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Automata and the susceptibility of the square lattice Ising model modulo powers of primes*

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Abstract

We study the full susceptibility of the Ising model modulo powers of primes. We find exact functional equations for the full susceptibility modulo these primes. Revisiting some lesser-known results on discrete finite automata, we show that these results can be seen as a consequence of the fact that, modulo 2^r , one cannot distinguish the full susceptibility from some simple diagonals of rational functions which reduce to algebraic functions modulo 2^r , and, consequently, satisfy exact functional equations modulo 2^r . We sketch a possible physical interpretation of these functional equations modulo 2^r as reductions of a master functional equation corresponding to infinite order symmetries such as the isogenies of elliptic curves. One relevant example is the Landen transformation which can be seen as an exact generator of the Ising model renormalization group. We underline the importance of studying a new class of functions corresponding to ratios of diagonals of rational functions: they reduce to algebraic functions modulo powers of primes and they may have solutions with natural boundaries.

Keywords: susceptibility of the Ising model, modulo prime calculation, automaton, functional equations, diagonals of rational functions, algebraic power series, congruence of combinatorial numbers

* Dedicated to R J Baxter, for his 75th birthday.

1. Introduction

Despite the enormous progress made over the last 75 years in the study of (Yang–Baxter) integrable models in lattice statistical mechanics and enumerative combinatorics, there still remain many important unanswered questions.

One of the most intriguing is the susceptibility of the two-dimensional Ising model. The closed form expression³ of the partition function was obtained by Onsager [2] in 1944, and the spontaneous magnetization was obtained a few years later by both Onsager (unpublished) and Yang [3]. However, after more than 70 years, a closed form expression for the full susceptibility still eludes us. Accordingly, understanding the nature of this function remains a challenging problem.

Forty years ago, Wu *et al* [4] showed that the full susceptibility of the square-lattice Ising model can be decomposed as the infinite sum of holonomic n -fold integrals [5–9], denoted $\chi^{(n)}$. In the last decade the linear differential operators corresponding to the first $\chi^{(n)}$'s, up to $n = 6$, were obtained, underlying the role of the elliptic curve parametrization [10], but showing also the emergence of (at least) one Calabi–Yau ODE, and beyond, of linear differential operators with selected differential Galois groups [11–13]. A complete description of the singular points of the linear differential operators corresponding to the first few $\chi^{(n)}$'s has also been obtained [6, 14–16]. Despite being an infinite sum of holonomic n -fold integrals, the full susceptibility is not a holonomic function [17].

Further, in a recent paper it has been shown that these n -fold integrals are actually diagonals of rational functions [18, 19]. Consequently their series expansions are such that modulo any prime, or power of a prime, they can be identified with the series expansions of an algebraic function [18, 19]. These properties were explicitly shown in the case of $\chi^{(3)}$, in section 3.1 of [18]. In particular it was shown that $H(w) = \tilde{\chi}^{(3)}/8$ (defined in section 2, see (15)), satisfies, modulo 2, the quadratic equation

$$H(w)^2 + w \cdot H(w) + w^{10} = 0 \pmod{2}, \tag{1}$$

and modulo 3, the polynomial equation of degree nine

$$p_9 \cdot H(w)^9 + w^6 \cdot p_3 \cdot H(w)^3 + w^{10} \cdot p_1 \cdot H(w) + w^{19} \cdot p_0^{(1)} \cdot p_0^{(2)} = 0, \tag{2}$$

where:

$$\begin{aligned} p_0^{(1)} &= w^6 + w^5 + w^4 - w^2 - w + 1, \\ p_0^{(2)} &= w^{37} - w^{36} + w^{35} - w^{33} + w^{31} - w^{30} + w^{28} + w^{27} + w^{24} - w^{23} + w^{22} \\ &\quad - w^{21} - w^{18} - w^{16} + w^{14} - w^{12} - w^{11} - w^{10} + w^7 - w^5 - w^3 - 1, \\ p_1 &= (w^2 + 1)^{20} (1 - w)^{13}, \quad p_3 = (w^2 + 1)^{18} (1 - w)^{15} (w^4 - w^2 - 1), \\ p_9 &= (w + 1)^3 (w^2 + 1)^{18} (w - 1)^{24}. \end{aligned} \tag{3}$$

Since all the $\chi^{(n)}$'s are diagonals of rational functions [18], similar results are expected for any $\chi^{(n)}$ modulo any prime p , and, beyond, modulo any power of a prime p^r . As a consequence of the Fermat relations, $a^p = a$, modulo p , one can expect relations, like (1) or (2), to be expressible as functional equations where $H(w)^p$ is replaced by $H(w^p)$. Now, the full susceptibility χ is not the diagonal of a rational function, indeed it is not even holonomic [15, 17]. Therefore, for the full susceptibility, one cannot expect relations like (1) or (2) to

³ It can be rewritten in a simpler ${}_4F_3$ hypergeometric form, see [1].

exist. Due to the complexity of this function⁴, one might not expect, at first sight, such functional equations for the full susceptibility.

However, as we show below, the full susceptibility, when expressed in the appropriate expansion variable, does satisfy some surprisingly simple functional equations modulo certain primes, or power of primes.

These exact results show that the full susceptibility reduces to an algebraic function, modulo certain primes, or powers of primes, and thus sheds new light on the integrable character of this very important function in physics. We consider this a surprising result: we certainly did not expect such simple results for the full susceptibility. This gives us considerable incentive to systematically study other non-holonomic physical series modulo primes or powers of primes. It will be interesting to see whether this is an exceptional result, in which case it sheds more light on the susceptibility, or a common occurrence, in which case we need to explain why.

2. Definitions and some known results on the full susceptibility

In 1976, Wu *et al* [4] showed that the susceptibility could be expressed as an infinite sum of contributions, known as n -particle contributions $\chi^{(n)}$. The low-temperature series were given by the case n even, and the high-temperature series by n odd. More precisely the low temperature susceptibility is given by [16]

$$\begin{aligned} kT \cdot \chi_L(w) &= (1 - 1/s^4)^{\frac{1}{4}} \cdot \tilde{\chi}_L(w) = (1 - 1/s^4)^{\frac{1}{4}} \cdot \sum \tilde{\chi}^{(2n)}(w) \\ &= (1 - s_L^4)^{\frac{1}{4}} \cdot \sum \tilde{\chi}^{(2n)}(w), \end{aligned} \tag{4}$$

in terms of the self-dual temperature variable $w = \frac{1}{2}s/(1 + s^2)$, where $s = \sinh(2J/kT)$, and $s_L = 1/\sinh(2J/kT)$. The high temperature susceptibility is given by [16]

$$kT \cdot \chi_H(w) = \frac{1}{s} \cdot (1 - s^4)^{\frac{1}{4}} \cdot \tilde{\chi}_H(w) = \frac{1}{s} \cdot (1 - s^4)^{\frac{1}{4}} \cdot \sum \tilde{\chi}^{(2n+1)}(w). \tag{5}$$

Remarkably long series expansions with respectively 2042 and 2043 coefficients⁵, have been obtained⁶ [20] for $\tilde{\chi}_L(w)$ and $\tilde{\chi}_H(w)$, namely

$$\begin{aligned} \tilde{\chi}_L(w) &= 4 w^4 + 80 w^6 + 1400 w^8 + 23\,520 w^{10} + 388\,080 w^{12} + 6342\,336 w^{14} \\ &\quad + 103\,062\,976 w^{16} + 1668\,639\,424 w^{18} + 26\,948\,549\,680 w^{20} \\ &\quad + \dots + \tilde{c}_{4086}^{(L)} w^{4086} + \dots \end{aligned} \tag{6}$$

and

$$\begin{aligned} \tilde{\chi}_H(w) &= 2 w + 8 w^2 + 32 w^3 + 128 w^4 + 512 w^5 + 2048 w^6 + 8192 w^7 + 32\,768 w^8 \\ &\quad + 131\,080 w^9 + 524\,288 w^{10} + 2097\,440 w^{11} + \dots + \tilde{c}_{2043}^{(H)} w^{2043} + \dots \end{aligned} \tag{7}$$

It is worth comparing these two series with the series corresponding to the first $\tilde{\chi}^{(n)}(w)$ in the two infinite sums (4) and (5), namely:

⁴ Which has, for instance, a natural boundary [15–17].

⁵ The low temperature series $\tilde{\chi}_L(w)$, being an even function, means that the expansion is known up to the coefficient of w^{4086} (see (6)).

