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ALMOST INTEGRABLE MAPPINGS*

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We analyze birational transformations obtained from very simple algebraic calculations, namely taking the inverse of $q \times q$ matrices and permuting some of the entries of these matrices. We concentrate on 4×4 matrices and elementary transpositions of two entries. This analysis brings out six classes of birational transformations. Three classes correspond to integrable mappings, their iteration yielding elliptic curves. Generically, the iterations corresponding to the three other classes are included in higher dimensional non-trivial algebraic varieties. Nevertheless some orbits of the parameter space lie on (transcendental) curves. These transformations act on fifteen (or $q^2 - 1$) variables, however one can associate to them remarkably simple non-linear recurrences bearing on a single variable. The study of these last recurrences gives a complementary understanding of these amazingly regular non-integrable mappings, which could provide interesting tools to analyze weak chaos.

1. Introduction

In previous publications, the study of integrability of lattice models in statistical mechanics has brought out the existence of an infinite discrete symmetry group of the Yang–Baxter equations,[1–6] which originates from the so-called *inversion relations.*[7–12] More generally this group corresponds to non-trivial symmetries of phase diagrams of lattice models in statistical mechanics.[13–15] The representations of this group are *birational* transformation groups, generated by *involutions*, acting on the parameter space of the model.[1,2] This analysis has been performed in detail for the sixteen vertex model associated to the two-dimensional square lattice and for a particular subcase of a sixty-four vertex model corresponding to the three-dimensional cubic lattice.[4,5,16–18] In both cases, the parameter space of the model

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can be represented by 4×4 matrices, one of the group generators I, coinciding with the matricial inverse and the other(s) being some permutation(s) of the entries of the 4×4 matrix, denoted t generically.[1] The study of this group brings to analyze these birational mappings and especially the (generically) *infinite order* transformation $t\,I$.[1,2] Remarkably, for the sixteen vertex model and some subcases of the sixty-four vertex model, the iterations of these birational transformations $t\,I$ actually *densify algebraic elliptic curves* in the parameter space, $\mathbb{C}\mathbf{P}_{15}$ (Ref. 5) (generically for the sixty-four vertex model subcase detailed in Refs. 4 and 17, they densify algebraic surfaces). These algebraic curves, (or algebraic surfaces), indeed define a foliation of *the whole parameter space*.[5] When elliptic curves occur in the whole parameter space, the mappings are integrable, though the associated lattice model may *not be integrable itself* (in the sense of Yang–Baxter equations[a] to be satisfied).[13] Such lattice models have been denoted quasi-integrable.[5]

Integrable mappings acting on fifteen inhomogeneous variables have thus emerged from this analysis of the symmetry of integrability for lattice models. This result is interesting in itself, as far as discrete dynamical systems are concerned, since the examples known in the literature were always systems bearing on few variables.[22,23] Besides, it suggests considering other birational mappings in $\mathbb{C}\mathbf{P}_{15}$, generated by the matricial inverse and some (involutive) permutations of the entries of the matrix, not even related to any symmetry of lattice models of statistical mechanics.[24,25] The results will be reported here and, in parallel, in a series of papers,[24–26] for the simplest examples of permutations: the transpositions of two entries. It is important to note that, though this paper deals mainly with 4×4 matrices and therefore mappings of $\mathbb{C}\mathbf{P}_{15}$, the results *can be generalized to $q \times q$ matrices*, the associated mappings acting in $\mathbb{C}\mathbf{P}_{q^2-1}$. (Refs. 24 and 25).

We will first show that one can reduce the study to *six classes* of such mappings. Their iterations often lie on *curves*, however this emergence of curves does not correspond to a unique situation, but on the contrary to two different ones. Three of these classes correspond to integrable mappings, their iterations actually yielding algebraic elliptic curves. The equations of these elliptic curves are given as intersections of fourteen quadrics in $\mathbb{C}\mathbf{P}_{15}$. On the other hand, the three remaining classes correspond to another kind of mappings, we will call "almost" integrable: they are *not generically integrable, even if their iterations stay on curves* in some regions of the projective space. Actually, these curves are *not algebraic, but transcendental*, though they may look very much like algebraic elliptic curves (see Secs. 4, 6 and 8 in the following). Indeed, for such "almost" integrable mappings, one can follow the evolution of an orbit from a transcendental curve close to an algebraic elliptic curve, up to an "explosion" into a spray of points. This is reminiscent of the KAM theorem.[23] More precisely, when iterating these mappings one gets orbits similar to the one described by Siegel's theorem,[27–30] though there is no associated complex structure.

[a]or of their higher dimensional generalizations.[19–21]

In other publications,[24–26] it is shown that the iterations of these birational transformations present *some remarkable factorization properties*, and that the polynomial factors occurring in these factorizations satisfy for some classes *non-trivial non-linear recurrences.*[24,25] It will be shown that these non-linear recurrences on one variable also describe algebraic elliptic curves or transcendental curves, and thus, can also be classified in two categories : integrable recurrences and "almost" integrable recurrences. The equations of these elliptic curves are *biquadratic relations*, and the transcendental curves look like deformations of these algebraic elliptic curves. When the deformation parameters become large enough one can again see the "explosion" of these transcendental curves (the KAM tori).

These recurrences can be generalized to build new recurrences on *a single variable*, yielding again either algebraic elliptic curves or transcendental curves (see Sec. 8.3).

Finally, one will sketch other 15-dimensional mappings, which are neither integrable nor "almost" integrable, their orbits lying on *surfaces or higher dimensional algebraic varieties.* These last algebraic varieties could also be deformed into transcendental manifolds. This analysis suggests some "graduation of chaos", from integrability strictly speaking to more and more chaotic situations.

2. Notations

Let us consider the 4×4 matrix

$$R = \begin{pmatrix} a_1 & a_2 & b_1 & b_2 \\ a_3 & a_4 & b_3 & b_4 \\ c_1 & c_2 & d_1 & d_2 \\ c_3 & c_4 & d_3 & d_4 \end{pmatrix} . \tag{2.1}$$

Let us also introduce the homogeneous matrix inverse I:

$$I : R \longrightarrow R^{-1} \cdot \det(R) . \tag{2.2}$$

The homogeneous inverse I is a *polynomial* transformation on each of the entries. It associates with each entry its corresponding cofactor. The homogeneous transformation I is an *involution* up to a multiplicative factor: it satisfies $I^2 = (\det(R))^2 \cdot \mathcal{I}d$, where $\mathcal{I}d$ denotes the identity transformation.

One will also introduce the involution t, which denotes an arbitrary transposition of two entries of matrix R, and $K_t = t \cdot I$, the infinite order transformation associated with each transposition t. Transformation K_t is a polynomial transformation of the sixteen homogeneous entries of the matrix. Γ_t will denote the infinite discrete group generated by I and t.[1,2,6,13] Such groups Γ_t, generated by two involutions are isomorphic to $\mathbb{Z}$, up to a semi-direct product by a two element group (the infinite dihedral group). The "infinite part" of the group (which is isomorphic to $\mathbb{Z}$) is generated by K_t, that is the simplest *infinite order* generator of the group. Notice that Γ_t is a *group of birational transformations.*[1,2]

3. Six Equivalence Classes

Let us analyze these groups Γ_t, of birational transformations. At first sight, one has to study as many groups of mappings as there are transpositions t, of two elements among the sixteen entries of the matrix, that is $\binom{2}{16} = 120$.

In fact, the 120 corresponding groups Γ_t, *fall in only six different classes.*

Let us first exhibit an equivalence relation on these 120 transpositions, which does not modify the structure of the corresponding group, defining seven different classes. Let us construct this equivalence relation by introducing transformations acting on the entries of the matrix, which commute with the matricial inverse I. Let us consider transformation T, which associates to a matrix, its transpose. Let us also consider transformations T_{ij} $(i,\, j = 1 \ldots 4)$, which permute rows i and j, and at the same time columns i and j of matrix (2.1). For example T_{12} reads:

$$T_{12}(R) = \begin{pmatrix} a_4 & a_3 & b_3 & b_4 \\ a_2 & a_1 & b_1 & b_2 \\ c_2 & c_1 & d_1 & d_2 \\ c_4 & c_3 & d_3 & d_4 \end{pmatrix} .$$

Obviously, these transformations are involutions which commute with the matricial inverse I (moreover T does commute with all the T_{ij}'s). A generic element of the finite group $\mathcal{G}$, generated by these involutions, reads:

$$g = \left(\prod_{i,\, j=1 \ldots 4} T^{\alpha_{ij}}_{\sigma(i)\sigma'(j)} \right) T^{\alpha}, \qquad \alpha,\, \alpha_{ij} = 0,\, 1$$

where σ and σ' are two permutations of $\{1,\, 2,\, 3,\, 4\}$.[b]

$\mathcal{G}$ is isomorphic to the group of permutations of four elements $\mathcal{S}_4$, up to the semi-direct product by the two element group $\{\mathcal{I}d,\, T\}$. One can now define the equivalence relation : two transpositions t and t', belong to the same equivalence class if, and only if, there exists an element g of $\mathcal{G}$ such that:

$$t' = g^{-1}\, t\, g\ . \tag{3.1}$$

For two such transpositions let us compare the two groups of transformations Γ_t and $\Gamma_{t'}$, generated respectively by $(I,\, t)$ and $(I,\, t')$. A generic element of $\Gamma_{t'}$ reads:

$$\begin{aligned} \gamma_{t'} &= I^{\alpha^1}(It')^n I^{\alpha^2} = I^{\alpha^1}(I\, g^{-1}tg)^n I^{\alpha^2} = I^{\alpha^1}(g^{-1}Itg)^n I^{\alpha^2} \\ &= I^{\alpha^1}\, g^{-1}\,(I\,t)^n\, g\, I^{\alpha^2} = g^{-1}\,(I^{\alpha^1}\,(I\,t)^n\, I^{\alpha^2})\, g, \qquad \text{with} \qquad \alpha_1,\, \alpha_2 = 0,\, 1 \\ &= g^{-1}\, \gamma_t\, g \end{aligned}$$

where γ_t is an element of Γ_t.

[b]The T_{ij}'s transformations do not commute with each other, thus their order in the product is relevant.

Thus, Γ_t and $\Gamma_{t'}$ are actually two (birational) representations of the same group. If Δ_t denotes a rational expression of the entries of matrix R, invariant under the group Γ_t, which means

$$\Delta_t(\gamma_t(R)) = \Delta_t(R) \tag{3.2}$$

for all γ_t in Γ_t, then the rational expression $\Delta_{t'}(R) = \Delta_t(g(R))$ is invariant under $\Gamma_{t'}$:

$$\begin{aligned}\Delta_{t'}(\gamma_{t'}(R)) &= \Delta_t(g(g^{-1}\,\gamma_t\,g(R))) = \Delta_t(g\,g^{-1}\,\gamma_t\,g(R)) = \Delta_t(\gamma_t\,(g(R))) = \Delta_t(g(R))\\ &= \Delta_{t'}(R)\end{aligned}$$

for all $\gamma_{t'}$ in $\Gamma_{t'}$.

In order to list all the classes and their elements, let us detail the correspondence between $\mathcal{G}$ and $\mathcal{S}_4$: one can associate to $g = \prod T_{ij}$ in $\mathcal{G}$ a permutation $\sigma = \prod t_{ij}$ in $\mathcal{S}_4$ (t_{ij} being the transposition of $\{1\ldots4\}$ exchanging i and j). If t denotes the transposition exchanging $R_{i_1j_1}$ and $R_{i_2j_2}$, then $t' = g^{-1}\,t\,g$ is the transposition exchanging $R_{\sigma(i_1)\sigma(j_1)}$ and $R_{\sigma(i_2)\sigma(j_2)}$, and obviously $t'' = T\,t\,T$ is the transposition exchanging $R_{j_1i_1}$ and $R_{j_2i_2}$.

With the notation $[r_{ij} - r_{kl}]$ for transposition exchanging the two entries r_{ij} and r_{kl} of matrix R (2.1), the seven classes read:

— Class $\mathcal{C}_1$ corresponds to all the 6 transpositions of the form $[r_{ij} - r_{ji}]$
— Class $\mathcal{C}_2$ corresponds to all the 6 transpositions of the form $[r_{ii} - r_{jj}]$
— Class $\mathcal{C}_3$ corresponds to all the 12 transpositions of the form $[r_{ij} - r_{kl}]$
— Class $\mathcal{C}_4$ corresponds to all the 24 transpositions of the form $[r_{ij} - r_{jk}]$ or $[r_{ji} - r_{kj}]$
— Class $\mathcal{C}_5$ corresponds to all the 24 transpositions of the form $[r_{ij} - r_{ik}]$ or $[r_{ji} - r_{ki}]$
— Class $\mathcal{C}_6$ corresponds to all the 24 transpositions of the form $[r_{ii} - r_{jk}]$
— Class $\mathcal{C}_7$ corresponds to all the 24 transpositions of the form $[r_{ii} - r_{ij}]$ or $[r_{ii} - r_{ji}]$

where all the indices i, j, k and l are different.

