Diagonals of rational functions, pullbacked ${}_2F_1$ hypergeometric functions and modular forms (unabrigded version)

Y. Abdelaziz[†] **S. Boukraa**[£], **C. Koutschan**[¶], **J-M. Maillard**[†]

[†] LPTMC, UMR 7600 CNRS, Université Pierre et Marie Curie, Sorbonne Université, Tour 23, 5ème étage, case 121, 4 Place Jussieu, 75252 Paris Cedex 05, France

[£] LPTHIRM and IAESB, Université de Blida, Algeria

[¶] Johann Radon Institute for Computational and Applied Mathematics, RICAM, Altenberger Strasse 69, A-4040 Linz, Austria

Abstract.

We recall that diagonals of rational functions naturally occur in lattice statistical mechanics and enumerative combinatorics. We find that a seven-parameter rational function of three variables with a numerator equal to one (reciprocal of a polynomial of degree two at most) can be expressed as a pullbacked ${}_2F_1$ hypergeometric function. This result can be seen as the simplest non-trivial family of diagonals of rational functions. We focus on some subcases such that the diagonals of the corresponding rational functions can be written as a pullbacked ${}_2F_1$ hypergeometric function with two possible rational functions pullbacks algebraically related by modular equations, thus showing explicitely that the diagonal is a modular form. We then generalise this result to eight, nine and ten parameters families adding some selected cubic terms at the denominator of the rational function defining the diagonal. We finally show that each of these previous rational functions yields an infinite number of rational functions whose diagonals are also pullbacked ${}_2F_1$ hypergeometric functions and modular forms.

PACS: 05.50.+q, 05.10.-a, 02.30.Hq, 02.30.Gp, 02.40.Xx

AMS Classification scheme numbers: 34M55, 47E05, 81Qxx, 32G34, 34Lxx, 34Mxx, 14Kxx

Key-words: Diagonals of rational functions, pullbacked hypergeometric functions, modular forms, modular equations, Hauptmoduls, creative telescoping, telescopers, series with integer coefficients, globally bounded series.

1. Introduction

It was shown in [1, 2] that different physical related quantities, like the n-fold integrals $\chi^{(n)}$, corresponding to the n-particle contributions of the magnetic susceptibility of the Ising model [3, 4, 5, 6], or the *lattice Green functions* [7, 8, 9, 10, 11], are *diagonals of rational functions* [12, 13, 14, 15, 16, 17].

While showing that the n-fold integrals $\chi^{(n)}$ of the susceptibility of the Ising model are diagonals of rational functions requires some effort, seeing that the lattice

Green functions are diagonals of rational functions nearly follows from their definition. For example, the lattice Green functions (LGF) of the d-dimensional face-centred cubic (fcc) lattice are given [10, 11] by:

$$\frac{1}{\pi^d} \int_0^{\pi} \cdots \int_0^{\pi} \frac{\mathrm{d}k_1 \, \cdots \, \mathrm{d}k_d}{1 \, - x \cdot \, \lambda_d}, \qquad \text{with:} \qquad \lambda_d = \binom{d}{2}^{-1} \sum_{i=1}^{d} \sum_{j=i+1}^{d} \cos(k_i) \, \cos(k_j). \quad (1)$$

The LGF can easily be seen to be a diagonal of a rational function: introducing the complex variables $z_j = e^{i \, k_j}$, $j = 1, \cdots, d$, the LGF (1) can be seen as a d-fold generalization of Cauchy's contour integral [1]:

$$\mathsf{Diag}(\mathcal{F}) \quad = \quad \frac{1}{2\pi i} \oint_{\gamma} \mathcal{F}(z_1, z/z_1) \, \frac{dz_1}{z_1}. \qquad (2)$$

Furthermore, the linear differential operators annihilating the physical quantities mentioned earlier $\chi^{(n)}$, are reducible operators. Being reducible they are "breakable" into smaller factors [4, 5] that happen to be elliptic functions, or generalizations thereof: *modular forms*, Calabi-Yau operators [18, 19]... Yet there exists a class of diagonals of rational functions in *three* variables†† whose diagonals are pullbacked ${}_2F_1$ hypergeometric functions, and in fact *modular forms* [21]. These sets of diagonals of rational functions in *three* variables in [21] were obtained by imposing the coefficients of the polynomial $P(x, y, z)$ appearing in the rational function $1/P(x, y, z)$ defining the diagonal to be 0 or 1¶.

While these constraints made room for exhaustivity, they were quite arbitrary, which raises the question of randomness of the sample : is the emergence of modular forms [20], with the constraints imposed in [21], an artefact of the sample?

Our aim in this paper is to show that *modular forms* emerge for a much larger set of rational functions of three variables, than the one previously introduced in [21], firstly because we obtain a whole family of rational functions whose diagonals give modular forms by adjoining parameters, and secondly through considerations of symmetry.

In particular, we will find that the *seven-parameter* rational function of three variables, with a numerator equal to one and a denominator being a polynomial of degree two at most, given by:

$$R(x, \, y, \, z) \quad = \quad \frac{1}{a \; + b_1 \, x \; + b_2 \, y \; + b_3 \, z \; + c_1 \, y \, z \; + c_2 \, x \, z \; + c_3 \, x \, y}, \qquad (3)$$

can be expressed as a particular pullbacked ${}_2F_1$ hypergeometric function†

$$\frac{1}{P_2(x)^{1/4}} \cdot {}_2F_1\Big([\frac{1}{12}, \, \frac{5}{12}], \, [1], \, 1 \, - \, \frac{P_4(x)^2}{P_2(x)^3}\Big), \qquad (4)$$

where $P_2(x)$ and $P_4(x)$ are two polynomials of degree two and four respectively. We then focus on subcases where the diagonals of the corresponding rational functions can

†† Diagonals of rational functions of two variables are just algebraic functions, so one must consider *at least three* variables to obtain special functions.

¶ Or 0 or ± 1 in the four variable case also examined in [21].

† The selected ${}_2F_1([1/12, 5/12], [1], \mathcal{P})$ hypergeometric function is closely related to modular forms [22, 23]. This can be seen as a consequence of the identity with the Eisenstein series E_4 and E_6 and this very ${}_2F_1([1/12, 5/12], [1], \mathcal{P})$ hypergeometric function (see Theorem 3 page 226 in [24] and page 216 of [25]): $E_4(\tau) = {}_2F_1([1/12, 5/12], [1], 1728/j(\tau))^4$ (see also equation (88) in [22] for E_6).

be written as a pullbacked ${}_2F_1$ hypergeometric function with two rational function pullbacks that are algebraically related by *modular equations*†.

This seven-parameter family will then be generalized into an eight, nine and finally ten parameters family of rational functions that are reciprocal of a polynomial of three variables of degree at most three. We will finally show that each of the previous results yields an *infinite number of new exact pullbacked ${}_2F_1$ hypergeometric function results*, through symmetry considerations on monomial transformations and some function-dependent rescaling transformations.

2. Diagonals of rational functions of three variables depending on seven parameters

2.1. Recalls on diagonals of rational functions

Let us recall the definition of the diagonal of a rational function in n variables $\mathcal{R}(x_1, \dots, x_n) = \mathcal{P}(x_1, \dots, x_n)/\mathcal{Q}(x_1, \dots, x_n)$, where $\mathcal{P}$ and $\mathcal{Q}$ are polynomials of $x_1, \cdots, x_n$ with *integer coefficients* such that $\mathcal{Q}(0, \dots, 0) \neq 0$. The diagonal of $\mathcal{R}$ is defined through its multi-Taylor expansion (for small x_i's)

$$\mathcal{R}\Big(x_1, \, x_2, \, \dots, \, x_n\Big) \quad = \quad \sum_{m_1 \, = 0}^{\infty} \cdots \sum_{m_n \, = 0}^{\infty} R_{m_1, \, \dots, \, m_n} \cdot x_1^{m_1} \, \cdots \, x_n^{m_n}, \qquad (5)$$

as the series in *one variable* x:

$$Diag\Big(\mathcal{R}\Big(x_1, \, x_2, \, \dots, \, x_n\Big)\Big) \quad = \quad \sum_{m \, = 0}^{\infty} R_{m, \, m, \, \dots, \, m} \cdot x^m. \qquad (6)$$

Diagonals of rational functions of two variables are algebraic functions [27, 28]. Interesting cases of diagonals of rational functions thus require to consider rational functions of at least *three* variables.

2.2. A seven parameters family of rational functions of three variables

We obtained the diagonal of the rational function in *three* variables depending on *seven* parameters:

$$R(x, \, y, \, z) \quad = \quad \frac{1}{a \; + b_1 \, x \; + b_2 \, y \; + b_3 \, z \; \; + c_1 \, y \, z \; + c_2 \, x \, z \; + c_3 \, x \, y}. \qquad (7)$$

This result was obtained by:

- Running the *HolonomicFunctions* [29] package in mathematica for arbitrary parameters $a, b_1, \cdots, c_1, \cdots$ and obtaining a large-sized second order linear differential operator L_2.
- Running the maple command "*hypergeometricsols*" [30] for different sets of values of the parameters on the operator L_2, and guessing¶ the Gauss hypergeometric function ${}_2F_1$ with general parameters solution of L_2.

† Thus providing a nice illustration of the fact that the diagonal is a modular form [23].
¶ The program "*hypergeometricsols*" [30] does not run for arbitrary parameters, hence our recourse to guessing.

2.3. The diagonal of the seven parameters family of rational functions: the general form

We find the following experimental results: all these diagonals are expressed in terms of *only one pullbacked hypergeometric function*. This is worth noticing since, in general, when an order-two linear differential operator has pullbacked ${}_2F_1$ hypergeometric function solutions, the "hypergeometricsols" command gives the two solutions as *sums of two* ${}_2F_1$ *hypergeometric functions*. Here, quite remarkably, the result is "encapsulated" in just one pullbacked hypergeometric function. Furthermore we find that all these diagonals are expressed as pullbacked hypergeometric functions of the form

$$\frac{1}{P_4(x)^{1/6}} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{7}{12}], [1], \frac{1728 \cdot x^3 \cdot P_5(x)}{P_4(x)^2}\Big), \tag{8}$$

where the two polynomials $P_4(x)$ and $P_5(x)$, in the $1728\, x^3\, P_5(x)/P_4(x)^2$ pullback, are polynomials of degree four and five in x respectively. The pullback in (8), given by $1728\, x^3\, P_5(x)/P_4(x)^2$, has the form $1 - \tilde{Q}$ where $\tilde{Q}$ is given by the simpler expression

$$\tilde{Q} \;=\; \frac{P_2(x)^3}{P_4(x)^2}, \tag{9}$$

where $P_2(x)$ is a polynomial of degree two in x. Recalling the identity

$${}_2F_1\Big([\frac{1}{12}, \frac{7}{12}], [1], x\Big) \;=\; (1\,-\,x)^{-1/12} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], \frac{-x}{1\,-\,x}\Big), \tag{10}$$

the previous pullbacked hypergeometric function (8) can be rewritten as

$$\frac{1}{P_2(x)^{1/4}} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], -\frac{1728 \cdot x^3 \cdot P_5(x)}{P_2(x)^3}\Big), \tag{11}$$

where $P_5(x)$ is the same polynomial of degree five as the one in the pullback in expression (8). This new pullback also has the form $1 - Q$ with Q given by‡:

$$-\,\frac{1728 \cdot x^3 \cdot P_5(x)}{P_2(x)^3} \;=\; 1\,-Q \qquad \text{where:} \qquad Q \;=\; \frac{P_4(x)^2}{P_2(x)^3}. \tag{12}$$

Finding the exact result for arbitrary values of the seven parameters now boils down to a guessing problem.

2.4. Exact expression of the diagonal for arbitrary parameters a, b_1, ..., c_1, ...

Now that the structure of the result is understood "experimentally" we obtain the result *for arbitrary parameters* a, b_1, b_2, b_3, c_1, c_2, c_3.

Assuming that the diagonal of the rational function (7) has the form explicited in the previous subsection

$$\frac{1}{P_2(x)^{1/4}} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], 1\,-\,\frac{P_4(x)^2}{P_2(x)^3}\Big), \tag{13}$$

where $P_2(x)$ and $P_4(x)$ are two polynomials of degree two and four respectively:

$$P_4(x) \;=\; A_4\, x^4 \,+\, A_3\, x^3 \,+\, A_2\, x^2 \,+\, A_1\, x \,+\, A_0, \tag{14}$$

$$P_2(x) \;=\; B_2\, x^2 \,+\, B_1\, x \,+\, B_0, \tag{15}$$

‡ Note that Q, given by (12), is the reciprocal of $\tilde{Q}$ given in (9): $Q \;=\; 1/\tilde{Q}$.

one can write the order-two linear differential operator having this eight-parameter solution (13), and identify this second order operator depending on eight arbitrary parameters, with the second order linear differential operator obtained using the *HolonomicFunctions* [29] program for arbitrary parameters. Using the results obtained for specific values of the parameters, one easily guesses that $A_0 = a^6$ and $B_0 = a^4$. One finally gets:

$$P_2(x) =$$
$$8 \cdot \Bigl(3\, a\, c_1\, c_2\, c_3 \, + 2 \cdot \, (b_1^2\, c_1^2 + \, b_2^2\, c_2^2 \, + \, b_3^2\, c_3^2 - \, b_1\, b_2\, c_1\, c_2 - \, b_1\, b_3\, c_1\, c_3 - \, b_2\, b_3\, c_2\, c_3)\Bigr) \cdot x^2$$
$$- 8 \cdot a \cdot \Bigl(a \cdot (b_1\, c_1 \, + \, b_2\, c_2 \, + \, b_3\, c_3) - 3\, b_1\, b_2\, b_3\Bigr) \cdot x \quad + a^4, \qquad (16)$$

and

$$P_4(x) \, = \, 216 \cdot c_1^2\, c_2^2\, c_3^2 \cdot x^4 \quad - 16 \cdot \Bigl(9 \cdot a\, c_1\, c_2\, c_3 \cdot (b_1\, c_1 + b_2\, c_2 \, + b_3\, c_3)$$
$$- 6 \cdot (b_1^2\, b_2\, c_1^2\, c_2 \, + b_1\, b_2^2\, c_1\, c_2^2 \, + b_1^2\, b_3\, c_1^2\, c_3 \, + b_1\, b_3^2\, c_1\, c_3^2 \, + b_2^2\, b_3\, c_2^2\, c_3 \, + \, b_2\, b_3^2\, c_2\, c_3^2)$$
$$+ 4 \cdot (b_1^3\, c_1^3 + \, b_2^3\, c_2^3 \, + \, b_3^3\, c_3^3) \, - 3\, b_1\, b_2\, b_3\, c_1\, c_2\, c_3\Bigr) \cdot x^3$$
$$+ 12 \cdot \Bigl(3\, a^3\, c_1\, c_2\, c_3 \, + 4 \cdot a^2 \cdot (b_1^2\, c_1^2 \, + b_2^2\, c_2^2 \, + b_3^2\, c_3^2)$$
$$+ 2 \cdot a^2 \cdot (b_1\, b_2\, c_1\, c_2 \, + b_1\, b_3\, c_1\, c_3 \, + b_2\, b_3\, c_2\, c_3)$$
$$- 12 \cdot a \cdot b_1\, b_2\, b_3 \cdot (b_1\, c_1 + b_2\, c_2 + b_3\, c_3) \, + 18 \cdot b_1^2\, b_2^2\, b_3^2\Bigr) \cdot x^2$$
$$- 12 \cdot a^3 \cdot \Bigl(a \cdot (b_1\, c_1 \, + \, b_2\, c_2 \, + \, b_3\, c_3) \, - 3\, b_1\, b_2\, b_3\Bigr) \cdot x \quad + a^6. \qquad (17)$$

The polynomial $P_5(x)$ in (12), given by $P_5(x) = (P_4(x)^2 - P_2(x)^3)/1728/x^3$, is a slightly larger polynomial of the form

$$P_5(x) = \, 27 \cdot c_1^4\, c_2^4\, c_3^4 \cdot x^5 \, + \, \cdots \, + q_1 \cdot x \, + q_0, \qquad \text{where:}$$
$$q_0 \, = \, - b_1\, b_2\, b_3\, a^3 \cdot (a\, c_1 \, - b_2\, b_3) \cdot (a\, c_2 \, - b_1\, b_3) \cdot (a\, c_3 \, - b_1\, b_2). \qquad (18)$$

The coefficient q_1 in x reads for instance:

$$q_1 \, = \, c_1\, c_2\, c_3\, (b_1\, b_2\, c_1\, c_2 + b_1\, b_3\, c_1\, c_3 + b_2\, b_3\, c_2\, c_3) \cdot a^5$$
$$- \Bigl(b_1^2\, b_2^2\, c_1^2\, c_2^2 + b_1^2\, b_3^2\, c_1^2\, c_3^2 + b_2^2\, b_3^2\, c_2^2\, c_3^2 \, - 8\, b_1\, b_2\, b_3\, c_1\, c_2\, c_3 \cdot (b_1\, c_1 + \, b_2\, c_2 + b_3\, c_3)\Bigr) \cdot a^4$$
$$- b_1\, b_2\, b_3 \cdot \Bigl(57\, b_1\, b_2\, b_3\, c_1\, c_2\, c_3$$
$$+ \, 8 \cdot (b_1^2\, b_2\, c_1^2\, c_2 + \, b_1^2\, b_3\, c_1^2\, c_3 + \, b_1\, b_2^2\, c_1\, c_2^2 \, + \, b_1\, b_3^2\, c_1\, c_3^2 + \, b_2^2\, b_3\, c_2^2\, c_3 + \, b_2\, b_3^2\, c_2\, c_3^2)\Bigr) \cdot a^3$$
$$+ 8\, b_1^2\, b_2^2\, b_3^2 \cdot (b_1^2\, c_1^2 + \, b_2^2\, c_2^2 + \, b_3^2\, c_3^2) \cdot a^2$$
$$+ 46 \cdot b_1^2\, b_2^2\, b_3^2 \cdot (b_1\, b_2\, c_1\, c_2 + \, b_1\, b_3\, c_1\, c_3 + \, b_2\, b_3\, c_2\, c_3) \cdot a^2$$
$$- 36 \cdot b_1^3\, b_2^3\, b_3^3 \cdot (b_1\, c_1 + b_2\, c_2 + b_3\, c_3) \cdot a \quad + 27\, b_1^4\, b_2^4\, b_3^4. \qquad (19)$$

Having "guessed" the exact result, one can easily verify directly that this exact pullbacked hypergeometric result is truly the solution of the large second order linear differential operator obtained using the "*HolonomicFunctions*" program [29].

2.5. Selected subcases of these results

When $P_2(x)^3 - P_3(x)^2 = -1728\, x^3 \cdot P_5(x) = 0$, the pullback in (13) (with (16), (17)) vanishes, and the previous exact result (13), for the diagonal of the rational function (7), degenerates into a simple algebraic function (see (11) and (13)):

$$\frac{1}{P_2(x)^{1/4}} \quad = \quad \frac{1}{P_4(x)^{1/6}}. \tag{20}$$

The condition $P_2^3 \,-\, P_3^2 \;=\; -1728\,x^3 \cdot P_5(x) \;=\; 0$ corresponds, for instance, to $c_3 \,=\, 0,\; b_1 \,=\, 0$, with the rational function

$$\frac{1}{a \,+ b_2\,y + b_3\,z \,+ c_1\,y\,z \,+ c_2\,x\,z}, \tag{21}$$

or to $c_3 \,=\, 0,\; c_1 = \, b_2\,b_3/a,\; c_2 = \, b_1\,b_3/a$, with the rational function:

$$\frac{1}{a^2 \,+\, a\,b_1\,x \,+\, a\,b_2\,y +\, a\,b_3\,z \,+ b_2\,b_3\,y\,z +\, b_1\,b_3\,x\,z}. \tag{22}$$

One easily verifies that the diagonals of the corresponding rational functions read respectively:

$$\frac{1}{\sqrt{a^2 \,-4\,b_2\,c_2 \cdot x}}, \qquad\qquad \frac{\sqrt{a}}{\sqrt{a^3 \,+4\,b_1\,b_2\,b_3 \cdot x}}. \tag{23}$$

2.6. Simple symmetries of this seven-parameter result

The different pullbacks

$$\mathcal{P}_1 \;=\; -\,\frac{1728 \cdot x^3 \cdot\, P_5(x)}{P_2(x)^3}, \qquad \frac{1728 \cdot x^3 \cdot\, P_5(x)}{P_4(x)^2}, \qquad 1 \,-\, \frac{P_4(x)^2}{P_2(x)^3}, \tag{24}$$

must be compatible with some obvious symmetries. They verify the relations

$$\begin{aligned} &\mathcal{P}_1(\lambda\cdot\, a,\;\lambda\cdot\, b_1,\;\lambda\cdot\, b_2,\;\lambda\cdot\, b_3,\;\lambda\cdot\, c_1,\;\lambda\cdot\, c_2,\;\lambda\cdot\, c_3,\;\;x) \\ &\qquad =\;\; \mathcal{P}_1(a,\;b_1,\;b_2,\;b_3,\;c_1,\;c_2,\;c_3,\;\;x). \end{aligned} \tag{25}$$

and

$$\begin{aligned} &\mathcal{P}_1\Big(a,\;\;\lambda_1\cdot\, b_1,\;\lambda_2\cdot\, b_2,\;\lambda_3\cdot\, b_3,\;\;\lambda_2\,\lambda_3\cdot\, c_1,\;\lambda_1\,\lambda_3\cdot\, c_2,\;\lambda_1\,\lambda_2\cdot\, c_3,\;\;\frac{x}{\lambda_1\,\lambda_2\,\lambda_3}\Big) \\ &\qquad =\;\; \mathcal{P}_1(a,\;b_1,\;b_2,\;b_3,\;\;c_1, c_2, c_3,\;\;x), \end{aligned} \tag{26}$$

where $\lambda,\;\lambda_1,\;\lambda_2$ and λ_3 are arbitrary complex numbers. A demonstration of these symmetry-invariance relations (25) and (26) is sketched in Appendix A.

