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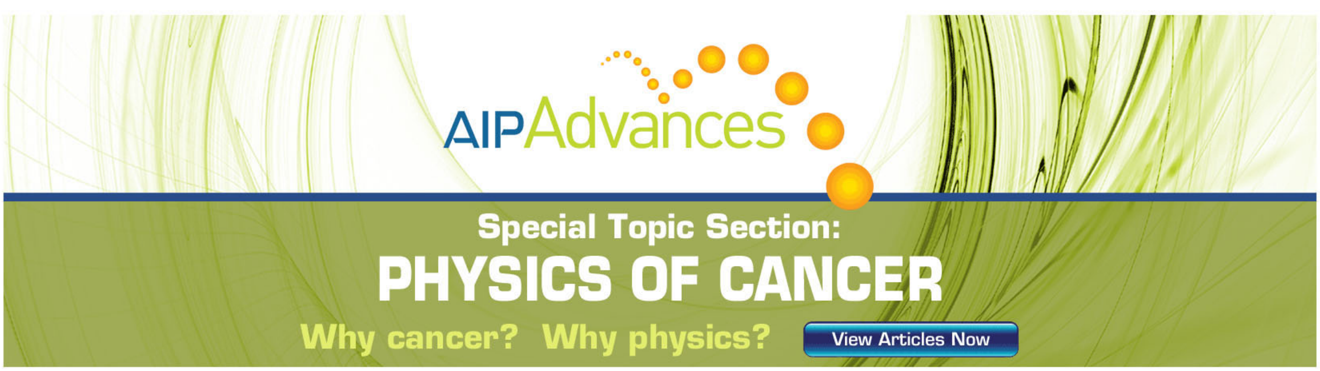

Automorphisms of algebraic varieties and Yang–Baxter equations

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It is shown that commuting transfer matrix models in statistical mechanics are parametrized by algebraic varieties having a set of automorphisms deduced from the so-called "inversion relation." In general this set of automorphisms is infinite: this shows that for algebraic varieties of dimension 1, the models are parametrized by algebraic curves of genus 0 or 1.

I. INTRODUCTION

Commuting transfer matrices provide the best known criterion for exactly solvable lattice models in statistical mechanics (or models of quantum field theory). A key role is played by a special system of algebraic equations, the so-called Yang–Baxter equations (or star triangle relation or factorization equations): the underlying reason is that the (local) star-triangle relation is a sufficient (and, to some extent, necessary[1]) condition for the commutation of (global) transfer matrices.

These Yang–Baxter equations can be seen as certain homological conditions that describe the structure of the exactly solvable models. A large number of solutions of the Yang–Baxter equations have been found and recorded.[2,3] One should, however, note that all these solutions are parametrized in terms of elliptic, trigonometric, or rational functions. The few examples that gave some hope to elaborating more sophisticated structures seem to confirm a somewhat disappointing situation: the two-dimensional vertex models for which a uniformization by theta functions of genus $g>1$ was introduced do not satisfy the Yang–Baxter equations despite the fact that a Zamolodchikov algebra does exist for these models (because of the Frobenius relation on theta functions)[4,5]; on the other hand, the remarkable solution to the three-dimensional generalization of the star triangle equation, namely the "tetrahedron equation," obtained by Zamolodchikov and Baxter, turned out to be closely related with the two-dimensional free fermion Ising model (for which an elliptic parametrization occurs).[6,7] The star-triangle relation appears to be a very stringent structure (overdetermined set of equations) and this fully legitimatizes the attempts to classify exhaustively these remarkable nontrivial solutions. Along this line one should recall the beautiful papers of Belavin–Drinfeld (in which an exhaustive classification of some "classical" limit of the Yang–Baxter equations related to simple Lie algebras is displayed[8]) as well as Jimbo's success at "quantizing" this classical limit by introducing a q-analog of the universal enveloping algebra and an associated Hecke algebra.[9] But an exhaustive list of solutions is still unavailable.

