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A new approach for lattice models in statistical mechanics

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Abstract

We analyze discrete groups of symmetry of vertex models in lattice statistical mechanics representing them as groups of birational transformations. They can be seen as generated by involutions corresponding respectively to two kinds of transformations on $q \times q$ matrices: the inversion of the $q \times q$ matrix and an (involutive) permutation of the entries of the matrix. We show that the analysis of the factorizations of the iterations of these transformations is a precious tool in the study of lattice models in statistical mechanics. This approach enables to analyze two-dimensional q^4-state vertex models as simply as three-dimensional vertex models, or higher dimensional vertex models. Various examples of birational symmetries of vertex models are analyzed. A particular emphasis is devoted to the 64-state cubic vertex model which exhibits a polynomial growth of the complexity of the calculations. We also concentrate on a specific two-dimensional vertex model to see how the generic exponential growth of the calculations reduces to a polynomial growth when the model becomes Yang-Baxter integrable. It is also underlined that a polynomial growth of the complexity of these iterations can occur even for transformations yielding algebraic surfaces, or higher dimensional algebraic varieties.

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1 Introduction

In previous papers [1, 2, 3], we have analyzed birational representations of discrete groups generated by involutions, having their origin in the theory of exactly solvable models in lattice statistical mechanics [4, 5, 6, 7, 8, 9, 10, 11]. These involutions correspond respectively to two kinds of transformations on $q \times q$ matrices: the inversion of the $q \times q$ matrix and an (involutive) permutation of the entries of the matrix. In [1, 2, 3], permutations of the entries corresponding to *transpositions of two entries* were analyzed. For these transpositions, it has been shown that the iteration of the associated birational transformations present some remarkable factorization properties [1, 2]. These factorization properties explain why the complexity of these iterations, instead of having the exponential growth one expects at first sight, may have a *polynomial growth* [12, 13, 14]. It has also been shown that the polynomial factors occurring in these remarkable factorizations *can satisfy remarkable non-linear recurrences* and that some of these recurrences were actually *integrable* recurrences yielding *elliptic curves* [1, 2, 3].

These papers have tried, on simple examples of permutations (namely *transpositions of two entries* [1, 2, 3]), to shed some light on the relation between various structures and properties, such as the factorization of the iterations, the polynomial growth of the complexity [1, 2, 12] and the integrability of the mappings, as well as the nature of the various algebraic varieties preserved by these mappings. The structures, concepts, properties and results that emerged in these studies will be used here for lattice models of statistical mechanics: vertex models and also (edge) spin models. The mappings that one considers here are birational representations of symmetries acting in the parameter space of vertex models in two, three, or even *arbitrary*, dimensions.

A special attention will be devoted to a three-dimensional vertex model, the 64-state cubic vertex model, which exhibits a polynomial growth of the complexity of the calculations. We will also concentrate on a specific two-dimensional q^4-vertex model, with a special attention to $q = 3$, to see *how the generic exponential growth of the calculations reduces to a polynomial growth when the model becomes Yang-Baxter integrable*. It will be shown that a polynomial growth of the complexity of these iterations can occur not only for transformations yielding algebraic curves but also for transformations yielding algebraic surfaces, or higher dimensional algebraic varieties[1].

We will show that this factorization analysis provides a new approach for q^d-vertex models with *arbitrary* number q of colors and *arbitrary* lattice dimension d.

2 General framework and notations

Let us consider the more general vertex model where one direction, denoted direction (1), is singled out. Pictorially this can be interpreted as follows:

(1)

(2.1)

where i and k (corresponding to direction (1)) can take q values while J and L take m values.

One can define a "partial" transposition on direction (1) denoted t_1. The action of t_1 on the R-matrix is given by [10]:

$$(t_1 R)^{iJ}_{kL} \;=\; R^{kJ}_{iL} \qquad (2.2)$$

[1] This happens at least when one can associate a Jacobian variety to these birational transformations as will be seen in the following.

The R-matrix is a $(q\,m) \times (q\,m)$ matrix which can be seen as q^2 blocks which are $m \times m$ matrices:

$$R = \begin{pmatrix} A[1,1] & A[1,2] & A[1,3] & \cdots & A[1,q] \\ A[2,1] & A[2,2] & A[2,3] & \cdots & A[2,q] \\ A[3,1] & A[3,2] & A[3,3] & \cdots & A[3,q] \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A[q,1] & A[q,2] & A[q,3] & \cdots & A[q,q] \end{pmatrix} \qquad (2.3)$$

where $A[1,1]$, $A[1,2]$, ..., $A[q,q]$ are $m \times m$ matrices. With these notations the partial transposition t_1 amounts to permute matrices $A[\alpha,\beta]$ and $A[\beta,\alpha]$.

We use the same notations as in [1, 2, 3], that is, we introduce the following transformations, the matrix inverse $\widehat{I}$ and the homogeneous matrix inverse I:

$$\widehat{I} : R \longrightarrow R^{-1} \qquad (2.4)$$

$$I : R \longrightarrow \det(R) \cdot R^{-1} \qquad (2.5)$$

The homogeneous inverse I is a polynomial transformation on each of the entries of matrix R, which associates with each entry its corresponding cofactor.

The two transformations t_1 and $\widehat{I}$ are *involutions* and $I^2 = (\det(R))^{q\,m-2} \cdot \mathcal{I}d$ where $\mathcal{I}d$ denotes the identity transformation.

We also introduce the (generically infinite order) transformations:

$$K = t_1 \cdot I \qquad \text{and} \qquad \widehat{K} = t_1 \cdot \widehat{I} \qquad (2.6)$$

Transformation $\widehat{K}$ is clearly a *birational transformation* on the entries of matrix R, since its inverse transformation is $\widehat{I} \cdot t_1$ which is obviously a rational transformation. K is a *homogeneous polynomial transformation* on the entries of matrix R.

3 Two-dimensional vertex models

3.1 Iterations associated with the sixteen-vertex model

The *sixteen-vertex model* corresponds to vertex of (2.1) and R-matrix (2.3) with $q = m = 2$. In this case of 4×4 matrices, permutation t_1, has already been introduced in the framework of the symmetries of the sixteen-vertex model [10]. Namely, t_1 amounts to permute two 2×2 (off-diagonal) submatrices of the 4×4 matrix R.

Remarkably, the symmetry group generated by the matrix inverse $\widehat{I}$ and transformation t_1, or the infinite generator $\widehat{K} = t_1 \cdot \widehat{I}$, has been shown to yield *algebraic elliptic curves* [10] in $\mathbb{C}P_{15}$.

Let us consider a 4×4 matrix M_0 and the successive matrices obtained by iteration of transformation $K = t_1 \cdot I$. Remarkably all the entries of the successive matrices obtained iterating K on M_0 do factorizes common polynomials. This enables to introduce at each step reduced matrices, denoted M_n. Moreover, the determinants of these M_n's also factorize. More precisely, similarly to factorizations described in [1, 2], one has the following factorizations for the iterations of K [1]:

$$M_1 = K(M_0), \quad M_2 = K(M_1), \quad F_1 = \det(M_0), \quad F_2 = \det(M_1), \quad F_3 = \frac{\det(M_2)}{F_1^3}, \quad M_3 = \frac{K(M_2)}{F_1^2}, \; \cdots$$

and for arbitrary n:

$$M_{n+2} = \frac{K(M_{n+1})}{F_n^2}, \qquad F_{n+2} = \frac{\det(M_{n+1})}{F_n^3} \qquad (3.1)$$

One also has the following relation:

$$\widehat{K}(M_{n+2}) = \frac{K(M_{n+2})}{\det(M_{n+2})} = \frac{M_{n+3}}{F_{n+1}F_{n+3}} \qquad (3.2)$$

One can also introduce a *right-action* of K on matrices M_n or on any homogeneous polynomial expression of their entries (such as the F_n's for instance): the entries of matrix M_0 are replaced by the entries of $K(M_0)$. Amazingly, the right action of K on the F_n's and on the matrices M_n factorizes F_1 *and only* F_1 [1]:

$$(M_n)_K = M_{n+1} \cdot F_1^{\nu_n}, \qquad (F_n)_K = F_{n+1} \cdot F_1^{\mu_n} \qquad (3.3)$$