2.7. A symmetric subcase $\tau\,\rightarrow\,3\,\tau$: ${}_2F_1([1/3, 2/3], [1], \mathcal{P})$

2.7.1. A few recalls on Maier's paper

We know from Maier [23] that the *modular equation* associated with† $\tau\,\rightarrow\,3\,\tau$ corresponds to the elimination of the z variable between the two rational pullbacks:

$$\mathcal{P}_1(z) \;\;=\;\; \frac{12^3\,\cdot\,z^3}{(z\,+27)\cdot\,(z\,+243)^3}, \qquad \mathcal{P}_2(z) \;\;=\;\; \frac{12^3\,\cdot\,z}{(z\,+27)\cdot\,(z\,+3)^3}. \tag{27}$$

† τ denotes the ratio of the two periods of the elliptic functions that naturally emerge in the problem [22].

Following Maier [23] one can also write the identities¶:

$$\Bigl(9 \cdot \Bigl({{z \, +27} \over {z \, +243}}\Bigr)\Bigr)^{1/4} \cdot {}_2F_1\Bigl([{{1} \over {12}}, {{5} \over {12}}], {{1728\, z^3} \over {(z \, +27) \cdot (z \, +243)^3}}\Bigr)$$

$$= \quad \Bigl({{1} \over {9}} \cdot \Bigl({{z \, +27} \over {z \, +3}}\Bigr)\Bigr)^{1/4} \cdot {}_2F_1\Bigl([{{1} \over {12}}, {{5} \over {12}}], {{1728\, z} \over {(z \, +27) \cdot (z \, +3)^3}}\Bigr) \qquad (28)$$

$$= \quad {}_2F_1\Bigl([{{1} \over {3}}, {{2} \over {3}}], [1], {{z} \over {z \, +27}}\Bigr). \qquad (29)$$

Having a hypergeometric function identity (28) with *two* rational pullbacks (27) related by a *modular equation* provides a good heuristic way to see that we have a *modular form* [22, 23]‡.

2.7.2. The symmetric subcase

Let us now consider the symmetric subcase $b_1 = b_2 = b_3 = b$ and $c_1 = c_2 = c_3 = c$. If we take that limit in our previous general expression (13), we obtain the solution of the order-two linear differential operator annihilating the diagonal†† in the form

$$\frac{1}{P_2(x)^{1/4}} \cdot {}_2F_1\Bigl([{{1} \over {12}}, {{5} \over {12}}], [1], \, 1 \, - \, {{P_4(x)^2} \over {P_2(x)^3}}\Bigr)$$

$$= \quad \frac{1}{P_2(x)^{1/4}} \cdot {}_2F_1\Bigl([{{1} \over {12}}, {{5} \over {12}}], [1], - \, {{1728 \cdot \, x^3 \, \cdot \, P_5(x)} \over {P_2(x)^3}}\Bigr), \qquad (30)$$

with

$$P_2(x) \quad = \quad a \cdot \, (24 \cdot \, c^3 \cdot \, x^2 \, - 24 \cdot \, b \cdot \, (a\, c \, - b^2) \cdot \, x \, \, + a^3), \qquad (31)$$

$$\begin{aligned} P_4(x) \quad = \quad & 216 \cdot \, c^6 \cdot \, x^4 \, - 432 \cdot \, b\, c^3 \cdot \, (a\, c \, - b^2) \cdot \, x^3 \\ & + 36 \cdot \, (a^3\, c^3 \, + 6 \cdot \, a^2\, b^2\, c^2 \, - 12 \cdot \, a\, b^4\, c + 6 \cdot \, b^6) \cdot \, x^2 \\ & \quad - 36 \cdot \, a^3\, b \cdot \, (a\, c \, - b^2) \cdot \, x \, \, + a^6. \end{aligned} \qquad (32)$$

and:

$$P_5(x) \quad = \quad (27\, c^3\, x^2 \, - 27\, b \cdot \, (a\, c - b^2) \cdot \, x \, \, + a^3) \cdot \, (c^3\, x \, - b \cdot \, (a\, c - b^2))^3. \qquad (33)$$

In this symmetric case, one can write the pullback in (30) as follows:

$$- \, {{1728 \cdot \, x^3 \, \cdot \, P_5(x)} \over {P_2(x)^3}} \quad = \quad {{12^3 \, \cdot \, z^3} \over {(z \, +27) \, \cdot \, (z \, +243)^3}}, \qquad (34)$$

where z reads:

$$z \quad = \quad - \, {{9^3 \, \cdot \, x \, \cdot \, (c^3 \, \cdot \, x \, \, - b \, \cdot \, (a\, c \, - b^2))} \over {27 \, \cdot \, c^3 \, \cdot \, x^2 \, \, - 27 \, \cdot \, b \, \cdot \, (a\, c \, - b^2) \, \cdot \, x \, \, + a^3}}. \qquad (35)$$

Injecting the expression (35) for z in $\mathcal{P}_2(z)$ given by (27), one gets another pullback

$$\mathcal{P}_2(z) \quad = \quad - \, 1728 \cdot \, x \, \cdot \, {{\tilde{P}_5} \over {\tilde{P}_2(x)^3}}, \qquad (36)$$

¶ One has hypergeometric identities on ${}_2F_1([1/3, 2/3], [1], \mathcal{P})$, however they are not associated with the involutive transformation $z \, \rightarrow \, 729/z$ as one could expect from the fact that the two Hauptmoduls in (28) are exchanged by this involution: see Appendix B.

‡ Something that is obvious here since we are dealing with a ${}_2F_1([1/12, 5/12], [1], x)$ hyperegeometric function which is known to be related modular functions [22, 23] due to its relation with the Eisenstein series E_4, but is less clear for other hypergeometric functions.

††Called the “telescoper” [31, 32].

with

$$\tilde{P}_5(x) \; = \; (27\, c^3\, x^2 \, - 27\, b \cdot (a\, c \, - b^2) \cdot x \, + a^3)^3 \cdot (c^3\, x \, - b \cdot (a\, c \, - b^2)). \qquad (37)$$

and:

$$\tilde{P}_2(x) \; = \; a \cdot (-216 \cdot c^3 \cdot x^2 \, + 216 \cdot b \cdot (a\, c \, - b^2) \cdot x \, + a^3), \qquad (38)$$

In this case the diagonal of the rational function can be written as a single hypergeometric function with two different pullbacks

$$\frac{1}{P_2(x)^{1/4}} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], -\frac{1728 \cdot x^3 \cdot P_5(x)}{P_2(x)^3}\Big)$$
$$= \; \frac{1}{\tilde{P}_2(x)^{1/4}} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], -\frac{1728 \cdot x \cdot \tilde{P}_5(x)}{\tilde{P}_2(x)^3}\Big), \qquad (39)$$

with the relation between the two pullbacks given by the *modular equation* associated with [22, 23] $\tau \, \rightarrow \, 3\,\tau$:

$$2^{27} \cdot 5^9 \cdot Y^3 Z^3 \cdot (Y+Z) \; + 2^{18} \cdot 5^6 \cdot Y^2 Z^2 \cdot (27\, Y^2 - 45946\, Y Z + 27\, Z^2)$$
$$+ 2^9 \cdot 5^3 \cdot 3^5 \cdot \; YZ \cdot (Y+Z) \cdot (Y^2 + 241433\, YZ + Z^2)$$
$$+ 729 \cdot (Y^4 \, + \, Z^4) \; - 779997924 \cdot (YZ^3 \, + Y^3 Z) \; + 31949606 \cdot 3^{10} \cdot Y^2 Z^2$$
$$+ 2^9 \cdot 3^{11} \cdot 31 \cdot Y\, Z \cdot (Y+Z) \; - 2^{12} \cdot 3^{12} \cdot YZ \quad = \quad 0.$$

2.7.3. Alternative expression for the symmetric subcase

Alternatively, we can obtain the exact expression of the diagonal using directly the "*HolonomicFunctions*" program [29] for arbitrary parameters a, b and c to get an order-two linear differential operator annihilating that diagonal. Then, using "*hypergeometricsols*"‡ we obtain the solution of this second order linear differential operator in the form

$$\frac{1}{a} \cdot {}_2F_1\Big([\frac{1}{3}, \frac{2}{3}], [1], \; -\frac{27}{a^3} \cdot x \cdot (c^3\, x \, - b \cdot (a\, c \, - b^2))\Big), \qquad (40)$$

which looks, at first sight, different from (30) with (31) and (32). This last expression (40) is compatible with the form (30) as a consequence of the identity:

$$\Big(\frac{9 \, - 8\, x}{9}\Big)^{1/4} \cdot {}_2F_1\Big([\frac{1}{3}, \frac{2}{3}], [1], x\Big) \quad = \quad {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], \frac{64\, x^3 \cdot (1 \, - x)}{(9 \, - 8\, x)^3}\Big). \qquad (41)$$

The reduction of the (generic) ${}_2F_1([1/12, 5/12], [1], \mathcal{P})$ hypergeometric function to a ${}_2F_1([1/3, 2/3], [1], \mathcal{P})$ form corresponds to a selected $\tau \, \rightarrow \, 3\,\tau$ modular equation situation (27) well described in [23].

These results can also be expressed in terms of ${}_2F_1([1/3, 1/3], [1], \, \mathcal{P})$ pullbacked hypergeometric functions [23] using the identities

$${}_2F_1\Big([\frac{1}{3}, \frac{1}{3}], [1], x) \;\; = \;\; (1 \, - x)^{-1/3} \cdot {}_2F_1\Big([\frac{1}{3}, \frac{2}{3}], [1], \; -\frac{x}{1 \, - x}) \qquad (42)$$
$$= \; \Big((1 \, - 9\, x)^3 \cdot (1 \, - \, x)\Big)^{-1/12} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], \; -\frac{64\; x}{(1 \, - 9\, x)^3 \cdot (1 \, - \, x)}\Big),$$

or:

$${}_2F_1\Big([\frac{1}{3}, \frac{1}{3}], [1], -\frac{x}{27}) \;\; = \;\; \Big(1 \, + \frac{x}{27}\Big)^{-1/3} \cdot {}_2F_1\Big([\frac{1}{3}, \frac{2}{3}], [1], \; \frac{x}{x \, + 27}\Big) \qquad (43)$$
$$= \; \Big(\frac{(x+3)^3 \cdot (x+27)}{729}\Big)^{-1/12} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], \frac{1728\, x}{(x+3)^3 \cdot (x+27)}\Big).$$

‡ We use M. van Hoeij "hypergeometricsols" program [30] for many values of a, b and c, and then perform some guessing.

2.8. A non-symmetric subcase $\tau \rightarrow 4\,\tau$: ${}_2F_1([1/2, 1/2], [1], \mathcal{P})$.

Let us consider the non-symmetric subcase $b_1 = b_2 = b_3 = b$ and $c_1 = c_2 = 0$, $c_3 = b^2/a$. The pullback in (30) reads:

$$\mathcal{P}_1 \, = \, -\, {{1728 \cdot x^3 \cdot \, P_5(x)} \over {P_2(x)^3}} \, = \, {{ 1728 \cdot \, a^3 \, b^{12} \cdot \, x^4 \cdot \, (16 \, b^3 \, x \, + \, a^3)} \over {(16 \, b^6 \, x^2 \, + \, 16 \, a^3 \, b^3 \, x \, + a^6)^3 }}. \qquad (44)$$

This pullback can be seen as the first of the two Hauptmoduls

$$\mathcal{P}_1 \, = \, {{1728 \cdot z^4 \cdot \, (z \, +16)} \over {(z^2 \, + \, 256 \, z \, + 4096)^3 }}, \qquad \mathcal{P}_2 \, = \, {{1728 \cdot z \cdot \, (z \, +16)} \over {(z^2 \, + \, 16 \, z \, + 16)^3 }}, \qquad (45)$$

provided z is given by‡:

$$z \, = \, {{256 \, b^3 \, x} \over {a^3}} \qquad \hbox{or:} \qquad z \, = \, {{-256 \, b^3 \cdot \, x} \over {a^3 \, + \, 16 \, b^3 \, x}}. \qquad (46)$$

These exact expressions (46) of z in terms of x give exact rational expressions of the second Hauptmodul $\mathcal{P}_2$ in terms of x:

$$\mathcal{P}_2^{(1)} \, = \, {{1728 \cdot \, a^{12} \, b^3 \cdot \, x \cdot \, (a^3 \, +16 \, b^3 \, x)^4 } \over {(4096 \, b^6 \, x^2 \, + 256 \, a^3 \, b^3 \, x \, + a^6)^3 }} \qquad \hbox{or:} \qquad (47)$$

$$\mathcal{P}_2^{(2)} \, = \, {{-1728 \cdot \, a^3 \, b^3 \cdot \, x \cdot \, (a^3 \, +16 \, b^3 \, x)^4 } \over {(256 \, b^6 \, x^2 \, - \, 224 \, a^3 \, b^3 \, x \, + a^6)^3 }}. \qquad (48)$$

These two pullbacks (44), (47) and (48) (or $\mathcal{P}_1$ and $\mathcal{P}_2$ in (45)) are related by a *modular equation* corresponding† to $\tau \rightarrow \, 4\,\tau$.

This subcase thus corresponds to the diagonal of the rational function being expressed in terms of a *modular form* associated to an identity on a hypergeometric function:

$$(16 \, b^6 \, x^2 \, + \, 16 \, a^3 \, b^3 \, x \, + a^6)^{-1/4} \cdot \, {}_2F_1\Bigl([{{1} \over {12}}, \, {{5} \over {12}}], \, [1], \, \mathcal{P}_1\Bigr)$$
$$= \, (4096 \, b^6 \, x^2 \, + 256 \, a^3 \, b^3 \, x \, + a^6)^{-1/4} \cdot \, {}_2F_1\Bigl([{{1} \over {12}}, \, {{5} \over {12}}], \, [1], \, \mathcal{P}_2^{(1)}\Bigr)$$
$$= \, (256 \, b^6 \, x^2 \, - \, 224 \, a^3 \, x \, + a^6)^{-1/4} \cdot \, {}_2F_1\Bigl([{{1} \over {12}}, \, {{5} \over {12}}], \, [1], \, \mathcal{P}_2^{(2)}\Bigr)$$
$$= \, {}_2F_1\Bigl([{{1} \over {2}}, \, {{1} \over {2}}], \, [1], \, -\, {{16 \cdot \, b^3} \over {a^3}} \cdot \, x\Bigr). \qquad (49)$$

The last equality is a consequence of the identity:

$${}_2F_1\Bigl([{{1} \over {2}}, \, {{1} \over {2}}], \, [1], \, -\, {{x} \over {16}}\Bigr) \qquad (50)$$
$$= \, 2 \cdot \, (x^2 \, +16 \, x \, +16)^{-1/4} \cdot \, {}_2F_1\Bigl([{{1} \over {12}}, \, {{5} \over {12}}], \, [1], \, {{1728 \cdot \, x \cdot \, (x+16)} \over {(x^2 \, +16 \, x \, +16)^3}}\Bigr).$$

Similarly, the elimination of x between the pullback $X \, = \, \mathcal{P}_1$ (given by (44)) and $Y \, = \, \mathcal{P}_2^{(1)}$ gives the *same modular equation* (representing $\tau \, \rightarrow \, 4\,\tau$) than the elimination of x between the pullback $X \, = \, \mathcal{P}_1$ (given by (44)) and $Y \, = \, \mathcal{P}_2^{(2)}$,

‡ These two expressions are related by the involution $z \, \leftrightarrow \, -16 \, z/(z \, + 16)$.
† See page 20 in [22].

namely:

$$825^9 \cdot X^6 Y^6 \quad - 389 \cdot 11^6 \cdot 5^{16} \cdot 3^{10} \cdot 2^6 \cdot X^5 Y^5 \cdot (X+Y)$$
$$+ 11^3 \cdot 5^{12} \cdot 3^7 \cdot 2^4 \cdot X^4 Y^4 \cdot \Big(26148290096 \cdot (X^2+Y^2) - 15599685235 \cdot X Y\Big)$$
$$- 105955481959 \cdot 5^{10} \cdot 3^7 \cdot 2^{15} \cdot X^3 Y^3 \cdot (X+Y) \cdot (X^2+Y^2)$$
$$+ 503027637092599 \cdot 5^{10} \cdot 3^7 \cdot 2^6 \cdot X^4 Y^4 \cdot (X+Y)$$
$$+ 5^6 \cdot 3^4 \cdot 2^{16} \cdot X^2 Y^2 \cdot \Big(1634268131 \cdot (X^4+Y^4) + 1788502080642816 \cdot X^2 Y^2$$
$$+ 848096080668355 \cdot (X^3 Y + X Y^3)\Big)$$
$$- 5^4 \cdot 3^4 \cdot 2^{22} \cdot X Y \cdot (X+Y) \cdot \Big(389 \cdot (X^4 + Y^4) + 41863592956503 \cdot X^2 Y^2$$
$$- 54605727143 \cdot (X^3 Y + X Y^3)\Big)$$
$$+ 2^{24} \cdot \Big(X^6 + Y^6 + 561444609 \cdot (X^5 Y + X Y^5)$$
$$+ 1425220456750080 \cdot (X^4 Y^2 + X^2 Y^4) + 2729942049541120 \cdot X^3 Y^3\Big)$$
$$- 5 \cdot 3^7 \cdot 2^{34} \cdot X Y \cdot (X+Y) \cdot (391 X^2 - 12495392 X Y + 391 Y^2) \qquad (51)$$
$$+ 31 \cdot 3^7 \cdot 2^{40} \cdot X Y \cdot (X + 2Y) \cdot (2X+Y) \quad - 3^9 \cdot 2^{42} \cdot X Y \cdot (X+Y) \quad = \quad 0.$$

The elimination of x between the pullback $X = \mathcal{P}_2^{(1)}$ (given by (44)) and the pullback $Y = \mathcal{P}_2^{(2)}$ also gives‡ the *same modular equation* (51).

2.9. ${}_2F_1([1/4, 3/4], [1], \mathcal{P})$ *subcases: walks in the quarter plane*

The diagonal of the rational function

$$\frac{2}{2 + (x+y+z) + x z + 1/2 \cdot x y} \quad = \quad \frac{4}{4 + 2 \cdot (x+y+z) + 2 x z + x y}, \qquad (52)$$

is given by the pullbacked hypergeometric function:

$$\Big(1 + \frac{3}{4} \cdot x^2\Big)^{-1/4} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], \frac{27 x^4 \cdot (x^2+1)}{(3 x^2 + 4)^3}\Big)$$
$$= \quad {}_2F_1\Big([\frac{1}{4}, \frac{3}{4}], [1], -x^2\Big), \qquad (53)$$

which is reminiscent of the hypergeometric series number 5 and 15 in Figure 10 of Bostan's HDR [33]. Such pullbacked hypergeometric function (53) corresponds to the rook walk problems [34, 35, 36].

Thus the diagonal of the rational function corresponding to the simple rescaling $(x, y, z) \longrightarrow (\pm\sqrt{-1}\, x, \pm\sqrt{-1}\, y, \pm\sqrt{-1}\, z)$ of (52) namely

$$R_\pm \quad = \quad \frac{2}{2 \pm \sqrt{-1} \cdot (x+y+z) - x z - 1/2 \cdot x y} \qquad (54)$$

or the diagonal of the rational function $(R_+ + R_-)/2$ reading

$$\frac{4 \cdot (4 - xy - 2 xz)}{y^2 x^2 + 4 x^2 yz + 4 x^2 z^2 + 4 x^2 - 8 xz + 4 y^2 + 8 yz + 4 z^2 + 16}, \qquad (55)$$

‡ This result can be also seen in the z variable (see (46)): see the details in Appendix C.

becomes (as a consequence of identity (53)):

$$\Big(1\,-\,\frac{3}{4}\cdot x^2\Big)^{-1/4} \cdot {}_2F_1\Big([\frac{1}{12},\, \frac{5}{12}],\, [1],\, \frac{27\,x^4\cdot\,(1\,-\,x^2)}{(4\,-\,3\,x^2)^3}\Big)$$
$$=\quad {}_2F_1\Big([\frac{1}{4},\, \frac{3}{4}],\, [1],\, x^2\Big). \qquad (56)$$

Remark: ${}_2F_1([1/4,3/4],[1],\mathcal{P})$ hypergeometric functions can be also seen as modular forms corresponding to identities with *two* pullbacks related by a modular equation. For example the following identity:

$${}_2F_1\Big([\frac{1}{4},\, \frac{3}{4}],\, [1],\, \frac{x^2}{(2\,-\,x)^2}\Big)$$
$$=\quad \Big(\frac{2\,-\,x}{2\cdot\,(1\,+\,x)}\Big)^{1/2} \cdot {}_2F_1\Big([\frac{1}{4},\, \frac{3}{4}],\, [1],\, \frac{4\,x}{(1\,+\,x)^2}\Big), \qquad (57)$$

where the two rational pullbacks

$$A \,\,=\,\, \frac{4\,x}{(1\,+\,x)^2}, \qquad\qquad B \,\,=\,\, \frac{x^2}{(2\,-\,x)^2}, \qquad (58)$$

are related by the *asymmetric*¶ modular equation:

$$81\cdot\, A^2\,B^2 \,-\,18\,A\,B\cdot\,(8\,B\,+A) \,\,+(A^2\,+80\cdot\,A\,B\,+64\,B^2) \,\,-\,64\,B \,\,=\,\,0. \qquad (59)$$

The modular equation (59) gives the following expansion for B seen as an *algebraic series*§ in A:

$$B \,\,=\,\, \frac{1}{64}A^2 \,\,+\frac{5}{256}A^3 \,\,+\frac{83}{4096}A^4 \,\,+\frac{163}{8192}A^5 \,\,+\frac{5013}{262144}A^6 \,\,+\,\,\cdots \qquad (60)$$

More details are given in Appendix D.

