Diagonals of rational functions, pullbacked \( \mathcal{F}_1 \) hypergeometric functions and modular forms
(unabridged version)

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Abstract.
We recall that diagonals of rational functions naturally occur in lattice statistical mechanics and enumerative combinatorics. We find that a seven-parameter rational function of three variables with a numerator equal to one (reciprocal of a polynomial of degree two at most) can be expressed as a pullbacked \( \mathcal{F}_1 \) hypergeometric function. This result can be seen as the simplest non-trivial family of diagonals of rational functions. We focus on some subcases such that the diagonals of the corresponding rational functions can be written as a pullbacked \( \mathcal{F}_1 \) hypergeometric function with two possible rational functions pullbacks algebraically related by modular equations, thus showing explicitly that the diagonal is a modular form. We then generalise this result to eight, nine and ten parameter families adding some selected cubic terms at the denominator of the rational function defining the diagonal. We finally show that each of these previous rational functions yields an infinite number of rational functions whose diagonals are also pullbacked \( \mathcal{F}_1 \) hypergeometric functions and modular forms.

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1. Introduction

It was shown in [1, 2] that different physical related quantities, like the \( n \)-fold integrals \( \chi^{(n)} \), corresponding to the \( n \)-particle contributions of the magnetic susceptibility of the Ising model [3, 4, 5, 6], or the lattice Green functions [7, 8, 9, 10, 11], are diagonals of rational functions [12, 13, 14, 15, 16, 17].

While showing that the \( n \)-fold integrals \( \chi^{(n)} \) of the susceptibility of the Ising model are diagonals of rational functions requires some effort, seeing that the lattice
Green functions are diagonals of rational functions nearly follows from their definition. For example, the lattice Green functions (LGF) of the $d$-dimensional face-centred cubic (fcc) lattice are given \cite{10, 11} by:
\[
\frac{1}{\pi^d} \int_0^\pi \cdots \int_0^\pi \frac{dk_1 \cdots dk_d}{1 - x : \lambda_d}, \quad \text{with:} \quad \lambda_d = \left(\frac{d}{2}\right)^{-1} \sum_{i=1}^{d} \sum_{j=i+1}^{d} \cos(k_i) \cos(k_j). \quad (1)
\]

The LGF can easily be seen to be a diagonal of a rational function: introducing the complex variables $z_j = e^{ik_j}$, $j = 1, \cdots, d$, the LGF (1) can be seen as a $d$-fold generalization of Cauchy's contour integral \cite{1}:
\[
\text{Diag}(\mathcal{F}) = \frac{1}{2\pi i} \oint_{\gamma} \mathcal{F}(z_1, z/z_1) \frac{dz_1}{z_1}. \quad (2)
\]

Furthermore, the linear differential operators annihilating the physical quantities mentioned earlier $\chi^{(n)}$, are reducible operators. Being reducible they are “breakable” into smaller factors \cite{4, 5} that happen to be elliptic functions, or generalizations thereof: modular forms, Calabi-Yau operators \cite{18, 19}... Yet there exists a class of diagonals of rational functions in three variables\footnote{Diagonals of rational functions of two variables are just algebraic functions, so one must consider at least three variables to obtain special functions.} whose diagonals are pullbacked \textunderscore{}$\text{2F}_1$ hypergeometric functions, and in fact modular forms \cite{21}. These sets of diagonals of rational functions in three variables in \cite{21} were obtained by imposing the coefficients of the polynomial $P(x, y, z)$ appearing in the rational function $1/P(x, y, z)$ defining the diagonal to be 0 or 1\footnote{Or 0 or $\pm 1$ in the four variable case also examined in \cite{21}.}.

While these constraints made room for exhaustivity, they were quite arbitrary, which raises the question of randomness of the sample: is the emergence of modular forms \cite{20}, with the constraints imposed in \cite{21}, an artefact of the sample?

Our aim in this paper is to show that modular forms emerge for a much larger set of rational functions of three variables, than the one previously introduced in \cite{21}, firstly because we obtain a whole family of rational functions whose diagonals give modular forms by adjoining parameters, and secondly through considerations of symmetry.

In particular, we will find that the seven-parameter rational function of three variables, with a numerator equal to one and a denominator being a polynomial of degree two at most, given by:
\[
R(x, y, z) = \frac{1}{a + b_1 x + b_2 y + b_3 z + c_1 y z + c_2 x z + c_3 x y}, \quad (3)
\]
can be expressed as a particular pullbacked \textunderscore{}$\text{2F}_1$ hypergeometric function\footnote{The selected $\text{2F}_1[1/12, 5/12, [1], \mathcal{P}]$ hypergeometric function is closely related to modular forms \cite{22, 23}. This can be seen as a consequence of the identity with the Eisenstein series $E_4$ and $E_6$ and this very $\text{2F}_1[1/12, 5/12, [1], \mathcal{P}]$ hypergeometric function \cite{24} page 226 in \cite{24} and page 216 of \cite{25}): $E_4(\tau) = \text{2F}_1[1/12, 5/12, [1], 1728/(24 \pi)^2]$,}:
\[
\frac{1}{P_2(x)^{1/4}} \cdot \text{2F}_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], 1 - \frac{P_3(x)^2}{P_2(x)^4}\right), \quad (4)
\]
where $P_2(x)$ and $P_3(x)$ are two polynomials of degree two and four respectively. We then focus on subcases where the diagonals of the corresponding rational functions can
be written as a pullbacked $_2F_1$ hypergeometric function with two rational function pullbacks that are algebraically related by modular equations$^\dagger$.

This seven-parameter family will then be generalized into an eight, nine and finally ten parameters family of rational functions that are reciprocal of a polynomial of three variables of degree at most three. We will finally show that each of the previous results yields an infinite number of new exact pullbacked $_2F_1$ hypergeometric function results, through symmetry considerations on monomial transformations and some function-dependent rescaling transformations.

2. Diagonals of rational functions of three variables depending on seven parameters

2.1. Recalls on diagonals of rational functions

Let us recall the definition of the diagonal of a rational function in $n$ variables $R(x_1, \ldots , x_n) = P(x_1, \ldots , x_n)/Q(x_1, \ldots , x_n)$, where $P$ and $Q$ are polynomials of $x_1, \ldots , x_n$ with integer coefficients such that $Q(0, \ldots , 0) \neq 0$. The diagonal of $R$ is defined through its multi-Taylor expansion (for small $x_i$’s)

$$R(x_1, x_2, \ldots , x_n) = \sum_{m_1 = 0}^{\infty} \cdots \sum_{m_n = 0}^{\infty} R_{m_1, \ldots , m_n} \cdot x_1^{m_1} \cdots x_n^{m_n},$$

as the series in one variable $x$:

$$\text{Diag}(R(x_1, x_2, \ldots , x_n)) = \sum_{m = 0}^{\infty} R_{m, \ldots , m} \cdot x^m.$$  

Diagonals of rational functions of two variables are algebraic functions [27, 28]. Interesting cases of diagonals of rational functions thus require to consider rational functions of at least three variables.

2.2. A seven parameters family of rational functions of three variables

We obtained the diagonal of the rational function in three variables depending on seven parameters:

$$R(x, y, z) = \frac{1}{a + b_1 x + b_2 y + b_3 z + c_1 y z + c_2 x z + c_3 x y}.$$  

This result was obtained by:

- Running the HolonomicFunctions [29] package in mathematica for arbitrary parameters $a, b_1, \ldots , c_1, \ldots$ and obtaining a large-sized second order linear differential operator $L_2$.

- Running the maple command “hypergeometricsols” [30] for different sets of values of the parameters on the operator $L_2$, and guessing$^\ddagger$ the Gauss hypergeometric function $_2F_1$ with general parameters solution of $L_2$.

$^\dagger$ Thus providing a nice illustration of the fact that the diagonal is a modular form [23].

$^\ddagger$ The program “hypergeometricsols” [30] does not run for arbitrary parameters, hence our recourse to guessing.
2.3. The diagonal of the seven parameters family of rational functions: the general form

We find the following experimental results: all these diagonals are expressed in terms of only one pullbacked hypergeometric function. This is worth noticing since, in general, when an order-two linear differential operator has pullbacked \( _2F_1 \) hypergeometric function solutions, the “hypergeometricsol” command gives the two solutions as sums of two \( _2F_1 \) hypergeometric functions. Here, quite remarkably, the result is “encapsulated” in just one pullbacked hypergeometric function. Furthermore we find that all these diagonals are expressed as pullbacked hypergeometric functions of the form

\[
\frac{1}{P_4(x)^{1/6}} \cdot _2F_1\left(\left[1 \frac{1}{12}, \frac{7}{12}\right], \left[1\right], \frac{1728 \cdot x^3 \cdot P_3(x)}{P_4(x)^2}\right),
\]

where the two polynomials \( P_4(x) \) and \( P_3(x) \), in the \( 1728 x^2 P_4(x)/P_4(x)^2 \) pullback, are polynomials of degree four and five in \( x \) respectively. The pullback in (8), given by \( 1728 x^2 P_3(x)/P_4(x)^2 \), has the form \( 1 - \tilde{Q} \) where \( \tilde{Q} \) is given by the simpler expression

\[
\tilde{Q} = \frac{P_3(x)^3}{P_4(x)^2},
\]

where \( P_2(x) \) is a polynomial of degree two in \( x \). Recalling the identity

\[
_2F_1\left(\left[1 \frac{1}{12}, \frac{7}{12}\right], \left[1\right], x\right) = (1 - x)^{-1/12} \cdot _2F_1\left(\left[1 \frac{5}{12}\right], \left[1\right], \frac{-x}{1-x}\right),
\]

the previous pullbacked hypergeometric function (8) can be rewritten as

\[
\frac{1}{P_2(x)^{1/4}} \cdot _2F_1\left(\left[1 \frac{1}{12}, \frac{5}{12}\right], \left[1\right], \frac{1728 \cdot x^3 \cdot P_3(x)}{P_2(x)^3}\right),
\]

where \( P_3(x) \) is the same polynomial of degree five as the one in the pullback in expression (8). This new pullback also has the form \( 1 - Q \) with \( Q \) given by †:

\[
- \frac{1728 \cdot x^3 \cdot P_3(x)}{P_2(x)^3} = 1 - Q \quad \text{where:} \quad Q = \frac{P_4(x)^2}{P_2(x)^3}.
\]

Finding the exact result for arbitrary values of the seven parameters now boils down to a guessing problem.

2.4. Exact expression of the diagonal for arbitrary parameters \( a, b_1, ..., c_1, ... \)

Now that the structure of the result is understood “experimentally” we obtain the result for arbitrary parameters \( a, b_1, b_2, b_3, c_1, c_2, c_3 \).

Assuming that the diagonal of the rational function (7) has the form explicited in the previous subsection

\[
\frac{1}{P_3(x)^{1/4}} \cdot _2F_1\left(\left[1 \frac{1}{12}, \frac{5}{12}\right], \left[1\right], 1 - \frac{P_4(x)^2}{P_2(x)^3}\right),
\]

where \( P_2(x) \) and \( P_4(x) \) are two polynomials of degree two and four respectively:

\[
P_2(x) = A_4 x^4 + A_3 x^3 + A_2 x^2 + A_1 x + A_0,
\]

\[
P_4(x) = B_2 x^2 + B_1 x + B_0.
\]

† Note that \( Q \), given by (12), is the reciprocal of \( \tilde{Q} \) given in (9): \( Q = 1/\tilde{Q} \).
The polynomial $A$ for specific values of the parameters, one easily guesses that one can write the order-two linear differential operator having this eight-parameter Diagonals of rational functions (17) vanishes, and the previous exact result (13), for the diagonal of the rational HolonomicFunctions [29] program for arbitrary parameters. Using the results obtained for specific values of the parameters, one easily guesses that $A_0 = a^6$ and $B_0 = a^4$. One finally gets:

$$P_2(x) = 8 \cdot \left(3a c_1 c_2 c_3 + 2 \cdot (b_1^2 c_1^2 + b_2^2 c_2^2 + b_3^2 c_3^2 - b_1 b_2 c_1 c_2 - b_1 b_3 c_1 c_3 - b_2 b_3 c_2 c_3)\right) \cdot x^2$$

$$-8 \cdot a \cdot \left(a \cdot (b_1 c_1 + b_2 c_2 + b_3 c_3) - 3b_1 b_2 b_3\right) \cdot x + a^4,$$

(16)

and

$$P_4(x) = 216 \cdot c_1^2 c_2^2 c_3^2 \cdot x^4 - 16 \cdot \left(9 \cdot a c_1 c_2 c_3 \cdot (b_1 c_1 + b_2 c_2 + b_3 c_3) - 6 \cdot (b_1^2 c_1^2 c_2^2 + b_1 b_2 c_1 c_2 c_3 + b_1 b_2 c_1 c_2 c_3 + b_2 b_3 c_2 c_3 + b_1 b_2 c_1 c_2 c_3 + b_1 b_2 c_1 c_2 c_3) + 4 \cdot (b_1^2 c_1^2 + b_1^2 c_2^2 + b_1^2 c_3^2) - 3b_1 b_2 c_1 c_2 c_3\right) \cdot x^3$$

$$+12 \cdot \left(3a^3 c_1 c_2 c_3 + 4 \cdot a^2 \cdot (b_1^2 c_1^2 + b_2^2 c_2^2 + b_3^2 c_3^2) + 2 \cdot a \cdot (b_1 b_2 c_1 c_2 + b_1 b_3 c_1 c_3 + b_2 b_3 c_2 c_3) - 12 \cdot a \cdot b_1 b_2 b_3 \cdot (b_1 c_1 + b_2 c_2 + b_3 c_3) + 18 \cdot b_1^2 b_2^2 b_3^2\right) \cdot x^2$$

$$-12 \cdot a^3 \cdot \left(a \cdot (b_1 c_1 + b_2 c_2 + b_3 c_3) - 3b_1 b_2 b_3\right) \cdot x + a^6.$$

(17)

The polynomial $P_5(x)$ in (12), given by $P_5(x) = (P_4(x)^2 - P_2(x)^3)/1728 x^3$, is a slightly larger polynomial of the form

$$P_5(x) = 27 \cdot c_1^4 c_2^4 c_3^4 \cdot x^5 + \cdots + q_1 \cdot x + \Phi_0,$$

where:

$$\Phi_0 = -b_1 b_2 b_3 a^3 \cdot (a c_1 - b_2 b_3) \cdot (a c_2 - b_1 b_3) \cdot (a c_3 - b_1 b_2).$$

(18)

The coefficient $q_1$ in $x$ reads for instance:

$$q_1 = c_1 c_2 c_3 (b_1 b_2 c_1 c_2 + b_1 b_3 c_1 c_3 + b_2 b_3 c_2 c_3) \cdot a^5$$

$$- (b_1^2 b_2^2 c_1^2 c_2^2 + b_1 b_2 b_3 c_1 c_2 c_3 + b_1 b_2 b_3 c_1 c_2 c_3 - 8b_1 b_2 b_3 c_1 c_2 c_3 \cdot (b_1 c_1 + b_2 c_2 + b_3 c_3)) \cdot a^4$$

$$- b_1 b_2 b_3 \cdot (57 b_1 b_2 b_3 c_1 c_2 c_3$$

$$+ 8 \cdot (b_1^2 b_2^2 c_1^2 c_2^2 + b_1 b_2 b_3 c_1 c_2 c_3 + b_1 b_2 c_1 c_2 c_3 + b_1 b_3 c_1 c_2 c_3 + b_2 b_3 c_2 c_3 + b_2 b_3 c_2 c_3)\right) \cdot a^3$$

$$+ 8 b_1^2 b_2^2 b_3^2 \cdot (b_1^2 c_1^2 + b_2^2 c_2^2 + b_3^2 c_3^2) \cdot a^2$$

$$+ 46 \cdot b_1^2 b_2^2 b_3^2 \cdot (b_1 b_2 c_1 c_2 + b_1 b_3 c_1 c_3 + b_2 b_3 c_2 c_3) \cdot a^2$$

$$- 36 \cdot b_1^2 b_2^2 b_3^2 \cdot (b_1 c_1 + b_2 c_2 + b_3 c_3) \cdot a + 27 b_1^4 b_2^4 b_3^4.$$

(19)

Having “guessed” the exact result, one can easily verify directly that this exact pullbacked hypergeometric result is truly the solution of the large second order linear differential operator obtained using the “HolonomicFunctions” program [29].