We will not deal in this paper with the (infinite-dimensional) Lie algebra aspects of the problem. The aim here is rather to suggest an approach to this classification problem that concentrates on the parametrization of the Yang–Baxter equations in the framework of algebraic geometry. We shall show that the parameter space of the exactly solvable models of statistical mechanics is naturally foliated by algebraic varieties that are stable under the action of a generically infinite number of birational transformations. Our problem then reduces to classical problems of algebraic geometry (algebraic varieties possessing an infinite set of automorphisms, diophantine equations, etc.) for which numerous results are available.

In that generic case, the existence of an infinite set of automorphisms does not allow these algebraic varieties to be of the so-called "general type." In particular when these are of dimension 1 it means that the model can be parametrized by curves of genus 0 or 1 only (elliptic or rational parametrization). The study of these varieties, which are not of the general type, will lead us to make a distinction between the varieties obtained by a complete and an incomplete intersection.

The requirement that the group of automorphisms be finite very sharply constrains the model: for instance, in the case of the anisotropic q-state Potts model this imposes a restriction to the values

$$q = 2 + 2\cos 2\pi m/n \quad (m,n\in\mathbb{Z}).$$

These particular values have already been singled out by many authors (Tutte–Beraha numbers, two-dimensional models with conformal covariance, rational critical exponents, etc.[10,11]).

The results of that paper are not restricted to two-dimensional exactly solvable models. No assumption is made on the existence of a particular classical limit for the Yang–Baxter (or tetrahedron) equations.

II. THE BAXTER MODEL

Let us recall briefly some basic results concerning one of the most important exactly solvable model: the symmetric eight-vertex Baxter model.[12] It is parametrized by four homogeneous variables $(a,b,c,d)\in\mathbb{P}_3$, and the Yang–Baxter equations take the form of six trilinear homogeneous equations for three sets of points in $\mathbb{P}_3$: (a,b,c,d), (a',b',c',d'), and (a'',b'',c'',d''). This system of homogeneous equations has nontrivial solutions if

$$F_1(a,b,c,d) = F_1(a',b',c',d') = F_1(a'',b'',c'',d'') \quad (1)$$

and

$$F_2(a,b,c,d) = F_2(a',b',c',d') = F_2(a'',b'',c'',d''), \quad (2)$$

where

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$$F_1 = (a^2 + b^2 - c^2 - d^2)/ab$$

and

$$F_2 = cd/ab.$$

The Yang–Baxter equations imply the commutation of the $2^N \times 2^N$ row-to-row transfer matrices for arbitrary N (N is the number of vertices in a row), that is,

$$[T_N(a,b,c,d), T_N(a',b',c',d)] = 0,$$

when Eqs. (1) and (2) are satisfied.

The integrability of the model leads thus to the following foliation of the parameter space:

$$F_1(a,b,c,d) = K_1 = \text{const},$$
$$F_2(a,b,c,d) = K_2 = \text{const}. \tag{3}$$

One recognizes the well-known projective representation of an elliptic curve as an intersection of quadrics in $\mathbf{P}_3$ (Clebsch's biquadratic). One can introduce the following elliptic parametrization:

$$\begin{aligned} a &= \rho\cdot\text{sn}(v+\eta,k),\\ b &= \rho\cdot\text{sn}(\eta - v,k),\\ c &= \rho\cdot\text{sn}(2\eta,k),\\ d &= \rho k\cdot\text{sn}(2\eta,k)\text{sn}(\eta - v,k)\text{sn}(v+\eta,k), \end{aligned} \tag{4}$$

with

$$K_1 = 2\,\text{cn}(2\eta,k)\cdot\text{dn}(2\eta,k),$$
$$K_2 = k\,\text{sn}^2(2\eta,k),$$

where sn, cn, and dn are the Jacobian elliptic functions of modulus k. With that elliptic parametrization the Yang–Baxter equations simply read

$$v + v' + v'' = \eta. \tag{5}$$

In this particular case we have an obvious connection between the Yang–Baxter structure and the Abelian character of the algebraic curve. There also exist exact symmetries on the model, the so-called inversion relations,[13] which correspond to rational transformations on the parameters of the model. These transformations are involutions and will be denoted by I and J:

$$I\text{: } a \to a/(a^2 - d^2), \quad b \to b/(b^2 - c^2),$$
$$c \to -c/(b^2 - c^2), \quad d \to -d/(a^2 - d^2), \tag{6a}$$
$$J\text{: } a \to a/(a^2 - c^2), \quad b \to b/(b^2 - d^2),$$
$$c \to -c/(a^2 - c^2), \quad d \to -d(b^2 - d^2). \tag{6b}$$