⁶ Using an algorithm adapted from the Fortran algorithm in [17].

$$\begin{aligned}\tilde{\chi}_L^{(2)}(w) &= 4w^4 \cdot {}_2F_1\left(\left[\frac{3}{2}, \frac{5}{2}\right], [3], 16w^2\right) \\ &= 4w^4 + 80w^6 + 1400w^8 + 23\,520w^{10} + 388\,080w^{12} + 6342\,336w^{14} \\ &\quad + 103\,062\,960w^{16} + 1668\,638\,400w^{18} + 26\,948\,510\,160w^{20} + \dots\end{aligned}\tag{8}$$

and

$$\begin{aligned}\tilde{\chi}_H^{(1)}(w) &= \frac{2w}{1-4w} = 2w + 8w^2 + 32w^3 + 128w^4 + 512w^5 + 2048w^6 + 8192w^7 \\ &\quad + 32\,768w^8 + 131\,072w^9 + 524\,288w^{10} + 2097\,152w^{11} + \dots\end{aligned}\tag{9}$$

It is known that $\tilde{\chi}^{(n)} = O(w^{n^2})$, so that the coefficients are the same up to w^{14} for the low-temperature series, and up to w^8 for the high-temperature series. Further, one observes that the ratio of the coefficients for $\tilde{\chi}_L$ and $\tilde{\chi}_L^{(2)}$ (resp. $\tilde{\chi}_H$ and $\tilde{\chi}_H^{(1)}$) is very close to 1.

The series expansion for $\tilde{\chi}_L^{(4)}(w)$ reads⁷

$$\begin{aligned}\frac{\tilde{\chi}_L^{(4)}(w)}{2^4} &= w^{16} + 64w^{18} + 2470w^{20} + 74\,724w^{22} + 1954\,688w^{24} + 46\,428\,552w^{26} \\ &\quad + 1029\,903\,288w^{28} + 21\,716\,367\,896w^{30} \\ &\quad + 440\,440\,693\,418w^{32} + 8663\,350\,828\,976w^{34} \\ &\quad + 166\,258\,457\,615\,526w^{36} + 3126\,949\,985\,578\,700w^{38} \\ &\quad + 57\,833\,406\,662\,680\,980w^{40} + 1054\,656\,431\,047\,823\,680w^{42} \\ &\quad + 19\,003\,412\,267\,837\,223\,432w^{44} + \dots\end{aligned}\tag{10}$$

The series expansion for $\tilde{\chi}_L^{(6)}(w)$ reads

$$\begin{aligned}\frac{\tilde{\chi}_L^{(6)}(w)}{2^6} &= w^{36} + 144w^{38} + 11\,306w^{40} + 641\,604w^{42} + 29\,455\,804w^{44} \\ &\quad + 1161\,654\,484w^{46} + 40\,827\,303\,872w^{48} + 1310\,513\,628\,660w^{50} \\ &\quad + 39\,090\,651\,539\,936w^{52} + 1097\,452\,668\,063\,296w^{54} \\ &\quad + 29\,281\,457\,807\,054\,052w^{56} + 748\,130\,523\,334\,531\,340w^{58} \\ &\quad + 18\,414\,177\,309\,344\,582\,452w^{60} + \dots\end{aligned}\tag{11}$$

The difference between $\tilde{\chi}_L(w)$ and $\tilde{\chi}_L^{(2)}(w)$ reads:

$$\begin{aligned}\tilde{\chi}_L - \tilde{\chi}_L^{(2)} &= 16w^{16} + 1024w^{18} + 39\,520w^{20} + 1195\,584w^{22} + 31\,275\,008w^{24} \\ &\quad + 742\,856\,832w^{26} + 16\,478\,452\,608w^{28} \\ &\quad + 347\,461\,886\,336w^{30} + 7047\,051\,094\,688w^{32} \\ &\quad + 138\,613\,613\,263\,616w^{34} + 2660\,135\,321\,848\,480w^{36} + \dots\end{aligned}\tag{12}$$

⁷ Since there is an overall integer of the form 2^n for all the coefficients of the $\tilde{\chi}_L^{(n)}(w)$ or $\tilde{\chi}_H^{(n)}(w)$ series, we divide them, in the following, by an appropriate power of 2^n factor. The series expansion remains an expansion with (smaller) integer coefficients.

The difference between $\tilde{\chi}_L(w)$ and $\tilde{\chi}_L^{(2)}(w) + \tilde{\chi}_L^{(4)}(w)$ reads:

$$\begin{aligned} \tilde{\chi}_L - \left(\tilde{\chi}_L^{(2)} + \tilde{\chi}_L^{(4)} \right) &= 64 w^{36} + 9216 w^{38} + 723\,584 w^{40} + 41\,062\,656 w^{42} \\ &\quad + 1885\,171\,456 w^{44} + 74\,345\,886\,976 w^{46} \\ &\quad + 2612\,947\,447\,808 w^{48} + \dots \end{aligned} \tag{13}$$

The difference between $\tilde{\chi}_L(w)$ and $\tilde{\chi}_L^{(2)}(w) + \tilde{\chi}_L^{(4)}(w) + \tilde{\chi}_L^{(6)}(w)$ reads:

$$\begin{aligned} \tilde{\chi}_L - \left(\tilde{\chi}_L^{(2)} + \tilde{\chi}_L^{(4)} + \tilde{\chi}_L^{(6)} \right) &= 256w^{64} + 65\,536 w^{66} + 8815\,104 w^{68} \\ &\quad + 829\,038\,592 w^{70} + 61\,219\,149\,824 w^{72} + 3779\,726\,083\,072 w^{74} \\ &\quad + 202\,925\,982\,372\,864 w^{76} + 9729\,999\,547\,422\,720 w^{78} \\ &\quad + 424\,756\,293\,921\,653\,248 w^{80} + 17\,127\,494\,149\,322\,319\,872 w^{82} \\ &\quad + 645\,117\,850\,681\,779\,326\,976 w^{84} + \dots \end{aligned} \tag{14}$$

The series expansion for $\tilde{\chi}_H^{(3)}(w)$ reads

$$\begin{aligned} \frac{\tilde{\chi}_H^{(3)}(w)}{2^3} &= w^9 + 36 w^{11} + 4 w^{12} + 884 w^{13} + 196 w^{14} + 18532 w^{15} + 6084 w^{16} \\ &\quad + 357\,391w^{17} + 153\,484 w^{18} + 6556\,516 w^{19} + 3440\,964 w^{20} \\ &\quad + 116\,449\,960 w^{21} \\ &\quad + 71\,553\,656 w^{22} + 2022\,814\,844 w^{23} + 1413\,292\,572 w^{24} \\ &\quad + 34\,583\,048\,616 w^{25} + 26\,900\,157\,072 w^{26} + 584\,324\,509\,812 w^{27} \\ &\quad + 498\,048\,104\,276 w^{28} + \dots \end{aligned} \tag{15}$$

The series expansion for $\tilde{\chi}_H^{(5)}(w)$ reads

$$\begin{aligned} \frac{\tilde{\chi}_H^{(5)}(w)}{2^5} &= w^{25} + 100 w^{27} + 5652 w^{29} + 4 w^{30} + 238\,032 w^{31} \\ &\quad + 484 w^{32} + 8323\,743 w^{33} + 32\,436 w^{34} + 255\,716\,632 w^{35} \\ &\quad + 1592\,488 w^{36} + 7139\,250\,236 w^{37} + 63\,994\,900 w^{38} \\ &\quad + 185\,181\,953\,320 w^{39} + 2231\,760\,988 w^{40} \\ &\quad + 4531\,508\,893\,397 w^{41} + 69\,986\,224\,204 w^{42} \\ &\quad + 105\,775\,797\,597\,812 w^{43} + 2020\,409\,460\,692 w^{44} \\ &\quad + 2374\,723\,605\,151\,320 w^{45} + 54\,584\,651\,129\,624 w^{46} \\ &\quad + 51\,602\,310\,149\,637\,388 w^{47} + 1396\,760\,803\,374\,712 w^{48} \\ &\quad + 1090\,696\,414\,153\,653\,447 w^{49} + \dots \end{aligned} \tag{16}$$

Comparing $\tilde{\chi}_H(w)$ and the sum $\tilde{\chi}_H^{(1)}(w) + \tilde{\chi}_H^{(3)}(w)$ one finds that these two series are the same up to $O(w^{25})$, as expected:

$$\begin{aligned} \tilde{\chi}_H - \left(\tilde{\chi}_H^{(1)} + \tilde{\chi}_H^{(3)} \right) &= 32 w^{25} + 3200 w^{27} + 180\,864 w^{29} + 128 w^{30} + 7617\,024 w^{31} \\ &\quad + 15\,488 w^{32} + 266\,359\,776 w^{33} + 1037\,952 w^{34} + 8182\,932\,224 w^{35} + \dots \end{aligned} \tag{17}$$

and

$$\begin{aligned} \tilde{\chi}_H &= \left(\tilde{\chi}_H^{(1)} + \tilde{\chi}_H^{(3)} + \tilde{\chi}_H^{(5)} \right) \\ &= 128 w^{49} + 25\,088 w^{51} + 2621\,952 w^{53} + 194185216 w^{55} \\ &\quad + 512 w^{56} + 11\,431\,676\,800 w^{57} + 115\,200 w^{58} \\ &\quad + 569\,065\,324\,032 w^{59} + 13\,709\,824 w^{60} + \dots \end{aligned} \quad (18)$$

