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The Ising correlation $C(M,N)$ for $\nu = -k$

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Abstract

We present Painlevé VI sigma form equations for the general Ising low and high temperature two-point correlation functions $C(M,N)$ with $M \leqslant N$ in the special case $\nu = -k$ where $\nu = \sinh 2E_h/k_BT/\sinh 2E_v/k_BT$. More specifically four different non-linear ODEs depending explicitly on the two integers M and N emerge: these four non-linear ODEs correspond to distinguish respectively low and high temperature, together with $M+N$ even or odd. These four different non-linear ODEs are also valid for $M \geqslant N$ when $\nu = -1/k$. For the low-temperature row correlation functions $C(0,N)$ with N odd, we exhibit again for this selected $\nu = -k$ condition, a remarkable phenomenon of a Painlevé VI sigma function being the sum of four Painlevé VI sigma functions having the same Okamoto parameters. We show in this $\nu = -k$ case for $T < T_c$ and also $T > T_c$, that $C(M,N)$ with $M \leqslant N$ is given as an $N \times N$ Toeplitz determinant.

Keywords: Ising correlation functions, sigma form of Painlevé VI, Okamoto parameters

1. Introduction

The anisotropic Ising model on the square lattice is defined by the interaction energy

$$\mathcal{E} \;=\; -\sum_{j,k} \{E_v \sigma_{j,k}\sigma_{j+1,k} + E_h \sigma_{j,k}\sigma_{j,k+1}\}, \qquad (1)$$

where $\sigma_{j,k} = \pm 1$ is the spin at row j and column k and the sum is over all lattice sites. The free energy in the thermodynamic limit was computed by Onsager [1] in 1944.

The investigation of the correlation functions was initiated by Kaufman and Onsager [2] in 1949 and in 1963 Montroll, Potts and Ward [3] extended and simplified these results to show

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that all correlations can be written as determinants in an infinite number of ways. The simplest of these cases is the row correlation

$$C(0,N) \;=\; \langle \sigma_{0,0}\sigma_{0,N} \rangle \;=\; \begin{vmatrix} a_0 & a_{-1} & \cdots & a_{-N+1} \\ a_1 & a_0 & \cdots & a_{-N+2} \\ \vdots & \vdots & \vdots & \\ a_{N-1} & a_{N-2} & \cdots & a_0 \end{vmatrix}, \tag{2}$$

with

$$a_n \;=\; \frac{1}{2\pi} \int_0^{2\pi} \left[\frac{(1-\alpha_1\, e^{i\theta})(1-\alpha_2\, e^{-i\theta})}{(1-\alpha_1\, e^{-i\theta})(1-\alpha_2\, e^{i\theta})} \right]^{1/2} e^{in\theta}\, d\theta, \tag{3}$$

where

$$\alpha_1 \;=\; e^{-2E_v/k_BT}\, \tanh\, E_h/k_BT, \qquad \alpha_2 \;=\; e^{-2E_v/k_BT}\, \coth\, E_h/k_BT, \tag{4}$$

and the diagonal correlation $C(N,N)$ also given by (2) and (3) with

$$\alpha_1 \;=\; 0, \qquad \alpha_2 \;=\; k \;=\; (\sinh 2E_v/k_BT\; \sinh 2E_h/k_BT)^{-1}. \tag{5}$$

Both the free energy and the correlations have singularites at the critical temperature T_c defined by

$$k \;=\; (\sinh 2E_v/k_BT_c\; \sinh 2E_h/k_BT_c)^{-1} \;=\; 1. \tag{6}$$

In 1976 Wu, McCoy, Tracy and Barouch [4] discovered, in the scaling limit $T \to T_c$ with $N \cdot (T - T_c)$ fixed, that the diagonal correlation $C(N,N)$ is given by a Painlevé III function. This was generalized in 1980 by Jimbo and Miwa [5] who defined for $T < T_c$

$$\sigma \;=\; t \cdot (t-1) \cdot \frac{d}{dt}\, \ln\, C(N,N) \;-\; \frac{t}{4} \quad \text{with} \quad t \;=\; k^2, \tag{7}$$

and for $T > T_c$

$$\sigma \;=\; t \cdot (t-1) \cdot \frac{d}{dt}\, \ln\, C(N,N) \;-\; \frac{1}{4} \quad \text{with} \quad t \;=\; k^{-2}, \tag{8}$$

and in both cases derived:

$$\left(t \cdot (t-1) \cdot \frac{d^2\sigma}{dt^2} \right)^2 \;=\; N^2 \cdot \left((t-1) \cdot \frac{d\sigma}{dt} - \sigma \right)^2 \\ - 4 \cdot \frac{d\sigma}{dt} \cdot \left((t-1) \cdot \frac{d\sigma}{dt} - \sigma - \frac{1}{4} \right) \cdot \left(t\frac{d\sigma}{dt} - \sigma \right). \tag{9}$$

For $T < T_c$ the boundary condition for (9) at $t = 0$ is

$$C(N,N;\, t) \;=\; (1-t)^{1/4} \cdot \left[1 \,+ \lambda^2 \cdot \frac{(1/2)_N\, (3/2)_N}{4[(N+1)!]^2} \cdot t^{N+1} \cdot (1 \,+ O(t)) \right], \tag{10}$$

with $\lambda = 1$, $(a)_n = a(a+1)\cdots(a+n-1)$ and $(a)_0 = 1$. For $T > T_c$ the boundary condition at $t = 0$ is

$$C(N,N;\,t) \;=\; (1-t)^{1/4}\cdot\, t^{N/2}\cdot\,\Big[\frac{(1/2)_N}{N!}\cdot\, {}_2F_1\Big([\frac{1}{2},\, N+\frac{1}{2}],\,[N+1],\,t\Big) \qquad\qquad$$

$$+\,\lambda^2\cdot\, \frac{(1/2)_N\,((3/2)_N)^2}{16\,(N+1)!\,(N+2)!}\cdot\, t^{N+2}\cdot\,(1\;+O(t))\Big], \qquad (11)$$

with $\lambda = 1$.

We note for both cases of $T < T_c$ and $T > T_c$ that there are solutions of (9) with boundary condition where $\lambda \neq 1$. Those solutions do not correspond to the determinants for $C(N,N)$ but rather for the lambda extended Fredholm determinants obtained from the form factor expansions [6, 7]. We also remark that for $T > T_c$ the term in (10) with $\lambda = 0$ is by itself an exact solution of (9) even though it is not a correlation function of the Ising model.

It is an outstanding open question to generalize (9) *to the general two-point correlation functions* $C(M,N) = \langle\sigma_{0,0}\sigma_{M,N}\rangle$.

In this paper we consider the correlation $C(M,N)$ with anisotropy

$$\nu \;=\; \frac{\sinh 2E_h/k_BT}{\sinh 2E_v/k_BT}, \qquad (12)$$

for the special case

$$\nu \;=\; -k, \qquad (13)$$

which corresponds to

$$\sinh 2E_h/k_BT \;=\; \pm i, \qquad\quad k \;=\; \mp\frac{i}{\sinh 2E_v/k_BT}. \qquad (14)$$

Because $k \to 0$ as $T \to 0$ we refer to this case as $T < T_c$ for ν and k real even though E_v and E_h are complex (and hence *unphysical*).

We also consider the special case

$$\nu \;=\; -1/k \;=\; -k_>, \qquad (15)$$

where

$$\sinh 2E_v/k_BT \;=\; \pm i, \qquad\quad k_> \;=\; \mp i\;\sinh 2E_h/k_BT \qquad (16)$$

and because $k_> \to 0$ corresponds to $T \to \infty$ we refer to this case as $T > T_c$. In both cases we find that there is indeed a generalization of (9).

For concreteness we consider $M \geqslant 0$ and $N \geqslant 0$. We note that the formalism for $M \leqslant N$ and $M \geqslant N$ is different but, in the general, the symmetry under $M \leftrightarrow N$ and $E_v \leftrightarrow E_h$ yields the relation:

$$C(M,\,N;\,k,\,\nu) \;=\; C(N,\,M;\,k,\,1/\nu). \qquad (17)$$

However the restrictions (13) and (15) are *not preserved* by (17) and we have instead:

$$C(M,\,N;\,\nu=-k) \;=\; C(N,\,M;\,\nu=-1/k). \qquad (18)$$

In this paper we consider only $M \leqslant N$ with some remarks about $M \geqslant N$ at the end of subsection 3.3 and in the discussion section 7.

We recall previous results [8] on $C(M,N)$ in section 2. In section 3 we use the program *guessfunc* developed by Jay Pantone [9] to find, from large series expansion, *nonlinear differential equations* for $C(M,N)$ with $M \leqslant N$, both for $T < T_c$ (see equation (71) below) and for $T > T_c$ (see equations (73) and (79) below). In section 4 we transform these nonlinear differential equations into the canonical form of Okamoto [10] for sigma form of Painlevé VI. In section 5 we compare our equations with the ones obtained by Forrester and Witte for determinants [11] as given in [12] and show in (125), for $\nu = -k$ and $T < T_c$, that $C(M,N)$ for $M \leqslant N$ is expressed as an $N \times N$ Toeplitz determinant. Appendix C shows when $T > T_c$ that $C(M,N)$ can also be expressed as an $N \times N$ Toeplitz determinant. In section 6 we show for $T < T_c$, $\nu = -k$ and $M + N$ odd that $C(M,N)$, $M \leqslant N$, factors, and for $C(0,N)$ that these factors *also satisfy* an Okamoto sigma form of Painlevé VI equation. We conclude in section 7 with a discussion of several open questions. In appendix B we give examples of $C(M,N)$ with $\nu = -k$ for both $T < T_c$ and $T > T_c$. In appendix D we present the one parameter family of boundary conditions for the general Painlevé VI sigma Okamoto form which are analytic at $k = 0$.

2. The correlation $C(M,N)$ for $\nu = -k$

In [8] it was shown for all M,N that the correlation $C(M,N)$ with $M \leqslant N$ can be written for all M,N as a *homogeneous polynomial* in the three elliptic integrals

$$\tilde{K}(k) \; = \; \frac{2}{\pi} \int_0^{\pi/2} \frac{d\theta}{(1 - k^2 \sin^2\theta)^{1/2}} \; = \; {}_2F_1\left([\frac{1}{2}, \frac{1}{2}], [1], k^2\right), \qquad (19)$$

$$\tilde{E}(k) \; = \; \frac{2}{\pi} \int_0^{\pi/2} d\theta \, (1 - k^2 \sin^2\theta)^{1/2} \; = \; {}_2F_1\left([\frac{1}{2}, -\frac{1}{2}], [1], k^2\right), \qquad (20)$$

$$\tilde{\Pi}(-k\nu, k) \; = \; \frac{2}{\pi} \int_0^{\pi/2} \frac{d\theta}{(1 + k\nu \sin^2\theta)(1 - k^2 \sin^2\theta)^{1/2}}, \qquad (21)$$

where

$$k \; = \; (\sinh 2E_v/kT \; \sinh 2E_h/kT)^{-1} \; = \; (s_v \, s_h)^{-1} \; = \; \frac{\alpha_2 - \alpha_1}{1 - \alpha_1\alpha_2}, \qquad (22)$$

$$\nu \; = \; \frac{\sinh 2E_h/kT}{\sinh 2E_v/kT} \; = \; \frac{s_h}{s_v} \; = \; \frac{4\,\alpha_1\alpha_2}{(\alpha_2 - \alpha_1)(1 - \alpha_1\alpha_2)}, \qquad k\,\nu \; = \; \frac{1}{s_v^2}, \qquad (23)$$

which are valid for

$$0 \leqslant k \leqslant 1 \quad \text{and} \quad -1 \leqslant k\,\nu, \qquad (24)$$

${}_2F_1([a,b], [c], z)$ being the hypergeometric function.

2.1. $C(0,1)$ for $\nu = -k$

It was shown in [1, 2], for $T < T_c$ where $\alpha_1 < \alpha_2 < 1$, that

$$C(0,1) \; = \; \sqrt{1 + \nu k} \cdot \left[\left(1 + \frac{k}{\nu}\right) \cdot \tilde{\Pi}(-\nu k, k) \; - \frac{k}{\nu} \cdot \tilde{K}(k) \right], \qquad (25)$$

which is conveniently rewritten as

$$C(0,1) \; = \; \sqrt{1 + \nu k} \cdot \frac{2}{\pi} \int_0^{\pi/2} d\theta \cdot \frac{(1 - k^2 \sin^2\theta)^{1/2}}{1 + k\nu \sin^2\theta}. \qquad (26)$$

For $T > T_c$ where $1 < \alpha_2$ and $k_> = 1/k$

$$C(0,1) = \frac{1}{\nu} \cdot \sqrt{1+\nu/k_>} \cdot \Big[(1+\nu k_>) \cdot \tilde{\Pi}(-\nu k_>, k_>) - \tilde{K}(k_>)\Big]$$

$$= k_> \cdot \sqrt{1+\nu/k_>} \cdot \frac{2}{\pi} \int_0^{\pi/2} \frac{1 - \sin^2\theta}{(1 + k_>\nu \sin^2\theta)(1 - k_>^2 \sin^2\theta)^{1/2}} \cdot \mathrm{d}\theta. \qquad (27)$$

In general $C(M,N)$ depends on two (complex) variables through the elliptic integral $\tilde{\Pi}(-k\nu, k)$. However, as is seen in (2) and (3) the row correlation $C(0,N)$ reduces to $C(N,N)$ when $\alpha_1 = 0$ (which from (23) is equivalent to $\nu = 0$) because $\tilde{\Pi}(-k\nu, k)$ degenerates to $\tilde{K}(k)$.

There are two other special cases where $\tilde{\Pi}(-k\nu, k)$ reduces to combinations of $\tilde{K}(k)$ and $\tilde{E}(k)$. One case is the *isotropic* case $\nu = 1$ where

$$\tilde{\Pi}(-k, k) = \frac{1}{2} \cdot \tilde{K}(k) + \frac{1}{2 \cdot (1+k)}. \qquad (28)$$

Many examples of this reduction of $C(M,N)$ have been given by Shrock and Ghosh [13, 14].