2.5. Selected subcases of these results

When $P_2(x)^3 - P_3(x)^2 = -1728 x^3 \cdot P_5(x) = 0$, the pullback in (13) (with (16), (17)) vanishes, and the previous exact result (13), for the diagonal of the rational
function (7), degenerates into a simple algebraic function (see (11) and (13)):

\[
\frac{1}{P_3(x)^{1/4}} = \frac{1}{P_4(x)^{1/6}}.
\]  

(20)

The condition \( P_2^3 - P_3^2 = -1728x^3 \cdot P_5(x) = 0 \) corresponds, for instance, to \( c_3 = 0, b_1 = 0 \), with the rational function

\[
\frac{1}{a + b_2 y + b_3 z + c_1 y z + c_2 x z},
\]

(21)

or to \( c_3 = 0, c_1 = b_2 b_3/a, c_2 = b_1 b_3/a \), with the rational function:

\[
\frac{1}{a^2 + a b_1 x + a b_2 y + a b_3 z + b_2 b_3 y z + b_1 b_3 x z}.
\]

(22)

One easily verifies that the diagonals of the corresponding rational functions read respectively:

\[
\frac{1}{\sqrt{a^2 - 4 b_2 c_2 \cdot x}}, \quad \frac{\sqrt{a}}{\sqrt{a^3 + 4 b_1 b_2 b_3 \cdot x}}.
\]

(23)

2.6. Simple symmetries of this seven-parameter result

The different pullbacks

\[
P_1 = -\frac{1728 \cdot x^3 \cdot P_3(x)}{P_2(x)^3}, \quad \frac{1728 \cdot x^3 \cdot P_5(x)}{P_4(x)^2}, \quad 1 - \frac{P_4(x)^2}{P_2(x)^3},
\]

(24)

must be compatible with some obvious symmetries. They verify the relations

\[
P_1(\lambda \cdot a, \lambda \cdot b_1, \lambda \cdot b_2, \lambda \cdot b_3, \lambda \cdot c_1, \lambda \cdot c_2, \lambda \cdot c_3, x)
= P_1(a, b_1, b_2, b_3, c_1, c_2, c_3, x).
\]

(25)

and

\[
P_1(a, \lambda_1 \cdot b_1, \lambda_2 \cdot b_2, \lambda_3 \cdot b_3, \lambda_2 \lambda_3 \cdot c_1, \lambda_1 \lambda_3 \cdot c_2, \lambda_1 \lambda_2 \cdot c_3, \frac{x}{\lambda_1 \lambda_2 \lambda_3})
= P_1(a, b_1, b_2, b_3, c_1, c_2, c_3, x),
\]

(26)

where \( \lambda, \lambda_1, \lambda_2 \) and \( \lambda_3 \) are arbitrary complex numbers. A demonstration of these symmetry-invariance relations (25) and (26) is sketched in Appendix A.

2.7. A symmetric subcase \( \tau \rightarrow 3 \tau: \ 2F_1([1/3, 2/3], [1], \mathcal{P}) \)

2.7.1. A few recalls on Maier’s paper

We know from Maier [23] that the modular equation associated with \( \tau \rightarrow 3 \tau \) corresponds to the elimination of the \( z \) variable between the two rational pullbacks:

\[
P_1(z) = \frac{12^3 \cdot z^3}{(z + 27) \cdot (z + 243)^3}, \quad P_2(z) = \frac{12^3 \cdot z}{(z + 27) \cdot (z + 3)^3}.
\]

(27)

\( \tau \) denotes the ratio of the two periods of the elliptic functions that naturally emerge in the problem [22].
Following Maier [23] one can also write the identities:

\[
\left( 9 \cdot \frac{z + 27}{z + 243} \right)^{1/4} \cdot 2F_1\left( \frac{1}{12}, \frac{5}{12}; \frac{1728}{z + 27}, \frac{1728}{z + 243} \right) = \left( \frac{1}{9} \cdot \frac{z + 27}{z + 3} \right)^{1/4} \cdot 2F_1\left( \frac{1}{12}, \frac{5}{12}; \frac{1728}{z + 27}, \frac{1728}{z + 3} \right) = 2F_1\left( \frac{1}{3}, \frac{2}{3}, [1], \frac{z}{z + 27} \right) \tag{28} \]

Having a hypergeometric function identity (28) with two rational pullbacks (27) related by a modular equation provides a good heuristic way to see that we have a modular form [22, 23].

2.7.2. The symmetric subcase

Let us now consider the symmetric subcase \( b_1 = b_2 = b_3 = b \) and \( c_1 = c_2 = c_3 = c \). If we take that limit in our previous general expression (13), we obtain the solution of the order-two linear differential operator annihilating the diagonal in the form

\[
\frac{1}{P_2(x)^{1/4}} \cdot 2F_1\left( \frac{1}{12}, \frac{5}{12}, [1], 1 - \frac{P_4(x)^2}{P_2(x)^2} \right) = \frac{1}{P_2(x)^{1/4}} \cdot 2F_1\left( \frac{1}{12}, \frac{5}{12}, [1], -\frac{1728 \cdot x^3 \cdot P_5(x)}{P_2(x)^3} \right), \tag{30} \]

with

\[
P_2(x) = a \cdot (24 \cdot c^3 \cdot x^2 - 24 \cdot b \cdot (a c - b^2) \cdot x + a^3),
\]

\[
P_4(x) = 216 \cdot c^6 \cdot x^4 - 432 \cdot b \cdot c^3 \cdot (a c - b^2) \cdot x^3
\]

\[
+ 36 \cdot (a^3 c^3 + 6 \cdot a^2 b^2 c^2 - 12 \cdot a b^4 c + 6 \cdot b^6) \cdot x^2
\]

\[
- 36 \cdot a^3 b \cdot (a c - b^2) \cdot x + a^6. \tag{32} \]

and:

\[
P_3(x) = (27 \cdot c^3 \cdot x^2 - 27 \cdot b \cdot (a c - b^2) \cdot x + a^3) \cdot (c^3 \cdot x - b \cdot (a c - b^2))^3. \tag{33} \]

In this symmetric case, one can write the pullback in (30) as follows:

\[
\frac{1728 \cdot x^3 \cdot P_5(x)}{P_2(x)^3} = \frac{12^3 \cdot z^3}{(z + 27) \cdot (z + 243)^3}, \tag{34} \]

where \( z \) reads:

\[
z = -\frac{9^3 \cdot x \cdot (c^3 \cdot x - b \cdot (a c - b^2))}{27 \cdot c^3 \cdot x^2 - 27 \cdot b \cdot (a c - b^2) \cdot x + a^3}. \tag{35} \]

Injecting the expression (35) for \( z \) in \( P_2(z) \) given by (27), one gets another pullback

\[
P_2(z) = -1728 \cdot x \cdot \frac{P_5}{P_2(x)^3}. \tag{36} \]

\* One has hypergeometric identities on \( _2F_1([1/3, 2/3], [1], P) \), however they are not associated with the involutive transformation \( z \rightarrow \frac{1729}{z} \) as one could expect from the fact that the two Hauptmoduls in (28) are exchanged by this involution; see Appendix B.

\*\* Something that is obvious here since we are dealing with a \( _2F_1([1/12, 5/12], [1], x) \) hypergeometric function which is known to be related modular functions [22, 23] due to its relation with the Eisenstein series \( E_4 \), but is less clear for other hypergeometric functions.

\*\*\* Called the “telescopers” [31, 32].
with
\[ \tilde{P}_5(x) = (27c^3x^2 - 27b \cdot (ac - b^2) \cdot x + a^3) \cdot (c^3x - b \cdot (ac - b^2)) \] (37)
and:
\[ \tilde{P}_3(x) = a \cdot (-216 \cdot c^3 \cdot x^2 + 216 \cdot b \cdot (ac - b^2) \cdot x + a^3) \] (38)
In this case the diagonal of the rational function can be written as a single hypergeometric function with two different pullbacks
\[
\frac{1}{P_3(x)^{1/4}} \cdot \frac{1}{P_2(x)^{1/3}} \cdot 
\begin{bmatrix}
\frac{1}{12}, \frac{5}{12}, [1], 
- \frac{1728 \cdot x^3 \cdot P_5(x)}{P_2(x)^3}
\end{bmatrix}
\]
with the relation between the two pullbacks given by the modular equation associated with [22, 23] \( \tau \to 3\tau \):
\[
2^{27} \cdot x^9 \cdot Y^2 \cdot Z^3 \cdot (Y + Z) + 2^{18} \cdot 3^6 \cdot Y^2 \cdot Z^2 \cdot (27Y^2 - 45946YZ + 27Z^2)
+ 2^9 \cdot 3^5 \cdot 5^2 \cdot YZ \cdot (Y + Z) \cdot (Y^2 + 241433YZ + 2Z^2)
+ 729 \cdot (Y^4 + Z^4) - 779997924 \cdot (YZ^3 + Y^3Z) + 31949606 \cdot 3^{10} \cdot Y^2 \cdot Z^2
+ 2^9 \cdot 3^{11} \cdot 31 \cdot Y \cdot Z \cdot (Y + Z) - 2^{12} \cdot 3^{12} \cdot YZ = 0.
\]

2.7.3. Alternative expression for the symmetric subcase
Alternatively, we can obtain the exact expression of the diagonal using directly the "HolonomicFunctions" program [29] for arbitrary parameters \( a, b \) and \( c \) to get an order-two linear differential operator annihilating that diagonal. Then, using "hypergeometricsols" we obtain the solution of this second order linear differential operator in the form
\[
\frac{1}{a} \cdot 2F1\left(\left[\frac{1}{3}, \frac{2}{3}\right], [1], \frac{-27}{a^2} \cdot x \cdot (c^3x - b \cdot (ac - b^2))\right),
\]
which looks, at first sight, different from (30) with (31) and (32). This last expression (40) is compatible with the form (30) as a consequence of the identity:
\[
\left(\frac{9 - 8x}{9}\right)^{1/4} \cdot 2F1\left(\left[\frac{1}{3}, \frac{2}{3}\right], [1], x\right) = 2F1\left(\left[\frac{1}{12}, \frac{5}{12}\right], \frac{64x^3 \cdot (1 - x)}{(9 - 8x)^3}, [1]\right).
\]
(41)
The reduction of the (generic) \( 2F1([1/12, 5/12], [1], \mathcal{P}) \) hypergeometric function to a \( 2F1([1/3, 2/3], [1], \mathcal{P}) \) form corresponds to a selected \( \tau \to 3\tau \) modular equation situation (27) well described in [23].

These results can also be expressed in terms of \( 2F1([1/3, 1/3], [1], \mathcal{P}) \) pullback hypergeometric functions [23] using the identities
\[ 2F1\left(\left[\frac{1}{3}, \frac{1}{3}\right], [1], x\right) = (1 - x)^{-1/3} \cdot 2F1\left(\left[\frac{1}{3}, \frac{2}{3}\right], [1], \frac{x}{1 - x}\right) \] (42)
or:
\[ 2F1\left(\left[\frac{1}{3}, \frac{1}{3}\right], [1], - \frac{x}{27}\right) = \left(\frac{x + 3}{27}\right)^{-1/3} \cdot 2F1\left(\left[\frac{1}{3}, \frac{2}{3}\right], [1], \frac{x}{x + 27}\right) \] (43)
\[ \frac{1}{12}, \frac{5}{12}, [1], \frac{1728x}{(x + 3)^3 \cdot (x + 27)}\]
2.8. A non-symmetric subcase $\tau \to 4 \tau$: $2F_1([1/2, 1/2], [1], \mathcal{P})$.

Let us consider the non-symmetric subcase $b_1 = b_2 = b_3 = b$ and $c_1 = c_2 = 0$, $c_3 = b^2/a$. The pullback in (30) reads:

$$\mathcal{P}_1 = -\frac{1728 \cdot x^3 \cdot P_1(x)}{P_2(x)^3} = \frac{1728 \cdot a^3 b^3 \cdot x^4 \cdot (16 b^3 x + a^3)}{(16 b^6 x^2 + 16 a^3 b^3 x + a^6)^3}. \quad (44)$$

This pullback can be seen as the first of the two Hauptmoduls

$$\mathcal{P}_1 = \frac{1728 \cdot z^4 \cdot (z + 16)}{(z^2 + 256 z + 4096)^3}, \quad \mathcal{P}_2 = \frac{1728 \cdot z \cdot (z + 16)}{(z^2 + 16 z + 16 z^3)}, \quad (45)$$

provided $z$ is given by:

$$z = \frac{256 b^3 x}{a^3} \quad \text{or} \quad z = \frac{-256 \cdot b^3 \cdot x}{a^3 + 16 b^3 x}. \quad (46)$$

These exact expressions (46) of $z$ in terms of $x$ give exact rational expressions of the second Hauptmodul $\mathcal{P}_2$ in terms of $x$:

$$\mathcal{P}_2^{(1)} = \frac{1728 \cdot a^3 b^3 \cdot x \cdot (a^3 + 16 b^3 x)^4}{(4096 b^6 x^2 + 256 a^3 b^3 x + a^6)^3} \quad \text{or} \quad \mathcal{P}_2^{(2)} = \frac{-1728 \cdot a^3 b^3 \cdot x \cdot (a^3 + 16 b^3 x)^4}{(256 b^6 x^2 - 224 a^3 b^3 x + a^6)^3}. \quad (47, 48)$$

These two pullbacks (44), (47) and (48) (or $\mathcal{P}_1$ and $\mathcal{P}_2$ in (45)) are related by a modular equation corresponding to $\tau \to 4 \tau$.

This subcase thus corresponds to the diagonal of the rational function being expressed in terms of a modular form associated to an identity on a hypergeometric function:

$$(16 b^6 x^2 + 16 a^3 b^3 x + a^6)^{-1/4} \cdot 2F_1\left([1/12, 5/12], [1], \mathcal{P}_1\right)$$

$$= (4096 b^6 x^2 + 256 a^3 b^3 x + a^6)^{-1/4} \cdot 2F_1\left([1/12, 5/12], [1], \mathcal{P}_2^{(1)}\right)$$

$$= (256 b^6 x^2 - 224 a^3 b^3 x + a^6)^{-1/4} \cdot 2F_1\left([1/12, 5/12], [1], \mathcal{P}_2^{(2)}\right)$$

$$= 2F_1\left([1/2, 1], [1], -\frac{16 \cdot b^3}{a^3} \cdot x\right). \quad (49)$$

The last equality is a consequence of the identity:

$$2F_1\left([1/2, 1], [1], -\frac{x}{16}\right) \quad (50)$$

$$= 2 \cdot (x^2 + 16 x + 16)^{-1/4} \cdot 2F_1\left([1/12, 5/12], [1], \frac{1728 \cdot x \cdot (x + 16)}{(x^2 + 16 x + 16)^3}\right).$$

Similarly, the elimination of $x$ between the pullback $X = \mathcal{P}_1$ (given by (44)) and $Y = \mathcal{P}_2^{(1)}$ gives the same modular equation (representing $\tau \to 4 \tau$) than the elimination of $x$ between the pullback $X = \mathcal{P}_1$ (given by (44)) and $Y = \mathcal{P}_2^{(2)}$.

† These two expressions are related by the involution $z \leftrightarrow -16 z/(z + 16)$.

†† See page 20 in [22].
namely:

\[ 825^3 \cdot X^6 Y^6 - 389 \cdot 11^6 \cdot 5^{16} \cdot 3^{10} \cdot 2^6 \cdot X^5 Y^5 \cdot (X + Y) + 11^3 \cdot 5^{12} \cdot 3^7 \cdot 2^4 \cdot X^4 Y^4 \cdot \left( 26148290096 \cdot (X^2 + Y^2) - 1559685235 \cdot X Y \right) - 10595481959 \cdot 5^{10} \cdot 3^7 \cdot 2^{15} \cdot X^3 Y^3 \cdot (X + Y) \cdot (X^2 + Y^2) + 53027637092599 \cdot 5^{10} \cdot 3^7 \cdot 2^6 \cdot X^4 Y^4 \cdot (X + Y) + 5^9 \cdot 3^4 \cdot 2^6 \cdot X^2 Y^2 \cdot \left( 1634268131 \cdot (X^4 + Y^4) + 1788502080642816 \cdot X^2 Y^2 + 84896080668355 \cdot (X^3 Y + X Y^3) \right) \]

\[ - 5^4 \cdot 3^3 \cdot 2^{22} \cdot X Y \cdot (X + Y) \cdot \left( 389 \cdot (X^4 + Y^4) + 41863592956503 \cdot X^2 Y^2 - 54605727143 \cdot (X^3 Y + X Y^3) \right) + 524 \cdot \left( X^6 + Y^6 + 561444609 \cdot (X^5 Y + X Y^5) + 1425220456750089 \cdot (X^4 Y^2 + X^2 Y^4) + 2729942049541120 \cdot X^3 Y^3 \right) - 5 \cdot 3^7 \cdot 2^{34} \cdot X Y \cdot (X + Y) \cdot (391 X^2 - 12495392 X Y + 391 Y^2) + 31 \cdot 3^7 \cdot 2^{40} \cdot X Y \cdot (X + Y) \cdot (2 X + Y) - 3^9 \cdot 2^{42} \cdot X Y \cdot (X + Y) = 0. \]

The elimination of \( x \) between the pullback \( X = \mathcal{P}_2^{(1)} \) (given by (44)) and the pullback \( Y = \mathcal{P}_2^{(2)} \) also gives the same modular equation (51).