Here F_1 and F_2 are invariant under I and J. With the elliptic parametrization I and J reduce to

$$I\text{: } v \to +2\eta - v, \quad J\text{: } v \to -2\eta - v.$$

They are conjugate via the "crossing" symmetry on the model

$$a \leftrightarrow b, \quad v \to -v.$$

These involutions generate an infinite discrete group G of symmetries of the model isomorphic to the semidirect product

$$\mathbf{Z}_2 \oplus \mathbf{Z} \quad (v \to \pm v \pm 2n\eta, \quad n \in \mathbf{Z}).$$

This infinite set of birational transformations preserve the elliptic curve (3) and the modulus of the elliptic functions. One should not confuse these transformations with the isogenies of the elliptic curve (Landen, Jacobi, Legendre transformations). One of these isogenies, the Landen transformation $k \to k_L = 2\sqrt{k}/(1+k)$ can be identified with a generator of the renormalization group for that model (a fixed point of that transformation is $k = 1$, the critical point of the model): the group G and the renormalization group act in an "orthogonal" way.

Finally the Baxter model trivializes on the so-called disorder varieties of the parameter space, on which the partition function reduces to that of an isolated vertex. For this model these varieties have a very simple expression; one of these varieties, for instance, reads

$$a + d = b + c. \tag{7}$$

The partition function per size Z is then very simple:

$$Z = a + d. \tag{8}$$

Of course these disorder varieties correspond to a trivialization of the parametrization: equation (7) corresponds to a relation between F_1 and F_2 and a value of the modulus of the elliptic functions for which this parametrization trivializes

$$F_1 = 2 - 2F_2 \Rightarrow k = -1 \quad (\text{or } k_L = \infty).$$

III. INTRODUCTION TO THE GENERAL SITUATION

For the sake of simplicity we restrict ourselves to the q-state IRF model[3] but the ideas we develop here also apply straightforwardly to two-dimensional vertex models, three- (or higher-) dimensional models. In order to fix the notations let us first recall the definition of the q-state IRF model. The spin variable associated to each site i of a square lattice are assumed to take q values: $W(\sigma_i, \sigma_j, \sigma_k, \sigma_l)$ is the Boltzmann weight associated to each of the q^4 spin configurations around a face with sites i,j,k,l (see Fig. 1). The model depends therefore on q^4 homogeneous parameters $(x_1,\ldots, x_i,\ldots, x_{q^4})$. The partition function per site Z is defined by

$$Z^N = \sum_{\{\sigma_i\}} \prod_{\square} W(\sigma_i,\sigma_j,\sigma_k,\sigma_l) \quad (\sigma_i \in \mathbb{Z}_q), \tag{9}$$

where the product is taken over all the elementary square of the lattice and N is the number of these squares.

More accurately the partition function (or even the transfer matrices) are invariant under some "gauge" transformations

$$W(\sigma_i,\sigma_j,\sigma_k,\sigma_l) \to W(\sigma_i,\sigma_j,\sigma_k,\sigma_l)\,\frac{D(\sigma_i,\sigma_l)}{D(\sigma_j,\sigma_k)}\,\frac{\Delta(\sigma_i,\sigma_j)}{\Delta(\sigma_l,\sigma_k)}. \tag{10}$$

The analysis made in this paper forgets these trivial transformations. There exist two inversion relations I and J. They act on the Boltzmann weight to give W_I and W_J defined by (see Fig. 2)

$$\sum_{\sigma_k} W(\sigma_i,\sigma_j,\sigma_k,\sigma_l)\cdot W_I(\sigma_l,\sigma_k,\sigma_j,\sigma_m) = \lambda\cdot\delta\sigma_i,\sigma_m, \tag{11a}$$

$$\sum_{\sigma_k} W(\sigma_i,\sigma_j,\sigma_k,\sigma_l)\cdot W_J(\sigma_j,\sigma_m,\sigma_l,\sigma_k) = \lambda\cdot\delta\sigma_i,\sigma_m. \tag{11b}$$

These transformations amount (*up to a rotation of the elementary square*) to looking at W, in two different ways, as q^2

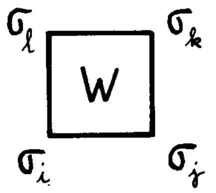

FIG. 1. The Boltzmann weight $W(\sigma_i, \sigma_j, \sigma_k, \sigma_l)$ associated to each of the q^4 spin configurations $(\sigma_i, \sigma_j, \sigma_k, \sigma_l)$ around a face with sites i,j,k,l.

