ fixed, that the isotropic diagonal correlation $C(N,N)$ is given by a Painlevé III equation. This was generalized in 1980 by Jimbo and Miwa [4] who defined for $T < T_c$

$$\sigma = t \cdot (t-1) \cdot \frac{d}{dt} \ln C(N,N) - \frac{t}{4} \qquad \text{with} \qquad t = k^2, \tag{1}$$

and for $T > T_c$

$$\sigma = t \cdot (t-1) \cdot \frac{d}{dt} \ln C(N,N) - \frac{1}{4} \qquad \text{with} \qquad t = k^{-2}, \tag{2}$$

and in both cases derived the sigma-form of Painlevé VI non-linear ordinary differential equation (ODE) satisfied by σ:

$$\begin{aligned} \Big(t \cdot (t-1) \cdot \frac{d^2\sigma}{dt^2} \Big)^2 &= N^2 \cdot \Big((t-1) \cdot \frac{d\sigma}{dt} - \sigma \Big)^2 \\ &\quad - 4 \cdot \frac{d\sigma}{dt} \cdot \Big((t-1) \cdot \frac{d\sigma}{dt} - \sigma - \frac{1}{4} \Big) \cdot \Big(t \frac{d\sigma}{dt} - \sigma \Big). \end{aligned} \tag{3}$$

The low-temperature diagonal two-point correlation functions $C(N,N)$ are (homogeneous) polynomial expressions [5, 6] in the complete elliptic integral of the first and second kind[3]:

$$K = {}_2F_1 \Big(\Big[\frac{1}{2}, \frac{1}{2} \Big], [1], t \Big), \qquad E = {}_2F_1 \Big(\Big[\frac{1}{2}, -\frac{1}{2} \Big], [1], t \Big). \tag{4}$$

In [1] it has been underlined that these correlation functions $C(N,N)$ have lambda extensions *which are also solutions of* (3), that can be defined using a 'form factor' low-temperature expansion [1, 7] (see (9) in [1]):

$$C_-(N,N;\lambda) = (1-t)^{1/4} \cdot \Big(1 + \sum_{n=1}^{\infty} \lambda^{2n} \cdot f_{N,N}^{(2n)} \Big), \tag{5}$$

where the form factors [1] $f_{N,N}^{(2n)}$ are also polynomial expressions [5, 6] integral of the first and second kind (4). For instance for the simplest low-temperature correlation function this form factor expansion reads

$$C_-(1,1;\lambda) = (1-t)^{1/4} \cdot \Big(1 + \sum_{n=1}^{\infty} \lambda^{2n} \cdot f_{1,1}^{(2n)} \Big), \tag{6}$$

where the first form factors read:

$$f_{1,1}^{(2)} = \frac{1}{2} \cdot \big(1 - 3EK - (t-2) \cdot K^2 \big), \tag{7}$$

$$\begin{aligned} f_{1,1}^{(4)} = \frac{1}{24} \cdot \big(& 9 - 30EK - 10 \cdot (t-2) \cdot K^2 \\ & + (t^2 - 6t + 6) \cdot K^4 + 15E^2K^2 + 10 \cdot (t-2) \cdot EK^3 \big). \end{aligned} \tag{8}$$

[3] In Maple K is $2/\pi \text{Elliptic}K(t^{1/2})$ and E is $2/\pi \text{Elliptic}E(t^{1/2})$. With that normalization one has $K = \theta_3(0,q)^2$ and $t^{1/2} = k = \theta_2(0,q)^2/\theta_3(0,q)^2$ and thus $k \cdot K = \theta_2(0,q)^2$, where q denotes the nome.

For $\lambda = 1$ we must recover, from (6), the well-known expression of the *low-temperature* two-point correlation function $C(1,1) = E$:

$$C_-(1,1;1) = E = 1 - \frac{1}{4} \cdot t - \frac{3}{64} \cdot t^2 - \frac{5}{256} \cdot t^3 - \frac{175}{16384} \cdot t^4 + \cdots$$
$$= (1-t)^{1/4} \cdot \left(1 + \sum_{n=1}^{\infty} f_{1,1}^{(2n)}\right), \tag{9}$$

which corresponds to write the ratio $E/(1-t)^{1/4}$ as an infinite sum of polynomial expressions of E and K.

2. Simple power series expansions and formal calculations

For pedagogical reasons we restrict our analysis to the low-temperature two-point correlation function $C(1,1)$ and its lambda extension. Since all these lambda extensions are power series in t, we can try to get, order by order, the series expansion of $C_-(1,1;\lambda)$ from the non-linear ODE (3). Recalling [1] the form factor expansion (6), we can either see the series expansion in t as a deformation of the simple algebraic function $(1-t)^{1/4}$, or *more naturally*, see the series expansion of the lambda-extension of the low-temperature two-point correlation function $C_-(1,1;\lambda)$ as a deformation of the exact expression $C_-(1,1) = E$ (M denotes here a difference to $\lambda^2 = 1$, see (16) below):

$$C_-(1,1;\lambda) = C_M(1,1;M) = E + M \cdot g_1(t) + M^2 \cdot g_2(t) + M^3 \cdot g_3(t) + \cdots. \tag{10}$$

Using the sigma-form of Painlevé VI equation (3) one can find that this expansion (10) reads as a series expansion in the variable t:

$$C_M(1,1;M) = 1 - \frac{1}{4} \cdot t - \left(\frac{3}{64} + \frac{3}{256} \cdot M\right) \cdot t^2 - \left(\frac{5}{256} + \frac{9}{1024} \cdot M\right) \cdot t^3$$
$$- \left(\frac{175}{16384} + \frac{441}{65536} \cdot M\right) \cdot t^4 - \left(\frac{441}{65536} + \frac{1407}{262144} \cdot M\right) \cdot t^5$$
$$- \left(\frac{4851}{1048576} + \frac{9281}{2097152} \cdot M - \frac{5}{16777216} \cdot M^2\right) \cdot t^6 + \cdots. \tag{11}$$

Note that this low-temperature expansion (11) gives for σ defined by (1):

$$\sigma = t \cdot (t-1) \cdot \frac{d}{dt} \ln C(1,1;M) - \frac{t}{4} = (M-4) \cdot \sigma_M, \tag{12}$$

where:

$$\sigma_M = \frac{3}{128} \cdot t^2 + \frac{3}{256} \cdot t^3 + \frac{3}{32768} \cdot (3M+74) \cdot t^4 + \frac{3}{65536} \cdot (9M+94) \cdot t^5$$
$$+ \frac{3}{8388608} \cdot (9M^2 + 1270M + 8176) \cdot t^6 + \cdots. \tag{13}$$

Recalling the expansions of $(1-t)^{1/4}$

$$(1-t)^{1/4} = 1 - \frac{1}{4} \cdot t - \frac{3}{32} \cdot t^2 - \frac{7}{128} \cdot t^3 - \frac{77}{2048} \cdot t^4 + \cdots \tag{14}$$

one can see that this series coincides (as it should) with the series (11) for $M = 4$ (i.e. $\lambda = 0$ in (6)). Recalling the expansions of $f^{(2)}_{1,1}$ and $f^{(4)}_{1,1}$:

$$f^{(2)}_{1,1} = \frac{3}{64} \cdot t^2 + \frac{3}{64} \cdot t^3 + \frac{705}{16384} \cdot t^4 + \frac{321}{8192} \cdot t^5 + \frac{18795}{524288} \cdot t^6 + \cdots,$$
$$f^{(4)}_{1,1} = \frac{5}{1048576} \cdot t^6 + \frac{15}{1048576} \cdot t^7 + \frac{7335}{268435456} \cdot t^8 + \frac{2855}{67108864} \cdot t^9 + \frac{4052025}{687194767366} \cdot t^{10} + \frac{5215005}{68719476736} \cdot t^{11} + \cdots \quad (15)$$

the series expansion (11) can be seen to match with the (form factor) expansion (6) with (7) and (8) (together with the previous expansions (15)) if one has the following correspondence:

$$M = 4 \cdot (1 - \lambda^2). \quad (16)$$

At the first order in λ^2 one gets from (11):

$$(1-t)^{1/4} \cdot f^{(2)}_{1,1} = \frac{3}{64} \cdot t^2 + \frac{9}{256} \cdot t^3 + \frac{441}{16384} \cdot t^4 + \frac{1407}{65536} \cdot t^5 + \frac{2319}{131072} \cdot t^6 + \cdots \quad (17)$$

in agreement with the exact expression (7). At the second order in λ^2 one gets from (11):

$$(1-t)^{1/4} \cdot f^{(4)}_{1,1} = \frac{5}{1048576} \cdot t^6 + \frac{55}{41943046} \cdot t^7 + \frac{6255}{2684354566} \cdot t^8 + \frac{36625}{10737418246} \cdot t^9 + \frac{3079025}{687194767366} \cdot t^{10} + \frac{15116115}{2748779069446} \cdot t^{11} + \cdots \quad (18)$$

in agreement with the exact expression (8). At the third order in λ^2 one gets from (11):

$$(1-t)^{1/4} \cdot f^{(6)}_{1,1} = \frac{7}{4398046511104} \cdot t^{12} + \frac{161}{17592186044416} \cdot t^{13} + \frac{33789}{1125899906842624} \cdot t^{14} + \frac{332703}{4503599627370496} \cdot t^{15} + \cdots. \quad (19)$$

Matching the form-factor expansion (6) with the series expansion (10) one gets the following (infinite ...) identities:

$$(1-t)^{1/4} = E + \sum_{n=1}^{\infty} 4^n \cdot g_n(t), \qquad (1-t)^{1/4} \cdot f^{(2)}_{1,1} = -\sum_{n=1}^{\infty} n \cdot 4^n \cdot g_n(t),$$
$$(1-t)^{1/4} \cdot f^{(4)}_{1,1} = \sum_{n=1}^{\infty} \frac{n \cdot (n-1)}{2} \cdot 4^n \cdot g_n(t), \cdots \quad (20)$$

and conversely:

$$E = (1-t)^{1/4} \cdot \left(1 + \sum_{n=1}^{\infty} f^{(2n)}_{1,1}\right), \qquad g_1(t) = -\frac{(1-t)^{1/4}}{4} \cdot \sum_{n=1}^{\infty} n \cdot f^{(2n)}_{1,1},$$
$$g_2(t) = \frac{(1-t)^{1/4}}{32} \cdot \sum_{n=1}^{\infty} n \cdot (n-1) \cdot f^{(2n)}_{1,1}, \cdots. \quad (21)$$

2.1. Algebraic subcases

It had been noticed [1], for $\lambda = \cos(\pi m/n)$ where m and n are integers, and $\lambda^2 \neq 1$, that the lambda extension (6) is not only D-finite[4], but is, in fact, an *algebraic function*.

2.1.1. $\lambda = \cos(\pi/4)$. For instance for $\lambda = \cos(\pi/4) = 1/\sqrt{2}$, i.e. for $M = 2$, one finds that (11) is actually the series expansion of an algebraic expression

$$(1-t)^{1/16} \cdot {}_2F_1\left(\left[-\frac{3}{8}, \frac{1}{8}\right], \left[\frac{1}{4}\right], t\right)$$
$$= (1-t)^{1/16} \cdot \left(\frac{1+(1-t)^{1/2}}{2}\right)^{3/4} = 1 - \frac{1}{4} \cdot t - \frac{9}{128} \cdot t^2 - \frac{19}{512} \cdot t^3 - \frac{791}{32768} \cdot t^4$$
$$- \frac{2289}{131072} \cdot t^5 - \frac{56523}{4194304} \cdot t^6 + \cdots \tag{22}$$

in agreement of the exact result given in equation (99) of [8].

2.1.2. $\lambda = \cos(\pi/6)$. Another example corresponds to $M = 1$ (i.e. $\lambda = \sqrt{3}/2 = \cos(\pi/6)$). The series (11) reads:

$$1 - \frac{1}{4} \cdot t - \frac{15}{256} \cdot t^2 - \frac{29}{1024} \cdot t^3 - \frac{1141}{65536} \cdot t^4 - \frac{3171}{262144} \cdot t^5$$
$$- \frac{151859}{16777216} \cdot t^6 - \frac{477697}{67108864} \cdot t^7 + \cdots. \tag{23}$$

One first finds that this series (23) is D-finite, being the solution of the order-*four* linear differential operator:

$$D_t^4 + \frac{1}{3} \cdot \frac{19t^3 - 30t^2 + 36t - 14}{(t-1)(t^2-t+1)t} \cdot D_t^3$$
$$+ \frac{1}{216} \cdot \frac{1625t^4 - 3439t^3 + 5091t^2 - 3628t + 680}{(t-1)^2(t^2-t+1)t^2} \cdot D_t^2$$
$$+ \frac{1}{11664} \cdot \frac{10033t^5 - 26608t^4 + 53854t^3 - 55334t^2 + 16160t + 880}{(t-1)^3(t^2-t+1)t^3} \cdot D_t$$
$$+ \frac{1}{186624} \cdot \frac{3689t^5 - 6725t^4 + 2573t^3 + 8t^2 + 5200t - 3520}{(t-1)^4(t^2-t+1)t^3}. \tag{24}$$

In fact the series (23) is not only D-finite, it is *an algebraic series*. Denoting $S(t)$ the series (23), and $S_{12} = S(t)^{12}$ its *twelfth* power, one can see that S_{12} is actually solution of the quartic equation

$$3^{36} \cdot t^8 \cdot S_{12}^4 + 2^{10} \cdot 3^{26} \cdot t^6 \cdot (t-1) \cdot p_6 \cdot S_{12}^3$$
$$+ 2^{17} \cdot 3^{15} \cdot t^4 \cdot p_{12} \cdot (t-1)^2 \cdot S_{12}^2 + 2^{26} \cdot (t-1) \cdot p_{24} \cdot S_{12}$$
$$+ 2^{32} \cdot (t-1)^4 \cdot (t^2-t+1)^{12} = 0, \tag{25}$$

[4] Like in the $\lambda^2 = 1$, $M = 0$ case.

where:

$$
\begin{aligned}
p_6 &= 5t^6 - 15t^5 + 138t^4 - 251t^3 + 138t^2 - 15t + 5, \\
p_{12} &= 113t^{12} - 678t^{11} + 5829t^{10} - 22\,930t^9 + 148\,410t^8 - 463\,518t^7 \\
&\quad + 665\,661t^6 - 463\,518t^5 + 148\,410t^4 - 22\,930t^3 + 5829t^2 - 678t + 113, \\
p_{24} &= 64t^{24} - 768t^{23} + 4965t^{22} - 22\,231t^{21} + 3243\,192t^{20} - 31\,880\,523t^{19} \\
&\quad + 66\,263\,383t^{18} + 309\,635\,262t^{17} - 1791\,331\,236t^{16} + 3209\,457\,458t^{15} \\
&\quad - 698\,769\,519t^{14} - 6199\,132\,605t^{13} + 10\,265\,065\,180t^{12} - 6199\,132\,605t^{11} \\
&\quad - 698\,769\,519t^{10} + 3209\,457\,458t^9 - 1791\,331\,236t^8 + 309\,635\,262t^7 \\
&\quad + 66\,263\,383t^6 - 31\,880\,523t^5 + 3243\,192t^4 - 22\,231t^3 + 4965t^2 - 768t + 64. \qquad (26)
\end{aligned}
$$