Another case of reduction of $\tilde{\Pi}(-k\nu, k)$ is

$$\nu = -k, \qquad (29)$$

where

$$\tilde{\Pi}(k^2, k) = \frac{\tilde{E}(k)}{1-k^2}, \qquad (30)$$

and the $C(M,N)$, $M \leqslant N$ *are reduced to homogeneous polynomials of the two ellptic integrals* $\tilde{E}(k)$ and $\tilde{K}(k)$. For example for $T < T_c$ when $\nu = -k$ we see from (26) that

$$C(0,1) = \sqrt{1-k^2} \cdot \frac{2}{\pi} \int_0^{\pi/2} \frac{1}{(1-k^2 \sin^2\theta)^{1/2}} \cdot \mathrm{d}\theta = \sqrt{1-k^2} \cdot \tilde{K}(k). \qquad (31)$$

For $T > T_c$ from (27) when $\nu = -k_> = -1/k$

$$C(0,1) = 0, \qquad (32)$$

because of the vanishing of the square root factor. If we remove this factor by writing for $T > T_c$

$$C(0,1) = \sqrt{1+\nu/k_>} \cdot \tilde{C}(0,1), \qquad (33)$$

we find, in the special case $\nu = -k_>$, that

$$\tilde{C}(0,1) = \frac{k_>}{2} \cdot {}_2F_1([\frac{3}{2}, \frac{1}{2}], [2], k_>^2) = \frac{\tilde{K}(k_>) - \tilde{E}(k_>)}{k_>}. \qquad (34)$$

2.2. $C(0,N)$ for $T < T_c$ at $\nu = -k$

More generally for $T < T_c$ we find, from (22) and (23), that if $\nu = -k$ then

$$\alpha_2 = -\alpha_1 = \alpha \quad \text{and} \quad k = \frac{2\alpha}{\alpha^2+1}, \qquad s_h = i, \qquad s_v = -\frac{i}{k}, \qquad (35)$$

where, for simplicity, we have defined $\alpha = \alpha_2$. Thus for $\nu = -k$ the a_n matrix elements (3) reduce to

$$a_n \;=\; \frac{1}{2\pi}\int_0^{2\pi} \frac{1\,-\alpha^2\,+\alpha\,(e^{i\theta}\,-e^{-i\theta})}{\{(1\,-\alpha^2\,e^{2i\theta})\,(1-\alpha^2\,e^{-2i\theta})\}^{1/2}}\cdot e^{in\theta}\cdot d\theta, \tag{36}$$

which, by sending $\alpha \to -\alpha$ and $\theta \to -\theta$ has the symmetry:

$$a_{-n}(k) \;=\; a_n(-k). \tag{37}$$

By considering invariance under $\theta \to \theta + \pi$ and setting $2\theta = \phi$, we see that

$$a_{2m} \;=\; \frac{1}{2\pi}\int_0^{2\pi} \frac{1-\alpha^2}{\{(1\,-\alpha^2\,e^{i\phi})\,(1\,-\alpha^2\,e^{-i\phi})\}^{1/2}}\cdot e^{i|m|\phi}\cdot d\phi, \tag{38}$$

and:

$$a_{\pm(2|m|+1)} \;=\; \frac{\pm}{2\pi}\int_0^{2\pi} \frac{\alpha(e^{i\phi}-1)}{\{(1-\alpha^2\,e^{i\phi})\,(1\,-\alpha^2 e^{-i\phi})\}^{1/2}}\cdot e^{i|m|\phi}\cdot d\phi. \tag{39}$$

We may reduce a_{2m} to a hypergeometric function as

$$a_{2m} \;=\; (1\,-\alpha^2)\cdot\,\alpha^{2|m|}\cdot\,\frac{\Gamma(|m|+1/2)}{\pi^{1/2}|m|!}\cdot\,{}_2F_1\left([|m|+\frac{1}{2},\frac{1}{2}],\,[|m|+1],\,\alpha^4\right), \tag{40}$$

which may be rewritten in terms of k using (5) of page 111 of [15]:

$${}_2F_1\left([m+\frac{1}{2},m+\frac{1}{2}],\,[,2m+1],\;k^2\right) \;=\; (1+\alpha^2)^{2m+1}\cdot\,{}_2F_1\left([m+\frac{1}{2},\frac{1}{2}],\,[m+1],\;\alpha^4\right). \tag{41}$$

We find that:

$$\begin{aligned} a_{2m} \;&=\; \left(\frac{\alpha}{1+\alpha^2}\right)^{2|m|}\frac{1-\alpha^2}{1+\alpha^2}\cdot\frac{\Gamma(|m|+1/2)}{\pi^{1/2}|m|!}\cdot{}_2F_1\left([|m|+\frac{1}{2},|m|+\frac{1}{2}],\,[2\,|m|+1],\;k^2\right)\\ &=\;\left(\frac{k}{2}\right)^{2|m|}\cdot\sqrt{1-k^2}\cdot\frac{\Gamma(|m|+1/2)}{\pi^{1/2}|m|!}\cdot{}_2F_1\left([|m|+\frac{1}{2},|m|+\frac{1}{2}],\,[2\,|m|+1],\;k^2\right). \end{aligned} \tag{42}$$

Similarly:

$$\begin{aligned} a_{2m+1} \;=\; \left(\frac{k}{2}\right)^{2m+1}\frac{\Gamma(m+1/2)}{\pi^{1/2}\,m!}\cdot\Bigg[\left(\frac{k}{2}\right)^2\frac{m+1/2}{m+1}\cdot{}_2F_1\left([m+\frac{3}{2},m+\frac{3}{2}],\,[2m+3],\,k^2\right)\\ -\;{}_2F_1\left([m+\frac{1}{2},m+\frac{1}{2}],\,[2m+1],\;k^2\right)\Bigg]. \end{aligned} \tag{43}$$

The two hypergeometric functions in (43) combine and thus, with the symmetry (37), give the final result:

$$a_{\pm(2m+1)} \;\;=\mp\left(\frac{k}{2}\right)^{2|m|+1}\cdot\,\frac{\Gamma(|m|+1/2)}{\pi^{1/2}|m|!}\cdot\,{}_2F_1\left([|m|+\frac{1}{2},|m|+\frac{1}{2}],\,[2\,|m|+2],\;k^2\right). \tag{44}$$

2.3. $C(0, N)$ for $T > T_c$ at $\nu = -k_> = -1/k$

In the following we will simply denote $\tilde{K}(k)$, $\tilde{E}(k)$ and $\tilde{\Pi}(-k\nu, k)$ (see (19), (20), (21)) by $\tilde{K}$, $\tilde{E}$ and $\tilde{\Pi}$. For $T > T_c$ we find, from (22) and (23), that if $\nu = -k_> = -1/k$ then

$$\alpha_1 \; = \; -\alpha_2^{-1} \; = \; -\alpha, \quad \text{and} \quad k_> \; = \; \frac{2\alpha}{\alpha^2+1}, \qquad s_h \; = \; -ik_>, \qquad s_v \; = \; i, \tag{45}$$

and we find the matrix elements (3) reduce to

$$a_n \; = \; -\frac{1}{2\pi}\int_0^{2\pi} \left[\frac{1-\alpha^2\,\mathrm{e}^{2\mathrm{i}\theta}}{1-\alpha^2\,\mathrm{e}^{-2\mathrm{i}\theta}}\right]^{1/2} \cdot \mathrm{e}^{(n-1)\mathrm{i}\theta}\cdot \mathrm{d}\theta. \tag{46}$$

By sending $\theta \to \theta + \pi$ we see that $a_n = (-1)^{n-1} a_n$, and thus

$$a_{2n} \; = \; 0, \tag{47}$$

and

$$\begin{aligned} a_{2n+1} \; &= \; -\frac{1}{2\pi}\int_0^{2\pi} \left[\frac{1-\alpha^2\,\mathrm{e}^{2\mathrm{i}\theta}}{1-\alpha^2\,\mathrm{e}^{-2\mathrm{i}\theta}}\right]^{1/2} \cdot \mathrm{e}^{2n\mathrm{i}\theta}\cdot \mathrm{d}\theta \\ &= \; -\frac{1}{2\pi}\int_0^{2\pi} \left[\frac{1-\alpha^2\,\mathrm{e}^{\mathrm{i}\phi}}{1-\alpha^2\,\mathrm{e}^{-\mathrm{i}\phi}}\right]^{1/2} \cdot \mathrm{e}^{n\mathrm{i}\phi}\cdot \mathrm{d}\phi, \end{aligned} \tag{48}$$

which we recognize as the matrix elements a_{-n} of the diagonal correlation (3) for $T < T_c$ with $\alpha_1 = 0$ and $\alpha_2 \to \alpha^2$.

We further recognize because of (47) that

$$C(0, 2N+1) \; = \; 0, \tag{49}$$

and that the $2N \times 2N$ determinants for $C(0, 2N)$ factorize as:

$$C(0,2N) \; = \; \begin{vmatrix} a_{-1} & a_1 & \cdots & a_{2N-3} \\ a_{-3} & a_{-1} & \cdots & a_{2N-5} \\ \vdots & \vdots & \vdots & \\ a_{-(2N-1)} & a_{-(2N-3)} & \cdots & a_{-1} \end{vmatrix} \times \begin{vmatrix} a_1 & a_3 & \cdots & a_{2N-1} \\ a_{-1} & a_1 & \cdots & a_{2N-3} \\ \vdots & \vdots & \vdots & \\ a_{-(2N-3)} & a_{-(2N-5)} & \cdots & a_1 \end{vmatrix}. \tag{50}$$

For $2n + 1 > 0$ we express (48) in terms of hypergeometric functions as

$$a_{2n+1} \; = \; \alpha^{2n} \cdot \frac{\Gamma(n+1/2)}{\sqrt{\pi} n!} \cdot {}_2F_1\left([-\frac{1}{2}, n+\frac{1}{2}], [n+1], \; \alpha^4\right), \tag{51}$$

and:

$$a_{-(2n+1)} \; = \; \alpha^{2n+2} \cdot \frac{\Gamma(n+1/2)}{2\,\sqrt{\pi}(n+1)!} \cdot {}_2F_1\left([\frac{1}{2}, n+\frac{1}{2}], \; [n+2], \; \alpha^4\right). \tag{52}$$

As a special case we note from (56) of [8] by use of (45) for $\nu = -k_>$ that

$$C(0,2) \; = \; k_>^{-2} \cdot \left(\tilde{E}^2 \; - (1 \; - k_>^2) \cdot \tilde{K}^2\right)$$
$$= \; k_>^{-2} \cdot \left(\tilde{E} \; - \sqrt{1 - k_>^2} \cdot \tilde{K}\right) \cdot \left(\tilde{E} \; + \sqrt{1 - k_>^2} \cdot \tilde{K}\right), \qquad (53)$$

which illustrates the factorization property of $C(0, 2N)$. For small k we have

$$C(0,2) \; = \; \frac{1}{8}\, k_>^2 \; + \frac{1}{16}\, k_>^4 \; + \frac{39}{1024}\, k_>^6 \; + \frac{53}{2048}\, k_>^8 \; + \frac{1235}{65\,536}\, k_>^{10}$$
$$+ \frac{1887}{131\,072}\, k_>^{12} \; + \frac{382\,291}{33\,554\,432}\, k_>^{14} \; + O(k^{16}), \qquad (54)$$

which using (45) is rewritten in terms of α as

$$C(0,2) \; = \; \frac{1}{2}\,\alpha^2 \; - \frac{1}{16}\,\alpha^6 \; - \frac{1}{64}\,\alpha^{10} \; - \frac{13}{2048}\,\alpha^{14} \; + O(\alpha^{18}). \qquad (55)$$

Using maple we find

$$\tilde{E} \; + \sqrt{1 - k_>^2} \cdot \tilde{K} \; = \; \frac{2}{1 + \alpha^2} \cdot a_1 \qquad (56)$$

$$\frac{1}{k_>^2} \cdot \left(\tilde{E} \; - \sqrt{1 - k_>^2} \cdot \tilde{K}\right) \; = \; \frac{1 + \alpha^2}{2} \cdot a_{-1}, \qquad (57)$$

or equivalently using

$$\alpha \; = \; \frac{1 - \sqrt{1 - k_>^2}}{k_>}, \qquad (58)$$

we have:

$$a_1 \; = \; \frac{1 - \sqrt{1 - k_>^2}}{k_>^2} \cdot \left(\tilde{E} \; + \sqrt{1 - k^2} \cdot \tilde{K}\right), \qquad (59)$$

$$a_{-1} \; = \; \frac{1 + \sqrt{1 - k_>^2}}{k_>^2} \cdot \left(\tilde{E} \; - \sqrt{1 \; - k_>^2} \cdot \tilde{K}\right). \qquad (60)$$

To generalize and derive (59) and (60) we treat a_{2n+1}/α and $\alpha a_{-(2n+1)}$ separately. These calculations are detailed in appendix A.

2.4. Quadratic difference equations for C(M, N)

In general for $M \neq 0$ the correlation $C(M, N)$ with $M < N$ can be written as an $N \times N$ determinant which is *not* Toeplitz. We will not use this determinant representation but, instead, use quadratic difference equations [16–19] which relate the (high-temperature) correlation functions $C(M, N)$ for $T > T_c$ to the *dual correlation* $C_d(M, N)$ for $T > T_c$, defined as the low temperature correlation with the replacement: $s_v \rightarrow 1/s_h$ and $s_h \rightarrow 1/s_v$

$$s_h^2 \cdot \, [C_d(M,N)^2 \, - C_d(M,N-1) \cdot \, C_d(M.N+1)] \\ +[C(M,N)^2 \, - C(M-1,N) \cdot \, C(M+1,N)] \, = \, 0, \tag{61}$$

$$s_v^2 \cdot \, [C_d(M,N)^2 \, - C_d(M-1,N) \cdot \, C_d(M+1,N)] \\ + [C(M,N)^2 \, - C(M,N-1) \cdot \, C(M,N+1)] \, = \, 0, \tag{62}$$

$$s_v s_h \cdot \, [C_d(M,N) \cdot \, C_d(M+1,N+1) \, - C_d(M,N+1) \cdot \, C_d(M+1,N)] \\ = \, C(M,N) \cdot \, C(M+1,N+1) \, - C(M,N+1) \cdot \, C(M+1,N), \tag{63}$$

which hold for all M and N, except $M = 0, N = 0$, where we have:

$$C(1,0) \, = \, (1+s_h^2)^{1/2} \, - s_h \cdot \, C_d(0,1), \tag{64}$$

$$C(0,1) \, = \, (1+s_v^2)^{1/2} \, - s_v \cdot \, C_d(1,0). \tag{65}$$

with $s_h = \sinh 2E_h/kT$ and $s_v = \sinh 2E_v/kT$.