$q \times q$ matrices, and taking the inverse of these q^2 matrices as

$$(W(\sigma_i,\sigma_j,\sigma_k,\sigma_l) \to W_{\sigma_i,\sigma_k}(\sigma_j,\sigma_l) \quad \text{or} \quad W_{\sigma_j,\sigma_l}(\sigma_i,\sigma_k)).$$

Because of this composition by a rotation these transformations are not involutions as the one previously introduced for the Baxter model; they are generally of infinite order.

These transformations I and J are both birational transformations

$$x_i \to \frac{P_i(x_1,...,x_{q^4})}{Q_i(x_1,...,x_{q^4})} = I(x_i) \text{ or } J(x_i), \tag{12}$$

where P_i and Q_i are two homogeneous polynomials of degree $q-1$ and q in the x_i's, respectively, with integer coefficients ($+1$ or -1).

This model may seem to be too general, depending on a too large number of parameters. The usual practice corresponds to imposing different symmetries or constraints on the model in order to restrict the number of homogeneous parameters of the model (equalities between different x_i's, exclusion of some configurations $x_j = 0$, etc) from q^4 to n. In the following we will restrict the parameter space to such a homogeneous space $\mathbb{P}_{n-1}$ with the condition that the (rational) transformations I and J leave that subspace invariant. Heuristic arguments based on the transfer matrix formalism enable us to show the partition function per site presents some automorphy properties with respect to these two transformations (11a) and (11b) and of course the group G generated by these two transformations[14]:

$$Z(x_1,...,x_n)\cdot Z(I(x_1),..., I(x_n)) = \lambda, \tag{13a}$$

$$Z(x_1,...,x_n)\cdot Z(J(x_1),...,J(x_n)) = \lambda'. \tag{13b}$$

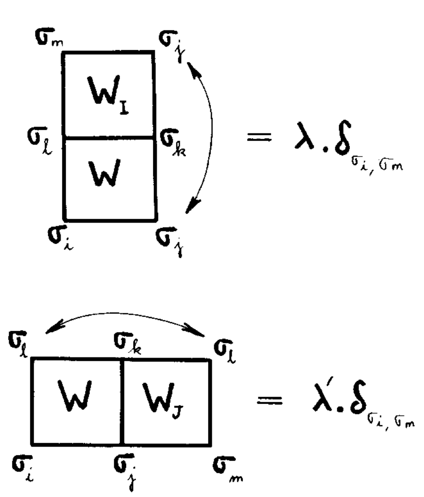

FIG. 2. Pictorial representation of the definition of the two inverse Boltzmann weights W_I and W_J.

The group G is, in general, an infinite discrete group. We now suppose that the model is exactly solvable in the sense that the Yang–Baxter equations are satisfied for the model. This leads to the commutation of the row-to-row (*and also column-to-column*) transfer matrices for arbitrary size N ($[T_N(W), T_N(W')] = 0$). The commutation of transfer matrices of specific sizes N leads to a set of algebraic equations[1,14,15] (see Appendix A for a simpler demonstration than in Ref. 1):

$$F_{\alpha,N}(x_1,...,x_n) = F_{\alpha,N}(x_1',...,x_n'), \tag{14}$$

where

$$F_{\alpha,N}(x_1,...,x_n) = \frac{U_{\alpha,N}(x_1,...,x_n)}{V_{\alpha,N}(x_1,...,x_n)},$$

where $U_{\alpha,N}$ and $V_{\alpha,N}$ are homogeneous polynomials (of degree $d_{\alpha,N}$) with integer coefficients. It can be shown that the algebraic varieties defined by the intersection of the expressions $F_{\alpha,N}$ corresponding to the row-to-row and column-to-column transfer matrices are invariant under the transformations I and J (see Refs. 14 and 16):

$$F_{\alpha,N}(x_1,...,x_n) = F_{\alpha,N}(I(x_1),...,I(x_n)) = \cdots . \tag{15}$$