Since the modulus of elliptic functions parametrising the Ising model [10, 15] is $k = s^2$, with the conjectured natural boundary [16] corresponding to the unit k or s circle, it is natural to introduce series expansions in the s or s_L variables. In fact, we have studied series expansions in the $v = s_L/2 = 1/(2s)$ variable in the low-temperature regime, and the $v = s/2$ variable in the high-temperature regime, in order to have series with integer coefficients (instead of rational coefficients with denominators of the form 2^n). The corresponding low and high temperatures series $\chi_L(v) = \chi_L(s_L/2)$ and $\chi_H(v) = \chi_H(s/2)$ read respectively⁸

$$\begin{aligned} \chi_L(v) &= 4 v^4 + 16 v^6 + 104 v^8 + 416 v^{10} + 2224 v^{12} + 8896 v^{14} + 43\,840 v^{16} \\ &\quad + 175\,296 v^{18} + 825\,648 v^{20} + 3300\,480 v^{22} + \dots + c_{4086}^{(L)} v^{4086} + \dots \end{aligned} \quad (19)$$

and

$$\begin{aligned} \chi_H(v) &= 1 + 4 v + 12v^2 + 32v^3 + 76 v^4 + 176 v^5 + 400 v^6 + 896v^7 + 1964 v^8 \\ &\quad + 4256 v^9 + 9184 v^{10} + 19728 v^{11} + 41952 v^{12} + \dots + c_{2043}^{(H)} v^{2043} + \dots \end{aligned} \quad (20)$$

which can be compared with $\chi_L^{(2)} = (1 - 16 v^4)^{1/4} \cdot \tilde{\chi}_L^{(2)}$

$$\begin{aligned} \chi_L^{(2)}(v) &= \frac{1}{4^3} \cdot (1 - 16 v^4)^{1/4} \cdot \left(\frac{4 v}{1 + 4 v^2} \right)^4 \cdot {}_2F_1 \left(\left[\frac{3}{2}, \frac{5}{2} \right], [3], \left(\frac{4 v}{1 + 4 v^2} \right)^2 \right) \\ &= 4 v^4 + 16 v^6 + 104 v^8 + 416 v^{10} + 2224 v^{12} + 8896 v^{14} + 43\,824 v^{16} \\ &\quad + 175\,296 v^{18} + 825\,104 v^{20} + 330\,0416 v^{22} + \dots \end{aligned} \quad (21)$$

and $\chi_H^{(1)} = (1 - s^4)^{1/4}/s \cdot \tilde{\chi}_H^{(1)}$

$$\begin{aligned} \chi_H^{(1)}(v) &= \frac{(1 - s^4)^{1/4}}{s} \cdot \frac{s}{(1 - s)^2} = \left(\frac{1 - 16 v^4}{(1 - 2v)^8} \right)^{1/4} \\ &= 1 + 4 v + 12v^2 + 32 v^3 + 76 v^4 + 176 v^5 + 400 v^6 + 896 v^7 + 1960 v^8 \\ &\quad + 4256 v^9 + 9184 v^{10} + 19\,712 v^{11} + 41\,888 v^{12} + \dots \end{aligned} \quad (22)$$

As must be the case, the coefficients are the same up to v^{14} for the low-temperature series, and up to v^8 for the high-temperature series, and, beyond, the ratio of the coefficients for $\chi_L(v)$ and $\chi_L^{(2)}(v)$ (resp. $\chi_H(v)$ and $\chi_H^{(1)}(v)$) are very close to 1. For the low-temperature series expansion the difference between $\chi_L(v)$ and $\chi_L^{(2)}(v)$ reads:

⁸ Throughout this paper the $\tilde{\chi}$ are functions of the variable w , while the χ are functions of the variable v .

$$\begin{aligned} \chi_L(v) - \chi_L^{(2)}(v) = & 16 v^{16} + 544 v^{20} + 64 v^{22} + 13\,056 v^{24} + 2944 v^{26} + 272\,512 v^{28} \\ & + 88\,448 v^{30} + 5286\,560 v^{32} + 2201\,856 v^{34} \\ & + 98\,136\,096 v^{36} + \dots \end{aligned} \tag{23}$$

It is worth recalling that the very long (low and high temperature) series expansions have been obtained for the full susceptibility as a consequence of a quadratic finite difference Painlevé functional equation [17], yielding an N^4 polynomial algorithm for calculating the series. This series is therefore ‘algorithmically integrable’. Furthermore the n -fold integrals of the infinite sum decomposition [4], the $\chi^{(n)}$ ’s, have been shown to be highly selected holonomic functions, namely diagonals of rational functions [18].

These properties (‘algorithmic integrability’, infinite sums of diagonals of rational functions, ...) suggest that transcendental non-holonomic functions such as the full susceptibility of the square Ising model, should correspond to a ‘rather special class’ of non-holonomic functions, which require new concepts and tools to characterize and analyze them.

Obtaining such remarkably long series for the full susceptibility was a computational ‘tour de force,’ and it is likely that these series have much more to tell us. To date they have only been used to obtain some results on $\chi^{(5)}$ and $\chi^{(6)}$, to confirm exact results [14, 15] on the singularities of the linear ODEs of the $\chi^{(n)}$ ’s, and to clarify the natural boundary scenario [16].

In the following sections we revisit these remarkably long series from a new finite automaton [21] perspective, which in effect means considering the various series modulo various integers, in particular, taking a ‘ p -adic’ perspective [22], modulo integers that are integer powers of primes.

3. Functional equations modulo 2^r for the full susceptibility

3.1. The low-temperature susceptibility

Consider the low temperature series (19) for the full susceptibility [20], for which 2043 coefficients have been obtained in the $u = v^2$ variable [20]. We denote this series $F(u)$, so that

$$\begin{aligned} F(u) = & 4 u^2 + 16 u^3 + 104 u^4 + 416 u^5 + 2224 u^6 + 8896 u^7 + 43\,840 u^8 \\ & + 175\,296 u^9 + 825\,104 u^{10} + \dots + a_{2043} \cdot u^{2043} + \dots \end{aligned} \tag{24}$$

Now consider this series modulo various integers $q = 2^r$, ($q = 2, 4, 8, 16, 32, 64, \dots$), where we denote by F_q the corresponding series modulo q . We found the following simple results:

$$F_2(u) = 0, \quad F_4(u) = 0, \quad F_8(u) = 4 u^2, \quad F_{16}(u) = 4 u^2 + 8 u^4, \tag{25}$$

where the first two results are of no significance, and just reflect the lattice symmetry. However for $q = 32$ and $q = 64$, we found the appearance of simple lacunary series, so that

$$F_{32}(u) = 20 u^2 + 24 u^4 + 16 \cdot u^2 \cdot L(u), \tag{26}$$

$$F_{64}(u) = 60 u^2 \cdot (11 + 8 u + 10 u^2 + 8 u^4 + 8 u^6) + (48 u^2 + 32 u^4) \cdot L(u), \tag{27}$$

where $L(u)$ corresponds to the first 1024 coefficients of the lacunary series with a natural boundary on the unit-circle $|u| = 1$:

$$\begin{aligned}
 L(u) &= \sum_{n=0}^{n=\infty} u^{2^n} \\
 &= 1 + u + u^2 + u^4 + u^8 + u^{16} + u^{32} + u^{64} + u^{128} + u^{256} + u^{512} + u^{1024} + \dots.
 \end{aligned}
 \tag{28}$$

This strongly suggests that $F_{32}(u)$ and $F_{64}(u)$ satisfy the modulo 32 and modulo 64 functional equations respectively:

$$u^2 \cdot F_{32}(u) = F_{32}(u^2) + 16 u^5 + 24 u^6 + 8 u^8, \tag{29}$$

$$\begin{aligned}
 &u^2 \cdot (3 + 2 u^4) \cdot F_{64}(u) \\
 &= (3 + 2 u^2) \cdot F_{64}(u^2) \\
 &\quad + 16 u^5 \cdot (3 + 4 u + 6 u^2 + 2 u^4 + 58 u^5 + 4 u^6 + 2 u^7 + 6 u^{11}).
 \end{aligned}
 \tag{30}$$

The series expansions F_{128}, F_{256}, \dots also satisfy similar functional equations, but they are more involved, the series having a less obvious lacunary series interpretation. For instance one finds that

$$\begin{aligned}
 &u^2 \cdot F_{128}(u) = F_{128}(u^2) + 32 u^6 \cdot (3 - u^2) \cdot L(u) + 8 \cdot u^5 \cdot p_{13}, \\
 \text{where: } &p_{13} = 2 - u - 8 u^2 - 15 u^3 + 12 u^4 - 4 u^5 \\
 &\quad + 8 u^6 + 4 u^7 - 8 u^9 + 4 u^{11} - 8 u^{13},
 \end{aligned}
 \tag{31}$$

where $L(v)$ is the lacunary series (28), which satisfies the functional equation $u + L(u^2) = L(u)$. Therefore one deduces the functional equation modulo 128:

$$\begin{aligned}
 &u^8 \cdot (u^4 - 3) \cdot F_{128}(u) - u^4 \cdot (u^6 - 2u^2 - 3) \cdot F_{128}(u^2) \\
 &= (u^2 - 3) \cdot F_{128}(u^4) + 16 u^{10} \cdot p_{28}, \quad \text{where:}
 \end{aligned}
 \tag{32}$$

$$\begin{aligned}
 p_{28} &= 4 u^{28} - 12 u^{26} - 2 u^{24} + 6 u^{22} + 4 u^{20} - 16 u^{18} + 10 u^{14} + 10 u^{12} + 4 u^{11} \\
 &\quad - 2 u^{10} + 4 u^9 + 12 u^8 - 10 u^7 - 13 u^6 - 11 u^5 + 11 u^4 - 6 u^3 - u^2 - 3 u + 3.
 \end{aligned}$$

Since we have seen that the full susceptibility series is quite close to the series expansion of $\chi^{(2)}$, it is natural to ask if one obtains similar results modulo 2^r , for $\chi^{(2)}$. From the series expansion (21), we find that one obtains the same series as the one displayed in (25) modulo 2, 4, 8, 16. Modulo 32 and 64 one obtains simple functional equations for $\chi^{(2)}$ which are similar to (29) and (30) but actually slightly different.