From these quadratic difference equations we find [8] for example for $T < T_{\rm c}$ where $k = (s_v s_h)^{-1}$:

$$C(1,2) \, = \, s_v^2 \cdot (s_v^{-2}+1)^{1/2} \cdot \Big(s_h^{-2} \cdot (s_v^{-2} s_h^{-2} - 1) \cdot \, \tilde{K}^2 \, + (s_h^{-2} - 1) \cdot \, \tilde{E}\tilde{K} \, + E^2 \\ + (s_v^{-2} - 1)\,(s_h^{-2} + 1) \cdot \, \tilde{E}\tilde{\Pi} \, - (s_h^{-2}+1) \cdot \, (s_v^{-2} s_h^{-2} - 1) \cdot \, \tilde{K}\tilde{\Pi} \Big), \tag{66}$$

and for $T > T_{\rm c}$ where $k_> = s_v s_h$:

$$C(1,2) \, = \, \frac{(s_v^2+1)^{1/2}}{s_h^2 s_v} \cdot \, \Big(\tilde{E}^2 \, - (s_h^2 s_v^2 - 1) \cdot \, \tilde{K}^2 \, + (s_h^2 s_v^2 \, + s_v^2 \, - 2) \cdot \, \tilde{E}\tilde{K} \\ + (s_h^2 \, + 1) \cdot \, (s_v^2 \, - 1) \cdot \, \tilde{E}\tilde{\Pi} \, + (s_h^2 \, + 1) \cdot \, (s_h^2 s_v^2 \, - 1) \cdot \, \tilde{K}\tilde{\Pi}\Big). \tag{67}$$

For $T < T_{\rm c}$ and $\nu = -k$, where $s_h = i, s_v = -i/k$, one has:

$$C(1,2) \, = \, -(1 \, - k^2)^{1/2} \cdot \, k^{-2} \cdot \, \big((1 \, - k^2) \cdot \, \tilde{K}^2 \, - 2 \cdot \, \tilde{E}\tilde{K} \, + \tilde{E}^2\big). \tag{68}$$

For $T > T_{\rm c}$ and $\nu = -k_>$ where $s_h = -ik_>, s_v = i$ one gets:

$$C(1,2) \, = \, 0. \tag{69}$$

Further special cases are given in appendix B.

3. Two-parameter family of nonlinear differential equations for $C(M, N)$ for $\nu = -k$

We have obtained a nonlinear equation with the Painlevé property (i.e. fixed critical points) which is satisfied by $C(M,N)$ for $\nu = -k$ by using the program *guessfunc* developed by J Pantone [9]. This program searches for nonlinear equations satisfied by series expansions. We have applied this program for many values of the integers M, N for the series expansions of $C(M,N)$ at $\nu = -k$ obtained from either the Toeplitz determinants for $C(0,N)$ of section 2, or from expressions deduced from the quadratic recursion relations of section 2.4. The results are as follows.

3.1. Nonlinear differential equations for $C(M,N)$ for $\nu = -k$ and $M \leqslant N$: the low-temperature case

For $T < T_c$ and $\nu = -k$ with $t = k^2$ and

$$\sigma \; = \; t \cdot (t-1) \cdot \frac{d \ln C(M,N)}{dt} \; - \frac{t}{4}, \qquad (70)$$

we have:

$$[t \cdot (t-1) \cdot \sigma'']^2 \; + 4 \cdot \{\sigma' \cdot (t\sigma' - \sigma) \cdot ((t-1)\sigma' - \sigma)\}$$
$$- M^2 \cdot (t\sigma' - \sigma)^2 \; - N^2 \cdot \sigma'^2$$
$$+ [M^2 + N^2 \; - \frac{1}{2}(1 \; + (-1)^{M+N})] \cdot \sigma' \cdot (t\sigma' - \sigma) \; = \; 0. \qquad (71)$$

Note that when $M = N$ the diagonal correlation $C(N,N)$ *does not depend on the anisotropy variable* ν. There is no difference between the diagonal correlation functions $C(N,N)$ for $\nu = -k$ and for arbitrary ν. As expected the two-parameters (M,N)-family of nonlinear differential equation (71) actually reduces when $M = N$ to the Jimbo–Miwa nonlinear differential equation (9) for the diagonal correlation $C(N,N)$ for $T < T_c$.

3.2. Nonlinear differential equations for $C(M,N)$ for $\nu = -k$, $M \leqslant N$ and $M+N$ even: the high-temperature case

For $T > T_c$ and $\nu = -k_>$ with $M \leqslant N$ and $M+N$ *even* with $t = k_>^2$ and

$$\sigma \; = \; t \cdot (t-1) \cdot \frac{d \ln C(M,N)}{dt} \; - \frac{1}{4}, \qquad (72)$$

we have:

$$[t \cdot (t-1) \cdot \sigma'']^2 \; + 4 \cdot \{\sigma' \cdot (t\sigma' - \sigma) \cdot ((t-1)\sigma' - \sigma)\}$$
$$- M^2 \cdot (t\sigma' - \sigma)^2 \; + (N^2 + M^2 - 1) \cdot \sigma' \cdot (t\sigma' - \sigma)$$
$$- N^2 \cdot \sigma'^2 \; - \frac{1}{4}(N^2 - M^2) \cdot (t\sigma' - \sigma)$$
$$- \frac{1}{4} \cdot (N^2 - M^2) \cdot \sigma' - \frac{1}{16} \cdot (N^2 - M^2)^2 \; = \; 0. \qquad (73)$$

As expected, when $M = N$, the two-parameters (M,N)-family of nonlinear differential equation (73) *also reduces to the Jimbo–Miwa nonlinear differential equation* (9) for the diagonal correlation $C(N,N)$ for $T > T_c$.

3.3. Nonlinear differential equation for $C(M,N)$ for $\nu = -k_>$, $M < N$ and $M+N$ odd: the high temperature case

For $T > T_c$ and $\nu = -k_>$ we found in (49) that $C(0, 2N+1) = 0$. For $C(0,1)$ this vanishing occurs because of the vanishing of the factor $\sqrt{1 + \nu/k_>}$ in (27) and we found, for instance in (34), that:

$$\lim_{\nu \to -k_>} \left(1 + \frac{\nu}{k_>}\right)^{-1/2} \cdot C(0,1) \; = \; \frac{\tilde{K} - \tilde{E}}{k_>}. \qquad (74)$$

We have examined this phenomenon more generally by considering $C(M,N)$ for low values of M,N and find that for $M+N$ odd

$$C(M,N) \;=\; 0, \tag{75}$$

and that the limit

$$\lim_{\nu\to -k_>} \left(1 + \frac{\nu}{k_>}\right)^{-1/2} \cdot C(M,N) \;=\; \tilde{C}(M,N), \tag{76}$$

exists and is nonzero. For example

$$\tilde{C}(0,3) \;=\; \frac{4}{k_>^5} \cdot \Big[(k_>^2-1)^2 \cdot \tilde{K}^3 \;-(2k_>^2-3)\cdot(k_>^2-1)\cdot \tilde{K}^2\,\tilde{E} \\ -(2k_>^2-3)\cdot \tilde{K}\,\tilde{E}^2 \;-(k_>^2+1)\cdot \tilde{E}^3\Big]. \tag{77}$$

We define

$$\sigma \;=\; t\cdot(t-1)\cdot \frac{\mathrm{d}\ln \tilde{C}(M,N)}{\mathrm{d}t} \;-\frac{1}{4}, \tag{78}$$

and find:

$$[t\cdot(t-1)\cdot\sigma'']^2 \;+4\cdot\{\sigma'\cdot(t\,\sigma'-\sigma)\cdot((t-1)\,\sigma'-\sigma)\} \\ -M^2\cdot(t\,\sigma'-\sigma)^2 \;+(N^2+M^2-2)\cdot\sigma'\cdot(t\,\sigma'-\sigma) \\ -N^2\cdot\sigma'^2 \;-\frac{1}{4}\cdot(N^2-M^2-1)\cdot(t\,\sigma'-\sigma) \;-\frac{1}{4}\cdot(N^2-M^2+1)\cdot\sigma' \\ -\frac{1}{16}\cdot(N^2-M^2)^2 \;+\frac{1}{8}\cdot(M^2+N^2-1) \;=\; 0. \tag{79}$$

Remark: All the previous sigma non-linear ODEs (71), (73) and (79) are valid for $\nu=-k$ and $M \leqslant N$. However recalling the symmetry relations (17) and especially (18), $C(M,N;\nu=-k)=C(N,M;\nu=-1/k)$, it is straightforward to see that these sigma non-linear ODEs (71), (73) and (79) *are also valid* for the $C(M,N)$ correlation functions for $M \geqslant N$ *but when* $\nu=-1/k$.

3.3.1. A Kramers–Wannier formal symmetry. In [7] a representation of the Kramers–Wannier duality on σ has been introduced[4]:

$$(t,\,\sigma,\,\sigma'\sigma'') \longrightarrow \left(\frac{1}{t},\;\frac{\sigma}{t},\;\sigma\,-\,t\cdot\sigma',\;t^3\cdot\sigma''\right). \tag{80}$$

It had been noticed that this involutive transformation (80) preserves the sigma form of Painlevé VI equation (9). This transformation just amounts to saying, for any function $F(t)$, that the change of variable $t \to 1/t$ changes

$$\sigma \;=\; t\cdot(t-1)\cdot\frac{\mathrm{d}\,\ln(F(t))}{\mathrm{d}t} \;-\frac{1}{4}, \tag{81}$$

[4] See equation (16) page 78 of [7].

into $\tilde{\sigma}/t$ where:

$$\tilde{\sigma} \,\, = \,\, t \cdot (t-1) \cdot \frac{\mathrm{d} \, \ln(F(t))}{\mathrm{d}t} \,\, - \frac{t}{4}. \tag{82}$$

and vice-versa.

- One first remarks that this involutive transformation (80) actually transforms the (low-temperature) nonlinear differential equation (71) into itself where M and N are permuted.
- One then remarks that this involutive transformation (80) *also transforms the* (*high-temperature, $M+N$ even*) *non-linear differential equation* (73) *into itself where M and N are permuted.*
- One finally remarks that this involutive transformation (80) *also transforms the* (*high-temperature, $M+N$ odd*) *non-linear differential equation* (79) *into itself where M and N are permuted.*

These three results must be seen as mathematical symmetries: the question of the physical interpretation of the non-linear differential equations (71), (73) and (79) when M and N are permuted, remains an open question.

Note that ν *is left invariant by the Kramers–Wannier duality*[5], in contrast with k which becomes its reciprocal $k \rightarrow 1/k$. Consequently, the selected condition $\nu = -k$ is *not left invariant by the Kramers–Wannier duality*. The high-temperature non-linear differential equations (73) and (79), valid at $\nu = -k$ have no reason to be deduced from the low-temperature non-linear differential equation (71), valid at $\nu = -k$, using a Kramers–Wannier-like duality (80).

Along this line, let us note, for $M+N$ even, that one can change the low-temperature non-linear differential equation (71) into the high-temperature non-linear differential equation (73) using the involutive transformation:

$$(\sigma, \, \sigma', \, \sigma'', \, M, \, N) \longrightarrow \left(\sigma \, + \, \frac{N^2 \, - M^2}{4} \cdot (t \, - 1), \,\, \sigma' \, + \, \frac{N^2 \, - M^2}{4}, \, \sigma'', \, N, \, M\right). \tag{83}$$

4. Sigma form of Painlevé VI: Okamoto parameters

The search for nonlinear differential equations with the Painlevé property is an ongoing field of research [20] and is far from being complete even for equations of second order. However for equations of the form

$$(y'')^2 \,\, = \,\, F(y, \, y', \, x), \tag{84}$$

with *fixed singularities* at $x = 0, 1, \infty$, a solution was given by Cosgrove and Scoufis in (4.9) of [21] where it is shown that the non-linear differential equation with six parameters

$$\begin{aligned} (x \cdot (x-1) \cdot y'')^2 \, &+ 4 \cdot \left\{ y' \cdot (xy' - y)^2 \, - y'^2 \cdot (xy' \, - y) \right. \\ &+ c_5 \cdot (xy' - y)^2 \, + c_6 \cdot y' \cdot (xy' - y) \, + c_7 \cdot (y')^2 \\ &\left. + c_8 \cdot (xy' - y) \, + c_9 \cdot y' \, + c_{10} \right\} \, = \, 0, \end{aligned} \tag{85}$$

[5] The Kramers–Wannier duality changes $s_h \rightarrow s_v^* \, = \, 1/s_v$, $s_v \rightarrow s_h^* \, = \, 1/s_h$ and thus $s_h/s_v \rightarrow s_h/s_v$.

has the Painlevé property[6] and is birationally equivalent to Painlevé VI. Both equations (71) and (79) are of the form (85) and hence are sigma forms of Painlevé VI. The non-linear differential equation (85) is invariant in form under the transformation

$$y \;=\; \bar{y} \;+A\cdot x \,+B, \tag{86}$$

which transforms the six parameters c_k into new parameters $\tilde{c}_k$ as follows

$$\tilde{c}_5 \;=\; c_5 \;+A, \qquad \tilde{c}_6 \;=\; c_6 \;-2B \;-2A, \tag{87}$$

$$\tilde{c}_7 \;=\; c_7 \;+B, \qquad \tilde{c}_8 \;=\; c_8 \;-2AB \;-A^2 \;-2B\cdot c_5 \;+A\cdot c_6, \tag{88}$$

$$\tilde{c}_9 \;=\; c_9 \;+B^2 \;+2AB \;-B\cdot c_6 \;+2A\cdot c_7, \tag{89}$$

$$\tilde{c}_{10} \;=\; c_{10} \;+AB^2 \;+A^2B \;+B^2\cdot c_5 \;-AB\cdot c_6 \;+A^2\cdot c_7 \;-B\cdot c_8 \;+A\cdot c_9. \tag{90}$$

The canonical form of sigma Painlevé VI given by Okamoto [10] which depends on four parameters n_1, n_2, n_3, n_4 reads

$$h'\cdot\,\{t\cdot\,(t-1)\cdot\,h''\}^2 \;+\{h'\cdot\,(2h\;-(2t-1)\,h')\;+n_1n_2n_3n_4\}^2$$
$$-\,(h'\;+n_1^2)\cdot\,(h'\;+n_2^2)\cdot\,(h'\;+n_3^2)\cdot\,(h'\;+n_4^2)\;\;=\;\;0, \tag{91}$$

which when expanded and removing the common factor of h' reads

$$\{t\cdot\,(t-1)\cdot\,h''\}^2 \;+4\,h'\cdot\,(t\,h'\;-h)\cdot\,((t\;-1)\,h'\;-h)$$
$$+4\,c_{10}\;\;+4\,c_9\cdot\,h'\;\;+4\,c_8\cdot\,(t\cdot\,h'\;-h)\;\;\;+4\,c_7\cdot\,h'^2\;\;=\;\;0, \tag{92}$$

which is of the Cosgrove form (85) with

$$c_7 \;=\; -(n_1^2+n_2^2+n_3^2+n_4^2)/4, \qquad c_8 \;=\; -n_1n_2n_3n_4, \tag{93}$$

$$c_9 \;=\; -(n_1^2n_2^2+n_1^2n_3^2\;+n_1^2n_4^2+n_2^2n_3^2\;+n_2^2n_4^2+n_3^2n_4^2\;-2n_1n_2n_3n_4)/4, \tag{94}$$

$$c_{10} \;=\; -(n_1^2n_2^2n_3^2\;+n_1^2n_2^2n_4^2\;+n_1^2n_3^2n_4^2\;+n_2^2n_3^2n_4^2)/4. \tag{95}$$

We see from (87) if we choose

$$A \;=\; -c_5, \qquad B \;=\; \frac{c_6}{2}\;+c_5, \tag{96}$$

that

$$\tilde{c}_5 \;=\; \tilde{c}_6 \;=\; 0, \tag{97}$$

and thus the general non-linear differential equation (85) is reduced to the Okamoto form with the Okamoto parameters determined from (93)–(95).