This is a consequence of the fact that if a Yang–Baxter equation exists for the Boltzmann weight (W,W',W'') there necessarily exists another one involving W_I' and W and in fact an infinite set of other triplets of Boltzmann weights corresponding to some transformations of the initial triplet (W,W',W'') under the action of the group G (see Ref. 14). In the previous example of the Baxter model this corresponds to saying that Eq. (5) is also satisfied if one replaces (v,v',v'') by $(2n_1\eta + v, 2n_2\eta + v', 2n_3\eta + v'')$ with $n_1 + n_2 + n_3 = 0$.

An integrable model must therefore present the two following remarkable features.

(i) The infinite set of equations (14) corresponding to the various values of N must be redundant and equivalent to a finite set of m equations ($m \leqslant n-2$) we will denote from now on by F_α ($\alpha = 1,...,m$) (if this is not the case we are reduced to the trivial commutation of a matrix with itself).

(ii) The algebraic variety $\mathscr{V}$ defined by the intersection of these m equations (of dimension $n-1-m$) has to be invariant under the infinite discrete group G of birational transformations in $\mathbb{P}_{n-1}$.

Therefore one sees that the exactly solvable models are naturally parametrized in terms of algebraic varieties that have (in the general case) an infinite group of automorphisms.

IV. RESULTS

A. Algebraic curves

In almost all the examples of exactly solvable models known in statistical mechanics the algebraic varieties $\mathscr{V}$ turn out to be of dimension 1 (i.e., an algebraic curve). The following result is well-known: the only algebraic curves with an infinite group of automorphisms are of genus 0 or 1 (see Ref. 17).

In other words, if the group G does not degenerate into a finite group G, one has to deal with a rational or elliptic parametrization. This result can be understood in the following heuristic way: the main distinction between the curves of genus 0 or 1 and curves of the general type of genus $g \geqslant 2$ (for which one would have to envisage a uniformization in terms of automorphic functions) lies in that there exists a finite number of particular points, called the Weierstrass points, for the curves of general type. (A point is called an ordinary point if the gap values are $1,2,...,g$; otherwise it is called a Weierstrass point.) The group G that leaves invariant the algebraic curve must leave invariant these points. One understands that it is difficult for an infinite discrete group to leave invariant such a finite set of points. An old demonstration of Hettner (and also Noether) is based on these ideas. It is amusing to notice that if we consider a rational point in $\mathbb{P}_{n-1}$ ($x_i \in \mathbb{Q} \Rightarrow F_\alpha \in \mathbb{Q}$), the images of that point by the infinite group G are also rational points. We are thus led to an algebraic curve with a (generically) infinite set of rational points: Falting's theorem confirms that the curve has to be of genus 0 or 1 (see Ref. 18).

Now that we have a precise characterization of the curves that can possibly arise in the context of exactly solvable models it is useful to study the projective representation of an elliptic curve (in $\mathbb{P}_n$); the results are the following: the only case when a curve of genus 1 is given by a complete intersection are the plane cubic in $\mathbb{P}_2$ and the previous Clebsch's biquadratic in $\mathbb{P}_3$; the other representations are in $\mathbb{P}_n$ ($n \geqslant 4$) and correspond to incomplete intersections. The case of incomplete intersection may, at first sight, seem rather academic as far as statistical mechanics is concerned. However, there does exist at least one interesting example of model corresponding to that situation: for the hard hexagon model[19] the elliptic curve that parametrizes the model is given by an incomplete intersection of a quadric $F_1 = \text{const} = C_1$, a cubic $F_2 = \text{const} = C_2$, and a quartic $F_3 = \text{const} = C_3$ in $\mathbb{P}_4$,

$$F_1 = \frac{x_1^2 - x_4 x_5}{x_2 x_3}, \quad F_2 = \frac{x_4 x_3^2 + x_5 x_2^2 - x_1 x_4 x_5}{x_1 x_2 x_3},$$
$$F_3 = \frac{x_1 x_2^2 x_5 + x_1 x_3^2 x_4 - x_4^2 x_5^2 - x_2^2 x_3^2}{x_2 x_3 x_4 x_5}. \tag{16}$$