This can be rewritten in terms of the difference (23). This difference (23) is zero modulo 2, 4, 8, 16. Modulo 32 it is just one term, namely $16 v^{16}$ (the series for $\chi^{(2)}$ being a non-trivial lacunary series) and modulo 64, it becomes the lacunary series

$$\begin{aligned}
 \chi_L(v) - \chi_L^{(2)}(v) &= 16 v^{16} + 32 v^{20} + 32 v^{32} + 32 v^{36} + 32 v^{68} + 32 v^{132} \\
 &\quad + 32 v^{260} + 32 v^{516} + 32 v^{1028} + 32 v^{2052} + \dots
 \end{aligned}
 \tag{33}$$

Remark. The low temperature series (24) relied on having coefficients up to the term in v^{2043} . Consequently the previous functional equations have been checked up to order 2043 in the the expansion (24). The previous calculations underline the crucial role played by the lacunary series (28) where the next term is u^{2048} . It would thus be interesting to validate a functional equation such as (32) up to the point where the term u^{2048} in (28) is expected to

emerge: this would require one to find just⁹ a few (less than 10) extra terms in the low temperature series (24). Without trying to get more coefficients for the full susceptibility in exact arithmetic (not modulo a prime, or a power of a prime) which requires very substantial computer resources, we can try to check all our previous functional equations modulo some integers of the form 2^r , seizing the opportunity of having a polynomial algorithm to get many more than 2000 coefficients (5000, 6000, 10 000, ...) but just modulo 2, 4, 8, 16 ...

3.2. The high-temperature susceptibility

Similarly, we now study the high-temperature expansion (20), modulo q with $q = 2, 4, 8, 16, 32, 64$, and compare these series with the ones obtained modulo q with $q = 2, 4, 8, 16, 32, 64$ for (22). Since, apart from the first constant coefficients, all the coefficients are divisible by 4, we introduce the series

$$G(v) = \frac{\chi(v) - 1}{4} = v + 3v^2 + 8v^3 + 19v^4 + 44v^5 + 100v^6 + 224v^7 + 491v^8 \\ + 1064v^9 + 2296v^{10} + 4932v^{11} + 10488v^{12} \\ + 22180v^{13} + \dots \quad (34)$$

and denote by G_q the corresponding series modulo q . We obtained the following results: modulo 2 the series G_2 is the lacunary series $L(v) - 1$ (where $L(v)$ is given by (28)):

$$G_2(v) = v + v^2 + v^4 + v^8 + v^{16} + v^{32} + v^{64} + v^{128} + v^{256} + v^{512} + v^{1024} + \dots \quad (35)$$

which is a solution of the functional equation

$$G_2(v) = G_2(v^2) + v. \quad (36)$$

Modulo 4 the series G_4 is the lacunary series $3L(v) - 3 - 2v$

$$G_4(v) = v + 3v^2 + 3v^4 + 3v^8 + 3v^{16} + 3v^{32} + 3v^{64} + 3v^{128} \\ + 3v^{256} + 3v^{512} + 3v^{1024} + \dots \quad (37)$$

which is a solution of the functional equation:

$$G_4(v) = G_4(v^2) + v + 2v^2. \quad (38)$$

Modulo 8 the series $G_8(v)$ becomes more difficult to recognise,

$$G_8(v) = v + 3v^2 + 3v^4 + 4v^5 + 4v^6 + 3v^8 + 4v^{11} + 4v^{13} + 4v^{14} + 4v^{15} + 7v^{16} \\ + 4v^{17} + 4v^{19} + 4v^{20} + 4v^{22} + 4v^{23} + 4v^{24} + 4v^{26} + 4v^{27} + \dots \quad (39)$$

though if we define

$$\hat{G}_8(v) = G_8(v) + L(v) - 1, \quad (40)$$

then

$$2 \cdot \hat{G}_8(v) = 4v \pmod{8}. \quad (41)$$

⁹ Even though obtaining more terms for the low temperature series (24) can be done with a polynomial time algorithm, getting more coefficients requires substantial computer resources: however, here the idea is that we just need a few extra terms.

Comparing the series (34) with the series $(\chi_H^{(1)} - 1)/4$ which is equal to

$$\begin{aligned} \frac{1}{4} \cdot \left(\left(\frac{1 - 16v^4}{(1 - 2v)^8} \right)^{1/4} - 1 \right) &= v + 3v^2 + 8v^3 + 19v^4 + 44v^5 + 100v^6 + 224v^7 \\ &+ 490v^8 + 1064v^9 + 2296v^{10} \\ &+ 4928v^{11} + 10488v^{12} + 22180v^{13} + \dots \end{aligned} \tag{42}$$

one gets mod.2, 4, 8, 16, 32 respectively:

$$\begin{aligned} v + v^2 + v^4q \quad \text{mod. } 2, \quad v + 3v^2 + 3v^4 + 2v^8 \quad \text{mod. } 4, \\ v + 3v^2 + 3v^4 + 4v^5 + 4v^6 + 2v^8 \quad \text{mod. } 8, \\ v + 3v^2 + 8v^3 + 3v^4 + 12v^5 + 4v^6 + 10v^8 + 8v^9 + 8v^{10} + 8v^{12} + 8v^{16} \\ \text{mod. } 16. \end{aligned} \tag{43}$$

We see that, in contrast to the low-temperature expansion, a simple rational function like $\chi_H^{(1)}$ yielding polynomial expressions modulo 2, 4, 8, 16, 32 cannot give rise to the emergence of lacunary series like (35) and (37). For high-temperature series, one must therefore rather ask whether, modulo 2^r , one can distinguish between the full susceptibility χ_H and $\chi^{(1)} + \chi^{(3)}$.

3.3. Functional equations mod. 2^r for $\tilde{\chi}$ in the variable w

For the low and high-temperature series for $\tilde{\chi}$ in the variable w (see (6)–(9), ...), we have obtained similar results and functional equations modulo $q = 2, 4, 8, 16, 32, 64$. These series, and the corresponding functional equations, are given in appendix 6.

3.3.1. High temperature for $\tilde{\chi}$ in the variable w . Let us consider the previous question of comparing χ_H with $\chi^{(1)} + \chi^{(3)}$, but in the variable w , so that we are comparing $\tilde{\chi}_H$ and $\tilde{\chi}^{(1)} + \tilde{\chi}^{(3)}$ modulo 2^r .

The series expansion of the difference $\Delta_H = \tilde{\chi}_H - (\tilde{\chi}_H^{(1)} + \tilde{\chi}_H^{(3)})$ is given in (17). The series expansion for $\tilde{\chi}^{(1)} + \tilde{\chi}^{(3)}$ can be obtained with an arbitrary number of exact coefficients, while 2043 coefficients of the series expansion of $\tilde{\chi}_H$ are known. Considering, modulo various integers, the 2043 coefficients of the series (17), we found that $\Delta_H = 0$ modulo 2^r , for $r \leq 5$.