2.1.3. $\lambda = \cos(\pi/3)$. Similarly, for $M = 3$ (i.e. $\lambda = 1/2 = \cos(\pi/3)$), the series (11) reads

$$
\begin{aligned}
& 1 - \frac{1}{4}\cdot t - \frac{21}{256}\cdot t^2 - \frac{47}{1024}\cdot t^3 - \frac{2023}{65\,536}\cdot t^4 - \frac{5985}{262\,144}\cdot t^5 \\
& \quad - \frac{300\,315}{16\,777\,216}\cdot t^6 - \frac{979\,737}{67\,108\,864}\cdot t^7 + \cdots \qquad (27)
\end{aligned}
$$

and can be seen to be solution of an order-*four* linear differential operator:

$$
\begin{aligned}
& D_t^4 + \frac{2}{3}\cdot\frac{11t-7}{(t-1)t}\cdot D_t^3 + \frac{1}{54}\cdot\frac{587t^2 - 737t + 170}{(t-1)^2t^2}\cdot D_t^2 \\
& \quad + \frac{1}{1458}\cdot\frac{2855t^3 - 5223t^2 + 2130t + 110}{(t-1)^3t^3}\cdot D_t \\
& \quad + \frac{1}{11\,664}\cdot\frac{161t^3 - 702t^2 + 1785t - 220}{(t-1)^4t^3}. \qquad (28)
\end{aligned}
$$

Again, the series (27) is not only D-finite, it is also an algebraic series. Denoting $S(t)$ the series (27), and $S_6 = S(t)^6$ its *sixth* power, one can see that S_6 is solution of the quartic equation

$$
\begin{aligned}
& 3^{27}\cdot t^4\cdot S_6^4 - 2^{10}\cdot 3^{20}\cdot t^4\cdot(t-1)\cdot(t-2)\cdot S_6^3 \\
& \quad + 2^9\cdot 3^{11}\cdot t^2\cdot p_4\cdot(t-1)^2\cdot S_6^2 + 2^{15}\cdot(t-2)\cdot p_8\cdot(t-1)^2\cdot S_6 \\
& \quad - 2^{16}\cdot(t-1)^8 = 0, \qquad (29)
\end{aligned}
$$

where:

$$
\begin{aligned}
p_8 &= 8192t^8 - 38\,912t^7 + 82\,304t^6 - 93\,704t^5 + 64\,151t^4 \\
&\quad - 20\,756t^3 + 6914t^2 + 4t - 1, \\
p_4 &= 3584t^4 + 5312t^3 - 5307t^2 - 10t + 5. \qquad (30)
\end{aligned}
$$

Actually (11) provides [1] an *infinite number of algebraic functions for selected values of* λ, namely $\lambda = \cos(\pi m/n)$, or $M = 4\cdot\sin^2(\pi m/n)$, with m and n integers.

2.2. The g_n's are, at first sight, DD-finite

The form factor expansion (6) is well-suited [1] to analyze the deformation of the $(1-t)^{1/4}$ algebraic solution of the sigma-form of Painlevé VI equation (3). We underlined in [1] the fact that all the form factors $f_{1,1}^{(2n)}$ are D-finite (polynomials in E and K).

Let us now see the series expansion (11) as a (one-parameter) deformation (10) of the $C(1,1) = E$ low-temperature exact expression:

$$C_M(1,1;M) = E + M \cdot g_1(t) + M^2 \cdot g_2(t) + M^3 \cdot g_3(t) + \cdots. \tag{31}$$

At first sight these $g_n(t)$'s *have no reason to be D-finite*. The series expansion of $g_1(t)$ reads:

$$g_1(t) = -\frac{3}{256} \cdot t^2 - \frac{9}{1024} \cdot t^3 - \frac{441}{65\,536} \cdot t^4 - \frac{1407}{262\,144} \cdot t^5 - \frac{9281}{2097\,152} \cdot t^6 - \frac{31\,405}{8388\,608} \cdot t^7 - \frac{13\,877\,397}{4294\,967\,296} \cdot t^8 + \cdots. \tag{32}$$

Inserting (31) in the sigma form of Painlevé VI non-linear ODE (3) (with σ defined by (1)), one gets straightforwardly, at the first order in M, that $g_1(t)$ is *DD-finite*[5] [9]: it is solution of an order-three linear differential operator $\mathcal{L}_3$ with coefficients *that are themselves D-finite* (they are polynomials of hypergeometric ${}_2F_1$ functions). This order-three linear differential operator is of the form $\mathcal{L}_3 = \mathcal{L}_1 \cdot L_E$ where the order-two linear differential operator L_E is the operator annihilating the complete elliptic integral of the second kind E, and where the order-one DD-finite operator $\mathcal{L}_1$ reads:

$$\begin{aligned}
\mathcal{L}_1 &= K^3 \cdot (t-1)^2 \cdot (2 \cdot (t-1) \cdot tD_t + 5t - 3) \\
&\quad - EK^2 \cdot (t-1) \cdot \big(4 \cdot (t-1) \cdot (t-2) \cdot tD_t + 10t^2 - 27t + 13\big) \\
&\quad - KE^2 \cdot (t-1) \cdot (10 \cdot (t-1) \cdot tD_t + 26t - 17) \\
&\quad + E^3 \cdot \big(2 \cdot (t-1) \cdot (t-2) \cdot tD_t + 3t^2 - 14t + 7\big) \\
&= 2 \cdot \big((t-2) \cdot E^3 - 5 \cdot (t-1) \cdot KE^2 \\
&\quad -2(t-1) \cdot (t-2) \cdot EK^2 + (t-1)^2 \cdot K^3\big) \cdot (t-1) \cdot t \cdot D_t \\
&\quad + (t-1)^2 \cdot (5t-3) \cdot K^3 - (t-1) \cdot (10t^2 - 27t + 13) \cdot EK^2 \\
&\quad - (t-1) \cdot (26t - 17) \cdot KE^2 + (3t^2 - 14t + 7) \cdot E^3.
\end{aligned} \tag{33}$$

At first sight $g_1(t)$ is DD-finite and one easily verifies that the series expansion (32) is actually solution of the order-three DD-finite linear differential operator $\mathcal{L}_3 = \mathcal{L}_1 \cdot L_E$. Could it be possible that $g_1(t)$ is, in fact, D-finite?

3. The $g_n(t)$'s are D-finite

In order to see that the $g_n(t)$'s are D-finite, let us recall that there actually exists an *exact closed expression* [8] for the lambda extension $C(1,1;\lambda)$. This requires to rewrite everything in terms of the *nome* [10] variable q and use extensively Jacobi theta functions. This exact expression has been given in equation (98) of [8]:

$$C_-(1,1;\lambda) = \frac{-\theta_2'(u,q)}{\sin(u) \cdot \theta_2(0,q) \cdot \theta_3(0,q)^2} \quad \text{where: } \lambda = \cos(u), \tag{34}$$

where $\theta_2'(u,q)$ denotes the partial derivative of $\theta_2(u,q)$ with respect to u. This exact expression, when rewritten in terms of the t variable, is, at first sight, a *differentially algebraic function*[6]. Let us write (34) as

[5] A D-finite function is a function solution of a linear ODE with polynomial coefficients. A DD-finite function is a function solution of a linear differential equation whose coefficients are D-finite functions [9].

[6] A differentially algebraic function [11] is a function $f(t)$ solution of a polynomial relation $P(t, f(t), f'(t), \cdots f^{(n)}(t)) = 0$, where $f^{(n)}(t)$ denotes the nth derivative of $f(t)$ with respect to t.

$$\frac{f(u)}{\sin(u)\cdot\theta_2(0,q)\cdot\theta_3(0,q)^2} \text{ where: } f(u) = -\theta_2'(u,q), \sin(u) = \left(\frac{M}{4}\right)^{1/2}, \qquad (35)$$

where M is defined by (16), and let us perform the Taylor expansion[7] of $f(u)/\sin(u)$ in M:

$$\frac{f\left(\arcsin((M/4)^{1/2})\right)}{(M/4)^{1/2}}$$
$$= f^{(1)}(0) + \frac{1}{24}\cdot\left(f^{(3)}(0) + f^{(1)}(0)\right)\cdot M + \frac{1}{1920}\cdot\left(f^{(5)}(0) + 10f^{(3)}(0) + 9f^{(1)}(0)\right)\cdot M^2$$
$$+ \frac{1}{322\,560}\cdot\left(f^{(7)}(0) + 35f^{(5)}(0) + 259f^{(3)}(0) + 225\cdot f^{(1)}(0)\right)\cdot M^3 + \cdots \qquad (36)$$

where $f^{(n)}(u)$ denotes the nth derivative[8] of $f(u)$ (with respect to u). From this Taylor expansion (36) one gets the following exact expressions for $g_1(t)$, $g_2(t)$, etc …(and even the first term $g_0(t) = E$):

$$g_0(t) = E = -\frac{\theta_2^{(2)}(0,q)}{\theta_2(0,q)\cdot\theta_3(0,q)^2},$$
$$g_1(t) = -\frac{1}{24}\cdot\frac{\theta_2^{(4)}(0,q) + \theta_2^{(2)}(0,q)}{\theta_2(0,q)\cdot\theta_3(0,q)^2},$$
$$g_2(t) = -\frac{1}{1920}\cdot\frac{\theta_2^{(6)}(0,q) + 10\cdot\theta_2^{(4)}(0,q) + 9\cdot\theta_2^{(2)}(0,q)}{\theta_2(0,q)\cdot\theta_3(0,q)^2},$$
$$g_3(t) = -\frac{1}{322\,560}\cdot\frac{\theta_2^{(8)}(0,q) + 35\cdot\theta_2^{(6)}(0,q) + 259\cdot\theta_2^{(4)}(0,q) + 225\cdot\theta_2^{(2)}(0,q)}{\theta_2(0,q)\cdot\theta_3(0,q)^2},$$
$$g_4(t) = -\frac{1}{92\,897\,280}\cdot\frac{N_4}{\theta_2(0,q)\cdot\theta_3(0,q)^2},$$
$$g_5(t) = -\frac{1}{40\,874\,803\,200}\cdot\frac{N_5}{\theta_2(0,q)\cdot\theta_3(0,q)^2}, \qquad (37)$$

where

$$N_4 = \theta_2^{(10)}(0,q) + 84\cdot\theta_2^{(8)}(0,q) + 1974\cdot\theta_2^{(6)}(0,q)$$
$$+ 12\,916\cdot\theta_2^{(4)}(0,q) + 11\,025\cdot\theta_2^{(2)}(0,q),$$
$$N_5 = \theta_2^{(12)}(0,q) + 165\cdot\theta_2^{(10)}(0,q) + 8778\cdot\theta_2^{(8)}(0,q) + 172\,810\cdot\theta_2^{(6)}(0,q)$$
$$+ 1057\,221\cdot\theta_2^{(4)}(0,q) + 893\,025\cdot\theta_2^{(2)}(0,q), \qquad (38)$$

and where $\theta_2^{(2n)}(u,q)$ denotes the $(2n)$th partial derivative of $\theta_2(u,q)$ with respect to u.

Let us recall that *ratios* of D-finite expressions are *not* (generically[9] …) D-finite: they are *differentially algebraic* [11]. Section (2.2) suggests that the $g_n(t)$'s are *DD-finite* (or DDD-finite, …): the previous expressions (37) of the $g_n(t)$'s as *ratio* of derivatives of theta functions confirms this prejudice. On the other hand, all these $g_n(t)$'s are *globally bounded series* [12] (see (32)), and we have seen, so many times in physics, and in particular the two-dimensional

[7] One has, at first sight, a Puiseux series in $M^{1/2}$ but all the coefficients for $M^{-1/2}, M^{1/2}, M^{3/2}, \ldots$ are here equal to zero because all the even derivative $f^{(2n)}(0)$ are equal to zero.

[8] Note, in this Taylor series (36), that the terms corresponding to even derivatives $f(0), f^{(2)}(0), \ldots, f^{(2n)}(0)$, are identically zero, since the odd derivatives of $\theta_2(u,q)$ with respect to u vanish: $\theta_2^{(2n+1)}(u,q)) = 0$.

[9] The denominator must not be an algebraic function.

Ising model, the emergence of globally bounded series as a consequence of the frequent occurrence of *diagonals of rational functions* [12–15] (or n-fold integrals [10, 16–21]). This may suggest, on the contrary, that the $g_n(t)$'s could be D-finite.

3.1. Expansions of the $g_n(t)$'s in the t variable

From the previous exact expressions (37) in terms of theta functions, one can obtain the series expansions of the $g_n(t)$'s in the t variable and try to see if these $g_n(t)$'s are solutions of linear differential operators.

From these expansions (37), rewritten in t, one can get large enough series in t to see that $g_1(t)$ is in fact solution of an order-six linear differential operator L_6 which is actually the direct sum (LCLM) of an order-four linear differential operator L_4 and of the order-two linear differential operator L_E having $E = {}_2F_1([\frac{1}{2}, -\frac{1}{2}], [1], t)$ as a solution. Furthermore one finds that this order-four linear differential operator L_4 is homomorphic to the *symmetric third power* of this order-two linear differential operator L_E, with an intertwiner reading:

$$\frac{3}{8}R_1 = (t-1)\cdot t^3\cdot D_t^3 + \frac{3}{2}\cdot(t-1)\cdot t^2\cdot D_t^2 - \frac{1}{4}\cdot(3t+1)\cdot t\cdot D_t + \frac{3}{8}\cdot\frac{t^2+1}{t-1}. \tag{39}$$

One finally finds that the series expansion (32) is exactly the linear combination of E and the order-three linear differential operator (39) acting on E^3:

$$g_1(t) = \frac{1}{24}\cdot E + \frac{1}{24}\cdot R_1(E^3) = \frac{1}{24}\cdot E - \frac{1}{8}\cdot KE^2 - \frac{t-1}{12}\cdot K^3. \tag{40}$$

Similar calculations can be performed for $g_2(t)$. The series $g_2(t)$ can also be seen to be D-finite, being solution of an order-*twelve* linear differential operator which turns out to be the direct-sum (LCLM) of the previous order-two linear differential operator L_E, of the previous order-four L_4, and of an order-six linear differential operator homomorphic to the *symmetric fifth power* of L_E with the following order-five intertwiner:

$$\begin{aligned} -\frac{5}{8}R_2 = &\frac{4}{3}\cdot(t-1)\cdot(t-2)\cdot t^5\cdot D_t^5 + \frac{5}{2}\cdot(t-1)\cdot(4t-9)\cdot t^4\cdot D_t^4 \\ &+ 5\cdot(2t-3)\cdot(t-3)\cdot t^3\cdot D_t^3 - \frac{5}{24}\frac{24t^3-122t^2+59t+103}{t-1}\cdot t^2\cdot D_t^2 \\ &+ \frac{1}{24}\cdot\frac{90t^4-488t^3-7t^2+774t-1}{(t-1)^2}\cdot t\cdot D_t \\ &- \frac{5}{96}\cdot\frac{36t^5-205t^4-59t^3+409t^2+23t-12}{(t-1)^3}. \end{aligned} \tag{41}$$

One finally finds that the series expansion for $g_2(t)$ is exactly the linear combination of E, of the order-three linear differential operator (39) acting on E^3, and of the order-five linear differential operator (41) acting on E^5:

$$\begin{aligned} g_2(t) &= \frac{3}{640}\cdot E + \frac{1}{192}\cdot R_1(E^3) + \frac{1}{1920}\cdot R_2(E^5) \\ &= \frac{3}{640}\cdot E - \frac{1}{64}\cdot E^2K - \frac{t-1}{96}\cdot K^3 \\ &\quad + \frac{1}{128}\cdot E^3K^2 + \frac{t-1}{64}\cdot EK^4 + \frac{(t-1)(t-2)}{240}\cdot K^5. \end{aligned} \tag{42}$$

Similar calculations can be performed for $g_3(t)$. They are displayed in appendix A.