Denoting α_n the degree of the determinant of the matrix M_n and β_n the degree of the polynomial F_n, one immediately gets from equations (3.1) and (3.2) the following linear relations (with integer coefficients):

$$\alpha_{n+1} = 3\,\beta_n + \beta_{n+2}, \qquad 3\,\alpha_{n+1} = \alpha_{n+2} + 8\,\beta_n, \qquad (3.4)$$
$$\alpha_{n+2} + \alpha_{n+3} = 4\,(\beta_{n+1} + \beta_{n+3}), \qquad 3\,\beta_n = \beta_{n+1} + 4\,\mu_n, \qquad 3\,\alpha_n = \alpha_{n+1} + 16\,\nu_n$$

yielding on the generating functions:

$$x\,\alpha(x) = \left(3\,x^2 + 1\right) \cdot \beta(x), \qquad (3\,x - 1) \cdot \alpha(x) + 4 = 8\,x^2 \cdot \beta(x),$$
$$(1 + x) \cdot \alpha(x) = 4\,(1 + x^2) \cdot \beta(x) + 4 \qquad (3.5)$$

From these factorizations, one can easily deduce linear recurrences on the series α_n, β_n, μ_n and ν_n, and, then, the following expressions for their generating functions:

$$\alpha(x) = \frac{4(1 + 3x^2)}{(1 - x)^3}, \quad \beta(x) = \frac{4x}{(1 - x)^3}, \quad \mu(x) = \frac{x^2(3 - x)}{(1 - x)^3}, \quad \nu(x) = \frac{2x^2}{(1 - x)^3} \qquad (3.6)$$

The expressions of the degrees and exponents α_n, β_n, μ_n and ν_n respectively read:

$$\alpha_n = 4\,(2\,n^2 + 1), \quad \beta_n = 2n\,(n + 1), \quad \mu_n = n^2 - 1, \quad \nu_n = n\,(n - 1) \qquad (3.7)$$

Let us also mention that, for a given initial matrix M_0, the successive iterates of M_0 under transformation K^2, move in a *three-dimensional affine projective space*:

$$K^{2\,n}(M_0) = a_0^{(n)} \cdot M_0 + a_1^{(n)} \cdot M_2 + a_2^{(n)} \cdot M_4 + a_3^{(n)} \cdot M_6 \qquad (3.8)$$

$$K^{2\,n+1}(M_0) = b_0^{(n)} \cdot M_1 + b_1^{(n)} \cdot M_3 + b_2^{(n)} \cdot M_5 + b_3^{(n)} \cdot M_7 \qquad (3.9)$$

In term of these homogeneous variables $a_0^n, a_1^n, a_2^n, a_3^n$ (or $b_0^n, b_1^n, b_2^n, b_3^n$) transformation K is represented as a *cubic* (birational) homogeneous transformation:

$$a_i^{(n)} \longrightarrow b_i^{(n)} = \qquad (3.10)$$
$$\sum_{N_0+N_1+N_2+N_3=3} B_i(M_0; N_0, N_1, N_2, N_3) \cdot (a_0^{(n)})^{N_0} \cdot (a_1^{(n)})^{N_1} \cdot (a_2^{(n)})^{N_2} \cdot (a_3^{(n)})^{N_3} \quad \text{with} \quad i = 0, 1, 2, 3$$

the N_i's being positive integers, and similarly:

$$b_i^{(n)} \longrightarrow a_i^{(n+1)} = \qquad (3.11)$$
$$\sum_{N_0+N_1+N_2+N_3=3} A_i(M_0; N_0, N_1, N_2, N_3) \cdot (b_0^{(n)})^{N_0} \cdot (b_1^{(n)})^{N_1} \cdot (b_2^{(n)})^{N_2} \cdot (b_3^{(n)})^{N_3} \quad \text{with} \quad i = 0, 1, 2, 3$$

Considering the points in $\mathbb{C}P_{15}$ associated with the successive 4×4 matrices corresponding to the iteration of M_0 under transformation K (instead of K^2), one thus gets sets of points (lying on elliptic curves) which be g to *two* three dimensional affine subspace of $\mathbb{C}P_{15}$, *which also depend on the initial matrix* M_0 in a quite involved way.

Amazingly, the F_n's do satisfy *a whole hierarchy of recurrences* [1] such as:

$$\frac{F_n F_{n+3}^2 - F_{n+4} F_{n+1}^2}{F_{n-1} F_{n+3} F_{n+4} - F_n F_{n+1} F_{n+5}} = \frac{F_{n-1} F_{n+2}^2 - F_{n+3} F_n^2}{F_{n-2} F_{n+2} F_{n+3} - F_{n-1} F_n F_{n+4}} \tag{3.12}$$

Let us recall that this very recurrence is integrable, yielding algebraic elliptic curves [1].

Let us remark that, for the sixteen-vertex model, the two directions are equivalent. Therefore, one can replace, in this factorization analysis, transposition t_1 by the transposition on direction (2), denoted t_2. It amounts to a relabelling of the rows and columns of the R-matrix. In fact, product $t_1 \cdot t_2$ is nothing but the "total" transposition of matrix R, and thus commute with I.

After these recalls let us now consider new examples of vertex models.

3.2 q^4-state two-dimensional vertex models

Let us consider a generalization of the sixteen-vertex model for an arbitrary number of spin values. It corresponds to $m = q$ for model (2.1). Matrix (2.3) is now a $q^2 \times q^2$ matrix.

Similarly to the factorizations described in (3.1), one has the following factorizations for the iterations of K acting on M_0, a $q^2 \times q^2$ matrix:

$$M_1 = K(M_0), \quad M_2 = K(M_1), \quad F_1 = \det(M_0), \quad F_2 = \det(M_1), \quad F_3 = \frac{\det(M_2)}{F_1^{q^2-1}}, \cdots$$

and for arbitrary n:

$$M_{n+2} = \frac{K(M_{n+1})}{F_n^{q^2-2}}, \quad F_{n+2} = \frac{\det(M_{n+1})}{F_n^{q^2-1}} \tag{3.13}$$

One recovers relation (3.2), *independently of q*:

$$\widehat{K}(M_{n+2}) = \frac{K(M_{n+2})}{\det(M_{n+2})} = \frac{M_{n+3}}{F_{n+1} F_{n+3}} \tag{3.14}$$

Moreover, one notices that the right action of transformation K again yields the factorization of F_1 and only F_1, enabling to define the exponents μ_n and ν_n.

It is clear that these factorizations are straight generalizations of the one described in section (3.1). From these factorizations, one can easily get linear recurrences on the exponents α_n, β_n, μ_n and ν_n:

$$\begin{aligned} \alpha_{n+3} + \alpha_{n+2} &= q^2 \cdot (\beta_{n+1} + \beta_{n+3}), \qquad \alpha_{n+1} = (q^2-1)\cdot\beta_n + \beta_{n+2}, \\ (q^2-1)\cdot\alpha_{n+1} &= \alpha_{n+2} + (q^2-2)\cdot\beta_n, \\ (q^2-1)\cdot\beta_n &= \beta_{n+1} + q^2\cdot\mu_n, \qquad (q^2-1)\cdot\alpha_n = \alpha_{n+1} + q^2\cdot\nu_n \end{aligned} \tag{3.15}$$

One deduces the relations on their generating functions:

$$\begin{aligned} &x\cdot\alpha(x) = (1+(q^2-1)x^2)\cdot\beta(x), \qquad (1+x)\cdot\alpha(x) = q^2(1+x^2)\cdot\beta(x) + q^2, \\ &q^2(q^2-2)\cdot x^2\beta(x) = ((q^2-1)x-1)\cdot\alpha(x) + q^2, \qquad q^2\cdot x\,\mu(x) - q^2 x = ((q^2-1)x-1)\cdot\beta(x), \\ &q^4\cdot x\,\nu(x) - q^2 = ((q^2-1)x-1)\cdot\alpha(x) \end{aligned} \tag{3.16}$$

and the following expressions for these generating functions:

$$\begin{aligned} \alpha(x) &= \frac{q^2\cdot(1+(q^2-1)x^2)}{(1-x)\cdot(1-(q^2-2)x+x^2)}, \qquad \beta(x) = \frac{q^2 x}{(1-x)\cdot(1-(q^2-2)x+x^2)}, \\ \mu(x) &= \frac{x^2\cdot((q^2-1)-x)}{(1-x)\cdot(1-(q^2-2)x+x^2)}, \qquad \nu(x) = \frac{(q^2-2)\cdot x^2}{(1-x)\cdot(1-(q^2-2)x+x^2)} \end{aligned} \tag{3.17}$$

The expressions of the exponents α_n, β_n, μ_n and ν_n clearly have (generically) an *exponential growth in terms of n* when q is different from 2. This suggests that the q^4-state vertex models are not generically "good candidates" for integrability when the number of colors q is not 2 anymore.