Modulo 16 one cannot distinguish $\tilde{\chi}_H$ and $\tilde{\chi}_H^{(1)} + \tilde{\chi}_H^{(3)}$, their series expansions being a very simple lacunary series:

$$\begin{aligned} \tilde{\chi}_H(w) &= \tilde{\chi}_H^{(1)} + \tilde{\chi}_H^{(3)} = 10w + 8w^3 + 8w^5 + 8w \cdot L(w) \\ &= 2w + 8w^2 + 8w^9 + 8w^{17} + 8w^{33} \\ &+ 8w^{65} + 8w^{129} + 8w^{257} + 8w^{513} + \dots \end{aligned} \tag{44}$$

yielding the simple functional equation modulo 16:

$$\tilde{\chi}_H(w^2) = w \cdot \tilde{\chi}_H(w) + 8w^3 \cdot (w^7 - w - 1). \tag{45}$$

Modulo 32, similarly, one cannot distinguish $\tilde{\chi}_H$ and $\tilde{\chi}_H^{(1)} + \tilde{\chi}_H^{(3)}$, their series expansions being a very simple lacunary series

$$\begin{aligned}
 & 2w + 8 w^2 + 8 w^9 + 24 w^{17} + 24 w^{33} + 24 w^{65} + 24 w^{129} + 24 w^{257} + 24 w^{513} + \dots \\
 & = 10w + 16 w^2 + 8 w^3 + 8 w^5 + 16 w^9 + 24w \cdot L(w),
 \end{aligned} \tag{46}$$

where $L(w)$ is the lacunary series (28). This yields the simple functional equation modulo 32:

$$\tilde{\chi}_H(w^2) = w \cdot \tilde{\chi}_H(w) + 8w^3 \cdot (2w^{15} - w^7 + w - 5). \tag{47}$$

Modulo 64, 128, similarly, one cannot distinguish between $\tilde{\chi}_H$ and $\tilde{\chi}_H^{(1)} + \tilde{\chi}_H^{(3)} + \tilde{\chi}_H^{(5)}$, their series expansions being, again, very simple lacunary series.

3.3.2. Low temperature for $\tilde{\chi}$ in the variable w . Similarly, if one compares the low-temperature full susceptibility with $\tilde{\chi}_L^{(2)}(w)$ modulo 32 one finds the lacunary series:

$$\begin{aligned}
 \tilde{\chi}_L(w) = & 4 w^4 + 16 w^6 + 24 w^8 + 16 w^{12} + 16 w^{20} + 16 w^{36} + 16 w^{68} \\
 & + 16 w^{132} + 16 w^{260} + 16 w^{516} + 16 w^{1028} + \dots,
 \end{aligned} \tag{48}$$

versus

$$\begin{aligned}
 \tilde{\chi}_L^{(2)}(w) = & 4 w^4 + 16 w^6 + 24 w^8 + 16 w^{12} + 16 w^{16} + 16 w^{20} + 16 w^{36} + 16 w^{68} \\
 & + 16 w^{132} + 16 w^{260} + 16 w^{516} + 16 w^{1028} + \dots,
 \end{aligned} \tag{49}$$

the difference being only $16 w^{16}$.

One finds that the difference between $\tilde{\chi}_L$ and $\tilde{\chi}_L^{(2)}$, given in equation (12), is zero modulo 2, 4, 8, 16, and equal to $16 w^{16}$ modulo 32. Modulo 64 it is given by a lacunary series

$$\begin{aligned}
 \tilde{\chi}_L - \tilde{\chi}_L^{(2)} = & 32 w^4 \cdot L(w) + 16w^{16} \\
 & + 32w^4 \cdot (w^{28} - w^8 - w^4 - w^2 - w - 1).
 \end{aligned} \tag{50}$$

One finds that the difference between $\tilde{\chi}_L$ and $\tilde{\chi}_L^{(2)} + \tilde{\chi}_L^{(4)}$, as given by (13), is zero modulo 2, 4, 8, 16, 32, 64 and is given by a lacunary series modulo 128:

$$\begin{aligned}
 \tilde{\chi}_L - (\tilde{\chi}_L^{(2)} + \tilde{\chi}_L^{(4)}) = & 64w^4 \cdot L(w) \\
 & - 64w^4 \cdot (w^{16} + w^8 + w^4 + w^2 + w + 1).
 \end{aligned} \tag{51}$$

If one includes $\tilde{\chi}_L^{(6)}$, the difference (14) between $\tilde{\chi}_L$ and $\tilde{\chi}_L^{(2)} + \tilde{\chi}_L^{(4)} + \tilde{\chi}_L^{(6)}$ is seen to be zero modulo 2, 4, 8, 16, 32, 64, 128, 256.

The scenario seems to be that one cannot distinguish between the series for $\tilde{\chi}_L$ and that for a finite sum like $\tilde{\chi}_L^{(2)} + \tilde{\chi}_L^{(4)} + \dots + \tilde{\chi}_L^{(2n)}$ modulo 2^r where r grows with n . The series expansion for the $\tilde{\chi}_L^{(n)}$ are given in [23], up to $n = 12$. This scenario has been checked and found to hold up to $\tilde{\chi}_L^{(12)}$. Recall that the finite sum $\tilde{\chi}_L^{(2)} + \tilde{\chi}_L^{(4)} + \dots + \tilde{\chi}_L^{(2n)}$ is also the diagonal of a rational function [18], implying that this finite sum reduces to algebraic functions modulo primes, or power of primes, and thus satisfies functional equations modulo primes, or power of primes. For instance, $\tilde{\chi}_L$, which cannot be distinguished from this sum modulo 2^r for some r , satisfies a functional equation modulo 2^r .

These functional equations can thus be seen as related to the functional equations for $\tilde{\chi}_L^{(n)}$. For instance modulo 2, $\tilde{\chi}_H^{(3)}(w)/8$ becomes the lacunary series

$$\frac{\tilde{\chi}_H^{(3)}(w)}{8} = w \cdot L(w) - w \cdot (w^4 + w^2 + w + 1), \tag{52}$$

from which one deduces the functional equation modulo 2:

$$w \cdot \frac{\tilde{\chi}_H^{(3)}(w)}{8} = \frac{\tilde{\chi}_H^{(3)}(w^2)}{8} + w^{10}. \tag{53}$$

Modulo 4 the series $\tilde{\chi}_H^{(3)}(w)/8$ becomes the lacunary series

$$\frac{\tilde{\chi}_H^{(3)}(w)}{8} = 3w \cdot L(w) + w \cdot (2w^8 + w^4 + w^2 + w + 1), \tag{54}$$

from which one deduces the functional equation modulo 4:

$$w \cdot \frac{\tilde{\chi}_H^{(3)}(w)}{8} = \frac{\tilde{\chi}_H^{(3)}(w^2)}{8} + w^3 \cdot (4 + w^7 - 2w^{15}). \tag{55}$$

For $\tilde{\chi}_L^{(4)}/16$ we have similar results. The series $\tilde{\chi}_L^{(4)}/16$ reduces, modulo 2, to w^{16} . Modulo 4 the series $\tilde{\chi}_L^{(4)}/16$ becomes¹⁰ the simple lacunary series

$$\frac{\tilde{\chi}_L^{(4)}(w)}{16} = 2w^4 \cdot L(w) + w^{16} + 2w^4 \cdot (w^{28} + w^8 + w^4 + w^2 + w + 1), \tag{56}$$

yielding the functional equation modulo 4

$$\frac{\tilde{\chi}_L^{(4)}(w^2)}{16} = w^4 \cdot \frac{\tilde{\chi}_L^{(4)}(w)}{16} + w^{20} \cdot (2w^{44} - 2w^{16} + w^{12} + 2w^4 - 1). \tag{57}$$

4. Automaton interpretation of the functional equations

Recalling the decomposition of the full susceptibility as an infinite sum of n -fold integrals, $\chi^{(n)}$, these striking results can be seen as a consequence of the fact that, modulo integers that are powers of the prime 2, the full susceptibility series is the same lacunary series as the series for the first $\chi^{(n)}$'s: for instance the low-temperature series modulo 64 of the full susceptibility series and of $\tilde{\chi}^{(2)} + \tilde{\chi}^{(4)}$ (which is the diagonal of a rational function) are the same. There is a not-widely-known discrete automaton [21, 34] result that, modulo a prime p , diagonals of rational functions [38] not only reduce to algebraic functions, but also satisfy [21] ‘functional equations modulo p^r ’ of the form $F(f(x), f(x^p), \dots, f(x^{p^r})) = 0$.

Let us recall some relevant results on discrete automaton [21, 24–26]: modulo a prime p , the diagonal of a rational function reduces to an algebraic function, and this is also true modulo p^r (p prime, r integer). Furthermore, these papers tell us that if $f(x)$ is algebraic modulo a prime p then $1, f(x), f^2(x), f^3(x), \dots$ are linearly dependent, and $1, f(x), f^p(x), f^{p^2}(x), f^{p^3}(x), \dots$ are also linearly dependent. From Fermat’s little theorem, namely that if p is a prime number, $a^p = a \pmod{p}$, one deduces for any series $f(x) = \sum a_n \cdot x^n$

$$\left(\sum a_n \cdot x^n\right)^p = \sum a_n^p \cdot x^{pn} = \sum a_n \cdot x^{pn}, \quad \text{mod. } p, \tag{58}$$

and, thus, $f(x)^p = f(x^p)$ modulo p , and, more generally, $f(x)^{p^r} = f(x^{p^r})$ modulo p^r . One deduces that the relations $F(f(x), f(x^p), \dots, f(x^{p^r})) = 0$ can, in fact be written linearly, as

¹⁰ Here, the calculations can be checked with an arbitrary number of coefficients. We did so with 6000 coefficients.

$$\sum_n p_n(x) \cdot f(x^{P^n}) = 0, \tag{59}$$

where the $p_n(x)$ are polynomials with integer coefficients, (see for instance section 2 in Lipshitz and van der Poorten [21]).