2.9. \( _2F_1([1/4, 3/4], [1], P) \) subcases: walks in the quarter plane

The diagonal of the rational function

\[ \frac{2}{2 + (x + y + z) + x z + 1/2 \cdot xy} = \frac{4}{4 + 2 \cdot (x + y + z) + 2 x z + x y}, \]  

is given by the pullbacked hypergeometric function:

\[ \left( 1 + \frac{3}{4} \cdot x^2 \right)^{-1/4} \cdot _2F_1 \left( \frac{1}{12}, \frac{5}{12}, [1], \frac{27 x^4 \cdot (x^2 + 1)}{(3 x^2 + 4)^3} \right) = _2F_1 \left( \frac{1}{4}, \frac{3}{4}, [1], -x^2 \right), \]  

which is reminiscent of the hypergeometric series number 5 and 15 in Figure 10 of Bostan’s HDR [33]. Such pullbacked hypergeometric function (53) corresponds to the rook walk problems [34, 35, 36].

Thus the diagonal of the rational function corresponding to the simple rescaling \((x, y, z) \rightarrow (\pm \sqrt{-1} x, \pm \sqrt{-1} y, \pm \sqrt{-1} z)\) of (52) namely

\[ R_\pm = \frac{2}{2 + \sqrt{-1} \cdot (x + y + z) - x z - 1/2 \cdot xy} \]  

or the diagonal of the rational function \((R_+ + R_-)/2\) reading

\[ \frac{4 \cdot (4 - x y - 2 x z)}{y^2 x^2 + 4 x^2 y z + 4 x^2 z^2 + 4 x^2 - 8 x z + 4 y^2 + 8 y z + 4 z^2 + 16}. \]

\[ \dagger \text{This result can be also seen in the} \ z \text{variable (see (46)): see the details in Appendix C.} \]
becomes (as a consequence of identity (53)):

\[
(1 - \frac{3}{4} \cdot x^2)^{-1/4} \cdot {}_2F_1 \left( \frac{1}{12}, \frac{5}{12}; 1, \frac{27 x^4 \cdot (1 - x^2)}{(4 - 3x^2)^3} \right) = {}_2F_1 \left( \frac{1}{12}, \frac{3}{4}; 1, x^2 \right).
\]

(56)

**Remark:** \(_2F_1([1/4, 3/4], [1], \mathcal{P})_2\) hypergeometric functions can be also seen as modular forms corresponding to identities with *two* pullbacks related by a modular equation. For example the following identity:

\[
{}_2F_1 \left( \frac{1}{4}, \frac{3}{4}; 1, \frac{x^2}{(2-x)^2} \right) = (\frac{2-x}{2\cdot(1+x)})^{1/2} \cdot {}_2F_1 \left( \frac{1}{4}, \frac{3}{4}; 1, \frac{4x}{(1+x)^2} \right),
\]

(57)

where the two rational pullbacks

\[
A = \frac{4x}{(1+x)^2}, \quad B = \frac{x^2}{(2-x)^2},
\]

(58)

are related by the asymmetrical modular equation:

\[
81 \cdot A^2B^2 - 18AB \cdot (8B + A) + (A^2 + 80 \cdot AB + 64B^2) - 64B = 0.
\]

(59)

The modular equation (59) gives the following expansion for \(B\) seen as an *algebraic series* in \(A\):

\[
B = \frac{1}{64} A^2 + \frac{5}{256} A^3 + \frac{83}{4096} A^4 + \frac{163}{8192} A^5 + \frac{5013}{262144} A^6 + \cdots
\]

(60)

More details are given in Appendix D.

2.10. The generic case: modular forms, with just one rational pullback

The previous pullbacks in the pullbacked \(_2F_1_\) hypergeometric functions can be seen (and should be seen) as *Hauptmoduls* [23]. It is only in certain cases like in sections (2.7) or (2.8) that we encounter the situation underlined by Maier [23] of a representation of a modular form as a pullbacked hypergeometric function with *two* possible *rational pullbacks*, related by a modular equation of *genus zero*. These selected situations are recalled in Appendix E.

Simple examples of modular equations of genus zero with *rational pullbacks* include reductions of the generic \(_2F_1([1/12, 5/12]; [1], \mathcal{P})_2\) hypergeometric function to selected hypergeometric functions like \(_2F_1([1/2, 1/2]; [1], \mathcal{P})_2\), \(_2F_1([1/3, 2/3]; [1], \mathcal{P})_2\), \(_2F_1([1/4, 3/4]; [1], \mathcal{P})_2\), and also [25] \(_2F_1([1/6, 5/6]; [1], \mathcal{P})_2\) (see for instance [26]).

However, in the generic situation corresponding to (13) we have a single hypergeometric function with two different pullbacks \(A\) and \(B\)

\[
{}_2F_1 \left( \frac{1}{12}, \frac{5}{12}; 1, A \right) = G \cdot {}_2F_1 \left( \frac{1}{12}, \frac{5}{12}; 1, B \right),
\]

(61)

with \(G\) an algebraic function of \(x\), and where \(A\) and \(B\) are *related by an algebraic modular equation*, but one of the two pullbacks say \(A\) is a rational function given by

\* At first sight one expects the two pullbacks \([58]_2\) in a relation like (59) to be on the same footing, the *modular equation* between these two pullbacks being *symmetric*: see for instance [22]. This paradox is explained in detail in Appendix D

\[ \text{§ We discard the other root expansion } B = 1 + A + \frac{5}{3} A^2 + \frac{27}{10} A^3 + \frac{91}{18} A^4 + \cdots \text{ since } B(0) \neq 0. \]
Diagonals of rational functions

(12) where \( P_2(x) \) and \( P_4(x) \) are given by (16), (17). The two pullbacks \( A \) and \( B \) are also related by a Schwarzian equation that can be written in a symmetric way in \( A \) and \( B \):

\[
\frac{1}{72} \frac{32 B^2 - 41 B + 36}{B^2 \cdot (B - 1)^2} \cdot \left( \frac{dB}{dx} \right)^2 + \{B, x\} = \frac{1}{72} \frac{32 A^2 - 41 A + 36}{A^2 \cdot (A - 1)^2} \cdot \left( \frac{dA}{dx} \right)^2 + \{A, x\}.
\]

One can rewrite the exact expression (13) in the form

\[
\frac{1}{P_2(x)^{1/4}} \cdot 2F_1 \left( \frac{1}{12}, \frac{5}{12}, [1], 1 - \frac{P_4(x)^2}{P_5(x)^3} \right) = B \cdot 2F_1 \left( \frac{1}{12}, \frac{5}{12}, [1], B \right),
\]

where \( B \) is an algebraic function, and \( B \) is another pullback related to the rational pullback \( A = 1 - \frac{P_4(x)^2}{P_5(x)^3} \) by a modular equation. The pullback \( B \) is an algebraic function. In the generic case, only one of the two pullbacks (63) can be expressed as a rational function: see Appendix E for more details.

3. Eight, nine and ten-parameters generalizations

Adding randomly terms in the denominator of (7) yields diagonals annihilated by minimal linear differential operators of order higher than two: this is what happens when quadratic terms like \( x^2 \), \( y^2 \) or \( z^2 \) are added for example. This leads to irreducible telescopers [31, 32] (i.e. minimal order linear differential operators annihilating the diagonals) of higher orders than the previous order two, or to telescopers [31] of quite high orders that are not irreducible, but factor into many irreducible factors, one of them being of order larger than two.

With the idea of keeping the linear differential operators annihilating the diagonal of order two, we were able to generalize the seven-parameter family (7) by carefully choosing the terms added to the quadratic terms in (7) and still keep the linear differential operator annihilating the diagonal of order two.

3.1. Eight-parameter rational functions giving pullbacked \( 2F_1 \) hypergeometric functions for their diagonals

Adding the cubic term \( x^2 y \) to the denominator of (7) yields the rational function:

\[
R(x, y, z) = \frac{1}{a + b_1 x + b_2 y + b_3 z + c_1 y z + c_2 x z + c_3 x y + d x^2 y}.
\]

After obtaining the diagonal of (64) for several sets of values of the parameters, one can make the educated guess that the diagonal of the rational function (64) has the form

\[
\frac{1}{P_3(x)^{1/4}} \cdot 2F_1 \left( \frac{1}{12}, \frac{5}{12}, [1], 1 - \frac{P_4(x)^2}{P_5(x)^3} \right),
\]

where \( P_3(x) \) and \( P_4(x) \) are two polynomials of degree three and four respectively:

\[
P_3(x) = A_4 x^4 + A_5 x^3 + A_2 x^2 + A_1 x + A_0,
\]

\[
P_3(x) = B_3 x^3 + B_2 x^2 + B_1 x + B_0.
\]
and where the coefficients $A_i$ and $B_j$ are at most quadratic expressions in the parameter $d$ appearing in the denominator of (64). The pullback in (65) has the form
\[
1 - \frac{P_4(x)^2}{P_3(x)^3} = \frac{1728 x^3 P_6}{P_3(x)^3},
\]
where
\[
P_4 = p_4 + 216 \cdot b_3^2 c_2^2 \cdot d^2 \cdot x^4 + d \cdot u_1 \cdot x^4 + a \cdot u_2 \cdot x^3 - 144 \cdot a b_3 c_2^2 d \cdot x^3 - 144 \cdot b_3^2 d^2 \cdot (b_1 c_1 + 4 b_2 c_2 - 2 b_3 c_3) \cdot x^3 + 36 a^2 \cdot (a b_3 c_1 - 2 b_2 b_3^2) \cdot d \cdot x^2,
\]
\[
P_3 = p_2 - 48 \cdot c_1^2 c_2 \cdot d \cdot x^3 + 24 b_3 \cdot (a c_1 - 2 b_2 b_3) \cdot d \cdot x^2,
\]
with
\[
u_1 = 144 \cdot (2 b_1 c_1^3 c_2 - 4 b_2 c_1^2 c_2^2 - b_3 c_1^2 c_2 c_3),
\]
\[
u_2 = 72 \cdot (10 b_2 b_3 c_1 c_2 - a c_1^2 c_2 - 2 b_2 b_3 c_1 c_3),
\]
and where the polynomials $p_2$ and $p_4$ denote the polynomials $P_2(x)$ and $P_4(x)$ given by (16) and (17) in section (2); $p_2$ and $p_4$ correspond to the $d = 0$ limit.

3.2. Nine-parameter rational functions giving pullbacked $\,_2F_1$ hypergeometric functions for their diagonals

Adding now another cubic term $y z^2$ to the denominator of (64)
\[
\frac{1}{a + b_1 x + b_2 y + b_3 z + c_1 y z + c_2 x z + c_3 x y + d x^2 y + e y z^2},
\]
also yields linear differential operator annihilating the diagonal of (71) of order two. After computing the second order linear differential operator annihilating the diagonal of (71) for several values of the parameters with the “HolonomicFunctions” program [29], and, in a second step, obtaining their pullbacked hypergeometric solutions using the maple command “hypergeometricsols” [30], we find that the diagonal of the rational function (71) has the form
\[
\frac{1}{P_4(x)^{1/4}} \cdot \,_2F_1 \left( \left[ \frac{1}{12}, \frac{5}{12} \right], [1], 1 - \frac{P_6(x)^2}{P_4(x)^3} \right),
\]
where $P_4(x)$ and $P_6(x)$ are two polynomials of degree four and six respectively:
\[
P_4(x) = p_2 + 16 \cdot a^2 \cdot e^2 \cdot x^4 - 16 \left( 3 \cdot c_2 \cdot (c_1^2 \cdot d + c_2^2 \cdot e) + (b_1 c_1 + b_2 c_2 - 14 b_3 c_3) \cdot d e \right) \cdot x^3 + 8 \cdot (3 a b_3 c_1 d + 3 a b_1 c_3 e - a^2 d e - 6 b_2 b_3^2 d - 6 b_2 b_3^2 e) \cdot x^2,
\]
and
\[
\begin{align*}
P_6(x) &= p_4 - 12 \cdot a^4 d e \cdot x^2 \\
&\quad + 36 \cdot a^2 \left( b_3 \cdot (ac_1 - 2b_2 b_3) \cdot d + b_1 \cdot (ac_3 - 2b_1 b_2) \cdot e \right) \cdot x^2 \\
&\quad - 72 \cdot a c_1 \cdot (ac_1 c_2 - 10b_2 b_3 c_3 + 2b_3^2 c_3) \cdot d \cdot x^3 \\
&\quad - 72 \cdot a c_3 \cdot (ac_2 c_3 - 10b_1 b_2 c_2 + 2b_1^2 c_1) \cdot e \cdot x^3 \\
&\quad - 144 \cdot b_2 b_3^2 \cdot (b_1 c_1 + 4b_2 c_2 - 2b_3 c_3) \cdot d \cdot x^3 \\
&\quad - 144 \cdot b_2 b_3^2 \cdot (b_3 c_3 + 4b_2 c_2 - 2b_1 c_1) \cdot e \cdot x^3 \\
&\quad - 144 \cdot a b_1 b_3 \cdot (c_1^2 \cdot d + c_3^2 \cdot e) \cdot x^3 \\
&\quad + 24 \cdot a (a b_1 c_3 + a b_1 c_1 - 20a b_2 c_2 + 30b_1 b_2 b_3) \cdot d \cdot e \cdot x^3 \\
&\quad + 216 \cdot (b_3^2 c_3^2 \cdot d^2 + b_3^2 c_3^2 \cdot e^2) \cdot x^4 \\
&\quad - 144 \cdot c_1^2 c_2 \cdot (b_3 c_3 + 4b_2 c_2 - 2b_1 c_1) \cdot d \cdot x^4 \\
&\quad - 144 \cdot c_3^2 c_2 \cdot (b_1 c_1 + 4b_2 c_2 - 2b_3 c_3) \cdot e \cdot x^4 \\
&\quad + 48 \cdot a^2 d^2 \cdot e^2 \cdot x^4 + 96 \cdot (b_1^2 c_1^2 + b_3^2 c_3^2 + 22b_2^2 c_2^2) \cdot d \cdot e \cdot x^4 \\
&\quad - 144 \cdot \left( (a b_3 c_1 + 4b_2 b_3^2) \cdot d + (a b_1 c_3 + 4b_2 b_1^2) \cdot e \right) \cdot d \cdot e \cdot x^4 \\
&\quad + 48 \cdot (b_1 b_3 c_1 c_3 + 15a c_1 c_2 c_3 - 20b_1 b_2 c_1 c_2 - 20b_2 b_3 c_2 c_3) \cdot d \cdot e \cdot x^4 \\
&\quad + 96 \cdot (b_1 c_1 + 22b_2 c_2 + b_3 c_3) \cdot d^2 \cdot e^2 \cdot x^5 \\
&\quad - 576 c_2 \cdot (c_1^2 \cdot e + c_3^2 \cdot d) \cdot d e \cdot x^5 \\
&\quad - 64 \cdot d^3 \cdot e^3 \cdot d^5,
\end{align*}
\]

where the polynomials \( p_2 \) and \( p_4 \) are the polynomials \( P_2(x) \) and \( P_4(x) \) of degree two and four in \( x \) given by (16) and (17) in section (2); \( p_2 \) and \( p_4 \) correspond to the \( d = e = 0 \) limit.

Note that the \( d \leftrightarrow e \) symmetry corresponds to keeping \( c_2 \) fixed, but changing \( c_1 \leftrightarrow c_3 \) (or equivalently \( y \) fixed, \( x \leftrightarrow z \)).