[6] Namely has fixed critical points.

4.1. Okamoto parameters for $T < T_c$ and $\nu = -k$

For $T < T_c$ and $\nu = -k$ we obtain the Okamoto parameters for (71) with the parameters which shift from (71) to the canonical Okamoto form determined from (96), to be

$$A \,=\, \frac{M^2}{4}, \qquad B \,=\, \frac{1}{8}\cdot\Big(N^2 - M^2 \,-\, \frac{1+(-1)^{M+N}}{2}\Big). \tag{98}$$

Thus we find from (88)–(90) that

$$\tilde{c}_7 \,=\, -\frac{1}{8}\cdot\Big(N^2 + M^2 + \frac{1+(-1)^{M+N}}{2}\Big), \qquad \tilde{c}_8 \,=\, \frac{M^2}{16}\cdot\Big(N^2 \,-\, \frac{1+(-1)^{M+N}}{2}\Big),$$
$$\tilde{c}_9 \,=\, -\frac{1}{64}\cdot\Big(N^2 - M^2 \,-\, \frac{1+(-1)^{M+N}}{2}\Big)^2 \,-\, \frac{N^2M^2}{8},$$
$$\tilde{c}_{10} \,=\, -\frac{M^2}{128}\cdot\Big[\Big(N^2 - M^2 \,-\, \frac{1+(-1)^{M+N}}{2}\Big)\cdot\Big(N^2 \,-\, \frac{1+(-1)^{M+N}}{2}\Big) \,+2\,M^2N^2\Big], \tag{99}$$

and thus, from (93)–(95), we obtain the Okamoto parameters (unique up to permutations and the change of an even number of signs)

$$n_1 \,=\, \frac{1}{2}\cdot\Big(N - \frac{1+(-1)^{M+N}}{2}\Big), \qquad n_2 \,=\, \frac{1}{2}\cdot\Big(N + \frac{1+(-1)^{M+N}}{2}\Big),$$
$$n_3 \,=\, \frac{M}{2}, \qquad n_4 \,=\, -\frac{M}{2}, \tag{100}$$

with

$$h \,=\, t\cdot(t-1)\cdot\frac{\mathrm{d}\ln C(M,N)}{\mathrm{d}t}$$
$$-\frac{M^2+1}{4}\cdot t \,-\, \frac{1}{8}\cdot\Big(N^2 - M^2 \,-\, \frac{1+(-1)^{M+N}}{2}\Big), \tag{101}$$

where $t = k^2$.

4.2. Okamoto parameters for $T > T_c$, $\nu = -k_>$ and $M + N$ even

For $T > T_c$, $\nu = -k_>$ and $M + N$ *even* we find for (73), from (96), that:

$$A \,=\, \frac{M^2}{4}, \qquad B \,=\, \frac{1}{8}\cdot(N^2 - M^2 - 1). \tag{102}$$

Thus with $t = k_>^2$

$$h \,=\, t\cdot(t-1)\cdot\frac{\mathrm{d}\ln C(M,N)}{\mathrm{d}t} \,-\, \frac{M^2}{4}\cdot t \,-\, \frac{1}{8}\cdot\big(N^2 - M^2 \,+1\big), \tag{103}$$

satisfies (91) with the Okamoto parameters:

$$n_1 \,=\, \frac{M-1}{2}, \qquad n_2 \,=\, \frac{M+1}{2}, \qquad n_3 \,=\, \frac{N}{2}, \qquad n_4 \,=\, -\frac{N}{2}. \tag{104}$$

4.3. Okamoto parameters for $T > T_c$, $\nu = -k_>$, $M < N$ and $M + N$ odd

For $T > T_{\rm c}$ we find for (79), from (96), that

$$A \;=\; \frac{M^2}{4}, \qquad B \;=\; \frac{1}{8}\cdot(N^2 - M^2 - 2). \tag{105}$$

Thus with $t = k_>^2$

$$h \;=\; t\cdot(t-1)\cdot\frac{\rm d}{{\rm d}t}\,\ln\tilde{C}(M,N) \;-\frac{M^2}{4}\cdot t \;-\frac{1}{8}\cdot(N^2-M^2), \tag{106}$$

satisfies the Okamoto equation (91) with:

$$n_1 \;=\; \frac{M-1}{2}, \qquad n_2 \;=\; \frac{M+1}{2}, \qquad n_3 \;=\; \frac{N+1}{2}, \qquad n_4 \;=\; -\frac{N-1}{2}. \tag{107}$$

4.4. Boundary conditions

In order to complete the specification $\nu = -k$ of $C(M,N)$ we must specify the boundary conditions for the nonlinear equations (71), (73) and (79). This is most systematically done by first determining the allowed boundary conditions on the canonical equation of Okamoto (91) which are analytic at $t = 0$. A detailed and explicit analysis of these boundary conditions is performed in appendix D.

5. Relation to the determinants of Forrester–Witte

These results should be compared with the results of Forrester–Witte [11] as given in [12] as

$$D_N^{(p,p',\eta,\xi)}(t) \;=\; \det\left[A_{j-k}^{(p,p',\eta,\xi)}(t)\right]_{j,k=0}^{N-1}, \tag{108}$$

where

$$A_m^{(p,p',\eta,\xi)}(t) \;=\; \frac{\Gamma(1+p')\,t^{(\eta-m)/2}\,(1-t)^p}{\Gamma(1+\eta-m)\,\Gamma(1-\eta+m+p')}\cdot{}_2F_1\left([-p,1+p'],\,[1+\eta-m],\,\frac{t}{t-1}\right)$$
$$+\frac{\xi\cdot\Gamma(1+p)t^{(m-\eta)/2}\,(1-t)^{p'}}{\Gamma(1-\eta+m)\,\Gamma(1+\eta-m+p)}\cdot{}_2F_1\left([-p',1+p],\,[1-\eta+m],\,\frac{t}{t-1}\right), \tag{109}$$

which, using the identity (6) on page 109 of [15]

$${}_2F_1([a,b],\,[c],\,t) \;=\; (1-t)^{-a}\cdot{}_2F_1\left([a,c-b],\,[c],\,\frac{t}{t-1}\right), \tag{110}$$

is rewritten as:

$$A_m^{(p,p',\eta,\xi)}(t) \;=\frac{\Gamma(1+p')\,t^{(\eta-m)/2}}{\Gamma(1+\eta-m)\,\Gamma(1-\eta+m+p')}\cdot{}_2F_1([-p,-p'+\eta-m],\,[1+\eta-m],\,t)$$
$$+\frac{\xi\cdot\Gamma(1+p)\,t^{(m-\eta)/2}}{\Gamma(1-\eta+m)\,\Gamma(1+\eta-m+p)}$$
$$\cdot\,{}_2F_1([-p',-p-\eta+m],\,[1-\eta+m],\,t). \tag{111}$$

From (2,27) of [12]

$$h \; = \; t \cdot (t-1) \cdot \frac{\mathrm{d}}{\mathrm{d}t} \; \ln\left(t^{(\theta_0^2+\theta_t^2-\theta_1^2-\theta_\infty^2)/2} \cdot (1-t)^{(\theta_t^2+\theta_1^2-\theta_0^2-\theta_\infty^2)/2} \cdot \tau(t) \right)$$

$$= \; t \cdot (t-1) \cdot \frac{\mathrm{d}}{\mathrm{d}t} \; \ln\left(t^{(n_1 n_2+n_3 n_4)/2} \cdot (1-t)^{(n_1 n_2-n_3 n_4)/2} \tau(t) \right), \qquad (112)$$

with

$$\tau_N^{(p,p',\eta,\xi)}(t) \; = \; (1\,-t)^{-N\cdot\,(N+p+p')/2} \cdot D_N^{(p,p',\eta,\xi)} \; = \; (1-t)^{(n_1+n_2)(n_3-n_4)/2} \cdot D_N^{(\nu,\nu',\eta,\xi)}, \qquad (113)$$

satisfies the Okamoto equation (91) with

$$n_1 \; = \; \theta_t - \theta_\infty, \qquad n_2 \; = \; \theta_t + \theta_\infty, \qquad n_3 \; = \; \theta_0 - \theta_1, \qquad n_4 \; = \; \theta_0 + \theta_1, \quad (114)$$

where

$$(\theta_0,\; \theta_t,\; \theta_1,\; \theta_\infty) \; = \; \frac{1}{2} \cdot (\eta,\, N,\, -N\, - p - p',\; p\, - p' + \eta), \qquad (115)$$

are the eigenvalues of the linear system for isomonodromic deformation

$$\frac{\mathrm{d}Y}{\mathrm{d}z} \; = \; \left(\frac{A_0}{z} + \frac{A_t}{z-t} + \frac{A_1}{z-1} \right) Y, \qquad (116)$$

where A_i are traceless 2×2 matrices and:

$$A_\infty \; = \; -A_0 \; - A_t \; - A_1. \qquad (117)$$

Using (115) in (114) we have:

$$n_1 \; = \; (N\, - p + p' - \eta)/2, \qquad n_2 \; = \; (N\, + p - p' + \eta)/2,$$

$$n_3 \; = \; (\eta + N + p + p')/2, \qquad n_4 \; = \; (\eta - N - p - p')/2. \qquad (118)$$

Thus we see, for $M+N$ *even*, that the parameters n_k of (100) agree with the parameters n_k (118) if

$$\eta \; = \; 0, \qquad p \; = \; \frac{M-N\, +1}{2}, \qquad p' \; = \; \frac{M-N\, -1}{2} \qquad (119)$$

and for $M+N$ *odd* the parameters n_k of (100) agree with the parameters n_k (118) if:

$$\eta \; = \; 0, \qquad p \; = \; p' \; = \; \frac{M-N}{2}. \qquad (120)$$

For either choice we see that (113) reduces to:

$$\tau_N \; = \; (1-t)^{-MN/2} \cdot D_N. \qquad (121)$$

5.1. The case $M = 0, N = 1$

When $M = 0$ and $N = 1$ we see from (111) with $\xi = 0$ that:

$$D_1 \;=\; A_0(t) \;=\; {}_2F_1\left([\frac{1}{2}, \frac{1}{2}], [1],\; t\right). \tag{122}$$

Thus from (31) and (121)

$$C(0,1) \;=\; (1-t)^{1/2} \cdot D_1 \;=\; (1-t)^{1/2} \cdot \tau_1, \tag{123}$$

and (112) reduces to

$$\begin{aligned} h \;&=\; t \cdot (t-1) \cdot \frac{\mathrm{d}}{\mathrm{d}t} \ln\left(t^{1/8} \cdot (1-t)^{1/8} \cdot (1-t)^{-1/2} \cdot C(0,1)\right) \\ &=\; t \cdot (t-1) \cdot \frac{\mathrm{d}}{\mathrm{d}t} \ln\; C(0,1) \;-\frac{t}{4} \;-\frac{1}{8}, \end{aligned} \tag{124}$$

which agrees with (101) as required.

5.2. The general case

For $C(M, N)$ for $\nu = -k$ the special case (123) generalizes to:

$$C(M,N) \;=\; (1-t)^{[(N-M)^2+1-(1+(-1)^{M+N})/2]/4} \cdot D_N. \tag{125}$$

To verify (125) we use (121) to write:

$$C(M,N) \;=\; (1-t)^{[N^2+M^2+1-(1+(-1)^{M+N})/2]/4} \cdot \tau_N = \; (1-t)^{n_1n_2-n_3n_4+1/4} \cdot \tau_N. \tag{126}$$

Thus, substituting into (112) we find

$$\begin{aligned} h \;&=\; t \cdot (t-1) \cdot \frac{\mathrm{d}}{\mathrm{d}t} \ln\left(C(M,N)\, t^{(n_1n_2+n_3n_4)/2}\, (1-t)^{-(n_1n_2-n_3n_4)/2-1/4}\right) \\ &=\; t \cdot (t-1) \cdot \frac{\mathrm{d}}{\mathrm{d}t} \ln\; C(M,N) \;+ (n_3n_4 - \frac{1}{4}) \cdot t \;-\frac{1}{2} \cdot (n_1n_2 + n_3n_4) \\ &=\; t \cdot (t-1) \cdot \frac{\mathrm{d}}{\mathrm{d}t} \ln\; C(M,N) \;-\frac{M^2+1}{4} \cdot t - \frac{1}{8} \cdot \left(N^2 - M^2 - \frac{1+(-1)^{M+N}}{2}\right), \end{aligned} \tag{127}$$

which agrees with (101) as required.

Finally it may be verified that (125) satisfies the boundary condition (D.40).

5.3. Specialization of D_N

It remains to specialize the matrix elements A_m of (111) to the special cases (119) and (120). For $m \leqslant 0$ we may directly set $\eta = 0$ in (111) and use the identity

$$\frac{\Gamma(1+p') \cdot \Gamma(-p')}{\Gamma(1+m+p') \cdot \Gamma(-p'-m)} \;=\; (-1)^m, \tag{128}$$

to find:

$$A_{-|m|} \;=\; \frac{\Gamma(|m|-p') \cdot (-1)^{|m|} \cdot t^{|m|/2}}{|m|! \cdot \Gamma(-p')} \cdot {}_2F_1([-p, -p'+|m|], [1+|m|],\; t). \tag{129}$$

For $m \geqslant 1$ the factor $\Gamma(1+\eta-m)$ in the denominator diverges for $\eta \to 0$ and consequently the limit $\eta \to 0$ must be taken carefully. Then, using the identity (128), we find for $m \geqslant 1$ as $\eta \to 0$:

$$A_m \,=\, \frac{\Gamma(-p+m)\cdot(-1)^m\cdot t^{m/2}}{\Gamma(-p)\cdot m!}\cdot {}_2F_1([m-p,-p'],\,[m+1],\,t). \tag{130}$$

For $M+N$ even we thus use (119) in (129) for $m \leqslant 0$ to find

$$A_{-|m|} \,=\, \frac{\Gamma(|m|+\frac{N-M+1}{2})\cdot(-1)^m\cdot t^{|m|/2}}{\Gamma(\frac{N-M+1}{2})\cdot|m|!} \times {}_2F_1\left([\frac{N-M-1}{2},\frac{N-M+1}{2}+|m|],\,[1+|m|],\,t\right), \tag{131}$$

and for $m \geqslant 1$ we use (119) in (130) to find:

$$A_m \,=\, \frac{\Gamma(m+\frac{N-M-1}{2})\cdot(-1)^m\cdot t^{m/2}}{\Gamma(\frac{N-M-1}{2})\cdot m!}\cdot {}_2F_1\left([\frac{N-M+1}{2},\frac{N-M-1}{2}+m],\,[1+m],\,t\right). \tag{132}$$

For $M+N$ *odd* we use (120) in (129) and (130) to obtain:

$$A_{-m} \,=\, A_m \,=\, \frac{\Gamma(m+\frac{N-M}{2})\cdot(-1)^m\cdot t^{m/2}}{\Gamma(\frac{N-M}{2})\cdot|m|!}{}_2F_1([\frac{N-M}{2},\frac{N-M}{2}+m],\,[1+m],\,t). \tag{133}$$

Similar Toeplitz elements can be found for the correlation functions obtained for $T > T_c$ and $\nu = -k_>$. The expressions of A_m for this case are given in appendix C.