Remark 1. All these $R_1(E^3), R_2(E^5), \ldots$ which are *homogeneous* polynomials in the complete elliptic integrals E and K, *can be directly expressed in terms of simple ratios of theta functions*:

$$R_1(E^3) = -\frac{\theta_2^{(4)}(0,q)}{\theta_2(0,q)\cdot\theta_3(0,q)^2}, \; R_2(E^5) = -\frac{\theta_2^{(6)}(0,q)}{\theta_2(0,q)\cdot\theta_3(0,q)^2},$$
$$R_3(E^7) = -\frac{\theta_2^{(8)}(0,q)}{\theta_2(0,q)\cdot\theta_3(0,q)^2}, \cdots \tag{43}$$

where $\theta_2^{(n)}(u,q)$ denote the nth derivative of $\theta_2(u,q)$ with respect to u.

One can conjecture the following expression for (11):

$$\begin{aligned}
C_M(1,1;M) &= E + M\cdot(c_1^{(1)}\cdot E + c_2^{(1)}\cdot R_1(E^3)) \\
&\quad + M^2\cdot(c_1^{(2)}\cdot E + c_2^{(2)}\cdot R_1(E^3) + c_3^{(2)}\cdot R_2(E^5)) \\
&\quad + M^3\cdot(c_1^{(3)}\cdot E + c_2^{(3)}\cdot R_1(E^3) + c_3^{(3)}\cdot R_2(E^5) + c_4^{(3)}\cdot R_3(E^7)) + \cdots \\
&= (1 + c_1^{(1)}\cdot M + c_1^{(2)}\cdot M^2 + c_1^{(3)}\cdot M^3 + \cdots)\cdot E \\
&\quad + (c_2^{(1)}\cdot M + c_2^{(2)}\cdot M^2 + c_2^{(3)}\cdot M^3 + \cdots)\cdot R_1(E^3) \\
&\quad + (c_3^{(2)}\cdot M^2 + c_3^{(3)}\cdot M^3 + \cdots)\cdot R_2(E^5) + (c_4^{(3)}\cdot M^3 + \cdots)\cdot R_3(E^7) \\
&\quad + \cdots
\end{aligned} \tag{44}$$

where the $c_i^{(j)}$'s are constants obtained from equations (37) and (43) (see (40) and (A.3)). One can encapsulate these results in the following closed formula, deduced from (34) and its Taylor expansion (see also (35)):

$$\begin{aligned}
C_M(1,1;M) &= -\frac{2}{\sqrt{M}}\cdot\frac{\theta_2'\left(\arcsin\frac{\sqrt{M}}{2}, q\right)}{\theta_2(0,q)\cdot\theta_3(0,q)^2} \\
&= -\frac{2}{\sqrt{M}}\cdot\sum_{p=0}^{\infty}\left(\arcsin\frac{\sqrt{M}}{2}\right)^{(2p+1)}\cdot\frac{\theta_2^{(2p+2)}(0,q)}{\theta_2(0,q)\cdot\theta_3(0,q)^2\cdot(2p+1)!} \\
&= \frac{2}{\sqrt{M}}\cdot\sum_{p=0}^{\infty}\left(\arcsin\frac{\sqrt{M}}{2}\right)^{(2p+1)}\cdot\frac{R_p(E^{(2p+1)})}{(2p+1)!}.
\end{aligned} \tag{45}$$

4. Lambda-extensions and globally bounded series

Let us consider the series expansion (11) for values of the parameter $M \neq 0$ not yielding the previous algebraic function series (i.e. $M \neq 4\cdot\sin^2(\pi m/n)$ where m and n are integers). These series are[10] *differentially algebraic* [11]: is it possible that such series could be D-finite for selected values of M?

Let us change t into $16\,t$ in the series expansion (11). One gets the following expansion:

$$\begin{aligned}
&1 - 4t - (12+3M)\cdot t^2 - (80+36M)\cdot t^3 - (700+441M)\cdot t^4 \\
&\quad - (7056+5628M)\cdot t^5 - (77\,616+74\,248M-5M^2)\cdot t^6 \\
&\quad - (906\,048+1004\,960M-220M^2)\cdot t^7
\end{aligned}$$

[10] They are solutions of a non-linear ODE, the sigma-form of Painlevé VI.

$$
\begin{aligned}
&- (11\,042\,460 + 13\,877\,397M - 6255M^2) \cdot t^8 \\
&- (139\,053\,200 + 194\,712\,812M - 146\,500M^2) \cdot t^9 \\
&- (1796\,567\,344 + 2767\,635\,832M - 3079\,025M^2) \cdot t^{10} + \cdots . \qquad (46)
\end{aligned}
$$

One sees immediately that this (generically) *differentially algebraic* series provides, *for any integer M, an infinite number of series with integer coefficients*. In fact one can see that the series expansion (11) (or the series expansion (46)) is a *globally bounded series*[11] when M is *any rational number*. One thus obtains the quite puzzling result that an *infinite number of* (at first sight ...) *differentially algebraic series* can be *globally bounded series*.

Quite often we see the emergence of *globally bounded series* [12] as solutions of D-finite linear differential operators, and more specifically as *diagonals of rational functions* [12–15] (this is related to the so-called Christol's conjecture [22]). Along this line it is tempting to imagine that such *globally bounded* situation could correspond to cases where the globally bounded series are in fact D-finite. If this is not the case, it will thus be tempting to imagine that such *globally bounded* situation could correspond to particular ratio of D-finite functions, namely ratio of diagonals of rational functions (or even rational functions of diagonals).

4.1. The $M = 5$ case

Let us restrict to simple integer values of M and see whether the corresponding globally bounded series (11) are D-finite.

Let us consider an integer M different from $M = 0$ (the D-finite solution $C(1,1)$), and different from $M = 1, 2, 3, 4$, which correspond to algebraic functions. For simplicity we will consider the integer coefficient series (46) for $M = 5$. The $M = 5$ series (46) reads:

$$
\begin{aligned}
&1 - 4t - 27t^2 - 260\,t^3 - 2905\,t^4 - 35\,196\,t^5 - 448\,731\,t^6 - 5925\,348\,t^7 \\
&- 80\,273\,070\,t^8 - 1108\,954\,760\,t^9 - 15\,557\,770\,879\,t^{10} - 220\,998\,916\,404\,t^{11} \\
&- 3171\,743\,667\,652\,t^{12} - 45\,915\,042\,520\,880\,t^{13} + \cdots . \qquad (47)
\end{aligned}
$$

One finds that this series (47) does not seem to be D-finite: one does not find any linear differential operator even with a thousand coefficients. Let us recall the strategy we have used in [11]: we study the series with integer coefficients modulo small increasing primes $p = 3, 5, 7, 11, 13, \cdots$ and seek for the linear differential operator annihilating these series modulo such a prime.

For the prime $p = 3$ the series (47) mod. 3 is solution of an order-one linear differential operator (of degree one in t):

$$2t + (t+2) \cdot tD_t. \qquad (48)$$

For the prime $p = 5$ the series (47) mod. 5 is solution of an order-one linear differential operator (of degree two in t):

$$3 \cdot t \cdot (t+1) + (t^2 + 2t + 2) \cdot tD_t. \qquad (49)$$

These series mod. 3 or 5, are not only D-finite, they are in fact *algebraic series* mod. 3 or 5:

$$p = 3, F^{16} + 2 \cdot (t^2 + t + 1) \cdot F^8 + (t+2) \cdot t^6 = 0, \qquad (50)$$

$$p = 5, (t^2 + 4t + 1)^5 \cdot F^4 + 4 \cdot (t+3)^4 \cdot (t+4)^4 = 0. \qquad (51)$$

[11] A series with rational coefficients and non-zero radius of convergence is a globally bounded series [12] if it can be recast into a series with integer coefficients with one rescaling $t \rightarrow Nt$ where N is an integer.

For the prime $p = 7$ the series (47) mod. 7 is solution of an order-three linear differential operator (of degree three in t):

$$2 \cdot t \cdot (t+2) + (9t^3 + 13t^2 + 4t + 9) \cdot tD_t + (5t^3 + 16t^2 + 9t + 19) \cdot t^2 D_t^2 \\ + (t^3 + 4t^2 + 3t + 6) \cdot t^3 D_t^3. \tag{52}$$

This mod. 7 series is also algebraic, but finding the corresponding characteristic polynomial equation (like (52) previously) requires more than one thousand coefficients.

For the next primes we get more and more involved linear differential operators of increasing orders and degrees of the polynomials in t. One finds for the prime $p = 11$ an order 5 and a degree in t also equal to five, and one gets for the following primes

$$\begin{aligned} &p = 13, \text{order} = \text{degree} = 6, \quad && p = 17, \text{order} = \text{degree} = 8, \\ &p = 19, \text{order} = \text{degree} = 9, \quad && p = 23, \text{order} = \text{degree} = 11, \\ &p = 29, \text{order} = \text{degree} = 14, \quad && p = 31, \text{order} = \text{degree} = 15, \\ &p = 37, \text{order} = \text{degree} = 18, \quad && p = 41, \text{order} = \text{degree} = 20, \cdots. \end{aligned}$$

An inspection of the corresponding linear differential operators strongly suggests that the orders and degrees of the polynomials in t of the linear differential operator grow (linearly) with the prime p according to the formula:

$$\text{order} = \text{degree} = \frac{p-1}{2}. \tag{53}$$

These results have to be compared with the same mod. prime calculations for the D-finite (possibly algebraic) series (46) for $M = 0, 1, 2, 3, 4$. In that case, since there is a linear differential operator (in characteristic zero), the series modulo a prime is solution of the mod. prime reduction of that linear differential operator, however for small primes the series modulo a prime can be solution of a linear differential operator of smaller order (order one, ...). Therefore the previous analysis modulo increasing primes provides linear differential operators of increasing orders, but very quickly saturating to the order of the linear differential operator in characteristic zero.

These calculations, thus, strongly suggest that the integer coefficient series (47) is *not D-finite* but is *only differentially algebraic*.

Similar calculations can be performed for any integer $M \geqslant 5$ (or any integer $M \leqslant -1$) with similar results. Similar calculations can be performed *for any rational number M* with similar results ruling out D-finiteness. Let us display miscellaneous algebraic equation for the series for various M and modulo various primes:

$$\begin{aligned} &M = 6, \quad p = 3, \qquad (t^3+1) \cdot F^2 + 2 \cdot (t^2 + t + 1) = 0, \\ &M = 7, \quad p = 3, \qquad F^4 + (t+2) = 0, \\ &M = 7, \quad p = 7, \qquad (t+1)^7 \cdot (t+3)^7 \cdot (t+5)^7 \cdot F^6 \\ &\qquad\qquad\qquad\qquad + 6 \cdot (t+6)^6 \cdot (t^2 + 2t + 5)^6 = 0, \\ &M = 11, \quad p = 3, \qquad F^{16} + 2 \cdot (t^2 + t + 1) \cdot F^8 + (t+2) \cdot t^6 = 0. \end{aligned}$$

All these calculations suggest that the infinite number of integer coefficient series (46), for any integer $M \geqslant 5$ (or any integer $M \leqslant -1$), are *not* D-finite, as well as the infinite number of globally bounded series (11) or (46) when M is any rational number, thus providing an *infinite set of globally bounded differentially algebraic series* (far beyond the D-finite diagonals of rational functions [12–15] providing so many globally bounded series, see Christol's conjecture [22]).

The question to see whether these globally bounded series could be ratio of particular D-finite functions, namely ratio of diagonals of rational functions[12] remains open.

Remark 2. Finding that a series is actually the ratio of particular D-finite functions can be a difficult task, possibly some tour-de-force, requiring a lot of (guessing) intuition. Conversely, there are very few papers, in the literature, addressing the question of ruling out the possibility that a series can be the ratio of D-finite functions, or even ruling out the possibility that a series can be DD-finite [9]. Here we have a prejudice that the series (46) for integer values $M \geqslant 5$ are not ratio of diagonals of rational functions, but we are not able to prove such a no-go result, even for specific integer vales of M.

5. Other one-parameter deformations: deformations of algebraic functions

The 'form factor' expansion (5) (see (9) in [1]) amounts to seeing the lambda-extension of the correlation function $C_{-}(N,N;\lambda)$ as a deformation of the algebraic solution $(1-t)^{1/4}$. With section (2.1) we have seen that there are many other (algebraic) values of the parameter λ for which the lambda-extension $C_{-}(N,N;\lambda)$ becomes an *algebraic function* [1]. Let us consider 'form factor' expansions [1] similar to (6), but corresponding to seeing the lambda-extension as a deformation around these other algebraic functions (see (22), (25) and (29)).

Recalling the exact expressions of the $g_n(t)$'s in terms of theta functions displayed in (37) and (38), it is worth noticing that similar expressions can also be obtained for the form factors $f_{1,1}^{(2n)}$. One gets respectively (with[13] $f_{1,1}^{(0)} = 1$):

$$\begin{aligned}
(1-t)^{1/4}\cdot f_{1,1}^{(0)} &= \frac{\theta_1^{(1)}(0;q)}{\theta_2(0,q)\cdot\theta_3(0,q)^2},\\
(1-t)^{1/4}\cdot f_{1,1}^{(2)} &= \frac{1}{2}\cdot\frac{\theta_1^{(3)}(0,q)+\theta_1^{(1)}(0,q)}{\theta_2(0,q)\cdot\theta_3(0,q)^2},\\
(1-t)^{1/4}\cdot f_{1,1}^{(4)} &= \frac{1}{24}\cdot\frac{\theta_1^{(5)}(0,q)+10\cdot\theta_1^{(3)}(0,q)+9\cdot\theta_1^{(1)}(0,q)}{\theta_2(0,q)\cdot\theta_3(0,q)^2},\\
(1-t)^{1/4}\cdot f_{1,1}^{(6)} &= \frac{1}{720}\cdot\frac{\theta_1^{(7)}(0,q)+35\cdot\theta_1^{(5)}(0,q)+259\cdot\theta_1^{(3)}(0,q)+225\cdot\theta_1^{(1)}(0,q)}{\theta_2(0,q)\cdot\theta_3(0,q)^2},\\
(1-t)^{1/4}\cdot f_{1,1}^{(8)} &= \frac{1}{40\,320}\cdot\frac{\mathcal{N}_9}{\theta_2(0,q)\cdot\theta_3(0,q)^2},\\
(1-t)^{1/4}\cdot f_{1,1}^{(10)} &= \frac{1}{3628\,800}\cdot\frac{\mathcal{N}_{11}}{\theta_2(0,q)\cdot\theta_3(0,q)^2},\\
(1-t)^{1/4}\cdot f_{1,1}^{(12)} &= \frac{1}{479\,001\,600}\cdot\frac{\mathcal{N}_{13}}{\theta_2(0,q)\cdot\theta_3(0,q)^2},\cdots
\end{aligned} \tag{54}$$

[12] Or more generally rational functions of diagonals of rational functions.