2.10. The generic case: modular forms, with just one rational pullback

The previous pullbacks in the pullbacked ${}_2F_1$ hypergeometric functions can be seen (and should be seen) as *Hauptmoduls* [23]. It is only in certain cases like in sections (2.7) or (2.8) that we encounter the situation underlined by Maier [23] of a representation of a modular form as a pullbacked hypergeometric function with *two* possible *rational pullbacks*, related by a modular equation of *genus zero*. These selected situations are recalled in Appendix E.

Simple examples of modular equations of genus zero with *rational pullbacks* include reductions of the generic ${}_2F_1([1/12,5/12],[1],\,\mathcal{P})$ hypergeometric function to selected hypergeometric functions like ${}_2F_1([1/2,1/2],[1],\,\mathcal{P})$, ${}_2F_1([1/3,2/3],[1],\,\mathcal{P})$, ${}_2F_1([1/4,3/4],[1],\,\mathcal{P})$, and also [25] ${}_2F_1([1/6,5/6],[1],\,\mathcal{P})$ (see for instance [26]).

However, in the generic situation corresponding to (13) we have a single hypergeometric function with two different pullbacks A and B

$${}_2F_1\Big([\frac{1}{12},\, \frac{5}{12}],\, [1],\, A\Big) \,\,=\,\, G\cdot\,{}_2F_1\Big([\frac{1}{12},\, \frac{5}{12}],\, [1],\, B\Big), \qquad (61)$$

with G an algebraic function of x, and where A and B *are related by an algebraic modular equation*, but one of the two pullbacks say A is a rational function given by

¶ At first sight one expects the two pullbacks (58) in a relation like (59) to be on the same footing, the *modular equation* between these two pullbacks being *symmetric*: see for instance [22]. This paradox is explained in detail in Appendix D

§ We discard the other root expansion $B \,=\, 1+A+\frac{5}{4}A^2+\frac{25}{16}A^3+\frac{31}{16}A^4\,+\cdots$ since $B(0)\,\neq\,0$.

(12) where $P_2(x)$ and $P_4(x)$ are given by (16), (17). The two pullbacks A and B are also related by a Schwarzian equation that can be written in a symmetric way in A and B:

$$\frac{1}{72} \, \frac{32\, B^2 \, - 41\, B \, + 36}{B^2 \cdot \, (B \, - 1)^2} \, \cdot \, \Bigl(\frac{dB}{dx} \Bigr)^2 \, + \, \{B, \, x\}$$
$$= \, \frac{1}{72} \, \frac{32\, A^2 \, - 41\, A \, + 36}{A^2 \cdot \, (A \, - 1)^2} \, \cdot \, \Bigl(\frac{dA}{dx} \Bigr)^2 \, + \, \{A, \, x\}. \qquad (62)$$

One can rewrite the exact expression (13) in the form

$$\frac{1}{P_2(x)^{1/4}} \cdot {}_2F_1\Bigl([\frac{1}{12}, \, \frac{5}{12}], \, [1], \, 1 \, - \, \frac{P_4(x)^2}{P_2(x)^3}\Bigr)$$
$$= \, \mathcal{B} \cdot {}_2F_1\Bigl([\frac{1}{12}, \, \frac{5}{12}], \, [1], \, B\Bigr), \qquad (63)$$

where $\mathcal{B}$ is an algebraic function, and B is another pullback related to the *rational* pullback $A \, = \, 1 \, - \, P_4(x)^2/P_2(x)^3$ by a modular equation. The pullback B is an *algebraic function*. In the generic case, *only one of the two pullbacks* (63) *can be expressed as a rational function*: see Appendix E for more details.

3. Eight, nine and ten-parameters generalizations

Adding randomly terms in the denominator of (7) yields diagonals annihilated by minimal linear differential operators of order higher than two: this is what happens when quadratic terms like x^2, y^2 or z^2 are added for example. This leads to irreducible telescopers [31, 32] (i.e. minimal order linear differential operators annihilating the diagonals) of *higher orders* than the previous order two, or to telescopers [31] of quite high orders that are not irreducible, but factor into many irreducible factors, one of them being of order larger than two.

With the idea of keeping the linear differential operators annihilating the diagonal of order two, we were able to generalize the seven-parameter family (7) by carefully choosing the terms added to the quadratic terms in (7) and still keep the linear differential operator annihilating the diagonal of order two.

3.1. Eight-parameter rational functions giving pullbacked ${}_2F_1$ hypergeometric functions for their diagonals

Adding the cubic term $x^2\, y$ to the denominator of (7) yields the rational function:

$$R(x, \, y, \, z) \, = \, \frac{1}{a \, + b_1\, x \, + \, b_2\, y \, + \, b_3\, z \, + \, c_1\, y\, z \, + \, c_2\, x\, z \, + \, c_3\, x\, y \, + \, d\, x^2\, y}. \qquad (64)$$

After obtaining the diagonal of (64) for several sets of values of the parameters, one can make the educated guess that the diagonal of the rational function (64) has the form

$$\frac{1}{P_3(x)^{1/4}} \cdot {}_2F_1\Bigl([\frac{1}{12}, \, \frac{5}{12}], \, [1], \, 1 \, - \, \frac{P_4(x)^2}{P_3(x)^3}\Bigr), \qquad (65)$$

where $P_3(x)$ and $P_4(x)$ are two polynomials of degree three and four respectively:

$$P_4(x) \, = \, \mathcal{A}_4\, x^4 \, + \, \mathcal{A}_3\, x^3 \, + \, \mathcal{A}_2\, x^2 \, + \mathcal{A}_1\, x \, + \mathcal{A}_0, \qquad (66)$$
$$P_3(x) \, = \, \mathcal{B}_3\, x^3 \, + \mathcal{B}_2\, x^2 \, + \mathcal{B}_1\, x \, + \mathcal{B}_0, \qquad (67)$$

and where the coefficients $\mathcal{A}_i$ and $\mathcal{B}_j$ are at most quadratic expressions in the parameter d appearing in the denominator of (64). The pullback in (65) has the form

$$1 \,-\, \frac{P_4(x)^2}{P_3(x)^3} \quad = \quad \frac{1728\, x^3\, P_6}{P_3(x)^3}, \tag{68}$$

where

$$\begin{aligned} P_4 \,=\,\,\,\, & p_4 \,\,\, + 216 \cdot \, b_3^2\, c_1^2 \cdot \, d^2 \cdot \, x^4 \,\,\, + d \cdot \, u_1 \cdot \, x^4 \\ & + a\, d \cdot \, u_2 \cdot \, x^3 \,\,\, - 144 \cdot \, a\, b_1\, b_3\, c_1^2\, d \cdot \, x^3 \\ & \quad - 144 \cdot \, b_2\, b_3^2\, d \cdot \, (b_1\, c_1 \, + 4\, b_2\, c_2 \, - 2\, b_3\, c_3) \cdot \, x^3 \\ & \quad + 36\, a^2 \, \cdot \, (a\, b_3\, c_1 \, - 2\, b_2\, b_3^2) \cdot \, d \cdot \, x^2, \\ P_3 \,=\,\,\,\, & p_2 \,\,\, - 48 \cdot \, c_1^2\, c_2 \cdot \, d \cdot \, x^3 \,\,\, + 24\, b_3 \cdot \, (a\, c_1 \, - 2\, b_2\, b_3) \cdot \, d \cdot \, x^2, \end{aligned} \tag{69}$$

with

$$\begin{aligned} u_1 \,=\,\, & 144 \cdot \, (2\, b_1\, c_1^3\, c_2 \,\,\, - 4\, b_2\, c_1^2\, c_2^2 \,\,\, - \,\, b_3\, c_1^2\, c_2\, c_3), \\ u_2 \,=\,\, & 72 \cdot \, (10\, b_2\, b_3\, c_1\, c_2 \,\, - \, a\, c_1^2\, c_2 \,\,\, - 2\, b_3^2\, c_1\, c_3), \end{aligned} \tag{70}$$

and where the polynomials p_2 and p_4 denote the polynomials $P_2(x)$ and $P_4(x)$ given by (16) and (17) in section (2): p_2 and p_4 correspond to the $d \,=\, 0$ limit.

3.2. Nine-parameter rational functions giving pullbacked ${}_2F_1$ hypergeometric functions for their diagonals

Adding now another cubic term $y\, z^2$ to the denominator of (64)

$$\frac{1}{a \,\, + b_1\, x \, + \, b_2\, y \, + \, b_3\, z \,\,\, + c_1\, y\, z \, + \, c_2\, x\, z \, + \, c_3\, x\, y \,\,\, + \,\, d\, x^2\, y \,\,\, + \,\, e\, y\, z^2}, \tag{71}$$

also yields linear differential operator annihilating the diagonal of (71) of order two. After computing the second order linear differential operator annihilating the diagonal of (71) for several values of the parameters with the "*HolonomicFunctions*" program [29], and, in a second step, obtaining their pullbacked hypergeometric solutions using the maple command "*hypergeometricsols*" [30], we find that the diagonal of the rational function (71) has the form

$$\frac{1}{P_4(x)^{1/4}} \cdot \, {}_2F_1\Big([\frac{1}{12}, \, \frac{5}{12}], \, [1], \,\, 1 \,-\, \frac{P_6(x)^2}{P_4(x)^3}\Big), \tag{72}$$

where $P_4(x)$ and $P_6(x)$ are two polynomials of degree four and six respectively:

$$\begin{aligned} P_4(x) \,\,=\,\,\,\, & p_2 \,\,\, + \,\, 16 \cdot \, d^2 \cdot \, e^2 \cdot \, x^4 \\ & - 16 \cdot \, \Big(3 \cdot \, c_2 \cdot \, (c_1^2 \cdot \, d \, + \, c_3^2 \cdot \, e) \,\,\, + (b_1\, c_1 \, + b_3\, c_3 \, - 14\, b_2\, c_2) \cdot \, d\, e\Big) \cdot \, x^3 \\ & \,\, + 8 \cdot \, (3\, a\, b_3\, c_1\, d \, + 3\, a\, b_1\, c_3\, e \,\, - \, a^2\, d\, e \,\, - 6\, b_2\, b_3^2\, d \, - 6\, b_2\, b_1^2\, e) \cdot \, x^2, \end{aligned} \tag{73}$$

and

$$
\begin{aligned}
P_6(x) \;=\;\;& p_4 \;\; - 12 \cdot a^4\, d\, e \, \cdot \, x^2 \\
& + 36 \cdot a^2 \Bigl(b_3 \cdot \; (a\, c_1 \, - 2\, b_2\, b_3) \cdot \, d \, + \, b_1 \cdot \, (a\, c_3 \, - 2\, b_1\, b_2) \cdot \, e \Bigr) \cdot \, x^2 \\
& - 72 \cdot \, a\, c_1 \cdot \, (a\, c_1\, c_2 \, - 10\, b_2\, b_3\, c_2 \, + 2\, b_3^2\, c_3) \cdot \, d \cdot \, x^3 \\
& - 72 \cdot \, a\, c_3 \cdot \, (a\, c_2\, c_3 \, - 10\, b_1\, b_2\, c_2 \, + 2\, b_1^2\, c_1) \cdot \, e \cdot \, x^3 \\
& - 144 \cdot \, b_2\, b_3^2 \cdot \, (b_1\, c_1 \, + 4\, b_2\, c_2 \, - 2\, b_3\, c_3) \cdot \, d \cdot \, x^3 \\
& - 144 \cdot \, b_2\, b_1^2 \cdot \, (b_3\, c_3 \, + 4\, b_2\, c_2 \, - 2\, b_1\, c_1) \cdot \, e \cdot \, x^3 \\
& - 144 \cdot \, a\, b_1\, b_3 \cdot \, (c_1^2 \cdot \, d \, + \, c_3^2 \cdot \, e) \cdot \, x^3 \\
& + 24 \cdot a \, (a\, b_3\, c_3 \, + \, a\, b_1\, c_1 \, - 20\, a\, b_2\, c_2 \, + \, 30\, b_1\, b_2\, b_3) \cdot \, d \cdot \, e \cdot \, x^3 \\
& + 216 \cdot \, (b_3^2\, c_1^2 \cdot \, d^2 \, + \, b_1^2\, c_3^2 \cdot \, e^2) \cdot \, x^4 \\
& - \, 144 \cdot \, c_1^2 \, c_2 \cdot \, (b_3\, c_3 \, + 4\, b_2\, c_2 \, - 2\, b_1\, c_1) \cdot \, d \cdot \, x^4 \\
& - \, 144 \cdot \, c_3^2 \, c_2 \cdot \, (b_1\, c_1 \, + 4\, b_2\, c_2 \, - 2\, b_3\, c_3) \cdot \, e \cdot \, x^4 \\
& + 48 \cdot \, a^2\, d^2 \cdot \, e^2 \cdot \, x^4 \;\; + 96 \cdot \, (b_1^2\, c_1^2 \, + b_3^2\, c_3^2 \, + 22\, b_2^2\, c_2^2) \cdot \, d \cdot \, e \cdot \, x^4 \\
& - 144 \cdot \, \Bigl((a\, b_3\, c_1 \, + 4\, b_2\, b_3^2) \cdot \, d \, + \, (a\, b_1\, c_3 \, + 4\, b_2\, b_1^2) \cdot \, e \Bigr) \cdot \, d \cdot \, e \cdot \, x^4 \\
& + 48 \cdot \, (b_1\, b_3\, c_1\, c_3 \, + 15\, a\, c_1\, c_2\, c_3 \, - 20\, b_1\, b_2\, c_1\, c_2 \, - 20\, b_2\, b_3\, c_2\, c_3) \cdot \, d \cdot \, e \cdot \, x^4 \\
& + 96 \cdot \, (b_1\, c_1 \, + 22\, b_2\, c_2 \, + b_3\, c_3) \cdot \, d^2 \cdot \, e^2 \cdot \, x^5 \\
& - 576\, c_2 \, \cdot \, (c_3^2 \cdot \, e \, + \, c_1^2 \cdot \, d) \cdot \, d\, e \, \cdot \, x^5 \\
& - 64 \cdot \, d^3 \cdot \, e^3 \cdot \, x^6,
\end{aligned} \tag{74}
$$

where the polynomials p_2 and p_4 are the polynomials $P_2(x)$ and $P_4(x)$ of degree two and four in x given by (16) and (17) in section (2): p_2 and p_4 correspond to the $d \, = \, e \, = \, 0$ limit

Note that the $d \, \leftrightarrow \, e$ symmetry corresponds to keeping c_2 fixed, but changing $c_1 \, \leftrightarrow \, c_3$ (or equivalently y fixed, $x \, \leftrightarrow \, z$).

Remark 1: The nine-parameter family (71) singles out x and y, but of course, similar families that single out x and z, or single out y and z exist, with similar results (that can be obtained permuting the three variables x, y and z).

Remark 2: Note that the simple symmetries arguments displayed in section (2.6) for the seven-parameter family straightforwardly generalize for this nine-parameter family. The pullback $\mathcal{H}$ in (72) verifies (as it should)

$$
\begin{aligned}
& \mathcal{H}\Bigl(a, \;\; \lambda_1 \cdot \, b_1, \; \lambda_2 \cdot \, b_2, \; \lambda_3 \cdot \, b_3, \;\; \lambda_2\, \lambda_3 \cdot \, c_1, \; \lambda_1\, \lambda_3 \cdot \, c_2, \; \lambda_1\, \lambda_2 \cdot \, c_3, \\
& \qquad\qquad \lambda_1^2\, \lambda_2 \cdot \, d, \;\;\; \lambda_3^2\, \lambda_2 \cdot \, e, \;\;\; {{x} \over {\lambda_1\, \lambda_2\, \lambda_3}} \Bigr) \\
& \quad = \;\; \mathcal{H}(a, \; b_1, \; b_2, \; b_3, \;\; c_1, c_2, c_3, \;\; d, \;\; e, \;\; x),
\end{aligned} \tag{75}
$$

and:

$$
\begin{aligned}
& \mathcal{H}\Bigl(\lambda \cdot \, a, \;\; \lambda \cdot \, b_1, \;\; \lambda \cdot \, b_2, \;\; \lambda \cdot \, b_3, \;\; \lambda \cdot \, c_1, \, \lambda \cdot \, c_2, \, \lambda \cdot \, c_3, \;\; \lambda \cdot \, d, \;\; \lambda \cdot \, e, \;\; x) \\
& \quad = \;\; \mathcal{H}(a, \; b_1, \; b_2, \; b_3, \;\; c_1, c_2, c_3, \;\; d, \;\; e, \;\; x).
\end{aligned} \tag{76}
$$

3.3. Ten-parameter rational functions giving pullbacked ${}_2F_1$ hypergeometric functions for their diagonals

Adding the *three cubic terms*‡ $x^2\, y$, $y^2\, z$ and $z^2\, x$ to the denominator of (7) we get the rational function:

$$R(x,\, y,\, z) \;=\; \frac{1}{a \;+ b_1\, x \,+\, b_2\, y \,+\, b_3\, z \;\, +\, c_1\, y\, z \,+\, c_2\, x\, z \,+\, c_3\, x\, y \;\,+\; d_1\, x^2\, y \;+\; d_2\, y^2\, z \,+\; d_3\, z^2\, x}. \tag{77}$$

While (77) is *not* a generalization of (71), it is a generalization of (64) .

After computing the second order linear differential operator annihilating the diagonal of (77) for several values of the parameters with the "*HolonomicFunctions*" program [29], and, in a second step, their pullbacked hypergeometric solutions using "*hypergeometricsols*" [30], we find that the diagonal of the rational function (77) has the experimentally observed form:

$$\frac{1}{P_3(x)^{1/4}} \cdot {}_2F_1\Big([\frac{1}{12},\, \frac{5}{12}],\, [1],\, 1\, -\, \frac{P_6(x)^2}{P_3(x)^3}\Big). \tag{78}$$

Furthermore, the pullback in (78) is seen to be of the form:

$$1\, -\, \frac{P_6(x)^2}{P_3(x)^3} \;\;=\;\; \frac{1728\, x^3 \cdot\, P_9}{P_3(x)^3}. \tag{79}$$

The polynomial $P_3(x)$ reads

$$\begin{aligned} P_3(x) \,=\, p_2 \,\; -\, 24 \cdot \Big(& 9 \cdot a \cdot d_1\, d_2\, d_3 \;-\, 6 \cdot (b_1\, c_3 \cdot d_2\, d_3 \,+\, b_2\, c_1 \cdot d_1\, d_3 \,+\, b_3\, c_2 \cdot d_1\, d_2) \\ & +\, 2 \cdot (c_1^2\, c_2\, d_1 \,+\, c_1\, c_3^2\, d_3 \,+\, c_2^2\, c_3\, d_2)\Big) \cdot x^3 \\ +\, 24 \cdot \Big(a \cdot (b_1\, c_2\, d_2 \,+\, b_2\, c_3\, d_3 \,&+\, b_3\, c_1\, d_1) \,-\, 2 \cdot (b_1^2\, b_3\, d_2 \,+\, b_1\, b_2^2\, d_3 \,+\, b_2\, b_3^2\, d_1)\Big) \cdot x^2, \end{aligned} \tag{80}$$

where p_2 is the polynomial $P_2(x)$ of degree two in x given by (16) in section (2): p_2 corresponds to the $d_1 \,=\, d_2 \,=\, d_3 \,=\, 0$ limit. The expression of the polynomial $P_6(x)$ is more involved. It reads:

$$P_6(x) \;=\; \, p_4 \;+\; \Delta_6(x), \tag{81}$$

where p_4 is the polynomial $P_4(x)$ of degree four in x given by (17) in section (2). The expression of polynomial $\Delta_6(x)$ of degree six in x is quite large and is given in Appendix F.