Moreover, one can show that classes $\mathcal{C}_1$ and $\mathcal{C}_2$ lead to the same behavior as far as iterations of their associated birational mappings are concerned: let us denote C_{ij} and L_{ij} the involutions respectively permuting columns i and j, or rows i and j ($C_{ij} \cdot L_{ij} = L_{ij} \cdot C_{ij} = T_{ij}$). In contrast with the T_{ij}'s, the C_{ij}'s and L_{ij}'s do not commute with I, but are intertwined by I:

$$C_{ij} \cdot I \cdot L_{ij} = L_{ij} \cdot I \cdot C_{ij} = I\ . \tag{3.3}$$

If t_{c_1} denotes a transposition of class $\mathcal{C}_1$, namely $[r_{ij} - r_{ji}]$ and t_{c_2} a transposition of class $\mathcal{C}_2$, namely $[r_{ii} - r_{jj}]$, one has

$$t_{c_2} = C_{ij} \cdot t_{c_1} \cdot C_{ij} = L_{ij} \cdot t_{c_1} \cdot L_{ij}\ . \tag{3.4}$$

Therefore a generic element of $\Gamma_{t_{c_2}}$ reads

$$\begin{aligned}
\gamma_{t_{c_2}} &= I^{\alpha_1}\,(I\,t_{c_2})^n\,I^{\alpha_2} = I^{\alpha_1}\,(I\,C_{ij}\,t_{c_1}\,C_{ij}\,I\,L_{ij}\,t_{c_1}\,L_{ij}\ldots)\,I^{\alpha_2}\\
&= I^{\alpha_1}\,L_{ij}\,((L_{ij}\,I\,C_{ij})\,t_{c_1}\,(C_{ij}\,I\,L_{ij})\,t_{c_1}\,(L_{ij}\,I\,C_{ij})\,t_{c_1}\,(C_{ij}\,I\,L_{ij})\ldots)\,I^{\alpha_2}\\
&= I^{\alpha_1}\,L_{ij}\,(I\,t_{c_1}\,I\,t_{c_1}\,I\,t_{c_1}\,I\ldots)\,W\,I^{\alpha_2} = W_1\,I^{\alpha_1}\,(I\,t_{c_1})^n\,I^{\alpha_2}\,W_2\\
&= W_1\,\gamma_{t_{c_1}}\,W_2
\end{aligned}$$

where $\gamma_{t_{c_1}}$ is an element of $\Gamma_{t_{c_1}}$, and where W, W_1 and W_2 denote either C_{ij} or L_{ij} depending on the parity of n and the values of α_1 and α_2 (0 or 1).

Therefore the iterations of $(I\,t_{c_1})^n$ and $(I\,t_{c_2})^n$ are equivalent up to some changes of variables. If n is even, one notices that the elements $(I\,t_{c_1})^n$ and $(I\,t_{c_2})^n$, respectively associated to classes $\mathcal{C}_1$ and $\mathcal{C}_2$, *are actually conjugated.*

Let us now consider $\Delta_{t_{c_1}}$, a rational expression of the entries of matrix R, invariant under the group $\Gamma_{t_{c_1}}$ and build a new rational expression invariant under the group $\Gamma_{t_{c_2}}$, denoted $\Delta_{t_{c_2}}$.

Let us first introduce $\Delta^L_{t_{c_2}}$ and $\Delta^C_{t_{c_2}}$:

$$\Delta^L_{t_{c_2}}(R) = \Delta_{t_{c_1}}(L_{ij}\,(R))\ , \quad \Delta^C_{t_{c_2}}(R) = \Delta_{t_{c_1}}(C_{ij}\,(R))\ . \tag{3.5}$$

One remarks that

$$\begin{aligned}
\Delta^L_{t_{c_2}}(t_{c_2}\,(R)) &= \Delta_{t_{c_1}}(L_{ij}\,t_{c_2}\,(R)) = \Delta_{t_{c_1}}(t_{c_1}\,L_{ij}\,(R)) = \Delta_{t_{c_1}}(L_{ij}\,(R)) = \Delta^L_{t_{c_2}}(R)\\
\Delta^C_{t_{c_2}}(t_{c_2}\,(R)) &= \Delta_{t_{c_1}}(C_{ij}\,t_{c_2}\,(R)) = \Delta_{t_{c_1}}(t_{c_1}\,C_{ij}\,(R)) = \Delta_{t_{c_1}}(C_{ij}\,(R)) = \Delta^C_{t_{c_2}}(R)
\end{aligned} \tag{3.6}$$

and also

$$\begin{aligned}
\Delta^L_{t_{c_2}}(I\,(R)) &= \Delta_{t_{c_1}}(L_{ij}\,I\,(R)) = \Delta_{t_{c_1}}(I\,C_{ij}\,(R)) = \Delta_{t_{c_1}}(C_{ij}\,(R)) = \Delta^C_{t_{c_2}}(R)\\
\Delta^C_{t_{c_2}}(I\,(R)) &= \Delta_{t_{c_1}}(C_{ij}\,I\,(R)) = \Delta_{t_{c_1}}(I\,L_{ij}\,(R)) = \Delta_{t_{c_1}}(L_{ij}\,(R)) = \Delta^L_{t_{c_2}}(R)\ .
\end{aligned} \tag{3.7}$$

One thus can define $\Delta_{t_{c_2}} = \Delta^L_{t_{c_2}} + \Delta^C_{t_{c_2}}$, which is invariant by t_{c_2} and I, and thus invariant under all the group $\Gamma_{t_{c_2}}$. Finally classes $\mathcal{C}_1$ and $\mathcal{C}_2$ *can be brought together in the same class*, we will denote class I in the following. The five other classes ($\mathcal{C}_3$, ..., $\mathcal{C}_7$) will be relabelled classes (II, ..., VI) in the same order.

Let us now give an exhaustive list of elements for all the six classes. The notation $[a_1 - a_2]$ still denotes the transposition exchanging the entries a_1 and a_2 in the 4×4 matrix (2.1).

— Class I: (12 elements)

$$[a_2 - a_3]\,,\ [b_1 - c_1]\,,\ [b_2 - c_3]\,,\ [b_4 - c_4]\,,\ [d_2 - d_3]\,,\ [c_2 - b_3]\,,\ [a_1 - a_4]\,,\ [a_1 - d_1]\,,$$
$$[a_1 - d_4]\,,\ [a_4 - d_1]\,,\ [a_4 - d_4]\,,\ [d_1 - d_4]\,,$$

— Class II: (12 elements)

$$[b_1 - b_4]\,,\ [b_1 - c_4]\,,\ [b_2 - b_3]\,,\ [b_2 - c_2]\,,\ [a_2 - d_2]\,,\ [a_2 - d_3]\,,\ [a_3 - d_2]\,,\ [a_3 - d_3]\,,$$
$$[c_1 - b_4]\,,\ [c_1 - c_4]\,,\ [c_3 - b_3]\,,\ [c_3 - c_2]\,,$$

— Class III: (24 elements)

$$\begin{array}{cccccc}
[a_2 - b_3]\,, & [a_2 - b_4]\,, & [a_2 - b_2]\,, & [a_2 - c_3]\,, & [d_2 - b_3]\,, & [d_2 - b_1]\,, \\
[d_2 - c_3]\,, & [d_2 - c_4]\,, & [a_3 - c_2]\,, & [a_3 - c_4]\,, & [a_3 - b_1]\,, & [a_3 - b_2]\,, \\
[d_3 - c_2]\,, & [d_3 - c_1]\,, & [d_3 - b_2]\,, & [d_3 - b_4]\,, & [c_2 - b_4]\,, & [c_2 - b_1]\,, \\
[b_2 - c_1]\,, & [b_2 - c_4]\,, & [b_3 - c_4]\,, & [b_3 - c_1]\,, & [c_3 - b_1]\,, & [c_3 - b_4]\,,
\end{array}$$

— Class IV: (24 elements)

$$\begin{array}{cccccc}
[a_2 - b_1]\,, & [a_2 - b_2]\,, & [b_1 - b_2]\,, & [a_3 - b_3]\,, & [a_3 - b_4]\,, & [b_3 - b_4]\,, \\
[c_1 - c_2]\,, & [c_1 - d_2]\,, & [c_2 - d_2]\,, & [c_3 - c_4]\,, & [c_3 - d_3]\,, & [c_4 - d_3]\,, \\
[a_3 - c_1]\,, & [a_3 - c_3]\,, & [c_1 - c_3]\,, & [a_2 - c_2]\,, & [a_2 - c_4]\,, & [c_2 - c_4]\,, \\
[b_1 - b_3]\,, & [b_1 - d_3]\,, & [b_3 - d_3]\,, & [b_2 - b_4]\,, & [b_2 - d_2]\,, & [b_4 - d_2]\,,
\end{array}$$

— Class V: (24 elements)

$$\begin{array}{cccccc}
[a_1 - b_3]\,, & [a_1 - b_4]\,, & [a_1 - d_2]\,, & [a_4 - b_1]\,, & [a_4 - b_2]\,, & [a_4 - d_2]\,, \\
[d_1 - a_2]\,, & [d_1 - b_2]\,, & [d_1 - b_4]\,, & [d_4 - a_2]\,, & [d_4 - b_1]\,, & [d_4 - b_3]\,, \\
[a_1 - c_2]\,, & [a_1 - c_4]\,, & [a_1 - d_3]\,, & [a_4 - c_1]\,, & [a_4 - c_3]\,, & [a_4 - d_3]\,, \\
[d_1 - a_3]\,, & [d_1 - c_3]\,, & [d_1 - c_4]\,, & [d_4 - a_3]\,, & [d_4 - c_1]\,, & [d_4 - c_2]\,,
\end{array}$$

— Class VI: (24 elements)

$$
\begin{array}{cccccc}
[a_4 - a_2]\,, & [a_1 - b_1]\,, & [a_1 - b_2]\,, & [a_4 - a_3]\,, & [a_4 - b_3]\,, & [a_4 - b_4]\,, \\
[d_1 - c_1]\,, & [d_1 - c_2]\,, & [d_1 - d_2]\,, & [d_4 - c_3]\,, & [d_4 - c_4]\,, & [d_4 - d_3]\,, \\
[a_1 - a_3]\,, & [a_1 - c_1]\,, & [a_1 - c_3]\,, & [a_1 - a_2]\,, & [a_4 - c_2]\,, & [a_4 - c_4]\,, \\
[d_1 - b_1]\,, & [d_1 - b_3]\,, & [d_1 - d_3]\,, & [d_4 - b_2]\,, & [d_4 - b_4]\,, & [d_4 - d_2]\,.
\end{array}
$$

It is important to note that this classification in six classes *holds for $q \times q$ matrices, for any value of $q \geq 4$*. For $q = 3$ one remarks that class II no longer exists and similarly, for $q = 2$, classes II, III, IV and V do not exist anymore. Also note that any transposition of two entries $R_{i_1j_1}$ and $R_{i_2j_2}$ of a $q \times q$ matrix can be associated with a transposition exchanging $R_{\sigma(i_1)\sigma(j_1)}$ and $R_{\sigma(i_2)\sigma(j_2)}$, where $\sigma(i_1)$, $\sigma(j_1)$, $\sigma(i_2)$ and $\sigma(j_2)$ run into $\{1, 2, 3, 4\}$. One can thus restrict the transposition to act in the 4×4 block-matrix corresponding to the first four rows and columns, and be back to the previous classification on 4×4 matrices.

One will now study a single mapping in each class and directly deduce the results concerning all the other transformations of the same class.

4. Numerical Study

An efficient method to analyze such transformations, especially when there are many variables, is to iterate numerically the action of the (generically) infinite order transformation K_t, on an arbitrary initial matrix and to visualize a two-dimensional projection of the orbit.[1,2]

The iterations are not performed on the sixteen homogeneous entries, but on fifteen inhomogeneous variables and thus K_t is represented in terms of *birational* transformations.[5] In fact the numerical calculations are not performed in $\mathbb{C}\mathbf{P}_{15}$, but in $\mathbb{C}^{15}$ (one of the variables being normalized to 1).[c] Since this representation of $\mathbb{C}\mathbf{P}_{15}$ is not compact, some points may apparently go to infinity. K_t, seen as an inhomogeneous transformation, being of degree -1, favors points going to infinity, while K_t^2 do not lead to such point spreading. One thus iterates transformation K_t^2 and obtains Figs. (1) to (6).

Figure (1) shows a two-dimensional projection in the 15-dimensional space, of a trajectory of the K_{t_I}-iteration, where t_I is a transposition of class I. Similarly Figs. 2 to 6 correspond to two-dimensional projections of trajectories of the K_t-iteration, where t is a transposition of classes II to VI, in the same order. These figures have been obtained with around 10^5 iterations.

[c]Since coefficients of birational transformations are integers all the iterates are in $\mathbb{R}^{15}$, if the initial matrix is taken real. All figures given in this paper correspond to iterations in $\mathbb{R}^{15}$.

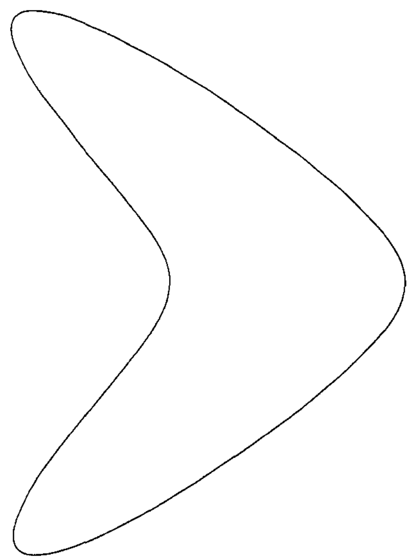


Fig. 1.

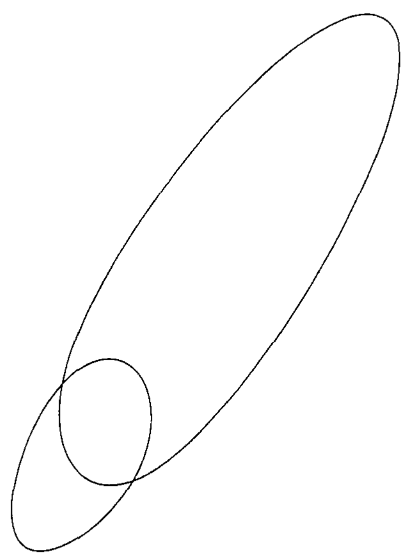


Fig. 2.

All these figures exhibit curves. This result is astonishing, if one takes into account the complexity of the birational transformation K_t^2. Moreover this complexity, due to the degree of the homogeneous transformation and to the number

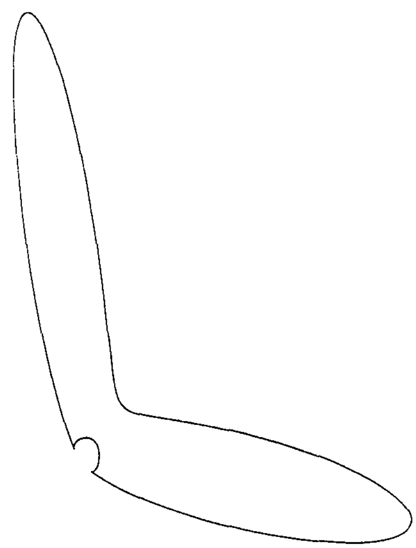


Fig. 3.