Series generated by a finite automaton correspond to a system of algebraic equations, which correspond, in turn (non-trivially) to being algebraic. All these functional equations occurring for discrete automata can be seen as functional equations associated with algebraic functions modulo integers, in particular diagonals of rational functions. This can be seen as the origin of the functional equations of this paper. The functional equations we have obtained can be interpreted¹¹ as consequences of the fact that, modulo some integers that are powers of the prime 2, one cannot really make a distinction between the full susceptibility and the diagonal of a rational function (like the sum of the first $\chi^{(n)}$ s), and consequently reduce to algebraic functions modulo 2^r . One could say that non-holonomic functions, like the full susceptibility of the Ising model, correspond to ‘almost diagonal functions’.

The automaton interpretation of this section can be revisited from a binomial viewpoint. Recall that the coefficients of the series expansion of diagonals of rational functions necessarily reduce to nested sums of products of binomials [27, 28]. Binomial coefficients modulo prime powers have been considered by many great mathematicians of the nineteenth century¹², yielding a large set of elegant results. Among the various prime powers, the powers of 2 seem to play a selected¹³ role [33]. Combining these two set of results is another approach to the main problem addressed in this paper, namely the study of (infinite sums of) diagonals of rational functions modulo prime powers.

For powers of the prime 2, the functional equations satisfied by the full susceptibility are quite simple ones, which are associated with the lacunary series¹⁴ (28). Of course for powers of other primes ($3^r, 5^r, \dots$), the functional equations satisfied by the full susceptibility should, if they exist, be much more involved, certainly not reducing to simple lacunary series. For powers of other primes, the scenario that modulo some powers of primes, one cannot disentangle the full susceptibility from some finite sum of $\chi^{(n)}$ s, is no longer valid. For instance, if one considers the series expansion (18) of the difference $\tilde{\chi}_H - (\tilde{\chi}_H^1 + \tilde{\chi}_H^3)$, one sees that the coefficient of w^{100} and w^{101} are, respectively, the following products of primes¹⁵:

$$\begin{aligned} &2^{12} \cdot 59 \cdot 1403\ 746\ 269\ 427 \cdot 1965\ 616\ 530\ 023 \cdot 269\ 691\ 689\ 798\ 741\ 092\ 891, \\ &2^8 \cdot 29 \cdot 2681\ 1049 \cdot 99\ 658\ 008\ 281\ 797\ 903\ 856\ 656\ 009\ 433\ 736\ 710\ 068\ 597. \end{aligned} \tag{60}$$

Similarly, if one considers the series expansion (15) of $\tilde{\chi}_H^{(3)}/8$, one finds that the coefficient of w^{100} and w^{101} are, respectively, the following products of primes:

¹¹ Equivalently, our conjectured functional equations can be seen as conjectures on the fact that, for instance, the non-holonomic infinite sum $\tilde{\chi}_L - (\tilde{\chi}_L^{(2)} + \tilde{\chi}_L^{(4)}) = \tilde{\chi}_L^{(6)} + \tilde{\chi}_L^{(8)} + \dots$ reduces to zero modulo 64, and possibly, that each series $\tilde{\chi}_L^{(2^n)}$ for $n \geq 3$, reduces to zero modulo 64, which corresponds to the (experimental) remark of section 2, that the $\tilde{\chi}_L^{(n)}(w)$ (resp. $\tilde{\chi}_H^{(n)}(w)$) have an overall factor 2^n .

¹² For instance Cauchy, Cayley, Gauss, Hensel, Hermite, Kummer, Legendre, see [29, 30]. The study of congruences of combinatorial numbers [31] usually starts with their p -adic order: it was first studied by Kummer [32].

¹³ In 1899 Glaisher observed that the number of odd entries in any given row of Pascal’s triangle is a power of 2.

¹⁴ Note that, as far as reduction to algebraic functions modulo powers of the prime 2 is concerned, a remarkably simple quadratic algebraic function corresponding to the Catalan number generating function also reduces to this lacunary series (28), as can be seen in appendix B.

¹⁵ Using the command ‘ifactor’ in Maple.

$$\begin{aligned}
 &2^2 \cdot 3^2 \cdot 263 \cdot 3291\ 604\ 173\ 673 \cdot 340\ 864\ 762\ 033 \\
 &\cdot 3935\ 416\ 959\ 987\ 419\ 344\ 918\ 432\ 619, 2^3 \cdot 5 \cdot 5581 \\
 &\cdot 140\ 051\ 834\ 806\ 578\ 509\ 195\ 477\ 348\ 569\ 596\ 056\ 396\ 208\ 311\ 876\ 1649. \tag{61}
 \end{aligned}$$

Besides powers of 2, there is no prime cancelling all the coefficients of the difference (18) or of $\tilde{\chi}_H^{(3)}$.

The question whether modulo primes different from 2, or powers of primes different from 2^f , the full susceptibility, that no longer reduces to the sum of the first $\tilde{\chi}^{(n)}$'s, satisfies (probably involved) functional equations remains open.

5. Comments and speculations

5.1. Towards a physical interpretation of the functional equations

We can view these exact functional equations modulo integers that are powers of the prime 2, as a finite discrete automata [34] result corresponding to the fact that, modulo such integers, one cannot disentangle the full susceptibility from the diagonal of a rational function. From a more speculative, but more physical perspective, one might hope that such functional equations are the ‘shadow’ modulo primes or powers of primes, of (probably very involved) functional equations¹⁶, the $x \rightarrow x^p$ Frobenius symmetry being, in fact, an infinite order transformation. Furthermore, such an infinite order discrete transformation might be seen as a symmetry of the model. Along this symmetry line, recall that, in the case of the square Ising model, any isogeny of the elliptic curve parametrizing the model [10] can be interpreted as an exact generator of the renormalization group [35].

We remark that $\tilde{\chi}_L^{(2)}$ (see (8)), can be written so that the Landen modulus clearly appears. Consider the ${}_2F_1$ hypergeometric function $\Phi(x)$, and recall the Landen modulus k_L :

$$\Phi(x) = \frac{1}{4^3} \cdot x^4 \cdot {}_2F_1\left(\left[\frac{3}{2}, \frac{5}{2}\right], [3], x^2\right), \quad k_L = 2\sqrt{k} \cdot 1 + k.$$

From (8) $\tilde{\chi}_L^{(2)}$ reads:

$$\begin{aligned}
 \tilde{\chi}_L^{(2)} &= \Phi(k_L) = \Phi(4w) = \Phi\left(\frac{4v}{1+4v^2}\right) = \Phi\left(\frac{2s}{1+s^2}\right) \\
 &= 4v^4 + 16v^6 + 120v^8 + 480v^{10} + 2800v^{12} + 11200v^{14} + 58800v^{16} + \dots \tag{62}
 \end{aligned}$$

One would like to see the Landen transformation $k \rightarrow k_L$, which can be viewed as an exact generator¹⁷ of the renormalization group [35], as a symmetry of $\tilde{\chi}_L^{(2)}$, before seeing it as a symmetry of the full susceptibility. Remarkably, this is the case: $\Phi(k_L)$ and $\Phi(k)$ are very closely and very simply related! This remarkable relation can be written in many ways, using the various variables we have introduced [9, 10, 36, 37] (s, k, w, v), but since our functional relations are mostly written as series in v , we will write this relation in v . In v the Landen transformation corresponds to

¹⁶ In characteristic zero, not ‘modulo primes or powers of primes’.

¹⁷ This highly selected infinite order transformation (isogeny of the elliptic curve parametrizing the model [10, 35]) has $k = 1$ as a fixed point, $k = 0, \infty$ being clearly special [35].

$$4v^2 \longrightarrow \frac{4v}{1+4v^2} \quad \text{or:} \quad v^2 \longrightarrow \frac{v}{1+4v^2}. \quad (63)$$

Let us introduce

$$\Psi(x) = \frac{1+x}{x} \cdot \frac{d\Phi(x)}{dx}. \quad (64)$$

One then has the remarkably simple (differential-functional equation) relation representing the Landen transformation as a symmetry of $\tilde{\chi}_L^{(2)}$:

$$\Phi\left(\frac{4v}{1+4v^2}\right) = 4 \cdot \Psi(4v^2). \quad (65)$$

Pursuing this line of argument on functional equations with an infinite order transformation (hopefully with a physical symmetry interpretation like the Landen transformation representing a generator of the renormalization group [35]), it is tempting to imagine the $v \rightarrow v^2$ infinite order transformation in functional equations like (26), (36), or (37), as a mod. 2^r reduction of an infinite order symmetry of the model. In such a scenario, since one cannot distinguish, modulo 2 or 4, between v and $v/(1+4v^2)$, a functional equation $G_2(v) = G_2(v^2) + v$, like (36), could also be written as

$$G_2\left(\frac{v}{1+4v^2}\right) = G_2(v^2) + v \quad \text{or:} \quad G_2\left(\frac{k_L}{4}\right) = G_2\left(\frac{k}{4}\right) + \frac{\sqrt{k}}{2}, \quad (66)$$

and one could expect that the functional equations we discover modulo 2^r , are also the restriction modulo 2^r of some (quite involved) functional equations where the infinite order transformations have some physical meaning. Keeping in mind the unit circle natural boundary of the full susceptibility of the Ising model, it is worth recalling that functional equations like $G(v) = G(v^2) + v$, not modulo integers but in characteristic zero, are the simplest examples to actually show that a series has a (unit circle) natural boundary.