We note that from (4.35) of [12] that for $\eta = 0$ that the matrix elements are obtained from:

$$A_m \,=\, \frac{1}{2\pi}\int_0^{2\pi} d\theta\; e^{im\theta}\cdot(1\,-k\;e^{i\theta})^p\,(1\,-k\;e^{-i\theta})^{p'}. \tag{134}$$

5.4. *Direct proof for $C(0,2)$ when $\nu = -k$*

The relation of $C(M,N)$ for $\nu = -k$ to the determinant D_N (125) was obtained from the nonlinear equations for $C(M,N)$ for $\nu = -k$ and D_N. In this section we give a direct proof of (125) for $C(0,2)$, when $\nu = -k$, by use of contiguous relations for hypergeometric functions. This provides a proof of the Okamoto equation (91) with (100) and (101) for $C(0,2)$ for $\nu = -k$ for $T < T_c$. The relation for $C(0,1)$ has already been shown in section 5.1.

To prove (125) for $C(0,2)$ we need to prove the following identity between the 2×2 determinants for $C(0,2)$ and D_2 obtained by using (42) and (44) for $C(0,2)$ and (131) and (132) for D_2

$$\begin{vmatrix} \sqrt{1-t}\cdot {}_2F_1([\frac{1}{2},\frac{1}{2}],\,[1],\,t) & -\frac{t^{1/2}}{2}\cdot {}_2F_1([\frac{1}{2},\frac{1}{2}],\,[2],\,t) \\ \frac{t^{1/2}}{2}\cdot {}_2F_1([\frac{1}{2},\frac{1}{2}],\,[2],\,t) & \sqrt{1-t}\cdot {}_2F_1([\frac{1}{2},\frac{1}{2}],\,[1],\,t) \end{vmatrix}$$
$$= (1-t)\cdot \begin{vmatrix} {}_2F_1([\frac{1}{2}.\frac{3}{2}],\,[1],\,t) & -\frac{t^{1/2}}{2}\cdot {}_2F_1([\frac{3}{2},\frac{3}{2}],\,[2],\,t) \\ -\frac{3\,t^{1/2}}{2}\cdot {}_2F_1([\frac{1}{2},\frac{5}{2}],\,[2],\,t) & {}_2F_1([\frac{1}{2},\frac{3}{2}],[1],\,t) \end{vmatrix}, \tag{135}$$

which we rewrite

$$\begin{vmatrix} {}_2F_1([\frac{1}{2},\frac{1}{2}],[1],\ t) & \frac{t}{2}\cdot {}_2F_1([\frac{1}{2},\frac{1}{2}],[2],\ t) \\ -\frac{1}{2}\cdot {}_2F_1([\frac{1}{2},\frac{1}{2}],[2],\ t) & (1-t)\cdot {}_2F_1([\frac{1}{2},\frac{1}{2}],[1],\ t) \end{vmatrix}$$

$$= \begin{vmatrix} (1-t)\cdot {}_2F_1([\frac{1}{2},\frac{3}{2}],[1],\ t) & \frac{t(1-t)}{2}\cdot {}_2F_1([\frac{3}{2},\frac{3}{2}],[2],\ t) \\ \frac{3}{2}\cdot {}_2F_1([\frac{1}{2},\frac{5}{2}],[2],\ t) & {}_2F_1([\frac{1}{2},\frac{3}{2}],[1],\ t) \end{vmatrix}. \qquad (136)$$

We then use (41) on page 103 of [15]

$$(1-t)\cdot {}_2F_1\left([\frac{1}{2},\frac{3}{2}],[1],\ t\right)$$
$$= (1-t)\cdot {}_2F_1\left([\frac{1}{2},\frac{1}{2}],[1],\ t\right) + \frac{t}{2}\cdot {}_2F_1\left([\frac{1}{2},\frac{1}{2}],[2],\ t\right), \qquad (137)$$

to rewrite the (1,1) element of the right-hand side of (136) and we use (33) on page 103 of [15] with $a = 1/2, b = 3/2, c = 2$

$$(1-t)\cdot {}_2F_1\left([\frac{3}{2},\frac{3}{2}],[2],\ t\right) = {}_2F_1\left([\frac{1}{2},\frac{1}{2}],[2],\ t\right), \qquad (138)$$

to rewrite the (1,2) element on the right-hand side. Then we subtract column 2 from column 1 to find that the right-hand side of (136) becomes:

$$\begin{vmatrix} (1-t)\cdot {}_2F_1([\frac{1}{2},\frac{1}{2}],[1],\ t) & \frac{t}{2}\cdot {}_2F_1([\frac{1}{2},\frac{1}{2}],[2],\ t) \\ \frac{3}{2}\cdot {}_2F_1([\frac{1}{2},\frac{5}{2}],[2],\ t) - {}_2F_1([\frac{1}{2},\frac{3}{2}],[1],\ t) & {}_2F_1([\frac{1}{2},\frac{3}{2}],[1],\ t) \end{vmatrix}$$

$$= \begin{vmatrix} {}_2F_1([\frac{1}{2},\frac{1}{2}],[1],\ t) & \frac{t}{2}\cdot {}_2F_1([\frac{1}{2},\frac{1}{2}],[2],\ t) \\ \frac{3}{2}\cdot {}_2F_1([\frac{1}{2},\frac{5}{2}],[2],\ t) - {}_2F_1([\frac{1}{2},\frac{3}{2}],[1],\ t) & (1-t)\cdot {}_2F_1([\frac{1}{2},\frac{3}{2}],[1],\ t) \end{vmatrix}. \qquad (139)$$

We then rewrite the (2,2) element on the right-hand side (139) using (137) and subtract row 1 from row 2 to obtain:

$$\begin{vmatrix} {}_2F_1([\frac{1}{2},\frac{1}{2}],[1],\ t) & \frac{t}{2}\cdot {}_2F_1([\frac{1}{2},\frac{1}{2}],[2],\ t) \\ \frac{3}{2}\cdot {}_2F_1([\frac{1}{2},\frac{5}{2}],[2],\ t) - {}_2F_1([\frac{1}{2},\frac{3}{2}],[1],\ t) - {}_2F_1([\frac{1}{2},\frac{1}{2}],[1],\ t) & (1-t)\cdot {}_2F_1([\frac{1}{2},\frac{1}{2}],[1],\ t) \end{vmatrix}.$$

Then we note that if

$$\frac{3}{2}\cdot {}_2F_1\left([\frac{1}{2},\frac{5}{2}],[2],\ t\right) - {}_2F_1\left([\frac{1}{2},\frac{3}{2}],[1],\ t\right) - {}_2F_1\left([\frac{1}{2},\frac{1}{2}],[1],\ t\right)$$
$$= -\frac{1}{2}\cdot {}_2F_1\left([\frac{1}{2},\frac{1}{2}],[2],\ t\right), \qquad (140)$$

then (140) agrees with the left-hand side of (136) as required. To prove (140) we use (43) on page 104 of [15] (with a missing factor of z restored in the last term) with $a = 1/2$, $b = 5/2, c = 1$

$$\frac{1-t}{2} \cdot {}_2F_1\left([\frac{1}{2},\frac{5}{2}],\,[1],\,t\right) \,-\, {}_2F_1\left([\frac{1}{2},\frac{3}{2}],\,[1],\,t\right) \,+\, \frac{t}{2} \cdot {}_2F_1\left([\frac{1}{2},\frac{5}{2}],\,[2],\,t\right) \,=\, 0, \tag{141}$$

and (41) on page 103 with $a = 1/2, b = 1/2, c = 1$

$$\frac{1-t}{2} \cdot {}_2F_1\left([\frac{1}{2},\frac{1}{2}],\,[1],\,t\right) - \frac{1-t}{2} \cdot {}_2F_1\left([\frac{1}{2},\frac{3}{2}],\,[1],\,t\right) + \frac{t}{4} \cdot {}_2F_1\left([\frac{1}{2},\frac{1}{2}],\,[2],\,t\right) = 0, \tag{142}$$

to eliminate ${}_2F_1([\frac{1}{2},\frac{5}{2}],\,[2],\,t)$ and ${}_2F_1([\frac{1}{2},\,\frac{1}{2}],\,[2],\,t)$. The desired result is then obtained by use of (29) on page (103) of [15] with $a = 1/2, b = 3/2, c = 1$:

$$\frac{1}{2} \cdot {}_2F_1\left([\frac{1}{2},\frac{1}{2}],\,[1],\,t\right) \,+ (2\,-t) \cdot {}_2F_1\left([\frac{1}{2},\frac{3}{2}],\,[1],\,t\right)$$
$$-\,\frac{3}{2} \cdot (1-t) \cdot {}_2F_1\left([\frac{1}{2},\frac{5}{2}],\,[1],\,t\right) \,=\, 0. \tag{143}$$

6. Factorizations

All symmetric $N \times N$ Toeplitz determinants can be factored into the product of two determinants by use of the procedure used by Wilf [22] for determinants with N even of subtracting column j from column $N+1-j$ for $1 \leqslant j \leqslant N/2$ and then adding row $N+1-j$ to row j for $1 \leqslant j \leqslant N/2$. Thus, for example, we find:

$$D_2 \,=\, (A_0 - A_1) \cdot (A_0 + A_1), \tag{144}$$

$$D_3 \,=\, (A_0 - A_2) \cdot \begin{vmatrix} A_0 + A_2 & 2A_1 \\ A_1 & A_0 \end{vmatrix}, \tag{145}$$

$$D_4 \,=\, \begin{vmatrix} A_0 + A_3 & A_1 + A_2 \\ A_1 + A_2 & A_0 + A_1 \end{vmatrix} \cdot \begin{vmatrix} A_0 - A_1 & A_1 - A_2 \\ A_1 - A_2 & A_0 - A_3 \end{vmatrix}, \tag{146}$$

$$D_5 \,=\, \begin{vmatrix} A_0 + A_4 & A_1 + A_3 & 2A_2 \\ A_1 + A_3 & A_0 + A_2 & 2A_1 \\ A_2 & A_1 & A_0 \end{vmatrix} \cdot \begin{vmatrix} A_0 - A_2 & A_1 - A_3 \\ A_1 - A_3 & A_0 - A_4 \end{vmatrix}. \tag{147}$$

where the D_n's and An's are given by (108) and (109). Thus for the special case (133) when $T < T_c$ with $M + N$ odd and $M \neq 0$, one finds that $C(M,N)$ factors into two terms.

Note that the factors D_2, D_3, D_4 can be put into a Toeplitz form.

For the special case (133) when $T < T_c$ with $M + N$ *odd*, we find when $M = 0$, that D_3 further factors into three factors so that

$$C(0,3) \,=\, -4 \cdot (1-t)^{1/2} \cdot t^{-2} \cdot \tilde{E} \cdot (\tilde{E} \,-\tilde{K}) \cdot (\tilde{E} \,-(1\,-t) \cdot \tilde{K}), \tag{148}$$

and D_N with N odd, $N \geqslant 5$, factors into *four* factors, so that

$$C(0,N) \,=\, \text{constant} \cdot (1\,-t)^{1/2} \cdot t^{(1-N^2)/4} \cdot f_1\, f_2\, f_3\, f_4, \tag{149}$$

For example, for $C(0,5)$, the f_i's read:

$$f_1 \;=\; (2t-1)\cdot \tilde{E} \;+(1-t)\cdot \tilde{K}, \qquad f_2 \;=\; (1+t)\cdot \tilde{E} \;-(1-t)\cdot \tilde{K}, \tag{150}$$

$$f_3 \;=\; (t-2)\cdot \tilde{E} \;+2\cdot (1-t)\cdot \tilde{K}, \quad f_4 \;=\; 3\tilde{E}^2 \;+2\cdot (t-2)\cdot \tilde{E}\tilde{K} \;+(1-t)\cdot \tilde{K}^2. \tag{151}$$

We have studied these four factors of $C(0,N)$ with N odd by the process previously described and found that all four factors satisfy the equation (91) with the identical Okamoto parameters

$$n_1 \;=\; \frac{N-1}{4}, \qquad n_2 \;=\; \frac{N+1}{4}, \qquad n_3 \;=\; -\frac{1}{2}, \qquad n_4 \;=\; 0, \tag{152}$$

where the relation of the factors f_j to h is given by:

$$h_1 \;=\; t\cdot(t-1)\cdot \frac{\mathrm{d}\ln f_1}{\mathrm{d}t} \;-\frac{N^2+3}{16}\cdot t \;+\frac{N^2+3}{32}, \tag{153}$$

$$h_2 \;=\; t\cdot(t-1)\cdot \frac{\mathrm{d}\ln f_2}{\mathrm{d}t} \;-\frac{N^2-1}{16}\cdot t \;+\frac{N^2+3}{32}, \tag{154}$$

$$h_3 \;=\; t\cdot(t-1)\cdot \frac{\mathrm{d}\ln f_3}{\mathrm{d}t} \;-\frac{N^2-1}{16}\cdot t \;+\frac{N^2-5}{32}, \tag{155}$$

$$h_4 \;=\; t\cdot(t-1)\cdot \frac{\mathrm{d}\ln f_4}{\mathrm{d}t} \;-\frac{N^2-5}{16}\cdot t \;+\frac{N^2-5}{32}. \tag{156}$$

By comparing (153)–(156) for $C(0,5)$ with the four cases of boundary conditions in appendix D, we see that the factors f_1 and f_2 are in case 1 with $c_0^{(1)}$ and $c_1^{(1)}$ given by (D.4) and (D.8) and from (D.20) the coefficient of $t^{(N+3)/2}$ is a constant which must be specified separately for f_1 and f_2. Similarly the factors f_3 and f_4 are in case 4 with $c_0^{(4)}$ and $c_1^{(4)}$ given by (D.7) and (D.11) and from (D.23) the coefficient of $t^{N+1/2}$ is a constant which must be specified separately for f_3 and f_4.

7. Discussion

In this paper we have discovered for the special case $\nu = -k$ and for arbitrary positive integers $M \leqslant N$, that the correlation $C(M,N)$ satisfies an Okamoto sigma form of the Painlevé VI equation (91) with parameters (100) for $T < T_c$ and (104), and (107) for $T > T_c$. These non-linear differential equations have been obtained using extensively Pantone's program and checked with a large set of exact expressions of the $C(M,N)$ in terms of $\tilde{E}$ and $\tilde{K}$. Moreover the nonlinear differential equation for $T < T_c$ is the same equation satisfied by a particular case of the $N \times N$ Toeplitz determinants of Forrester–Witte [11] and Gamayun, Igorov and Lisovyy [12]. This is perhaps surprising because no Toeplitz form for $C(M,N)$ is in the literature except for $M = 0$, $M = N$ and the results of Au-Yang and Perk [23] for $C(N-1,N)$. We have also investigated in appendix D the boundary conditions which must be applied to the nonlinear differential equations to obtain solutions which are analytic at $k = 0$.