Remark 1: A set of results and subcases (sections (3.3.2) and (3.3.3)), can be used to "guess" the general exact expressions of the polynomials $P_3(x)$ and $P_6(x)$ in (78) for the ten-parameters family (77). From the subcase $d_3 \,=\, 0$ of section (3.3.1) below, it is easy to see that one can deduce similar exact results for $d_1 \,= 0$ or $d_2 \,= 0$: it just amounts to performing some cyclic transformation $x \,\rightarrow\, y \,\rightarrow\, z \,\rightarrow\, x$ which corresponds to transformation $b_1 \,\rightarrow\, b_2 \,\rightarrow\, b_3 \,\rightarrow\, b_1$, $c_1 \,\rightarrow\, c_2 \,\rightarrow\, c_3 \,\rightarrow\, c_1$, $d_1 \,\rightarrow\, d_2 \,\rightarrow\, d_3 \,\rightarrow\, d_1$. One can see P_3 and $P_6(x)$ as p_2 and p_4 given by (16) and (17) plus some corrections given, in Appendix G, by (G.1) and (G.2) for $d_3 \,=\, 0$, and similar corrections† for $d_1 \,=\, 0$ and $d_2 \,=\, 0$, plus corrections of the form $d_1\, d_2\, d_3 \times$ something. These last terms are the most difficult to get. We already know some of

‡ An equivalent family of ten-parameter rational functions amounts to adding $x\, y^2$, $y\, z^2$ and $z\, x^2$.
† Taking care of the double counting !

these terms from (88) and (89) in section (3.3.2) below. Furthermore, the symmetry constraints (83) and (82) below, as well as other constraints corresponding to the symmetric subcase of section (3.3.3), give additional constraints on the kind of allowed final correction terms.

Remark 2: Note, again, that the simple symmetries arguments displayed in section (2.6) for the seven-parameter family straightforwardly generalize for this ten-parameter family. The $\mathcal{H}$ pullback (79) in (78) verifies (as it should):

$$\mathcal{H}\Bigl(a, \quad \lambda_1 \cdot b_1, \; \lambda_2 \cdot b_2, \; \lambda_3 \cdot b_3, \quad \lambda_2\,\lambda_3 \cdot c_1, \; \lambda_1\,\lambda_3 \cdot c_2, \; \lambda_1\,\lambda_2 \cdot c_3,$$
$$\lambda_1^2\,\lambda_2 \cdot d_1, \quad \lambda_2^2\,\lambda_3 \cdot d_2, \; \lambda_3^2\,\lambda_1 \cdot d_3, \quad \frac{x}{\lambda_1\,\lambda_2\,\lambda_3}\Bigr)$$
$$= \quad \mathcal{H}(a, \; b_1, \; b_2, \; b_3, \; c_1, c_2, c_3, \; d_1, \; d_2, \; d_3, \; x), \qquad (82)$$

and:

$$\mathcal{H}\Bigl(\lambda \cdot a, \; \lambda \cdot b_1, \; \lambda \cdot b_2, \; \lambda \cdot b_3, \; \lambda \cdot c_1, \lambda \cdot c_2, \lambda \cdot c_3, \; \lambda \cdot d_1, \; \lambda \cdot d_2, \; \lambda \cdot d_3, \; x)$$
$$= \quad \mathcal{H}(a, \; b_1, \; b_2, \; b_3, \; c_1, c_2, c_3, \; d_1, \; d_2, \; d_3, \; x). \qquad (83)$$

Remark 3: Do note that adding arbitrary sets of cubic terms yields telescopers [31] of order larger than two: *the corresponding diagonals are no longer pullbacked* ${}_2F_1$ *hypergeometric functions.*

Let us just now focus on simpler subcases whose results are easier to obtain than in the general case (77).

3.3.1. Noticeable subcases of (77): a nine-parameter rational function

Instead of adding three cubic terms, let us add two cubic terms. This amounts to restricting the rational function (77) to the $d_3 = 0$ subcase

$$\frac{1}{a \; + b_1\,x \, + b_2\,y \; + b_3\,z \; + c_1\,y\,z \; + c_2\,x\,z \; + c_3\,x\,y \; + d_1\,x^2\,y \; + d_2\,y^2\,z}, \qquad (84)$$

which *cannot be reduced to the nine parameter family* (71) even if it looks similar. The diagonal of the rational function (84) has the *experimentally observed* form

$$\frac{1}{P_3(x)^{1/4}} \cdot {}_2F_1\Bigl([\frac{1}{12}, \frac{5}{12}], [1], \; 1 \, - \, \frac{P_5(x)^2}{P_3(x)^3}\Bigr), \qquad (85)$$

where $P_3(x)$ and $P_5(x)$ are two polynomials of degree respectively three and five in x. Furthermore the pullback in (85) has the form:

$$1 \, - \, \frac{P_5(x)^2}{P_3(x)^3} \quad = \quad \frac{1728\,x^3 \cdot \, P_7}{P_3(x)^3}. \qquad (86)$$

The two polynomials $P_3(x)$ and $P_5(x)$ are given in Appendix G.

3.3.2. Cubic terms subcase of (77)

A simple subcase of (77) corresponds to $b_1 \, = \, b_2 \, = b_3 \, = \, c_1 \, = \, c_2 \, = c_3 \, = \, 0$, namely to the rational function:

$$R(x, \, y, \, z) \quad = \quad \frac{1}{a \;\; + d_1 \cdot \, x^2\,y \;\; + d_2 \cdot \, y^2\,z \; + d_3 \cdot \, z^2\,x},$$

whose diagonal reads

$$ {}_2F_1\Bigl([\frac{1}{3}, \frac{2}{3}], [1], \quad -27 \cdot \frac{d_1\, d_2\, d_3}{a^3} \cdot x^3\Bigr) \tag{87}$$
$$ = \quad \Bigl(1\, -\, 216 \, \cdot \, \frac{d_1\, d_2\, d_3}{a^3} \cdot x^3\Bigr)^{-1/4} \cdot {}_2F_1\Bigl([\frac{1}{12}, \frac{5}{12}], [1], \ 1\, -\, \frac{P_6(x)^2}{P_3(x)^3}\Bigr), $$

with:

$$ P_3(x) \ = \ -216 \cdot a\ d_1\ d_2\ d_3 \cdot x^3 \ + a^4, \tag{88}$$
$$ P_6(x) \ = \ -\, 5832 \cdot d_1^2\ d_2^2\ d_3^2 \cdot x^6 \ + 540 \cdot a^3\ d_1\, d_2\, d_3 \cdot x^3 \ + a^6. \tag{89}$$

Relation (87) actually corresponds to the hypergeometric identities:

$$ {}_2F_1\Bigl([\frac{1}{3}, \frac{2}{3}], [1], -27\, X\Bigr) \tag{90}$$
$$ = \ \Bigl(1\, -\, 216\, X\Bigr)^{-1/4} \cdot {}_2F_1\Bigl([\frac{1}{12}, \frac{5}{12}], [1], -\, \frac{1728\, X\, \cdot\, (1\, +\, 27\, X)^3}{(1\, -\, 216\, X)^3}\Bigr) $$
$$ = \ \Bigl(1\, -\, 216\, \cdot\, X\Bigr)^{-1/4} \cdot {}_2F_1\Bigl([\frac{1}{12}, \frac{5}{12}], [1], \ 1\, -\, \frac{(1\, +540\, X\, -\, 5832\, X^2)^2}{(1\, -\, 216\, X)^3}\Bigr). $$

3.3.3. A symmetric subcase of (77)

Let us also consider another simple very symmetric subcase of (77). For $b_1 \, = \, b_2 \, = \, b_3 \, = \, b$, $c_1 \, = \, c_2 \, = \, c_3 \, = \, c$, $d_1 \, = \, d_2 \, = \, d_3 \, = \, d$, the diagonal reads‡

$$ \frac{1}{a\, -\, 6\ d\cdot\ x} \cdot {}_2F_1\Bigl([\frac{1}{3}, \frac{2}{3}], [1], \ \mathcal{P}\Bigr), \tag{91}$$

where the pullback $\mathcal{P}$ reads:

$$ \mathcal{P} \ = \ -\, \frac{27\, x\, \cdot\, \Bigl(a^2\, d\, -\, a\, b\, c\, +\, b^3 \ + (c^3\, -\, 3\, b\, c\, d\, -\, 3\, a\, d^2) \cdot\, x \ + 9\ d^3 \cdot\, x^2\Bigr)}{(a\, -\, 6\ d\, \cdot\, x)^3}. \tag{92}$$

At first sight the hypergeometric result (91) with the pullback (92) does not seem to be in agreement with the hypergeometric result (87) of section (3.3.2). In fact these two results are in agreement as a consequence of the hypergeometric identity:

$$ \frac{1}{1\, -\, 6\, X} \cdot {}_2F_1\Bigl([\frac{1}{3}, \frac{2}{3}], [1], \quad -\frac{27\, \cdot\, X\, \cdot\, (1\, -\, 3\, X\, +\, 9\, X^2)}{(1\, -\, 6\, X)^3}\Bigr) $$
$$ = \ {}_2F_1\Bigl([\frac{1}{3}, \frac{2}{3}], [1], \quad -27\, \cdot\, X^3\Bigr) \qquad \text{with:} \qquad X \ = \ \frac{d\, \cdot\, x}{a}. \tag{93}$$

This hypergeometric result (87) can also be rewritten in the form (78) where the two polynomials $P_3(x)$ and $P_6(x)$ read respectively:

$$ P_3(x) \ = \ -\, 72 \cdot\, d \cdot\, (3\, ad^2 \, -\, 6\, bcd\, +\, 2\, c^3) \cdot\, x^3 \ + 24 \cdot\, (3\, abc\, d\, +\, ac^3 \, -\, 6\, b^3\, d) \cdot\, x^2 $$
$$ -\, 24 \cdot\, a\, b \cdot\, (ac\, -\, b^2) \cdot\, x \quad + a^4, \tag{94}$$
$$ P_6(x) \ = \ -\, 5832 \cdot\, d^6 \cdot\, x^6 \ + 3888 \cdot\, c\, d^3 \cdot\, (3\, b\, d\, -\, c^2) \cdot\, x^5 $$
$$ -\, 216 \cdot\, (18\, abc\, d^3 \, +\, 18\, b^3\, d^3 \, -\, 12\, ac^3 d^2 \, -\, 9\, b^2 c^2\, d^2 \ + 6\, bc^4\, d \ \, -\, c^6) \cdot\, x^4 $$
$$ +\, 108 \cdot\, (5\, a^3\, d^3 \ -\, 18\, a^2 bc\, d^2 \ -\, 2\, a^2 c^3\, d\, +\, 12\, ab^2 c^2\, d \ + 24\, ab^3\, d^2 \, -\, 4\, a\, bc^4 $$
$$ -\, 12\, b^4 c\, d\, +\, 4\, b^3 c^3) \cdot\, x^3 $$
$$ +\, 36 \cdot\, (3\, a^3 bc\, d\, -\, 6\, a^2 b^3\, d\, +\, a^3 c^3 \, +\, 6\, a^2 b^2 c^2 \ -\, 12\, ab^4 c\, +\, 6\, b^6) \cdot\, x^2 $$
$$ -\, 36 \cdot\, a^3\, b \cdot\, (ac\, -\, b^2) \cdot\, x \quad + a^6. \tag{95}$$

‡ Note that trying to mix the two previous subcases imposing $b_1 = b_2 = b_3 = b$, $c_1 = c_2 = c_3 = c$ with d_1, d_2 , d_3 no longer equal, do not yield a ${}_2F_1([1/3, 2/3], [1], \mathcal{P})$ hypergeometric function.

4. Transformation symmetries of the diagonals of rational functions

The previous results can be expanded through symmetry considerations.

We are first going to see that performing monomial transformations on each of the previous (seven-parameter, eight, nine or ten-parameter) rational functions yields an *infinite number* of rational functions whose diagonals are pullbacked ${}_2F_1$ hypergeometric functions.

4.1. $(x, y, z) \rightarrow (x^n, y^n, z^n)$ symmetries

We have a first remark: once we have an exact result for a diagonal, we immediately get another diagonal by changing (x, y, z) into (x^n, y^n, z^n) for any positive integer n in the rational function. As a result we obtain a new expression for the diagonal changing x into x^n.

A simple example amounts to revisiting the fact that the diagonal of (54) given below is the hypergeometric function (56). Changing (x, y, z) into $(8\,x^2, 8\,y^2, 8\,z^2)$ in (54), one obtains the pullbacked ${}_2F_1$ hypergeometric function number 5 or 15 in Figure 10 of Bostan's HDR [33] (see also [34, 35, 36])

$$
{}_2F_1\Big([\frac{1}{4}, \frac{3}{4}], [1], 64\,x^4\Big), \tag{96}
$$

can be seen as the diagonal of

$$
\frac{1}{2 \; + 8\sqrt{-1} \cdot (x^2+y^2+z^2) \; - \; 64\,x^2\,z^2 \; - \; 32 \cdot x^2\,y^2}, \tag{97}
$$

which is tantamount to saying that the transformation $(x, y, z) \rightarrow (x^n, y^n, z^n)$ is a symmetry.

4.2. Monomial transformations on rational functions

More generally, let us consider the monomial transformation

$$
\begin{aligned}
(x, y, z) \quad &\longrightarrow \quad M(x, y, z) \; = \; (x_M, y_M, z_M) \\
&= \; \Big(x^{A_1} \cdot y^{A_2} \cdot z^{A_3}, \; x^{B_1} \cdot y^{B_2} \cdot z^{B_3}, \; x^{C_1} \cdot y^{C_2} \cdot z^{C_3}\Big),
\end{aligned} \tag{98}
$$

where the A_i's, B_i's and C_i's are positive integers such that $A_1 = A_2 = A_3$ is excluded (as well as $B_1 = B_2 = B_3$ as well as $C_1 = C_2 = C_3$), and that the determinant of the 3×3 matrix

$$
\left[\begin{array}{ccc} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{array} \right], \tag{99}
$$

is not equal to zero††, and that:

$$
A_1 \, + B_1 \, + C_1 \; = \; A_2 \, + B_2 \, + C_2 \; = \; A_3 \, + B_3 \, + C_3. \tag{100}
$$

We will denote by n the integer in these three equal† sums (100): $n \, = \, A_i \, + B_i \, + C_i$. The condition (100) is introduced in order to impose that the product¶ of $x_M\,y_M\,z_M$ is an integer power of the product of $x\,y\,z$: $x_M\,y_M\,z_M \; = \; (x\,y\,z)^n$.

††We want the rational function $\tilde{\mathcal{R}} \, = \, \mathcal{R}(M(x, y, z))$ deduced from the monomial transformation (98) to remain a rational function of three variables and not of two, or one, variables.

† For $n \, = \, 1$ the 3×3 matrix (99) is stochastic and transformation (98) is a *birational transformation.*

¶ Recall that taking the diagonal of a rational function of three variables extracts, in the multi-Taylor expansion (5), only the terms that are n-th power of the *product* $x\,y\,z$.

If we take a rational function $\mathcal{R}(x,\, y,\, z)$ in three variables and perform a monomial transformation (98) $(x,\, y,\, z) \, \rightarrow \, M(x,\, y,\, z)$, on the rational function $\mathcal{R}(x,\, y,\, z)$, we get another rational function that we denote by $\tilde{\mathcal{R}} \, = \, \mathcal{R}(M(x,\, y,\, z))$. Now the diagonal of $\tilde{\mathcal{R}}$ is the diagonal of $\mathcal{R}(x,\, y,\, z)$ where we have changed x into x^n:

$$\Phi(x) \, = \, Diag\Big(\mathcal{R}\Big(x,\, y,\, z\Big)\Big), \qquad Diag\Big(\tilde{\mathcal{R}}\Big(x,\, y,\, z\Big)\Big) \, = \, \Phi(x^n). \qquad (101)$$

A demonstration of this result is sketched in Appendix H.

From the fact that the diagonal of the rational function

$$\frac{1}{1 \, +x \, +y \, +z \, +3 \cdot \, (x\, y \, + \, y\, z + \, x\, z)}, \qquad (102)$$

is the hypergeometric function

$${}_2F_1\Big([\frac{1}{3},\, \frac{2}{3}],\, [1], \, \, 27\, x \cdot \, (2 \, - 27\, x)\Big), \qquad (103)$$

one deduces immediately that the diagonal of the rational function (104) transformed by the monomial transformation $(x,\, y,\, z) \, \longrightarrow \, (z, \, \, x^2\, y, \, \, y\, z)$

$$\frac{1}{1 \, +y\, z \, +x^2\, y \, +3 \cdot \, (y\, z^2 \, + \, x^2\, y\, z \, + \, x^2\, y^2\, z)}, \qquad (104)$$

is the pullbacked hypergeometric function

$${}_2F_1\Big([\frac{1}{3},\, \frac{2}{3}],\, [1], \, \, 27\, x^2 \cdot \, (2 \, - 27\, x^2)\Big), \qquad (105)$$

which is (103) where $x \, \rightarrow \, x^2$.

To illustrate the point further, from the fact that the diagonal of the rational function

$$\frac{1}{1 \, +x \, +y \, +z \, + \, 3\, x\, y \, + \, 5\, y\, z + \, 7\, x\, z}, \qquad (106)$$

is the hypergeometric function

$$\frac{1}{(2712 \, x^2 \, - 96 \, x \, +1)^{1/4}} \qquad (107)$$
$$\times \, {}_2F_1\Big([\frac{1}{12},\, \frac{5}{12}],\, [1], \, \, 1 \, - \, \frac{(2381400\, x^4 \, - 181440\, x^3 \, + 7524\, x^2 - 144\, x + 1)^2}{(2712 \, x^2 \, - 96 \, x \, +1)^3}\Big),$$

one deduces immediately that the diagonal of the rational function (106) transformed by the monomial transformation $(x,\, y,\, z) \, \rightarrow \, (x\, z, \, \, x^2\, y, \, \, y^2\, z^2)$

$$\frac{1}{1 \, +x\, z \, +x^2\, y \, +y^2\, z^2 \, + \, 3\, x^2\, y^3 \, + \, 5\, x\, y^2\, z^3 + \, 7\, x^3\, y\, z}, \qquad (108)$$

is the hypergeometric function

$$\frac{1}{(2712 \, x^6 \, - 96 \, x^3 \, +1)^{1/4}} \qquad (109)$$
$$\times \, {}_2F_1\Big([\frac{1}{12},\, \frac{5}{12}],\, [1], \, \, 1 \, - \, \frac{(2381400\, x^{12} \, - 181440\, x^9 \, + 7524\, x^6 \, - 144\, x^3 \, + 1)^2}{(2712 \, x^6 \, - 96 \, x^3 \, +1)^3}\Big),$$

which is nothing but (107) where x has been changed into x^3.

We have the same result for more involved rational functions and more involved monomial transformations.

4.3. More symmetries on diagonals

Other transformation symmetries of the diagonals include the function-dependent rescaling transformation

$$(x,\, y,\, z) \quad \longrightarrow \quad \Big(F(x\,y\,z)\cdot\, x, \quad F(x\,y\,z)\cdot\, y, \quad F(x\,y\,z)\cdot\, z\Big), \qquad (110)$$

where $F(x\,y\,z)$ is a rational function† of the product of the three variables $x,\ y$ and z. Under such a transformation the previous diagonal $\Delta(x)$ becomes $\Delta(x\cdot F(x)^3)$.

For instance, changing

$$(x,\, y,\, z) \quad \longrightarrow \quad \Big(\frac{x}{1\,+\,7\,x\,y\,z},\ \frac{y}{1\,+\,7\,x\,y\,z},\ \frac{z}{1\,+\,7\,x\,y\,z}\Big), \qquad (111)$$

the rational function

$$\frac{1}{1\ -x\ -y-z\ +y\,z}, \qquad (112)$$

whose diagonal is ${}_2F_1([1/2,\,1/2],\,[1],\,16\,x)$, becomes the rational function

$$\frac{(1\,+\,7\,x\,y\,z)^2}{1\ -x-y-z\ +y\,z\ +14\,x\,y\,z\,-7\,x^2\,y\,z\,-7\,x\,y^2\,z\,-7\,x\,y\,z^2\ +49\,x^2\,y^2\,z^2}, \qquad (113)$$

which has the following diagonal:

$$\begin{aligned} {}_2F_1\Big([\frac{1}{2},\,\frac{1}{2}],\,[1],\,\frac{16\,x}{(1\,+\,7\,x)^3}\Big) \ =\ & 1\ +4\,x\ -48\,x^2\ +64\,x^3\ +3024\,x^4 \\ & -13524\,x^5\ -245196\,x^6\ +1933152\,x^7\ +21288192\,x^8\ -263440460\,x^9 \\ & -1758664568\,x^{10}\ +34575759792\,x^{11}\ +\ \cdots \qquad (114) \end{aligned}$$

To illustrate the point further take

$$(x,\, y,\, z) \quad \longrightarrow \quad \Big(x\cdot\, F,\ \ y\cdot\, F,\ \ z\cdot\, F\Big), \qquad \text{with:} \qquad (115)$$

$$F \ =\ \frac{1\,+\,2\,x\,y\,z}{1\,+\,3\,x\,y\,z\,+\,5\,x^2\,y^2\,z^2}\ =\ \Phi(x\,y\,z), \qquad (116)$$

$$\text{where:} \qquad \Phi(x)\ =\ \frac{1\,+\,2\,x}{1\,+\,3\,x\,+\,5\,x^2}, \qquad (117)$$

the rational function

$$\frac{1}{1\ +x\ +y\ +z\ \ +y\,z+\,x\,z\,+\,x\,y}, \qquad (118)$$

whose diagonal is ${}_2F_1([1/3,\,2/3],\,[1],\,-27\,x^2)$, becomes the rational function $P(x,\,y,\,z)/Q(x,\,y,\,z)$, where the numerator $P(x,\,y,\,z)$ and the denominator $Q(x,\,y,\,z)$, read respectively:

$$P(x,\,y,\,z)\ =\ (1\,+\,3\,x\,y\,z\,+\,5\,x^2\,y^2\,z^2)^2, \qquad (119)$$

$$\begin{aligned} Q(x,\,y,\,z)\ =\ & 25\,x^4y^4z^4\ +10\cdot\,(x^4y^3z^3\,+\,x^3y^4z^3\,+\,x^3y^3z^4)\ +30\,x^3y^3z^3 \\ & +4\cdot\,(x^3y^3z^2\,+\,x^3y^2z^3\,+\,x^2y^3z^3)\ +11\cdot\,(x^3y^2z^2\,+\,x^2y^3z^2\,+\,x^2y^2z^3) \\ & +19\,x^2y^2z^2\ +4\cdot\,(x^2y^2z\,+\,x^2yz^2\,+\,xy^2z^2)\ +5\cdot\,(x^2yz\,+\,xy^2z\,+\,xyz^2) \\ & +6\,xyz\ \ +xy+xz+yz\ \ +x+y+z\ \ +1. \qquad (120) \end{aligned}$$

† More generally one can imagine that $F(x\,y\,z)$ is the series expansion of an algebraic function.