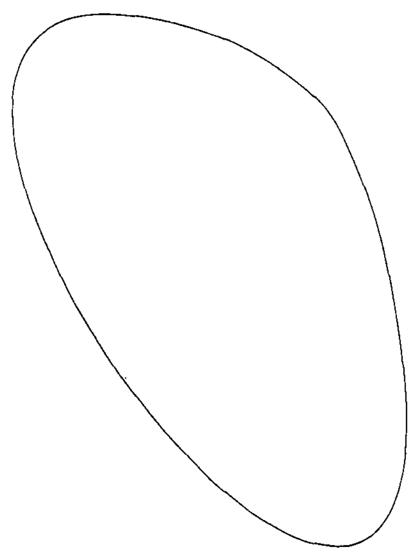


Fig. 4.

of variables, *does not yield any numerical instability of the curves.* One can also remark that fewer number of iterations still densify these curves in a quite uniform way.

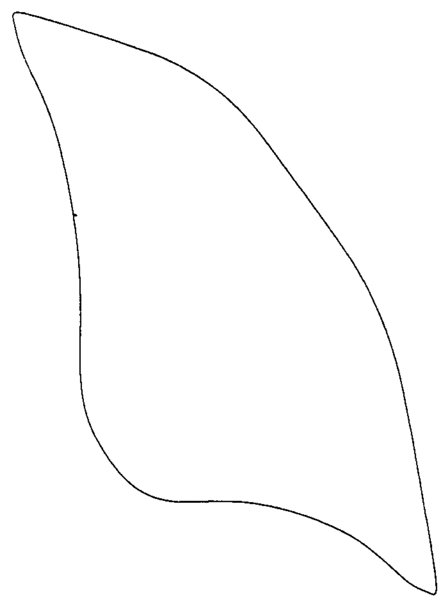


Fig. 5.

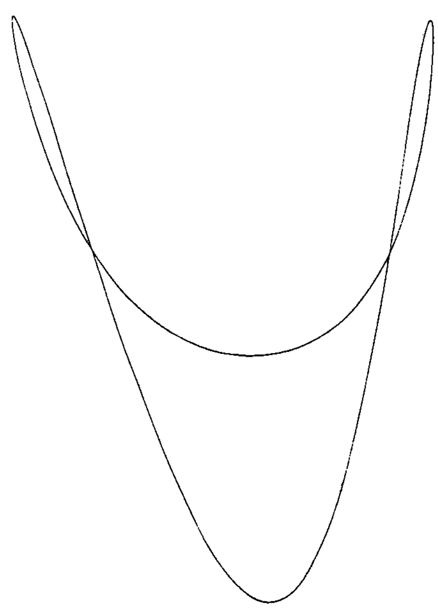


Fig. 6.

5. Algebraic Invariants

The above numerical study tends to show that the orbits of the groups of birational transformations Γ_t associated with any transposition are *curves* (at least in some domain of the parameter space). One can thus try to get the equations of these curves in $\mathbb{C}\mathbf{P}_{15}$, for instance from explicit algebraic expressions, invariant under Γ_t.

Let us first recall an important property of $k \times k$ minors of a generic matrix A: $a_{(I_k)(J_k)} = a_{i_1,\ldots,i_k,j_1,\ldots,j_k}$ denotes the $k \times k$ minor of matrix A corresponding to rows $i_1, \ldots, i_k$ and columns $j_1, \ldots, j_k$ and $\widehat{a}_{(I_k)(J_k)} = \widehat{a}_{i_1,\ldots,i_k,j_1,\ldots,j_k}$ denotes the minor corresponding to the $(n-k)$ complementary rows and columns in $\{1, \ldots, n\}$. Let us also introduce $B = A^{-1}$ the inverse matrix of A, $b_{(I_k)(J_k)}$ the minor of matrix B corresponding to rows $i_1, \ldots, i_k$ and columns $j_1, \ldots, j_k$ and $\widehat{b}_{(I_k)(J_k)}$ the minor corresponding to the $(n-k)$ complementary rows and columns in $\{1, \ldots, n\}$. Some elementary considerations on exterior algebra yield the relation

$$a_{(I_k)(J_k)} = \frac{(-1)^{i_1+\ldots+i_k+j_1+\ldots+j_k}}{\det(A)} \quad \widehat{b}_{(I_k)(J_k)} \ . \tag{5.1}$$

In particular if A denotes a $2p \times 2p$ matrix, relation (5.1), for $k = p$, becomes an equation relating two $p \times p$ minors, one of A and the other one of A^{-1}. Hence the set of $p \times p$ minors is globally invariant, up to a multiplicative factor, under the matricial inverse I. Let us restrict to 4×4 matrices, that is $p = 2$. The set of all 2×2 minors of the 4×4 matrix R reads

$$\begin{array}{lll}
m_1 = a_1\, a_4 - a_2\, a_3 \ , & m_2 = a_1\, b_3 - a_3\, b_1 \ , & m_3 = a_1\, b_4 - a_3\, b_2 \ , \\
m_4 = a_2\, b_3 - a_4\, b_1 \ , & m_5 = a_2\, b_4 - a_4\, b_2 \ , & m_6 = b_1\, b_4 - b_3\, b_2 \ , \\
m_7 = a_1\, c_2 - c_1\, a_2 \ , & m_8 = a_1\, d_1 - c_1\, b_1 \ , & m_9 = a_1\, d_2 - c_1\, b_2 \ , \\
m_{10} = a_2\, d_1 - c_2\, b_1 \ , & m_{11} = a_2\, d_2 - c_2\, b_2 \ , & m_{12} = b_1\, d_2 - b_2\, d_1 \ , \\
m_{13} = a_1\, c_4 - c_3\, a_2 \ , & m_{14} = a_1\, d_3 - c_3\, b_1 \ , & m_{15} = a_1\, d_4 - c_3\, b_2 \ , \\
m_{16} = a_2\, d_3 - c_4\, b_1 \ , & m_{17} = a_2\, d_4 - c_4\, b_2 \ , & m_{18} = b_1\, d_4 - d_3\, b_2 \ , \\
m_{19} = a_3\, c_4 - c_3\, a_4 \ , & m_{20} = a_3\, d_3 - c_3\, b_3 \ , & m_{21} = a_3\, d_4 - c_3\, b_4 \ , \\
m_{22} = a_4\, d_3 - c_4\, b_3 \ , & m_{23} = a_4\, d_4 - c_4\, b_4 \ , & m_{24} = b_3\, d_4 - b_4\, d_3 \ , \\
m_{25} = a_3\, c_2 - c_1\, a_4 \ , & m_{26} = a_3\, d_1 - c_1\, b_3 \ , & m_{27} = a_3\, d_2 - c_1\, b_4 \ , \\
m_{28} = a_4\, d_1 - c_2\, b_3 \ , & m_{29} = a_4\, d_2 - c_2\, b_4 \ , & m_{30} = b_3\, d_2 - d_1\, b_4 \ , \\
m_{31} = c_1\, c_4 - c_3\, c_2 \ , & m_{32} = c_1\, d_3 - c_3\, d_1 \ , & m_{33} = c_1\, d_4 - c_3\, d_2 \ , \\
m_{34} = c_2\, d_3 - c_4\, d_1 \ , & m_{35} = c_2\, d_4 - c_4\, d_2 \ , & m_{36} = d_1\, d_4 - d_3\, d_2 \ .
\end{array} \tag{5.2}$$

This property of global invariance of these minors by I reads in that particular case

$$I(m_1) = \eta\, m_{36}\ , \quad I(m_2) = -\eta\, m_{24}, \quad I(m_3) = \eta\, m_{30}\ ,$$
$$I(m_4) = \eta\, m_{18}\ , \quad I(m_5) = -\eta\, m_{12}\ , \quad I(m_6) = \eta\, m_6, \ldots$$

where $\eta = det(R)$.

The non-linear transformation I, thus has a linear representation in terms of the 2×2 minors.

We will thus barter the sixteen homogeneous variables for these thirty-six quadratic minors (5.2). Of course the number of homogeneous variables being sixteen, there must exist many relations between these minors.

To find out these relations, let us first recall an important property of projective spaces and *Plücker coordinates*[31]: if π is an m-dimensional subspace of an n-dimensional projective space $\mathbb{C}\mathbf{P_n}$ spanned by $m+1$ independent points $A_0, A_1, \ldots, A_m$ of $\mathbb{C}\mathbf{P_n}$. Let us denote by $(a_{00}, \ldots, a_{0n})$, $(a_{10}, \ldots, a_{1n})$, ..., $(a_{m0}, \ldots, a_{mn})$ projective coordinates of these points, and by $p_{j_0,j_1,\ldots,j_m}$ the determinants

$$p_{j_0,j_1,\ldots,j_m} = \begin{vmatrix} a_{0j_0} & \cdots & a_{0j_m} \\ \cdots & \cdots & \cdots \\ a_{mj_0} & \cdots & a_{mj_m} \end{vmatrix}$$

where $0 \le j_0, \ldots, j_m \le n$.

The subspace π can be represented by these homogeneous coordinates $p_{j_0,j_1,\ldots,j_m}$, which are independent of the choice of $m+1$ points spanning π. These Plücker coordinates actually satisfy *quadratic relations*, the so-called *Plücker relations.*[31] In the well-known subcase[31] corresponding to $m = 1$, $n = 3$ this set of equations reduces to the relation

$$p_{01}\, p_{23} - p_{02}\, p_{13} + p_{03}\, p_{12} = 0\ . \qquad (5.3)$$

With each couple of rows, or columns, of the matrix R, are associated six minors, which are actually Plücker coordinates. One thus obtains twelve quadratic relations like (5.3).

Besides, let us recall another important property of $k \times k$ minors of a generic matrix A: if $a_{(I_k)(J_k)} = a_{i_1,\ldots,i_k,j_1,\ldots,j_k}$ still denotes the minor corresponding to rows $i_1, \ldots, i_k$ and columns $j_1, \ldots, j_k$, and $\widehat{a}_{(I_k)(J_k)} = \widehat{a}_{i_1,\ldots,i_k,j_1,\ldots,j_k}$ the minor corresponding to the $(n-k)$ complementary rows and columns in $\{1, \ldots, n\}$, let us denote $\mathcal{M}_k$ the matrix of entries $a_{(I_k)(J_k)}$, and C_k the matrix of entries

$(-1)^{i_1+\ldots+i_k+j_1+\ldots+j_k}\,\widehat{a}_{(I_k)(J_k)}$, the Laplace theorem then reads

$$\mathcal{M}_k \cdot (C_k)^t = (C_k)^t \cdot \mathcal{M}_k = det(A)\,\mathcal{I}d_{\binom{k}{n}} \qquad (5.4)$$

where $(C_k)^t$ denotes the transpose of matrix C_k.[d]

In the case, mainly studied in this paper, where $n = 4$ and $k = 2$, one obtains a set of thirty-six more quadratic equations on the 2×2 minors. Obviously, these equations and the twelve Plücker relations (5.3) are not independent.

These well-suited Plücker-like variables enable one to get the algebraic invariants under the various groups Γ_i. Since I is *linear* with respect to these new variables, it can be "diagonalized" in terms of linear combinations of these minors. Since I satisfies $I^2 = \eta^2\,\mathcal{I}d$ (η still denotes $\det(R)$), the only possible "eigenvalues" are $+\eta$ and $-\eta$. One can thus calculate the linear combinations of these minors covariant, up to $\pm\eta$, under the matricial inverse I, as "eigenvectors" associated to each "eigenvalue".

Let us first give the minors directly covariant under I, with cofactor $+\eta$:

$$l_1 = m_6\,, \quad l_2 = m_{11}\,, \quad l_3 = m_{16}\,, \quad l_4 = m_{20}\,, \quad l_5 = m_{27}\,, \quad l_6 = m_{31}\,. \qquad (5.5)$$

Moreover, *the following fifteen linear combinations are also covariant under I with cofactor $+\eta$*:

$$\begin{array}{llll}
l_7 = m_1 + m_{36}\,, & l_8 = m_3 + m_{30}\,, & l_9 = m_4 + m_{18}\,, & l_{10} = m_8 + m_{23}\,, \\
l_{11} = m_{13} + m_{34}\,, & l_{12} = m_{15} + m_{28}\,, & l_{13} = m_{25} + m_{33}\,, & l_{14} = m_2 - m_{24}\,, \\
l_{15} = m_5 - m_{12}\,, & l_{16} = m_7 - m_{35}\,, & l_{17} = m_9 - m_{29}\,, & l_{18} = m_{10} - m_{17}\,, \\
l_{19} = m_{14} - m_{22}\,, & l_{20} = m_{19} - m_{32}\,, & l_{21} = m_{21} - m_{26}\,. &
\end{array} \qquad (5.6)$$

Finally, the following fifteen linear combinations of minors, are covariant under I, with cofactor $-\eta$:

$$\begin{array}{llll}
l_{22} = m_1 - m_{36}\,, & l_{23} = m_3 - m_{30}\,, & l_{24} = m_4 - m_{18}\,, & l_{25} = m_8 - m_{23}\,, \\
l_{26} = m_{13} - m_{34}\,, & l_{27} = m_{15} - m_{28}\,, & l_{28} = m_{25} - m_{33}\,, & l_{29} = m_2 + m_{24}\,, \\
l_{30} = m_5 + m_{12}\,, & l_{31} = m_7 + m_{35}\,, & l_{32} = m_9 + m_{29}\,, & l_{33} = m_{10} + m_{17}\,, \\
l_{34} = m_{14} + m_{22}\,, & l_{35} = m_{19} + m_{32}\,, & l_{36} = m_{21} + m_{26}\,. &
\end{array} \qquad (5.7)$$

[d]Relation (5.4) is a generalization of the well-known relation, which expresses the inverse matrix as the transpose of the matrix of cofactors up to a determinant factor (this particular case corresponds to $k = 1$).