Recalling the expression of $\tilde{\chi}_L^{(2)}(w)$ given by (8), the previous functional equation (65) reads:

$$\tilde{\chi}_L^{(2)}\left(\frac{v}{1+4v^2}\right) = \frac{1}{8} \cdot \frac{1+4v^2}{v^3} \cdot \frac{d\tilde{\chi}_L^{(2)}(v^2)}{dv}, \quad (67)$$

This might suggest replacing $\tilde{\chi}_L^{(2)}(w)$ by $\tilde{\chi}_L(w)$ so as to consider this differential-functional equation (67) for the full susceptibility given by (6) modulo 2, 4, 8. Unfortunately, in contrast with the calculations performed in section 3.1, one finds that a differential¹⁸ functional equation like (67) is not satisfied by the full susceptibility modulo 2, 4, 8. This seems to suggest that, if a ‘master’ functional equation with an infinite order transformation symmetry exists (in exact arithmetic, not modulo some integers) for the full susceptibility, it is certainly much more involved than any simple generalization of (67).

Of course, all these ideas are quite speculative. It is reasonable to imagine that there must be some (probably involved) representation of the renormalization group [35] of the full susceptibility. In particular, for this integrable model which has an elliptic parametrization, one might expect a representation of the action of the Landen transformation on the full susceptibility.

¹⁸ Therefore different from the functional equations in section 3.1 .

5.2. Non-holonomic functions that are ratios of diagonals of rational functions, and beyond

Almost everything remains to be done to understand and describe this class of ‘nice’ non-holonomic functions reducing to algebraic functions modulo some powers of primes. It is worth recalling that, while the product of two holonomic functions is a holonomic function, the ratio of two holonomic functions is, in general¹⁹, non-holonomic! The class of functions that are expressible as a ratio of two holonomic functions, and, further, the ratio of diagonals of rational functions, is clearly a very interesting class of functions: they are such that their series can be recast into series with integer coefficients [18, 38] (the ratio of series with integer coefficients is up to an overall integer a series with integer coefficients), and that their series, modulo primes, or modulo powers of primes, reduce to algebraic functions²⁰ (the ratio of series reducing to algebraic functions reduces to the ratio of algebraic functions, and thus reduces to algebraic functions). Keeping in mind the (natural boundary of the) susceptibility of the Ising model, it is worth recalling that the ratio of holonomic functions can also yield a natural boundary, as can be seen from the solutions of nonlinear Chazy III equations [39, 40].

The solutions of a particular nonlinear third order differential equations having the Painlevé property, the Chazy III equations [39, 40], have (circular) natural boundaries, and this can be seen as a direct consequence of the fact that the solutions correspond to the ratio of two holonomic functions, as shown by Chazy in crystal clear papers.

The Chazy III equation [39, 40] is a third-order nonlinear differential equation with a movable singularity that has a natural boundary for its solutions [41]:

$$\frac{d^3y}{dx^3} = 2y \cdot \frac{d^2y}{dx^2} - 3\left(\frac{dy}{dx}\right)^2. \tag{68}$$

It can be rewritten in terms of a Schwarzian derivative:

$$f^{(4)} = 2f'^2 \cdot \{f, x\} = 2f'f''' - 3f''^2 \text{ with: } y = \frac{df}{dx}. \tag{69}$$

Similarly, it is important to recall that the ratio of two holonomic functions, which is in general a non-holonomic function, is the solution of a nonlinear Schwarzian derivative ODE:

$$\frac{d^2y}{dx} + R(x) \cdot y = 0, \quad \tau(x) = \frac{y_1}{y_2}, \quad \{\tau(x), x\} = 2R(x). \tag{70}$$

The Chazy III nonlinear differential equation (68) has the quasi-modular form Eisenstein series $E_2/2$. It can also be written as a log-derivative²¹, namely a ratio Δ'/Δ , where Ramanujan’s modular discriminant function [42, 43] Δ is actually a selected holonomic function: a modular form.

It is worth recalling, with the example of the enumeration of three-dimensional convex polygons [44], that we have already encountered, in enumerative combinatorics, the emergence of ratios of holonomic functions. The class of functions characterized by ratios of holonomic functions and ratios of diagonals of rational functions is certainly an over-simplified scenario for the susceptibility of the Ising model. It is however an interesting ‘toy class’ for the susceptibility of the Ising model, the class of the algebraic functions of diagonals of rational functions being much too large to reasonably explore.

¹⁹ Except when the holonomic function in the denominator is an algebraic function: in that case the ratio is also holonomic.

²⁰ More generally, a rational or even algebraic function (with integer coefficients) of holonomic functions, is such that it reduces modulo primes, or modulo powers of primes, to an algebraic function.

²¹ One takes the derivative with respect to the nome q (see equation (6) in [41]).

6. Conclusion

This paper underlines the central role of discrete finite automata, or diagonals of rational functions, in lattice statistical mechanics and enumerative combinatorics, in particular regarding the challenging problem of the full susceptibility of the two-dimensional Ising model [17].

The natural emergence of diagonals of rational functions in an extremely large set of lattice statistical mechanics and enumerative combinatorics models, has been emphasized and explained in [18]. That paper explains why a large class of functions describing lattice models that can be expressed as n -fold integrals of an algebraic²² integrand [18], which are, consequently, solutions of linear differential equations, and, thus, at first sight, transcendental functions, is in fact a remarkable class of transcendental holonomic functions, namely diagonals of rational functions [18].

The corresponding selected linear differential operators are not only Fuchsian, but also [5] globally nilpotent²³, and, since these transcendental functions are diagonals of rational functions, they reduce to algebraic functions modulo any prime [18]. They even reduce to algebraic functions modulo any integral power of a prime number. We may call this class of transcendental holonomic functions, that quite naturally occur in so many problems of theoretical physics [18], ‘almost algebraic functions’.

As far as transcendental non-holonomic functions are concerned, the full susceptibility of the two-dimensional Ising model is ‘algorithmically integrable’ (with an $O(N^4)$ polynomial algorithm) and can be decomposed as an infinite sum of n -fold integrals, that have been shown to be diagonals of rational functions [18]. Such nice transcendental non-holonomic functions emerging in physics require further concepts and tools to characterize and analyze them.

In this paper we have obtained exact functional equations for low and high temperature series of the full susceptibility modulo integers that are powers of the prime 2, the series being associated with simple lacunary series. Since these exact results come from remarkably long low- and high-temperature series [20] with more than 2000 coefficients, these exact functional equations are currently not yet proved but extremely plausible conjectures. Recalling the decomposition of the full susceptibility as an infinite sum of n -fold integrals $\chi^{(n)}$, these striking results can, in fact, be seen as a consequence of the fact that, modulo integers that are powers of the prime 2, the full susceptibility series are the same series as the series for the sum of the first $\chi^{(n)}$ ’s: for instance the low-temperature series modulo 16 of the full susceptibility series and of $\chi^{(2)}$ (which is the diagonal of a rational function) are the same.

Modulo a prime p , diagonals of a rational function not only reduce to algebraic functions, but also satisfy equations of the form $F(f(x), f(x^p), \dots, f(x^{p^h})) = 0$. In other words, the functional equations we have obtained, can be interpreted as the fact that modulo some integers that are powers of the prime 2, one cannot really distinguish between the full susceptibility and the diagonal of a rational function (like, for instance, $\chi^{(2)} + \chi^{(4)}, \dots$). The scenario seems to be that one cannot distinguish the series for $\tilde{\chi}_L$ and for a finite sum like $\tilde{\chi}_L^{(2)} + \tilde{\chi}_L^{(4)} + \dots + \tilde{\chi}_L^{(2n)}$ modulo 2^r where r grows with n . The series expansion for the $\tilde{\chi}_L^{(n)}$ are given in [23], up to $n = 12$. This scenario can be checked up to $\tilde{\chi}_L^{(12)}$. Recall that the finite sum $\tilde{\chi}_L^{(2)} + \tilde{\chi}_L^{(4)} + \dots + \tilde{\chi}_L^{(2n)}$ is also a diagonal of a rational function [18], therefore this finite sum reduces to algebraic functions modulo powers of primes, and, thus, satisfies

²² Or even holonomic.