Let us recall that a *polynomial growth* of the calculations corresponds to cases where the roots of the denominators of the generating functions $\alpha(x)$, $\beta(x)$, ... are *Nth root of unity*. On the explicit expressions (3.17) one sees that such a situation can only occur when q^2 is a *Tutte-Beraha number* [15, 16]:

$$q^2 = 2 + t + \frac{1}{t}, \quad \text{with} \quad t^N = 1 \quad \text{for some integer } N \tag{3.18}$$

A polynomial growth behaviour cannot *generically* occur when q is an *integer* different from 2 (or 0...). In general, one does not expect the birational transformations defined in [1, 2, 3] from permutations of entries of a $q \times q$ matrix to be integrable mappings[2]. It is however important to recall that integrability cases are not ruled out even when q is an integer different from 2.

3.3 From exponential growth to integrability

It is known that there does exist "Yang-Baxter-integrable" subcases of the generic 9×9 matrix (and more generally of the generic $q^2 \times q^2$ matrix [18]): how is it possible for such integrable cases to survive in such a "hostile framework" (exponential growth of the complexity ...). Does these restrictions on the 9×9 matrices change the (generic) exponential growth of the calculations into a polynomial one ?

For heuristic reasons let us, for example, consider a simple pattern for a 9×9 matrix corresponding to a vertex model introduced by Stroganov [19]:

$$R_{Strog} = \begin{pmatrix} 1 & 0 & 0 & 0 & b & 0 & 0 & 0 & b \\ 0 & 0 & 0 & c & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 \\ 0 & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ b & 0 & 0 & 0 & 1 & 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & c & 0 \\ 0 & 0 & c & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & c & 0 & 0 & 0 \\ b & 0 & 0 & 0 & b & 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.19}$$

This model is known to possess two "Yang-Baxter-integrable" subcases [19]:

$$c = 1 - b, \quad \text{and also} \quad c = \frac{1-b}{1+b} \tag{3.20}$$

When restricted to one of the two integrability subcases (3.20), the partition function of this model can easily be calculated using the *inversion trick* [19]. We consider this very example because it is simple (only two parameters and a single one in the integrable subcases) and yields a *rational* parameterization of the integrable subcases of the model[3]. We restrict to the first integrable subcase (3.20): $c = 1 - b$.

Let us first remark that, in this (rational) subcase, all the birational transformation-symmetries we consider are just *homographic* transformations. For instance transformations t_1, I and K^n read:

$$t_1 : b \to c = 1-b, \quad I : b \to \frac{-b}{1+b}, \quad K : b \to \frac{2b+1}{b+1}, \quad K^n : b \to \frac{N_n(b)}{D_n(b)} \tag{3.21}$$

[2] It is however worth recalling the example of the q-state standard scalar Potts model for which, using the Lieb-Temperley algebra, a matrix representation can be given in terms of matrices of sizes *independent of q*, q becoming a parameter in the entries of these matrices [17]. Therefore one can also imagine to be able to define the birational transformations K for *non-integer values* of q.

[3] Of course the reader can, as an exercise, replace this model and this very form of the 9×9 matrix (3.19) by other "Yang-Baxter-integrable" 9×9 matrix patterns, for instance the solvable q^4-state models introduced in [18] among many other possibilities ...

where the numerators and denominators of the first successive homography K^n respectively read:

$$N_1 = 2b+1,\ N_2 = 5b+3,\ N_3 = 13b+8,\ N_4 = 34b+21 \cdots \qquad (3.22)$$
$$D_1 = b+1,\ D_2 = 3b+2,\ D_3 = 8b+5,\ D_4 = 21b+13,\ D_5 = 55b+34, \quad \cdots$$

These successive polynomials can be shown to satisfy the following recurrences:

$$N_{n+1} \;=\; 2N_n + D_n, \qquad D_{n+1} \;=\; N_n + D_n \qquad (3.23)$$

The F_n's and the entries of the successive matrices M_n's, previously defined for generic 9×9 matrices (see (3.13) with $q=3$), do factorize:

$$F_1 \;=\; -N_1\cdot(b-1)^8, \quad F_2 = -N_2\cdot(b-1)^{63}\cdot b^8, \quad F_3 = -N_3\cdot(b-1)^{440}\cdot b^{63},$$

and for arbitrary n:

$$F_n \;=\; -N_n\cdot(b-1)^{r_n}\cdot b^{r_{n-1}} \qquad (3.24)$$

where the r_n's are the coefficients of the rational function:

$$r(x) \;=\; 1 + r_1\cdot x + r_2\cdot x^2 + r_3\cdot x^3 + \cdots \;=\; \frac{1+7x^2-x^3}{(1-x)(1-7x+x^2)} \qquad (3.25)$$

All the entries of the M_n's factorize the same polynomial which enables to introduce new matrices M_n^{int}'s which entries are *polynomial* expressions in b:

$$M_1 \;=\; -M_1^{int}\cdot(b-1)^7, \quad M_2 = -M_2^{Strog}\cdot(b-1)^{56}\cdot b^7, \quad M_3 = -M_3^{int}\cdot(b-1)^{392}\cdot b^{56}, \cdots$$

and for arbitrary n:

$$M_n \;=\; -M_n^{int}\cdot(b-1)^{s_n}\cdot b^{s_{n-1}} \qquad (3.26)$$

where the s_n's are the coefficients of the rational function:

$$s(x) \;=\; 1 + s_1\cdot x + s_2\cdot x^2 + s_3\cdot x^3 + \cdots \;=\; \frac{1-x+8x^2-x^3}{(1-x)(1-7x+x^2)} \qquad (3.27)$$

Let us see how these new factorizations (3.24) and (3.26) *are actually compatible with the generic ones* (3.13).

The new (highly) factorized matrices M_n^{int}'s have a remarkably simple form in terms of the N_n's and D_n's:

$$M_n^{int} \;=\; \begin{pmatrix} D_n & 0 & 0 & 0 & N_n & 0 & 0 & 0 & N_n \\ 0 & 0 & 0 & -N_{n-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -N_{n-1} & 0 & 0 \\ 0 & -N_{n-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ N_n & 0 & 0 & 0 & D_n & 0 & 0 & 0 & N_n \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -N_{n-1} & 0 \\ 0 & 0 & -N_{n-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ D_n & 0 & 0 & 0 & 0 & -N_{n-1} & 0 & 0 & 0 \\ N_n & 0 & 0 & 0 & N_n & 0 & 0 & 0 & D_n \end{pmatrix} \qquad (3.28)$$

As it should, this matrix (up to the normalization of $R[1,1]$) has exactly the same form as (3.19) where b has been changed into $K^n(b)$ (taking relation (3.23) into account). The determinant of M_n^{int} can easily be calculated:

$$\det(M_n^{int}) \;=\; -N_{n-1}^6\cdot(2N_n+D_n)\cdot(N_n-D_n)^2 \qquad (3.29)$$

Recalling recurrences (3.23) this expression also reads:

$$\det(M_n^{int}) \;=\; -N_{n+1}\cdot N_{n-1}^8 \qquad (3.30)$$

Recalling (3.24), (3.26), (3.30) and factorization (3.13) for $q=3$, one can write:

$$F_{n+2} \;=\; \frac{\det(M_{n+1})}{F_n^8} \;=\; \frac{\det(M_{n+1}^{int})\cdot(b-1)^{9s_{n+1}}\cdot b^{9s_n}}{(-N_n\cdot(b-1)^{r_n}\cdot b^{r_{n-1}})^8} \qquad (3.31)$$
$$\;=\; -N_{n+2}\cdot(b-1)^{9s_{n+1}-8r_n}\cdot b^{9s_n-8r_{n-1}}$$

This *compatibility between the factorizations for the generic 9×9 matrices (exponential growth) and the one for the Stroganov's model* (3.19) *corresponds to the following relation on the r_n's, the s_n's and the* β_n's:

$$\beta_{n+2} \;=\; 9\cdot(s_{n+1}+s_n) - 8\cdot(r_n+r_{n-1}) + 1 \qquad (3.32)$$

or, in terms of the associated generating functions $\beta(x)$, $r(x)$ and $s(x)$:

$$\beta(x) \;=\; 9x\cdot(1+x)\cdot s(x) \;-\; 8x^2\cdot(1+x)\cdot r(x) \;+\; \frac{1}{1-x} \;-1-2x+7x^2+8x^3 \qquad (3.33)$$

which is actually verified.