Let us denote $\mathcal{S}_+$, the set of l_i's, i running from 1 to 21 (see (5.5) and (5.6)), and $\mathcal{S}_-$, the set of l_i's, i running from 22 to 36 (see (5.7)). $\mathcal{S}_+$ and $\mathcal{S}_-$ respectively generate the eigenspaces associated to $+\eta$ and $-\eta$. One can thus choose any linear combination of elements of $\mathcal{S}_+$, or any linear combination of elements of $\mathcal{S}_-$ (but no mixed one), to build algebraic invariants under a transposition t, without losing covariance under I. As previously explained, one can reduce this study to one representent in each class.

Let us take a representative in each of the six classes.

Let us consider for instance for class I, transposition t_I exchanging a_2 and a_3. Similarly the other classes will be represented respectively by the following transpositions: transposition t_{II} exchanging b_1 and b_4, transposition t_{III} exchanging a_2 and b_3, transposition t_{IV} exchanging a_2 and b_1, transposition t_V exchanging a_1 and b_3, and finally transposition t_{VI} exchanging a_2 and a_4.

The following polynomials are respectively invariant under $t_I, \ldots, t_{VI}$:

— t_I-invariants:

$$p_1 = l_1 , \quad p_2 = l_6 , \quad p_3 = l_7 , \quad p_4 = l_{10} , \quad p_5 = l_{12} , \quad p_6 = l_{17} , \quad p_7 = l_{19} ,$$
$$p_8 = l_{22} , \quad p_9 = l_{25} , \quad p_{10} = l_{27} , \quad p_{11} = l_{32} , \quad p_{12} = l_{34} , \quad p_{13} = l_2 + l_5 ,$$
$$p_{14} = l_3 + l_4 , \quad p_{15} = l_{18} - l_{21} , \quad p_{16} = l_{33} + l_{36} ,$$

— t_{II}-invariants:

$$p_1 = l_1 , \quad p_2 = l_2 , \quad p_3 = l_4 , \quad p_4 = l_6 , \quad p_5 = l_7 , \quad p_6 = l_{11} , \quad p_7 = l_{12} ,$$
$$p_8 = l_{13} , \quad p_9 = l_{16} , \quad p_{10} = l_{20} , \quad p_{11} = l_{22} , \quad p_{12} = l_{26} , \quad p_{13} = l_{27} ,$$
$$p_{14} = l_{28} , \quad p_{15} = l_{31} , \quad p_{16} = l_{35} , \quad p_{17} = l_{17} - l_{18} , \quad p_{18} = l_{19} + l_{21} ,$$
$$p_{19} = l_{32} + l_{33} , \quad p_{20} = l_{34} + l_{36} , \quad p_{21} = l_3 + l_5 + l_{10} ,$$

— t_{III}-invariants:

$$p_1 = l_5 , \quad p_2 = l_6 , \quad p_3 = l_9 , \quad p_4 = l_{10} , \quad p_5 = l_{13} , \quad p_6 = l_{17} , \quad p_7 = l_{20} ,$$
$$p_8 = l_{24} , \quad p_9 = l_{25} , \quad p_{10} = l_{28} , \quad p_{11} = l_{32} , \quad p_{12} = l_{35} , \quad p_{13} = l_2 + l_8 ,$$
$$p_{14} = l_4 + l_{11} , \quad p_{15} = l_{16} - l_{21} , \quad p_{16} = l_{31} + l_{36} ,$$

— t_{IV}-invariants:

$$p_1 = l_4 , \quad p_2 = l_5 , \quad p_3 = l_6 , \quad p_4 = l_8 , \quad p_5 = l_{12} , \quad p_6 = l_{13} , \quad p_7 = l_{17} ,$$
$$p_8 = l_{20} , \quad p_9 = l_{21} , \quad p_{10} = l_{23} , \quad p_{11} = l_{27} , \quad p_{12} = l_{28} , \quad p_{13} = l_{32} ,$$
$$p_{14} = l_{35} , \quad p_{15} = l_{36} , \quad p_{16} = l_7 + l_{14} , \quad p_{17} = l_{10} + l_{16} , \quad p_{18} = l_{11} + l_{19} ,$$
$$p_{19} = l_{22} + l_{29} , \quad p_{20} = l_{25} + l_{31} , \quad p_{21} = l_{26} + l_{34} , \quad p_{22} = l_1 + l_{15} - l_2 ,$$

— t_V-invariants:

$$p_1 = l_2 , \quad p_2 = l_3 , \quad p_3 = l_5 , \quad p_4 = l_6 , \quad p_5 = l_{13} , \quad p_6 = l_{15} , \quad p_7 = l_{18} ,$$
$$p_8 = l_{20} , \quad p_9 = l_{28} , \quad p_{10} = l_{30} , \quad p_{11} = l_{33} , \quad p_{12} = l_{35} ,$$
$$p_{13} = l_{12} - l_{14} - l_{16} , \quad p_{14} = l_{27} + l_{29} + l_{31} ,$$

— t_{VI}-invariants:

$$p_1 = l_1 , \quad p_2 = l_4 , \quad p_3 = l_5 , \quad p_4 = l_6 , \quad p_5 = l_8 , \quad p_6 = l_{14} , \quad p_7 = l_{21} ,$$
$$p_8 = l_{23} , \quad p_9 = l_{29} , \quad p_{10} = l_{36} , \quad p_{11} = l_2 - l_{17} , \quad p_{12} = l_3 - l_{19} ,$$
$$p_{13} = l_{11} + l_{20} , \quad p_{14} = l_{13} + l_{16} , \quad p_{15} = l_{26} + l_{35} , \quad p_{16} = l_{28} + l_{31} ,$$
$$p_{17} = l_{10} - l_{12} - l_{18} , \quad p_{18} = l_{25} + l_{27} - l_{33} .$$

We have obtained in each class, a set of homogeneous polynomials invariant under transposition t and covariant under the homogeneous matricial inverse I, all with the same cofactor (the determinant of the matrix R), up to a sign. Thus the algebraic invariants (up to a sign), under the group of transformation Γ_t are *the ratios* of these covariants. For example, one can take in each case the ratios: p_i/p_1.

Let us now sketch how one can get this set of covariants, for a transposition of class $\mathcal{C}_2$, from covariants of transposition t_I (since this last transposition belongs to class $\mathcal{C}_1$).

For instance let us take in class $\mathcal{C}_2$ transposition t_{c_2} exchanging a_1 and a_4. Notice that when dealing with covariants (instead of invariants as in Sec. 5), one has to consider the set of $p_i^L + p_i^C$ (with the same notations as in (3.5)) *together* with the set of $p_i^L - p_i^C$ (where the p_i's denote the covariants corresponding to t_I). The family $p_i^L + p_i^C$ reads

$$l_{22} , \quad l_{33} + l_{36} , \quad l_7 , \quad l_{18} - l_{21} , \quad l_2 + l_5 , \quad l_3 + l_4 , \quad l_{32} , \quad l_{34} , \quad l_{25} - l_{27} , \quad l_{10} + l_{12}$$

and the $p_i^L - p_i^C$'s read

$$l_1 , \quad l_6 , \quad l_{18} + l_{21} , \quad l_5 - l_2 , \quad l_4 - l_3 , \quad l_{36} - l_{33} .$$

Merging these results together one gets the t_{c_2}-invariants

$$p_1 = l_1 , \quad p_2 = l_2 , \quad p_3 = l_3 , \quad p_4 = l_4 , \quad p_5 = l_5 , \quad p_6 = l_6 , \quad p_7 = l_7 ,$$
$$p_8 = l_{18} , \quad p_9 = l_{21} , \quad p_{10} = l_{22} , \quad p_{11} = l_{32} , \quad p_{12} = l_{33} ,$$
$$p_{13} = l_{34} , \quad p_{14} = l_{36} , \quad p_{15} = l_{10} + l_{12} , \quad p_{16} = l_{25} - l_{27} .$$

Since quadratic relations like (5.3) do exist, the problem of the algebraic independence of these sets of quadratic covariants is not obvious. It is analyzed in the next section.

6. Almost Integrable Mappings

The orbits of a particular group of birational transformations Γ_t, are included into the intersection of quadrics defined by the invariants ($p_i/p_1 = k_i$, k_i being arbitrary constants). One has to calculate the dimension of this intersection, to confirm whether they actually are curves or not, as the numerical study suggested in Sec. 4. Thus one has to calculate the "algebraic rank" of the different sets of invariant polynomials.[e]

The results are:

- the orbits of the groups of classes I, II and III are actually included into *algebraic curves*,
- in contrast, for class IV, one invariant is missing, the orbits are only included into *algebraic surfaces*,[f]
- finally for classes V and VI, only thirteen algebraically independent polynomials are covariant, the orbits of the corresponding mappings are only assigned to lie on *three-dimensional algebraic varieties.*

In all these cases, the different algebraic varieties *foliate the whole parameter space.*

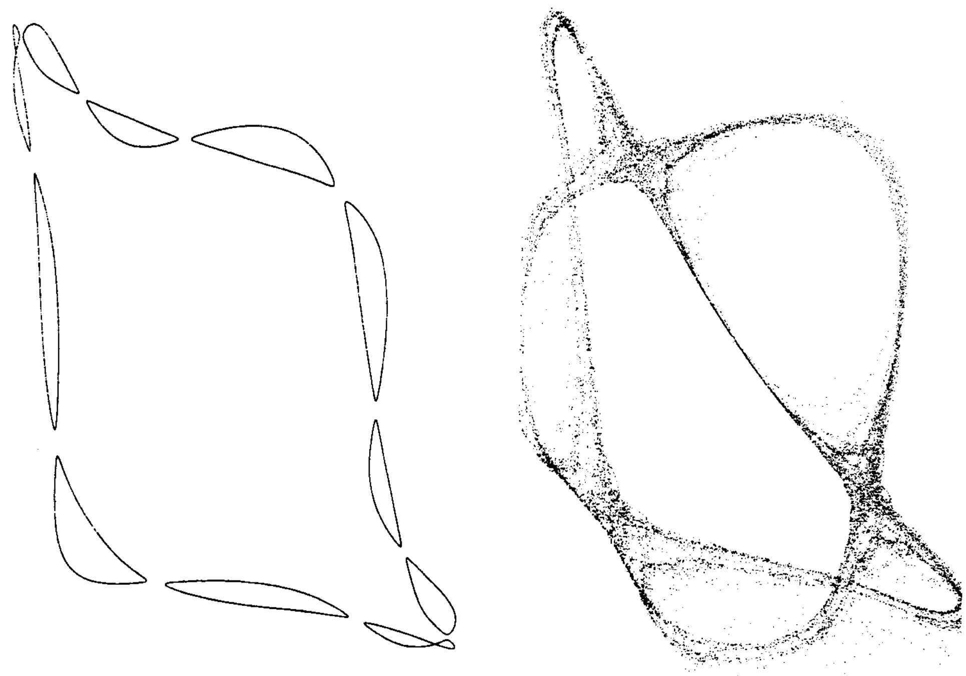
The orbits corresponding to classes I, II and III are actually *elliptic curves since they are algebraic curves stable under an infinite number of automorphisms*[6] (or even they may degenerate into rational curves).[1,2] Thus the corresponding mappings are integrable. This situation is very similar to the one encountered with the birational mappings associated to the sixteen-vertex model[5].[g] In terms of discrete dynamical systems, these three classes provide new interesting examples of integrable mappings, since the parameter space is a 15-dimensional one.

In contrast algebraic calculations of Sec. 5 for classes IV, V and VI, do not enable one to understand the numerical calculations of Sec. 4: either the curves of Figs. 4, 5 and 6 are not really curves (but for instance fractal-like set of points with Hausdorff dimension very close to 1), either one does have algebraic curves and some covariants are missing, and have to be hinted among polynomials of higher degrees, or the curves are *not algebraic* anymore but *transcendental.* To clarify this point let us come back to a more detailed numerical study on classes IV, V and VI.

[e]The "algebraic rank" of a set, $\mathcal{S}$, of polynomials denotes the cardinal of the biggest subset of $\mathcal{S}$, whose elements do not satisfy any algebraic relation. It corresponds to the rank (in the common linear sense) of the jacobian matrix of the polynomials with respect to its variables (which are here the entries of matrix R). Since the homogeneous parameter space is a sixteen dimensional one, an algebraic curve corresponds to fifteen algebraically independent covariants.

[f]It is shown in Ref. 24, that these surfaces are *actually planes.*

[g]For lattice models of statistical mechanics one should not confuse the Yang–Baxter like integrability and the integrability of the birational transformations associated with the inversion symmetries, acting on the parameter space of the model.[1,2] The sixteen-vertex model is said to be "quasi-integrable", and not integrable.[5]



(a) (b)

Fig. 7.

Iteration calculations with high precision does not exhibit any subtle fractal-like difference with curves: *Figs.* 1 to 6 *are actually curves* as far as high precise computations are concerned. These numerical figures are highly *stable* under very large number of iterations (more than 10^9), moreover they also remain stable under perturbations of the initial matrix. Nevertheless strong enough perturbations can make some of them explode, as shown on Figs. 7a and b. This rules out the existence of some additional algebraic invariant, and shows that, in some domain of $\mathbb{C}\mathbf{P}_{15}$, *one actually has non-algebraic (transcendental) curves.*[h] Such a situation, where one gets transcendental curves in some regions of the parameter space, will be denoted "almost" integrable.