In the course of this investigation we have found several open questions:

1) We have seen that all the correlations $C(M,N)$ considered are members of a one parameter family of Painlevé VI functions but the principle for determining the specific value of the boundary condition is not known.

2) In section 6 we exhibited, for the row correlation functions, a remarkable phenomenon of a Painlevé VI sigma function which satisfies an equation with one set of Okamoto parameters and a specific boundary condition constant is equal to a *sum of four*[7] Painlevé VI sigma functions which all have the same Okamoto parameters (which are different from the previous set) and have four specific boundary condition constants. One would like to find the conditions yielding such a remarkable phenomenon.

3) We also saw in section 6 for $T < T_c$, $M + N$ odd and $M \leqslant N$, that $C(M, N)$ with $M \neq 0$, always factors into *two* terms. It is not known if these terms have the Painlevé property of having no movable critical points and, if they do have this property, are they expressible as known Painlevé functions ? These factorizable cases need much further study.

4) The case $M \geqslant N$ with $\nu = -k$ remains to be understood. By (18) this is equivalent to $M \leqslant N$ with $\nu = -1/k$ and, when this constraint holds, we see from (21) that $\tilde{\Pi}(-\nu\, k,\, k)$ becomes singular. This has the effect that some (but not all) correlations for $M \geqslant N$ are no longer homogeneous polynomials in $\tilde{K}$ and $\tilde{E}$. For example for $T < T_c$

$$C(2,0) \;=\; 1-t \;\; + (1-t)\cdot \tilde{K}^2 \;\; - 2\cdot(1-t)\cdot \tilde{E}\tilde{K} \;\; + \tilde{E}^2, \tag{157}$$

$$C(3,0) \;=\; \sqrt{1-t}\cdot \Big[(1-t)^2 + 2\cdot(t-1)^2\cdot \tilde{K}^2 \;\; + 4\cdot(t-1)\cdot \tilde{E}\tilde{K} \;\; + 2\cdot(1+t)\cdot \tilde{E}^2\Big]. \tag{158}$$

Consequently the corresponding non-linear ODEs are much more involved than Okamoto sigma form of Painlevé VI equations. It is not even clear that all these non-linear ODEs can be encapsulated in closed formulae depending on M and N, like this was the case with the two-parameters families of equations (71), (73) and (79).

More generally, the discovery of these four two-parameters families of Okamoto sigma form of Painlevé VI equations is a strong incentive to find non-linear ODEs with the Painlevé property, for two-point correlation functions $C(M, N)$ that are not restricted to selected conditions like $\nu = -k$ or $\nu = -1/k$ for the anisotropic model.

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Appendix A. Calculations of the a_n's

For a_{2n+1} of (51) we first use (36) on page 103 of [15] to write

$${}_2F_1\left([n+\frac{1}{2}, -\frac{1}{2}],\, [n+1],\; \alpha^4\right) \;=\; \frac{1}{2}\cdot\left[{}_2F_1\left([n-\frac{1}{2}, -\frac{1}{2}],\, [n+1],\; \alpha^4\right) + (1-\alpha^4)\cdot {}_2F_1\left([n+\frac{1}{2}, \frac{1}{2}], [n+1],\; \alpha^4\right)\right], \tag{A.1}$$

[7] A similar phenomenon for Painlevé V with a sigma function being the sum of *two* (see equation (6.23) in [24]) sigma functions was found by Tracy and Widom [24].

and then by use of (5) on page 111 of [15] and

$$k_> \;=\; \frac{2\,\alpha}{1+\alpha^2}, \tag{A.2}$$

we find:

$$\begin{aligned} {}_2F_1\Big([n+\frac{1}{2},-\frac{1}{2}],\,[n+1],\;\alpha^4\Big) \;&=\; \frac{1}{2\cdot(1+\alpha^2)^{2n-1}}\cdot{}_2F_1\Big([n-\frac{1}{2},n+\frac{1}{2}],\,[2n+1],\;k_>^2\Big)\\ &\quad+\frac{1-\alpha^4}{2\cdot(1+\alpha^2)^{2n+1}}\cdot{}_2F_1\Big([n+\frac{1}{2},n+\frac{1}{2}],\,[2n+1],\;k_>^2\Big). \end{aligned} \tag{A.3}$$

Using (A.3) in (51) we obtain:

$$\begin{aligned} \frac{a_{2n+1}}{\alpha} \;&=\; \frac{\Gamma(n+\frac{1}{2})}{2\sqrt{\pi}n!}\cdot\Bigg[\Big(\frac{\alpha}{1+\alpha^2}\Big)^{2n-1}\cdot{}_2F_1\Big([n-\frac{1}{2},n+\frac{1}{2}],\,[2n+1],\;k_>^2\Big)\\ &\quad+\Big(\frac{\alpha}{1+\alpha^2}\Big)^{2n+1}(\alpha^{-2}-\alpha^2)\cdot{}_2F_1\Big([n+\frac{1}{2},n+\frac{1}{2}],\,[2n+1],\;k_>^2\Big)\Bigg]. \end{aligned} \tag{A.4}$$

Thus, by using (A.2) and the inverse

$$\alpha^{\pm 2} \;=\; \frac{2\,-k_>^2\;\mp 2\sqrt{1-k_>^2}}{k_>^2}, \tag{A.5}$$

we obtain the final result:

$$\begin{aligned} a_{2n+1} \;&=\; \frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}n!}\cdot\Big(\frac{k_>}{2}\Big)^{2(n-1)}\cdot 4\cdot(1-\sqrt{1\,-k_>^2})\\ &\quad\times\Big[{}_2F_1\Big([n-\frac{1}{2},n+\frac{1}{2}],\,[2n+1],\;k_>^2\Big)+\sqrt{1-k_>^2}\\ &\quad\cdot{}_2F_1\Big([n+\frac{1}{2},n+\frac{1}{2}],\,[2n+1],\;k_>^2\Big)\Big]. \end{aligned} \tag{A.6}$$

For $\alpha a_{-(2n+1)}$ (see (52)) we proceed in a similar fashion, and use (33) on page 103 of [15] to write:

$$\begin{aligned} {}_2F_1\Big([n+\frac{1}{2},\frac{1}{2}],\,[n+2],\;\alpha^4\Big) \;&=\; \Big(n+\frac{n}{2}\Big)\cdot{}_2F_1\Big([n+\frac{1}{2},-\frac{1}{2}],\,[n+2],\;\alpha^4\Big)\\ &\quad-\Big(n+\frac{1}{2}\Big)\cdot(1-\alpha^4)\cdot{}_2F_1\Big([n+\frac{3}{2},\frac{1}{2}],\,[n+2],\;\alpha^4\Big). \end{aligned} \tag{A.7}$$

Then we use (5) on page 111 of [15] to obtain

$$
\begin{aligned}
&{}_2F_1\left([n+\frac{1}{2},\frac{1}{2}],\,[n+2],\,\alpha^4\right),\\
&\quad = \left(n+\frac{3}{2}\right)\cdot\,(1\,+\alpha^2)^{-2n-1}\cdot\,{}_2F_1\left([n+\frac{1}{2},n+\frac{3}{2}],\,[2n+3],\;k_>^2\right)\\
&\quad\quad -\left(n+\frac{1}{2}\right)\cdot\,\frac{1-\alpha^4}{(1+\alpha^2)^{2n+3}}\cdot\,{}_2F_1\left([n+\frac{3}{2},n+\frac{3}{2}],\,[2n+3],\;k_>^2\right),
\end{aligned}
\tag{A.8}
$$

and thus

$$
\begin{aligned}
\alpha\,a_{-(2n+1)} &= \alpha^2\cdot\,\frac{\Gamma(n+\frac{1}{2})}{2\sqrt{\pi}(n+1)!}\cdot\\
&\times\left[\left(\frac{\alpha}{1+\alpha^2}\right)^{2n+1}\cdot\,\left(n+\frac{3}{2}\right)\cdot\,{}_2F_1\left([n+\frac{1}{2},n+\frac{3}{2}],\,[2n+1],\;k_>^2\right)\right.\\
&-\left(\frac{\alpha}{1+\alpha^2}\right)^{2n+3}\cdot\,\left(n+\frac{1}{2}\right)\cdot\,(\alpha^{-2}-\alpha^2)\\
&\left.\cdot\,{}_2F_1\left([n+\frac{3}{2},n+\frac{3}{2}],\,[2n+3],\;k_>^2\right)\right],
\end{aligned}
\tag{A.9}
$$

which by use of (A.2) and (A.5) becomes:

$$
\begin{aligned}
\alpha\,a_{-(2n+1)} \;=\; &\frac{\Gamma(n+\frac{1}{2})}{2\sqrt{\pi}}\cdot\,\left(\frac{k_>}{2}\right)^{2n+1}\cdot\,\left(\frac{2\,-k^2\,-2\,\sqrt{1-k_>^2}}{k_>^2}\right),\\
&\times\left[\left(n+\frac{3}{2}\right)\cdot\,{}_2F_1\left([n+\frac{1}{2},n+\frac{3}{2}],\,[2n+3],\;k_>^2\right)\right.\\
&\left.-\left(n+\frac{1}{2}\right)\cdot\,\sqrt{1-k^2}\cdot\,{}_2F_1\left([n+\frac{3}{2},n+\frac{3}{2}],\,[2n+3],\;k_>^2\right)\right].
\end{aligned}
\tag{A.10}
$$

To obtain the final desired result we first carry out the multiplication to write

$$
\alpha\cdot\,a_{-(2n+1)} \;=\; \frac{\Gamma(n+\frac{1}{2})}{8\,\sqrt{\pi}(n+1)!}\cdot\,\left(\frac{k_>}{2}\right)^{2n-1}\cdot\,\left(T_1\;-\sqrt{1\,-k_>^2}\cdot\;T_2\right),
\tag{A.11}
$$

with

$$
\begin{aligned}
T_1 \;=\; &(2\,-k_>^2)\cdot\,\left(n+\frac{3}{2}\right)\cdot\,{}_2F_1\left([n+\frac{1}{2},n+\frac{3}{2}],\,[2n+3],\;k_>^2\right)\\
&+2\cdot\,(1\,-k^2)\cdot\,\left(n+\frac{1}{2}\right)\cdot\,{}_2F_1\left([n+\frac{3}{2},n+\frac{3}{2}],\,[2n+3],\;k_>^2\right),
\end{aligned}
\tag{A.12}
$$

and

$$
\begin{aligned}
T_2 \;=\; &2\cdot\,\left(n+\frac{3}{2}\right)\cdot\,{}_2F_1\left([n+\frac{1}{2},n+\frac{3}{2}],[2n+3],\;k_>^2\right)\\
&+(2-k_>^2)\cdot\,\left(n+\frac{1}{2}\right)\cdot\,{}_2F_1\left([n+\frac{3}{2},n+\frac{3}{2}],\,[2n+3],\;k_>^2\right),
\end{aligned}
\tag{A.13}
$$

and then note the identities which may be discovered by use of series expansions on Maple and then proven by the use of contiguous identities

$$T_1 \;=\; 4\cdot (n+1)\cdot {}_2F_1\left([n-\frac{1}{2}, n+\frac{1}{2}],\, [2n+1],\; k_>^2\right), \tag{A.14}$$

$$T_2 \;=\; 4\cdot (n+1)\cdot {}_2F_1\left([n+\frac{1}{2}, n+\frac{1}{2}],\, [2n+1],\; k_>^2\right), \tag{A.15}$$

to find the desired result

$$\begin{aligned} a_{-(2n+1)} \;=\; & \frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi} n!}\cdot\left(\frac{k_>}{2}\right)^{2(n-1)}\cdot 4\cdot(1+\sqrt{1-k_>^2}) \\ & \times\Big[{}_2F_1\left([n-\frac{1}{2}, n+\frac{1}{2}],\, [2n+1],\, k_>^2\right) - \sqrt{1-k_>^2} \\ & \cdot\, {}_2F_1\left([n+\frac{1}{2}, n+\frac{1}{2}],\, [2n+1],\, k_>^2\right)\Big], \end{aligned} \tag{A.16}$$

which is to be compared with the result for a_{2n+1} of (A.6).

Appendix B. Examples of $C(M,N)$ and $\tilde{C}(M,N)$ for $\nu = -k$

For $T < T_c$

$$C(0,2) \;=\; \frac{1}{t}\cdot\left[\tilde{E}^2 - 2\cdot(1-t)\cdot\tilde{E}\tilde{K} + (1-t)\cdot\tilde{K}^2\right], \tag{B.1}$$

$$C(0,3) \;=\; -\frac{4\sqrt{1-t}}{t^2}\cdot\tilde{E}\cdot(\tilde{E}-\tilde{K})\cdot(\tilde{E}+(t-1)\cdot\tilde{K}) \tag{B.2}$$

$$\begin{aligned} C(1,3) \;=\; & \frac{4}{3t^2}\cdot\left[(2-t)\cdot\tilde{E}^3 - 5\cdot(1-t)\cdot\tilde{E}^2\tilde{K}\right. \\ & \left. + (1-t)\cdot(2-t)\cdot\tilde{E}\tilde{K}^2 - (1-t)^2\cdot\tilde{K}^3\right], \end{aligned} \tag{B.3}$$

$$\begin{aligned} C(0,5) \;=\; & \frac{256\sqrt{1-t}}{81\,t^6}\cdot\left[(1+t)\cdot\tilde{E} + (t-1)\cdot\tilde{K}\right]\cdot\left[(t-2)\cdot\tilde{E} + 2\,(1-t)\cdot\tilde{K}\right] \\ & \times\left[(2t-1)\cdot\tilde{E} + (1-t)\cdot\tilde{K}\right]\cdot\left[3\,\tilde{E}^2 + (2t-4)\cdot\tilde{E}\,\tilde{K} + (1-t)\cdot\tilde{K}^2\right]. \end{aligned} \tag{B.4}$$

For $T > T_c$

$$\begin{aligned} C(1,3) \;=\; & -\frac{4}{3\,t^{5/2}}\cdot\left[(1-2t)\cdot\tilde{E}^3 - (1-t)\cdot(3-t)\cdot\tilde{E}^2\,\tilde{K},\right. \\ & \left. + (1-t)\cdot(3-t)\cdot\tilde{E}\,\tilde{K}^2 - (1-t)^2\cdot\tilde{K}^3\right], \end{aligned} \tag{B.5}$$

$$\begin{aligned} C(0,4) \;=\; & -\frac{16}{9\,t^4}\cdot\left[(5-5t-t^2)\cdot\tilde{E}^4 - 8\cdot(2-t)\cdot(1-t)\cdot\tilde{E}^3\,\tilde{K} + (1-t)^3\cdot\tilde{K}^4\right. \\ & \left. + 2\cdot(1-t)\cdot(3-t)\cdot(3-2t)\cdot\tilde{E}^2\,\tilde{K}^2 - 4\cdot(2-t)\cdot(1-t)^2\cdot\tilde{E}\,\tilde{K}^3\right] \\ =\; & \frac{16}{9\,t^4}\cdot C_+\cdot C_-, \quad \text{where:} \end{aligned} \tag{B.6}$$

$$C_\pm \;=\; \Big(2 - t \;\pm 3 \cdot (1 - t)^{1/2}\Big) \cdot \tilde{E}^2 \;\pm (1 - t)^{3/2} \cdot \tilde{K}^2$$
$$- \Big(2 \cdot (1 - t) \;\pm 2 \cdot (2 - t) \cdot (1 - t)^{1/2}\Big) \cdot \tilde{E}\,\tilde{K}. \qquad \text{(B.7)}$$