[13] Note that $(1-t)^{1/4} = \theta_4(0,q)/\theta_3(0,q)$with $\theta_1'(0,q) = \theta_2(0,q)\theta_3(0,q)\theta_4(0,q)$.

where

$$\begin{aligned}
\mathcal{N}_9 \;&= \theta_1^{(9)}(0,q) + 84 \cdot \theta_1^{(7)}(0,q) + 1974 \cdot \theta_1^{(5)}(0,q) \\
&\quad + 12\,916 \cdot \theta_1^{(3)}(0,q) + 11\,025 \cdot \theta_1^{(1)}(0,q), \\
\mathcal{N}_{11} &= \theta_1^{(11)}(0,q) + 165 \cdot \theta_1^{(9)}(0,q) + 8778 \cdot \theta_1^{(7)}(0,q) + 172\,810 \cdot \theta_1^{(5)}(0,q) \\
&\quad + 1057\,221 \cdot \theta_1^{(3)}(0,q) + 893\,025 \cdot \theta_1^{(1)}(0,q), \\
\mathcal{N}_{13} &= \theta_1^{(13)}(0,q) + 286 \cdot \theta_1^{(11)}(0,q) + 28\,743 \cdot \theta_1^{(9)}(0,q) + 1234\,948 \cdot \theta_1^{(7)}(0,q) \\
&\quad + 21\,967\,231 \cdot \theta_1^{(5)}(0,q) + 128\,816\,766 \cdot \theta_1^{(3)}(0,q) + 108\,056\,025 \cdot \theta_1^{(1)}(0,q),
\end{aligned}$$

and where $\theta_1^{(2n+1)}(u,q)$ denotes the $(2n+1)$th partial derivative of the Jacobi theta function $\theta_1(u,q)$ with respect to u. Let us remark that these terms can be obtained similarly to (37) and (38), using now the expansion of $f(\arccos(\lambda))/\sqrt{1-\lambda^2}$ around $\lambda = 0$, which corresponds to $u = \pi/2$, and, then, use $\theta_2^{\rm odd}(\pi/2,q) = -\theta_1^{\rm odd}(0,q)$ and $\theta_2^{\rm even}(\pi/2,q) = 0$.

Remark 1. Similarly to (45) one can encapsulate the previous results in the following closed formula, deduced from (34) and its Taylor expansion:

$$\begin{aligned}
C_-(1,1;\lambda) &= -\frac{\theta_2'(\arccos\lambda, q)}{\sqrt{1-\lambda^2}\cdot\theta_2(0,q)\cdot\theta_3(0,q)^2} \\
&= \frac{1}{\sqrt{1-\lambda^2}} \cdot \sum_{p=0}^{\infty} (\arcsin\lambda)^{(2p)} \cdot \frac{\theta_1^{(2p+1)}(0,q)}{\theta_2(0,q)\cdot\theta_3(0,q)^2\cdot(2p)!}.
\end{aligned} \tag{55}$$

Remark 2. Introducing ratios of theta functions $S^{(2n+1)}$ by:

$$S^{(2n+1)} = \frac{\theta_1^{(2n+1)}(0,q)}{\theta_1^{(1)}(0,q)}, \tag{56}$$

and the quantities $\kappa^{(2n+1)}$'s related to the form factors $f_{1,1}^{(2n)}$'s introduced in (6):

$$f_{1,1}^{(2n)} = (2n+1)\cdot\kappa^{(2n+1)}, \tag{57}$$

one can deduce, from the previous relations (54), the expression of the $S^{(2n+1)}$'s in terms of these $\kappa^{(2n+1)}$'s:

$$\begin{aligned}
&S^{(1)} = \kappa^{(1)}, \\
&\frac{S^{(3)}}{3!} = \kappa^{(3)} - \frac{1}{6}\cdot\kappa^{(1)}, \\
&\frac{S^{(5)}}{5!} = \kappa^{(5)} - \frac{1}{2}\cdot\kappa^{(3)} + \frac{1}{120}\cdot\kappa^{(1)}, \\
&\frac{S^{(7)}}{7!} = \kappa^{(7)} - \frac{5}{6}\cdot\kappa^{(5)} + \frac{13}{120}\cdot\kappa^{(3)} - \frac{1}{5040}\cdot\kappa^{(1)}, \\
&\frac{S^{(9)}}{9!} = \kappa^{(9)} - \frac{7}{6}\cdot\kappa^{(7)} + \frac{23}{72}\cdot\kappa^{(5)} - \frac{41}{3024}\cdot\kappa^{(3)} + \frac{1}{362\,880}\cdot\kappa^{(1)}, \cdots.
\end{aligned} \tag{58}$$

The coefficients in these linear combinations (58) correspond exactly to the linear combinations we had to introduce for the (n-fold integrals) $\tilde{\chi}^{(2n+1)}$'s in the analysis of the susceptibility of the square Ising model, see for instance equation (8) in [23], *but in the high temperature regime*:

$$\Phi^{(5)} = \tilde{\chi}^{(5)} - \frac{1}{2} \cdot \tilde{\chi}^{(3)} + \frac{1}{120} \cdot \tilde{\chi}^{(1)}. \tag{59}$$

Along these lines we give, in B, a Taylor expansion similar to (55) but for the lambda extension of $C(0,0,\lambda)$, instead of $C(1,1,\lambda)$ in (55). From these expansions one deduces linear combinations (B.3) (similar to (58)), corresponding exactly to the linear combinations we had to introduce for the (n-fold integrals) $\tilde{\chi}^{(2n)}$'s in the analysis of the susceptibility of the square Ising model, see for instance Equation (26) in [23], *in the low temperature regime*:

$$\Phi^{(6)} = \tilde{\chi}^{(6)} - \frac{2}{3} \cdot \tilde{\chi}^{(4)} + \frac{2}{45} \cdot \tilde{\chi}^{(2)}. \tag{60}$$

5.1. Other one-parameter deformations: deformation of $M = 2$ (i.e. $u = \pi/4$)

Recalling that one finds that (11) is actually, for $M = 2$, the series expansion of an algebraic function (22), one can try to write the series (11) as a deformation of this $M = 2$ algebraic function (22):

$$C_\rho(1,1;\rho) = G_0(t) + \rho \cdot G_1(t) + \rho^2 \cdot G_2(t) + \cdots \tag{61}$$

where

$$G_0(t) = (1-t)^{1/16} \cdot \left(\frac{1+(1-t)^{1/2}}{2}\right)^{3/4} \tag{62}$$

$$= 1 - \frac{1}{4} \cdot t - \frac{9}{128} \cdot t^2 - \frac{19}{512} \cdot t^3 - \frac{791}{32768} \cdot t^4 - \frac{2289}{131072} \cdot t^5 - \frac{56523}{4194304} \cdot t^6 - \frac{182193}{16777216} \cdot t^7 + \cdots \tag{63}$$

and where $\rho = M - 2$.

Let us introduce

$$G_0(t) = -\sqrt{2} \cdot \frac{\theta_2^{(1)}(\pi/4, q)}{\theta_2(0,q) \cdot \theta_3(0,q)^2}, \tag{64}$$

which actually coincides with the algebraic expression (62). Let us also introduce the S_n's defined as

$$S_n = \frac{\theta_2^{(n)}(\pi/4, q)}{\theta_2'(\pi/4, q)}, \tag{65}$$

where $\theta_2^{(n)}(u,q)$ denotes the nth partial derivative with respect to u of $\theta_2(u,q)$. Similarly to (45) one can write (61) as

$$C_\rho(1,1;\rho) = \frac{\sqrt{2} \cdot G_0(t)}{\sqrt{\rho+2}} \cdot \sum_{p=0}^{\infty} \left(\arcsin\left(\frac{\sqrt{\rho+2}}{2}\right) - \frac{\pi}{4}\right)^{(p-1)} \cdot \frac{S_p}{(p-1)!}. \tag{66}$$

Again one can ask whether the $G_n(t)$'s in (61) are D-finite, and, again, polynomials in the complete elliptic integrals E and K. One can find that (61), or (66), can be written as

$$\begin{aligned}
\frac{C_\rho(1,1;\rho)}{G_0(t)} &= 1+\rho\cdot\Big(\frac{1}{4}\cdot S_2-\frac{1}{4}\Big)+\rho^2\cdot\Big(\frac{1}{32}\cdot S_3-\frac{1}{16}\cdot S_2+\frac{3}{32}\Big)\\
&+\rho^3\cdot\Big(\frac{1}{384}\cdot S_4-\frac{1}{128}\cdot S_3+\frac{13}{384}\cdot S_2-\frac{5}{128}\Big)\\
&+\rho^4\cdot\Big(\frac{1}{6144}\cdot S_5-\frac{1}{1536}\cdot S_4+\frac{17}{3072}\cdot S_3-\frac{19}{1536}\cdot S_2+\frac{35}{2048}\Big)\\
&+\rho^5\cdot\Big(\frac{1}{122\,880}\cdot S_6-\frac{1}{24\,576}\cdot S_5+\frac{7}{12\,288}\cdot S_4-\frac{23}{12\,288}\cdot S_3\\
&+\frac{263}{40\,960}\cdot S_2-\frac{63}{8192}\Big)+\cdots
\end{aligned}\tag{67}$$

where the S_n's are defined by (65). It is crucial to note that *all these ratio* (65) *are actually polynomial expressions* in the complete elliptic integrals E and K. The first S_n's read:

$$\begin{aligned}
S_2 &= \frac{2}{t}\cdot\Big(1-(1-t)^{1/2}\Big)\cdot E-\frac{1}{2t}\cdot\Big((t-4)\cdot(1-t)^{1/2}-(3t-4)\Big)\cdot K,\\
S_3 &= \frac{1}{4}\cdot\Big(6\cdot(1-t)^{1/2}-(t-2)\Big)\cdot K^2-3EK,\\
S_4 &= \frac{3}{t}\cdot\Big((t-4)\cdot(1-t)^{1/2}-(3t-4)\Big)\cdot EK^2-\frac{6}{t}\cdot(1-(1-t)^{1/2})\cdot E^2K\\
&\quad+\frac{1}{8t}\cdot\Big((t^2-28t+48)\cdot(1-t)^{1/2}-(21t^2-68t+48)\Big)\cdot K^3,\\
S_5 &= 15E^2K^2-\frac{5}{2}\cdot\Big(6\cdot(1-t)^{1/2}-(t-2)\Big)\cdot EK^3\\
&\quad-\frac{1}{16}\cdot\Big(60\cdot(t-2)\cdot(1-t)^{1/2}-(t^2+24t-24)\Big)\cdot K^4,\\
S_6 &= -\frac{1}{32t}\cdot\Big((t^3-168t^2+944t-960)\cdot(1-t)^{1/2}-(183t^3-1160t^2+1936t-960)\Big)\cdot K^5\\
&\quad-\frac{15}{8t}\cdot\Big((t^2-28t+48)\cdot(1-t)^{1/2}-(21t^2-68t+48)\Big)\cdot EK^4\\
&\quad-\frac{45}{2t}\cdot\Big((t-4)\cdot(1-t)^{1/2}-(3t-4)\Big)\cdot E^2K^3\\
&\quad+\frac{30}{t}\cdot\Big(1-(1-t)^{1/2}\Big)\cdot E^3K^2.
\end{aligned}\tag{68}$$

Let us note that these selected ratio of theta functions (65) are not only polynomials in E and K, but *homogeneous polynomials* in E and K. The $G_n(t)$'s will be D-finite, and again polynomials in E and K, as a consequence of the fact that the S_n's are polynomial expressions of E and K.

The expansion of $G_1(t)$ reads:

$$G_1(t) = -\frac{3}{256}t^2 - \frac{9}{1024}t^3 - \frac{441}{65\,536}t^4 - \frac{1407}{262\,144}t^5 - \frac{18\,557}{4194\,304}t^6 - \frac{62\,755}{16\,777\,216}t^7 - \frac{13\,852\,377}{4294\,967\,296}t^8 - \frac{48\,531\,703}{17\,179\,869\,184}t^9 + \cdots. \quad (69)$$