More precisely Figs. 5 and 7a correspond to the iteration of K_{t_V}, where t_V still denotes transposition $[a_1 - b_3]$ of class V. Figure 7a is clearly a curve (as well as Fig. 5) made of a set of "bubbles", and Fig. 7b corresponds to an "explosion" of these curves in a spray-like set of points. Such a situation, detailed here for class

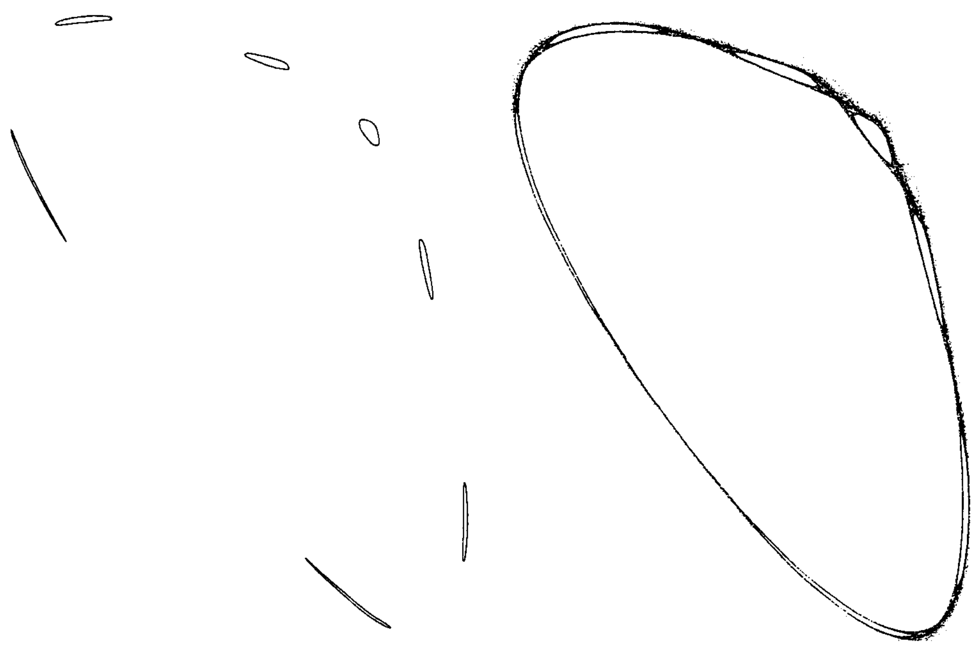
[h] At this point it is worth recalling Serre's GAGA theorem[32,33,31] (pp. 164–171). In particular, every analytical curve in a compact space (like $\mathbb{C}\mathbf{P}_n$ projective spaces) is necessarily an algebraic curve. The "smooth" curves one sees on Figs. 4, 5, 6, 7a and 8a, cannot be $\mathbb{C}$-analytic, since, if analytic, they would be algebraic, and then one would have a foliation of the whole parameter space, which is not (see Figs. 7b and 8b).

V, also occurs for class IV and VI, with similar occurrence of curves and "bubble-curves", and breaking of the curves into chaotic figures. For instance for class IV, Figs. 8a and 8b correspond respectively to a curve made of seven "bubbles" and to an explosion into a spray of points.[i]

The situation one encounters here, is *visually* similar to the one described by *Siegel's theorem*.[27–30] This theorem corresponds to describing the iteration of a quadratic transformation on one complex variable, namely

$$z \longrightarrow \lambda\, z + z^2 \tag{6.1}$$

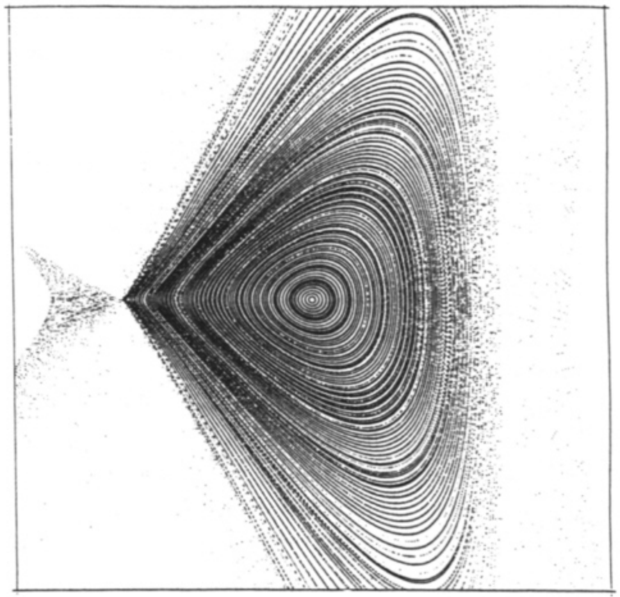
where $\lambda = e^{2i\pi\theta}$, θ being diophantian. Siegel's theorem shows that, in some neighbourhood of $z = 0$, *these iterations yield curves holomorphically conjugated to circles.* These curves are $\mathbb{R}$-analytic *transcendental* curves and are included in some domain with an involved Julia set-like frontier.[34–36] The situation encountered here with these birational mappings seems, as far as the visualization of the orbits is concerned, more related to Siegel's theorem than to the KAM theorem. This is



(a) (b)

Fig. 8.

[i]We would like to thank C. M. Viallet for large number of iteration calculations with high precision (150, 200, 250 digits), which confirm these curves and "bubble-curves" are actually curves, and that the chaos occurring in Figs. 7b and 8b is not a numerical artefact.



(c)

Fig. 8. (*Continued*)

well illustrated by Fig. 8c,[j] which represents a set of orbits corresponding to the iteration of $K_{t_{IV}}^2$, near a fixed point of this transformation, in the very plane stable under $K_{t_{IV}}^2$ (in contrast with arbitrary two-dimensional projection). One does not see any rapid succession of ordered and disordered regions like in the KAM dynamics (Ref. 22).[k] However more detailed analysis seems to show the occurrence of "bubbles-curves" in any neighborhood of the fixed point(s). Hence one does not really have foliation of some continuous domain of the parameter space in curves conjugated to circles. In fact it will be shown in Ref. 24 that, at least for class IV, one does not have any hidden complex structure enabling one to introduce a complex variable z, but that transformation $K_{t_{IV}}^2$ actually reduces to a birational transformation in some (a, b)-plane reading (with origin taken at some fixed point of $K_{t_{IV}}^2$):

$$\begin{pmatrix} a \\ b \end{pmatrix} \rightarrow \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} R_1(a, b) \\ R_2(a, b) \end{pmatrix} \tag{6.2}$$

[j] We thank M. P. Bellon for providing this figure and for numerous parallel calculations.

[k] In our examples the dimension of the parameter space is arbitrary (odd or even) and one has no obvious symplectic structure. One is not exactly in the framework of the KAM theorem.

where $R_1(a, b)$ and $R_2(a, b)$ are simple rational expressions of a and b, the lowest degree of their numerators being greater than two.

In order to study from another point of view, this distinction between integrable and "almost" integrable mappings, let us recall in the next section some results obtained in Refs. 24–26: one can associate to these birational mappings new recurrences *in a single variable.* This makes the numerical iterations simpler and numerically much more controlled.

7. From Birational Mappings in $\mathbf{CP}_{q^2-1}$ to Recurrences in One Variable

7.1. *Factorizations and integrability*

With the method described in Refs. 24 and 25, one can exactly study the iteration of $K_t = t\,I$ on an arbitrary $q \times q$ matrix. This exact study brings out remarkable factorization properties for these (q^2-1)-dimensional mappings.[24] In terms of homogeneous variables, K_t is a homogeneous polynomial transformation of degree $(q-1)$, thus $(K_t)^n(R)$ is *a priori* a matrix whose entries are homogeneous polynomials of degree $(q-1)^n$. In the cases considered here, the entries of $(K_t)^n(R)$ matrices *happen to factorize*, and since they are homogeneous, they can all be divided by their greatest common polynomial divisor. These homogeneous polynomials which factorize in all entries can be expressed in terms of some elementary polynomials (related to determinants), we will denote the f_n's (see Refs. 24–26). Thus, the degree of transformations $(K_t)^n$ falls sometimes to the point of being *polynomial in terms of the variable* n.[24,25] In other publications,[24–26,37,38] the link between this polynomial growth instead of the generic exponential growth, and integrability is detailed.[24–26,37,38] Indeed, if one can easily imagine that the integrable mappings do have a polynomial growth of the "complexity", the reciprocal statement is far from being obvious.[37]

In fact the degree of transformation $(K_t)^n$ becomes *polynomial for all the mappings of classes* I, II and III,[24] and even for the mapping associated with the sixteen vertex model of statistical mechanics,[26] that is for all cases corresponding to integrable mappings (see Sec. 5). This degree is exponential for the three remaining classes IV, V and VI, which does not correspond to integrable mappings (Ref. 24).[l]

Let us illustrate these factorization properties on a lattice model of statistical mechanics. For instance, let us consider the symmetry group of the sixteen vertex model.[5] This infinite discrete group is generated by the matricial inverse I and a permutation of entries denoted t_1.[5,26] Namely t_1 permutes the two off-diagonal 2×2 blocks of the 4×4 matrix.[5,26]

When dealing with birational transformations associated to the sixteen vertex model[5] the polynomials which factorize are given in terms of some elementary homogeneous polynomials, denoted F_n (instead of f_n).[24,25] Let us denote M_n the successive "reduced" matrices equal to $(K_{t_1})^n(R)$ divided by the greatest polyno-

[l]However this exponential growth is independent of q and in fact strictly bounded by 3^n (in comparison with $(q-1)^n$ generically).

mial which factorizes in all entries of $(K_{t_1})^n(R)$. M_0 denotes the initial matrix R. One does not have any factorization for the first two iterates

$$M_1 = K_{t_1}(M_0) , \quad M_2 = K_{t_1}(M_1) \tag{7.1}$$

but *one does have factorizations for the next iterations*[24,25]:

$$K_{t_1}(M_{n+1}) = F_n^2 \cdot M_{n+2} . \tag{7.2}$$

Moreover the F_n's *do satisfy the relation*[24,25]

$$F_{n+2} = \frac{\det(M_{n+1})}{F_n^3} . \tag{7.3}$$

When it is possible to find some particular patterns for matrix R, compatible with the action of K,[14] one can consider a "restricted factorization" problem where all the iterated matrices $K^n(R)$ are actually of the same form.[1,2,14] Because of the particular forms of the matrices some additional factorizations may occur.[26] The analysis of such "restricted factorization" problems deserves some attention, especially because many such patterns correspond to interesting lattice models of statistical mechanics.[14] Let us take a simple example in order to explicit the relation between these factorizations and the Plücker-like K_{t_1}-invariants.[5] For instance let us consider a well-known subcase of the sixteen vextex model: the Baxter model.[39]

7.2. *Relation Between Invariants and Factorizations for the Baxter Model*

Let us show how the various "determinantal" quantities F_n's, (see Eq. (7.3)) which emerges from the analysis of the iterations of transformation K_{t_1},[24–26] are actually related to well-known algebraic invariants of the Baxter model.[39,40]

Let us recall the R-matrix of the symmetric eight-vertex model[41,39]:

$$R = \begin{pmatrix} a & 0 & 0 & d \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ d & 0 & 0 & a \end{pmatrix} . \tag{7.4}$$

Introducing the row-to-row transfer matrix $T(a, b, c, d)$ deduced from the previous R-matrix (see Ref. 39), the Yang–Baxter equations show that one has a family of commuting transfer matrices[39,40,42]:

$$[T(a, b, c, d), T(a', b', c', d')] = 0 \tag{7.5}$$

if

$$\frac{a^2 + b^2 - c^2 - d^2}{ab} = \frac{a'^2 + b'^2 - c'^2 - d'^2}{a'b'} , \qquad \text{and} \qquad \frac{ab}{cd} = \frac{a'b'}{c'd'} . \tag{7.6}$$

One also has a Hamiltonian which commutes with the whole family of commuting transfer matrices.[42]

$$H = \sum_n (J_x\, \sigma_n^x\, \sigma_{n+1}^x + J_y\, \sigma_n^y\, \sigma_{n+1}^y + J_z\, \sigma_n^z\, \sigma_{n+1}^z) \tag{7.7}$$

with

$$J_x = ab + cd\ , \quad J_y = ab - cd\ , \quad J_z = (a^2 + b^2 - c^2 - d^2)/2\ . \tag{7.8}$$

These three quadratic polynomials J_x, J_y and J_z are *exactly the equivalent*, for the Baxter model, of the p_α's previously introduced (see Sec. 5), and also introduced for the sixteen vertex model.[5] The two algebraic invariants (7.6) are obviously related to the above three homogeneous invariants J_x, J_y and J_z (7.8).

It is important to note that the study of the iterations of K_{t_1} not only brings out factorizations of determinantal-like polynomials F_n (see (7.2), (7.3)), *but also show that these very expressions do satisfy recurrences.*[24,25] For the sixteen vertex model, and of course for the Baxter subcase, these recurrences happen to have a particular form: namely some rational expression of the F_n's is invariant under the shift $n \longrightarrow n+1$. The corresponding (homogeneous) invariant $\mathcal{I}$ reads[24,25]:

$$\mathcal{I} = \frac{F_{n-2}F_{n+2}F_{n+3} - F_{n-1}F_nF_{n+4}}{F_{n-1}F_{n+2}^2 - F_{n+3}F_n^2} = \frac{F_{n-1}F_{n+3}F_{n+4} - F_nF_{n+1}F_{n+5}}{F_nF_{n+3}^2 - F_{n+4}F_{n+1}^2}\ . \tag{7.9}$$

In terms of these various well-known homogeneous invariants (7.8), the expressions of our new K_{t_1}-invariant $\mathcal{I}$ is remarkably simple and symmetric:

$$\begin{aligned} \mathcal{I} = {} & (J_zJ_x + J_zJ_y + J_xJ_y)(J_xJ_y - J_zJ_x + J_zJ_y) \\ & \times (J_xJ_y - J_zJ_x - J_zJ_y)(J_xJ_y + J_zJ_x - J_zJ_y)\ . \end{aligned} \tag{7.10}$$

At this point one realizes that one has, at least, two approaches of the integrability of birational mappings:

— one previously developed (see Sec. 5) which amounts to a study of the algebraic invariants of the birational mappings in terms of Plücker-like variables or from a systematic search of the lowest degree invariants.[1,2]

— another one which concentrates on the analysis of the factorization properties of the iterates of transformation K_t (see Eqs. (7.2) and (7.3)). This section, and in particular Eq. (7.10), clearly shows that these two approaches are closely related.