On these expressions one verifies immediately that the intersection is incomplete (as it should) because it contains the spurious varieties $x_1 = x_2 = x_4 = 0$ and $x_1 = x_3 = x_5 = 0$. The genus of the algebraic curve defined by this intersection can be calculated from the formula of addition of the characteristic of Euler–Poincaré:

$$\begin{aligned} 1 - g = \chi(O_{\mathbb{P}_4}) &- \chi(O(-2)) \\ &- \chi(O(-3)) - \chi(O(-4)) + \chi(O(-5)) \\ &+ \chi(O(-6)) + \chi(O(-7)) - \chi(O(-9)), \end{aligned} \tag{17}$$

with

$$\chi(O(n)) = [(n+1)(n+2)(n+3)(n+4)]/4!$$

leading to a rather high genus if there were no singularities. The $g = 1$ case of the hard hexagon model corresponds to two relations between the previous constants C_i that raise the number of singularities to a maximum and thus reduce the genus to a minimum ($g = 1$):

$$C_1 \cdot C_2 = 1 \text{ and } C_1 + C_2 = C_3.$$

B. Algebraic surfaces

The problem of the classification of algebraic surfaces is much more complicated.[20] There exist invariants playing a role similar to the genus for curves (Kodeira's dimension, etc.) One can sketch the classification that way: first come surfaces of "general type," which have only a finite number of automorphisms. This case is excluded when G is infinite.

The surfaces that are not of the general type fall into five different classes (up to a birational correspondence): (a) the rational surfaces birationally isomorphic to $\mathbb{P}_2$; (b) the ruled surfaces ($\Gamma \times \mathbb{P}_1$) (these are surfaces that can be mapped onto a curve in such a way that all fibers of this mapping are isomorphic to $\mathbb{P}_1$); (c) the elliptic surfaces (fibrations by elliptic curves); (d) Abelian surfaces; and (e) K 3 surfaces. The K 3 surfaces have the property in common with Abelian surfaces that their canonical class is 0. However, in contrast with Abelian surfaces there are no regular one-dimensional forms on them.

These five sets of surfaces can all admit an infinite set of automorphisms.

Let us now assume that the algebraic variety $\mathscr{V}$ is given by a complete intersection (this corresponds *a priori* to the simplest situation in statistical mechanics).

A classical theorem (see Ref. 21, pp. 401 and 402) shows that complete intersection of dimension 2 has a trivial homotopy group ($\pi_1 = 0$). Thus the assumption of complete intersection excludes the Abelian surfaces and imposes that the variety $\mathscr{V}$ has singularities. To be more specific, this situation of complete intersection occurs for a cubic or a quartic in $\mathbb{P}_3$, for the intersection of two quadrics in $\mathbb{P}_4$ corresponding to a rational surface, and for the intersection of a quadric and a cubic in $\mathbb{P}_4$ or the intersection of three quadrics in $\mathbb{P}_5$ that correspond to a surface of type K 3.

In the case of a surface of type K 3 any explicit parametrization of the surface is, of course, hopeless.

C. Algebraic varieties of dimension > 2

Little information is available concerning the classification of these varieties. However, remarkable progress has been made during the past few years.[21] It is possible to define some invariants that unfortunately play only partially the role of the genus for algebraic curves (Betti numbers, etc.). Despite this complexity it is possible to single out varieties of a "general type" for which the number of automorphisms is finite.

The varieties that are not of a general type constitute a jungle, which is, however, fairly well understood in the simplest case of complete intersection.

Thus the situation seems rather unsatisfactory: one would like to be able to find other algebraic varieties invariant under the action of the group G that would make it possible (by taking the intersection with the algebraic varieties $\mathscr{V}$) to restrict the problem to an algebraic variety of lower dimension (eventually of dimension 1, leading to a foliation of the algebraic variety by curves of genus 0 or 1).