3.4 Stroganov's model outside the Yang-Baxter integrability

When Stroganov's model is not restricted to the Yang-Baxter integrability conditions (3.20) anymore, the model, despite its simplicity (only two parameters, and a very simple form for the $q^2\times q^2$ R-matrix (3.19) ...) is not known to be integrable.

Let us examine the factorization properties *outside* the integrability conditions (3.20) (that is in the whole (b,c) parameter space). The first factorizations read:

$$F_1 \;=\; -c^6\cdot(2b+1)\cdot(b-1)^2, \qquad F_2 \;=\; c^{51}\,b^6\,(b-1)^9\cdot h_1\cdot g_1^2, \quad \cdots \qquad (3.34)$$

with:

$$g_1 \;=\; -1-b+2b^2+c+\,b\,c, \qquad h_1 \;=\; -2-2b+4b^2-\,bc-c \qquad (3.35)$$

All the entries of matrix M_n factorize the same polynomial which enables to introduce new matrices M_n^{Strog}'s which entries are *polynomial* expressions in b *and* c:

$$M_1 \;=\; M_1^{Strog}\cdot(b-1)\cdot c^5, \qquad M_2 \;=\; -M_2^{Strog}\cdot(b-1)^8\cdot c^{45}\cdot g_1, \;\cdots \qquad (3.36)$$

Furthermore:

$$det(M_1^{Strog}) \;=\; c^6\,b^6\cdot h_1\cdot g_1^2, \qquad det(M_2^{Strog}) \;=\; c^6\,b^6\,(b-1)^6\cdot(2b+1)^8\cdot h_2\cdot g_2^2, \;\cdots \qquad (3.37)$$

with

$$g_2 \;=\; 2-6b^2+4b^3-c-cb+2b^2c-c^2-bc^2, \qquad (3.38)$$
$$h_2 \;=\; 4+8b-12b^2-16b^3+16b^4-2c-3cb+3b^2c+2b^3c-2c^2-3bc^2-b^2c^2$$

The successive "reduced" matrices M_n^{Strog}'s also have a simple form slightly generalizing (3.28):

$$M_n^{Strog} \;=\; \begin{pmatrix} A_n & 0 & 0 & 0 & B_n & 0 & 0 & 0 & B_n \\ 0 & 0 & 0 & -C_n & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -C_n & 0 & 0 \\ 0 & -C_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ B_n & 0 & 0 & 0 & A_n & 0 & 0 & 0 & B_n \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -C_n & 0 \\ 0 & 0 & -C_n & 0 & 0 & 0 & 0 & 0 & 0 \\ A_n & 0 & 0 & 0 & 0 & -C_n & 0 & 0 & 0 \\ B_n & 0 & 0 & 0 & B_n & 0 & 0 & 0 & A_n \end{pmatrix} \qquad (3.39)$$

There is no relation between the A_n's, B_n's, C_n's anymore (like $C_n = B_{n-1}$ for (3.28)). With this particular form (3.39) the determinant of the "reduced" matrix factorizes, at least, as follows:

$$det(M_n^{Strog}) = - C_n^6 \cdot \Big(2B_n + A_n\Big) \cdot \Big(B_n - A_n\Big)^2 \qquad (3.40)$$

The previous expressions g_1, h_1, g_2, h_2 simply read:

$$g_1 = B_1 - A_1, \quad h_1 = 2B_1 + A_1, \quad g_2 = B_2 - A_2, \quad h_2 = 2B_2 + A_2 \qquad (3.41)$$

Therefore one has a representation of K as a *birational transformation in* $\mathbb{C}P_2$:

$$(A_n, B_n, C_n) \longrightarrow (A_{n+1}, B_{n+1}, C_{n+1}) = \Big(\frac{C_n \cdot (B_n + A_n)}{H_n}, \frac{(B_n - A_n)\cdot(2B_n + A_n)}{H_n}, \frac{B_n \cdot C_n}{H_n}\Big) \qquad (3.42)$$

and a representation of K^{-1}:

$$(A_n, B_n, C_n) \longrightarrow (A_{n-1}, B_{n-1}, C_{n-1}) = \Big(\frac{B_n \cdot (C_n - A_n)}{L_n}, \frac{-B_n \cdot C_n}{L_n}, \frac{(A_n + C_n)\cdot(A_n - 2C_n)}{L_n}\Big) \qquad (3.43)$$

where H_n is the GCD polynomial of $C_n \cdot (B_n + A_n)$, $(B_n - A_n)\cdot(2B_n + A_n)$ and $B_n \cdot C_n$ and, similarly, L_n is the GCD polynomial of $B_n \cdot (C_n - A_n)$, $-B_n \cdot C_n$, $(A_n + C_n)\cdot(A_n - 2C_n)$ respectively. Transformation (3.42) can be written in a compact way:

$$M_{n+1}^{Strog} = \frac{K(M_n^{Strog})}{H_n} \qquad (3.44)$$

The first A_n's, B_n's, C_n's read:

$$A_0 = 1, \quad B_0 = b, \quad C_0 = -c, \quad A_1 = -c\cdot(b+1), \quad B_1 = (b-1)\cdot(2b+1), \quad C_1 = -bc,$$
$$A_2 = -bc\cdot(2b^2 - b - c - bc - 1), \quad B_2 = (2b^2 - b + c + bc - 1)\cdot(4b^2 - 2b - c - bc - 2),$$
$$C_2 = -bc\cdot(b-1)\cdot(2b+1), \quad \cdots \qquad (3.45)$$

Introducing other polynomials:

$$X_n = B_{n-1} - A_{n-1}, \quad Y_n = 2B_{n-1} + A_{n-1}, \quad Z_n = B_{n-1} + A_{n-1} \qquad (3.46)$$

polynomials $A_{n+1}, B_{n+1}, C_{n+1}$ simply read:

$$A_{n+1} = \frac{C_n}{H_n}\cdot Z_n, \quad B_{n+1} = \frac{X_n}{H_n}\cdot Y_n, \quad C_{n+1} = \frac{C_n}{H_n}\cdot B_n \qquad (3.47)$$

Polynomials Y_n and Z_n do not factorize, while polynomials X_n's, as well as polynomials C_n's (more precisely the C_{n-1}'s ...), *are divisible by* H_n. Defining polynomial X_n^H by:

$$X_n^H = \frac{X_n}{H_n} \qquad (3.48)$$

one remarks that these X_n^H's do not factorize. *Moreover one also remarks that H_n is actually equal to* Y_{n-2}. The first expressions of the GCD H_n's (or equivalently of the Y_{n-2}'s) read:

$$H_3 = Y_1 = 2b + 1, \quad H_4 = Y_2 = 4b^2 - 2b - 2 - bc - c, \qquad (3.49)$$
$$H_5 = Y_3 = 4 - 3bc - 12b^2 + 8b - 2c + 2b^3c + 3b^2c - b^2c^2 - 3bc^2 - 2c^2 + 16b^4 - 16b^3, \cdots$$

In terms of these X_n^H's, Y_n's and Z_n's, the A_n's, B_n's and C_n's are:

$$A_{n+1} = X_{n-2}^H \cdot X_{n-3}^H \cdot X_{n-4}^H \cdots X_1^H \cdot B_0 \cdot C_0 \cdot Z_{n+1}, \quad B_{n+1} = X_{n+1}^H \cdot Y_{n+1}, \qquad (3.50)$$
$$C_{n+1} = X_{n-1}^H \cdot X_{n-2}^H \cdot X_{n-3}^H \cdots X_1^H \cdot B_0 \cdot C_0 \cdot Y_n$$

A more complete list of the successive expressions of X_n^H, Y_n, Z_n is given in Appendix A.