Moreover the F_n's can be expressed in terms of determinants (see relation (7.3)). Hence one can get easily convinced that $\mathcal{I}$ is invariant under conjugations of the initial matrix $M_0 = R$, which are t_1-compatible. This property is reminiscent of the weak-graph duality symmetries[43] of the sixteen vertex model: actually it has been shown in Ref. 5 that one can construct from the previous Γ_t-covariants

p_α's (see Sec. 5) more "sophisticated" (and higher degree) covariants taking also into account the weak-graph duality invariance. This correspondence between these higher degree Γ_t-*and*-weak-graph duality invariants and the determinantal invariant $\mathcal{I}$ can be detailed explicitly for the Baxter subcase. These results are given in the Appendix.

8. Recurrences

Factorizations of the $K^n(R)$ matrices (see Sec. 7.1), of course, yield factorizations of their determinants. However these determinants factorize even more: one also has similar factorization equations as Eq. (7.3) for all the six classes I, ..., VI.[24–26] For class I, and *surprisingly for the generically non-integrable mappings of class* IV, the determinants of the $K^n(R)$ matrices *actually satisfy recurrences* of a very simple form.[24,25] Let us give here some results obtained in Ref. 24 concerning these recurrences.

Actually the recurrence obtained for class I is exactly the same as the one corresponding to the sixteen vertex model (see Eq. (7.9)) up to a simple change of variables[25] and can be written in a more compact form in terms of some homogeneous variables q_n's (simply related to the f_n's),[25] namely

$$\frac{q_n - q_{n+1}}{q_{n-1} - q_{n+2}} \cdot \frac{1}{q_n\, q_{n+1}} = \frac{q_{n+1} - q_{n+2}}{q_n - q_{n+3}} \cdot \frac{1}{q_{n+1} q_{n+2}} \,. \tag{8.1}$$

The recurrence obtained for class IV is actually different and reads[24]

$$\frac{q_{n+3} - q_{n+1}}{q_{n+4} - q_n} \cdot \frac{1}{q_{n+3}\, q_{n+1}} = \frac{q_{n+5} - q_{n+3}}{q_{n+6} - q_{n+2}} \cdot \frac{1}{q_{n+5}\, q_{n+3}} \,. \tag{8.2}$$

In this last case, the relation between the f_n's and q_n's is not as simple as for class I.[24]

Note that recurrences (8.1) and (8.2) are *still valid for classes* I and IV *generalized to* $q \times q$ *matrices*. One thus has *universal recurrences independent of* q.[24]

Let us now study the possible integrability of these recurrences.

8.1. *One integrable recurrence: class I*

In fact, Eq. (8.1) can be "integrated"

$$q_{n+2} - q_{n-1} = -\lambda \cdot \left(\frac{1}{q_{n+1}} - \frac{1}{q_n} \right) \tag{8.3}$$

which can be rewritten as

$$q_{n+2} + \frac{\lambda}{q_{n+1}} = q_{n-1} + \frac{\lambda}{q_n} \tag{8.4}$$

or, equivalently:

$$q_n + q_{n+1} + q_{n+2} + \frac{\lambda}{q_{n+1}} = q_{n-1} + q_n + q_{n+1} + \frac{\lambda}{q_n} \; . \tag{8.5}$$

The value of the left or the right-hand side of Eq. (8.5) is denoted ρ.

On the other hand, one has

$$q_{n+2} + \frac{\lambda}{q_{n+1}} = q_{n-1} + \frac{\lambda}{q_n} = \left(q_n + \frac{\lambda}{q_{n-1}}\right) \frac{q_{n-1}}{q_n} \; . \tag{8.6}$$

Denoting

$$Q_{n+1} = q_{n+1} + \frac{\lambda}{q_n} \tag{8.7}$$

Eq. (8.6) can be rewritten

$$q_n \cdot Q_{n+2} = q_{n-1} \cdot Q_n \tag{8.8}$$

or equivalently:

$$q_n \cdot Q_{n+2} \cdot Q_{n+1} = q_{n-1} \cdot Q_n \cdot Q_{n+1} \; . \tag{8.9}$$

The value of the left or the right-hand side of Eq. (8.9) is denoted μ.

One also has

$$q_n \cdot Q_{n+1} = q_n q_{n+1} + \lambda \tag{8.10}$$

and

$$\rho - q_n - q_{n+1} = \frac{\mu}{q_n \cdot Q_{n+1}} = \frac{\mu}{q_n \, q_{n+1} + \lambda} \; , \tag{8.11}$$

finally giving a *biquadratic equation* relating q_n and q_{n+1}:

$$(\rho - q_n - q_{n+1})(q_n \, q_{n+1} + \lambda) = \mu \; . \tag{8.12}$$

It is well known that biquadratic equations are associated with elliptic curves.[5] Hence this recurrence on one variable has an *elliptic parametrization*, corresponding to the biquadratic relation (8.12).[m]

To keep track that recurrence (8.1) enables to calculate q_{n+4} in terms of q_n, q_{n+1}, q_{n+2} and q_{n+3}, one can, for instance, express λ, μ and ρ in terms of the first four q_n's[24]:

$$\lambda = \frac{q_1 \, q_2 \, (q_3 - q_0)}{q_2 - q_1} \tag{8.13}$$

[m] Recently systematic approaches have made their appearance in the analysis of integrable mappings. Among these, one should mention the work of Quispel and other authors who have presented a wide class of mappings of the form $x_{n+1} = f(x_n, x_{n-1})$ which can be seen as discretizations of second-order ordinary differential equations related to elliptic functions.[44-50]

$$\mu = \frac{q_1\, q_2\, (q_0 + q_1 - q_2 - q_3)(q_0\, q_1 - q_2\, q_3)}{(q_1 - q_2)^2} \tag{8.14}$$

$$\rho = \frac{q_0\, q_1 - q_2\, q_3 + q_1^2 - q_2^2}{q_1 - q_2}\,. \tag{8.15}$$

Note that there is no longer any ambiguity in the determination of q_{n+1} in terms of the previous q_n's (because of some remarkable factorizations) in contrast with what could have been imagined considering the second order (in q_{n+1}) Eq. (8.12). For instance, one gets q_4 in terms of q_0, q_1, q_2 and q_3 as follows:

$$q_4 = \frac{q_1\, (q_0\, q_3 + q_1\, q_3 - q_3^3 - q_0\, q_2)}{q_3\, (q_1 - q_2)}\,.$$

8.2. *One "almost" integrable recurrence: class IV*

The left-hand side and the right-hand side of Eq. (8.2) are the same up to a *shift of two.* Therefore introducing the two constants of integration λ_1 and λ_2, one can see that Eq. (8.2) is equivalent to equation

$$q_{n+4} - q_n = \lambda_n \cdot \left(\frac{1}{q_{n+1}} - \frac{1}{q_{n+3}} \right) \tag{8.16}$$

where $\lambda_{2n+1} = \lambda_1$ and $\lambda_{2n} = \lambda_2$.

Since $\lambda_{n+2} = \lambda_n$ one can rewrite (8.16) as

$$q_{n+4} + q_{n+2} + \frac{\lambda_{n+2}}{q_{n+3}} = q_{n+2} + q_n + \frac{\lambda_n}{q_{n+1}}\,. \tag{8.17}$$

One sees that Eq. (8.17) relates the left hand-side and the right-hand side by a shift of two, leading one to introduce two new constants of integration

$$q_{n+2} + q_n + \frac{\lambda_n}{q_{n+1}} = \rho_n \tag{8.18}$$

where $\rho_{2n+1} = \rho_1$ and $\rho_{2n} = \rho_2$.[n]

To go on with this integration one has to write an equation similar to (8.6), which reads

$$q_{n+4} + \frac{\lambda_{n+2}}{q_{n+3}} = q_n + \frac{\lambda_n}{q_{n+1}} = \left(q_{n+1} + \frac{\lambda_n}{q_n} \right) \frac{q_n}{q_{n+1}}\,. \tag{8.19}$$

Unfortunately $\lambda_{n+2} \neq \lambda_{n+3}$ and one does not have any other covariant expression.

[n] Similar recurrences depending on the parity have been analyzed in Ref. 48 (see p. 1827).

Nevertheless, one can study the restricted recurrence, corresponding to $\lambda_1 = \lambda_2 = \lambda$. In that subcase this recurrence is *actually integrable* and closely related to recurrences and biquadratic relations studied in the previous subsection (see Eqs. (8.4), (8.6) and (8.11)). Equation (8.19) becomes

$$q_{n+4} + \frac{\lambda}{q_{n+3}} = q_n + \frac{\lambda}{q_{n+1}} = \left(q_{n+1} + \frac{\lambda}{q_n}\right)\frac{q_n}{q_{n+1}} \,. \tag{8.20}$$

With the same variables Q_n as in previous section, Eq. (8.20) reads

$$Q_{n+4} = \frac{q_n}{q_{n+1}}\, Q_{n+1} \,, \tag{8.21}$$

which can be rewritten as

$$q_{n+1}\, Q_{n+4}\, Q_{n+3}\, Q_{n+2} = q_n\, Q_{n+3}\, Q_{n+2}\, Q_{n+1} \,. \tag{8.22}$$

This last equation enables one to introduce a new constant of integration μ.

With variables Q_n equations (8.18) yield

$$Q_{n+2} = \rho_n - q_n \,. \tag{8.23}$$

From Eqs. (8.22) and (8.23), one gets

$$\begin{aligned} \mu &= q_n \cdot Q_{n+1} \cdot Q_{n+2} \cdot Q_{n+3} \\ &= (\rho_n - q_n)\,(\rho_{n+1} - q_{n+1})\,(q_n\, q_{n+1} + \lambda) \end{aligned} \tag{8.24}$$

yielding *two* biquadratic equations

$$\begin{aligned} (\rho_2 - q_{2n})\,(\rho_1 - q_{2n+1})(q_{2n}\, q_{2n+1} + \lambda) - \mu &= 0 \\ (\rho_2 - q_{2n+2})\,(\rho_1 - q_{2n+1})\,(q_{2n+2}\, q_{2n+1} + \lambda) - \mu &= 0 \,. \end{aligned} \tag{8.25}$$

Let us introduce the biquadratic polynomial $B(x, y)$:

$$B(x, y) = (xy + \lambda)(x - \rho_1)(y - \rho_2) - \mu \,. \tag{8.26}$$

In the (q_n, q_{n+1})-plane, these successive points respectively *lie on two biquadratic curves* depending on the parity of n, namely, if n is odd $B(x, y) = 0$, and $B(y, x) = 0$ if n is even.

This shows, in the $\lambda_1 = \lambda_2 = \lambda$ subcase, that recurrence (8.2), or equivalently recurrence (8.16), is *an integrable recurrence*. The algebraic elliptic curve corresponding to the biquadratic Eqs. (8.25) is well illustrated on Fig. 9a which shows the iterations of transformation $K_{t_{IV}}$ in the (q_n, q_{n+1})-plane. Figure 9a corresponding to a recurrence in one variable in the (q_n, q_{n+1})-plane has to be compared with Fig. 4, which corresponds to the iteration of transformation $K^2_{t_{IV}}$ seen as a birational transformation in a 15-dimensional space. This $\lambda_1 = \lambda_2$ integrable subcase can be rewritten as a condition bearing on the q_n's. This integrability condition is obviously stable under the recurrence, since it is similar to Eq. (8.2) with a shift of one instead of a shift of two. One can express this condition in terms of the initial values of the q_n's :

$$q_4\, q_2\, q_3\, q_5 - q_4\, q_2\, q_1\, q_5 + q_4\, q_2\, q_1^2 - q_3\, q_1\, q_4^2 + q_3\, q_1\, q_0\, q_4 - q_3\, q_1\, q_0\, q_2 = 0 \ . \qquad (8.27)$$

For 4×4 matrices it is possible to rewrite condition (8.27) as an integrable codimension one subvariety of the parameter space $\mathbb{C}\mathbf{P}_{15}$.This integrable subvariety will not be written here.