The first $G_1(t)$ reads

$$G_1(t) = G_0(t)\cdot\tilde{G}_1(t) \qquad \text{where:}$$

$$\tilde{G}_1(t) = \frac{1}{4}\cdot S_2 - \frac{1}{4} = -\frac{1}{4} + \left(\frac{1-(1-t)^{1/2}}{2t}\right)\cdot E - \frac{(t-4)\cdot(1-t)^{1/2} - (3t-4)}{8t}\cdot K \quad (70)$$

$$= -\frac{1}{4} + \frac{E}{2t} + \frac{(3t-4)}{8t}\cdot K - (1-t)^{1/2}\cdot\left(\frac{E}{2t} + \frac{(t-4)}{8t}\cdot K\right)$$

$$= -\frac{3}{256}\cdot t^2 - \frac{3}{256}\cdot t^3 - \frac{687}{65\,536}\cdot t^4 - \frac{303}{32\,768}\cdot t^5 - \frac{34\,355}{4194\,304}\cdot t^6 - \frac{30\,681}{4194\,304}\cdot t^7 - \frac{28\,298\,151}{4294\,967\,296}\cdot t^8 - \frac{6422\,951}{1073\,741\,824}\cdot t^9 + \cdots \quad (71)$$

and the next two read

$$G_2(t) = G_0(t)\cdot\tilde{G}_2(t) = \frac{5}{16\,777\,216}\cdot t^6 + \frac{55}{67\,108\,864}\cdot t^7 + \frac{6255}{4294\,967\,296}\cdot t^8 + \frac{36\,625}{17\,179\,869\,184}\cdot t^9 + \frac{3079\,025}{1099\,511\,627\,776}\cdot t^{10} + \frac{15\,116\,115}{4398\,046\,511\,104}\cdot t^{11} + \cdots \quad (72)$$

where

$$\tilde{G}_2(t) = \frac{1}{32}\cdot S_3 - \frac{1}{16}\cdot S_2 + \frac{3}{32} = \frac{3}{32} - \frac{1-(1-t)^{1/2}}{8t}\cdot E - \frac{(t-4)\cdot(1-t)^{1/2} - (3t-4)}{32t}\cdot K - \frac{3}{32}\cdot EK + \frac{6\cdot(1-t)^{1/2} - (t-2)}{128t}\cdot K^2$$

$$= \frac{5}{67\,108\,864}\cdot t^6 + \frac{15}{67\,108\,864}\cdot t^7 + \frac{7305}{17\,179\,869\,184}\cdot t^8 + \frac{2825}{4294\,967\,296}\cdot t^9 + \frac{3978\,105}{4398\,046\,511\,104}\cdot t^{10} + \frac{5075\,805}{4398\,046\,511\,104}\cdot t^{11} + \frac{1575\,278\,229}{1125\,899\,906\,842\,624}\cdot t^{12} + \cdots$$

and

$$G_3(t) = G_0(t)\cdot\tilde{G}_3(t) = -\frac{7}{281\,474\,976\,710\,656}\cdot t^{12} - \frac{161}{1125\,899\,906\,842\,624}\cdot t^{13} - \frac{33\,789}{72\,057\,594\,037\,927\,936}\cdot t^{14} - \frac{332\,703}{288\,230\,376\,151\,711\,744}\cdot t^{15} - \frac{43\,793\,127}{18\,446\,744\,073\,709\,551\,616}\cdot t^{16} - \frac{318\,184\,713}{73\,786\,976\,294\,838\,206\,464}\cdot t^{17} + \cdots \quad (73)$$

where:

$$\begin{aligned}
\tilde{G}_3(t) &= \frac{1}{384}\cdot S_4 - \frac{1}{128}\cdot S_3 + \frac{13}{384}\cdot S_2 - \frac{5}{128} \\
&= -\frac{5}{128} + \frac{13}{192}\cdot\frac{1-(1-t)^{1/2}}{t}\cdot E \\
&\quad - \frac{13}{768}\cdot\frac{(t-4)\cdot(1-t)^{1/2}-(3t-4)}{t}\cdot K + \frac{3}{128}\cdot EK - \frac{6\cdot(1-t)^{1/2}\ -(t-2)}{512}\cdot K^2 \\
&\quad - \frac{1-(1-t)^{1/2}}{64t}\cdot E^2K + \frac{(t-4)\cdot(1-t)^{1/2}-(3t-4)}{128t}\cdot EK^2 \\
&\quad + \frac{(1-t)^{1/2}\cdot(t^2-28t+48)-(21t^2-68t+48)}{3072t}\cdot K^3 \\
&= -\frac{7}{281\,474\,976\,710\,656}\cdot t^{12} - \frac{21}{140\,737\,488\,355\,328}\cdot t^{13} \\
&\quad - \frac{36\,603}{72\,057\,594\,037\,927\,936}\cdot t^{14} - \frac{93\,149}{72\,057\,594\,037\,927\,936}\cdot t^{15} + \cdots. \qquad (74)
\end{aligned}$$

We have obtained similar results for the next $G_n(t)$'s, namely polynomial expressions in E and K with algebraic function coefficients.

Similar results can be obtained for the other values $\lambda = \cos(\pi m/n)$ (m and n integers) yielding algebraic functions for the lambda-extension $C(1,1;\lambda)$. Again, the (form-factor-like) expansion (61) around each of these algebraic functions can be written in a similar way as (68) in terms of the corresponding ratio of theta functions

$$S_n = \frac{\theta_2^{(n)}(\pi m/n,q)}{\theta_2'(\pi m/n,q)}, \qquad (75)$$

where $\theta_2^{(n)}(u,q)$ denotes the nth partial derivative with respect to u of $\theta_2(u,q)$. It becomes much more difficult to see whether these new S_n's are actually polynomial expressions in E and K with more and more involved algebraic coefficients. One finds that these new S_n's are D-finite, but the reduction to polynomial expressions in E and K becomes a difficult task, in general. Let us display a few examples.

5.2. *Other one-parameter deformations: deformation of $u = \pi/6$*

For $u = \pi/6$ we find that the corresponding S_2

$$\frac{1}{\sqrt{3}}\cdot S_2 = \frac{1}{\sqrt{3}}\cdot\frac{\theta_2^{(2)}(\pi/6,q)}{\theta_2'(\pi/6,q)} = 1 - \frac{3}{128}\cdot t^2 - \frac{3}{128}\cdot t^3 - \frac{339}{16\,384}\cdot t^4 + \cdots \qquad (76)$$

is solution of an order-*eight* linear differential operator which is the LCLM (direct-sum) of two order-four linear differential operators L_4 and M_4. The first order-four linear differential operator L_4 is the *symmetric product*[14] of the two order-two linear differential operators

[14] This paper belonging to the symbolic computation literature and not pure mathematics, we use the standard Maple (DEtools) terminology of symmetric powers and symmetric products of linear differential operators [24]. Note that "symmetric product" is not a proper mathematical name for this construction on the solution space; it is a homomorphic

$$D_t^2 + \frac{1}{3} \cdot \frac{10t^3 - 15t^2 + 9t - 2}{(t^2 - t + 1)t(t-1)} \cdot D_t + \frac{1}{12} \cdot \frac{11t^6 - 33t^5 + 47t^4 - 39t^3 + 3t^2 + 11t - 5}{t^2(t-1)^2(t^2 - t + 1)^2},$$

$$D_t^2 + \frac{1}{4} \cdot \frac{t^6 - 3t^5 + 15t^4 - 25t^3 + 15t^2 - 3t + 1}{t^2(t-1)^2(t^2 - t + 1)^2}, \tag{77}$$

having, respectively, the two hypergeometric solutions:

$$t^{5/6} \cdot (1-t)^{5/6} \cdot (t^2 - t + 1)^{-1/2} \cdot {}_2F_1\left(\left[\frac{7}{6}, \frac{5}{2}\right], \left[\frac{7}{3}\right], t\right), \tag{78}$$

$$t^{1/2} \cdot (1-t)^{1/2} \cdot (t^2 - t + 1)^{-1/4} \cdot {}_2F_1\left(\left[-\frac{1}{12}, \frac{7}{12}\right], [1], \frac{27}{4} \frac{t^2 \cdot (1-t)^2}{(1 - t + t^2)^3}\right). \tag{79}$$

Let us first note that the first hypergeometric function $H = {}_2F_1([7/6, 5/2], [7/3], t)$ is actually an *algebraic function*. It is solution of the polynomial equation:

$$\begin{aligned} &3^{21} \cdot t^8 (t-1)^8 \cdot H^8 + 2^{17} \cdot 3^{11} \cdot t^4 \cdot (t^2 - t + 1) \cdot (t-1)^4 \cdot H^4 \\ &+ 2^{26} \cdot (t-2) \cdot (2t-1) \cdot (t+1) \cdot (32t^6 - 96t^5 + 219t^4 - 278t^3 \\ &+ 219t^2 - 96t + 32) \cdot H^2 - 2^{32} \cdot (t^2 - t + 1)^2 = 0. \end{aligned} \tag{80}$$

For the second solution (79), we use the identities

$$\begin{aligned} &{}_2F_1\left(\left[-\frac{1}{12}, \frac{7}{12}\right], [1], \frac{27}{4} \frac{t^2 \cdot (1-t)^2}{(1 - t + t^2)^3}\right) \\ &= -6 \cdot \frac{t \cdot (t-1)}{(t^2 - t + 1)^{1/2}} \cdot \frac{dH_2}{dt} + \frac{1}{2} \frac{(2t-1) \cdot (t^2 - t - 2)}{(t^2 - t + 1)^{3/2}} \cdot H_2, \end{aligned} \tag{81}$$

where the pullbacked hypergeometric function H_2 reads:

$$H_2 = {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], \frac{27}{4} \frac{t^2 \cdot (1-t)^2}{(1 - t + t^2)^3}\right) = (t^2 - t + 1)^{1/4} \cdot {}_2F_1\left(\left[\frac{1}{2}, \frac{1}{2}\right], [1], t\right). \tag{82}$$

Consequently, the relevant solution of the order-four linear differential operator L_4 will be a linear combination $\alpha(t) \cdot E + \beta(t) \cdot K$ of the two complete elliptic integrals E, K, $\alpha(t)$ and $\beta(t)$ being (quite) involved algebraic functions.

The other order-four linear differential operator M_4 is, at first sight, slightly more difficult to analyze. In fact we are in the typical situation of an *absolute factorization*[15] of this order-four linear differential operator, and this can be seen performing the *exterior square* of that order-four linear differential operator [25]. Some calculations are displayed in appendix C. These calculations strongly suggest that the relevant solution of the order-four linear differential operator M_4 will also be of the form $\alpha(t) \cdot E + \beta(t) \cdot K$, the functions $\alpha(t)$ and $\beta(t)$ being (very) involved algebraic functions of t.

Fortunately, one can get that result in a much more straight way, if one remarks that the two order-four linear differential operators L_4 and M_4 are actually (non-trivially) homomorphic. Introducing $\rho = t^{2/3} \cdot (1-t)^{2/3}$, one finds that a conjugate of M_4 is actually homomorphic to the first order-four linear differential operator L_4:

$$L_4 \cdot I_3 = J_3 \cdot \left(\frac{1}{\rho} \cdot M_4 \cdot \rho\right), \tag{83}$$

image of the tensor product. The (Maple/DEtools) reason for choosing the name symmetric_product is the resemblance with the function symmetric_power.

[15] A linear differential operator $L \in \mathbb{C}(x)[d/dx]$ is called absolutely reducible [25] if it admits a factorization over an *algebraic extension* of $\mathbb{C}(x)$.

where I_3 and J_3 are (slightly involved) order-three intertwiners.

Therefore we have shown that the relevant solution of the order-eight linear differential operator will be of the form $\alpha(t)\cdot E+\beta(t)\cdot K$, $\alpha(t)$ and $\beta(t)$ being (quite) involved algebraic functions of t.

Again, one finds that S_2 is D-finite, but the reduction to polynomials in the complete elliptic integrals E and K is far from obvious.

5.2.1. Deformation of $u=\pi/6$: the S_3 term. The next S_n, namely

$$S_3=\frac{\theta_2^{(3)}(\pi/6,q)}{\theta_2'(\pi/6,q)}, \tag{84}$$

is solution of a linear differential operator of order *twelve* with coefficient polynomials in t of degree 67. This is a quite large order (twelve) linear differential operator, that we will not give here. This order-twelve linear differential operator is actually the direct sum of an order-three operator and an order-nine operator L_9. The order-three linear differential operator L_3 reads

$$\begin{aligned}L_3=D_t^3&+6\cdot\frac{q_{12}}{q_6\cdot(t-1)\cdot(t+1)\cdot(t-2)\cdot(2t-1)\cdot(t^2-t+1)\cdot t}\cdot D_t^2\\&+\frac{r_{12}}{q_6\cdot(t-1)^2\cdot(t+1)\cdot(t-2)\cdot(t^2-t+1)\cdot t^2}\cdot D_t\\&+\frac{3}{2}\cdot\frac{r_6}{q_6\cdot(t-1)\cdot(t+1)\cdot(t-2)\cdot(2t-1)\cdot t},\end{aligned} \tag{85}$$

where:

$$\begin{aligned}q_{12}&=t^{12}-6t^{11}+2536t^{10}-12\,625t^9+18\,414t^8+2028t^7-31\,302t^6\\&\quad+33\,849t^5-16\,458t^4+4084t^3-528t^2+7t-1,\\q_6&=t^6-3t^5+1518t^4-3031t^3+1518t^2-3t+1,\\r_{12}&=t^{12}-6t^{11}+4881t^{10}-24\,350t^9+24\,459t^8+48\,198t^7-120\,498t^6\\&\quad+90\,597t^5-20\,496t^4-5105t^3+2304t^2+15t-2,\\r_6&=59t^6-177t^5+4512t^4-8729t^3+4512t^2-177t+59.\end{aligned} \tag{86}$$

Let us denote L_K the order-two linear differential operator annihilating the complete elliptic integral of the first kind $K={}_2F_1([1/2,1/2],[1],t)$:

$$L_K=D_t^2+\frac{2t-1}{t\cdot(t-1)}\cdot D_t+\frac{1}{4t\cdot(t-1)}. \tag{87}$$

This order-three linear differential operator (85) is actually homomorphic to the symmetric square of operator L_K, with order-two intertwiners. Consequently the solutions of L_3 are (quadratic) homogeneous polynomials in E and K. Actually one finds that the solution of L_3 given by (85) reads:

$$\begin{aligned}\mathrm{Sol}(L_3)&=\frac{(t-2)^3}{(t^2-t+1)}\cdot K^2+9\cdot EK\\&=1+\frac{177}{32}t^2+\frac{177}{32}t^3+\frac{1095}{8192}t^4-\frac{21\,561}{4096}t^5-\frac{1384\,095}{262\,144}t^6+\frac{22\,467}{262\,144}t^7\\&\quad+\frac{2927\,958\,291}{536\,870\,912}t^8+\frac{730\,823\,955}{134\,217\,728}t^9+\cdots.\end{aligned} \tag{88}$$

The order-nine linear differential operator L_9 can be seen to be the *symmetric product* of an order-three linear differential operator A_3, and of the order-three linear differential operator, which is the *symmetric square* of the order-two linear differential operator L_K annihilating $K = {}_2F_1([1/2, 1/2], [1], t)$

$$L_9 = \mathrm{SymProd}\left(\mathrm{Sym}^2(L_K), A_3\right). \tag{89}$$

The order-three linear differential operator A_3 reads

$$A_3 = D_t^3 + \frac{r_8 \cdot (2t-1)}{q_6 \cdot t \cdot (t-1) \cdot (t^2 - t + 1)} \cdot D_t^2 - \frac{5}{9} \cdot \frac{r_6 \cdot (t^2 - t + 1)}{q_6 \cdot t^2 \cdot (t-1)^2} \cdot D_t + \frac{5}{18} \cdot \frac{r_6' \cdot (2t-1)}{q_6 \cdot t^2 \cdot (t-1)^2}, \tag{90}$$

where:

$$\begin{aligned}
r_6 &= 52 - 156t - 3009t^2 + 6278t^3 - 3009t^4 - 156t^5 + 52t^6, \\
r_6' &= r_6 - 2106 \cdot t \cdot (t-1) \cdot (t-2) \cdot (t+1), \\
q_6 &= 5r_6 + 16\,038 \cdot t^2 \cdot (t-1)^2, \\
r_8 &= 5r_6 \cdot (t^2 - t + 1) + 17172 \cdot t^2 \cdot (t-1)^2 + 15\,471 \cdot t^3 \cdot (t-1)^3.
\end{aligned} \tag{91}$$