Representations (3.42) and (3.43) are nothing but transformation K (or K^{-1}) represented as a birational transformation on (b, c). Let us, for instance, give the representation of I and t_1 as *birational* transformations on (b, c):

$$I: (b,c) \longrightarrow \Big(\frac{-b}{1+b}, \frac{(1+2b)\cdot(1-b)}{(1+b)\cdot c}\Big), \qquad t_1: (b,c) \longrightarrow (c,b) \qquad (3.51)$$

Let us denote d_n the degree of polynomials A_n (or B_n or C_n) and $d(x)$ the associated generating function. The degree of the X_n^H's, Y_n's and Z_n's, denoted d_n^X, d_n^Y, d_n^Z respectively, satisfy:

$$d_n^X = 1 + d_{n-2}, \qquad d_n^Y = d_n^Z = d_{n-1} \qquad (3.52)$$

Relation $B_{n+1} = X_{n+1}^H \cdot Y_{n+1}$ yields:

$$d_n = d_n^X + d_n^Y = 1 + d_{n-2} + d_{n-1}, \qquad d(x)\cdot(1 - x - x^2) - \frac{1}{1-x} = 0 \qquad (3.53)$$

Thus $d(x)$ reads:

$$d(x) = 1 + d_1\cdot x + d_2\cdot x^2 + d_3 \cdot x^3 + \cdots = \frac{1}{(1-x)\cdot(1-x-x^2)} \qquad (3.54)$$
$$= 1 + 2x + 4x^2 + 7x^3 + 12x^4 + 20x^5 + 33x^6 + 54x^7 + 88x^8 + 143x^9 + 232x^{10} + \cdots$$

The generating function $d(x)$ corresponds to an exponential growth of the complexity of the iterations like z^n where z is the largest root of $1 + z - z^2$ ($z = 1.618033989$ ...). This growth has to be compared with the exponential growth corresponding to the generic 9×9 matrices (see section (3.2)) for which one has an exponential growth like z^n where z is the largest root of $1 - 7z + z^2$ ($z = 6.854101966$ ...).

This exponential growth for model (3.19) is confirmed by the fact that, seeking for algebraic expressions of b and c ($P_1(b,c)/Q_1(b,c)$) invariant under transformation K (or transformation I and t_1 see (3.51)), we have not found any such expressions up degree ten in b and c for $P_1(b,c)$ and $Q_1(b,c)$. One expects a quite "chaotic" behaviour for the iterations of K outside the two integrability conditions (3.20).

3.5 Back to integrability: the inversion trick

Since we claim that the various polynomials of the two variables (b, c) previously introduced (A_n, B_n, C_n, X_n^H, Y_n, Z_n ...) may be useful for a better understanding of Stroganov's model *outside* the Yang-Baxter-integrability conditions (3.20), it is natural to look at these polynomials when restricted to the Yang-Baxter-integrability conditions (3.20). For this purpose let us recall the (rational) well-suited parameterization of the model restricted to (3.20) (more precisely $c = 1 - b$):

$$b = \frac{\omega x - \omega^{-1}}{1 + x} \qquad \text{where} \qquad \omega = \frac{1 + \sqrt{5}}{2} \qquad (3.55)$$

and the expression of the partition function per site deduced from the *inversion trick* [19]:

$$Z(b, 1-b) = \omega^2 \cdot \frac{\sqrt{x}}{1+x}\cdot F(x)\cdot F(1/x) \qquad (3.56)$$

where $F(x)$ is an eulerian product:

$$F(x) = \prod_{k=1,\cdots\infty} \Big(1 - \frac{x}{\omega^{8k-2}}\Big)\cdot\Big(1 - \frac{x}{\omega^{8k+2}}\Big)^{-1} \qquad (3.57)$$

The expressions of the first X_n^H's, Y_n's, Z_n's read in terms of the variable x (3.55):

$$X_1^H = \frac{x-\omega^2}{\omega\,(1+x)}, \quad Y_1 = \frac{\left(\omega^4 x - \omega^{-2}\right)}{\omega(1+x)}, \quad Z_1 = \frac{\omega^3 x + \omega^{-1}}{\omega\,(1+x)}, \tag{3.58}$$

$$X_2^H = \frac{(x-\omega^2)\,(\omega^2 x - 1)}{(1+x)^2\,\omega^2}, \quad Y_2 = \frac{(x-\omega^2)\,(\omega^8 x - \omega^{-4})}{(1+x)^2\,\omega^2}, \quad Z_2 = \frac{(x-\omega^2)\,(\omega^5 x + \omega^{-3})}{(1+x)^2\,\omega^2},$$

$$X_3^H = \frac{(x-\omega^2)^2\,(x-\omega^{-2})\,(x-\omega^{-6})\,\omega^5}{(1+x)^3\,(\omega^6 x - 1)}, \quad Y_3 = \frac{(x-\omega^2)^2\,(x-\omega^{-2})\,(x-\omega^{-14})\,\omega^6}{(1+x)^4},$$

$$Z_3 = \frac{(x-\omega^2)^2\,(x+\omega^{-12})\,(x-\omega^{-2})\,\omega^5}{(1+x)^4}, \quad \cdots$$

It is clear that polynomials X_n^H, Y_n, Z_n ... are closely related to the various factors occurring in the partition function (3.56), and more precisely in the "eulerian" product (3.57).

When one considers weak-graph expansions [20] of this vertex model when it is no longer restricted to the "Yang-Baxter-integrability" conditions (3.20), the "complexity" of the polynomials in b and c is very similar to the one encountered with polynomials X_n^H, Y_n, Z_n ... seen as polynomials of the two variables b and c (see Appendix A). One can hope that these polynomials could be well-suited to "decipher" the complexity encountered in weak-graph expansions of models which are not Yang-Baxter-integrable [21, 22].

Therefore the following question pops out: is it possible that the inversion trick [4, 19, 23] *could, using such polynomials well-suited for the factorization analysis, yield an expansion in agreement with the weak-graph expansion ?* This would open a new class of model in lattice statistical mechanics: models which are "computable" without being "Yang-Baxter-integrable"[4]. We will address this very important question in forthcoming publications. We however have a negative prejudice on this very model, since the birational transformations K are not generically integrable (no foliation of the (b,c)-plane in algebraic elliptic curves, chaotic behaviour of the iteration of K outside the integrability conditions, exponential growth ...). Therefore we will address this "computability-versus-Yang-Baxter-integrability" question on better suited models for which a foliation of the *whole parameter space* in terms of (algebraic) elliptic curves does exist[5]. The very existence of these elliptic curves yields *analyticity properties* in *one variable* which *are known to be a key ingredient for the inversion trick to work* [26, 27, 28].

3.6 Stroganov's model for $q \geq 4$

These calculations can straightforwardly be generalized to arbitrary q, that is $q^2 \times q^2$ matrices.

The way the exponential growth "degenerates" into a polynomial or linear growth (here, for model (3.28), the situation is even more drastic: there is no growth; seen as a homogeneous transformations the degree of the N_n's or D_n's is 1) is exactly the same as for $q = 3$. We have again factorizations (3.24) and (3.26) but now the generating functions $r(x)$ and $s(x)$ respectively for arbitrary q:

$$r(x) = \frac{1+(q^2-2)\,x^2 - x^3}{(1-x)\,(1-(q^2-2)\,x+x^2)} = 1 + \frac{x\cdot((q^2-1)-x)}{(1-x)\,(1-(q^2-2)\,x+x^2)} \tag{3.59}$$

$$s(x) = \frac{1-x+(q^2-1)\,x^2-x^3}{(1-x)\,(1-(q^2-2)\,x+x^2)} = 1 + \frac{(q^2-2)\cdot x}{(1-x)\,(1-(q^2-2)\,x+x^2)} \tag{3.60}$$

[4]Such models do exist: for instance, disorder solutions [24, 25] provide some examples of "computable" models that are not Yang-Baxter-integrable. However such disorder solutions correspond to dimensional reductions of the model. We are seeking here for two-dimensional (or higher dimensional) models with a genuine two-dimensional complexity .