²³ Their critical exponents are rational numbers, their Wronskian are N -th roots of rational functions, etc.

functional equations modulo powers of primes. Therefore $\tilde{\chi}_L$ which cannot be distinguished from this sum modulo some 2^r satisfies a functional equation modulo some 2^r .

The question whether the full susceptibility satisfies (probably involved) functional equations modulo primes different from 2, or powers of primes different from 2^r , remains open (even if, given (60) and (61), it may seem unlikely).

Much remains to be done to understand, and describe, this class of ‘nice’ non-holonomic functions. It is worth recalling that, while the product of two holonomic functions is a holonomic function, the ratio of two holonomic functions is, in general, non-holonomic. The class of functions that are expressible as ratios of diagonals of rational functions, is clearly a very interesting and important class of functions: they are such that their series (i) can be recast into series with integer coefficients, and (ii) modulo primes, or modulo powers of primes, reduce to algebraic functions²⁴. Concerning the susceptibility of the Ising model, it is worth recalling that ratios of holonomic functions can also yield²⁵ a natural boundary. The ratio of diagonals of rational functions is probably an overly-simple scenario for the susceptibility of the Ising model. However it is clearly important to start studying this class of functions, and further, to study algebraic expressions of diagonals of rational functions, per se, before introducing them as a well-suited and powerful framework in which to study models of lattice statistical mechanics or enumerative combinatorics [44].

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Appendix. Full susceptibility expansions in w

A.1. Low-temperature expansion in w

Let us consider (6), the low-temperature expansion for $\tilde{\chi}_L$ in the $w = \frac{1}{2}s/(1 + s^2)$ variable and introduce the series $\tilde{F}(w) = \tilde{\chi}_L/4$:

$$\begin{aligned} \tilde{F}(w) = \frac{\tilde{\chi}_L}{4} = & w^4 + 20 w^6 + 350 w^8 + 5880 w^{10} + 97\,020 w^{12} + 1585\,584 w^{14} \\ & + 25\,765\,744 w^{16} + 417\,159\,856 w^{18} + \dots \end{aligned} \tag{A.1}$$

Modulo 2 and 4 the series (A.1) becomes simple polynomials:

$$\tilde{F}_2(w) = w^4 \pmod{2}, \quad \tilde{F}_4(w) = w^4 + 2w^8 \pmod{4}. \tag{A.2}$$

²⁴ More generally, algebraic expressions of diagonal of rational functions are such that they reduce modulo primes, or modulo power of primes, to algebraic functions.

²⁵ The fact that solutions of a particular Painlevé-like nonlinear third order differential equations, the Chazy III equations [39, 40], have (circular) natural boundaries is a direct consequence of the fact that the solutions correspond to the ratio of two holonomic functions, as shown by Chazy.

Modulo 8, this series becomes the lacunary series

$$\tilde{F}_8(w) = w^4 + 4 w^6 + 6 w^8 + 4 w^{12} + 4 w^{20} + 4 w^{36} + 4 w^{68} + 4 w^{132} + 4 w^{260} + \dots$$

which satisfies the functional equation modulo 8:

$$\tilde{F}_8(w^2) - w^4 \cdot \tilde{F}_8(w) + 2w^{10} \cdot (2 + w^2 - w^6) = 0 \pmod{8}. \quad (\text{A.3})$$

Comparing these results with $\tilde{\chi}_L^{(2)}/4$, the series expansion (8) divided by 4

$$\begin{aligned} \frac{\tilde{\chi}_L^{(2)}}{4} &= w^4 \cdot {}_2F_1\left(\left[\frac{3}{2}, \frac{5}{2}\right], [3], 16w^2\right) \\ &= w^4 + 20 w^6 + 350 w^8 + 5880 w^{10} + 97\,020 w^{12} + 1585\,584 w^{14} \\ &\quad + 25\,765\,740 w^{16} + 417\,159\,600 w^{18} + 6737\,127\,540 w^{20} + \dots, \end{aligned} \quad (\text{A.4})$$

one finds that this series (A.4) gives, modulo 2, 4, 8, the same series expansions as (A.2) and (A.1), and consequently satisfies the same functional equation as (A.3). Again, similarly to the results displayed in section 3.1, in the variable v , one cannot make, modulo 2, 4, 8, a distinction, for low-temperature expansions, between $\tilde{\chi}_L$ and $\tilde{\chi}_L^{(2)}$.

A.2. High-temperature expansion in w

Let us consider (7), the high-temperature expansion for $\tilde{\chi}_H$ in the $w = \frac{1}{2}s/(1 + s^2)$ variable, and introduce the series $\tilde{F}(w) = \tilde{\chi}_H/2$:

$$\begin{aligned} \tilde{F}(w) = \frac{\tilde{\chi}_H}{2} &= w + 4 w^2 + 16 w^3 + 64 w^4 + 256 w^5 + 1024 w^6 + 4096 w^7 \\ &\quad + 16384 w^8 + 65540 w^9 + 262144 w^{10} + 1048720 w^{11} + \dots \end{aligned} \quad (\text{A.5})$$

This series modulo 2 and 4 reads:

$$\tilde{F}_2(w) = w \pmod{2}, \quad \tilde{F}_4(w) = w \pmod{4}, \quad (\text{A.6})$$

This series modulo 8 reads:

$$\begin{aligned} \tilde{F}_8(w) &= w + 4 w^2 + 4 w^9 + 4 w^{17} + 4 w^{33} + 4 w^{65} + 4 w^{129} \\ &\quad + 4 w^{257} + 4 w^{513} + \dots \pmod{8}, \end{aligned}$$

which satisfies the functional equation:

$$\tilde{F}_8(w^2) + 4 w^3 \cdot (w^7 - w + 1) = w \cdot \tilde{F}_8(w) \pmod{8}. \quad (\text{A.7})$$

Modulo 16 it reads:

$$\begin{aligned} \tilde{F}_{16}(w) &= w + 4 w^2 + 4 w^9 + 12 w^{17} + 12 w^{33} + 12 w^{65} \\ &\quad + 12 w^{129} + 12 w^{257} + 12 w^{513} + \dots \pmod{16}, \end{aligned} \quad (\text{A.8})$$

from which one deduces the functional relation:

$$\tilde{F}_{16}(w^2) + 4 w^3 \cdot (2 w^{15} + w^7 - w + 1) = w \cdot \tilde{F}_{16}(w) \pmod{16}. \quad (\text{A.9})$$

Comparing the series (A.5) with the series $\tilde{\chi}_H^{(1)}/2$, namely the series (9) divided by 2

$$\begin{aligned} \frac{\tilde{\chi}_H^{(1)}}{2} &= \frac{w}{1 - 4w} = w + 4 w^2 + 16 w^3 + 64 w^4 + 256 w^5 + 1024 w^6 \\ &\quad + 4096 w^7 + 16384 w^8 + 65536 w^9 + 262144 w^{10} + 1048576 w^{11} + \dots \end{aligned} \quad (\text{A.10})$$

one gets respectively, mod.2, 4, 8, 16, 32:

$$\begin{aligned} w \pmod{2}, & \quad w \pmod{4}, & \quad w + 4w^2 \pmod{8}, \\ w + 4w^2 + 16w^3 \pmod{16}, & & \quad w + 4w^2 + 16w^3 \pmod{32}, \\ w + 4w^2 + 16w^3 \pmod{64}, & & \quad w + 4w^2 + 16w^3 + 64w^4 \pmod{128}. \end{aligned} \quad (\text{A.11})$$

Appendix B. A very simple algebraic function example illustrating the emergence of a lacunary series

Let us consider a very simple algebraic function, the Catalan number generating function [33]:

$$C(x) = \frac{1 - \sqrt{1 - 4x}}{2x}. \quad (\text{B.1})$$

It is the solution of the quadratic equation $x \cdot C(x)^2 - C(x) + 1 = 0$. Modulo 2 the series $x \cdot C(x)$ reduces to $L(x) - 1$, where $L(x)$ is the lacunary series :

$$\begin{aligned} L(x) = 1 + x + x^2 + x^4 + x^8 + x^{16} + x^{32} + x^{64} + x^{128} + x^{256} \\ + x^{512} + x^{1024} + x^{2048} + \dots \end{aligned} \quad (\text{B.2})$$

This Catalan number generating function (B.1) satisfies the functional equation

$$x \cdot C(x^2) - C(x) + 1 = 0 \pmod{2}. \quad (\text{B.3})$$

which can be seen as the consequence of $C(x^2) = C(x)^2 \pmod{2}$, or as the consequence of the functional equation $L(x^2) = L(x) + x$.

This generating function (B.1) yields many other lacunary series modulo 2^r (see for instance [33]), for instance, modulo 8 the series expansion of $4 + 4/(1 - x \cdot C(x^2))$ reduces to $4 \cdot L(x)$ where $L(x)$ is given by (B.2). This result, namely the emergence of lacunary series, can be seen as a simple example of the previous finite automaton results, or, equivalently, congruence on algebraic functions, in the simplest case where only square roots occur.

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