The solutions of this order-three linear differential operator A_3 are actually *algebraic functions* satisfying

$$\begin{aligned}
&432 \cdot (t^2 - t + 1)^4 \cdot F^4 - 72 \cdot P_6 \cdot (t^2 - t + 1)^2 \cdot F^2 \\
&\quad - 16 \cdot (t-2) \cdot (2t-1) \cdot (t+1) \cdot (t^2 - t + 1) \cdot \left(P_6 + 972 \cdot t^2 \cdot (t-1)^2\right) \cdot F \\
&\quad + 6480 \cdot t^2 \cdot (t-1)^2 \cdot (t^2 - t + 1)^3 - P_6^2 = 0,
\end{aligned} \tag{92}$$

where the polynomial P_6 reads:

$$P_6 = 4 \cdot (t^2 - t + 1)^3 - 243 \cdot t^2 \cdot (1-t)^2. \tag{93}$$

The well-suited solution of the order-three linear differential operator A_3 reads:

$$\begin{aligned}
Sol(A_3) = 1 - \frac{1}{2}t - \frac{165}{64}t^2 - \frac{165}{128}t^3 + \frac{26\,655}{16\,384}t^4 + \frac{101\,085}{32\,768}t^5 + \frac{6546\,741}{4194\,304}t^6 \\
- \frac{12\,198\,135}{8388\,608}t^7 - \frac{3182\,706\,057}{1073\,741\,824}t^8 - \frac{3159\,215\,679}{2147\,483\,648}t^9 + \cdots.
\end{aligned} \tag{94}$$

The solution of the order-nine linear differential operator L_9 reads:

$$\begin{aligned}
\mathrm{Sol}(L_9) = \mathrm{Sol}(A_3) \cdot K^2 = 1 - \frac{159}{64}t^2 - \frac{159}{64}t^3 + \frac{2973}{16\,384}t^4 + \frac{23\,325}{8192}t^5 \\
+ \frac{11\,858\,901}{4194\,304}t^6 + \frac{510\,591}{4194\,304}t^7 - \frac{2771\,276\,211}{1073\,741\,824}t^8 - \frac{695\,778\,099}{268\,435\,456}t^9 + \cdots.
\end{aligned} \tag{95}$$

The series expansion of (84) reads:

$$\begin{aligned}
-S_3 = -\frac{\theta_2^{(3)}(\pi/6, q)}{\theta_2'(\pi/6, q)} = 1 + \frac{3}{16}t^2 + \frac{3}{16}t^3 + \frac{339}{2048}t^4 + \frac{147}{1024}t^5 \\
+ \frac{262\,047}{2097\,152}t^6 + \frac{230\,109}{2097\,152}t^7 + \frac{1632\,105}{16\,777\,216}t^8 + \frac{365\,061}{4194\,304}t^9 + \cdots.
\end{aligned} \tag{96}$$

Recalling the series expansions (88) and (95), one actually finds that this series (96) is exactly:

$$\begin{aligned}
-S_3 &= -\frac{\theta_2^{(3)}(\pi/6, q)}{\theta_2'(\pi/6, q)} = \frac{1}{3} \cdot \mathrm{Sol}(L_3) + \frac{2}{3} \cdot \mathrm{Sol}(L_9) \\
&= \frac{1}{3} \cdot \left(\frac{(t-2)^3}{(t^2 - t + 1)} \cdot K^2 + 9 \cdot EK\right) + \frac{2}{3} \cdot \mathrm{Sol}(A_3) \cdot K^2.
\end{aligned} \tag{97}$$

Remark 3. More generally, for $u = \pi/6$, one has

$$C_\rho(1,1;\rho) = -2 \cdot \frac{\theta_2'\left(\frac{\pi}{6},q\right)}{\sqrt{\rho+1}\cdot\theta_2(0,q)\cdot\theta_3(0,q)^2} \times \sum_{p=0}^{\infty} \left(\arcsin\left(\frac{\sqrt{\rho+1}}{2}\right) - \frac{\pi}{6}\right)^p \cdot \frac{S_{(p+1)}}{(p)!}, \tag{98}$$

where:

$$S_n = \frac{\theta_2^{(n)}(\pi/6,q)}{\theta_2'(\pi/6,q)}. \tag{99}$$

5.3. Other one-parameter deformations: deformation of $u = \pi/3$

Note: To avoid any confusion with the linear differential operators introduced in the $u = \pi/3$ case (see sections 5.2 and C.2) we will add an extra (3) subscript for the linear differential operators of this $u = \pi/3$ case.

For $u = \pi/3$ we also find that

$$\sqrt{3}\cdot S_2 = \sqrt{3}\cdot\frac{\theta_2^{(2)}(\pi/3,q)}{\theta_2'(\pi/3,q)} = 1 - \frac{9}{128}\cdot t^2 - \frac{9}{128}\cdot t^3 - \frac{261}{4096}\cdot t^4 + \cdots \tag{100}$$

is actually D-finite: it is solution of a (slightly involved) *order-eight linear differential operator* $L_8^{(3)}$. In fact, revisiting the calculations performed in section 2.2, but this time with a perturbation around an algebraic solution $A(t)$ (see (29)), one easily finds, using the sigma-form of Painlevé VI non-linear differential equation (3), that the first correction term $G_1(t)$ is solution of an order-three linear differential operator, with very involved algebraic coefficients depending on the algebraic solution $A(t)$ and its derivatives. This provides lower order linear differential operators, but with a price to pay, namely very involved algebraic coefficients. In fact one can study directly the previous order-eight linear differential operator.

If one conjugates this order-eight linear differential operator $L_8^{(3)}$ by $t^{4/3}$, changing $L_8^{(3)}$ into $\tilde{L}_8^{(3)} = t^{-4/3}\cdot L_8^{(3)}\cdot t^{4/3}$, one can easily see that this new order-eight linear differential operator $\tilde{L}_8^{(3)}$ is actually the direct-sum (LCLM) of two order-four linear differential operators: $\tilde{L}_8^{(3)} = \mathrm{LCLM}(L_4^{(3)}, M_4^{(3)}) = L_4^{(3)} \oplus M_4^{(3)}$. Furthermore, one finds that these two order-four linear differential operators are non-trivially homomorphic, after performing a conjugation of one of the two linear differential operator by $\rho = t^{1/3}\cdot(1-t)^{1/3}$

$$M_4^{(3)}\cdot I_3 = J_3\cdot\left(\frac{1}{\rho}\cdot L_4^{(3)}\cdot\rho\right), \tag{101}$$

where I_3 and J_3 are order-three intertwiners. Let us focus on the simplest order-four linear differential operator, namely $L_4^{(3)}$:

$$L_4^{(3)} = D_t^4 + \frac{4}{3}\cdot\frac{9t-5}{(t-1)\cdot t}\cdot D_t^3 + \frac{1}{9}\cdot\frac{337t^2-373t+73}{(t-1)^2\cdot t^2}\cdot D_t^2$$
$$+ \frac{1}{54}\cdot\frac{1590t^3-2627t^2+1085t-42}{(t-1)^3\cdot t^3}\cdot D_t + \frac{1}{162}\cdot\frac{350t^3-769t^2+485t-84}{(t-1)^4\cdot t^3}. \tag{102}$$

We have a prejudice that this order-four linear differential operator could correspond to an absolute factorization [25], and could be written[16] as a *symmetric product* of two order-two

[16] This prejudice comes from section (5.2), see (77).

linear differential operators (see also appendix C). In order to check this scenario, let us calculate the *exterior square* of that order-four linear differential operator. One finds that it is actually the direct-sum (LCLM) of two order-three linear differential operators

$$\mathrm{Ext}^2\left(L_4^{(3)}\right) = \mathrm{LCLM}(A_3^{(3)}, B_3^{(3)}) = A_3^{(3)} \oplus B_3^{(3)}, \tag{103}$$

where the second order-three linear differential operator $B_3^{(3)}$ is exactly the *symmetric square* of an order-two linear differential operator $A_2^{(3)}$

$$A_2^{(3)} = D_t^2 + \frac{2}{3} \cdot \frac{7t-4}{t \cdot (t-1)} \cdot D_t + \frac{1}{36} \cdot \frac{117t^2 - 133t + 21}{t^2 \cdot (t-1)^2}, \tag{104}$$

which has the two *algebraic function* solutions:

$$t^{-1/2} \cdot (1-t)^{-1/6} \cdot {}_2F_1\left(\left[\frac{5}{6}, \frac{3}{2}\right], \left[\frac{5}{3}\right], t\right), t^{-7/6} \cdot (1-t)^{-1/6} \cdot {}_2F_1\left(\left[\frac{1}{6}, \frac{5}{6}\right], \left[\frac{1}{3}\right], t\right).$$

Recalling (105) the order-two linear differential operator L_K annihilating the complete elliptic integral of the first kind $K = {}_2F_1([1/2, 1/2], [1], t)$, let us consider the *symmetric product* of the order-two linear differential operator A_2 and of L_K. One finds that this symmetric product is non-trivially homomorphic to some conjugate of L_4

$$\mathrm{SymProd}(A_2^{(3)}, L_K) \cdot I_2 = J_2 \cdot \left(\frac{1}{\rho} \cdot L_4^{(3)} \cdot \rho\right), \tag{105}$$

where $\rho = t^{1/6} \cdot (1-t)^{1/6}$, and where I_2 and J_2 are order-two intertwiners. This shows that the solution of $L_4^{(3)}$ (and thus $M_4^{(3)}$), and therefore the solution of the order-eight linear differential operator $L_8^{(3)}$, are actually of the form $\alpha(t) \cdot E + \beta(t) \cdot K$ where $\alpha(t)$ and $\beta(t)$ are *algebraic functions.*

Remark 4. Note, eventually, that these two order-four linear differential operators $L_4^{(3)}$ and $M_4^{(3)}$ can, in fact, be seen to be (non-trivially) homomorphic to some well-suited conjugates of the two order-four operators L_4 and M_4 emerging for $u = \pi/6$ in the previous section (5.2).

5.3.1. Deformation of $u = \pi/3$: the S_3 term. The next S_n, namely

$$S_3 = \frac{\theta_2^{(3)}(\pi/3, q)}{\theta_2'(\pi/3, q)}, \tag{106}$$

is solution of a linear differential operator of order *twelve* with coefficient polynomials in t of degree 52. This is a quite large order twelve linear differential operator, that we will not give here. This order twelve linear differential operator is actually the direct sum of an order-three operator and an order-nine linear differential operator L_9. The order-three linear differential operator L_3 reads:

$$\begin{aligned} L_3^{(3)} = D_t^3 &+ \frac{6 \cdot (64t^4 - 170t^3 + 40t^2 + 3t - 1)}{(128t^2 + t - 1) \cdot (t-1) \cdot (t-2) \cdot t} \cdot D_t^2 \\ &+ \frac{(128t^5 - 410t^4 - 55t^3 + 218t^2 - 11t + 2)}{(128t^2 + t - 1) \cdot (t-1)^2 \cdot (t-2) \cdot t^2} \cdot D_t \\ &- \frac{3 \cdot (32t^2 + 5t - 5)}{2 \cdot (128t^2 + t - 1) \cdot (t-1)^2 \cdot (t-2) \cdot t}. \end{aligned} \tag{107}$$

This order-three linear differential operator (107) is actually homomorphic to the symmetric square of the order-two linear differential operator L_K, annihilating $K = {}_2F_1([1/2,1/2],[1],t)$, with order-two intertwiners. Consequently the solutions of L_3 are (quadratic) homogeneous polynomials in E and K. Actually one finds that the solution of $L_3^{(3)}$ given by (107) reads:

$$\begin{aligned}\mathrm{Sol}(L_3^{(3)}) &= 4\cdot(t-2)\cdot K^2 + 9\cdot EK \\ &= 1 - \frac{15}{32}t^2 - \frac{15}{32}t^3 - \frac{3513}{8192}t^4 - \frac{1593}{4096}t^5 \quad - \frac{92\,895}{262\,144}t^6 - \frac{85\,245}{262\,144}t^7 \\ &\quad - \frac{161\,330\,925}{536\,870\,912}t^8 - \frac{37\,507\,821}{134\,217\,728}t^9 + \cdots. \end{aligned} \tag{108}$$

The order-nine linear differential operator $L_9^{(3)}$ can be seen to be the *symmetric product* of an order-three linear differential operator $A_3^{(3)}$ and of the order-three linear differential operator which is the symmetric square of the order-two linear differential operator L_K annihilating $K = {}_2F_1([1/2,1/2],[1],t)$:

$$L_9^{(3)} = \mathrm{SymProd}\left(\mathrm{Sym}^2(L_K), A_3^{(3)}\right). \tag{109}$$

The order-three linear differential operator A_3 reads:

$$\begin{aligned} A_3^{(3)} &= D_t^3 + \frac{16t^3 - 94t^2 + 165t - 55}{t\cdot(t-1)\cdot(16t^2 - 55t + 55)}\cdot D_t^2 \\ &\quad + \frac{32t^4 - 130t^3 + 75t^2 + 110t - 55}{9\cdot t^2\cdot(t-1)^2\cdot(16t^2 - 55t + 55)}\cdot D_t - \frac{64t^3 - 240t^2 + 165t + 55}{18\cdot t^2\cdot(t-1)^2\cdot(16t^2 - 55t + 55)}. \end{aligned} \tag{110}$$

The solutions of this order-three linear differential operator $A_3^{(3)}$ are actually *algebraic functions* satisfying the algebraic equation:

$$\begin{aligned} &27\cdot F^4 - 18\cdot(16t^2 - t + 1)\cdot F^2 - 4(t-2)\cdot(128t^2 + t - 1)\cdot F \\ &\quad - (256t^4 - 752t^3 + 753t^2 - 2t + 1) = 0. \end{aligned} \tag{111}$$

The well-suited solution of the order-three linear differential operator A_3 reads:

$$\begin{aligned} \mathrm{Sol}(A_3^{(3)}) &= 1 - \frac{1}{2}t + \frac{9}{64}t^2 + \frac{9}{128}t^3 + \frac{747}{16\,384}t^4 + \frac{1089}{32\,768}t^5 + \frac{108\,603}{4194\,304}t^6 \\ &\quad + \frac{176\,679}{8388\,608}t^7 + \frac{18\,959\,247}{1073\,741\,824}t^8 + \frac{32\,508\,009}{2147\,483\,648}t^9 + \cdots. \end{aligned} \tag{112}$$

The solution of the order-nine linear differential operator reads:

$$\begin{aligned} \mathrm{Sol}(L_9^{(3)}) &= \mathrm{Sol}(A_3^{(3)})\cdot K^2 = 1 + \frac{15}{64}t^2 + \frac{15}{64}t^3 + \frac{3513}{16\,384}t^4 + \frac{1593}{8192}t^5 \\ &\quad + \frac{743\,115}{4194\,304}t^6 + \frac{681\,825}{4194\,304}t^7 + \frac{161\,265\,045}{1073\,741\,824}t^8 + \frac{37\,482\,261}{268\,435\,456}t^9 + \cdots. \end{aligned} \tag{113}$$