**Remark 1:** The nine-parameter family (71) singles out \( x \) and \( y \), but of course, similar families that single out \( x \) and \( z \), or single out \( y \) and \( z \) exist, with similar results (that can be obtained permuting the three variables \( x, y \) and \( z \)).

**Remark 2:** Note that the simple symmetries arguments displayed in section (2.6) for the seven-parameter family straightforwardly generalize for this nine-parameter family. The pullback \( \mathcal{H} \) in (72) verifies (as it should)
\[
\begin{align*}
\mathcal{H}(a, \lambda_1 \cdot b_1, \lambda_2 \cdot b_2, \lambda_3 \cdot b_3, \lambda_2 \lambda_3 \cdot c_1, \lambda_1 \lambda_3 \cdot c_2, \lambda_1 \lambda_2 \cdot c_3, \\
\lambda_1^2 \lambda_2 \cdot d, \lambda_3^2 \lambda_2 \cdot e, \frac{x}{\lambda_1 \lambda_2 \lambda_3})
&= \mathcal{H}(a, b_1, b_2, b_3, c_1, c_2, c_3, d, e, x),
\end{align*}
\]
and:
\[
\begin{align*}
\mathcal{H}(\lambda \cdot a, \lambda \cdot b_1, \lambda \cdot b_2, \lambda \cdot b_3, \lambda \cdot c_1, \lambda \cdot c_2, \lambda \cdot c_3, \lambda \cdot d, \lambda \cdot e, x)
&= \mathcal{H}(a, b_1, b_2, b_3, c_1, c_2, c_3, d, e, x).
\end{align*}
\]
3.3. Ten-parameter rational functions giving pullbacked \( _2F_1 \) hypergeometric functions for their diagonals

Adding the three cubic terms\(^\dagger\) \( x^2 y, \ y^2 z \) and \( z^2 x \) to the denominator of (7) we get the rational function:

\[
R(x, y, z) = \frac{1}{a + b_1 x + b_2 y + b_3 z + c_1 y z + c_2 x z + c_3 x y + d_1 x^2 y + d_2 y^2 z + d_3 z^2 x}.
\]

While (77) is not a generalization of (71), it is a generalization of (64).

After computing the second order linear differential operator annihilating the diagonal of (77) for several values of the parameters with the “HolonomicFunctions” program [29], and, in a second step, their pullbacked hypergeometric solutions using “hypergeometricsols” [30], we find that the diagonal of the rational function (77) has the experimentally observed form:

\[
\frac{1}{P_3(x)^{1/4}} \cdot _2F_1\left(\frac{1}{12}, \frac{5}{12}, 1, 1 - \frac{P_6(x)^2}{P_3(x)^3}\right).
\]  

Furthermore, the pullback in (78) is seen to be of the form:

\[
1 - \frac{P_6(x)^2}{P_3(x)^3} = \frac{1728 x^3 \cdot P_9}{P_3(x)^3}.
\]

The polynomial \( P_3(x) \) reads

\[
P_3(x) = p_2 - 24 \cdot \left(9 \cdot a \cdot d_1 d_2 d_3 - 6 \cdot (b_1 c_3 \cdot d_2 d_3 + b_2 c_1 \cdot d_1 d_3 + b_3 c_2 \cdot d_1 d_2) + 2 \cdot (c_1^2 c_2 d_1 + c_1 c_2^2 d_3 + c_2^2 c_3 d_2)\right) \cdot x^3
\]  

\[
+ 24 \cdot \left(a \cdot (b_1 c_2 d_2 + b_2 c_3 d_3 + b_3 c_1 d_1) - 2 \cdot (b_1^2 b_3 d_2 + b_1 b_2^2 d_3 + b_2 b_3^2 d_1)\right) \cdot x^2,
\]

where \( p_2 \) is the polynomial \( P_3(x) \) of degree two in \( x \) given by (16) in section (2): \( p_2 \) corresponds to the \( d_1 = d_2 = d_3 = 0 \) limit. The expression of the polynomial \( P_6(x) \) is more involved. It reads:

\[
P_6(x) = p_4 + \Delta_6(x),
\]

where \( p_4 \) is the polynomial \( P_4(x) \) of degree four in \( x \) given by (17) in section (2). The expression of polynomial \( \Delta_6(x) \) of degree six in \( x \) is quite large and is given in Appendix F.

Remark 1: A set of results and subcases (sections (3.3.2) and (3.3.3)), can be used to “guess” the general exact expressions of the polynomials \( P_3(x) \) and \( P_6(x) \) in (78) for the ten-parameters family (77). From the subcase \( d_3 = 0 \) of section (3.3.1) below, it is easy to see that one can deduce similar exact results for \( d_1 = 0 \) or \( d_2 = 0 \): it just amounts to performing some cyclic transformation \( x \to y \to z \to x \) which corresponds to transformation \( b_1 \to b_2 \to b_3 \to b_1, \ c_1 \to c_2 \to c_3 \to c_1, \ d_1 \to d_2 \to d_3 \to d_1 \). One can see \( P_3 \) and \( P_6(x) \) as \( p_2 \) and \( p_4 \) given by (16) and (17) plus some corrections given, in Appendix G, by (G.1) and (G.2) for \( d_3 = 0 \), and similar corrections\(^\dagger\) for \( d_1 = 0 \) and \( d_2 = 0 \), plus corrections of the form \( d_1 d_2 d_3 \times \) something. These last terms are the most difficult to get. We already know some of

\[^\dagger\text{An equivalent family of ten-parameter rational functions amounts to adding } xy^2, \ yz^2 \text{ and } zx^2.\]

\[^\dagger\text{Taking care of the double counting!}\]
these terms from (88) and (89) in section (3.3.2) below. Furthermore, the symmetry constraints (83) and (82) below, as well as other constraints corresponding to the symmetric subcase of section (3.3.3), give additional constraints on the kind of allowed final correction terms.

Remark 2: Note, again, that the simple symmetries arguments displayed in section (2.6) for the seven-parameter family straightforwardly generalize for this ten-parameter family. The $\mathcal{H}$ pullback (79) in (78) verifies (as it should):

$$\mathcal{H}(a, \lambda_1 b_1, \lambda_2 b_2, \lambda_3 b_3, \lambda_2 \lambda_3 c_1, \lambda_4 \lambda_3 c_2, \lambda_1 \lambda_2 c_3, \lambda_1^2 \lambda_2 \lambda_3, \frac{x}{\lambda_1 \lambda_2 \lambda_3}) = \mathcal{H}(a, b_1, b_2, b_3, c_1, c_2, c_3, d_1, d_2, d_3, x),$$

and:

$$\mathcal{H}(\lambda \cdot a, \lambda \cdot b_1, \lambda \cdot b_2, \lambda \cdot b_3, \lambda \cdot c_1, \lambda \cdot c_2, \lambda \cdot c_3, \lambda \cdot d_1, \lambda \cdot d_2, \lambda \cdot d_3, x) = \mathcal{H}(a, b_1, b_2, b_3, c_1, c_2, c_3, d_1, d_2, d_3, x).$$

Remark 3: Do note that adding arbitrary sets of cubic terms yields telescopes [31] of order larger than two: the corresponding diagonals are no longer pulled back $\, _2F_1$ hypergeometric functions.

Let us just now focus on simpler subcases whose results are easier to obtain than in the general case (77).

3.3.1. Noticeable subcases of (77): a nine-parameter rational function

Instead of adding three cubic terms, let us add two cubic terms. This amounts to restricting the rational function (77) to the $d_3 = 0$ subcase

$$\mathcal{R}(x, y, z) = \frac{1}{a + b_1 x + b_2 y + b_3 z + c_1 y z + c_2 x z + c_3 x y + d_1 x^2 y + d_2 y^2 z},$$

which cannot be reduced to the nine parameter family (71) even if it looks similar. The diagonal of the rational function (84) has the experimentally observed form

$$\frac{1}{P_3(x)^3} \cdot _2F_1\left(1, \frac{5}{12}, 1, \frac{P_3(x)^2}{P_3(x)^3}\right),$$

where $P_3(x)$ and $P_5(x)$ are two polynomials of degree respectively three and five in $x$. Furthermore the pullback in (85) has the form:

$$1 - \frac{P_3(x)^2}{P_5(x)^3} = \frac{1728 x^3 \cdot P_7}{P_3(x)^3}.$$

The two polynomials $P_3(x)$ and $P_5(x)$ are given in Appendix G.

3.3.2. Cubic terms subcase of (77)

A simple subcase of (77) corresponds to $b_1 = b_2 = b_3 = c_1 = c_2 = c_3 = 0$, namely to the rational function:

$$R(x, y, z) = \frac{1}{a + d_1 \cdot x^2 y + d_2 \cdot y^2 z + d_3 \cdot z^2 x}.$$
whose diagonal reads

\[ 2F1\left(\frac{1}{3}, \frac{2}{3}, [1], -27 \cdot \frac{d_1 d_2 d_3}{a^3} \cdot x^3\right) \]

\[ = \left(1 - 216 \cdot \frac{d_1 d_2 d_3}{a^3} \cdot x^3\right)^{-1/4} \cdot 2F1\left(\frac{1}{12}, \frac{5}{12}, [1], 1 - \frac{P_6(x)^2}{P_3(x)^3}\right), \quad (87) \]

with:

\[ P_3(x) = -216 \cdot a \cdot d_1 \cdot d_2 \cdot d_3 \cdot x^3 + a^4, \quad (88) \]

\[ P_6(x) = -5832 \cdot d_1^2 \cdot d_2^2 \cdot d_3^2 \cdot x^6 + 540 \cdot a^3 \cdot d_1 \cdot d_2 \cdot d_3 \cdot x^3 + a^6. \quad (89) \]

Relation (87) actually corresponds to the hypergeometric identities:

\[ 2F1\left(\frac{1}{3}, \frac{2}{3}, [1], -27 X\right) \]

\[ = \left(1 - 216 X\right)^{-1/4} \cdot 2F1\left(\frac{1}{12}, \frac{5}{12}, [1], -\frac{1728 X \cdot (1 + 27 X^3)}{(1 - 216 X)^3}\right), \quad (90) \]

\[ = \left(1 - 216 \cdot X\right)^{-1/4} \cdot 2F1\left(\frac{1}{12}, \frac{5}{12}, [1], 1 - \frac{(1 + 540 X - 5832 X^2)^2}{(1 - 216 X)^3}\right). \]

3.3.3. A symmetric subcase of (77)

Let us also consider another simple very symmetric subcase of (77). For \( b_1 = b_2 = b_3 = b, c_1 = c_2 = c_3 = c, d_1 = d_2 = d_3 = d \), the diagonal reads\(^\dagger\)

\[ \frac{1}{a - 6 d \cdot x} \cdot 2F1\left(\frac{1}{3}, \frac{2}{3}, [1], P\right), \quad (91) \]

where the pullback \( P \) reads:

\[ P = -\frac{27 \cdot \left(a^2 d - a b c + b^3 + (c^3 - 3 b c d - 3 a d^2) \cdot x + 9 a d^2 \cdot x^2\right)}{(a - 6 d \cdot x)^3}. \quad (92) \]

At first sight the hypergeometric result (91) with the pullback (92) does not seem to be in agreement with the hypergeometric result (87) of section (3.3.2). In fact these two results are in agreement as a consequence of the hypergeometric identity:

\[ \frac{1}{1 - 6 X} \cdot 2F1\left(\frac{1}{3}, \frac{2}{3}, [1], -\frac{27 \cdot X \cdot (1 - 3 X + 9 X^2)}{(1 - 6 X)^3}\right) \]

\[ = 2F1\left(\frac{1}{3}, \frac{2}{3}, [1], -27 \cdot X^3\right) \quad \text{with:} \quad X = \frac{d \cdot x}{a}. \quad (93) \]

This hypergeometric result (87) can also be rewritten in the form (78) where the two polynomials \( P_3(x) \) and \( P_6(x) \) read respectively:

\[ P_3(x) = -72 \cdot d \cdot (3 a d^2 - 6 b c d + 2 c^3) \cdot x^3 + 24 \cdot (3 a b c d + a c^3 - 6 b^2 d) \cdot x^2 + 24 \cdot a b \cdot (ac - b^2) \cdot x + a^4, \quad (94) \]

\[ P_6(x) = -5832 \cdot d^3 \cdot x^6 + 3888 \cdot c d^3 \cdot (3 b d - c^2) \cdot x^5 - 216 \cdot (18 a b c d^2 + 18 b^2 c d^2 - 12 a c d^2 - 9 b^2 c^2 d^2 + 6 b c^4 d - c^6) \cdot x^4 + 108 \cdot (5 a^3 d^2 - 18 a^2 b c d^2 - 2 a^2 c^3 d + 12 a b^2 c^2 d + 24 a b^3 d^2 - 4 a b c^4 + 12 b c^3 d + 4 b^3 c^3) \cdot x^3 + 36 \cdot (3 a^3 b c d - 6 b^2 c^3 d + a^2 c^3 + 6 a^2 b^2 c^2 - 12 a b^2 c + 6 b^3) \cdot x^2 - 36 \cdot a^3 b \cdot (ac - b^2) \cdot x + a^6. \quad (95) \]

\(^\dagger\) Note that trying to mix the two previous subcases imposing \( b_1 = b_2 = b_3 = b, c_1 = c_2 = c_3 = c \) with \( d_1, d_2, d_3 \) no longer equal, do not yield a \( 2F1([1/3, 2/3], [1], P) \) hypergeometric function.
4. Transformation symmetries of the diagonals of rational functions

The previous results can be expanded through symmetry considerations.

We are first going to see that performing monomial transformations on each of the previous (seven-parameter, eight, nine or ten-parameter) rational functions yields an infinite number of rational functions whose diagonals are pullbacked $_2F_1$ hypergeometric functions.

4.1. $(x, y, z) \rightarrow (x^n, y^n, z^n)$ symmetries

We have a first remark: once we have an exact result for a diagonal, we immediately get another diagonal by changing $(x, y, z)$ into $(x^n, y^n, z^n)$ for any positive integer $n$ in the rational function. As a result we obtain a new expression for the diagonal changing $x$ into $x^n$.

A simple example amounts to revisiting the fact that the diagonal of (54) given below is the hypergeometric function (36). Changing $(x, y, z)$ into $(8x^2, 8y^2, 8z^2)$ in (54), one obtains the pullbacked $_2F_1$ hypergeometric function number 5 or 15 in Figure 10 of Bostan’s HDR [33] (see also [34, 35, 36])

\[ _2F_1\left(\frac{1}{4}, \frac{3}{4}, 1, 64x^4\right), \]

(96)

can be seen as the diagonal of

\[ \frac{1}{2 + 8 \sqrt{x^2 + y^2 + z^2}} - 64 x^2 z^2 - 32 x^2 y^2, \]

(97)

which is tantamount to saying that the transformation $(x, y, z) \rightarrow (x^n, y^n, z^n)$ is a symmetry.

4.2. Monomial transformations on rational functions

More generally, let us consider the monomial transformation

\[ (x, y, z) \rightarrow M(x, y, z) = \left(x_M, y_M, z_M\right) \]

\[ = \left(x^{A_1} \cdot y^{A_2} \cdot z^{A_3}, x^{B_1} \cdot y^{B_2} \cdot z^{B_3}, x^{C_1} \cdot y^{C_2} \cdot z^{C_3}\right), \]

(98)

where the $A_i$’s, $B_i$’s and $C_i$’s are positive integers such that $A_1 = A_2 = A_3$ is excluded (as well as $B_1 = B_2 = B_3$ as well as $C_1 = C_2 = C_3$), and that the determinant of the $3 \times 3$ matrix

\[ \begin{bmatrix}
A_1 & B_1 & C_1 \\
A_2 & B_2 & C_2 \\
A_3 & B_3 & C_3
\end{bmatrix}, \]

(99)

is not equal to zero†‡, and that:

\[ A_1 + B_1 + C_1 = A_2 + B_2 + C_2 = A_3 + B_3 + C_3. \]

(100)

We will denote by $n$ the integer in these three equal† sums (100): $n = A_i + B_i + C_i$. The condition (100) is introduced in order to impose that the product $\Phi$ of $x_M y_M z_M$ is an integer power of the product $xyz$: $x_M y_M z_M = (xyz)^n$.

†† We want the rational function $\tilde{R} = R(M(x, y, z))$ deduced from the monomial transformation (98) to remain a rational function of three variables and not of two, or one, variables.

‡ For $n = 1$ the $3 \times 3$ matrix (99) is stochastic and transformation (98) is a birational transformation.