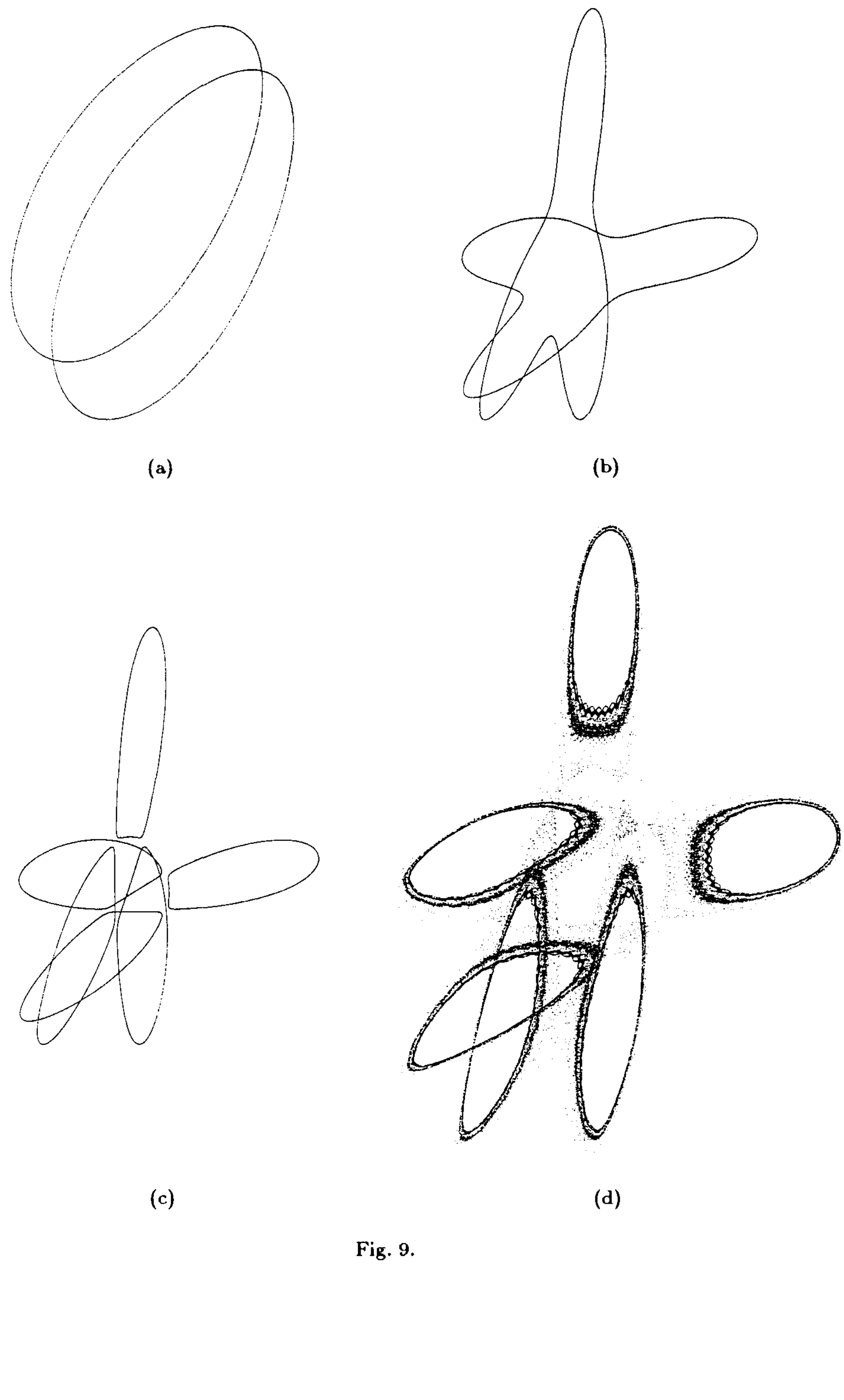
For $\lambda_1 \neq \lambda_2$, we have systematically considered the iterations of transformation $K_{t_{IV}}$ in the (q_n, q_{n+1})-plane. Remarkably, for a large set of initial conditions for the iterations, *one still gets curves*: these curves are *highly stable even after more than one* 10^9 *iterations*!!

When leaving the $\lambda_1 = \lambda_2$ integrability condition, one recovers the same phenomenon as the one described in Sec. 4: on Fig. 9b the "trajectories" of the iteration *still lie on curves* (up to the computer precision), *even when the deformation parameter* $\delta = \lambda_1 - \lambda_2$ *is not small anymore*. For increasing values of δ, one recovers some features of the well-known route to chaos.[52–55] These curves can deform more and more, and sometimes a finite number p, of "bubble-curves" pop out (see Fig. 9c). This finite number p, corresponds to *the occurrence of a specific period*[56] *in the dynamics of the problem*. Instead of iterating the birational mapping $K_{t_{IV}}$, one can iterate $K^p_{t_{IV}}$: this new iteration "extracts" *one bubble out of the p ones*. One notes again that, whatever the number of bubbles is, these "trajectories" *actually do lie on curves*, also stable even after a large number of iterations (greater than 10^9).

For sufficiently large deformation parameter δ, one can see these curves or "bubble-curves" explode in Cantor-like spray figures (see Fig. 9d), clearly showing that recurrence (8.2) *is not integrable*, though it is integrable in the $\lambda_1 = \lambda_2$ subcase, and though corresponding iterations very often yield highly stable (transcendental) curves. For not too large "deformation" parameter δ, the "trajectory" curves (see Fig. 9b) are similar to the biquadratic elliptic curves (8.26) (see Fig. 9a). Figure 9c and chaotic Fig. 9d again show that the curve of Fig. 9b *is not algebraic*.[o]

These examples are reminiscent of the KAM theorem, where the range of stability of the KAM tori is quite large[22,53,54] like in Siegel's theorem.[27–30] This situation justifies further analysis to understand the occurrence of *curves* and the remarkable regularities of these mappings, which are not a consequence of some hidden complex structure like in Siegel's theorem.[24] This study of a one variable recurrence associated with class IV is complementary of the one performed in fifteen variables (see subsection 6) and actually confirms the "almost" integrability of this class of mappings.

[o]We thank B. Grammaticos and A. Ramani for providing an analysis of (8.2) and (8.16), with the help of their singularity confinement method,[48,44,49] which confirms our visualization analysis showing the generic non-integrability of (8.2) and also that one algebraic invariant is actually missing (see Sec. 5).



(a) (b)

(c) (d)

Fig. 9.

Note that one can also associate to class IV a mapping in $\mathbb{CP}_2$ as follows:

$$q_{n+2} = \rho_1 - q_n - \frac{\lambda_1}{q_{n+1}}$$
$$q_{n+3} = \rho_2 - q_{n+1} - \frac{\lambda_2 \, q_{n+1}}{(\rho_1 - q_n) \, q_{n+1} - \lambda_1} \, . \tag{8.28}$$

Remark that mapping (8.28) in $\mathbb{CP}_2$ is also a birational mapping. One gets (q_n, q_{n+1}) in terms of (q_{n+2}, q_{n+3}) from Eqs. (8.28), with the changes

$$\lambda_1 \leftrightarrow \lambda_2 \, , \quad \rho_1 \leftrightarrow \rho_2 \, , \quad q_n \leftrightarrow q_{n+3} \, , \quad q_{n+1} \leftrightarrow q_{n+2} \, .$$

This complementary approach, which associates with 15-dimensional birational mappings, non-linear recurrences *on a single variable*, can also be envisaged for the four other classes (II, III, V and VI). Actually, we have been able to associate to these four classes either recurrences on two variables or even (for class III) a recurrence on a single variable (which is not simply related to the determinant of the 4×4 matrix).[24] The elimination of the second variable can, in theory, be performed but calculations are much too large.[24] Moreover, similarly to class IV (Eq. (8.28)), one can theoretically associate, with each class, mappings in various projective spaces $\mathbb{CP}_n$, n running from 1 to 15.

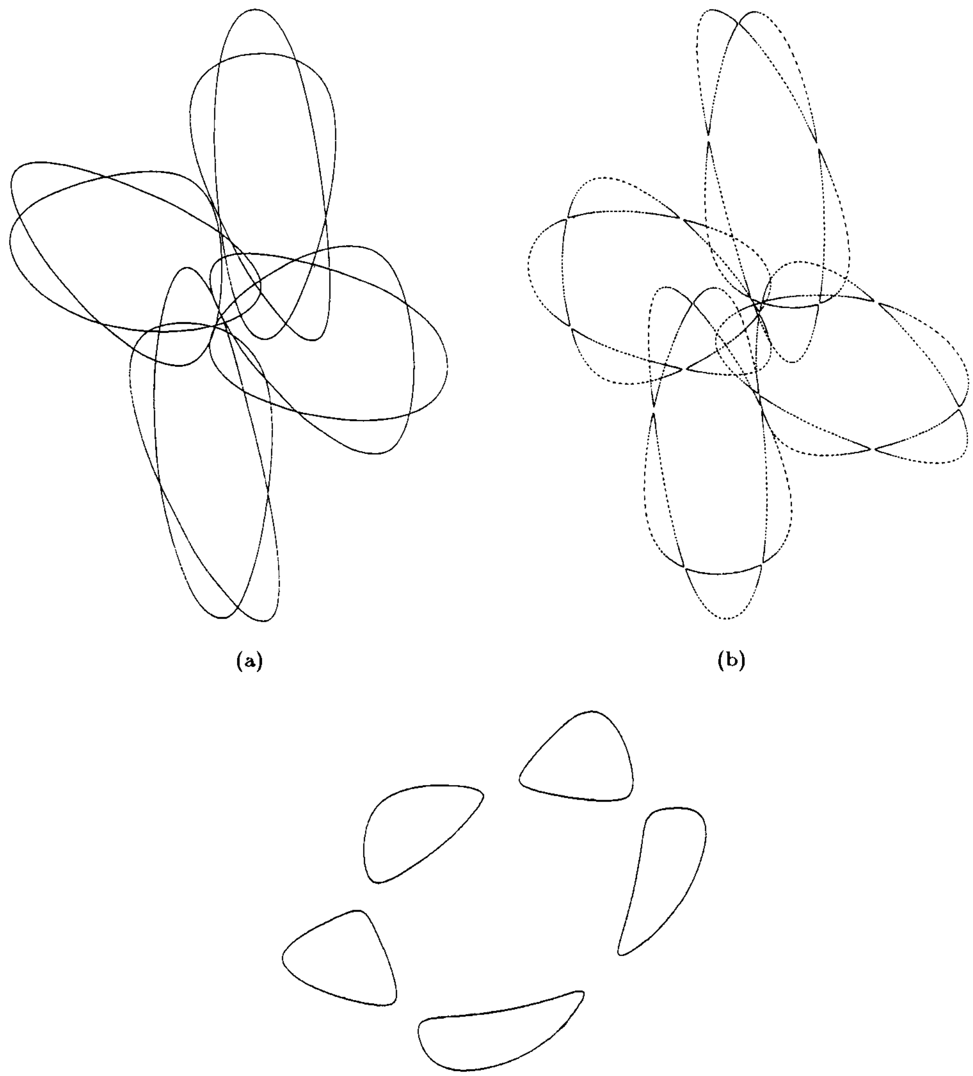
8.3. *Generalization of the recurrences: more "almost" integrable iterations*

These ideas naturally lead to consider a generalized problem (which is the straight generalization of the system of recurrences (8.18)): the analysis of the iterations of the p-recurrence

$$q_{n+4} + q_{n+2} + \frac{\lambda_n}{q_{n+3}} = \rho_n \tag{8.29}$$

with $\lambda_{n+p} = \lambda_n$ and $\rho_{n+p} = \rho_n$.

Figures 10a and 10b correspond to $p = 4$, and Fig. 10c illustrates $p = 5$. One of the most striking features of such systems of recurrences is that the visualization of the corresponding iteration *again gives curves*. These orbits of course reduce, in the $\lambda_1 = \lambda_2 = \cdots = \lambda_p$ and $\rho_1 = \rho_2 = \cdots = \rho_p$ limit, to algebraic elliptic curves associated with biquadratic equations such as (8.25). Again for quite large perturbations from this integrable limit, *one can get non-algebraic curves*. Such p-recurrences induce a natural period p, even before any "bubbles", like the one previously described in subsection (8.2), occur. Once more, one should remark *the astonishing numerical stability of these calculations*. This again provides new examples depending on an arbitrary number of parameters $(\lambda_1, \ldots, \lambda_p, \rho_1, \ldots, \rho_p)$ of KAM theorem where the KAM tori (curves here) remain actually stable for large perturbations from the integrable situation[23] like in Siegel's theorem.[27–30]



(a) (b)

(c)

Fig. 10.

Again, let us underline that this situation is remarkable: most of the generalizations of (8.18) yield more standard KAM-explosion.[22] For instance, one can easily verify with the iteration of the recurrence

$$q_{n+2} + (1+\epsilon)\, q_n + \frac{\lambda}{q_{n+1}} = \rho \qquad (8.30)$$

that the integrable curves (corresponding to $\epsilon = 0$) explode in chaotic set of points even for very small deformation parameter ϵ.

9. Generalization to Other Permutations

Let us come back to transformations acting in $\mathbb{C}\mathbf{P}_{15}$. Birational mappings associated with more general permutations of the entries of the 4×4 matrix can also be studied systematically.[26] Some examples related to lattice statistical mechanics in dimension two or three have already been analyzed in detail[4,5,17] yielding algebraic elliptic curves or *algebraic surfaces*. Since the permutation group of the sixteen entries of a 4×4 matrix is very large the associated birational mappings cannot be analyzed systematically[p, 26]: for the sake of simplicity one can first imagine restricting to permutations which are the product of two elementary transpositions.

Even restricting to such permutations one has to deal with too large number of permutations and therefore of associated birational mappings. Of course, one still has relabelling of the rows and columns of the 4×4 matrices and, as a consequence, equivalence classes of permutations. Since this number is still too large, one will not try to be exhaustive, but only consider some specific examples. Let us consider the permutations

$$\{a_2 \leftrightarrow a_3,\ d_2 \leftrightarrow d_3\},\ \{b_4 \leftrightarrow c_4,\ b_1 \leftrightarrow c_1\}$$

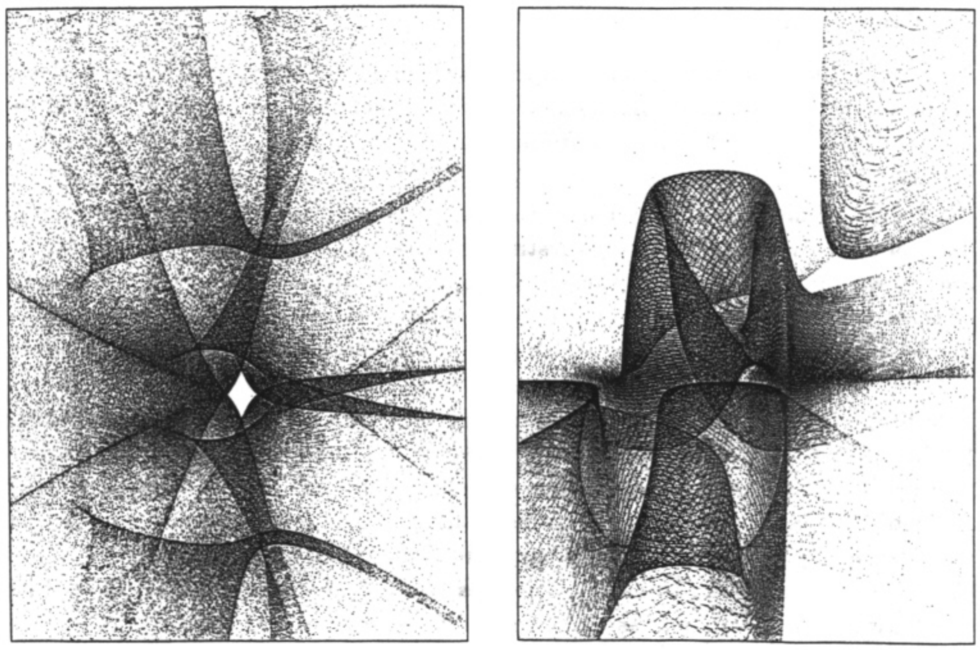
The iterations of the birational transformations in $\mathbb{C}\mathbf{P}_{15}$ respectively associated with these permutations correspond to Figs. 11a and 11b in the same order.