Fortunately such varieties can be obtained taking into account the fact that the inversion relations correspond (up to rotations of the elementary square) to taking the inverse of a set of matrices (see Appendix B). Of course, this approach applies only for algebraic varieties of dimension $\geqslant 3$. In the case of the Baxter model one can, for instance, exhibit in this way algebraic varieties defined by an intersection actually invariant under the group G:

$$abcd/[(a^2-c^2)(b^2-d^2)] = \text{const}, \tag{18a}$$

$$abcd/[(a^2-d^2)(b^2-c^2)] = \text{const}. \tag{18b}$$

However, the curve given by the intersection of these two quartics has, in general, no intersection with the elliptic curve (3).

V. *G* IS A FINITE GROUP

The previous analysis is based on the infinite character of the group G. When the group G is finite this leads to algebraic constraints on the parameter space that characterize the model very precisely. For every element g of G there exists an integer p such that g^P is equal to the unit element of G. This equality translated on the homogeneous parameters x_i means that the model is restricted to some very particular algebraic varieties.

Let us now recall the hexagon model, which can be seen as a subcase of the S.O.S. eight-vertex Baxter model[22]: despite the fact that this model has a finite group G, it presents (as we have mentioned already) an elliptic uniformization, which can be seen as a restriction of the elliptic uniformization of the Baxter model.[23] Nevertheless, it is true that it is difficult to specify the algebraic varieties corresponding to a model, with a finite group G, that is not obviously embedded into a larger model with an already known uniformization. It is, however, possible, in the case of algebraic curves of genus g, to give an upper bound of the order of the finite group G (see Ref. 17): $[G]\leqslant 84g-3$.

VI. DISORDER VARIETIES

We have already remarked that the Baxter model trivializes on a simple disorder variety (7). In fact such disorder varieties are quite easy to find[24] and their corresponding codimension is rather low. For instance, in the case of the 16-vertex model, there exist disorder varieties of codimension 1 in the parameter space. This should be compared with the codimension of the parameter space of the exactly solvable subcase of that model, the Fan and Wu free-fermion model[7] and the Baxter model of codimensions 4 and 5, respectively.

For instance let us consider a subcase of the 16-vertex model that has the two previous integrable models as subcases (but is not integrable in general): the asymmetric eight-vertex model. The homogeneous parameters of that model are usually denoted $a,a',b,b',c,c',d,$ and d' (the symmetric eight-vertex model corresponds to $a=a'$, $b=b'$, $c=c'$, and $d=d'$). That model has a disorder solution on the (disorder) variety given by the quartic equation (this result has also been obtained recently by Giacomini[25])

$$(a+a')+((a+a')^2-4(aa'-dd'))^{1/2} = (b+b')+((b+b'')^2-4(bb'-cc'))^{1/2}. \tag{19}$$

If the model were integrable, there should occur a trivialization of the parametrization on this disorder variety and also on the images of this variety by the infinite group G generated by the two inversion relations. It is a simple and instructive exercise to verify that there does, in fact, exist an infinite number of such images except in the two already-mentioned cases of the Baxter model and the free-fermion model ($aa'+bb'=cc'+dd'$), where the number of images of (19) under the action of the infinite group G is finite. The checkerboard Potts is another example of an infinite number of images of a disorder variety under the action of G (see Ref. 26); moreover one has a remarkable and instructive agreement between the exact expressions of the (analytical continuation of the) partition function on this infinite set of algebraic varieties and the exact expression of the partition function on the critical variety where the model is exactly solvable.[27]

The existence of such an infinite set of varieties at first seems hardly compatible with the exact solvability of the model. An obvious situation where this set is finite is when the group G is itself finite. Let us consider the checkerboard Ising model: this model has an elliptic uniformization and the modulus of the elliptic functions that occur is given (in terms of the high-temperature variables $t_i = th\, K_i$ and the dual variable

$$t_i^* = \frac{1-t_i}{1+t_i}, \quad \text{by Eq. (20):}$$

$$k = \prod_{i=1}^{4} \frac{t_i(t_i^* + t_j^* t_k^* t_l^*)(1-t_i^2)}{t_i^*(t_i + t_j t_k t_l)(1-t_i^{*2})}, \quad (i,j,k,l) = (1,2,3,4). \tag{20}$$

This algebraic expression trivializes on the disorder varieties of the model, on the dual of these disorder varieties (and of course when the coupling constant of the model trivializes $t_i=0$, $t_i=\pm 1$, $t_i^*=1$). From this example it is rather tempting (if one is willing to bet on the exact solvability of the noncritical three-state Potts model) to guess an algebraic expression k associated to that model from the known equations of the disorder varieties and their images under the group G (see Refs. 28 and 29).