The series expansion of (106) reads:

$$\begin{aligned} -S_3 &= -\frac{\theta_2^{(3)}(\pi/3,q)}{\theta_2'(\pi/63q)} = 1 - \frac{15}{2097\,152}t^6 - \frac{45}{2097\,152}t^7 - \frac{2745}{67\,108\,864}t^8 \\ &\quad - \frac{1065}{16\,777\,216}t^9 - \frac{3011\,265}{34\,359\,738\,368}t^{10} - \frac{3858\,885}{34\,359\,738\,368}t^{11} + \cdots. \end{aligned} \tag{114}$$

Recalling the series expansions (108) and (113), one actually finds that this series (114) is exactly

$$-S_3 = -\frac{\theta_2^{(3)}(\pi/3,q)}{\theta_2'(\pi/3,q)} = \frac{1}{3}\cdot \mathrm{Sol}(L_3^{(3)}) + \frac{2}{3}\cdot \mathrm{Sol}(L_9^{(3)})$$
$$= \frac{1}{3}\cdot \big(4\cdot(t-2)\cdot K^2 + 9\cdot EK\big) + \frac{2}{3}\cdot \mathrm{Sol}(A_3^{(3)})\cdot K^2. \qquad (115)$$

Remark 1. Let us recall the hypergeometric function $t^{-7/6}\cdot(1-t)^{-1/6}\cdot {}_2F_1([5/6,1/6],[1/3],t)$ which is an algebraic function and its order-two linear differential operator $A_2^{(3)}$ (see (104)). Let us also recall the order-two linear differential operator L_K annihilating $K = {}_2F_1([1/2,1/2],[1],t)$. Let us consider the order-three linear differential operators corresponding to the symmetric square of these two order-two linear differential operators, and let us consider the *symmetric product* of these two symmetric squares. One gets that way an order-*nine* linear differential operator:

$$\Omega_9 = \mathrm{SymProd}\Big(\mathrm{Sym}^2(L_K), \mathrm{Sym}^2(A_2^{(3)})\Big). \qquad (116)$$

This order-nine linear differential operator Ω_9 has a structure of solutions very similar to the one of the order-nine linear differential operator L_9. One finds, in fact, that this order-nine linear differential operator (116) is actually non-trivially homomorphic to the order-nine linear differential operator L_9:

$$I_8\cdot\Big(t^{-7/3}\cdot\Omega_9\cdot t^{7/3}\Big) = L_9^{(3)}\cdot J_8, \qquad (117)$$

where I_8 and J_8 are order-eight intertwiners. In conclusion the solution of the order-twelve operator corresponding to S_3 and thus annihilating (106), is a homogeneous (quadratic) polynomial of E and K with involved algebraic coefficients.

Remark 2. More generally, for $u = \pi/3$ one has:

$$C_\rho(1,1;\rho) = -2\cdot\frac{\theta_2'\left(\frac{\pi}{3},q\right)}{\sqrt{\rho+3}\cdot\theta_2(0,q)\cdot\theta_3(0,q)^2}\times\sum_{p=0}^{\infty}\left(\arcsin\left(\frac{\sqrt{\rho+3}}{2}\right)-\frac{\pi}{3}\right)^p\cdot\frac{S_{(p+1)}}{p!}, \qquad (118)$$

where:

$$S_n = \frac{\theta_2^{(n)}(\pi/3,q)}{\theta_2'(\pi/3,q)}. \qquad (119)$$

6. λ corresponds to the critical exponent at $t = 1$

The lambda extensions $C(1,1;\lambda)$ are a one-parameter family of solutions of the Okamoto-Painlevé VI equation (3). It is worth noticing that the parameter lambda cannot be seen in the non-linear ODE (3). *It is not a parameter of the non-linear ODE* (3). The parameter lambda *actually fixes the critical exponent at* $t = 1$ of the solution $C(1,1;\lambda)$.

Paper [26] gives, in equation (13) and (14), the behavior of the lambda extensions $C(N,N,\lambda)$ near[17] $t=1$:

$$C(N,N,\lambda) \simeq K(N,\sigma)\cdot(1-t)^{\sigma^2/4} \qquad \text{where:} \quad \sigma = \frac{2}{\pi}\cdot\arccos(\lambda), \tag{120}$$

or denoting $\lambda = \cos(u)$:

$$C(N,N,\lambda) \simeq K(N,\sigma)\cdot(1-t)^{(u/\pi)^2}. \tag{121}$$

One verifies that this power-law formula[18] (121) is actually valid for all the algebraic expressions displayed in section 2.1.1 (see (22)), section 2.1.2 (see (25)) and section 2.1.3 (see (29)):

- For $\lambda = 0$, i.e. $u = \pi/2$ one has a $(1-t)^{1/4}$ behavior.
- For $\lambda = 1/\sqrt{2}$, i.e. $u = \pi/4$ one has a $(1-t)^{1/16}$ behavior (see (22) and (62)).
- For $\lambda = \sqrt{3}/2$, i.e. $u = \pi/6$ one has a $(1-t)^{1/36}$ behavior: from (25) one actually gets: $S(t) = 2^{8/9}/3\cdot(1-t)^{1/36} + \cdots$.
- For $\lambda = 1/2$, i.e. $u = \pi/3$, one has a $(1-t)^{1/9}$ behavior: from (29) one actually gets: $S(t) = 2^{14/9}\cdot 3^{-3/2}\cdot(1-t)^{1/9} + \cdots$.

7. Comments and speculations

All these calculations, displayed on the low-temperature correlation function $C(1,1)$, illustrate the extremely rich structures of the lambda extensions of the two-point square[19] Ising correlation functions $C(M,N)$. For an *infinite set* of values of lambda ($\lambda = \cos(\pi m/n)$, m and n integers) these lambda extensions become *algebraic functions* and for another infinite set of values of lambda ($\lambda = (m/n)^{1/2}$, m and n integers) the series expansions of the lambda extension are *globally bounded series* [12] that are *not* D-finite[20] but *only differentially algebraic* (the corresponding σ are solutions of a sigma-form of Painlevé VI).

Furthermore we have seen, in section 2, that the 'form-factor-like' expansions (10) around the (D-finite) two-point correlation function $C(1,1) = E$, yield new 'form factors' $g_n(t)$'s which, at first sight, should be DD-finite expressions (see section (2.2)), *are, actually, D-finite expressions*. The $g_n(t)$'s are, in fact, *polynomial expressions in E* and *K*.

The 'form-factor-like' [1] expansions around the infinite set of algebraic functions at $\lambda = \cos(\pi m/n)$ yield new 'form factors' $G_n(t)$'s (see (61)) which turned out to be *D-finite expressions*: they are solutions of linear differential operators with (quite involved) algebraic functions coefficients. We showed that the first $G_n(t)$'s are actually *polynomial expressions in* E and K and, hopefully, one can expect that all these $G_n(t)$'s are polynomial expressions in E and K (with involved algebraic functions coefficients).

These results correspond to the (quite puzzling) fact that rational expressions of the derivatives (at selected values of u) of *Jacobi theta functions* (like (37)) can, in fact, be expressed as polynomial expressions in E and K, thus providing *an infinite set of remarkable identities between theta functions and complete elliptic integrals of the first and second kind*[21].

[17] Here σ is an exponent, which has nothing to do with the σ functions (1) or (2). Painlevé papers are famous for their terrible notations.

[18] It is very hard to get this result from the exact expression (34) of $C(1,1;\lambda)$ in terms of theta functions.

[19] One has similar results for the triangular, honeycomb, ...lattices. One has similar results for the high-temperature correlation functions. One has similar results for the anisotropic correlation functions $C(M,N)$ for $\nu = -k$.

[20] Except when $\lambda = 0, 1/\sqrt{2}, 3/\sqrt{2}, 1$ where λ is also of the form $\lambda = \cos(\pi m/n)$.

[21] For identities on products of ratio of Jacobi theta functions see for instance [27].

Such calculations provide an infinite set of new D-finite expressions on the two-dimensional Ising model that will join together with all the previous D-finite expressions we have already encountered on the two-dimensional Ising model as n-fold integrals that are diagonals of rational functions [12–15]. This corresponds to the kind of holonomic (i.e. D-finite) studies we are used to perform on the two-dimensional Ising model [1, 28] in the variable $t = k^2$. These D-finite expressions emerge from form factor-like perturbation theory (the kind of perturbation theory physicists are used to with Feynman diagrams, Periods of algebraic varieties, ...). However, we also see that the lambda extension $C(1,1;\lambda)$ which is differentially algebraic (solution of a *non-linear* ODE (3) with the Painlevé property of fixed critical points [29]), can be understood 'holistically', globally, and not using the bread and butter perturbative physicist's approach, if one switches to a description *in terms of the nome q* (or the ratio τ of the two periods of the elliptic function) requiring to introduce intensively Jacobi theta functions [1, 8, 29]. With that alternative holistic description one has a rather simple exact closed formula for the lambda extension (see (34)). The 'price to pay' is that this exact and elegant holistic expression of the lambda extension (like (34)) is solution of a *non-linear* ODE (3) and, for instance, the emergence of all the D-finite expressions, displayed in this paper, is not obvious from that non-linear differential equations or Jacobi theta functions viewpoint [30].

7.1. Painlevé VI transcendentals as deformations of elliptic functions and why theta functions are well-suited: Jacobi forms

The occurrence of *Jacobi theta functions* [31, 32] for the exact closed expression (37) of the lambda extension solution of sigma-form of Painlevé VI is, in fact, highly relevant as far as all the symmetries of the model are concerned.

Let us first recall that Painlevé VI transcendents should be seen as *deformations of elliptic functions* [33]. Along this line it is worth recalling Manin's idea [33] that the Painlevé VI equation for a particular choice of the four Okamoto parameters, can be written extremely simply in terms of the ratio of periods τ. Let us denote $\mathcal{P}(z,\tau)$ the $\mathcal{P}$-Weierstrass function and $\mathcal{P}_z(z,\tau) = \frac{\partial \mathcal{P}(z,\tau)}{\partial z}$. Manin's result means that the Painlevé VI equation can be written in a form (see equation (1.16) in [33]):

$$\frac{d^2 z(\tau)}{d\tau^2} = \left(\frac{1}{2\pi i}\right)^2 \cdot \sum_{i=0}^{3} \alpha_i \cdot \mathcal{P}_z\left(z + \frac{T_i}{2}, \tau\right). \qquad (122)$$

In previous studies of the $C(M,N)$ correlation functions and their non-linear Painlevé ODEs, we have underlined the fundamental role of Landen transformations [29]. The crucial role of *Landen transformations* is underlined in [16, 29, 33]. It is also worth recalling that the Weierstrass $\mathcal{P}$-function is simply related to theta functions. The Weierstrass $\mathcal{P}$-function is related[22] to the second log derivative of $\theta_1(u,q)$:

$$\mathcal{P}(u,\tau) = -\frac{\partial^2 \ln(\theta_1(u,\tau))}{\partial u^2} + c = -\frac{\partial^2 \ln(\theta_1(u,\tau))}{\partial u^2} + \frac{1}{3}\,\frac{\theta_1'''(0,q)}{\theta_1'(0,q)}. \qquad (123)$$

The closed expressions (37) for the lambda-extension $C(1,1;\lambda)$ underlines the occurrence of the partial derivative with respect to the u-deformation parameter (or equivalently the lambda

[22] The constant c is defined so that the Laurent expansion of $\mathcal{P}(u,\tau)$ at $u=0$ has zero constant term ($\theta_1'(0,q)$ is the derivative with respect to u), see (B.7), (B.8) in [34]. See for instance https://handwiki.org/wiki/Theta_function in the paragraph Relation to the Weierstrass elliptic function. See also [35].

parameter). Along this line one can recall another interesting property of the theta functions. They are solutions of the heat equation:

$$\frac{\partial \theta(u,\tau)}{\partial \tau} = q \cdot \frac{\partial \theta(u,q)}{\partial q} = \frac{\partial^2 \theta(u,q)}{\partial u^2}. \tag{124}$$

Consequently, and to some extent, the partial derivatives in u can be replaced by partial derivatives in τ.

It is also worth mentioning the *modular group relations* on the Weirstrass $\mathcal{P}$-functions as well as the similar 'modular group transformations' on the theta functions [36, 37]:

$$\mathcal{P}\left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 \cdot \mathcal{P}(z,\tau), \tag{125}$$

$$\mathcal{P}_z\left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^3 \cdot \mathcal{P}(z,\tau), \tag{126}$$

and[23]

$$\kappa \cdot (c\tau+d)^{1/2} \cdot \theta_\alpha(u,\tau) = \exp\left(-i\pi \frac{cu^2}{c\tau+d}\right) \cdot \theta_\beta\left(\frac{u}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right), \tag{127}$$

where κ is a constant, and where the integers a,b,c,d are such that $ad-bc=1$. For $u=0$ the previous modular group transformations (125) and (127) is reminiscent of the modular forms of weight k:

$$(c\tau+d)^k \cdot f(\tau) = f\left(\frac{a\tau+b}{c\tau+d}\right). \tag{128}$$

With some abuse of language we could say that the theta functions are 'some kind' of 'modular forms of weight $1/2$'.

Recalling the relation (34) between λ and u, *the theta functions thus provide, because of* (127), *some natural u-extension, and thus lambda-extension, of the modular forms* (Jacobi forms). From the closed expression (37) it is clear that the lambda-extension will naturally inheritate from (127), some symmetry properties *with respect to the modular group*. This kind of global (holistic) symmetry is almost impossible to see in the holonomic (D-finite) world of the linear differential operators in the variable t. Conversely all the D-finite results, we have displayed in this paper, are not an obvious consequence of the emergence of $\theta_2'(u,q)$ in (37). All these D-finite results are 'hidden' in the theta functions (considered at selected values of u). This is similar to the situation one encounters with *modular forms* [14, 38, 39] where the fact that they are D-finite in the variable t is not totally straightforward[24].

8. Conclusion

The lambda-extensions of the two-point correlation functions $C(M,N)$ of the square Ising model are a good illustration of the mirror-map $t \leftrightarrow q$ duality in mirror symmetries [42–44], where all the holonomic (D-finite) structures are well seen in the t variable but are hard to see in the nome [8, 10] q (or in the ratio of periods τ), and conversely the modular

[23] See equation (2.16) in [36].

[24] See in particular proposition 21 page 61 in [40]. One can find in [41] why automorphic forms are solutions of linear differential equations

group, modular forms structures are easily seen in the nome q variable (or in the ratio of periods τ) but are very hard to see in the original t variable. In the t variable the perturbative approach provides a large set of D-finite expressions which are n-fold integrals (and in fact diagonals of rational functions [12]), when the description in the nome variable (or the τ variable) provides a holistic understanding (see (34)) which makes crystal clear modular group symmetries and the emergence of Landen transformations [16, 29], and of modular forms [14, 35, 38], but requires to consider non-linear ODEs [29, 38, 39]. Both descriptions are complementary and necessary to describe efficiently these lambda-extensions.