[5]We have called these models "quasi-integrable" [10]. The most spectacular example of such a quasi-integrable, but not (generically) Yang-Baxter-integrable, model is the *sixteen vertex model* [10].

The compatibility relation between factorizations (3.24) and (3.26) and the generic factorizations (3.13) reads:

$$\beta_{n+2} = q^2\cdot(s_{n+1}+s_n) - (q^2-1)\cdot(r_n+r_{n-1}) + 1 \tag{3.61}$$

yielding on the associated generating functions $\alpha(x)$, $r(x)$ and $s(x)$:

$$\beta(x) = (1+x)\cdot\left(q^2 x\cdot s(x) - (q^2-1)\,x^2\cdot r(x)\right) + \frac{1}{1-x} - 1 - 2x + (q^2-2)\,x^2 + (q^2-1)\,x^3 \tag{3.62}$$

Let us note that $s(x)$ has a remarkably simple form for $q^2 = 2$. Let us also note that the difference between $r(x)$ and $s(x)$ is quite simple:

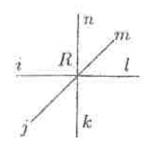
$$r(x) = s(x) + \frac{x}{1-(q^2-2)\,x+x^2} \tag{3.63}$$

Of course this does not completely rules out integrability for q different from 2: many integrable subcases of the q^4-state vertex model are known in the literature, but the corresponding patterns of the R-matrices are very specific [29].

In contrast it will be seen, in a forthcoming section (5), that *polynomial growth* occurs when some of the " arrows" of the vertex models takes 2 colors. It will be seen that this *polynomial growth* is closely related to the fact that transformation K *can thus be represented as a shift on a Jacobian variety* naturally associated with K.

4 Three-dimensional vertex models

Let us now recall that, for a *three-dimensional cubic vertex model* [8, 9], transposition t_1 associated with one of the three directions of the cubic lattice has already been introduced [8, 9]:



The action of t_1 on the three-dimensional R-matrix is given by:

$$(t_1 R)^{i_1 i_2 i_3}_{j_1 j_2 j_3} = R^{j_1 i_2 i_3}_{i_1 j_2 j_3} \tag{4.1}$$

and similar definitions for t_2 and t_3 [8, 9].

Such a situation corresponds to $m = q^2$ in framework described in section (2). Of course, one can define t_2 and t_3 on this model because the q^2-dimensional space decomposes into the tensorial product of two q-dimensional spaces.

We will restrict in this section to $q = 2$, the results for an arbitrary value of q will be given in the following (see section (5)). The analysis of the factorizations corresponding to the iterations of transformation K for t_1 for a general 64-state three-dimensional model (generic 8×8 matrix) gives the following factorizations:

$$M_1 = K(M_0), \quad f_1 = \det(M_0), \quad f_2 = \frac{\det(M_1)}{f_1^4}, \quad M_2 = \frac{K(M_1)}{f_1^3}, \quad f_3 = \frac{\det(M_2)}{f_1^7\cdot f_2^4},$$

$$M_3 = \frac{K(M_2)}{f_1^5\cdot f_2^3}, \quad f_4 = \frac{\det(M_3)}{f_1^8\cdot f_2^7\cdot f_3^4}, \quad M_4 = \frac{K(M_3)}{f_1^6\cdot f_2^5\cdot f_3^3}, \quad f_5 = \frac{\det(M_4)}{f_1^8\cdot f_2^8\cdot f_3^7\cdot f_4^4} \quad \cdots \tag{4.2}$$

and, for arbitrary n, the following *"string-like" factorizations*:

$$K(M_n) = M_{n+1} \cdot f_n^3 \cdot f_{n-1}^5 \cdot \Big(f_{n-2} \cdot f_{n-3} \cdots f_1\Big)^6 \qquad (4.3)$$

$$det(M_n) = f_{n+1} \cdot f_n^4 \cdot f_{n-1}^7 \cdot \Big(f_{n-2} \cdot f_{n-3} \cdot f_{n-4} \cdots f_1\Big)^8 \qquad (4.4)$$

yielding:

$$\widehat{K}(M_n) = \frac{K(M_n)}{det(M_n)} = \frac{M_{n+1}}{(f_1 \cdot f_2 \cdots f_{n-1})^2 \cdot f_n \cdot f_{n+1}} \qquad (4.5)$$

From factorization (4.5), one easily gets a relation between the generating functions $\alpha(x)$ and $\beta(x)$:

$$(1+x) \cdot \alpha(x) - \frac{8\,(1+x^2)}{1-x} \cdot \beta(x) - 8 = 0 \qquad (4.6)$$

leading:

$$\alpha(x) = \frac{8\,(1+x)^3}{(1-x)^4}, \qquad \beta(x) = \frac{8\,x}{(1-x)^3} \qquad (4.7)$$

The *"right-action"* of K also yields factorizations of f_1 and only f_1: one gets again equations (3.3) (with of course different expressions for the μ_n's and ν_n's) . These equations, combined with (4.6), give the following expressions for $\mu(x)$ and $\nu(x)$:

$$\mu(x) = \frac{x\,(1+x)(4-x)}{(1-x)^3}, \qquad \nu(x) = \frac{x\,(3+2\,x+x^2)}{(1-x)^4} \qquad (4.8)$$

One notes that α_n and β_n are respectively *cubic and quadratic functions* of n (to be compared with (3.7)):

$$\alpha_n = \frac{8}{3}\,(2n+1)\,(2n^2+2n+3)\,, \qquad \beta_n = 4\,n\,(n+1) \qquad (4.9)$$

At first sight it is amazing that such a polynomial growth occurs with involved "string-like" factorizations, such as (4.3) and (4.4).

The occurrence of *polynomial growth of the calculations of the iterations could correspond to situations where the algebraic varieties generated by K are abelian varieties* [30]. One knows that algebraic varieties having an *infinite set of automorphisms* cannot be of the so-called *general type*[6] [28]. This is the case here: we actually use the symmetries of the algebraic varieties (birational automorphisms) to visualize them [5, 6, 8, 9, 11].

The analysis of the iterations of transformation K has been performed in more details for a particular 8×8 matrix corresponding to a three-dimensional generalization of the Baxter model [8, 9]. This analysis shows that the orbits of the iterations lie, in this subcase [8, 9], on an *algebraic surface* given by intersection of *quadrics* [8, 9]. We will come back to this very model in a forthcoming section (4.3).

For the *general* 8×8-matrix considered here, the orbits do not lie on algebraic surfaces but on higher dimensional varieties [8, 9]. Introducing *Plücker*-like variables closely related to the *minors* of the R-matrix [1, 3], here 4×4 minors, one can, for this three-dimensional vertex model, explicitly write down the equations of these algebraic varieties as intersection of *quartics*. In fact the analysis of these algebraic varieties [7, 8, 9] is difficult to perform: are these varieties *abelian varieties*, or even *products of elliptic curves*[7], or any other algebraic varieties which are *not of the so-called "general type"* [28] (like

[6] Examples of algebraic varieties which are not of the general type are, for instance in the case of surfaces, abelian surfaces, hyperelliptic surfaces (surface fibred over $\mathbb{C}P_1$ by a pencil of elliptic curves), Enriques surfaces ...

[7] There exist some systematic procedures to see if an algebraic surface is a product of curves but they are extremely difficult to implement.

$K3$ surfaces [8] ...). We hope that the occurrence of polynomial growth of the associated iterations could help to clarify the kind of algebraic varieties associated with these birational transformations.

On the other hand, this could provide *a new way to analyse three or higher dimensional vertex models.* Of course, it is necessary to analyse *simultaneously* not only K but also K_{t_2} and K_{t_3}, the birational transformations corresponding to the two other directions of the cubic lattice and to their associated partial transposition t_2 and t_3.

Let us try to better understand the relation between polynomial growth and the occurrence of various examples of algebraic varieties which not of the "general type".

For this purpose we now examine *five different subcases* of this three-dimensional vertex model.