VII. CONCLUSION, PROSPECTS

The exactly solvable models are parametrized by means of algebraic varieties having a group of automorphisms deduced from the so-called "inversion relations." It is very constraining for a model of statistical mechanics to ask for this group to be finite. It is, in general, infinite and this shows that these algebraic varieties are not of the "general type" (but this does not prove that they should be Abelian varieties). For algebraic varieties of dimension 1, this sheds a new light on the occurence of curves of genus 0 or 1 only for all the exact models known at the present moment. Of course this is just a preliminary work and these ideas will be pursued in forthcoming publications. The ideas we have developed here also apply, *mutatis mutandis*, to statistical models in d dimensions with the difference that the number of inversion relations that generate the group G grows with the dimension d. *A priori* there is no relation between these generators. Therefore the group G is in general a very "large" one (infinite discrete of course): is it possible for algebraic varieties to have such a large group of automorphisms?

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APPENDIX A: ALGEBRAIC CONDITIONS FOR COMMUTATION

The commutation of two $n\times n$ matrices T and T' leads to the existence of algebraic expressions in the coefficients of these two matrices $F_\alpha(T)=F_\alpha(T')$. They can be seen as some symmetric functions of the eigenvectors shared by T and T'. We sketch here a simple way to get these F_α's: Let us denote by C and C' two matrices that are linear combinations of powers of T and T',

$$C=\sum_{p=0}^{n-1}\alpha_p\cdot T^p,\quad C'=\sum_{p=0}^{n-1}\alpha'_p\cdot T'^p.$$

We have

$$[T,T']=0\Rightarrow[C,C']=0. \tag{A1}$$

Let us denote by C_{ij}, C'_{ij}, T_{ij}, and T'_{ij} the coefficients of these matrices. We can choose α_p and α'_p some algebraic expressions of the T_{ij} and T'_{ij} such that

$$C_{1j}=0,\quad j=1,\dots,n-1,\quad C_{1n}\neq 0,$$

$$C'_{1j}=0,\quad j=1,\dots,n-1,\quad C'_{1n}\neq 0.$$

Equation (20) then leads to

$$\forall i:\quad C'_{1n}/C'_{ni}=C_{1n}/C_{ni}. \tag{A2}$$

Similar algebraic expressions can be obtained imposing C such that

$$C_{ij}=0,\quad j=1,\dots,n-1,\quad C_{in}\neq 0.$$

APPENDIX B: *G*-INVARIANT VARIETIES

The characteristic polynomial $P_M(\lambda)$ of an $n\times n$ matrix M and of its inverse matrix M^{-1} are related:

$$\lambda^n P_{M^{-1}}(1/\lambda)=P_M(\lambda). \tag{B1}$$

We denote by c_i the coefficients of $P_M(\lambda)$ and obtain

$$P_M(\lambda)=\lambda^n+c_1\cdot\lambda^{n-1}+\cdots+c_i\lambda^{n-i}+\cdots+c_n.$$

An immediate consequence of (B1) is that the expressions $\phi_i=c_ic_{n-i}/c_n$ are invariant under the transformation $M\to M^{-1}$. These expressions are the ratio of two homogeneous polynomials of degree n in the coefficients of the matrix M. The inversion relations I and J correspond (up to a permutation of the homogeneous parameters of the model x_i) to taking the inverse of a set of q^2 matrices M_α. One can associate to each of these matrices the corresponding expressions ϕ_i^α.

Let us consider ϕ_i the product of the ϕ_i^α and algebraic expressions A_j invariant under the previous permutation of the x_i; the algebraic variety defined by the intersection of equations

$$\phi_i(x_1,\dots,x_n)=\text{const},\quad A_j(x_1,\dots,x_n)=\text{const}$$

is invariant under the inversion relations I and J.

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