The diagonal of this last rational function is equal to:

$$ {}_2F_1\Big([\frac{1}{3}, \frac{2}{3}], [1], \ -27 \cdot \Big(x \cdot \Phi(x)^3\Big)^2\Big) $$
$$ = \ {}_2F_1\Big([\frac{1}{3}, \frac{2}{3}], [1], \ -27\, x^2 \cdot \Big(\frac{1 \, + \, 2\,x}{1 \, + \, 3\,x \, + \, 5\,x^2}\Big)^6\Big). \qquad (121) $$

Let us give a final example: let us consider again the rational function (106) whose diagonal is (107), and let us consider the same function-rescaling transformation (115) with (116). One finds that the diagonal of the rational function

$$ \frac{1}{1 \, + F \cdot x \, + F \cdot y \, + F \cdot z \ + 3 \cdot F^2 \cdot x\,y \, + \, 5 \cdot F^2 \cdot y\,z + \, 7 \cdot F^2 \cdot x\,z}, \qquad (122) $$

is the hypergeometric function

$$ \frac{1}{(2712 \ x^2 \, \Phi(x)^6 \ - 96 \ x \ \Phi(x)^3 \ + 1)^{1/4}} \ \times \ {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], 1 \, - \mathcal{H}\Big), \qquad (123) $$

where the pullback $1 \, - \mathcal{H}$ reads:

$$ 1 \ - \ \frac{(2381400\, x^4 \, \Phi(x)^{12} \ - 181440\, x^3 \, \Phi(x)^9 \ + 7524\, x^2 \, \Phi(x)^6 \ - 144\, x \, \Phi(x)^3 \ + 1)^2}{(2712 \ x^2 \, \Phi(x)^6 \ - 96 \ x \, \Phi(x)^3 \ + 1)^3}\Big). $$

The pullbacked hypergeometric function (124) is nothing but (107) where x has been changed into $x \, \Phi(x)^3$.

A demonstration of these results is sketched in Appendix I.

Thus for each rational function belonging to one of the seven, eight, nine or ten parameters families of rational functions yielding a pullbacked ${}_2F_1$ hypergeometric function one can deduce from the transformations (110) an *infinite number* of other rational functions, with denominators of degree much *higher* than two or three.

One can combine these two sets of transformations, the monomial transformations (98) and the function-dependent rescaling transformations (110), thus yielding from each of the (seven, eight, nine or ten parameters) rational functions of the paper an *infinite number* of rational functions of quite high degree yielding pullbacked ${}_2F_1$ hypergeometric (modular form) exact results for their diagonals.

5. Conclusion

We found here that a seven-parameter rational function of three variables with a numerator equal to one and a of polynomial denominator of degree two at most, can be expressed as a pullbacked ${}_2F_1$ hypergeometric function. We generalized that result to eight, then nine and ten parameters, by adding specific cubic terms. We focused on subcases where the diagonals of the corresponding rational functions are pullbacked ${}_2F_1$ hypergeometric function with two possible rational function pullbacks algebraically related by *modular equations*, thus obtaining the result that the diagonal is a *modular form*†.

We have finally seen that simple monomial transformations, as well as a simple function rescaling of the three (resp. N) variables, are symmetries of the diagonals of rational functions of three (resp. N) variables. Consequently each of our previous families of rational functions, once transformed by these symmetries yield an *infinite*

† Differently from the usual definition of modular forms in the τ variables.

number of families of rational functions of three variables (of higher degree) whose diagonals are also pullbacked ${}_2F_1$ hypergeometric functions and, in fact, modular forms.

Since diagonals of rational functions emerge naturally in integrable lattice statistical mechanics and enumerative combinatorics, exploring the kind of exact results we obtain for diagonals of rational functions (modular forms, Calabi-Yau operators, pullbacked ${}_nF_{n-1}$ hypergeometric functions, ...) is an important systematic work to be performed to provide results and tools in integrable lattice statistical mechanics and enumerative combinatorics.

Acknowledgments. J-M. M. would like to thank G. Christol for many enlightening discussions on diagonals of rational functions. S. B. would like to thank the LPTMC and the CNRS for kind support. We would like to thank A. Bostan for useful discussions on creative telescoping. C.K. was supported by the Austrian Science Fund (FWF): P29467-N32. Y. A. was supported by the Austrian Science Fund (FWF): F5011-N15. We thank the Research Institute for Symbolic Computation, for access to the RISC software packages. We thank M. Quaggetto for technical support. This work has been performed without any support of the ANR, the ERC or the MAE, or any PES of the CNRS.

Appendix A. Simple symmetries of the diagonal of the rational function (7)

Let us recall the pullbacks (24) in section (2.6), that we denote $\mathcal{P}_1$.

Appendix A.1. Overall parameter symmetry

The seven parameters are defined up to an overall parameter (they must be seen as homogeneous variables). Changing $(a,\, b_1,\, b_2,\, b_3,\, c_1,\, c_2,\, c_3)$ into $(\lambda\cdot\, a,\, \lambda\cdot\, b_1,\, \lambda\cdot\, b_2,\, \lambda\cdot b_3,\, \lambda\cdot c_1,\, \lambda\cdot c_2,\, \lambda\cdot c_3)$ the rational function R given by (7) and its diagonal $Diag(R)$ are changed into R/λ and $Diag(R)/\lambda$. It is thus clear that the previous pullbacks (24), which totally "encode" the exact expression of the diagonal as a pullbacked hypergeometric function, must be invariant under this transformation. This is actually the case:

$$\begin{aligned}&\mathcal{P}_1(\lambda\cdot\, a,\;\lambda\cdot\, b_1,\;\lambda\cdot\, b_2,\;\lambda\cdot\, b_3,\;\lambda\cdot\, c_1,\;\lambda\cdot\, c_2,\;\lambda\cdot\, c_3,\;\; x)\\ &\qquad\qquad =\;\; \mathcal{P}_1(a,\; b_1,\; b_2,\; b_3,\; c_1,\; c_2,\; c_3,\;\; x).\end{aligned} \tag{A.1}$$

This result corresponds to the fact that $P_2(x)$ (resp. $P_4(x)$) is a *homogeneous polynomial* in the seven parameters $a,\, b_1, \cdots, c_1, \cdots$ of degree two (resp. four).

Appendix A.2. Variable rescaling symmetry

On the other hand, the rescaling of the three variables $(x,\, y,\, z)$ in (7), $(x,\, y,\, z)\, \rightarrow$ $(\lambda_1\cdot x,\;\lambda_2\cdot y,\;\lambda_3\cdot z)$ is a change of variables that is compatible with the operation of taking the diagonal of the rational function R.

When taking the diagonal and performing this change of variables, the monomials in the multi-Taylor expansion of (7) transform as:

$$a_{m,\,n,\,p}\cdot\, x^m\, y^n\, z^p \quad\longrightarrow\quad a_{m,\,n,\,p}\cdot\,\lambda_1^m\cdot\,\lambda_2^n\cdot\,\lambda_3^p\cdot\, x^m\, y^n\, z^p. \tag{A.2}$$

Taking the diagonal yields

$$a_{m,\,m,\,m} \cdot x^m \qquad \longrightarrow \qquad a_{m,\,m,\,m} \cdot (\lambda_1\,\lambda_2\,\lambda_3)^m \cdot x^m. \tag{A.3}$$

Therefore it amounts to changing $x \,\rightarrow\, \lambda_1\,\lambda_2\,\lambda_3 \cdot x$. With that rescaling $(x,\, y,\, z) \rightarrow (\lambda_1 \cdot x,\; \lambda_2 \cdot y,\; \lambda_3 \cdot z)$ the diagonal of the rational function remains invariant if one changes the seven parameters as follows:

$$(a,\;\; b_1,\, b_2,\, b_3,\;\; c_1,\, c_2,\, c_3) \qquad \longrightarrow$$
$$(a,\;\lambda_1 \cdot b_1,\;\lambda_2 \cdot b_2,\;\lambda_3 \cdot b_3,\;\lambda_2\,\lambda_3 \cdot c_1,\;\lambda_1\,\lambda_3 \cdot c_2,\;\lambda_1\,\lambda_2 \cdot c_3). \tag{A.4}$$

One deduces that the pullbacks (24) verify:

$$\mathcal{P}_1\Big(a,\;\;\lambda_1 \cdot b_1,\;\lambda_2 \cdot b_2,\;\lambda_3 \cdot b_3,\;\;\lambda_2\,\lambda_3 \cdot c_1,\;\lambda_1\,\lambda_3 \cdot c_2,\;\lambda_1\,\lambda_2 \cdot c_3,\;\;\frac{x}{\lambda_1\,\lambda_2\,\lambda_3}\Big)$$
$$=\;\;\mathcal{P}_1(a,\;b_1,\;b_2,\;b_3,\;\;c_1,\,c_2,\,c_3,\;\;x). \tag{A.5}$$

Appendix B. Comment on ${}_2F_1([1/3, 2/3], [1], \mathcal{P})$ as a modular form

From identity (28) of section (2.7)

$${}_2F_1\Big([\frac{1}{3},\,\frac{2}{3}],\,[1],\,\frac{z}{z\,+\,27}\Big) \tag{B.1}$$
$$=\;\;\Big(9 \cdot \Big(\frac{z\,+\,27}{z\,+\,243}\Big)\Big)^{1/4} \cdot\, {}_2F_1\Big([\frac{1}{12},\,\frac{5}{12}],\,\frac{1728\,z^3}{(z\,+\,27) \cdot\,(z\,+\,243)^3}\Big) \tag{B.2}$$
$$=\;\;\Big(\frac{1}{9} \cdot \Big(\frac{z\,+\,27}{z\,+\,3}\Big)\Big)^{1/4} \cdot\, {}_2F_1\Big([\frac{1}{12},\,\frac{5}{12}],\,\frac{1728\,z}{(z\,+\,27) \cdot\,(z\,+\,3)^3}\Big), \tag{B.3}$$

it is tempting to imagine an identity relating ${}_2F_1([1/3, 2/3], [1], \mathcal{P})$ with two different pullbacks.

Since switching the last two Hauptmoduls in (B.2) and (B.3) amounts to performing the involutive transformation $z \rightarrow 729/z$, it is tempting to imagine that the first ${}_2F_1$ hypergeometric function (B.1) is related to itself with $z \;\rightarrow\; 729/z$, namely that

$${}_2F_1\Big([\frac{1}{3},\,\frac{2}{3}],\,[1],\,\frac{27}{z\,+\,27}\Big) \;\;=\;\; {}_2F_1\Big([\frac{1}{3},\,\frac{2}{3}],\,[1],\,1\,-\,\frac{z}{z\,+\,27}\Big), \tag{B.4}$$

is related to‡

$${}_2F_1\Big([\frac{1}{3},\,\frac{2}{3}],\,[1],\,\frac{27}{729/z\,+\,27}\Big) \;\;=\;\; {}_2F_1\Big([\frac{1}{3},\,\frac{2}{3}],\,[1],\,\;\frac{z}{z\,+\,27}\Big). \tag{B.5}$$

This is the case, since (B.4) and (B.5) are solutions of the same linear ODE, but this does not mean that one can deduce an identity on the different pullbacks (B.4) and (B.5): the relation between these two hypergeometric functions (B.4) and (B.5) corresponds to a connection matrix [37]. A direct identity on ${}_2F_1([1/3, 2/3], [1], \mathcal{P})$, does however exist:

$${}_2F_1([\frac{1}{3},\,\frac{2}{3}],\,[1],\,x^3)$$
$$=\;\;\frac{1}{1\,+\,2\,x} \cdot\, {}_2F_1\Big([\frac{1}{3},\,\frac{2}{3}],\,[1],\;\frac{9\,x \cdot\,(1\,+\,x\,+\,x^2)}{(1\,+\,2\,x)^3}\Big). \tag{B.6}$$

‡ It corresponds to a trivial pullback change $p \,=\, z/(z+27) \,\rightarrow\, 1\,-p$.

Identity (B.6) corresponds¶ to the identity on ${}_2F_1([1/3, 1/3], [1], \mathcal{P})$:

$$ {}_2F_1\Bigl([\frac{1}{3}, \frac{1}{3}], [1], -\frac{x^3}{1-x^3}\Bigr) $$
$$ = \Bigl(\frac{1+x+x^2}{(1-x)^2}\Bigr)^{1/3} \cdot {}_2F_1\Bigl([\frac{1}{3}, \frac{1}{3}], [1], -\frac{9x \cdot (1+x+x^2)}{(1-x)^3}\Bigr). \qquad \text{(B.7)} $$

Appendix C. Comments on the $\tau \to 4\tau$ modular equation (51).

The fact that in section (2.8), the three Hauptmoduls (44), (47) and (48) can be introduced for the $\tau \to 4\tau$ modular equation (51), can be revisited in the z variable. (see equation (45). Recalling $\mathcal{P}_1$ and $\mathcal{P}_2$ given in (45), and performing the (involutive) change of variable $z \to -16\,z/(z+16)$, on $\mathcal{P}_2$, we get a third Hauptmodul $\mathcal{P}_3$

$$ \mathcal{P}_3 = -\frac{1728 \cdot z \cdot (z+16)^4}{(z^2-224\,z+256)^3} = \Bigl(\frac{1728 \cdot z}{(z+16)^3}\Bigr) \circ \Bigl(-\frac{4096\,z}{(z+16)^2}\Bigr), \qquad \text{(C.1)} $$

to be compared† with:

$$ \mathcal{P}_2 = \frac{1728 \cdot z \cdot (z+16)}{(z^2+16\,z+16)^3} = \Bigl(\frac{1728 \cdot z}{(z+16)^3}\Bigr) \circ \Bigl(z \cdot (z+16)\Bigr). \qquad \text{(C.2)} $$

One also has:

$$ \mathcal{P}_1 = \frac{1728 \cdot z^4 \cdot (z+16)}{(z^2+256\,z+4096)^3} = \Bigl(\frac{1728 \cdot z}{(z+16)^3}\Bigr) \circ \Bigl(\frac{4096 \cdot (z+16)}{z^2}\Bigr) \qquad \text{(C.3)} $$
$$ = \Bigl(\frac{1728 \cdot z^2}{(z+256)^3}\Bigr) \circ \Bigl(\frac{z^2}{z+16}\Bigr). \qquad \text{(C.4)} $$

These three Hauptmoduls have to be compared with the Hauptmodul:

$$ \mathcal{P}_0 = \frac{1728 \cdot z^2 \cdot (z+16)^2}{(z^2+16\,z+256)^3} = \Bigl(\frac{1728 \cdot z}{(z+16)^3}\Bigr) \circ \Bigl(\frac{z^2}{z+16}\Bigr) \qquad \text{(C.5)} $$
$$ = \Bigl(\frac{1728 \cdot z^2}{(z+256)^3}\Bigr) \circ \Bigl(z \cdot (z+16)\Bigr). \qquad \text{(C.6)} $$

Note that the elimination of z, between this last Hauptmodul $\mathcal{P}_0$ *and each of the three Hauptmoduls* $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$, gives the $\tau \to 2\tau$ modular equation (see (E.9) below) instead of the $\tau \to 4\tau$ modular equation (51).

The decomposition of the Hauptmoduls $\mathcal{P}_3$, $\mathcal{P}_2$ and $\mathcal{P}_1$ given by (C.1), (C.2) and (C.3) suggests to substitute

$$ X = \frac{1728\tilde{X}}{(\tilde{X}+16)^3}, \qquad Y = \frac{1728\tilde{Y}}{(\tilde{Y}+16)^3}, \qquad \text{(C.7)} $$

in the $\tau \to 4\tau$ modular equation (51). This change of variable transforms the LHS of the modular equation (51) into the product of four polynomials:

$$ \hat{p}_1(\tilde{X}, \tilde{Y}) = \tilde{X}^2\,\tilde{Y}^2 - 3 \cdot 2^{16} \cdot \tilde{X}\,\tilde{Y} - 2^{24} \cdot (\tilde{X}+\tilde{Y}), \qquad \text{(C.8)} $$

$$ \hat{p}_{2,1}(\tilde{X}, \tilde{Y}) = \hat{p}_{2,2}(\tilde{Y}, \tilde{X}) = \tilde{X}^4\,\tilde{Y}^3 + 96\,\tilde{X}^3\,\tilde{Y}^3 + 196608\,\tilde{X}^3\,\tilde{Y}^2 + 2352\,\tilde{X}^2\,\tilde{Y}^3 $$
$$ + 16777216\,\tilde{X}^3\,\tilde{Y} - 7335936\,\tilde{X}^2\,\tilde{Y}^2 + 10496\,\tilde{X}\,\tilde{Y}^3 - \tilde{Y}^4 + 805306368\,\tilde{X}^2\,\tilde{Y} $$
$$ + 9633792\,\tilde{X}\,\tilde{Y}^2 + 1610612736\,\tilde{X}\,\tilde{Y} + 68719476736\,\tilde{X}, \qquad \text{(C.9)} $$

¶ Using the relation (43).
† As it should $z \to -16\,z/(z+16)$ changes $z \cdot (z+16)$ into $-4096\,z/(z+16)^2$.

$$
\begin{aligned}
\hat{p}_4(\tilde{X}, \tilde{Y}) \; = \; & \tilde{X}^7 \tilde{Y}^5 + \tilde{X}^5 \tilde{Y}^7 + 96 \tilde{X}^7 \tilde{Y}^4 + 144 \tilde{X}^6 \tilde{Y}^5 + 144 \tilde{X}^5 \tilde{Y}^6 + 96 \tilde{X}^4 \tilde{Y}^7 \\
& + 2352 \tilde{X}^7 \tilde{Y}^3 - 182784 \tilde{X}^6 \tilde{Y}^4 + 13968 \tilde{X}^5 \tilde{Y}^5 - 182784 \tilde{X}^4 \tilde{Y}^6 + 2352 \tilde{X}^3 \tilde{Y}^7 \\
& + \tilde{X}^8 \tilde{Y} + 10496 \tilde{X}^7 \tilde{Y}^2 + 7674625 \tilde{X}^6 \tilde{Y}^3 - 1300992 \tilde{X}^5 \tilde{Y}^4 - 1300992 \tilde{X}^4 \tilde{Y}^5 \\
& + 7674625 \tilde{X}^3 \tilde{Y}^6 + 10496 + 7674625 \tilde{X}^3 \tilde{Y}^6 + 10496 \tilde{X}^2 \tilde{Y}^7 + \tilde{X} \tilde{Y}^8 + 192 \tilde{X}^7 \tilde{Y} \\
& - 8122320 \tilde{X}^6 \tilde{Y}^2 + 1526542992 \tilde{X}^5 \tilde{Y}^3 + 700465152 \tilde{X}^4 \tilde{Y}^4 + 1526542992 \tilde{X}^3 \tilde{Y}^5 \\
& - 8122320 \tilde{X}^2 \tilde{Y}^6 + 192 \tilde{X} \tilde{Y}^7 + 13920 \tilde{X}^6 \tilde{Y} + 759331584 \tilde{X}^5 \tilde{Y}^2 \\
& + 56157592368 \tilde{X}^4 \tilde{Y}^3 + 56157592368 \tilde{X}^3 \tilde{Y}^4 + 759331584 \tilde{X}^2 \tilde{Y}^5 + 13920 \tilde{X} \tilde{Y}^6 \\
& + 472576 \tilde{X}^5 \tilde{Y} - 13144356607 \tilde{X}^4 \tilde{Y}^2 + 229377672192 \tilde{X}^3 \tilde{Y}^3 \\
& - 13144356607 \tilde{X}^2 \tilde{Y}^4 + 472576 \tilde{X} \tilde{Y}^5 + 7547184 \tilde{X}^4 \tilde{Y} + 39849037920 \tilde{X}^3 \tilde{Y}^2 \\
& + 39849037920 \tilde{X}^2 \tilde{Y}^3 + 7547184 \tilde{X} \tilde{Y}^4 + 49771008 \tilde{X}^3 \tilde{Y} \\
& - 13195144656 \tilde{X}^2 \tilde{Y}^2 + 49771008 \tilde{X} \tilde{Y}^3 + 95607040 \tilde{X}^2 \tilde{Y} + 95607040 \tilde{X} \tilde{Y}^2 \\
& + 19771392 \tilde{X} \tilde{Y} - 4096. \qquad \text{(C.10)}
\end{aligned}
$$

The elimination of z in

$$
\tilde{X} \; = \; -\frac{4096\, z}{(z+16)^2}, \qquad \qquad \tilde{Y} \; = \; z \cdot (z+16), \qquad \text{(C.11)}
$$

or in

$$
\tilde{X} \; = \; -\frac{4096 \cdot z}{(z+16)^2}, \qquad \qquad \tilde{Y} \; = \; \frac{4096 \cdot (z+16)}{z^2}, \qquad \text{(C.12)}
$$

or in

$$
\tilde{X} \; = \; \frac{4096 \cdot (z+16)}{z^2}, \qquad \qquad \tilde{Y} \; = \; z \cdot (z+16), \qquad \text{(C.13)}
$$

corresponds to (C.8), the first polynomial $\hat{p}_1(\tilde{X}, \tilde{Y}) \, = \, 0$.