$\Phi$ Recall that taking the diagonal of a rational function of three variables extracts, in the multi-Taylor expansion (5), only the terms that are $n$-th power of the product $xyz$. 

Diagonals of rational functions

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If we take a rational function $R(x, y, z)$ in three variables and perform a monomial transformation $(98) (x, y, z) \rightarrow M(x, y, z)$, on the rational function $R(x, y, z)$, we get another rational function that we denote by $\tilde{R} = R(M(x, y, z))$. Now the diagonal of $\tilde{R}$ is the diagonal of $R(x, y, z)$ where we have changed $x$ into $x^n$:

$$\Phi(x) = \text{Diag}(R(x, y, z)), \quad \text{Diag}(\tilde{R}(x, y, z)) = \Phi(x^n). \quad (101)$$

A demonstration of this result is sketched in Appendix H.

From the fact that the diagonal of the rational function

$$\frac{1}{1 + x + y + z + 3 \cdot (xy + yz + xz)}; \quad (102)$$

is the hypergeometric function

$$2F_1\left(\frac{1}{3}, \frac{2}{3}, [1], 27x \cdot (2 - 27x)\right); \quad (103)$$

one deduces immediately that the diagonal of the rational function (104) transformed by the monomial transformation $(x, y, z) \rightarrow (z, x^2y, yz)$

$$\frac{1}{1 + yz + x^2y + 3 \cdot (yz^2 + x^2yz + x^2y^2z)}, \quad (104)$$

is the pullbacked hypergeometric function

$$2F_1\left(\frac{1}{3}, \frac{2}{3}, [1], 27x^2 \cdot (2 - 27x^2)\right); \quad (105)$$

which is (103) where $x \rightarrow x^2$.

To illustrate the point further, from the fact that the diagonal of the rational function

$$\frac{1}{1 + x + y + z + 3xy + 5yz + 7xz}; \quad (106)$$

is the hypergeometric function

$$\frac{1}{(2712x^2 - 96x + 1)^{1/4}} \times 2F_1\left(\frac{1}{12}, \frac{5}{12}, [1], 1 - \frac{(2381400x^4 - 181440x^3 + 7524x^2 - 144x + 1)^2}{(2712x^2 - 96x + 1)^3}\right); \quad (107)$$

one deduces immediately that the diagonal of the rational function (106) transformed by the monomial transformation $(x, y, z) \rightarrow (xz, x^2y, y^2z^2)$

$$\frac{1}{1 + xz + x^2y + y^2z^2 + 3x^2y^3 + 5xy^2z^3 + 7xz^4yz}; \quad (108)$$

is the hypergeometric function

$$\frac{1}{(2712x^6 - 96x^5 + 1)^{1/4}} \times 2F_1\left(\frac{1}{12}, \frac{5}{12}, [1], 1 - \frac{(2381400x^{12} - 181440x^9 + 7524x^6 - 144x^3 + 1)^2}{(2712x^6 - 96x^5 + 1)^3}\right); \quad (109)$$

which is nothing but (107) where $x$ has been changed into $x^3$.

We have the same result for more involved rational functions and more involved monomial transformations.
4.3. More symmetries on diagonals

Other transformation symmetries of the diagonals include the function-dependent rescaling transformation

\[(x, y, z) \mapsto \left( \frac{x}{1 + 7xyz}, \frac{y}{1 + 7xyz}, \frac{z}{1 + 7xyz} \right), \]

where \( F(xyz) \) is a rational function\(^\dagger \) of the product of the three variables \( x, y \) and \( z \). Under such a transformation the previous diagonal \( \Delta(x) \) becomes \( \Delta(x \cdot F(x)^3) \).

For instance, changing \((x, y, z) \mapsto \left( \frac{x}{1 + 7xyz}, \frac{y}{1 + 7xyz}, \frac{z}{1 + 7xyz} \right)\), the rational function

\[ \frac{1}{1 - x - y - z + yz}, \]

whose diagonal is \( \sum_1 \left(1/2, 1/2, 1, 16x \right), \) becomes the rational function

\[ \frac{1}{1 - x - y - z + yz + 14xyz - 7x^2yz - 7xy^2z - 7xy^2z + 49x^2y^2z^2}, \]

which has the following diagonal:

\[ \text{\textit{\(2F1\left(1/2, 1/2, 1, \frac{16x}{(1 + 7x)^3} \right) \)}} \]

\[ = 1 + 4x - 48x^2 + 64x^3 + 3024x^4 \]
\[ - 13524x^5 - 245196x^6 + 1933152x^7 + 21288192x^8 - 263440460x^9 \]
\[ - 1758664568x^{10} + 34575759792x^{11} + \ldots \]

To illustrate the point further take

\[(x, y, z) \mapsto \left( x \cdot F, y \cdot F, z \cdot F \right), \quad \text{with:} \]

\[ F = \frac{1 + 2xyz}{1 + 3xyz + 5x^2y^2z^2} = \Phi(xyz), \]

where

\[ \Phi(x) = \frac{1 + 2x}{1 + 3x + 5x^2}. \]

the rational function

\[ \frac{1}{1 + x + y + z + yz + xz + xy}, \]

whose diagonal is \( \sum_1 \left(1/3, 2/3, 1, -27x^2 \right), \) becomes the rational function \( P(x, y, z)/Q(x, y, z) \), where the numerator \( P(x, y, z) \) and the denominator \( Q(x, y, z) \), read respectively:

\[ P(x, y, z) = (1 + 3xyz + 5x^2y^2z^2)^3, \]
\[ Q(x, y, z) = 25x^4y^4z^4 + 10 \cdot (x^4y^3z^3 + x^3y^4z^3 + x^2y^3z^4) + 30x^3y^3z^3 \]
\[ + 4 \cdot (x^3y^3z^2 + x^3y^2z^3 + x^2y^3z^2) + 11 \cdot (3x^3y^2z^2 + x^3y^2z^2 + x^3y^2z^2) \]
\[ + 19x^2y^2z^2 + 4 \cdot (x^2y^2z + x^2y^2z + x^2y^2z) + 5 \cdot (x^2yz + xy^2z + x^2z^2) \]
\[ + 6xyz + x + y + z + 1. \]

\( \dagger \) More generally one can imagine that \( F(xyz) \) is the series expansion of an algebraic function.
The diagonal of this last rational function is equal to:

\[
2F_1\left(\left[\frac{1}{3}, \frac{2}{3}\right], [1], -27 \cdot \left(x \cdot \Phi(x)^3\right)^2\right) = 2F_1\left(\left[\frac{1}{3}, \frac{2}{3}\right], [1], -27x^2 \cdot \left(\frac{1 + 2x}{1 + 3x + 5x^2}\right)^6\right). \tag{121}
\]

Let us give a final example: let us consider again the rational function (106) whose diagonal is (107), and let us consider the same function-rescaling transformation (115) with (116). One finds that the diagonal of the rational function

\[
1 + F \cdot x + F \cdot y + F \cdot z + 3 \cdot F^2 \cdot xy + 5 \cdot F^2 \cdot yz + 7 \cdot F^2 \cdot xz;
\tag{122}
\]

is the hypergeometric function

\[
\frac{1}{\left(2712 x^2 \Phi(x)^6 - 96 x \Phi(x)^3 + 1\right)^{1/4}} \times 2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], 1 - \mathcal{H}\right), \tag{123}
\]

where the pullback \(1 - \mathcal{H}\) reads:

\[
1 - \left(\frac{2381400 x^4 \Phi(x)^{12} - 181440 x^3 \Phi(x)^9 + 7524 x^2 \Phi(x)^6 - 144 x \Phi(x)^3 + 1}{2712 x^2 \Phi(x)^6 - 96 x \Phi(x)^3 + 1}\right)^{1/3}.
\]

The pullbacked hypergeometric function (124) is nothing but (107) where \(x\) has been changed into \(x \Phi(x)^3\).

A demonstration of these results is sketched in Appendix I.

Thus for each rational function belonging to one of the seven, eight, nine or ten parameters families of rational functions yielding a pullbacked \(2F_1\) hypergeometric function one can deduce from the transformations (110) an infinite number of other rational functions, with denominators of degree much higher than two or three.

One can combine these two sets of transformations, the monomial transformations (98) and the function-dependent rescaling transformations (110), thus yielding from each of the (seven, eight, nine or ten parameters) rational functions of the paper an infinite number of rational functions of quite high degree yielding pullbacked \(2F_1\) hypergeometric (modular form) exact results for their diagonals.

5. Conclusion

We found here that a seven-parameter rational function of three variables with a numerator equal to one and a polynomial denominator of degree two at most, can be expressed as a pullbacked \(2F_1\) hypergeometric function. We generalized that result to eight, then nine and ten parameters, by adding specific cubic terms. We focused on subcases where the diagonals of the corresponding rational functions are pullbacked \(2F_1\) hypergeometric function with two possible rational function pullbacks algebraically related by modular equations, thus obtaining the result that the diagonal is a modular form\(^\dagger\).

We have finally seen that simple monomial transformations, as well as a simple function rescaling of the three (resp. \(N\)) variables, are symmetries of the diagonals of rational functions of three (resp. \(N\)) variables. Consequently each of our previous families of rational functions, once transformed by these symmetries yield an infinite

\(^\dagger\) Differently from the usual definition of modular forms in the \(\tau\) variables.
number of families of rational functions of three variables (of higher degree) whose
diagonals are also pullbacked \( _2F_1 \) hypergeometric functions and, in fact, modular
forms.

Since diagonals of rational functions emerge naturally in integrable lattice
statistical mechanics and enumerative combinatorics, exploring the kind of exact
results we obtain for diagonals of rational functions (modular forms, Calabi-Yau
operators, pullbacked \( nF_{n-1} \) hypergeometric functions, ...) is an important systematic
work to be performed to provide results and tools in integrable lattice statistical
mechanics and enumerative combinatorics.

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Appendix A. Simple symmetries of the diagonal of the rational function
(7)

Let us recall the pullbacks (24) in section (2.6), that we denote \( P_1 \).

Appendix A.1. Overall parameter symmetry

The seven parameters are defined up to an overall parameter (they must be seen as
homogeneous variables). Changing \((a, b_1, b_2, b_3, c_1, c_2, c_3)\) into \((\lambda \cdot a, \lambda \cdot b_1, \lambda \cdot b_2, \\
\lambda \cdot b_3, \lambda \cdot c_1, \lambda \cdot c_2, \lambda \cdot c_3)\) the rational function \( R \) given by (7) and its diagonal \( \text{Diag}(R) \)
are changed into \( R/\lambda \) and \( \text{Diag}(R)/\lambda \). It is thus clear that the previous pullbacks
(24), which totally “encode” the exact expression of the diagonal as a pullbacked
hypergeometric function, must be invariant under this transformation. This is actually
the case:

\[
P_1(\lambda \cdot a, \lambda \cdot b_1, \lambda \cdot b_2, \lambda \cdot b_3, \lambda \cdot c_1, \lambda \cdot c_2, \lambda \cdot c_3, x) \\
= P_1(a, b_1, b_2, b_3, c_1, c_2, c_3, x). \tag{A.1}
\]

This result corresponds to the fact that \( P_2(x) \) (resp. \( P_4(x) \)) is a \textit{homogeneous polynomial}
in the seven parameters \( a, b_1, \cdots, c_1, \cdots \) of degree two (resp. four).

Appendix A.2. Variable rescaling symmetry

On the other hand, the rescaling of the three variables \((x, y, z)\) in (7), \((x, y, z) \rightarrow \\
(\lambda_1 \cdot x, \lambda_2 \cdot y, \lambda_3 \cdot z)\) is a change of variables that is compatible with the operation of
taking the diagonal of the rational function \( R \).

When taking the diagonal and performing this change of variables, the monomials
in the multi-Taylor expansion of (7) transform as:

\[
a_{m,n,p} \cdot x^m y^n z^p \rightarrow a_{m,n,p} \cdot \lambda_1^m \cdot \lambda_2^n \cdot \lambda_3^p \cdot x^m y^n z^p. \tag{A.2}
\]
Taking the diagonal yields
\[a_{m,m,m} \cdot x^m \longrightarrow a_{m,m,m} \cdot (\lambda_1 \lambda_2 \lambda_3)^m \cdot x^m.\] (A.3)

Therefore it amounts to changing \(x \rightarrow \lambda_1 \lambda_2 \lambda_3 \cdot x\). With that rescaling \((x, y, z) \rightarrow (\lambda_1 \cdot x, \lambda_2 \cdot y, \lambda_3 \cdot z)\) the diagonal of the rational function remains invariant if one changes the seven parameters as follows:

\[(a, \ b_1, \ b_2, \ b_3, \ c_1, \ c_2, \ c_3) \longrightarrow (a, \ \lambda_1 \cdot b_1, \ \lambda_2 \cdot b_2, \ \lambda_3 \cdot b_3, \ \lambda_2 \lambda_3 \cdot c_1, \ \lambda_1 \lambda_3 \cdot c_2, \ \lambda_1 \lambda_2 \cdot c_3).\] (A.4)

One deduces that the pullbacks (24) verify:

\[\mathcal{P}_1(a, \ \lambda_1 \cdot b_1, \ \lambda_2 \cdot b_2, \ \lambda_3 \cdot b_3, \ \lambda_2 \lambda_3 \cdot c_1, \ \lambda_1 \lambda_3 \cdot c_2, \ \lambda_1 \lambda_2 \cdot c_3, \ \frac{x}{\lambda_1 \lambda_2 \lambda_3}) = \mathcal{P}_1(a, \ b_1, \ b_2, \ b_3, \ c_1, \ c_2, \ c_3, \ x).\] (A.5)

**Appendix B. Comment on \(\, \!_2 \! \! \!_F \!_1([1/3, 2/3], [1], \mathcal{P})\) as a modular form**

From identity (28) of section (2.7)

\[\, \!_2 \! \! \!_F \!_1 \left( \frac{1}{3}, \frac{2}{3}, [1], \frac{z}{z + 27} \right) = \left( \frac{9 \cdot (z + 27)}{z + 243} \right)^{1/4} \cdot \, \!_2 \! \! \!_F \!_1 \left( \frac{1}{12}, \frac{5}{12}, \frac{1728 z^3}{(z + 27) \cdot (z + 243)^3} \right)\] (B.1)

\[= \left( \frac{1/9 \cdot (z + 27)}{z + 3} \right)^{1/4} \cdot \, \!_2 \! \! \!_F \!_1 \left( \frac{1}{12}, \frac{5}{12}, \frac{1728 z}{(z + 27) \cdot (z + 3)^3} \right),\] (B.2)

it is tempting to imagine that the first hypergeometric function (B.1) is related to itself with \(z \rightarrow 729/z\), namely that

\[\, \!_2 \! \! \!_F \!_1 \left( \frac{1}{3}, \frac{2}{3}, [1], \frac{27}{z + 27} \right) = \, \!_2 \! \! \!_F \!_1 \left( \frac{1}{3}, \frac{2}{3}, [1], 1 - \frac{z}{z + 27} \right),\] (B.4)

is related to:

\[\, \!_2 \! \! \!_F \!_1 \left( \frac{1}{3}, \frac{2}{3}, [1], \frac{27}{729/z + 27} \right) = \, \!_2 \! \! \!_F \!_1 \left( \frac{1}{3}, \frac{2}{3}, [1], \frac{z}{z + 27} \right).\] (B.5)

This is the case, since (B.4) and (B.5) are solutions of the same linear ODE, but this does not mean that one can deduce an identity on the different pullbacks (B.4) and (B.5): the relation between these two hypergeometric functions (B.4) and (B.5) corresponds to a connection matrix [37]. A direct identity on \(\, \!_2 \! \! \!_F \!_1([1/3, 2/3], [1], \mathcal{P})\), does however exist:

\[\, \!_2 \! \! \!_F \!_1 \left( \frac{1}{3}, \frac{2}{3}, [1], x^3 \right) = \frac{1}{1 + 2x} \cdot \, \!_2 \! \! \!_F \!_1 \left( \frac{1}{3}, \frac{2}{3}, [1], \frac{9x \cdot (1 + x + x^2)}{(1 + 2x)^3} \right).\] (B.6)

\(\dagger\) It corresponds to a trivial pullback change \(p = z/(z + 27) \rightarrow 1 - p\).
Identity (B.6) corresponds\(^\dagger\) to the identity on \(_2F_1([1/3, 1/3], [1], P)\):
\[
_2F_1\left([\frac{1}{3}, \frac{1}{3}], [1], -\frac{x^3}{1-x^3}\right) = \left(\frac{1 + x + x^2}{(1-x)^2}\right)^{1/3} \cdot _2F_1\left([\frac{1}{3}, \frac{1}{3}], [1], -\frac{9x \cdot (1 + x + x^2)}{(1-x)^3}\right). \tag{B.7}
\]