Focusing, for pedagogical reasons, on a very simple example of lambda-extension, namely $C(1,1;\lambda)$, we have considered the series expansion in t as different form-factor-like expansions around the D-finite subcase $C(1,1) = E$ or a large set of algebraic functions subcases (see (15), (22), (25) and (29)). For the first form-factor-like expansion (10), the corresponding form-factors $g_n(t)$, which should, at first sight, be DD-finite, turn out to be D-finite and simple polynomials of the complete elliptic integrals of the first and second kind K and E. On the other hand, the form-factors $G_n(t)$, corresponding to a deformation around the algebraic functions subcases of the lambda-extension, have been seen to be D-finite, and, either, shown to be polynomials of K and E, or can be very reasonably conjectured to be polynomials of K and E. These results can be seen as remarkable, non-trivial (and rather unexpected ...), identities between ratio of Jacobi theta functions and the complete elliptic integrals of the first and second kind K and E.

These identities are a nice illustration of this complementary description of the D-finite t-variable (elliptic integrals) viewpoint and the non-linear (modular group, Jacobi theta functions [1, 8, 35]) nome viewpoint.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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Appendix A. Calculation of the coefficient $g_3(t)$

The series $g_3(t)$ can also be seen to be D-finite, being solution of an order-*twelve* linear differential operator which turns out to be the direct-sum (LCLM) of the previous order-two linear differential operator L_E, of the previous order-four L_4, of the previous order-six linear differential operator homomorphic to the symmetric fifth power of L_E, and of an order-*eight* linear differential operator homomorphic to the *symmetric seventh power* of L_E, with the following order-seven intertwiner

$$
\begin{aligned}
\frac{256}{315}\cdot R_3 = & \frac{1}{8}\cdot(t-1)\cdot(8t^2-33t+33)\cdot t^7\cdot D_t^7 \\
& +\frac{7}{16}\cdot(t-1)\cdot(40t^2-173t+181)\cdot t^6\cdot D_t^6 \\
& +\frac{7}{32}\cdot(360t^3-2077t^2+3795t-2166))\cdot t^5\cdot D_t^5 \\
& +\frac{35}{64}\cdot\frac{120t^4-975t^3+2968t^2-3933t+1900}{t-1}\cdot t^4D_t^4-\frac{7}{128}\cdot\frac{q_5}{(t-1)^2}\cdot t^3D_t^3 \\
& +\frac{7}{256}\cdot\frac{q_6}{(t-1)^3}\cdot t^2\cdot D_t^2-\frac{1}{512}\cdot\frac{q_7}{(t-1)^4}\cdot t\cdot D_t+\frac{7}{1024}\cdot\frac{q_8}{(t-1)^5},
\end{aligned}
\tag{A.1}
$$

where the q_n polynomials read:

$$
\begin{aligned}
q_5 &= 600t^5-5379t^4+16\,550t^3-15\,061t^2-8708t+13\,854, \\
q_6 &= 1080t^6-10\,287t^5+30\,197t^4-9695t^3-59\,739t^2+51\,338t+4402, \\
q_7 &= 12\,600t^7-12\,5991t^6+346\,295t^5+108\,127t^4-1210\,745t^3 \\
&\quad +868\,060t^2+142\,022t+4016, \\
q_8 &= 1800t^8-18\,801t^7+47\,986t^6+43\,466t^5-233\,350t^4+147\,125t^3 \\
&\quad +409\,36t^2+1378t+180.
\end{aligned}
\tag{A.2}
$$

One finally finds that the series expansion for $g_3(t)$ is exactly the linear combination of E, of the order-three linear differential operator (39) acting on E^3, of an order-five linear differential operator (41) acting on E^5 and the order-*seven* linear differential operator (A.1) acting on E^5:

$$
\begin{aligned}
g_3(t) &= \frac{5}{7168}\cdot E+\frac{37}{46\,080}\cdot R_1(E^3)-\frac{1}{9216}\cdot R_2(E^5)+\frac{1}{322\,560}\cdot R_3(E^7) \\
&= \frac{5}{7168}\cdot E-\frac{37}{15\,360}\cdot KE^2-\frac{37}{23\,040}\cdot(t-1)\cdot K^3 \\
&\quad +\frac{5}{3072}\cdot K^2E^3+\frac{5}{1536}\cdot(t-1)\cdot K^4E+\frac{1}{1152}\cdot(t-1)\cdot(t-2)\cdot K^5 \\
&\quad -\frac{1}{3072}\cdot K^3E^4-\frac{1}{768}\cdot(t-1)\cdot K^5E^2-\frac{1}{1440}\cdot(t-1)\cdot(t-2)\cdot K^6E \\
&\quad -\frac{1}{80\,640}\cdot(t-1)\cdot(8t^2-33t+33)\cdot K^7.
\end{aligned}
\tag{A.3}
$$

Appendix B. Low temperature lambda extension $C_-(0,0,\lambda)$

Similarly to the Taylor expansion (55), we can write a similar identity for the lambda extension $C(0,0,\lambda)$. Introducing

$$
S_n = \frac{\theta_4^{(n)}(0,q)}{\theta_4(0,q)},
\tag{B.1}
$$

the lambda extension $C_-(0,0,\lambda)$ can be written

$$
\begin{aligned}
C_-(0,0;\lambda) &= \frac{\theta_3(\arccos\lambda, q)}{\theta_3(0,q)} \\
&= (1-t)^{1/4} \cdot \sum_{p=0}^{\infty} (\arcsin\lambda)^{2p} \cdot \frac{S_{2p}}{(2p)!} \\
&= (1-t)^{1/4} \cdot \Bigg(1 + \frac{S_2}{2} \cdot \lambda^2 + \Big(\frac{S_2}{6} + \frac{S_4}{24}\Big) \cdot \lambda^4 + \Big(\frac{4S_2}{45} + \frac{S_4}{36} + \frac{S_6}{720}\Big) \cdot \lambda^6 \\
&\quad + \Big(\frac{2S_2}{35} + \frac{7S_4}{360} + \frac{S_6}{720} + \frac{S_8}{40\,320}\Big) \cdot \lambda^8 + \cdots \Bigg) \\
&= (1-t)^{1/4} \cdot \sum_{p=0}^{\infty} \kappa^{(2p)} \cdot \lambda^{2p},
\end{aligned}
\tag{B.2}
$$

where $\kappa^{(2p)} = f_{0,0}^{(2p)}$. From (B.2) one can deduce the expression of the S_{2p}'s in terms of the $\kappa^{(2p)}$'s:

$$
\begin{aligned}
\frac{S_2}{2!} &= \kappa^{(2)}, \\
\frac{S_4}{4!} &= \kappa^{(4)} - \frac{1}{3}\kappa^{(2)}, \\
\frac{S_6}{6!} &= \kappa^{(6)} - \frac{2}{3}\kappa^{(4)} + \frac{2}{45}\kappa^{(2)}, \\
\frac{S_8}{8!} &= \kappa^{(8)} - \kappa^{(6)} + \frac{1}{5}\kappa^{(4)} - \frac{1}{315}\kappa^{(2)}, \\
\frac{S_{10}}{10!} &= \kappa^{(10)} - \frac{4}{3}\kappa^{(8)} + \frac{7}{15}\kappa^{(6)} - \frac{34}{945}\kappa^{(4)} + \frac{2}{14\,175}\kappa^{(2)}, \cdots
\end{aligned}
\tag{B.3}
$$

Appendix C. Exterior squares and absolute factorization

C.1. Absolute factorization

Let us recall a simple example of an *absolute factorization* of an order-four linear differential operator given in [25]:

$$
A_4 = D_t^4 - \frac{1}{t} \cdot D_t^3 + \frac{3}{4t^2} \cdot D_t^2 - t = \Big(D_t^2 - \frac{1}{t} \cdot D_t + \frac{3}{4t^2} + \sqrt{t}\Big) \cdot \big(D_t^2 - \sqrt{t}\big). \tag{C.1}
$$

The fact that such a *factorization over an algebraic extension* of $\mathbb{C}(t)$ exists can be deduced [25] from the fact that one has a direct-sum (LCLM) decomposition of the (order-five) exterior square of the order-four linear differential operator A_4:

$$
\mathrm{Ext}^2(A_4) = D_t \oplus \Big(D_t^4 - \frac{3}{2t} \cdot D_t^3 + \frac{9}{4t^2} \cdot D_t^2 - \frac{15}{8t^3} \cdot D_t + 4t\Big). \tag{C.2}
$$

C.2. Exterior square of M_4 and absolute factorization of M_4

Let us now study here the order-four linear differential operator M_4 occurring in section 5.2 for the deformations of $u = \pi/6$.

The order-four linear differential operator M_4 is slightly more difficult to analyze than the first order-four linear differential operator L_4 in (5.2). We seem to have a solution of this order-four linear differential operator M_4 of the form $\alpha(t) \cdot E + \beta(t) \cdot K$, $\alpha(t)$ and $\beta(t)$ being (very) involved algebraic functions, however finding a symmetric product form, like in the previous order-four linear differential operator L_4, is difficult. Let us show, in a quite indirect way, that this is probably the case. Let us consider the exterior square of this order-four linear differential operator M_4. This is an order-six linear differential operator M_6, which is actually the direct-sum (LCLM) of two order-three linear differential operators A_3 and B_3

$$M_6 = \mathrm{Ext}^2(M_4) = \ \mathrm{LCLM}(A_3, B_3) = A_3 \oplus B_3, \tag{C.3}$$

where one finds easily that the first order-three linear differential operator A_3 corresponds to *algebraic solutions* associated with the polynomial equation:

$$\begin{aligned}
&(16t^{17} - 184t^{16} - 135\,149t^{15} + 1128\,329t^{14} - 6708\,683t^{13} + 26\,956\,928t^{12}\\
&\quad - 65\,809\,991t^{11} + 96\,341\,783t^{10} - 88\,006\,226t^{9} + 63\,929\,329t^{8} - 60\,215\,242t^{7}\\
&\quad + 59\,165\,527t^{6} - 37\,633\,087t^{5} + 12\,783\,832t^{4} - 1787\,515t^{3} - 7679t^{2} - 1957t - 32)\\
&\quad + 4(t-1)(t^2-t+1)(20t^{15} - 186t^{14} - 20\,481t^{13} + 138\,367t^{12} - 473\,685t^{11}\\
&\quad + 1069\,635t^{10} - 1516\,399t^{9} + 1115\,037t^{8} - 53\,199t^{7} - 617\,857t^{6} + 547\,761t^{5}\\
&\quad - 255\,237t^{4} + 78\,967t^{3} - 12\,885t^{2} + 156t - 16) \cdot y(t)\\
&\quad + 18 \cdot (8t^6 - 33t^5 - 447t^4 + 943t^3 - 447t^2 - 33t + 8)(t-1)^2(t^2-t+1)^5 t \cdot y(t)^2\\
&\quad + 108 \cdot (t-1)^4(t^2-t+1)^7 \cdot y(t)^3 + 27t^3(t-1)^4(t^2-t+1)^7 t^2 \cdot y(t)^4 = 0.
\end{aligned} \tag{C.4}$$

The second order-three linear differential operator B_3 is *homomorphic*[25] to the symmetric square of an order-two linear differential operator L_2 which is simply conjugated to the order-two linear differential operator L_K annihilating the complete elliptic integral of the first kind $K = {}_2F_1([1/2, 1/2], [1], t)$

$$B_3 \cdot I_2 = J_2 \cdot \mathrm{Sym}^2(L_2), \ \text{where:} \tag{C.5}$$

$$L_2 = \frac{1}{\rho(t)} \cdot L_K \cdot \rho(t) = D_t^2 + \frac{4}{3} \cdot \frac{2t-1}{t(t-1)} \cdot D_t + \frac{25t^2 - 25t + 1}{36t^2(t-1)^2}, \tag{C.6}$$

where $\rho(t) = t^{1/6} \cdot (1-t)^{1/6}$. It is worth comparing these results with similar calculations (see C.3 for a general identity on exterior square of symmetric products and direct sum of symmetric square) for the first order-four linear differential operator L_4 in section 5.2 which was the direct-sum of two linear differential operators (77). In that case the exterior square of L_4 is an order-six linear differential operator

$$L_6 = \mathrm{Ext}^2(L_4) = \mathrm{LCLM}(\tilde{A}_3, \tilde{B}_3) = \tilde{A}_3 \oplus \tilde{B}_3, \tag{C.7}$$

where the two order-three linear differential operators $\tilde{A}_3$ and $\tilde{B}_3$ are both *symmetric squares* of order-two linear differential operators having respectively the solutions

$$\begin{aligned}
&t^{5/6} \cdot (1-t)^{5/6} \cdot (t^2-t+1)^{-1/2} \cdot {}_2F_1\left([\frac{7}{6}, \frac{5}{2}], [\frac{7}{3}], t\right),\\
&t^{1/6} \cdot (1-t)^{1/6} \cdot (t^2-t+1)^{-3/4} \cdot {}_2F_1\left([-\frac{1}{12}, \frac{7}{12}], [1], \frac{27}{4}\frac{t^2 \cdot (1-t)^2}{(1-t+t^2)^3}\right),
\end{aligned} \tag{C.8}$$

totally reminiscent of the two solutions (78) and (79).

[25] With order-two intertwiners I_2 and J_2.

According to [25] the direct-sum decomposition (C.7) means that the order-four operator M_4 is *absolutely reducible*, i.e. it admits a *factorization over an algebraic extension* of $\mathbb{C}(t)$. This is confirmed by relation (83) in section 5.2

$$L_4 \cdot I_3 = J_3 \cdot \left(\frac{1}{\rho} \cdot M_4 \cdot \rho \right) \qquad \text{with:} \quad \rho = t^{2/3} \cdot (1-t)^{2/3}, \tag{C.9}$$

where I_3 and J_3 are order-three intertwiners and where the order-four operator L_4 is a symmetric product of two order-two linear differential operators (77).

C.3. Exterior square of symmetric products and direct sum of symmetric squares

Let us consider two order-two linear differential operators

$$\begin{aligned} L_2 &= D_t^2 - \frac{1}{w_L(t)} \cdot \frac{dw_L(t)}{dt} \cdot D_t + l(t), \\ M_2 &= D_t^2 - \frac{1}{w_M(t)} \cdot \frac{dw_M(t)}{dt} \cdot D_t + m(t), \end{aligned} \tag{C.10}$$

where $w_L(t)$ is the Wronskian of L_2 and $w_M(t)$ is the Wronskian of M_2. We have the following identity between the exterior square of symmetric product of these two linear differential operators and the LCLM (i.e. direct sum) of the symmetric squares of these two linear differential operators[26]:

$$\mathrm{Ext}^2\left(\mathrm{SymProd}(L_2, M_2)\right) = \left(w_M(t) \cdot \mathrm{Sym}^2(L_2) \cdot \frac{1}{w_M(t)} \right) \oplus \left(w_L(t) \cdot \mathrm{Sym}^2(M_2) \cdot \frac{1}{w_L(t)} \right). \tag{C.11}$$

In a more general framework, like in (C.5), we do not have an identity but an *equivalence* (homomorphisms) between the LHS and the RHS: see for instance Lemma 8 in [25].

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