The elimination of z between any two Hauptmoduls among the three Hauptmoduls $\mathcal{P}_1$, $\mathcal{P}_2$, or $\mathcal{P}_3$, yields the *same modular equation* (51). In general for modular equations representing $\tau \, \rightarrow \, N \tau$, one Hauptmodul is of the form $\alpha \cdot z + \, \cdots$ when the other one is of the form $\alpha \cdot z^N \, + \, \cdots$ (see [22]). Here with $\mathcal{P}_2$ and $\mathcal{P}_3$, we have two Hauptmoduls algebraically related by the modular equation (51) representing $\tau \, \rightarrow \, 4 \tau$, but each of them is of the form $\pm \, \alpha \cdot z \, + \, \cdots$ This result is reminiscent of the *involutive* series solution of (51), (given by equation (104) in [22]):

$$
\begin{aligned}
Y \; = \; & -X \; - \frac{31 X^2}{36} \; - \frac{961}{1296} \cdot X^3 \; - \frac{203713}{314928} \cdot X^4 \; - \frac{4318517}{7558272} \cdot X^5 \\
& - \frac{832777775}{1632586752} \cdot X^6 \; - \frac{729205556393}{1586874322944} \cdot X^7 - \frac{2978790628903}{7140934453248} \cdot X^8 \; + \, \ldots \qquad \text{(C.14)}
\end{aligned}
$$

Replacing in (C.14), (X, Y) by† $(\mathcal{P}_2, \mathcal{P}_3)$, one verifies that the expansions in z, of LHS and RHS of (C.14) are equal.

† Or replacing (X, Y) by $(\mathcal{P}_2^{(1)}, \mathcal{P}_2^{(1)})$.

Appendix D. ${}_2F_1([1/4,3/4],[1],\mathcal{P})$ hypergeometric as modular forms

Appendix D.1. ${}_2F_1([1/4,1/4],[1],\mathcal{P})$ and ${}_2F_1([1/2,1/2],[1],\mathcal{P})$ as modular forms

In Table 15 of Maier [23], one sees that ${}_2F_1([1/4,1/4],[1],x)$ hypergeometric functions are related to $\tau \, \rightarrow \, 2\,\tau$:

$$
{}_2F_1\Bigl([\frac{1}{4},\, \frac{1}{4}],\, [1],\, -\,\frac{x}{64}\Bigr) \;=\; \Bigl(\frac{x\,+16}{16}\Bigr)^{-1/4} \cdot\, {}_2F_1\Bigl([\frac{1}{12},\, \frac{5}{12}],\, [1],\, \frac{1728\cdot\, x}{(x\,+16)^3}\Bigr)
$$
$$
=\; \Bigl(\frac{x\,+256}{256}\Bigr)^{-1/4} \cdot\, {}_2F_1\Bigl([\frac{1}{12},\, \frac{5}{12}],\, [1],\, \frac{1728\cdot\, x^2}{(x\,+256)^3}\Bigr). \qquad (D.1)
$$

One has the following identity†:

$$
{}_2F_1\Bigl([\frac{1}{4},\, \frac{1}{4}],\, [1],\; -\,\frac{1}{64}\cdot\, \frac{x^2}{x\,+16}\Bigr)
$$
$$
=\; \Bigl(\frac{x+16}{16}\Bigr)^{1/4} \cdot\, {}_2F_1\Bigl([\frac{1}{4},\, \frac{1}{4}],\, [1],\; -\,\frac{x\,\cdot\, (x\,+16)}{64}\Bigr). \qquad (D.2)
$$

One also sees in Table 15 of Maier [23] that ${}_2F_1([1/2,1/2],[1],x)$ hypergeometric functions are related to a $\tau \, \rightarrow \, 4\,\tau$ isogeny[22]:

$$
{}_2F_1\Bigl([\frac{1}{2},\, \frac{1}{2}],\, [1],\, -\,\frac{x}{16}\Bigr) \qquad (D.3)
$$
$$
=\; \Bigl(\frac{x^2\,+16\,x\,+16}{16}\Bigr)^{-1/4} \cdot\, {}_2F_1\Bigl([\frac{1}{12},\, \frac{5}{12}],\, [1],\, \frac{1728\cdot\, x\,\cdot\, (x+16)}{(x^2\,+16\,x\,+16)^3}\Bigr)
$$
$$
=\; \Bigl(\frac{x^2\,+256\,x\,+4096}{4096}\Bigr)^{-1/4} \cdot\, {}_2F_1\Bigl([\frac{1}{12},\, \frac{5}{12}],\, [1],\, \frac{1728\cdot\, x^4\,\cdot\, (x+16)}{(x^2\,+256\,x\,+4096)^3}\Bigr).
$$

One has the following identity:

$$
{}_2F_1\Bigl([\frac{1}{2},\, \frac{1}{2}],\, [1],\, \frac{8\,x\,\cdot\, (1\,+x^2)}{(1+x)^4}\Bigr) \;=\; (1\,+x)^2\cdot\, {}_2F_1\Bigl([\frac{1}{2},\, \frac{1}{2}],\, [1],\, x^4\Bigr). \qquad (D.4)
$$

Appendix D.2. ${}_2F_1([1/4,3/4],[1],\mathcal{P})$ hypergeometric as modular forms

Let us now focus on the ${}_2F_1([1/4,3/4],[1],\mathcal{P})$ hypergeometric function:

$$
{}_2F_1\Bigl([\frac{1}{4},\, \frac{3}{4}],\, [1],\, x\Bigr) \;=\; (1\,+3\,x)^{-1/4}\cdot\, {}_2F_1\Bigl([\frac{1}{12},\, \frac{5}{12}],\, \frac{27\,x\,\cdot\, (1\,-x)^2}{(1\,+3\,x)^3}\Bigr). \qquad (D.5)
$$

The emergence of ${}_2F_1([1/4,3/4],[1],\mathcal{P})$ hypergeometric functions in physics, walk problems in the quarter of a plane [34, 35, 36] in enumerative combinatorics, or in interesting subcases of diagonals (see section (2.9)), raises the question if ${}_2F_1([1/4,3/4],[1],\mathcal{P})$ should be seen as associated to the isogenies [22] $\tau \, \rightarrow \, 2\,\tau$ or $\tau \, \rightarrow \, 4\,\tau$. The identity

$$
{}_2F_1\Bigl([\frac{1}{4},\, \frac{3}{4}],\, [1],\, 64\,x^2\Bigr) \;=\; (1\,+8\,x)^{-1/2}\cdot\, {}_2F_1\Bigl([\frac{1}{2},\, \frac{1}{2}],\, [1],\, \frac{16\,x}{1\,+8\,x}\Bigr), \qquad (D.6)
$$

or equivalently

$$
{}_2F_1\Bigl([\frac{1}{4},\, \frac{3}{4}],\, [1],\, \Bigl(\frac{x}{2\,-\,x}\Bigr)^2\Bigr) \;=\; \Bigl(\frac{2\,-\,x}{2}\Bigr)^{1/2}\cdot\, {}_2F_1\Bigl([\frac{1}{2},\, \frac{1}{2}],\, [1],\, x\Bigr), \qquad (D.7)
$$

† It can be deduced from (D.1) together with (C.2) with (C.5), or (C.4) with (C.6).

seems to relate ${}_2F_1([1/4, 3/4], [1], \mathcal{P})$ to ${}_2F_1([1/2, 1/2], [1], x)$, and thus seems to relate ${}_2F_1([1/4, 3/4], [1], \mathcal{P})$ rather $\tau \rightarrow 4\,\tau$. Yet things are more subtle.

Let us see how ${}_2F_1([1/4, 3/4], [1], \mathcal{P})$ can be described as a modular form corresponding to pullbacked ${}_2F_1([1/4, 3/4], [1], \mathcal{P})$ hypergeometric functions with two different rational pullbacks. For instance, one deduces from (D.4) combined with (D.7), several identities on the hypergeometric function ${}_2F_1([1/4, 3/4], [1], \mathcal{P})$ like

$$ {}_2F_1\Bigl([\frac{1}{4}, \frac{3}{4}], [1], \frac{x^2}{(2\,-x)^2}\Bigr) $$
$$ = \quad \Bigl(\frac{2\,-x}{2 \cdot\,(1\,-2\,x)}\Bigr)^{1/2} \cdot {}_2F_1\Bigl([\frac{1}{4}, \frac{3}{4}], [1], \,-4 \cdot \frac{x \cdot\,(1\,-\,x)}{(1\,-\,2\,x)^2}\Bigr), \tag{D.8} $$

or

$$ {}_2F_1\Bigl([\frac{1}{4}, \frac{3}{4}], [1], \frac{x^2}{(2\,-x)^2}\Bigr) $$
$$ = \quad \Bigl(\frac{2\,-x}{2 \cdot\,(1\,+\,x)}\Bigr)^{1/2} \cdot {}_2F_1\Bigl([\frac{1}{4}, \frac{3}{4}], [1], \frac{4\,x}{(1\,+\,x)^2}\Bigr). \tag{D.9} $$

and thus:

$$ {}_2F_1\Bigl([\frac{1}{4}, \frac{3}{4}], [1], \frac{4\,x}{(1\,+\,x)^2}\Bigr) $$
$$ = \quad \Bigl(\frac{1\,+x}{1\,-2\,x}\Bigr)^{1/2} \cdot {}_2F_1\Bigl([\frac{1}{4}, \frac{3}{4}], [1], \,-4 \cdot \frac{x \cdot\,(1\,-\,x)}{(1\,-\,2\,x)^2}\Bigr). \tag{D.10} $$

One also has the identity:

$$ {}_2F_1\Bigl([\frac{1}{4}, \frac{3}{4}], [1], \frac{x^4}{(2\,-x^2)^2}\Bigr) \tag{D.11} $$
$$ = \quad \Bigl(\frac{2\,-x^2}{2 \cdot\,(1\,+6\,x\,\,+x^2)}\Bigr)^{1/2} \cdot {}_2F_1\Bigl([\frac{1}{4}, \frac{3}{4}], [1], \frac{16 \cdot\,x \cdot\,(1+x)^2}{(1\,+6\,x\,\,+x^2)^2}\Bigr). $$

Recalling the viewpoint developed in our previous paper [22] these identities can be seen to be of the form

$$ {}_2F_1\Bigl([\frac{1}{4}, \frac{3}{4}], [1], B\Bigr) \quad = \quad G \cdot {}_2F_1\Bigl([\frac{1}{4}, \frac{3}{4}], [1], \, A\Bigr), $$

where G is some algebraic factor. For instance in the case of the last identity (D.11)

$$ A \,\,=\,\, \frac{16 \cdot\,x \cdot\,(1+x)^2}{(1\,+6\,x\,\,+x^2)^2}, \qquad\qquad B \,\,=\,\, \frac{x^4}{(2\,-x^2)^2}, \tag{D.12} $$

we have

$$ B \,\,=\,\, \frac{A^4}{262144} \,+\, \frac{5\,A^5}{524288} \,+\, \frac{1069\,A^6}{67108864} \,+\, \frac{6003\,A^7}{268435456} \,+\, \frac{1961123\,A^8}{68719476736} \,+\, \cdots \tag{D.13} $$

and G is an algebraic factor

$$ G \,\,=\,\, 1 \,-\, \frac{3}{16}\,A \,-\, \frac{69\,A^2}{1024} \,-\, \frac{633\,A^3}{16384} \,-\, \frac{55209\,A^4}{2097152} \,-\, \frac{659109\,A^5}{33554432} \,+\, \cdots \tag{D.14} $$

solution of:

$$ 65536 \cdot G^8 \,-16384 \cdot G^6 \,+1536 \cdot\,(27\,A\,-26) \cdot G^4 \,+64 \cdot\,(135\,A\,-136) \cdot G^2 $$
$$ +\,3969\,A^2 \,-3456\,A \,\,-512 \,\,=\,\, 0. \tag{D.15} $$

The important result of [22] is that after elimination of the algebraic factor G one finds that the two pullbacks A and B verify the following *Schwarzian equation*:

$$-\frac{1}{8}\frac{3A^2-3A+4}{A^2(A-1)^2} + \frac{1}{8}\frac{3B^2-3B+4}{B^2(B-1)^2}\cdot\Big(\frac{dB}{dA}\Big)^2 + \{B, A\} = 0, \qquad (D.16)$$

where $\{B, A\}$ denotes the Schwarzian derivative.

Do note that ${}_2F_1([1/4,3/4],[1],\mathcal{P})$ is a selected hypergeometric function since the rational function in the Schwarzian derivative (D.16)

$$W(A) = -\frac{1}{8}\frac{3A^2-3A+4}{A^2\cdot(A-1)^2}, \qquad (D.17)$$

is invariant under the $A \rightarrow 1-A$ transformation: $W(A) = W(1-A)$.

This Schwarzian equation can be written in a more symmetric way between A and B, namely:

$$\frac{1}{8}\frac{3B^2-3B+4}{B^2(B-1)^2}\cdot\Big(\frac{dB}{dx}\Big)^2 + \{B, x\}$$
$$= \frac{1}{8}\frac{3A^2-3A+4}{A^2(A-1)^2}\cdot\Big(\frac{dA}{dx}\Big)^2 + \{A, x\}. \qquad (D.18)$$

Let us denote $\rho(x)$ the rational function of the LHS or the RHS of equality (D.18). For the three identities (D.8), (D.9), (D.10) this rational function is (of course†) the same rational function, namely

$$\rho(x) = \frac{1}{2}\cdot\frac{x^2-x+1}{x\cdot(x-1)^2}, \qquad (D.19)$$

when the last identity (D.11) corresponds to:

$$\rho(x) = \frac{1}{2}\cdot\frac{(x^2+1)^2}{x^2\cdot(x^2-1)^2}. \qquad (D.20)$$

Let us consider the first two identities (D.8) and (D.9), denoting by A and B the corresponding pullbacks:

$$A = -4\cdot\frac{x\cdot(1-x)}{(1-2x)^2}, \quad \text{or:} \quad \frac{4x}{(1+x)^2}, \qquad B = \frac{x^2}{(2-x)^2} \qquad (D.21)$$

These two pullbacks are related by the *asymmetric* modular equation:

$$81\cdot A^2B^2 - 18AB\cdot(8B+A) + (A^2+80\cdot AB+64B^2) - 64B = 0. \qquad (D.22)$$

giving the following expansion for A seen as an *algebraic series*‡ in B:

$$B = \frac{1}{64}A^2 + \frac{5}{256}A^3 + \frac{83}{4096}A^4 + \frac{163}{8192}A^5 + \frac{5013}{262144}A^6 + \cdots \qquad (D.23)$$

We will denote $\mathcal{M}_2(A, B)$ the LHS of the modular equation (D.22): such an algebraic series is clearly†† a $\tau \rightarrow 2\tau$ (or $q \rightarrow q^2$ in the nome q) isogeny [22]. Composing this algebraic transformation with itself in order to have a $\tau \rightarrow 4\tau$ (or $q \rightarrow q^4$) representation, amounts to eliminating¶ X between $\mathcal{M}_2(A, X) = 0$ and

† Since these identities share one pullback.
‡ We discard the other root expansion $B = 1 + A + \frac{5}{4}A^2 + \frac{25}{16}A^3 + \frac{31}{16}A^4 + \cdots$
††From (D.23) see [22].
¶ Performing the resultant: $resultant(\mathcal{M}_2(A, X), \mathcal{M}_2(X, B), X)$.

$\mathcal{M}_2(X,\,B) \;=\; 0$ (i.e. two times the modular equation (59)). This elimination gives the following *asymmetric* modular curve corresponding to identity (D.11):

$$\begin{aligned}
&15752961\,A^4B^4 \;\; - 428652\,A^3B^3\cdot\,(64\,B + 83\,A) \\
&\quad + 162\,A^2B^2\cdot\,(48640\,B^2 + 494208\,AB + 124051\,A^2) \\
&\quad + 108\;A\,B\cdot\,(32768\,B^3 - 500480\,B^2A - 491200\,BA^2\; - 83\,A^3) \\
&\quad + 262144\,B^4 + 10354688\,B^3A + 46715904\,B^2A^2 + 159488\,BA^3 + A^4 \\
&\quad - 3072\,B\cdot\,(256\,B^2 + 4736\,AB + 177\,A^2) \\
&\qquad\qquad + 131072\,B\cdot\,(6\,B + 5\,A)\;\; - 262144\,B \;\;=\;\; 0,
\end{aligned} \tag{D.24}$$

parametrised by

$$A \;=\; \frac{16\cdot\,x\cdot\,(1+x)^2}{(1\,+6\,x\;\,+x^2)^2}, \qquad\qquad B \;=\; \frac{x^4}{(2\,-x^2)^2}, \tag{D.25}$$

where:

$$B \;=\; \frac{A^4}{262144} \;+\; \frac{5\,A^5}{524288} \;+\; \frac{1069\,A^6}{67108864} \;+\; \frac{6003\,A^7}{268435456} \;+\; \frac{1961123\,A^8}{68719476736} \;+\;\cdots \tag{D.26}$$

Note that B in (D.25) is nothing but the composition of B in (58) by $x\,\rightarrow\,x^2$ and that A in (D.25) in nothing but the composition of A in (D.25) with itself:

$$\frac{16\cdot\,x\cdot\,(1+x)^2}{(1\,+6\,x\;\,+x^2)^2} \;\;=\;\; \frac{4\,x}{(1\,+\,x)^2}\,\circ\,\frac{4\,x}{(1\,+\,x)^2}. \tag{D.27}$$

The modular curve (59) is unpleasantly asymmetric: the two pullbacks are not on the same footing. Note that, using the $A\,\leftrightarrow\,1\,-\,A$ symmetry (see (D.17)) on the Schwarzian equations (D.18), and changing $A\,\rightarrow\,1\,-\,A$ in the asymmetric modular curve (59), one gets the *symmetric* modular curve:

$$\begin{aligned}
&81\cdot\;A^2\,B^2\;\; - 18\cdot\,(A^2\,B + A\,B^2)\;\; + A^2\; - 44\,A\,B\; + B^2 \\
&\qquad - 2\cdot\,(A + B)\;\; + 1 \;\;=\;\; 0.
\end{aligned} \tag{D.28}$$

Changing $B\,\rightarrow\,1\,-\,B$ in the asymmetric modular curve (59), one also gets another symmetric modular curve:

$$\begin{aligned}
&81\cdot\;A^2\,B^2\;\; - 144\cdot\,(A^2\,B + A\,B^2) \\
&\qquad + 208\,A\,B\;\; + 64\cdot\,(A^2\; + B^2\; - A\; - B) \;\;=\;\; 0.
\end{aligned} \tag{D.29}$$

The two pullbacks for (D.29) read:

$$A \;=\; \frac{4\,x}{(1\,+\,x)^2}, \qquad\qquad B \;=\; \frac{4\cdot\,(1\,-\,x)}{(2\,-\,x)^2}. \tag{D.30}$$

Similarly the *asymmetric* modular curve (D.24) can be turned back into a *symmetric* modular curve by changing $A\,\leftrightarrow\,1\,-\,A$, or $B\,\leftrightarrow\,1\,-\,B$. The price to pay to restore the symmetry between the two pullbacks (D.30) is that the corresponding pullbacks do not yield hypergeometric identities *expandable for x small*.

Finally, the identity (D.10) corresponds to a symmetric relation between these two-pullbacks which reads:

$$\begin{aligned}
&81\cdot\,C^2D^2\;\; - 144\cdot\,(C^2\,D\,+\,CD^2)\;\; + 16\cdot\,(4\,C^2\; + 13\,C\,D\; + 4\,D^2) \\
&\qquad\qquad - 64\cdot\,(C\,+\,D) \;\;=\;\; 0.
\end{aligned} \tag{D.31}$$

The corresponding series expansion

$$D \;=\; -C \;-\frac{5}{4}\,C^2 \;-\frac{25\,C^3}{16} \;-\frac{31\,C^4}{16} \;-\frac{305\,C^5}{128} \;-\frac{2979\,C^6}{1024} \;-\frac{14457\,C^7}{4096} \;-\frac{17445\,C^8}{4096} \;-\frac{167615\,C^9}{32768} \;-\frac{801941\,C^{10}}{131072} \;-\frac{3822989\,C^{11}}{524288} \;+\;\cdots \tag{D.32}$$

is *an involutive series.*

Appendix E. Modular forms: recalls on Maier's paper [23] and the associated Schwarzian equations

In fact, the previous pullbacks in the pullbacked ${}_2F_1$ hypergeometric functions can be seen (and should be seen) as *Hauptmoduls* [23].

In [23], Maier underlined the representation of a selected set of modular forms as pullbacked hypergeometric functions with two possible *rational pullbacks* (related by a genus zero modular equation). In [22], we revisited that viewpoint: an identity on a hypergeometric function with a pullback and the same hypergeometric function with another pullback, the (algebraic) map‡, changing one pullback into the other one, being a symmetry of *infinite order*†, is such a strong constraint that it is almost characteristic of modular forms [22]: the hypergeometric functions can be seen as *automorphic functions* with respect to these *infinite order symmetries*.