**Appendix C. Comments on the \(\tau \rightarrow 4\tau\) modular equation (51).**

The fact that in section (2.8), the three Hauptmoduls \((44), (47)\) and \((48)\) can be introduced for the \(\tau \rightarrow 4\tau\) modular equation (51), can be revisited in the \(z\) variable. (see equation (45). Recalling \(P_1\) and \(P_2\) given in (45), and performing the (involutive) change of variable \(z \rightarrow -16z/(z + 16)\), on \(P_2\), we get a third Hauptmodul \(P_3\)
\[
P_3 = \frac{1728 \cdot z \cdot (z + 16)^4}{(z^2 - 224z + 256)^3} = \left(\frac{1728 \cdot z}{(z + 16)^3}\right) \circ \left(\frac{4096z}{(z + 16)^2}\right), \tag{C.1}
\]
to be compared\(^\dagger\) with:
\[
P_2 = \frac{1728 \cdot z \cdot (z + 16)}{(z^2 + 16z + 16)^3} = \left(\frac{1728 \cdot z}{(z + 16)^3}\right) \circ \left(z \cdot (z + 16)\right). \tag{C.2}
\]
One also has:
\[
P_1 = \frac{1728 \cdot z^4 \cdot (z + 16)}{(z^2 + 256z + 4096)^3} = \left(\frac{1728 \cdot z}{(z + 16)^3}\right) \circ \left(\frac{4096 \cdot (z + 16)}{z^2}\right) \tag{C.3}
\]
\[
= \left(\frac{1728 \cdot z^2}{(z + 256)^3}\right) \circ \left(z^2 \cdot (z + 16)\right). \tag{C.4}
\]
These three Hauptmoduls have to be compared with the Hauptmodul:
\[
P_0 = \frac{1728 \cdot z^2 \cdot (z + 16)^2}{(z^2 + 16z + 256)^3} = \left(\frac{1728 \cdot z}{(z + 16)^3}\right) \circ \left(\frac{z^2}{z + 16}\right) \tag{C.5}
\]
\[
= \left(\frac{1728 \cdot z^2}{(z + 256)^3}\right) \circ \left(z \cdot (z + 16)\right). \tag{C.6}
\]
Note that the elimination of \(z\), between this last Hauptmodul \(P_0\) and each of the three Hauptmoduls \(P_1\), \(P_2\), \(P_3\), gives the \(\tau \rightarrow 2\tau\) modular equation (see (E.9) below) instead of the \(\tau \rightarrow 4\tau\) modular equation (51).

The decomposition of the Hauptmoduls \(P_3\), \(P_2\) and \(P_1\) given by (C.1), (C.2) and (C.3) suggests to substitute
\[
X = \frac{1728\bar{X}}{(\bar{X} + 16)^3}, \quad Y = \frac{1728\bar{Y}}{(\bar{Y} + 16)^3}, \tag{C.7}
\]
in the \(\tau \rightarrow 4\tau\) modular equation (51). This change of variable transforms the LHS of the modular equation (51) into the product of four polynomials:
\[
\hat{p}_1(\bar{X}, \bar{Y}) = \bar{X}^2\bar{Y}^2 + 3 \cdot 2^{16} \cdot \bar{X}\bar{Y} - 2^{24} \cdot (\bar{X} + \bar{Y}), \tag{C.8}
\]
\[
\hat{p}_{2,1}(\bar{X}, \bar{Y}) = \hat{p}_{2,2}(\bar{Y}, \bar{X}) = \bar{X}^4\bar{Y}^3 + 96\bar{X}^3\bar{Y}^3 + 196608\bar{X}^3\bar{Y}^2 + 2352\bar{X}^2\bar{Y}^3 + 16777216\bar{X}^2\bar{Y}^2 + 733936\bar{X}^2\bar{Y} + 10496\bar{X}\bar{Y}^3 - \bar{Y}^4 + 805306368\bar{X}^2\bar{Y} + 9633792\bar{X}\bar{Y}^2 + 1610612736\bar{X}\bar{Y} + 6871947636\bar{X}, \tag{C.9}
\]
\(^\dagger\) Using the relation (43).
\(^\dagger\) As it should \(z \rightarrow -16z/(z + 16)\) changes \(z \cdot (z + 16)\) into \(-4096z/(z + 16)^2\).
The elimination of $z$ between any two Hauptmoduls among the three Hauptmoduls $P_1$, $P_2$, or $P_3$, yields the same modular equation (51). In general for modular equations representing $\tau \to N \tau$, one Hauptmodul is of the form $\alpha \cdot z + \cdots$ when the other one is of the form $\alpha \cdot z^N + \cdots$ (see [22]). Here with $P_2$ and $P_3$, we have two Hauptmoduls algebraically related by the modular equation (51) representing $\tau \to 4 \tau$, but each of them is of the form $\pm \alpha \cdot z + \cdots$. This result is reminiscent of the involutive series solution of (51), (given by equation (104) in [22]):

$$Y = \frac{-X - \frac{31}{36} X^2 - \frac{961}{1296} X^3 - \frac{203713}{314928} X^4 - \frac{4318517}{7558272} X^5}{\frac{832777775}{1632586752} X^6 - \frac{729205556393}{1586874322944} X^7 - \frac{2978790628903}{7140934453248} X^8 + \cdots}.$$  

(14)

Replacing in (14), $(X, Y)$ by $(P_2, P_3)$, one verifies that the expansions in $z$, of LHS and RHS of (14) are equal.

† Or replacing $(X, Y)$ by $(P_2^{(1)}, P_2^{(1)})$.  

\[ \hat{\ell}_4(\hat{X}, \hat{Y}) = \hat{X}^7 \hat{Y}^5 + \hat{X}^5 \hat{Y}^7 + 96 \hat{X}^2 \hat{Y}^4 + 144 \hat{X}^{a} \hat{Y}^5 + 144 \hat{X}^5 \hat{Y}^6 + 96 \hat{X}^4 \hat{Y}^7 + 2352 \hat{X}^7 \hat{Y}^3 - 182784 \hat{X}^6 \hat{Y}^4 + 13968 \hat{X}^5 \hat{Y}^5 - 182784 \hat{X}^4 \hat{Y}^6 + 2352 \hat{X}^3 \hat{Y}^7 + \hat{X}^6 \hat{Y}^5 + 10496 \hat{X}^7 \hat{Y}^2 + 7674625 \hat{X}^5 \hat{Y}^3 - 1300992 \hat{X}^5 \hat{Y}^4 - 1300992 \hat{X}^4 \hat{Y}^5 + 7674625 \hat{X}^3 \hat{Y}^6 + 10496 \hat{X}^2 \hat{Y}^7 + \hat{X}^6 \hat{Y}^8 + 192 \hat{X}^7 \hat{Y}^3 - 81 \hat{X}^2 \hat{Y}^6 + 192 \hat{X} \hat{Y}^7 + 13920 \hat{X}^6 \hat{Y} + 759331584 \hat{X}^3 \hat{Y}^2 - 1314435607 \hat{X}^5 \hat{Y}^3 + 472576 \hat{X}^5 \hat{Y} - 13144356607 \hat{X}^4 \hat{Y}^2 + 229377672192 \hat{X}^3 \hat{Y}^3 - 1314435607 \hat{X}^2 \hat{Y}^4 + 472576 \hat{X}^6 \hat{Y} + 7547184 \hat{X}^4 \hat{Y} + 39849037920 \hat{X}^3 \hat{Y}^2 + 39849037920 \hat{X}^2 \hat{Y}^3 + 7547184 \hat{X}^5 \hat{Y} + 49771008 \hat{X}^3 \hat{Y} - 1319514656 \hat{X}^2 \hat{Y}^2 + 49771008 \hat{X}^3 \hat{Y}^3 + 95607040 \hat{X}^2 \hat{Y} + 95607040 \hat{X} \hat{Y}^2 + 19771392 \hat{X} \hat{Y} - 4096. \quad (C.10)

The elimination of $z$ in

$$\hat{X} = -\frac{4096 z}{(z + 16)^2}, \quad \hat{Y} = z \cdot (z + 16), \quad (C.11)$$

or in

$$\hat{X} = -\frac{4096 \cdot z}{(z + 16)^2}, \quad \hat{Y} = \frac{4096 \cdot (z + 16)}{z^2}, \quad (C.12)$$

or in

$$\hat{X} = \frac{4096 \cdot (z + 16)}{z^2}, \quad \hat{Y} = z \cdot (z + 16), \quad (C.13)$$

corresponds to (C.8), the first polynomial $\hat{\ell}_1(\hat{X}, \hat{Y}) = 0$.  

Diagonals of rational functions
Appendix D. $2F_1([1/4, 3/4],[1], \mathcal{P})$ hypergeometric as modular forms

Appendix D.1. $2F_1([1/4, 1/4],[1], \mathcal{P})$ and $2F_1([1/2, 1/2],[1], \mathcal{P})$ as modular forms

In Table 15 of Maier [23], one sees that $2F_1([1/4, 1/4],[1], x)$ hypergeometric functions are related to $\tau \to 2\tau$:

$$2F_1\left([\frac{1}{4}, \frac{1}{4}], [1], -\frac{x}{64}\right) = \left(\frac{x + 16}{16}\right)^{-1/4} \cdot 2F_1\left([\frac{1}{12}, \frac{5}{12}], [1], \frac{1728 \cdot x}{(x + 16)^3}\right)$$

$$= \left(\frac{x + 256}{256}\right)^{-1/4} \cdot 2F_1\left([\frac{1}{12}, \frac{5}{12}], [1], \frac{1728 \cdot x^2}{(x + 256)^3}\right). \quad (D.1)$$

One has the following identity†:

$$2F_1\left([\frac{1}{4}, \frac{1}{4}], [1], -\frac{x}{64} \cdot \frac{x^2}{x + 16}\right) = \left(\frac{x + 16}{16}\right)^{1/4} \cdot 2F_1\left([\frac{1}{4}, \frac{1}{4}], [1], -\frac{(x + 16)}{64}\right). \quad (D.2)$$

One also sees in Table 15 of Maier [23] that $2F_1([1/2, 1/2],[1], x)$ hypergeometric functions are related to $\tau \to 4\tau$ isogeny[22]:

$$2F_1\left([\frac{1}{2}, \frac{1}{2}], [1], -\frac{x}{16}\right) = \left(\frac{x^2 + 16x + 16}{16}\right)^{-1/4} \cdot 2F_1\left([\frac{1}{12}, \frac{5}{12}], [1], \frac{1728 \cdot x \cdot (x + 16)}{(x^2 + 16x + 16)^3}\right)$$

$$= \left(\frac{x^2 + 256x + 4096}{4096}\right)^{-1/4} \cdot 2F_1\left([\frac{1}{12}, \frac{5}{12}], [1], \frac{1728 \cdot x^4 \cdot (x + 16)}{(x^2 + 256x + 4096)^3}\right). \quad (D.3)$$

One has the following identity:

$$2F_1\left([\frac{1}{2}, \frac{1}{2}], [1], \frac{8x \cdot (1 + x^2)}{(1 + x)^4}\right) = (1 + x)^2 \cdot 2F_1\left([\frac{1}{2}, \frac{1}{2}], [1], x^4\right). \quad (D.4)$$

Appendix D.2. $2F_1([1/4, 3/4],[1], \mathcal{P})$ hypergeometric as modular forms

Let us now focus on the $2F_1([1/4, 3/4],[1], \mathcal{P})$ hypergeometric function:

$$2F_1\left([\frac{1}{4}, \frac{3}{4}], [1], x\right) = (1 + 3x)^{-1/4} \cdot 2F_1\left([\frac{1}{12}, \frac{5}{12}], \frac{27x \cdot (1 - x)^2}{(1 + 3x)^3}\right). \quad (D.5)$$

The emergence of $2F_1([1/4, 3/4],[1], \mathcal{P})$ hypergeometric functions in physics, walk problems in the quarter of a plane [34, 35, 36] in enumerative combinatorics, or in interesting subcases of diagonals (see section (2.9)), raises the question if $2F_1([1/4, 3/4],[1], \mathcal{P})$ should be seen as associated to the isogenies [22] $\tau \to 2\tau$ or $\tau \to 4\tau$. The identity

$$2F_1\left([\frac{1}{4}, \frac{3}{4}], [1], 64x^2\right) = (1 + 8x)^{-1/2} \cdot 2F_1\left([\frac{1}{2}, \frac{1}{2}], [1], \frac{16x}{1 + 8x}\right), \quad (D.6)$$

or equivalently

$$2F_1\left([\frac{1}{4}, \frac{3}{4}], [1], \left(\frac{x}{2 - x}\right)^2\right) = \left(\frac{2 - x}{2}\right)^{1/2} \cdot 2F_1\left([\frac{1}{2}, \frac{1}{2}], [1], x\right). \quad (D.7)$$

† It can be deduced from (D.1) together with (C.2) with (C.5), or (C.4) with (C.6).
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seems to relate \( _2F_1([1/4, 3/4], [1], \mathcal{P}) \) to \( _2F_1([1/2, 1/2], [1], x) \), and thus seems to relate \( _2F_1([1/4, 3/4], [1], \mathcal{P}) \) rather \( \tau \to 4 \tau \). Yet things are more subtle.

Let us see how \( _2F_1([1/4, 3/4], [1], \mathcal{P}) \) can be described as a modular form corresponding to pulled back \( _2F_1([1/4, 3/4], [1], \mathcal{P}) \) hypergeometric functions with two different rational pullbacks. For instance, one deduces from (D.4) combined with (D.7), several identities on the hypergeometric function \( _2F_1([1/4, 3/4], [1], \mathcal{P}) \) like

\[
_2F_1\left(\left[\frac{1}{4}, \frac{3}{4}\right], [1], \frac{x^2}{(2-x)^2}\right) = \left(\frac{2-x}{2 \cdot (1-x)}\right)^{1/2} \cdot _2F_1\left(\left[\frac{1}{4}, \frac{3}{4}\right], [1], \frac{4x}{(1+x)^2}\right).
\]

or

\[
_2F_1\left(\left[\frac{1}{4}, \frac{3}{4}\right], [1], \frac{x^2}{(2-x)^2}\right) = \left(\frac{2-x}{2 \cdot (1+x)}\right)^{1/2} \cdot _2F_1\left(\left[\frac{1}{4}, \frac{3}{4}\right], [1], \frac{4x}{(1+x)^2}\right).
\]

and thus:

\[
_2F_1\left(\left[\frac{1}{4}, \frac{3}{4}\right], [1], \frac{4x}{(1+x)^2}\right) = \left(\frac{1+x}{1-2x}\right)^{1/2} \cdot _2F_1\left(\left[\frac{1}{4}, \frac{3}{4}\right], [1], \frac{4x}{(1-x)^2}\right).
\]

One also has the identity:

\[
_2F_1\left(\left[\frac{1}{4}, \frac{3}{4}\right], [1], \frac{x^4}{(2-x^2)^2}\right) = \left(\frac{2-x^2}{2 \cdot (1+6x+x^2)}\right)^{1/2} \cdot _2F_1\left(\left[\frac{1}{4}, \frac{3}{4}\right], [1], \frac{16 \cdot x \cdot (1+x)^2}{(1+6x+x^2)^2}\right)
\]

Recalling the viewpoint developed in our previous paper [22] these identities can be seen to be of the form

\[
_2F_1\left(\left[\frac{1}{4}, \frac{3}{4}\right], [1], B\right) = G \cdot _2F_1\left(\left[\frac{1}{4}, \frac{3}{4}\right], [1], A\right),
\]

where \( G \) is some algebraic factor. For instance in the case of the last identity (D.11)

\[
A = \frac{16 \cdot x \cdot (1+x)^2}{(1+6x+x^2)^2}, \quad B = \frac{x^4}{(2-x^2)^2},
\]

we have

\[
B = \frac{A^4}{262144} + \frac{5 A^6}{524288} + \frac{1069 A^6}{67108864} + \frac{6003 A^7}{268435456} + \frac{1961123 A^8}{68719476736} + \cdots
\]

and \( G \) is an algebraic factor

\[
G = 1 - \frac{3}{16} A - \frac{69 A^2}{1024} - \frac{633 A^3}{16384} - \frac{55209 A^4}{2097152} - \frac{659109 A^5}{33554432} + \cdots
\]

solution of:

\[
65536 \cdot G^8 - 16384 \cdot G^6 + 1536 \cdot (27 A - 26) \cdot G^4 + 64 \cdot (135 A - 136) \cdot G^2 + 3969 A^2 - 3456 A - 512 = 0.
\]

(D.15)
The important result of [22] is that after elimination of the algebraic factor \( G \) one finds that the two pullbacks \( A \) and \( B \) verify the following *Schwarzian equation*:

\[
-\frac{1}{8} \frac{3A^2 - 3A + 4}{A^2 (A - 1)^2} + \frac{1}{8} \frac{3B^2 - 3B + 4}{B^2 (B - 1)^2} \cdot \left( \frac{dB}{dA} \right)^2 + \{B, A\} = 0, \tag{D.16}
\]

where \( \{B, A\} \) denotes the Schwarzian derivative.