For $T > T_{\rm c}$ with $M + N$ odd examples of $\tilde{C}(M, N)$ of (76):

$$\tilde{C}(1,2) \;=\; \frac{1}{t} \cdot \Big[(t - 1) \cdot \tilde{K}^2 \;- 2 \cdot (t - 2) \cdot \tilde{K}\,\tilde{E} \;- 3 \cdot \tilde{E}^2\Big], \qquad \text{(B.8)}$$

$$\tilde{C}(1,4) \;=\; \frac{16}{9\,t^4} \cdot \Big[5 \cdot (t-1)^3 \cdot \tilde{K}^4 \;- 12 \cdot (t - 2) \cdot (t-1)^2 \cdot \tilde{E}\tilde{K}^3 \;- 4 \cdot (t - 2)^3$$
$$\times \cdot \tilde{E}^3\,\tilde{K} + 6 \cdot (t - 1) \cdot (t^2 - 7t + 7) \cdot \tilde{E}^2\,\tilde{K}^2 \;- 9 \cdot (1 - t + t^2) \cdot \tilde{E}^4\Big], \qquad \text{(B.9)}$$

$$\tilde{C}(2,3) \;=\; \frac{4}{9\,t^{5/2}} \cdot \Big[(3t - 1) \cdot (t - 1)^2 \cdot \tilde{K}^3 \;- (t - 1) \cdot (6t^2 - 17t + 3) \cdot \tilde{E}\,\tilde{K}^2$$
$$- (20t^2 - 31t + 3) \cdot \tilde{E}^2\,\tilde{K} \;+ (t^2 - 16t + 1) \cdot \tilde{E}^3\Big]. \qquad \text{(B.10)}$$

Appendix C. Correlation functions as Toeplitz determinants for $T > T_{\rm c}$

We have shown in section 5 that the correlation functions for $T < T_{\rm c}$ and $\nu = -k$ are given in terms of Toeplitz determinants (see equations (125), (131)–(133)) using the results of Forrester–Witte [11].

Using the same method, one can easily generalise these equations to $T > T_{\rm c}$ and $\nu = -k_>$. One verifies that

- For $T > T_{\rm c}$ and $M + N$ even

$$C(M,N) \;=\; (-1)^{(N-M)/2} \cdot (1 - t)^{(N-M)^2/4} \cdot D_N, \qquad \text{(C.1)}$$

the Toeplitz matrix elements are given for $m \geqslant 1$ by:

$$A_m \;=\; \frac{\Gamma(\frac{N-M-1}{2} + m)}{\Gamma(\frac{N-M+1}{2})\,(m-1)!} \cdot (-1)^{m-1} \cdot t^{(m-1)/2}$$
$$\times \;{}_2F_1\left([\frac{N-M-1}{2}, \frac{N-M-1}{2} + m], [m], \; t\right) \qquad \text{(C.2)}$$

and for $m < 1$ by

$$A_m \;=\; \frac{\Gamma(\frac{N-M+1}{2} - m)}{\Gamma(\frac{N-M-1}{2})\,(1-m)!} \cdot (-1)^{m-1} \cdot t^{(1-m)/2}$$
$$\times \;{}_2F_1\left([\frac{N-M+1}{2}, \frac{N-M+1}{2} - m], [2-m], \; t\right). \qquad \text{(C.3)}$$

- For $T > T_{\rm c}$ and $M + N$ odd, we know that $C(M, N) = 0$. After removing the vanishing factor $(1 + \nu/k_>)^{1/2}$, one obtains:

$$\tilde{C}(M,N) \;=\; (-1)^{(N-M+1)/2} \cdot (M+N) \cdot (1 - t)^{((N-M)^2-1)/4} \cdot D_N \qquad \text{(C.4)}$$

The Toeplitz matrix elements for $m \geqslant 1$ are

$$A_m \;=\; \frac{\Gamma(\frac{N-M}{2}-1+m)}{\Gamma(\frac{N-M}{2})\,(m-1)!}\cdot(-1)^{m-1}\cdot\; t^{(m-1)/2}$$
$$\times\;{}_2F_1([\frac{N-M}{2},\frac{N-M}{2}-1+m],\,[m],\;t), \qquad \text{(C.5)}$$

and for $m < 1$

$$A_m \;=\; \frac{\Gamma(\frac{N-M}{2}+1-m)}{\Gamma(\frac{N-M}{2})\,(1-m)!}\cdot(-1)^{m-1}\cdot\; t^{(1-m)/2}$$
$$\times\;{}_2F_1\left([\frac{N-M}{2},\frac{N-M}{2}+1-m],\,[2-m],\;t\right). \qquad \text{(C.6)}$$

One verifies that these expressions for the correlation functions are totally compatible with the ones given in appendix B obtained by the quadratic difference equations (see section 2.4).

Appendix D. Boundary conditions

The study of solutions of PVI analytic at $t = 0$ was first done be Kaneko [25] for generic parameters in the Hamiltonian formalism where four solutions were found. Here we concentrate on the non-generic cases which allow one parameter families of solutions. This is most systematically done by determining the allowed boundary conditions on the canonical equation of Okamoto (91) which are analytic at $t = 0$. Thus we set in (91)

$$h(t) \;=\; \sum_{n=0} c_n\cdot\, t^n. \qquad \text{(D.1)}$$

From the constant term we find

$$n_1^2n_2^2n_3^2\;+n_1^2n_2^2n_4^2\;+n_1^2n_3^2n_4^2\;+n_2^2n_3^2n_4^2\;\;-4\,c_0\cdot\,n_1n_2n_3n_4$$
$$+\,c_1\cdot\,(n_1^2n_2^2+n_1^2n_3^2+n_1^2n_4^2+n_2^2n_3^2+n_2^2n_4^2+n_3^2n_4^2\;-2\,n_1n_2n_3n_4)$$
$$+\,c_1^2\cdot\,(n_1^2+n_2^2+n_3^2+n_4^2)\;\;-4\cdot\,c_0c_1\cdot\,(c_0+c_1)\;=\;0, \qquad \text{(D.2)}$$

and from the t term:

$$-\,2\cdot\,c_2\cdot\,\big[4\,c_0^2\;+8\,c_0c_1\;-2\,c_1\cdot\,(n_1^2+n_2^2+n_3^2+n_4^2)$$
$$-\,n_1^2n_2^2-n_1^2n_3^2-n_1^2n_4^2-n_2^2n_3^2-n_2^2n_4^2-n_3^2n_4^2\;+2\,n_1n_2n_3n_4\big]\;=\;0. \qquad \text{(D.3)}$$

We solve (D.3) for c_1 in terms of c_0 and n_k by setting the term in brackets to zero and using this in (D.2) we find that the resulting fourth order equation in c_0 factors into four factors linear in c_0 and thus we have the four solutions for c_0 of

$$c_0^{(1)} \;=\; (-n_1n_2-n_3n_4\;+(n_1+n_2)\,(n_3+n_4))\,/2, \qquad \text{(D.4)}$$

$$c_0^{(2)} \;=\; (n_1n_2+n_3n_4\;+(n_1-n_2)\,(n_3-n_4))\,/2, \qquad \text{(D.5)}$$

$$c_0^{(3)} \;=\; (n_1n_2+n_3n_4\;-(n_1-n_2)\,(n_3-n_4))\,/2, \qquad \text{(D.6)}$$

$$c_0^{(4)} \;=\; \left(-n_1 n_2 - n_3 n_4 \,-(n_1+n_2)\,(n_3+n_4)\right)/2, \tag{D.7}$$

and thus for c_1 the companion values are:

$$c_1^{(1)} \;=\; \frac{(n_1+n_2)\cdot\, n_3 n_4 \;\, - n_1 n_2 \cdot\, (n_3+n_4)}{n_1+n_2-n_3-n_4}, \tag{D.8}$$

$$c_1^{(2)} \;=\; \frac{(n_1-n_2)\cdot\, n_3 n_4 \;\, - n_1 n_2 \cdot\, (n_3-n_4)}{-n_1+n_2+n_3-n_4}, \tag{D.9}$$

$$c_1^{(3)} \;=\; \frac{(n_1-n_2)\cdot\, n_3 n_4 \;\, + n_1 n_2 \cdot\, (n_3-n_4)}{-n_1+n_2-n_3+n_4}, \tag{D.10}$$

$$c_1^{(4)} \;=\; \frac{(n_1+n_2)\cdot\, n_3 n_4 \;\, + n_1 n_2 \cdot\, (n_3+n_4)}{n_1+n_2+n_3+n_4}. \tag{D.11}$$

Case 4 is invariant under all permutations of the n_k. Case 1 is obtained from case 4 by changing the signs of n_3 and n_4. Case 2 is obtained from case 4 by changing the signs of n_1 and n_4. Case 3 is obtained from case 4 by changing the signs of n_2 and n_4. These sign changes are symmetries of the equation but not of the solutions.

The values of c_0 and c_1 for each of the four solutions may now be used to compute the term of order t^2 in the series expansion of (91) and we find that (91) holds if c_2 satisfies a linear equation. Thus we obtain:

$$c_2^{(1)} = -\frac{(n_1+n_2)(n_1-n_3)(n_1-n_4)(n_2-n_3)(n_2-n_4)(n_3+n_4)}{(n_1+n_2-n_3-n_4)^2(n_1+n_2-n_3-n_4+1)(n_1+n_2-n_3-n_4-1)}, \tag{D.12}$$

$$c_2^{(2)} = -\frac{(n_1-n_2)(n_1-n_3)(n_1+n_4)(n_2+n_3)(n_2-n_4)(n_3-n_4)}{(n_1-n_2-n_3+n_4)^2(n_1-n_2-n_3+n_4+1)(n_1-n_2-n_3+n_4-1)}, \tag{D.13}$$

$$c_2^{(3)} = \frac{(n_1-n_2)(n_1+n_3)(n_1-n_4)(n_2-n_3)(n_2+n_4)(n_3-n_4)}{(n_1-n_2+n_3-n_4)^2(n_1-n_2+n_3-n_4+1)(n_1-n_2+n_3-n_4-1)}, \tag{D.14}$$

$$c_2^{(4)} = \frac{(n_1+n_2)(n_1+n_3)(n_1+n_4)(n_2+n_3)(n_2+n_4)(n_3+n_4)}{(n_1+n_2+n_3+n_4)^2(n_1+n_2+n_3+n_4+1)(n_1+n_2+n_3+n_4-1)}. \tag{D.15}$$

Continuing the recursive procedure we find from the t^3 term in (91)

$$c_3^{(1)} \;=\; 2\cdot\,\frac{N_3^{(1)}\,\cdot\, c_2^{(1)}}{(n_1+n_2-n_3-n_4)\,(n_1+n_2-n_3-n_4+2)\,(n_1+n_2-n_3-n_4-2)},$$

$$\begin{aligned} N_3^{(1)} \;=\;& n_1^2 n_2 - n_1^2 n_3 - n_1^2 n_4 - n_2^2 n_3 - n_2^2 n_4 - n_3^2 n_4 \\ &+ n_1 n_2^2 + n_1 n_3^2 + n_1 n_4^2 + n_2 n_3^2 + n_2 n_4^2 - n_3 n_4^2 \\ &- n_1 n_2 n_3 - n_1 n_2 n_4 + n_1 n_3 n_4 + n_2 n_3 n_4 \, - n_1 - n_2 + n_3 + n_4, \end{aligned} \tag{D.16}$$

$$c_3^{(2)} \;=\; -2\cdot\,\frac{N_3^{(2)}\,\cdot\, c_2^{(2)}}{(n_1-n_2-n_3+n_4)(n_1-n_2-n_3+n_4+2)(n_1-n_2-n_3+n_4-2)},$$

$$\begin{aligned} N_3^{(2)} \;=\;& n_1^2 n_2 + n_1^2 n_3 - n_1^2 n_4 + n_2^2 n_3 - n_2^2 n_4 - n_3^2 n_4 \\ &- n_1 n_2^2 - n_1 n_3^2 - n_1 n_4^2 + n_2 n_3^2 + n_2 n_4^2 + n_3 n_4^2 \\ &- n_1 n_2 n_3 + n_1 n_2 n_4 + n_1 n_3 n_4 - n_2 n_3 n_4 \, + n_1 - n_2 - n_3 + n_4, \end{aligned} \tag{D.17}$$

$$c_3^{(3)} \;=\; -2 \cdot \frac{N_3^{(3)} \cdot c_2^{(3)}}{(n_1-n_2+n_3-n_4)\,(n_1-n_2+n_3-n_4\;+2)\,(n_1-n_2+n_3-n_4\;-2)},$$

$$\begin{aligned} N_3^{(3)} \;=\;& n_1^2n_2 - n_1^2n_3 + n_1^2n_4 - n_2^2n_3 + n_2^2n_4 + n_3^2n_4 \\ & - n_1n_2^2 - n_1n_3^2 - n_1n_4^2 + n_2n_3^2 + n_2n_4^2 - n_3n_4^2 \\ & + n_1n_2n_3 - n_1n_2n_4 + n_1n_3n_4 - n_2n_3n_4 + n_1 - n_2 + n_3 - n_4, \end{aligned} \qquad \text{(D.18)}$$

$$c_3^{(4)} \;=\; 2 \cdot \frac{N_3^{(4)} \cdot c_2^{(4)}}{(n_1+n_2+n_3+n_4)\,(n_1+n_2+n_3+n_4\;+2)\,(n_1+n_2+n_3+n_4\;-2)},$$

$$\begin{aligned} N_3^{(4)} \;=\;& n_1^2n_2 + n_1^2n_3 + n_1^2n_4 + n_2^2n_3 + n_2^2n_4 + n_3^2n_4 \\ & + n_1n_2^2 + n_1n_3^2 + n_1n_4^2 + n_2n_3^2 + n_2n_4^2 + n_3n_4^2 \\ & + n_1n_2n_3 + n_1n_2n_4 + n_1n_3n_4 + n_2n_3n_4 \;\; - n_1 - n_2 - n_3 - n_4. \end{aligned} \qquad \text{(D.19)}$$

This recursive solution may be extended to arbitrary order and for the general case $c_{n+1}^{(i)}$ will have factors in the denominator of

$$c_{n+1}^{(1)}: \quad n_1+n_2-n_3-n_4\;\pm n, \qquad \text{(D.20)}$$

$$c_{n+1}^{(2)}: \quad n_1-n_2-n_3-n_4\;\pm n, \qquad \text{(D.21)}$$

$$c_{n+1}^{(3)}: \quad n_1-n_2+n_3-n_4\;\pm n, \qquad \text{(D.22)}$$

$$c_{n+1}^{(4)}: \quad n_1+n_2+n_3+n_4\;\pm n. \qquad \text{(D.23)}$$

We thus conclude that as long as there are no vanishing factors in the denominator there are four distinct solutions of the equation (91) which are analytic at $t = 0$ and have no arbitrary constants. These form a four dimensional representation of the symmetry group of the equation. The statement that the solution is analytic at $t = 0$ is the boundary condition. This corresponds to the case (18) on page 7 of [26] where the subgroup of the monodromy group generated by M_0M_t and M_1 is reducible.