Considering these two "trajectories", one remarks that the "density" of points is similar to the one given in Refs. 4 and 17, where algebraic surfaces occur. Actually one can exhibit algebraic quadratic invariants similar to those given in Sec. 5, and show that *these orbits also lie on algebraic surfaces.*

10. Comment: "Almost Curves" Versus Surfaces

Let us compare the "trajectories" corresponding to permutations associated with class IV (see Figs. 4, 8a and 8b) and the "trajectories" corresponding to one of the two examples of the previous subsection (9) (see Figs. 11a and 11b). In both cases, one has the same number of independent algebraic invariants (namely thirteen). Figure 8b corresponds to a chaotic regime of the "trajectories" of class IV : the points actually wander in a chaotic manner on the algebraic surface given by the

[p] The previous exhaustive analysis on the transpositions cannot be generalized straightforwardly to arbitrary permutations of the entries, since there is no compatibility between the product in $\mathcal{S}_{16}$ and our groups: $\Gamma_{t\,t'}$ properties cannot be generically deduced from the analysis of Γ_t and $\Gamma_{t'}$. The only simple compatibility is the equivalence relation yielding the six classes of transpositions. Of course this equivalence relation also yield classes in all $\mathcal{S}_{16}$, but the number of permutations to study still remains too large.



(a) (b)

Fig. 11.

intersection of these thirteen quadrics. This chaotic trajectory is reminiscent of the transcendental curves one had before this explosion (see Fig. 4). Clearly the density of points is drastically different from the ones on Figs. 11a and 11b, or from the examples detailed in Refs. 4, 5 and 17, which correspond to some *uniform-like density* on algebraic surfaces seen as two-dimensional tori.

One can imagine to "upgrade" the distinction between integrable and "almost" integrable mappings, illustrated by the emergence of transcendental curves, to a distinction between mappings yielding either algebraic surfaces foliating the whole parameter space, or transcendental surfaces in some domain of the parameter space (see Fig. 8c). Again these surfaces can then explode into some higher dimensional algebraic varieties.

The examples detailed in this paper (in particular classes IV, V and VI) suggest to introduce some "double graduation of this almost integrability" d, d' ($d \leq d'$), d corresponding to the dimension of the manifold the almost integrable mappings densify (apparently) uniformly, and d' the dimension of the algebraic variety, in which the previous manifold will pop out. Integrable mappings correspond to $d = d' = 1$, while conditions $d = 1$ and $d' > 1$ define "almost" integrable mappings (for instance $d' = 2$ for mappings of class IV, $d' = 3$ for mappings of classes V and VI). In contrast, for the mappings described in Sec. 9, d is no longer equal to 1 and one has $d = d' = 2$.

11. Conclusion

We have studied a set of birational mappings which have first emerged in the analysis of the symmetries of lattice model in statistical mechanics. We think they could be powerful tools to study the route from integrability to chaos in discrete dynamical systems.

The set of such transformations is very large, as large as the number of permutations of q^2 elements: we have thus restricted ourselves to elementary transpositions and shown that this restricted subset of mappings fall in six classes for $q \geq 4$. Three of these six classes (namely I, II and III) are integrable mappings, their iterations giving *algebraic elliptic curves*, which can be written as intersections of *fourteen quadrics*. Even when these mappings are not integrable, they do present remarkable properties: their iterations actually lie on curves for some domain of initial points. For class IV one even has an integrable subcase (on some codimension-one algebraic variety) with again algebraic elliptic curves. These mappings exhibit many of the well-known chaotic features of discrete dynamical systems, such as the occurrence of periods, explosion of curves in Cantor-like spray of points ... However one notes several regularity properties of these mappings. First, the initial points, whose orbits are curves are not rare. This provides an illustration of a transition from integrable dynamical systems strictly speaking, with their associated algebraic elliptic curves, to more chaotic situations *through highly stable curves*, or set of points which cannot be numerically distinguished from curves. When curves they are transcendental and thus one would like to understand, in the absence of any algebraicity or any conformal structure, what is the hidden structure (symmetry) which forces the iteration to lie on such curves.[q] It will be shown in parallel publications that, at least near some fixed points of transformation K_t^2, this situation is visually similar to the one described in Siegel's theorem[27–30] but is definitely different[24] (see Eq. (6.2)).

Moreover, the calculations of these iterations happen to be amazingly stable, when one gets curves. It is important to note that *such a stability phenomenon is not the classical situation of stable fixed points* (or even curves or varieties), with negative Lyapounov exponents. One has no isolated contractant curve : in some domain of $\mathbb{C}\mathbf{P}_{15}$ one has stable curves, which seems to be arbitrarily close. The stability one encounters is a *global* stability of the curves : one has a stability property when leaving the curve, however, the calculations are *quite unstable along the curves*. This "dual" stable behavior (stable in the neighborhood of the curve, unstable along the curve) is the reason why our visualization approach works so well. We will try in forthcoming publications to better understand the remarkable numerical stability of such birational mappings.

[q]One could imagine that these curves are solutions of some hidden PDE or differential equations having these birational transformations as symmetries. In fact, for the same reason that Figs. 7b and 8b ruled out the existence of some additional algebraic invariant, one can certainly also rule out the existence of such hidden PDE or differential equations.

Finally associating with these mappings in $\mathbb{C}\mathbf{P}_{q^2-1}$ a non-linear recurrence bearing on *a single variable* enables to cross-check the numerical and analytical analysis. In particular the algebraic elliptic curves in $\mathbb{C}\mathbf{P}_{15}$ correspond to biquadratic curves in the (q_n, q_{n+1})-plane, for the sixteen-vertex model, for class I and for class IV when integrable. The analysis of the "route to chaos" is easier to perform on these non-linear recurrences, while the "graduation" (d, d') can only be defined for mappings in $\mathbb{C}\mathbf{P}_{q^2-1}$ and enables to discriminate between various chaotic situations occurring for recurrences on a single variable. For instance, the chaotic Fig. (9d) corresponds to an explosion in an algebraic surface (see Fig. 8b), that is, to a "less" chaotic situation than the one corresponding to class V (see Fig. 7b), for which the "explosion" takes place in a three-dimensional algebraic variety.

This remarkable correspondence between birational transformations in $\mathbb{C}\mathbf{P}_{q^2-1}$ and non-trivial non-linear recurrences in one variable should certainly be interpreted in terms of Grassmannian structures[31] associated with elliptic curves. The introduction of Plücker-like variables (see Sec. 5) for the $\mathbb{C}\mathbf{P}_{15}$ mappings, and of determinantal variables as far as recurrences are concerned (see (8.1) and (8.2)), strongly suggests such structures. This approach is clearly a fruitful one since these structures are not restricted to integrable mappings, but can also be introduced for more general mappings and their corresponding algebraic varieties (see Sec. 5).

In parallel publications[24–26,37] one will further analyze the six classes of mappings defined here in order to shed some light on the relations between different properties and structures, such as the polynomial growth of the complexity of their iterations, the occurrence of factorizations (see Eqs. (7.2) and (7.3)), the existence of recurrences bearing on the factorized polynomials f_n's (see Eq. (7.9)) and finally the relation with the integrability of these mappings.

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Appendix

Let us consider the Baxter model (see Sec. 7.2) and Eqs. (7.8), (7.9) and (7.10)). Let us also introduce the symmetric expressions of the variables J_x, J_y and J_z:

$$s_1 = J_x + J_y + J_z\,, \quad s_2 = J_xJ_y + J_yJ_z + J_xJ_z\,, \quad s_3 = J_xJ_yJ_z\;. \tag{11.1}$$

A set of algebraic homogeneous expressions denoted $\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_8$ and $\mathcal{K}_2$, $\mathcal{K}_3$ and $\mathcal{K}_4$ has been introduced in Ref. 5. They are covariants under transformation K_{t_1} and also have covariant properties with respect to the so-called weak-graph transformations.[43] These new invariants $\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_8$ read in the terms of the symmetric expressions (11.1).

$$\mathcal{I}_1 = \mathcal{I}_2 = 2\,s_1\ , \quad \mathcal{I}_3 = \mathcal{I}_4 = 0\ , \quad \mathcal{I}_5 = \mathcal{I}_6 = 96\,(s_1^2 - 3\,s_2)\ ,$$
$$\mathcal{I}_7 = \mathcal{I}_8 = 64\,(9\,s_1 s_2 - 2\,s_1^3 - 27\,s_3)\ . \qquad (11.2)$$

The K_{t_1}-and-weak-graph invariants $\mathcal{K}_2$, $\mathcal{K}_3$ and $\mathcal{K}_4$ (Ref. 5) read:

$$\mathcal{K}_2 = -8\,(C + J_z^2)\ , \quad \mathcal{K}_3 = -16\,s_3\ , \qquad \mathcal{K}_4 = 16\,(C^2 - 3\,D) \qquad (11.3)$$

with

$$C = J_x^2 + J_y^2 + J_z^2\ , \quad D = J_x^2 J_y^2 + J_y^2 J_z^2 + J_z^2 J_x^2 \qquad (11.4)$$

and also

$$\begin{aligned}\mathcal{I} &= s_2^4 - 4\,s_2\,s_3\,(s_1\,s_2 - 2\,s_3)\\ &= (\mathcal{I}_5 - 24\,\mathcal{I}_1^2)\,(\mathcal{I}_5^3 - 40\,\mathcal{I}_1^2\,\mathcal{I}_5^2 - 32\,\mathcal{I}_5\,\mathcal{I}_1\,\mathcal{I}_7 - 320\,\mathcal{I}_1^4\,\mathcal{I}_5\\ &\quad - 1280\mathcal{I}_1^3\mathcal{I}_7 + 512\mathcal{I}_1^6 - 64\mathcal{I}_7^2)\ . \end{aligned} \qquad (11.5)$$

The expression of $\mathcal{I}$ is related to the K_{t_1}-and-weak-graph invariants $\mathcal{K}_2$, $\mathcal{K}_3$ and $\mathcal{K}_4$ (Ref. 5) as follows:

$$A_0 + A_1\,\mathcal{I} + A_2\,\mathcal{I}^2 + A_3\,\mathcal{I}^3 = 0\ . \qquad (11.6)$$

The expressions of the A_n's of (11.6) in terms of the K_{t_1}-and-weak-graph invariants $\mathcal{K}_2$, $\mathcal{K}_3$ and $\mathcal{K}_4$ read:

$$A_3 = -1082535236962615296$$

$$\begin{aligned}A_2 &= 37180995010560\,\mathcal{K}_2^4 + 2062653748936704\,\mathcal{K}_4^2 - 452955840970752\,\mathcal{K}_2^2\,\mathcal{K}_4\\ &\quad + 5062877383753728\,\mathcal{K}_2\,\mathcal{K}_3^2\end{aligned}$$

$$\begin{aligned}A_1 &= -238878720\,\mathcal{K}_2^8 - 1294932639744\,\mathcal{K}_4^4 - 7080152924160\,\mathcal{K}_4^2\,\mathcal{K}_2\,\mathcal{K}_3^2\\ &\quad + 1269942976512\,\mathcal{K}_2^3\,\mathcal{K}_4\,\mathcal{K}_3^2 - 104979234816\,\mathcal{K}_4^2\,\mathcal{K}_2^4 + 626025037824\,\mathcal{K}_4^3\,\mathcal{K}_2^2\\ &\quad + 7484866560\,\mathcal{K}_2^6\,\mathcal{K}_4 + 1743496151040\mathcal{K}_3^4\mathcal{K}_4 - 7253886763008\,\mathcal{K}_2^2\,\mathcal{K}_3^4\\ &\quad - 92398288896\,\mathcal{K}_2^5\,\mathcal{K}_3^2\end{aligned}$$

$$\begin{aligned}A_0 &= 373248\,\mathcal{K}_2^9\,\mathcal{K}_3^2 + 66187368\,\mathcal{K}_2^6\,\mathcal{K}_3^4 + 3712464576\,\mathcal{K}_2^3\,\mathcal{K}_3^6 + 76087296\,\mathcal{K}_4^4\,\mathcal{K}_2^4\\ &\quad - 226492416\,\mathcal{K}_4^5\,\mathcal{K}_2^2 + 1119744\,\mathcal{K}_4^2\,\mathcal{K}_2^8 - 12828672\,\mathcal{K}_4^3\,\mathcal{K}_2^6 - 46656\,\mathcal{K}_2^{10}\,\mathcal{K}_4\\ &\quad - 1461583872\,\mathcal{K}_4^3\,\mathcal{K}_3^4 + 729\,\mathcal{K}_2^{12} + 7163154000\,\mathcal{K}_3^8 + 268435456\,\mathcal{K}_4^6\\ &\quad - 13934592\,\mathcal{K}_2^7\,\mathcal{K}_4\,\mathcal{K}_3^2 + 207028224\,\mathcal{K}_4^2\,\mathcal{K}_2^5\,\mathcal{K}_3^2 + 7622387712\,\mathcal{K}_4^2\,\mathcal{K}_2^2\,\mathcal{K}_3^4\\ &\quad - 1157504256\,\mathcal{K}_2^4\,\mathcal{K}_3^4\,\mathcal{K}_4 - 8396835840\,\mathcal{K}_2\,\mathcal{K}_3^6\,\mathcal{K}_4 - 1245708288\,\mathcal{K}_4^3\,\mathcal{K}_2^3\,\mathcal{K}_3^2\\ &\quad + 2566914048\,\mathcal{K}_4^4\,\mathcal{K}_2\,\mathcal{K}_3^2\ .\end{aligned}$$

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