Do note that \( _2F_1([1/4, 3/4], [1], \tau) \) is a selected hypergeometric function since the rational function in the Schwarzian derivative (D.16)

\[
W(A) = -\frac{1}{8} \frac{3A^2 - 3A + 4}{A^2 (A - 1)^2}, \tag{D.17}
\]

is invariant under the \( A \rightarrow 1 - A \) transformation: \( W(A) = W(1 - A) \).

This Schwarzian equation can be written in a more symmetric way between \( A \) and \( B \), namely:

\[
-\frac{1}{8} \frac{3B^2 - 3B + 4}{B^2 (B - 1)^2} \cdot \left( \frac{dB}{dx} \right)^2 + \{B, x\}
= \frac{1}{8} \frac{3A^2 - 3A + 4}{A^2 (A - 1)^2} \cdot \left( \frac{dA}{dx} \right)^2 + \{A, x\}. \tag{D.18}
\]

Let us denote \( \rho(x) \) the rational function of the LHS or the RHS of equality (D.18). For the three identities (D.8), (D.9), (D.10) this rational function is (of course) the same rational function, namely

\[
\rho(x) = \frac{1}{2} \frac{x^2 - x + 1}{x \cdot (x - 1)^2}, \tag{D.19}
\]

when the last identity (D.11) corresponds to:

\[
\rho(x) = \frac{1}{2} \frac{(x^2 + 1)^2}{x^2 \cdot (x^2 - 1)^2}. \tag{D.20}
\]

Let us consider the first two identities (D.8) and (D.9), denoting by \( A \) and \( B \) the corresponding pullbacks:

\[
A = -4 \frac{x \cdot (1 - x)}{(1 - 2x)^2}, \quad \text{or:} \quad B = \frac{4x}{(1 + x)^2}, \tag{D.21}
\]

These two pullbacks are related by the *asymmetric* modular equation:

\[
81 \cdot A^2 B^2 - 18A B \cdot (8B + A) + (A^2 + 80 \cdot A B + 64B^2) - 64B = 0. \tag{D.22}
\]

giving the following expansion for \( A \) seen as an *algebraic series*\( ^\dagger \) in \( B \):

\[
B = \frac{1}{64} A^2 + \frac{5}{256} A^3 + \frac{83}{4096} A^4 + \frac{163}{8192} A^5 + \frac{5013}{262144} A^6 + \cdots \tag{D.23}
\]

We will denote \( \mathcal{M}_2(A, B) \) the LHS of the modular equation (D.22): such an algebraic series is clearly\( ^\dagger \) \( a \tau \rightarrow 2 \tau \) (or \( q \rightarrow q^2 \) in the nome \( q \) isogery [22]).

Composing this algebraic transformation with itself in order to have a \( \tau \rightarrow 4 \tau \) (or \( q \rightarrow q^4 \) ) representation, amounts to eliminating\( ^\ddagger \) \( X \) between \( \mathcal{M}_2(A, X) = 0 \) and

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\( ^\dagger \) Since these identities share one pullback.

\( ^\ddagger \) We discard the other root expansion \( B = 1 + A + \frac{1}{4} A^2 + \frac{3}{16} A^3 + \frac{21}{64} A^4 + \cdots \).

\( ^\ddagger \) From (D.23) see [22].
\( \mathcal{M}_2(X, B) = 0 \) (i.e. two times the modular equation (59)). This elimination gives the following asymmetric modular curve corresponding to identity (D.11):

\[
15752961 A^4 B^4 - 428652 A^3 B^3 \cdot (64 B + 83 A) \\
+ 162 A^2 B^2 \cdot (48640 B^2 + 494208 A B + 124051 A^2) \\
+ 108 A B \cdot (32768 B^3 - 500480 B^2 A - 491200 BA^2 - 83 A^3) \\
+ 262144 B^4 + 1035468 B^3 A + 4671904 B^2 A^2 + 159488 B A^3 + A^4 \\
- 3072 B \cdot (256 B^2 + 4736 A B + 177 A^2) \\
+ 131072 B \cdot (6 B + 5 A) - 262144 B = 0, \quad (D.24)
\]

parametrised by

\[
A = \frac{16 \cdot x \cdot (1 + x)^2}{(1 + 6 x + x^2)^2}, \quad B = \frac{x^4}{(2 - x^2)^2}, \quad (D.25)
\]

where:

\[
B = \frac{A^4}{262144} + \frac{5 A^5}{524288} + \frac{1069 A^6}{67108864} + \frac{6003 A^7}{268435456} + \frac{1961123 A^8}{68719476736} + \cdots \quad (D.26)
\]

Note that \( B \) in (D.25) is nothing but the composition of \( B \) in (58) by \( x \to x^2 \) and that \( A \) in (D.25) is nothing but the composition of \( A \) in (D.25) with itself:

\[
\frac{16 \cdot x \cdot (1 + x)^2}{(1 + 6 x + x^2)^2} = \frac{4 x}{(1 + x)^2} \circ \frac{4 x}{(1 + x)^2}. \quad (D.27)
\]

The modular curve (59) is unpleasantly asymmetric: the two pullbacks are not on the same footing. Note that, using the \( A \leftrightarrow 1 - A \) symmetry (see (D.17)) on the Schwarzian equations (D.18), and changing \( A \to 1 - A \) in the asymmetric modular curve (59), one gets the symmetric modular curve:

\[
81 \cdot A^2 B^2 - 18 \cdot (A^2 B + AB^2) + A^2 - 44 AB + B^2 \\
- 2 \cdot (A + B) + 1 = 0. \quad (D.28)
\]

Changing \( B \to 1 - B \) in the asymmetric modular curve (59), one also gets another symmetric modular curve:

\[
81 \cdot A^2 B^2 - 144 \cdot (A^2 B + AB^2) \\
+ 208 AB + 64 \cdot (A^2 + B^2 - A - B) = 0. \quad (D.29)
\]

The two pullbacks for (D.29) read:

\[
A = \frac{4 x}{(1 + x)^2}, \quad B = \frac{4 \cdot (1 - x)}{(2 - x)^2}. \quad (D.30)
\]

Similarly the asymmetric modular curve (D.24) can be turned back into a symmetric modular curve by changing \( A \leftrightarrow 1 - A \), or \( B \leftrightarrow 1 - B \). The price to pay to restore the symmetry between the two pullbacks (D.30) is that the corresponding pullbacks do not yield hypergeometric identities expandable for \( x \) small.

Finally, the identity (D.10) corresponds to a symmetric relation between these two-pullbacks which reads:

\[
81 \cdot C^2 D^2 - 144 \cdot (C^2 D + CD^2) + 16 \cdot (4 C^2 + 13 CD + 4 D^2) \\
- 64 \cdot (C + D) = 0. \quad (D.31)
\]
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The corresponding series expansion

\[
D = -C - \frac{5}{4} C^2 - \frac{25}{16} C^3 - \frac{31}{16} C^4 - \frac{305}{128} C^5 - \frac{2979}{1024} C^6 - \frac{14457}{4096} C^7 - \frac{17445}{32768} C^8 - \frac{167615}{80191} C^9 - \frac{382289}{131072} C^{10} - \frac{524289}{524288} C^{11} + \ldots
\]  

(D.32)

is an involutive series.

Appendix E. Modular forms: recalls on Maier’s paper [23] and the associated Schwarzian equations

In fact, the previous pullbacks in the pullbacked \( _2F_1 \) hypergeometric functions can be seen (and should be seen) as Hauptmoduls [23].

In [23], Maier underlined the representation of a selected set of modular forms as pullbacked hypergeometric functions with two possible rational pullbacks (related by a genus zero modular equation). In [22], we revisited that viewpoint: an identity on a hypergeometric function with a pullback and the same hypergeometric function with another pullback, the (algebraic) map \( \tau \), changing one pullback into the other one, being a symmetry of infinite order, is such a strong constraint that it is almost characteristic of modular forms [22]: the hypergeometric functions can be seen as automorphic functions with respect to these infinite order symmetries.

The two different modular equations (40), (51) corresponding respectively to \( \tau \rightarrow 3\tau \) and \( \tau \rightarrow 4\tau \), suggest that a genus zero modular equation, corresponding to \( \tau \rightarrow N\tau \), could encapsulate these two subcases. In such a scenario, \( N \) must be a multiple of 3, 4, 5, \ldots. In fact, the set of values of \( N \) corresponding to modular equations with a (genus zero) rational parametrization is obtained for a finite set [23, 41, 42] of integer values: 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 16, 18 and 25. Some canonical rational parametrizations of these selected genus zero modular equations are given in [23]. The two Hauptmoduls read respectively for these selected values 2, 3, 4, 5, \ldots, 25:

\[
N = 2 : \quad 1728 \cdot (z + 256)^3, \quad \frac{1728}{(z + 16)^3}, \quad \frac{1728 \cdot z}{(z + 27) \cdot (z + 243)^3}, \quad \frac{1728 \cdot z}{(z + 27) \cdot (z + 3)^3}, \quad (E.1)
\]

\[
N = 3 : \quad 1728 \cdot z^3, \quad \frac{1728}{(z + 27) \cdot (z + 243)^3}, \quad \frac{1728 \cdot z}{(z + 27) \cdot (z + 3)^3}, \quad (E.2)
\]

\[
N = 4 : \quad 1728 \cdot z^4, \quad (z + 16), \quad \frac{1728 \cdot z \cdot (z + 16)}{(z^2 + 256z + 4096)^3}, \quad \frac{1728 \cdot z}{(z^2 + 16z + 16)^3}, \quad (z^2 + 16z + 16)^3, \quad (E.3)
\]

\[
N = 5 : \quad 1728 \cdot z^5, \quad (z^2 + 250z + 3125)^3, \quad \frac{1728 \cdot z}{(z^2 + 10z + 5)^3}, \quad (z^2 + 10z + 5)^3, \quad (E.4)
\]

\[
N = 6 : \quad \frac{1728 \cdot z^6 \cdot (z + 8)^3 \cdot (z + 9)^3}{(z + 12)^3 \cdot (z^3 + 252z^2 + 3888z + 15552)^3}, \quad \frac{1728 \cdot z \cdot (z + 8)^3 \cdot (z + 9)^2}{(z^2 + 18z^2 + 84z + 24)^3}, \quad (E.5)
\]

† Called by Veselov [40] in a mapping framework, “correspondence”.

† Of course hypergeometric functions have finite order symmetries like \( x \rightarrow 1 - x \), that we discard. With infinite order symmetries one can associate some discrete dynamical map: in these particular cases algebraic function maps [22, 38, 40].
The relation between \( A \)
where \( A \) is a multiple of 3, 4, 5, ... In the generic seven-parameters case one gets the expression (11) for the diagonal of the seven-parameters rational function (7) as a pullbacked \( 2F_1 \) hypergeometric function with a rational pullback (12), the other pullback being algebraic and deduced from the various modular equations [22] (see section (2.10)).

\[ x = \frac{4}{125}, \quad (1 - A) \cdot (1 - 16A)^2, \quad y = -\frac{1}{500} \cdot \frac{(1 - A)^2 \cdot (1 - 16A)}{A^3}. \]
Note that \( y \), in terms of \( A \) is nothing but \( x \), in terms of \( A \). \( A \) taken to be \( 1/16/A \). The two variables \( x \) and \( y \) are thus on the same footing; permuting \( x \) and \( y \) corresponds to the involutive transformation \( A \leftrightarrow 1/16/A \). Finally changing \( A \) into \( A = (z + 256)/(z + 16)/16 \), the previous parametrization (E.13) becomes the known parametrization [22, 23] of the fundamental modular equation (E.9), namely \( x = 1728 z/(z + 16)^3 \) and \( y = 1728 z^2/(z + 256)^3 \).

**Appendix E.2. Schwarzian equations**

In general, one can rewrite a remarkable hypergeometric identity like (39), (49), in the form

\[
A(x) \cdot {}_2F_1\left(\begin{array}{c} \alpha, \beta, \gamma \\ \alpha, \beta, \gamma \end{array}; x \right) = {}_2F_1\left(\begin{array}{c} \alpha, \beta, \gamma \\ \alpha, \beta, \gamma \end{array}; y(x) \right),
\]

(E.14)

where \( A(x) \) is an algebraic function and where \( y(x) \) is an algebraic function (more precisely an algebraic series) corresponding to the previous modular equation \( M(x, y(x)) = 0 \).

The Gauss hypergeometric function \( {}_2F_1[\alpha, \beta, \gamma, x] \) is solution of the second order linear differential operator\(^\dagger\):

\[
\Omega = D_x^2 + A(x) \cdot D_x + B(x),
\]

where:

\[
A(x) = \frac{(\alpha + \beta + 1) \cdot x - \gamma}{x \cdot (x - 1)} = \frac{u'(x)}{u(x)}, \quad B(x) = \frac{\alpha \beta}{x \cdot (x - 1)}. \quad (E.15)
\]

A straightforward calculation enables us to find the algebraic function \( A(x) \) in terms of the algebraic function pullback \( y(x) \) in (E.14):

\[
A(x) = \left( \frac{u(x)}{u(y(x))} \cdot y'(x) \right)^{-1/2}. \quad (E.16)
\]

The identification of the two operators, \( 1/\nu(x) \cdot \Omega \cdot v(x) \) and \( \Omega^{\text{pull}} \) (the pullback of operator \( \Omega \) for a pullback \( y(x) \)), thus corresponds (beyond (E.16)) to just one condition that can be rewritten (after some algebra ...) in the following Schwarzian form [22, 38]:

\[
W(x) = W(y(x)) \cdot y'(x)^2 + \{y(x), x\} = 0, \quad (E.17)
\]

or:

\[
\frac{W(x)}{y'(x)} = W(y(x)) \cdot y'(x) + \frac{\{y(x), x\}}{y'(x)} = 0, \quad (E.18)
\]

where

\[
W(x) = A'(x) + \frac{A(x)^2}{2} - 2 \cdot B(x), \quad (E.19)
\]

and where \( \{y(x), x\} \) denotes the Schwarzian derivative [39]:

\[
\{y(x), x\} = \frac{y''''(x)}{y'(x)} - \frac{3}{2} \left( \frac{y''(x)}{y'(x)} \right)^2 = \frac{d}{dx} \left( \frac{y''(x)}{y'(x)} \right) - \frac{1}{2} \left( \frac{y''(x)}{y'(x)} \right)^2. \quad (E.20)
\]

For (E.15) the function \( W(x) \) reads:

\[
(\alpha - \beta + 1) \cdot (\alpha - \beta - 1) \cdot x^2 + 2 \cdot (2\alpha \beta - \alpha \gamma - \beta \gamma + \gamma) \cdot x + \gamma \cdot (\gamma - 2) \quad (E.21)
\]

\( \dagger \) Note that \( A(x) \) is the log-derivative of \( u(x) = x^{\gamma} \cdot (1 - x)^{\alpha + \beta + 1 - \gamma} \).
The hypergeometric functions such that \( W(x) = W(1-x) \) correspond to the two conditions:

\[
\alpha + \beta = 1 \quad \text{or} \quad \alpha + \beta = 2\gamma - 1. \tag{E.22}
\]

This is the case, for instance [26], with the hypergeometric functions \( {}_2F_1([1/2,1/2][1], x) \), \( {}_2F_1([1/3,2/3][1], x) \), \( {}_2F_1([1/4,3/4][1], x) \), \( {}_2F_1([1/6,5/6][1], x) \).