The two different modular equations (40), (51) corresponding respectively to $\tau \,\rightarrow\, 3\,\tau$ and $\tau \,\rightarrow\, 4\,\tau$, suggest that a genus zero modular equation, corresponding to $\tau \,\rightarrow\, N\,\tau$, could encapsulate these two subcases. In such a scenario, N must be a multiple of 3, 4, 5, ... In fact, the set of values of N corresponding to *modular equations* with a (genus zero) rational parametrization is obtained for a *finite set* [23, 41, 42] of integer values: 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 16, 18 and 25. Some canonical rational parametrizations of these selected genus zero modular equations are given in [23]. The two Hauptmoduls read respectively for these selected values 2, 3, 4, 5, ..., 25:

$$N \,=\, 2: \qquad \frac{1728\cdot\, z^2}{(z+256)^3}, \qquad \frac{1728\cdot\, z}{(z+16)^3}, \tag{E.1}$$

$$N \,=\, 3: \qquad \frac{1728\cdot\, z^3}{(z\,+27)\cdot\,(z\,+243)^3}, \qquad \frac{1728\cdot\, z}{(z\,+27)\cdot\,(z\,+3)^3}, \tag{E.2}$$

$$N \,=\, 4: \qquad \frac{1728\cdot z^4\cdot\,(z\,+16)}{(z^2\,+256\,z\,+4096)^3}, \qquad \frac{1728\cdot z\cdot\,(z\,+16)}{(z^2\,+16\,z\,+16)^3}, \tag{E.3}$$

$$N \,=\, 5: \qquad \frac{1728\cdot\, z^5}{(z^2\,+250\,z\,+3125)^3}, \qquad \frac{1728\cdot\, z}{(z^2\,+10\,z\,+5)^3}, \tag{E.4}$$

$$N \,=\, 6: \qquad \frac{1728\cdot\, z^6\cdot\,(z+8)^2\cdot\,(z+9)^3}{(z+12)^3\cdot\,(z^3+252\,z^2+3888\,z+15552)^3}, \qquad \frac{1728\cdot\, z\cdot\,(z+8)^3\,(z+9)^2}{(z+6)^3\cdot\,(z^3+18\,z^2+84\,z+24)^3}, \tag{E.5}$$

‡ Called by Veselov [40] in a mapping framework, "correspondence".

† Of course hypergeometric functions have finite order symmetries like $x \,\rightarrow\, 1-x$, that we discard. With infinite order symmetries one can associate some discrete dynamical map: in these particular cases algebraic function maps [22, 38, 40].

$$N \,=\, 7: \qquad {{1728 \cdot \, z^7} \over {(z^2 \, +13\, z \, +49) \cdot \, (z^2 +245\, z +2401)^3}}, \qquad {{1728 \cdot \, z} \over {(z^2 \, +13\, z \, +49) \cdot \, (z^2 +5\, z +1)^3}}, \qquad \cdots \tag{E.6}$$

The seven-parameter pullbacked $_2F_1$ hypergeometric function (11) cannot correspond *generically* to rationally parametrized (genus zero) modular equations. One cannot imagine to identify the Hauptmodul pullback in (11) with expressions [23] like (27) or (45) for N a multiple of 3, 4, 5, ... In the generic seven-parameters case one gets the expression (11) for the diagonal of the seven-parameters rational function (7) as a pullbacked $_2F_1$ hypergeometric function with a rational pullback (12), *the other pullback being algebraic* and deduced from the various modular equations [22] (see section (2.10)).

Appendix E.1. Landen transformation: $\tau \, \rightarrow \, 2\,\tau$.

To describe this situation let us recall the result detailed in [22] for $\tau \, \rightarrow \, 2\,\tau$. The emergence of a *modular form* [1, 19] corresponds to the identity on the *same* hypergeometric function but where the pullback x is changed $x \, \rightarrow \, y(x) \, = \, y$ according to *modular equations* [43, 44, 45, 46, 47, 48]. Let us consider the modular equation (E.9) below corresponding to the Landen transformation [49, 50], or inverse Landen transformation, and consider the corresponding $_2F_1$ hypergeometric identity

$${}_2F_1\Big([{{1} \over {12}}, {{5} \over {12}}], [1], y\Big) \,\, = \,\, \mathcal{A}(x) \cdot {}_2F_1\Big([{{1} \over {12}}, {{5} \over {12}}], [1], x\Big), \tag{E.7}$$

where $\mathcal{A}(x)$ is an algebraic function given by:

$$1024 \, \mathcal{A}(x)^{12} \,\, - 1152 \, \mathcal{A}(x)^8 \,\, + 132 \, \mathcal{A}(x)^4 \,\, + 125\, x \,\, - 4 \,\, = \,\, 0. \tag{E.8}$$

The relation between x and y in (E.7) is given by the *modular equation* [43, 44, 45, 46, 47, 48]:

$$\begin{aligned} 1953125 \cdot \, x^3 y^3 \,\, - 187500 \cdot \, x^2 y^2 \cdot \, (x+y) \,\, + 375 \cdot \, xy \cdot \, (16\, x^2 \, - 4027\, xy + 16\, y^2) \\ - 64 \cdot \, (x+y) \cdot \, (x^2 + 1487\, xy + y^2) \,\, + 110592 \cdot \, xy \,\, = \,\, 0. \end{aligned} \tag{E.9}$$

Using this algebraic relation between x and y one can rewrite (E.7) with (E.8) as

$${}_2F_1\Big([{{1} \over {12}}, {{5} \over {12}}], [1], x\Big) \,\, = \,\, \tilde{A}(y) \cdot {}_2F_1\Big([{{1} \over {12}}, {{5} \over {12}}], [1], y\Big), \tag{E.10}$$

where $\tilde{A}(y)$ is an algebraic function given¶ by:

$$\tilde{A}(y)^{12} \,\, - 18 \, \tilde{A}(y)^8 \,\, + 33 \, \tilde{A}(y)^4 \,\, + 500\, y \,\, - 16 \,\, = \,\, 0. \tag{E.11}$$

Using the fact that $\tilde{A}(y)$ is the reciprocal of $\mathcal{A}(x)$, one can rewrite (E.11) as:

$$4 \, \mathcal{A}(x)^{12} \, \cdot (125\, y \, - 4) \,\, + 33 \, \mathcal{A}(x)^8 \,\, - 18\, \mathcal{A}(x)^4 \,\,\, + 1 \,\, = \,\, 0. \tag{E.12}$$

In other words the elimination of $\mathcal{A}(x)$ between (E.8) and (E.12) gives the modular curve (E.9). Thus, introducing $A \, = \, \mathcal{A}(x)^4$, (E.8) and (E.12) can be seen as alternative rational parametrizations of the modular equation (E.9):

$$x \,\, = \,\, {{4} \over {125}} \cdot (1 \, - A) \cdot \, (1 \, - 16\, A)^2, \qquad y \,\, = \,\, - {{1} \over {500}} \cdot {{(1 \, - A)^2 \cdot \, (1 \, - 16\, A)} \over {A^3}}. \tag{E.13}$$

¶ This result breaking the symmetry between the two variables x and y may look paradoxical. In fact assuming (E.8) and (E.11) and the modular equation (E.9) one has $\tilde{A}(y) \cdot \mathcal{A}(x) \,\, = \,\, 1$.

Note that y, in terms of A is nothing but x, in terms of A, A taken to be $1/16/A$. The two variables x and y *are thus on the same footing*: permuting x and y corresponds to the involutive transformation $A \leftrightarrow 1/16/A$. Finally changing A into $A = (z+256)/(z+16)/16$, the previous parametrization (E.13) becomes the known parametrization [22, 23] of the fundamental modular equation (E.9), namely $x = 1728\, z/(z+16)^3$ and $y = 1728\, z^2/(z+256)^3$.

Appendix E.2. Schwarzian equations

In general, one can rewrite a remarkable hypergeometric identity like (39), (49), in the form

$$\mathcal{A}(x) \cdot {}_2F_1\Big([\alpha,\, \beta],\, [\gamma],\, x\Big) \;=\; {}_2F_1\Big([\alpha,\, \beta],\, [\gamma],\, y(x)\Big), \qquad \text{(E.14)}$$

where $\mathcal{A}(x)$ is an algebraic function and where $y(x)$ is an algebraic function (more precisely an *algebraic series*) corresponding to the previous modular equation $M(x,\, y(x)) = 0$.

The Gauss hypergeometric function ${}_2F_1([\alpha,\, \beta],\, [\gamma],\, x)$ is solution of the second order linear differential operator†:

$$\Omega \;=\; D_x^2 \;+\; A(x) \cdot D_x \;+\; B(x), \qquad\qquad \text{where:}$$

$$A(x) \;=\; \frac{(\alpha+\beta+1) \cdot x \;-\gamma}{x \cdot (x-1)} \;=\; \frac{u'(x)}{u(x)}, \qquad B(x) \;=\; \frac{\alpha\,\beta}{x \cdot (x-1)}. \qquad \text{(E.15)}$$

A straightforward calculation enables us to find the algebraic function $\mathcal{A}(x)$ in terms of the algebraic function pullback $y(x)$ in (E.14):

$$\mathcal{A}(x) \;=\; \Big(\frac{u(x)}{u(y(x))} \cdot y'(x)\Big)^{-1/2}. \qquad \text{(E.16)}$$

The identification of the two operators, $1/v(x) \cdot \Omega \cdot v(x)$ and Ω^{pull} (the pullback of operator Ω for a pullback $y(x)$), thus corresponds (beyond (E.16)) to just one condition that can be rewritten (after some algebra ...) in the following Schwarzian form [22, 38]:

$$W(x) \;\; - W(y(x)) \cdot y'(x)^2 \;\; + \{y(x),\, x\} \;=\; 0, \qquad \text{(E.17)}$$

or:

$$\frac{W(x)}{y'(x)} \;\; - W(y(x)) \cdot y'(x) \;\; + \; \frac{\{y(x),\, x\}}{y'(x)} \;=\; 0, \qquad \text{(E.18)}$$

where

$$W(x) \;=\; A'(x) \;+\; \frac{A(x)^2}{2} \;\; - 2 \cdot B(x), \qquad \text{(E.19)}$$

and where $\{y(x),\, x\}$ denotes the *Schwarzian derivative* [39]:

$$\{y(x),\, x\} \;=\; \frac{y'''(x)}{y'(x)} \;-\; \frac{3}{2} \cdot \Big(\frac{y''(x)}{y'(x)}\Big)^2 \;=\; \frac{d}{dx}\Big(\frac{y''(x)}{y'(x)}\Big) \;-\; \frac{1}{2} \cdot \Big(\frac{y''(x)}{y'(x)}\Big)^2. \qquad \text{(E.20)}$$

For (E.15) the function $W(x)$ reads:

$$\frac{(\alpha-\beta+1) \cdot (\alpha-\beta-1) \cdot x^2 \;+ 2 \cdot (2\,\alpha\,\beta - \alpha\,\gamma - \beta\,\gamma + \gamma) \cdot x \;+ \gamma \cdot (\gamma-2)}{2 \cdot x^2 \cdot (x-1)^2}. \qquad \text{(E.21)}$$

† Note that $A(x)$ is the log-derivative of $u(x) = x^{\gamma} \cdot (1-x)^{\alpha+\beta+1-\gamma}$.

The hypergeometric functions such that $W(x) \, = \, W(1\, - \, x)$ correspond to the two conditions:

$$\alpha \, + \, \beta \, = \, 1 \qquad \hbox{or:} \qquad \alpha \, + \, \beta \, = \, 2\gamma \, - 1. \qquad \hbox{(E.22)}$$

This is the case, for instance [26], with the hypergeometric functions ${}_2F_1([1/2, 1/2][1], \, x)$, ${}_2F_1([1/3, 2/3][1], \, x)$, ${}_2F_1([1/4, 3/4][1], \, x)$, ${}_2F_1([1/6, 5/6][1], \, x)$.

Remark: Denoting $W_H(x)$ the rational function $W(x)$ given by (E.19) for the second order linear differential operator annihilating (13), i.e. ${}_2F_1([1/12, 5/12], [1], \mathcal{H}(x))$ where $\mathcal{H}(x)$ is the Hauptmodul $1 - P_4(x)^2/P_2(x)^3$ in (13), with P_2 and P_4 given by (16) and (17). $W_H(x)$ can be deduced from the $W(x)$ for the order-two linear differential operator annihilating ${}_2F_1([1/12, 5/12], [1], \, x)$, from the relation [38]

$$W_H(x) \, = \, W\Big(\mathcal{H}(x)\Big) \, - \, \{\mathcal{H}(x), \, x\}, \qquad W(x) \, = \, -\frac{32\,x^2 \, - \, 41\,x \, + \, 36}{72 \cdot \, (x\, - \, 1)^2 \cdot \, x^2}, \qquad \hbox{(E.23)}$$

where $\{\mathcal{H}(x), \, x\}$ denotes the Schwarzian derivative of $\mathcal{H}(x)$. $W_H(x)$ is of the form

$$W_H(x) \, = \, \frac{p_{16}(x)}{x^2 \cdot \, p_3(x)^2 \cdot \, p_5(x)^2} \, = \, -\frac{1}{2\,x^2} \, + \, \cdots \qquad \hbox{(E.24)}$$

where $p_{16}(x)$, $p_3(x)$ and $p_5(x)$ are polynomials of degree respectively sixteen, three and five in x. These polynomials are *homomogeneous polynomials* in the seven parameters a, b_i, c_i of (7). For instance $p_{16}(x)$ is a homogeneous polynomial of homogeneous degree 44, $p_3(x)$ is a homogeneous polynomial of homogeneous degree 10 and $p_5(x)$ is a homogeneous polynomial of homogeneous degree 12.

Appendix F. Exact expression of polynomial P_6 for the ten-parameter rational function (77)

The diagonal of the ten-parameters rational function (77) is the pullbacked hypergeometric function

$$\frac{1}{P_3(x)^{1/4}} \cdot \, {}_2F_1\Big([\frac{1}{12}, \, \frac{5}{12}], \, [1], \, 1\, - \, \frac{P_6(x)^2}{P_3(x)^3}\Big), \qquad \hbox{(F.1)}$$

where $P_3(x)$ is given by (80) and $P_6(x)$ is a polynomial of degree six in x of the form

$$P_6(x) \, = \, p_4 \, + \, \Delta_6(x), \qquad \hbox{(F.2)}$$

where p_4 is the polynomial $P_4(x)$ given by (17) in section (2), and where $\Delta_6(x)$ is the following polynomial of degree six in x:

$$\begin{aligned} \Delta_6(x) \, = \, & -5832 \cdot \, d_1^2\, d_2^2\, d_3^2 \cdot \, x^6 \\ & + 3888 \cdot \, d_1\, d_2\, d_3 \cdot \, (b_1\, c_2\, d_2 \, + \, b_2\, c_3\, d_3 + \, b_3\, c_1\, d_1) \cdot \, x^5 \\ & - 864 \cdot \, (c_1^3\, d_1^2\, d_3 + \, c_2^3\, d_1\, d_2^2 \, + \, c_3^3\, d_2\, d_3^2) \cdot \, x^5 \\ & - 1296 \cdot \, c_1\, c_2\, c_3\, d_1\, d_2\, d_3 \cdot \, x^5 \end{aligned}$$

$$
\begin{aligned}
&- 1296 \cdot \, b_1\, b_2\, b_3\, d_1\, d_2\, d_3 \cdot \, x^4 \\
&- 1296 \cdot \, a \cdot \, d_1\, d_2\, d_3\, (b_1\, c_1 + b_2\, c_2 + b_3\, c_3) \cdot \, x^4 \\
&- 1296 \cdot \, (b_1\, b_2\, c_2\, c_3\, d_2\, d_3 + b_1\, b_3\, c_1\, c_2\, d_1\, d_2 + b_2\, b_3\, c_1\, c_3\, d_1\, d_3) \cdot \, x^4 \\
&+ 864 \cdot \, (c_1^2\, c_3\, d_1\, d_3 + c_1\, c_2^2\, d_1\, d_2 + c_2\, c_3^2\, d_2\, d_3) \cdot \, a \, \cdot \, x^4 \\
&- 864 \cdot \, (b_1^3\, d_2^2\, d_3 + b_2^3\, d_1\, d_3^2 + b_3^3\, d_1^2\, d_2) \cdot \, x^4 \\
&+ 864 \cdot \, \Big(b_1^2\, c_1\, c_3\, d_2\, d_3 + b_1\, b_2\, c_1^2\, d_1\, d_3 + b_1\, b_3\, c_3^2\, d_2\, d_3 \\
&\qquad + b_2^2\, c_1\, c_2\, d_1\, d_3 + b_2\, b_3\, c_2^2\, d_1\, d_2 + b_3^2\, c_2\, c_3\, d_1\, d_2 \Big) \cdot \, x^4 \\
&+ 216 \cdot \, (b_1^2\, c_2^2\, d_2^2 + b_2^2\, c_3^2\, d_3^2 + b_3^2\, c_1^2\, d_1^2) \cdot \, x^4 \\
&+ 288 \cdot \, (b_1\, c_1^3\, c_2\, d_1 + b_2\, c_2^3\, c_3\, d_2 + b_3\, c_1\, c_3^3\, d_3) \cdot \, x^4 \\
&- 576 \cdot \, (b_1\, c_1^2\, c_3^2\, d_3 + b_2\, c_1^2\, c_2^2\, d_1 + b_3\, c_2^2\, c_3^2\, d_2) \cdot \, x^4 \\
&- 144 \cdot \, c_1\, c_2\, c_3 \cdot \, (b_1\, c_2\, d_2 + b_2\, c_3\, d_3 + b_3\, c_1\, d_1) \cdot \, x^4 \\
&\quad + 540 \cdot \, d_1\, d_2\, d_3\, a^3 \cdot \, x^3 \\
&\quad - 648 \cdot \, (b_1\, c_3\, d_2\, d_3 + b_2\, c_1\, d_1\, d_3 + b_3\, c_2\, d_1\, d_2) \cdot \, a^2 \cdot \, x^3 \\
&\qquad - 72 \cdot \, (c_1^2\, c_2\, d_1 + c_1\, c_3^2\, d_3 + c_2^2\, c_3\, d_2) \cdot \, a^2 \cdot \, x^3 \\
&\quad + 288 \cdot \, (b_1^3\, b_3\, c_1\, d_2 + b_1\, b_2^3\, c_2\, d_3 + b_2\, b_3^3\, c_3\, d_1) \cdot \, x^3 \\
&\quad - 576 \cdot \, (b_1^2\, b_2^2\, c_1\, d_3 + b_1^2\, b_3^2\, c_3\, d_2 + b_2^2\, b_3^2\, c_2\, d_1) \cdot \, x^3 \\
&\quad - 144 \cdot \, b_1\, b_2\, b_3\, (b_1\, c_2\, d_2 + b_2\, c_3\, d_3 + b_3\, c_1\, d_1) \cdot \, x^3 \\
&\quad + 864 \cdot \, (b_1^2\, b_2\, d_2\, d_3 + b_1\, b_3^2\, d_1\, d_2 + b_2^2\, b_3\, d_1\, d_3) \cdot \, a \cdot \, x^3 \\
&\quad - 144 \cdot \, \Big(b_1^2\, c_1\, c_2\, d_2 + b_1\, b_2\, c_2^2\, d_2 + b_1\, b_3\, c_1^2\, d_1 \\
&\qquad\quad + b_2^2\, c_2\, c_3\, d_3 + b_2\, b_3\, c_3^2\, d_3 + b_3^2\, c_1\, c_3\, d_1 \Big) \cdot \, a \cdot \, x^3 \\
&\quad + 720 \cdot \, (b_1\, b_2\, c_1\, c_3\, d_3 + b_1\, b_3\, c_2\, c_3\, d_2 + b_2\, b_3\, c_1\, c_2\, d_1) \cdot \, a \cdot \, x^3 \\
&\quad + 36 \cdot \, a^3 \cdot \, (b_1\, c_2\, d_2 + \, b_2\, c_3\, d_3 + \, b_3\, c_1\, d_1) \cdot \, x^2 \\
&\qquad - 72 \cdot \, a^2 \cdot \, (b_1^2\, b_3\, d_2 + \, b_1\, b_2^2\, d_3 + \, b_2\, b_3^2\, d_1) \cdot \, x^2.
\end{aligned} \tag{F.3}
$$

Appendix G. Polynomials $P_3(x)$ and $P_5(x)$ for the nine-parameter rational function (77)