4.1 Restricted factorizations in dimension three: product of elliptic curves

Similarly to what has been seen in section (3.3), one can consider the "restricted factorization problem" corresponding to the following initial matrix:

$$R_{3D} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad (4.10)$$

where the 4×4 submatrices A, B, C and D are of the form:

$$A = \begin{pmatrix} A_1 & 0 \\ A_2 & A_3 \end{pmatrix} \qquad (4.11)$$

with A_1, A_2 and A_3 are 2×2 matrices and "0" denotes the 2×2 matrix with zero entries, and the form for matrices B, C and D is similar to (4.11). It is straightforward to see that a form like (4.10), together with (4.11), is actually compatible with the action of the group generated by the matrix inverse $\widehat{I}$ and t_1 [7, 8].

For such matrices ((4.10), (4.11)) one can see (permuting rows and columns 3-4 and 5-6 of the 8×8 matrix R_{3D}) that the polynomials f_n defined by equations (4.2) factorize in the product of two polynomials. One can show that these two polynomials $F_n^{(1)}$ and $F_n^{(3)}$ actually correspond to the action the birational transformation K associated with two sixteen vertex models (see section (3.1)) associated with the two following 4×4 matrices:

$$M_0^{(1)} = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \qquad \text{and:} \qquad M_0^{(3)} = \begin{pmatrix} A_3 & B_3 \\ C_3 & D_3 \end{pmatrix} \qquad (4.12)$$

One gets therefore on the f_n's:

$$f_n = F_n^{(1)} \cdot F_n^{(3)} \qquad (4.13)$$

where each of the $F_n^{(1)}$'s and $F_n^{(3)}$'s satisfy *independently* the *same* recurrence which is actually the recurrence occurring for the sixteen vertex model (see section (3.1)). This non-linear recurrence has been shown to yield *algebraic elliptic curves* $\mathcal{E}$ [1, 2, 3]. It is thus clear, at least in subcase (4.11), that the f_n's do not satisfy a recurrence (like (3.12)), but that the orbits of the iteration of K are naturally associated with *algebraic surfaces* which are the *product of two algebraic elliptic curves*: $S = \mathcal{E} \times \mathcal{E}$.

From equation (4.13), one easily gets in this subcase ((4.10), (4.11)), that the degree of the f_n's and $\det(M_n)$'s, namely β_n and α_n can be written as a sum of two terms:

$$\beta_n = \beta_n^{(1)} + \beta_n^{(3)}, \qquad \alpha_n = \alpha_n^{(1)} + \alpha_n^{(3)} \qquad (4.14)$$

where the $\beta_n^{(i)}$'s (resp. the $\alpha_n^{(i)}$'s) are the degree of the $F_n^{(i)}$'s (resp. the $\det(M_n)^{(i)}$'s) with $i = 1, 3$.

[8] The birational transformations considered here, actually densify in a quite "uniform way" the algebraic surfaces we get (see figures (1a), (1b) and (1c)): this seems to exclude automorphisms of $K3$ surfaces.

From section (3.1) (see equations (3.7)), one immediately gets that $\beta_n^{(1)} = \beta_n^{(3)} = 2n\,(n+1)$ and $\alpha_n^{(1)} = \alpha_n^{(3)} = 4\,(2\,n^2+1)$. This provides an example of a *quadratic growth associated with an algebraic surface* (namely: $\mathcal{S} = \mathcal{E} \times \mathcal{E}$).

4.2 A three-dimensional generalization of the Baxter model

Another (less academical) example of "restricted factorization" corresponds to vertex-models defined in [8, 9, 31] which can be seen as a three-dimensional generalization of the Baxter model. This model corresponds to the following K-compatible conditions:

$$R^{i_1 i_2 i_3}_{j_1 j_2 j_3} \;=\; R^{-i_1,-i_2,-i_3}_{-j_1,-j_2,-j_3} \tag{4.15}$$

$$\text{and} \qquad R^{i_1 i_2 i_3}_{j_1 j_2 j_3} \;=\; 0 \quad \text{if} \quad i_1 i_2 i_3 j_1 j_2 j_3 \;=\; -1 \tag{4.16}$$

Let us assume that the order for the "in" triplet (i_1,i_2,i_3), as well as the "out" triplet (j_1,j_2,j_3) is as follows:

$$\Big[(1,2,3,4,5,6,7,8)\Big] \;= \tag{4.17}$$
$$\Big[(+1,+1,+1)(+1,+1,-1)(+1,-1,+1)(+1,-1,-1)(-1,+1,+1)(-1,+1,-1)(-1,-1,+1)(-1,-1,-1)\Big]$$

This order singles out direction 1, and is therefore well-suited to analyse transformation K.

With this ordering, conditions ((4.15), (4.16)) yield the following 8×8 matrix:

$$R^{3d} \;=\; \begin{pmatrix} a & 0 & 0 & k & 0 & l & m & 0 \\ 0 & b & c & 0 & n & 0 & 0 & d \\ 0 & e & f & 0 & p & 0 & 0 & g \\ q & 0 & 0 & h & 0 & i & j & 0 \\ 0 & j & i & 0 & h & 0 & 0 & q \\ g & 0 & 0 & p & 0 & f & e & 0 \\ d & 0 & 0 & n & 0 & c & b & 0 \\ 0 & m & l & 0 & k & 0 & 0 & a \end{pmatrix} \tag{4.18}$$

With another order for the rows and columns (namely $(1,2,3,4,5,6,7,8) \rightarrow (1,4,6,7,8,5,3,2)$), this 8×8 matrix can be seen as two identical 4×4 block-matrices:

$$B^{3d} \;=\; \begin{pmatrix} a & k & l & m \\ q & h & i & j \\ g & p & f & e \\ d & n & c & b \end{pmatrix} \tag{4.19}$$

These two block matrices are respectively associated to the two "odd and even" subspaces: $\Big[(+1,+1,+1)\,(+1,-1,-1)\,(-1,+1,-1)\,(-1,-1,+1)\Big]$ and $\Big[(-1,-1,-1)\,(-1,+1,+1)\,(+1,-1,+1)\,(+1,+1,-1)\Big]$. With this new order the three directions 1, 2 and 3 are on the same footing: it is better suited to analyse the group generated by *all* the (four) inversion relations of this three dimensional vertex model [8, 9] (of course transformation t_1 becomes a more involved permutation of the entries).

For this three-dimensional generalization of the eight-vertex model [9, 31], introducing the same f_n's as the ones given by (4.2), (4.3) and (4.4), one actually verifies the factorizations:

$$M_1 \;=\; K(M_0), \quad f_1 \;=\; \det(M_0), \quad f_2 \;=\; \frac{\det(M_1)}{f_1^4}, \quad M_2 \;=\; \frac{K(M_1)}{f_1^3}, \quad f_3 \;=\; \frac{\det(M_2)}{f_1^7 \cdot f_2^4},$$
$$M_3 \;=\; \frac{K(M_2)}{f_1^6 \cdot f_2^3}, \quad f_4 \;=\; \frac{\det(M_3)}{f_2^7 \cdot f_3^4}, \quad M_4 \;=\; \frac{K(M_3)}{f_2^6 \cdot f_3^3}, \quad f_5 \;=\; \frac{\det(M_4)}{f_3^7 \cdot f_4^4} \quad \cdots \tag{4.20}$$

Let us note that one has more factorizations than in the generic case (4.2). Moreover, for arbitrary n, one has the following factorizations but now with a *fixed* number of polynomials f_n, instead of the "string-like" factorizations ((4.3), (4.4)):

$$K(M_n) \;=\; M_{n+1} \cdot f_n^3 \cdot f_{n-1}^6\,, \qquad det(M_n) \;=\; f_{n+1} \cdot f_n^4 \cdot f_{n-1}^7 \tag{4.21}$$

yielding:

$$\widehat{K}(M_n) \;=\; \frac{K(M_n)}{det(M_n)} \;=\; \frac{M_{n+1}}{f_{n-1} \cdot f_n \cdot f_{n+1}} \tag{4.22}$$

Let us note that, since the 8×8 matrix (4.18) is, after a relabelling of the rows and columns, *the direct product of two times the same 4×4 matrix*, and since the homogeneous transformation K acts in the same way on these two blocks, all these f_n's *are exactly perfect squares*.