**Remark:** Denoting \( W_H(x) \) the rational function \( W(x) \) given by \((E.19)\) for the second order linear differential operator annihilating \((13)\), i.e. \( {}_2F_1([1/12,5/12][1], H(x)) \) where \( H(x) \) is the Hauptmodul \( 1 - P_4(x)^3/P_5(x)^3 \) in \((13)\), with \( P_2 \) and \( P_4 \) given by \((16)\) and \((17)\). \( W_H(x) \) can be deduced from the \( W(x) \) for the order-two linear differential operator annihilating \( {}_2F_1([1/12,5/12][1], x) \), from the relation \([38]\)

\[
W_H(x) = W\left(H(x)\right) - \{H(x), x\}, \quad W(x) = \frac{-32x^2 - 41x + 36}{72 \cdot (x-1)^2 \cdot x^2}, \tag{E.23}
\]

where \( \{H(x), x\} \) denotes the Schwarzian derivative of \( H(x) \). \( W_H(x) \) is of the form

\[
W_H(x) = \frac{p_{10}(x)}{x^2 \cdot p_3(x)^2 \cdot p_5(x)^2} = -\frac{1}{2x^2} + \cdots \tag{E.24}
\]

where \( p_{10}(x) \), \( p_3(x) \) and \( p_5(x) \) are polynomials of degree respectively sixteen, three and five in \( x \). These polynomials are *homogeneous polynomials* in the seven parameters \( a, b, c_i \) of \((7)\). For instance \( p_{10}(x) \) is a homogeneous polynomial of homogeneous degree 44, \( p_3(x) \) is a homogeneous polynomial of homogeneous degree 10 and \( p_5(x) \) is a homogeneous polynomial of homogeneous degree 12.

**Appendix F. Exact expression of polynomial \( P_6 \) for the ten-parameter rational function \((77)\)**

The diagonal of the ten-parameters rational function \((77)\) is the pullbacked hypergeometric function

\[
\frac{1}{P_3(x)^{1/4}} \cdot {}_2F_1\left([1, 5/12], [1], 1 - \frac{P_6(x)^2}{P_5(x)^3}\right), \tag{F.1}
\]

where \( P_3(x) \) is given by \((80)\) and \( P_6(x) \) is a polynomial of degree six in \( x \) of the form

\[
P_6(x) = p_4 + \Delta_6(x), \tag{F.2}
\]

where \( p_4 \) is the polynomial \( P_4(x) \) given by \((17)\) in section \((2)\), and where \( \Delta_6(x) \) is the following polynomial of degree six in \( x \):

\[
\Delta_6(x) = -5832 \cdot d_1^2 d_2 d_3^2 \cdot x^6 \\
+ 3888 \cdot d_1 d_2 d_3 \cdot (b_1 c_2 d_2 + b_2 c_3 d_3 + b_3 c_1 d_1) \cdot x^5 \\
- 864 \cdot (c_1^3 d_1^3 d_3 + c_2^3 d_1^3 d_2 + c_3^3 d_2^3 d_3) \cdot x^5 \\
- 1296 \cdot c_1 c_2 c_3 d_1 d_2 d_3 \cdot x^5
\]
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\[-1296 \cdot b_1 b_2 b_3 d_1 d_2 d_3 \cdot x^4\]
\[-1296 \cdot a \cdot d_1 d_2 d_3 (b_1 c_1 + b_2 c_2 + b_3 c_3) \cdot x^4\]
\[-1296 \cdot (b_1 b_2 b_3 c_2 d_2 d_3 + b_1 b_3 c_1 b_2 d_1 d_2 + b_2 b_3 c_1 d_1 d_3) \cdot x^4\]
\[+ 864 \cdot (c_1^2 c_3 d_1 d_3 + c_1 c_2^2 d_1 d_2 + c_2 c_3^2 d_2 d_3) \cdot a \cdot x^4\]
\[-864 \cdot (b_1^2 d_2^2 d_3 + b_3^3 d_1 d_3^3 + b_3^3 d_1^3 d_2) \cdot x^4\]
\[+ 864 \cdot (b_1^2 c_1 c_3 d_2 d_3 + b_1 b_2^2 c_1 d_1 d_3 + b_1 b_3 c_3^2 d_2 d_3\]
\[+ b_2^2 c_1 c_2 d_1 d_3 + b_2 b_3 c_2^2 d_1 d_2 + b_3^2 c_2 c_3 d_1 d_2) \cdot x^4\]
\[+ 216 \cdot (b_1^2 c_2^2 d_2^2 + b_2^2 c_2^2 d_3^2 + b_3^2 c_2^2 d_2^2) \cdot x^4\]
\[+ 288 \cdot (b_1 c_1^3 c_2 d_1 + b_2 c_2^3 c_3 d_2 + b_3 c_1 c_3^3 d_3) \cdot x^4\]
\[-576 \cdot (b_1 c_1^2 c_2^2 d_3 + b_2 c_1^2 c_2^2 d_1 + b_3 c_2^2 c_3^2 d_2) \cdot x^4\]
\[-144 \cdot c_1 c_2 c_3 \cdot (b_1 c_2 d_2 + b_2 c_3 d_3 + b_3 c_1 d_1) \cdot x^4\]
\[+ 540 \cdot d_1 d_2 d_3 a^3 \cdot x^3\]
\[-648 \cdot (b_1 c_3 d_2 d_3 + b_2 c_1 d_1 d_3 + b_3 c_2 d_1 d_2) \cdot a^2 \cdot x^3\]
\[-72 \cdot (c_1^2 c_2 d_1 + c_1 c_2^2 d_3 + c_2 c_3 d_2) \cdot a^2 \cdot x^3\]
\[+ 288 \cdot (b_1^2 b_3 c_1 d_2 + b_1 b_2^2 c_3 d_1 + b_2 b_3^2 c_3 d_1) \cdot x^3\]
\[-576 \cdot (b_1^2 b_3^2 c_1 d_3 + b_1 b_2^2 c_3 d_2 + b_2 b_3^2 c_3 d_1) \cdot x^3\]
\[-144 \cdot b_1 b_2 b_3 (b_1 c_2 d_2 + b_2 c_3 d_3 + b_3 c_1 d_1) \cdot x^3\]
\[+ 864 \cdot (b_1^2 b_2 d_2 d_3 + b_1 b_2^2 d_1 d_2 + b_2 b_3 d_1 d_3) \cdot a \cdot x^3\]
\[-144 \cdot (b_1^2 c_1 c_2 d_2 + b_1 b_2^2 c_2 d_2 + b_1 b_3^2 d_1 d_1\]
\[+ b_2^2 c_2 c_3 d_3 + b_2 b_3 c_2^2 d_2 + b_3^2 c_1 c_3 d_1) \cdot a \cdot x^3\]
\[+ 720 \cdot (b_1 b_2 c_1 c_2 d_3 + b_1 b_3 c_2 c_3 d_2 + b_2 b_3 c_1 c_2 d_1) \cdot a \cdot x^3\]
\[+ 36 \cdot a^3 \cdot (b_1 c_2 d_2 + b_2 c_3 d_3 + b_3 c_1 d_1) \cdot x^2\]
\[-72 \cdot a^2 \cdot (b_1^2 b_2 d_2 + b_1 b_2^2 d_2 + b_2 b_3^2 d_1) \cdot x^2\].

\[(F.3)\]

Appendix G. Polynomials $P_3(x)$ and $P_5(x)$ for the nine-parameter rational function (77)

The two polynomials $P_3(x)$ and $P_5(x)$ encoding the pullback of the pullbacked hypergeometric function (85) for the nine-parameter rational function (77) in section (3.3.1), read

\[P_3(x) = p_2 + 48 \cdot c_2 \cdot (3 b_3 d_1 d_2 - c_2^2 d_1 - c_2 c_3 d_2) \cdot x^3\]
\[+ 24 \cdot (a b_1 c_2 d_2 + a b_3 c_1 d_1 - 2 b_1^2 b_3 d_2 + 2 b_2 b_3^2 d_1) \cdot x^2, \quad (G.1)\]
and

\[ P_3(x) = p_4 - 864 \cdot c_1^3 d_1 d_2^2 \cdot x^5 \]
\[ + 864 \cdot (a c_1^2 d_1 d_2 + b_2 b_3 c_2 d_1 d_2 + b_3^2 c_2 c_3 d_1 d_2 - b_3^2 d_1^2 d_2) \cdot x^4 \]
\[ - 576 \cdot (b_2 c_1^2 c_2 d_1 + b_3^2 c_1^2 c_3 d_1) \cdot x^4 \]
\[ + 288 \cdot (b_1 c_1^2 c_2 d_1 + b_2^2 c_2 c_3 d_2) \cdot x^4 \]
\[ - 144 \cdot (b_1 c_1^2 c_3 d_2 + b_3^2 c_1^2 c_3 d_1) \cdot x^4 \]
\[ + 216 \cdot (b_1^2 c_1^2 d_2^2 + b_3^2 c_1^2 c_3 d_1 - 6 b_1 b_3 c_1 c_2 d_1 d_2) \cdot x^4 \]
\[ - 72 \cdot (9a^2 b_3 c_2 d_1 d_2 + a^2 c_1^2 c_2 d_1 + a^2 c_2^2 c_3 d_2) \cdot x^3 \]
\[ - 144 \cdot a \cdot (b_1^2 c_1 c_2 d_2 + b_1 b_2 c_2^2 d_2 + b_1 b_3 c_1 c_3 d_1 + b_3^2 c_1 c_3 d_1) \cdot x^3 \]
\[ - 144 \cdot (b_1^2 b_2 b_3 c_2 d_2 + b_1 b_2 b_3^2 c_1 d_1) \cdot x^3 \]
\[ + 720 \cdot (a b_1 b_3 c_2 c_3 d_2 + a b_2 b_3 c_1 c_2 d_1) \cdot x^3 \]
\[ - 576 \cdot (b_1^2 b_2^2 c_3 d_2 + b_2^2 b_3^2 c_2 d_1) \cdot x^3 \]
\[ + 288 \cdot (b_1^2 b_3 c_1 d_2 + b_2^2 b_3^2 c_3 d_1 + 3 a b_1 b_3^2 d_1 d_2) \cdot x^3 \]
\[ + 78 \cdot a^2 \cdot (a b_1 c_2 d_2 + a b_3 c_1 d_1 - 2 b_1^3 b_3 d_2 - 2 b_2 b_3^2 d_1) \cdot x^2, \quad (G.2) \]

where the polynomials \( p_2 \) and \( p_4 \) are the polynomials \( P_2(x) \) and \( P_4(x) \) of degree two and four in \( x \) given by (16) and (17) in section (2); \( p_2 \) and \( p_4 \) correspond to the \( d_1 = d_2 = 0 \) limit.

**Appendix H. Monomial symmetries on diagonals**

Let us sketch the demonstration of the monomial symmetry results of section (98), with the condition that the determinant of (99) is not zero and the conditions (100) are verified. We will denote by \( n \) the integer in the three equal sums (100): \( n = A_i + B_i + C_i \). The diagonal of the rational function of three variables \( R \) is defined through its multi-Taylor expansion (for small \( x \), \( y \), and \( z \)): 

\[ R(x, y, z) = \sum_{m_{1}=0}^{\infty} \sum_{m_{2}=0}^{\infty} \sum_{m_{3}=0}^{\infty} R_{m_{1}, \ldots, m_{n}} \cdot x^{m_{1}} \cdot y^{m_{2}} \cdot z^{m_{3}}, \quad (H.1) \]

as the series in one variable \( x \):

\[ \Phi(x) = \text{Diag} \left( R(x, y, z) \right) = \sum_{m=0}^{\infty} R_{m, m, m} \cdot x^{m}. \quad (H.2) \]

The monomial transformation (98) changes the multi-Taylor expansion (H.1) into

\[ \tilde{R}(x, y, z) = \sum_{M_{1}=0}^{\infty} \sum_{M_{2}=0}^{\infty} \sum_{M_{3}=0}^{\infty} \tilde{R}_{M_{1}, M_{2}, M_{3}} \cdot x^{M_{1}} \cdot y^{M_{2}} \cdot z^{M_{3}} = \]
\[ \sum_{m_{1}=0}^{\infty} \sum_{m_{2}=0}^{\infty} \sum_{m_{3}=0}^{\infty} R_{m_{1}, m_{2}, m_{3}} \cdot (x^{A_{1}} y^{A_{2}} z^{A_{3}})^{m_{1}} (x^{B_{1}} y^{B_{2}} z^{B_{3}})^{m_{2}} (x^{C_{1}} y^{C_{2}} z^{C_{3}})^{m_{3}} \]
\[ = \sum_{m_{1}=0}^{\infty} \sum_{m_{2}=0}^{\infty} \sum_{m_{3}=0}^{\infty} R_{m_{1}, m_{2}, m_{3}} \cdot x^{M_{1}} \cdot y^{M_{2}} \cdot z^{M_{3}} \]

where:

\[ M_{1} = A_{1} \cdot m_{1} + B_{1} \cdot m_{2} + C_{1} \cdot m_{3}, \quad (H.3) \]
\[ M_{2} = A_{2} \cdot m_{1} + B_{2} \cdot m_{2} + C_{2} \cdot m_{3}, \quad (H.4) \]
\[ M_{3} = A_{3} \cdot m_{1} + B_{3} \cdot m_{2} + C_{3} \cdot m_{3}. \quad (H.5) \]
Taking the diagonal amounts to forcing the exponents $m_1$, $m_2$ and $m_3$ to be equal. It is easy to see that when condition (100) is verified, $m_1 = m_2 = m_3$ yields $M_1 = M_2 = M_3$. Conversely if the determinant of (99) is not zero it is straightforward to see that the conditions $M_1 = M_2 = M_3$ yield $m_1 = m_2 = m_3$.

Then if one knows an exact expression for the diagonal of a rational function, the diagonal of this rational function changed by the monomial transformation (98) reads

$$
\text{Diag} \left( \tilde{R}(x, y, z) \right) = \sum_{M=0}^{\infty} \tilde{R}_{M, M, M} \cdot x^M = \sum_{m=0}^{\infty} R_{m, m, m} \cdot x^{n \cdot m} = \Phi(x^n), \quad (H.6)
$$

and thus equal to the previous exact expression $\Phi(x)$, where we have changed $x \to x^n$, where $n$ is the integer $n = A_1 + B_1 + C_1 = A_2 + B_2 + C_2 = A_3 + B_3 + C_3$. These monomial symmetries for diagonal of rational functions are not specific of rational functions of three variables: they can be straightforwardly generalized to an arbitrary number of variables.

**Appendix I. Rescaling symmetries on diagonals**

We sketch the demonstration of the result in section (4.3). One recalls that the diagonal of the rational function of three variables $\mathcal{R}$ is defined through its multi-Taylor expansion (for small $x$, $y$ and $z$)

$$
\mathcal{R}(x, y, z) = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \sum_{m_3=0}^{\infty} R_{m_1,...,m_n} \cdot x^{m_1} \cdot y^{m_2} \cdot z^{m_3}, \quad (I.1)
$$

as the series in one variable $x$:

$$
\Phi(x) = \text{Diag} \left( \mathcal{R}(x, y, z) \right) = \sum_{m=0}^{\infty} R_{m, m, m} \cdot x^{m}. \quad (I.2)
$$

The (function rescaling) transformation (110) transforms the multi-Taylor expansion (I.1) into:

$$
\mathcal{R}(x, y, z) = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \sum_{m_3=0}^{\infty} R_{m_1,...,m_n} \cdot x^{m_1} \cdot y^{m_2} \cdot z^{m_3} \cdot F(x y z)^{m_1 + m_2 + m_3}. \quad (I.3)
$$

We assume that the function $F(x)$ has some simple Taylor series expansion. Each time taking the diagonal of (I.3) forces the exponents $m_1$, $m_2$ and $m_3$ to be equal in the term $x^{m_1} \cdot y^{m_2} \cdot z^{m_3}$ of the multi-Taylor expansion (I.3), one gets a factor $F(x y z)^{m_1 + m_2 + m_3} = F(x y z)^{3m}$. Consequently, the diagonal of (I.3) becomes:

$$
\text{Diag} \left( \tilde{R}(x, y, z) \right) = \sum_{m=0}^{\infty} R_{m, m, m} \cdot x^{n} \cdot F(x)^{3n}
\quad = \text{Diag} \left( \mathcal{R}(x, y, z) \right) \left( x \cdot F(x)^{3} \right). \quad (I.4)
$$

Clearly, these function-dependent rescaling symmetries for diagonals of rational functions are not specific of rational functions of three variables: they can be straightforwardly generalized to an arbitrary number of variables.
Diagonals of rational functions

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