The two polynomials $P_3(x)$ and $P_5(x)$ encoding the pullback of the pullbacked hypergeometric function (85) for the nine-parameter rational function (77) in section (3.3.1), read

$$
\begin{aligned}
P_3(x) \; = \; \; & p_2 \; + \; 48 \cdot \, c_2 \cdot \, (3\, b_3\, d_1\, d_2 \, - c_1^2\, d_1 \, - c_2\, c_3\, d_2) \cdot \, x^3 \\
& + 24 \cdot \, (a\, b_1\, c_2\, d_2 \, + a\, b_3\, c_1\, d_1 \, - 2\, b_1^2\, b_3\, d_2 \, - 2\, b_2\, b_3^2\, d_1) \cdot \, x^2,
\end{aligned} \tag{G.1}
$$

and

$$
\begin{aligned}
P_5(x) \, = \, & p_4 \, - 864 \cdot c_2^3 \, d_1 \, d_2^2 \cdot x^5 \\
& + 864 \cdot (a \, c_1 \, c_2^2 \, d_1 \, d_2 + b_2 \, b_3 \, c_2^2 \, d_1 \, d_2 + b_3^2 \, c_2 \, c_3 \, d_1 \, d_2 - b_3^3 \, d_1^2 \, d_2) \cdot x^4 \\
& - 576 \cdot (b_2 \, c_1^2 \, c_2^2 \, d_1 + b_3 \, c_2^2 \, c_3^2 \, d_2) \cdot x^4 \\
& + 288 \cdot (b_1 \, c_1^3 \, c_2 \, d_1 + b_2 \, c_2^3 \, c_3 \, d_2) \cdot x^4 \\
& - 144 \cdot (b_1 \, c_1 \, c_2^2 \, c_3 \, d_2 + b_3 \, c_1^2 \, c_2 \, c_3 \, d_1) \cdot x^4 \\
& + 216 \cdot (b_1^2 \, c_2^2 \, d_2^2 + b_3^2 \, c_1^2 \, d_1^2 - 6 \, b_1 \, b_3 \, c_1 \, c_2 \, d_1 \, d_2) \cdot x^4 \\
& - 72 \cdot (9 \, a^2 \, b_3 \, c_2 \, d_1 \, d_2 + a^2 \, c_1^2 \, c_2 \, d_1 + a^2 \, c_2^2 \, c_3 \, d_2) \cdot x^3 \\
& - 144 \cdot a \cdot (b_1^2 \, c_1 \, c_2 \, d_2 + b_1 \, b_2 \, c_2^2 \, d_2 + b_1 \, b_3 \, c_1^2 \, d_1 + b_3^2 \, c_1 \, c_3 \, d_1) \cdot x^3 \\
& \qquad - 144 \cdot (b_1^2 \, b_2 \, b_3 \, c_2 \, d_2 + b_1 \, b_2 \, b_3^2 \, c_1 \, d_1) \cdot x^3 \\
& + 720 \cdot (a \, b_1 \, b_3 \, c_2 \, c_3 \, d_2 + a \, b_2 \, b_3 \, c_1 \, c_2 \, d_1) \cdot x^3 \\
& - 576 \cdot (b_1^2 \, b_3^2 \, c_3 \, d_2 + b_2^2 \, b_3^2 \, c_2 \, d_1) \cdot x^3 \\
& + 288 \cdot (b_1^3 \, b_3 \, c_1 \, d_2 + b_2 \, b_3^3 \, c_3 \, d_1 + 3 \, a \, b_1 \, b_3^2 \, d_1 \, d_2) \cdot x^3 \\
& + 36 \cdot a^2 \cdot (a \, b_1 \, c_2 \, d_2 + a \, b_3 \, c_1 \, d_1 - 2 \, b_1^2 \, b_3 \, d_2 - 2 \, b_2 \, b_3^2 \, d_1) \cdot x^2, \qquad \text{(G.2)}
\end{aligned}
$$

where the polynomials p_2 and p_4 are the polynomials $P_2(x)$ and $P_4(x)$ of degree two and four in x given by (16) and (17) in section (2): p_2 and p_4 correspond to the $d_1 \, = \, d_2 \, = \, 0$ limit.

Appendix H. Monomial symmetries on diagonals

Let us sketch the demonstration of the monomial symmetry results of section (98), with the condition that the determinant of (99) is not zero and the conditions (100) are verified. We will denote by n the integer in the three equal sums (100): $n \, = \, A_i \, + B_i \, + C_i$. The diagonal of the rational function of three variables $\mathcal{R}$ is defined through its multi-Taylor expansion (for small x, y and z):

$$
\mathcal{R}\Big(x, \, y, \, z\Big) \, = \, \sum_{m_1 \, = 0}^{\infty} \, \sum_{m_2 \, = 0}^{\infty} \, \sum_{m_3 \, = 0}^{\infty} \, R_{m_1, \, \dots, \, m_n} \cdot x^{m_1} \cdot y^{m_2} \cdot z^{m_3}, \qquad \text{(H.1)}
$$

as the series in one variable x:

$$
\Phi(x) \, = \, Diag\Big(\mathcal{R}\Big(x, \, y, \, z\Big)\Big) \, = \, \sum_{m \, = 0}^{\infty} \, R_{m, \, m, \, m} \cdot x^m. \qquad \text{(H.2)}
$$

The monomial transformation (98) changes the multi-Taylor expansion (H.1) into

$$
\begin{aligned}
\tilde{\mathcal{R}}\Big(x, \, y, \, z\Big) \, = \, & \sum_{M_1 \, = 0}^{\infty} \, \sum_{M_2 \, = 0}^{\infty} \, \sum_{M_3 \, = 0}^{\infty} \, \tilde{R}_{M_1, \, M_2 \, M_3} \cdot x^{M_1} \cdot y^{M_2} \cdot z^{M_3} \, = \\
\sum_{m_1 \, = 0}^{\infty} \, \sum_{m_2 \, = 0}^{\infty} \, \sum_{m_3 \, = 0}^{\infty} \, & R_{m_1, \, m_2 \, m_3} \cdot \Big(x^{A_1} \, y^{A_2} \, z^{A_3}\Big)^{m_1} \, \Big(x^{B_1} \, y^{B_2} \, z^{B_3}\Big)^{m_2} \, \Big(x^{C_1} \, y^{C_2} \, z^{C_3}\Big)^{m_3} \\
= \, & \sum_{m_1 \, = 0}^{\infty} \, \sum_{m_2 \, = 0}^{\infty} \, \sum_{m_3 \, = 0}^{\infty} \, R_{m_1, \, m_2 \, m_3} \cdot x^{M_1} \cdot y^{M_2} \cdot z^{M_3}
\end{aligned}
$$

where:

$$
M_1 \, = \, A_1 \cdot m_1 \, + B_1 \cdot m_2 \, + \, C_1 \cdot m_3, \qquad \text{(H.3)}
$$

$$
M_2 \, = \, A_2 \cdot m_1 \, + B_2 \cdot m_2 \, + \, C_2 \cdot m_3, \qquad \text{(H.4)}
$$

$$
M_3 \, = \, A_3 \cdot m_1 \, + B_3 \cdot m_2 \, + \, C_3 \cdot m_3. \qquad \text{(H.5)}
$$

Taking the diagonal amounts to forcing the exponents m_1, m_2 and m_3 to be equal. It is easy to see that when condition (100) is verified, $m_1 = m_2 = m_3$ yields $M_1 = M_2 = M_3$. Conversely if the determinant of (99) is not zero it is straightforward to see that the conditions $M_1 = M_2 = M_3$ yield $m_1 = m_2 = m_3$.

Then if one knows an exact expression for the diagonal of a rational function, the diagonal of this rational function changed by the monomial transformation (98) reads

$$Diag\Big(\tilde{\mathcal{R}}\Big(x,\, y,\, z\Big)\Big) \,=\, \sum_{M=0}^{\infty} \tilde{R}_{M,\,M,\,M} \cdot x^M \,=\, \sum_{m=0}^{\infty} R_{m,\,m,\,m} \cdot x^{n \cdot m} \,=\, \Phi(x^n), \quad \text{(H.6)}$$

and thus equal to the previous exact expression $\Phi(x)$, where we have changed $x \rightarrow x^n$, where n is the integer $n = A_1 + B_1 + C_1 = A_2 + B_2 + C_2 = A_3 + B_3 + C_3$. These monomial symmetries for diagonal of rational functions are not specific of rational functions of three variables: they can be straightforwardly generalized to an arbitrary number of variables.

Appendix I. Rescaling symmetries on diagonals

We sketch the demonstration of the result in section (4.3). One recalls that the diagonal of the rational function of three variables $\mathcal{R}$ is defined through its multi-Taylor expansion (for small x, y and z)

$$\mathcal{R}\Big(x,\, y,\, z\Big) \,=\, \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \sum_{m_3=0}^{\infty} R_{m_1,\,\dots,\,m_n} \cdot x^{m_1} \cdot y^{m_2} \cdot z^{m_3}, \qquad \text{(I.1)}$$

as the series in one variable x:

$$\Phi(x) \,=\, Diag\Big(\mathcal{R}\Big(x,\, y,\, z\Big)\Big) \,=\, \sum_{m=0}^{\infty} R_{m,\,m,\,m} \cdot x^m. \qquad \text{(I.2)}$$

The (function rescaling) transformation (110) transforms the multi-Taylor expansion (I.1) into:

$$\mathcal{R}\Big(x,\, y,\, z\Big) \,= \qquad \text{(I.3)}$$

$$\sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \sum_{m_3=0}^{\infty} R_{m_1,\,\dots,\,m_n} \cdot x^{m_1} \cdot y^{m_2} \cdot z^{m_3} \cdot F(x\,y\,z)^{m_1+m_2+m_3}.$$

We assume that the function $F(x)$ has some simple Taylor series expansion. Each time taking the diagonal of (I.3) forces the exponents m_1, m_2 and m_3 to be equal in the term $x^{m_1} \cdot y^{m_2} \cdot z^{m_3}$ of the multi-Taylor expansion (I.3), one gets a factor $F(x\,y\,z)^{m_1+m_2+m_3} = F(x\,y\,z)^{3\,m}$. Consequently, the diagonal of (I.3) becomes:

$$\begin{aligned} Diag\Big(\tilde{\mathcal{R}}\Big(x,\, y,\, z\Big)\Big) \,&=\, \sum_{m=0}^{\infty} R_{m,\,m,\,m} \cdot x^n \cdot F(x)^{3\,n} \\ &=\, Diag\Big(\mathcal{R}\Big(x,\, y,\, z\Big)\Big)\Big(x \cdot F(x)^3\Big). \qquad \text{(I.4)} \end{aligned}$$

Clearly, these function-dependent rescaling symmetries for diagonals of rational functions are not specific of rational functions of three variables: they can be straightforwardly generalized to an *arbitrary number of variables*.

References

[1] A. Bostan, S. Boukraa, G. Christol, S. Hassani, J-M. Maillard, *Ising n-fold integrals as diagonals of rational functions and integrality of series expansions*, (2013), J. Phys. **A 46**: Math. Theor. 185202 (44 pages), `http://arxiv.org/abs/1211.6645v2`

[2] Bostan A, Boukraa S, Christol G, Hassani S and Maillard J-M Ising n-fold integrals as diagonals of rational functions and integrality of series expansions: integrality versus modularity Preprint, `http://arxiv.org/abs/1211.6031`

[3] Zenine N, Boukraa S, Hassani S and Maillard J-M 2005 The Fuchsian differential equation of the square lattice Ising model $\chi^{(3)}$ susceptibility, *J. Phys. A* **37** pp. 9651-9668 `https://arxiv.org/abs/math-ph/0407060` arXiv:math-ph/0407060v1

[4] Bostan A, Boukraa S, Guttmann A J, Hassani S, Jensen I, Maillard J-M and Zenine N 2009 High order Fuchsian equations for the square lattice Ising model: $\tilde{\chi}^{(5)}$ *J. Phys. A* **42** 275209, 32, `http://arxiv.org/abs/0904.1601`

[5] Boukraa S, Hassani S, Jensen I, Maillard J-M and Zenine N 2010 High-order Fuchsian equations for the square lattice Ising model: $\chi^{(6)}$ *J. Phys. A* **43** 115201, 22, `http://arxiv.org/abs/0912.4968`

[6] Boukraa S, Hassani S, Maillard J-M and Zenine N 2007 Singularities of n-fold integrals of the Ising class and the theory of elliptic curves *J. Phys. A* **40** pp. 11713-11748, `http://arxiv.org/abs/0706.3367`

[7] Guttmann A J 2010 Lattice Green's functions in all dimensions *J. Phys. A* **43** 305205, 26, `http://arxiv.org/abs/1004.1435`

[8] Glasser M L and Guttmann A J 1994 Lattice Green function (at 0) for the 4D hypercubic lattice *J. Phys. A* **27** 7011–7014, `http://arxiv.org/abs/cond-mat/9408097`

[9] S. Boukraa, S. Hassani, J-M. Maillard, J-A. Weil, *Differential algebra on lattice Green functions and Calabi-Yau operators*, (2014), J. Phys. **A 48**: Math. Theor. 095203 (37 pages)

[10] N. Zenine, S. Hassani, J-M. Maillard, *Lattice Green Functions: the seven-dimensional face-centred cubic lattice*, (2015), J. Phys. **A 48**: Math. Theor 035205 (18 pages) arXiv:1409.8615v1 [math-ph]

[11] S. Hassani, C. Koutschan, J-M. Maillard and N. Zenine, *Lattice Green Functions: the d-dimensional face-centred cubic lattice,* d = 8, 9, 10, 11, 12, (2016), J. Phys. **A 49**: Math. Theor 164003 (30 pages) arXiv:1601.05657v2 [math-ph]

[12] Lipshitz L and van der Poorten A J 1990 Rational functions, diagonals, automata and arithmetic *Number theory (Banff, AB, 1988)* (Berlin: de Gruyter) pp. 339-358

[13] Lipshitz L 1988 The diagonal of a D-finite power series is D-finite *J. Algebra* **113** 373–378 `http://dx.doi.org/10.1016/0021-8693(88)90166-4`

[14] Christol G 1984 Diagonales de fractions rationnelles et équations différentielles *Study group on ultrametric analysis, 10th year: 1982/83, No. 2, Exp. No. 18* (Paris: Inst. Henri Poincaré) pp 1–10 `http://archive.numdam.org/article/GAU_1982-1983__10_2_A4_0.pdf`

[15] Christol G 1985 Diagonales de fractions rationnelles et équations de Picard-Fuchs *Study group on ultrametric analysis, 12th year, 1984/85, No. 1, Exp. No. 13* (Paris: Secrétariat Math.) pp 1–12 `http://archive.numdam.org/article/GAU_1984-1985__12_1_A8_0.pdf`

[16] Christol G 1988 Diagonales de fractions rationnelles *Séminaire de Théorie des Nombres, Paris 1986–87* (*Progr. Math.* vol 75) (Boston, MA: Birkhäuser Boston) pp. 65-90

[17] G. Christol. 1990 *Globally bounded solutions of differential equations*, *Analytic number theory (Tokyo, 1988)* (*Lecture Notes in Math.* **1434** (Berlin: Springer) pp. 45-64 `http://dx.doi.org/10.1007/BFb0097124`

[18] Assis M, Boukraa S, Hassani S, van Hoeij M, Maillard J-M and McCoy B M 2012 Diagonal Ising susceptibility: elliptic integrals, modular forms and Calabi-Yau equations *J. Phys. A* **45** 075205, 32 `http://arxiv.org/abs/1110.1705`

[19] A. Bostan, S. Boukraa, S. Hassani, M. van Hoeij, J-M. Maillard, J-A. Weil, N. J. Zenine, *The Ising model: from elliptic curves to modular forms and Calabi-Yau equations*, J. Phys. **A 44**: Math. Theor. (2011) (43 pp) 045204 and arXiv: 1007.69804 v1 [math-ph] and hal-00684883, version 1

[20] M. Assis, M. van Hoeij, J-M. Maillard, *The perimeter generating functions of three-choice, imperfect, and one-punctured staircase polygons*, (2016), J. Phys. **A 49**: Math. Theor. 214002 (29 pages), `https://arxiv.org/abs/1602.00868`

[21] A. Bostan, S. Boukraa, J-M. Maillard, J-A. Weil, *Diagonal of rational functions and selected differential Galois groups*, (2015), J. Phys. **A 48**: Math. Theor. 504001 (29 pages) arXiv:1507.03227v2 [math-ph]

[22] Y. Abdelaziz, J-M. Maillard, *Modular forms, Schwarzian conditions, and symmetries of*

differential equations in physics, (2016), J. Phys. **A 49**: Math. Theor. 215203 (44 pages), arXiv:1611.08493v3 [math-ph]

[23] Maier R S 2009 On rationally parametrized modular equations *J. Ramanujan Math. Soc.* **24** pp. 1-73 `http://arxiv.org/abs/math/0611041`

[24] P. F. Stiller, *Classical Automorphic Forms and Hypergeometric Functions*, Journ. of Number Theory, **28**, num. 2, pp. 219-232 (1988).

[25] L. C. Shen, *A note on Ramanujan's identities involving the hypergeometric function* ${}_2F_1(1/6, 5/6; 1, z)$, The Ramanujan Journal **30**, Issue 2, (2013), pp. 211-222

[26] F. G. Garvan, *Ramanujan's Theories of Elliptic Functions to Alternative Bases - a Symbolic Excursion*, (1995), J. Symbolic Computation, **20**, pp. 517-536.

[27] Furstenberg H 1967 Algebraic functions over finite fields *J. Algebra* **7** pp. 271-277 `http://dx.doi.org/10.1016/0021-8693(67)90061-0`

[28] Denef J and Lipshitz L 1987 Algebraic power series and diagonals *J. Number Theory* **26** 46–67 `http://dx.doi.org/10.1016/0022-314X(87)90095-3`

[29] HolonomicFunctions Package version 1.7.1 (09-Oct-2013) written by Christoph Koutschan, Copyright 2007-2013, Research Institute for Symbolic Computation (RISC), Johannes Kepler University, Linz, Austria

[30] "hypergeometricsols" Mark van Hoeij, https://www.math.fsu.edu/ hoeij/maple.html, https://www.math.fsu.edu/ hoeij/comparison-hypergeomsols

[31] A. Bostan, P. Lairez and B. Salvy, *Creative telescoping for rational functions using the Griffiths-Dwork method*, Proceedings ISSAC'13, pp. 93-100, ACM Press, 2013. `http://specfun.inria.fr/bostan/publications/BoLaSa13.pdf`

[32] S. Chen, M. Kauers and M.F. Singer, Telescopers for Rational and Algebraic Functions via Residues, 2012, pp. 130-137. https://arxiv.org/abs/1201.1954

[33] A. Bostan, Calcul Formel pour la Combinatoire des Marches, Habilitation à Diriger des Recherches, (15 décembre 2017) Université de Paris 13,

[34] A. Bostan, F. Chyzak, M. van Hoej and L. Pech, Explicit formula for the generating series of Diagonal 3D rook paths, Sém. Lothar. Combin., vol **66**, (2011), 27 pages, and http://www.math.univie.ac.at/s/s66bochhope.html

[35] A. Bostan, F. Chyzak, M. van Hoeij and L. Pech, Explicit formula for the generating functions of walks with small steps in the quarter plane, European J. Combin. **61**, (2017) pp. 242-275.

[36] A. Bostan, F. Chyzak, M. van Hoeij and L. Pech, Explicit formula for the generating series of diagonal 3D rook paths, Sém. Lotar. Combin. **66**, (2011) 27 pages and http://www.mat.univie.ac.at/ slc/s/s66bochhope.html.

[37] Zenine N, Boukraa S, Hassani S and Maillard J-M 2005 Square lattice Ising model susceptibility: connection matrices and singular behaviour of $\chi^{(3)}$ and $\chi^{(4)}$ *J. Phys. A* **38** pp. 9439-9474 `http://arxiv.org/abs/math-ph/0506065`

[38] Y. Abdelaziz, J-M. Maillard, *Schwarzian conditions for linear differential operators with selected differential Galois groups*, (2017), J. Phys. **A 50**: Math. Theor. 465201 (39 pages), arXiv:1706.07423v2 [math-ph]

[39] V. Ovsienko and S. Tabachnikov, *What is ... the schwarzian Derivative ?*, (2009), Notice of the AMS, **56**, pp. 34-36

[40] A. P. Veselov, Growth and integrabillity in the dynamics of mappings, Com. Math. Phys. **145**, (1992) p 181.

[41] https://en.wikipedia.org/wiki/Classical-modular-curve

[42] Erich Hecke, Die eindeutige Bestimmung der Modulfunktionen q-ter Stufe durch algebraische Eigenschaften, Math. Ann. **111** (1935), 293-301, reprinted in Mathematische Werke, third edition, Vandenhoeck and Ruprecht, Göttingen, 1983, 568-576

[43] G.E. Andrews and B.C. Berndt, Chapter 17 pp. 373-393, in *Ramanujan's Lost Notebook*, Part I, 52005) Springer

[44] H. H. Chan and M.-L. Lang, *Ramanujan's modular equations and Atkin-Lehner involutions*, Israel Journal of Mathematics, **103**, (1998) pp. 1–16.

[45] C. Hermite, *Sur la théorie des équations modulaires*, Comptes Rendus Acad. Sci. Paris **49**, 16-24, 110-118, and 141-144, 1859 Oeuvres complètes, Tome II. Paris: Hermann, p. 61, 1912.

[46] M. Hanna, *The Modular Equations*, Proc. London Math. Soc. **28**, 46-52, 1928.

[47] F. Morain, *Calcul du nombre de points sur une courbe elliptique dans un corps fini: aspects algorithmiques*, Journal de Théorie des Nombres de Bordeaux, tome 7, (1995) pp.255-282 and https://eudml.org/doc/247643

[48] Weisstein, Eric W. "Modular Equation." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/ModularEquation.html

[49] Abramowitz, M. and Stegun, I. A. (Eds.). "Ascending Landen Transformation" and "Landen's

Transformation." §16.14 and 17.5 in Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing. New York: Dover, pp. 573-574 and 597-598, 1972.

[50] Landen Transformations: https://dlmf.nist.gov/22.7