D.1. Arbitrary constants

This recursive solution will break down at an order n where $c_n^{(i)}$ has a zero in the denominator. This will give a solution only if there is a corresponding zero in the numerator which indicates that the corresponding recursive equation is automatically satisfied independently of the value of $c_n^{(i)}$ which now becomes an arbitrary parameter that must be specified as an additional boundary condition.

There are two ways in which these vanishing factors in the numerator can happen. Either $c_2^{(i)} = 0$ or $N_n^{(i)} = 0$. In this paper we will apply this analysis to the correlations $C(M,N)$ with $\nu = -k$. The behaviors of solutions of cases 1 and 4 are quite different from cases 2 and 3 and we treat them separately.

D.1.1. Cases 1 and 4 for $T < T_c$. For $T < T_c$ the Okamoto parameters for $C(M,N)$ are given by (100) so that

$$n_1 + n_2 \;=\; N, \quad \text{and} \quad n_3 + n_4 \;=\; 0, \qquad \text{(D.24)}$$

and thus from (D.4) and (D.7)

$$c_0^{(1)} \;=\; c_0^{(4)} \;=\; -\frac{1}{8}\cdot\left(N^2-M^2\;-\;\frac{1+(-1)^{M+N}}{2}\right), \tag{D.25}$$

and from (D.8) and (D.11):

$$c_1^{(1)} \;=\; c_1^{(4)} \;=\; -\frac{M^2}{4}. \tag{D.26}$$

Furthermore we see, from (D.12), (D.15), (D.16) and (D.19), that, because of the factor of n_3+n_4, the recursive solutions for cases 1 and 4, $c_k^{(1)}$ and $c_k^{(4)}$, will always vanish unless there is also a vanishing factor in the denominator. When for some k the denominator in $c_k^{(1,4)}$ does vanish then $c_k^{(1,4)}$ for that k is not determined from the recursive procedure and is an arbitrary constant. For $k=2$ and $T<T_{\rm c}$ we explicitly see from (D.12) and (D.15), and for $k=3$ from (D.16) and (D.19), that the factor in the denominator

$$n_1\;+n_2\;\;\pm\,(n_3+n_4)\;\;-k\,+1\;=\;N\,+1\;-k, \tag{D.27}$$

vanishes for $k=N+1$. This pattern continues for all k and thus

$$c_k^{(1)} \;=\; c_k^{(4)} \;=\; 0 \quad \text{for } 2\leqslant k\leqslant N, \tag{D.28}$$

and the coefficients $c_{N+1}^{(1)}$ and $c_{N+1}^{(4)}$ are arbitrary.

To compute the coefficients $c_k^{(1,4)}$ for $k>N+1$ a new recursive solution must be computed which uses the term $c_{N+1}\cdot t^{N+1}$ as input.

Thus for $t\to 0$

$$h \;=\; -\frac{1}{8}\cdot\left(N^2-M^2-\frac{1+(-1)^{M+N}}{2}\right)$$
$$-\frac{M^2}{4}\cdot t\quad +c_{N+1}\cdot t^{N+1}\quad +O(t^{N+2}) \tag{D.29}$$

Thus from (101) we find for $t\to 0$

$$C(M,N)\;=\;(1-t)^{1/4}\quad +K(M,N)\cdot t^{N+1}\quad +O(t^{N+2}), \tag{D.30}$$

which agrees with the series expansions of $C(M,N)$ for $T<T_{\rm c}$.

D.1.2. Cases 2 and 3 for $T>T_{\rm c}$ with $M+N$ even. For $T>T_{\rm c}$ with $M+N$ even we see with the Okamoto parameters for $C(M,N)$ of (104) that for cases 2 and 3 we have from (D.5) and (D.6)

$$c_0^{(2)}\;=\;\frac{1}{8}\cdot(M^2-N^2-1)\;-\frac{N}{2},\qquad c_0^{(3)}\;=\;\frac{1}{8}\cdot(M^2-N^2-1)\;+\frac{N}{2}, \tag{D.31}$$

and from (D.9) and (D.10)

$$c_1^{(2)}\;=\;\frac{N}{4}\cdot\frac{N+1-M^2}{1+N},\qquad c_1^{(3)}\;=\;\frac{N}{4}\cdot\frac{N-1+M^2}{1-N}. \tag{D.32}$$

Furthermore the denominator in $c_n^{(2)}$ vanishes when $n=N+2$ and the denominator of $c_n^{(3)}$ vanishes when $n=N$ and thus $c_{N+2}^{(2)}$ and $c_N^{(3)}$ are arbitrary. By using the definition of h in (103) and comparing with the series expansions of $C(M,N)$ we see for $T>T_{\rm c}$ and $M+N$ even that $C(M,N)$ is in case 2.

D.1.3. Cases 2 and 3 for $\tilde{C}(M,N)$ for $T > T_c$ with $M+N$ odd. For $T > T_c$ and $M+N$ odd we see with the Okamoto parameters of (107) that for cases 2 and 3 we have from (D.5) and (D.6)

$$c_0^{(2)} \; = \; \frac{1}{8} \cdot (M^2 - N^2) - \frac{N}{2}, \qquad c_0^{(3)} \; = \; \frac{1}{8} \cdot (M^2 - N^2) + \frac{N}{2}, \tag{D.33}$$

and from (D.9) and (D.10)

$$c_1^{(2)} \; = \; \frac{N^2 - 1 - (M^2-1)N}{4 \cdot (1+N)}, \qquad c_1^{(3)} \; = \; \frac{N^2 - 1 + (M^2-1)N}{4 \cdot (1-N)}. \tag{D.34}$$

Furthermore the denominator in $c_n^{(2)}$ also vanishes when $n = N+2$ and the denominator of $c_n^{(3)}$ vanishes when $n = N$ and thus $c_{N+2}^{(2)}$ and $c_N^{(3)}$ are arbitrary. By using the definition of h in (106) and comparing with the series expansions of $\tilde{C}(M,N)$ we see for $T > T_c$ and $M+N$ odd that $\tilde{C}(M,N)$ is in case 2.

D.2. Determination of λ when $\nu = -k$

It remains to determine the values of the arbitrary parameter which are appropriate for $C(M,N)$ when $\nu = -k$. To do this we first examine the behavior $C(M,N)$ at $t \to 0$ for several values of M and N. We consider $T < T_c$ and $T > T_c$ separately.

D.2.1. $T < T_c$. For $T < T_c$ the correlations are in case $1 = 4$ where the arbitrary constant is at order t^{N+1}. Several examples are

$$C(0,1;\,\lambda) \; = \; (1-t)^{1/4} \cdot \left[1 \; - \lambda^2 \cdot \left(\frac{1}{2^6}\,t^2 \; + O(t^3)\right)\right], \tag{D.35}$$

$$C(0,2;\,\lambda) \; = \; (1-t)^{1/4} \cdot \left[1 \; + \lambda^2 \cdot \left(\frac{1}{2^8}\,t^3 \; + O(t^4)\right)\right], \tag{D.36}$$

$$C(1,2;\,\lambda) \; = \; (1-t)^{1/4} \cdot \left[1 \; - \lambda^2 \cdot \left(\frac{1}{2^8}\,t^3 \; + O(t^4)\right)\right], \tag{D.37}$$

$$C(0,3;\,\lambda) \; = \; (1-t)^{1/4} \cdot \left[1 \; - \lambda^2 \cdot \left(\frac{9}{2^{14}}\,t^4 \; + O(t^5)\right)\right], \tag{D.38}$$

$$C(1,3;\,\lambda) \; = \; (1\,-t)^{1/4} \cdot \left[1 \; + \lambda^2 \left(\frac{3 \cdot 5}{2^{14}}\,t^4 \; + O(t^5)\right)\right], \tag{D.39}$$

where λ is chosen so that $\lambda = 1$ agrees with $C(M,N)$.

In general one has:

$$C(M,N;\;\lambda) \; = \; (1-t)^{1/4} \cdot \left[1 \; + (-1)^{M+N} \cdot \lambda^2 \cdot \left(K_{M,N}\,t^{N+1} \; + O(t^{N+2})\right)\right]. \tag{D.40}$$

We note for $\lambda = 0$ that

$$C(M,N;\;0) \; = \; (1\,-t)^{1/4}, \tag{D.41}$$

is an exact solution to the non-linear differential equation.

D.2.2. $T > T_c$ for $M+N$ even. These correlations are in the class 2 where the arbitrary constant is at order $t^{3N/2+2}$. Several examples are

$$C(0,2;\ \lambda) \ = \ (1\ -t)^{1/4} \cdot \frac{t}{8} \cdot \left[1\ +\frac{3}{4}t\ +\frac{3\cdot 5^2}{2^7}t^2\ +\frac{5\cdot 7^2}{2^9}t^3\ +\tilde{\lambda}^2\,\frac{3309}{2^{13}}t^4\ +O(t^5)\right]$$
$$= \ (1\ -t)^{1/4} \cdot \frac{t}{8} \cdot \left[{}_2F_1\left([\frac{3}{2},\frac{3}{2}],\,[3],\ t\right)\ +\lambda^2 \cdot \left(\frac{3}{2^{14}}t^4\ +O(t^5)\right)\right], \quad \text{(D.42)}$$

$$C(1,3;\ \lambda) \ = \ (1\ -t)^{1/4} \cdot t^{3/2} \cdot \frac{1}{16} \cdot \left[1\ +\frac{3\cdot 5}{2^4}t\ +\frac{3\cdot 5\cdot 7}{2^7}t^2\ +\frac{3\cdot 5\cdot 7^2}{2^{10}}t^3,\right.$$
$$\left. +\frac{3^3\cdot 5\cdot 7\cdot 11}{2^{14}}t^4\ +\tilde{\lambda}^2\,\frac{297\,315}{2^{19}}t^5\ +O(t^6)\right]$$
$$= \ (1\ -t)^{1/4} \cdot t^{3/2} \cdot \frac{1}{2^4} \cdot \left[{}_2F_1\left([\frac{3}{2},\frac{5}{2}],\,[4],\ t\right)\ +\lambda^2\left(\frac{3}{2^{18}}t^5\ +O(t^6)\right)\right] \quad \text{(D.43)}$$

$$C(0,4;\ \lambda) \ = \ (1\ -t)^{1/4} \cdot t^2 \cdot \frac{3}{2^7} \cdot \left[1\ +\frac{5}{2^2}\,t\ +\frac{5\cdot 7^2}{3\cdot 2^6}\,t^2\ +\frac{3^2\cdot 5\cdot 7}{2^8}\,t^3\right.$$
$$\left. +\frac{3^2\cdot 5\cdot 7\cdot 11^2}{2^{15}}t^4\ +\frac{7\cdot 11^2\cdot 13^2}{2^{17}}t^5\ +\tilde{\lambda}^2 \cdot \frac{5\cdot 429\,431}{2^{21}}t^6\right]$$
$$= \ (1\ -t)^{1/4} \cdot t^2 \cdot \frac{3}{2^7} \cdot \left[{}_2F_1\left([\frac{5}{2},\frac{5}{2}],\,[5],\ t\right)\ +\lambda^2 \cdot \left(\frac{5}{2^{20}}t^6\ +O(t^7)\right)\right], \quad \text{(D.44)}$$

where both λ and $\tilde{\lambda}$ are chosen such that, when $\lambda = 1$ and $\tilde{\lambda} = 1$, there is agreement with $C(M,N)$. In general:

$$C(M,N;\ \lambda) \ = (1\ -t)^{1/4} \cdot t^{N/2} \cdot K^{(1)}_{M,N} \cdot \left[{}_2F_1\left([\frac{N-M+1}{2},\frac{N+M+1}{2}],\,[N+1],\ t\right)\right.$$
$$\left. +\lambda^2 \cdot \left(K^{(2)}_{M.N} \cdot t^{N+2}\ +O(t^{N+3})\right)\right]. \quad \text{(D.45)}$$

We note that $\tilde{\lambda}$ and λ are not the same. When $\tilde{\lambda} = 0$ the term $O(t^{N+3})$ does not in general vanish. This is in contrast with the case $\lambda = 0$ where, for some constant ρ

$$C(M,N;\ 0) \ = \rho \cdot (1\ -t)^{1/4} \cdot t^{N/2} \cdot {}_2F_1\left([\frac{N-M+1}{2},\frac{N+M+1}{2}],\,[N+1],\ t\right), \quad \text{(D.46)}$$

is an exact solution to the nonlinear differential equation. The constant $K^{(1)}_{M,N}$ is a normalization constant which cannot be determined from the non linear equation.

In the specific examples (D.35)–(D.39), (D.42)–(D.44) the numerical coefficient of λ^2 has been chosen so that $\lambda = 1$ is the desired correlation where $C(M,N)$ is given as a finite homogenerous polynomial in $\tilde{K}$ and $\tilde{E}$, and in the case of $C(0,N)$ as an $N \times N$ Toeplitz determinant. However, in general no explicit formula for $K_{M,N}$ or $K^{(2)}_{M,N}$ is known which allows $\lambda = 1$ to reduce to the desired result $C(M,N)$.

D.2.3. $T > T_c$ for $M+N$ odd. Similarly, one can see that

$$\tilde{C}(M,N;\ \lambda) \ = \ (1\ -t)^{-1/4} \cdot t^{N/2} \cdot K^{(1)}_{M,N} \cdot \Big[{}_2F_1\Big([\frac{N-M}{2}, \frac{N+M}{2}], [N+1], \ t \Big) + \lambda^2 \cdot \Big(K^{(2)}_{M,N} \cdot t^{N+2} \ + O(t^{N+3}) \Big) \Big], \tag{D.47}$$

and, for some constant ρ, that

$$\tilde{C}(M,N;\ 0) \ = \ \rho \cdot (1\ -t)^{-1/4} \cdot t^{N/2} \cdot {}_2F_1\Big([\frac{N-M}{2}, \frac{N+M}{2}], [N+1], \ t \Big), \tag{D.48}$$

is an exact solution to the nonlinear differential equation (79).

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