It is illuminating to see how factorizations like (4.2), (4.3), (4.4) and (4.5) become (4.20), (4.21) and (4.22). One actually has the same first factorizations up to M_2 and f_3. They first become different with M_3 for which *one gets an extra factorization of f_1*. Obviously, in the factorization of f_4, one does not have a factorization of f_1^8 anymore (because an extra factorization of f_1 in all the entries of M_3 yield an extra factorization of f_1^8 in $\det(M_3)$). These slight modifications however have the amazing consequence to change the "string-like" factorizations (4.3) and (4.5) *into factorizations with a fixed number of terms* (see relations (4.21) and (4.22)).

The generating functions $\alpha(x)$ and $\beta(x)$ verify:

$$(1+x)\cdot\alpha(x) \,-\, 8\,(1+x+x^2)\cdot\beta(x) \,-\, 8 \;=\; 0 \tag{4.23}$$

leading:

$$\alpha(x) \;=\; \frac{8\,(1+4\,x+7\,x^2)}{(1-x)^3}, \quad \beta(x) \;=\; \frac{8\,x}{(1-x)^3}, \quad \mu(x) \;=\; \frac{x(1+x)(4-x)}{(1-x)^3}, \quad \nu(x) \;=\; \frac{3\,x\,(1+2\,x)}{(1-x)^3} \tag{4.24}$$

One notes that α_n and β_n are *both quadratic functions* of n:

$$\alpha_n \;=\; 8\,(6\,n^2+1)\,, \qquad \beta_n \;=\; 4\,n\,(n+1) \tag{4.25}$$

One remarks that the two generating functions $\beta(x)$ and $\mu(x)$ *are the same as for the general 8×8 matrix* (see (4)), the difference being on the α_n's or the ν_n's (or equivalently on the generating functions $\alpha(x)$ and $\nu(x)$): the *cubic* growth of the α_n's or the ν_n's (see relation (4.9)) being replaced by a *quadratic growth* (see relation (4.25)).

In fact this modification of a "string-like" factorization into factorizations with a *fixed* number of terms is not as drastic as it looks at first sight. Let us for instance define, for a 8×8 matrix of the form (4.18), the variables f_n^{string}'s and the successive matrices M_n^{string}'s from the "string-like" factorization relations (4.3) and (4.5), which are valid for general 8×8 matrices, and therefore, a fortiori, for matrices of the form (4.18). We have just seen that an extra factorization occur for model (4.18) (M_3 ...). It is amusing to remark *that the variables f_n^{string}'s defined from* (4.3) and (4.5) *and the variables f_n's defined from* (4.21) *actually coincide* ! This can be proved recursively. Let us denote y_n the multiplicative factor between M_n (defined by (4.21)) and M_n^{string}: $M_n^{string} = y_n \cdot M_n$. One immediately gets, from (4.3) and (4.21), the following relations:

$$\frac{1}{y_{n+1}} \;=\; \frac{\big(f_1 \cdot f_2 \cdots f_{n-2}\big)^6}{f_{n-1}\cdot y_n^2} \qquad \text{and also:} \qquad \frac{f_n}{f_n^{string}} \;=\; \frac{f_1 \cdot f_2 \cdots f_{n-3}}{y_{n-1}} \tag{4.26}$$

It is then simple to show recursively that $y_n = f_1 \cdot f_2 \cdots f_{n-2}$ and therefore that $f_n = f_n^{string}$. This means that this string-like factorization may be seen, to some extend, just as a "propagation" of the extra factorization occurring with M_3. This explains that the generating functions $\mu(x)$, $\beta(x)$ *are actually identical* for matrices of the form (4.18) and for general 8×8 matrices (see (4.7), (4.8) and (4.24)).

Let us note that if one relaxes the spin reversal constraint (4.15) (for instance just relaxing the equality between $M_0[1,1]$ and $M_0[8,8]$ in (4.18)), which means that, after relabelling, the 8×8 matrix can be written as two *non identical* 4×4 block matrices, *one gets back to the above detailed "string-like" factorizations* of section (4). Of course if one relaxes the "charge-conservation" constraint (4.16) (for instance just make $M_0[1,2]$ non zero in (4.18))) one also gets back to the above detailed "string-like" factorizations of section (4).

Let us note that the orbits of K can be shown to yield algebraic varieties in $\mathbb{C}P_{15}$ given by *intersection of quadrics* [1][9].

4.3 A nine parameter three-dimensional generalization of the Baxter model

The visualization of the orbits of K have been performed in [8, 9] for the particular subcase which amounts to imposing, together with (4.15) and (4.16), matrix R (4.18) to be *symmetric*:

$$R^{i_1 i_2 i_3}_{j_1 j_2 j_3} \quad = \quad R^{j_1, j_2, j_3}_{i_1, i_2, i_3} \tag{4.27}$$

In this subcase one clearly gets *surfaces* and it has been shown that these surfaces are *algebraic surfaces* given by *intersection of quadrics* [1, 9, 31][10].

The explicit expressions of these quadrics has actually been written down in the particular subcase of the model defined by (4.15) and (4.16) together with (4.27) [1, 9, 31]. Condition (4.27) is clearly preserved by transformation I and $t_1 \cdot t_2 \cdot t_3$ (the matrix inversion and the matrix transposition). More remarkably condition (4.27) is actually *preserved by the three other inversions* [8, 9] of this three dimensional vertex model namely: $I_1 = t_1 \cdot I \cdot t_2 \cdot t_3$, $I_2 = t_2 \cdot I \cdot t_1 \cdot t_3$ and $I_3 = t_3 \cdot I \cdot t_1 \cdot t_2$. This is a consequence of the fact that condition (4.27) is preserved by the partial transpositions t_1, t_2 and t_3. With this last condition the three dimensional vertex model looks even closer to a generalization in three dimension of the symmetric eight vertex Baxter model [9, 8]. When condition (4.27) is satisfied, together with conditions (4.15) and (4.16), the two identical block matrices (4.19) depend only on *ten* homogeneous parameters. Using the notations introduced in [9] or [31], one can introduce ten homogeneous parameters:

$$B^{3d} = \begin{pmatrix} a & k & l & m \\ q & h & i & j \\ g & p & f & e \\ d & n & c & b \end{pmatrix} = \begin{pmatrix} a & d_1 & d_2 & d_3 \\ d_1 & b_1 & c_3 & c_2 \\ d_2 & c_3 & b_2 & c_1 \\ d_3 & c_2 & c_1 & b_3 \end{pmatrix} \tag{4.28}$$

Of course transformation t_1, which is the block transposition of the two off-diagonal 4×4 matrices, as well as transformations t_2 and t_3 become, as a consequence of the relabelling, new permutations of the entries of this 4×4 matrix B^{3d} [31]:

$$t_i \;:\; c_j \leftrightarrow d_j \,, \; c_k \leftrightarrow d_k \,, \qquad (i,j,k) = (1,2,3) \tag{4.29}$$

Actually, and quite remarkably, there exist four quantities which are invariant by *all the four generating involutions* I, I_1, I_2, I_3 *and therefore the whole group* Γ_{3D} *they generate*. Let us recall the results of [31].

Let us introduce:

$$ab_1 + b_2 b_3 - c_1^2 - d_1^2, \qquad c_2 d_2 - c_3 d_3 \tag{4.30}$$

and the polynomials obtained by permutations of 1, 2 and 3. They form a five dimensional space of polynomials. Any ratio of the five independent polynomials is invariant under all the four generating involutions I, I_1, I_2, I_3. The parameter space $\mathbb{C}P_9$ is thus foliated by *five dimensional* algebraic varieties invariant under *the whole group* Γ_{3D}:

$$\frac{P_i(a, \cdots, d_3)}{Q_i(a, \cdots, d_3)} \quad = \quad \text{constant} \tag{4.31}$$

[9]The occurrence of *quadrics* is closely related [1, 3] to the occurrence of 4×4 matrices like (4.19) for model (4.18).

[10]When conditions (4.15) and (4.16) are relaxed one does not get (algebraic) surfaces anymore, but higher dimensional varieties.

where P_i and Q_i are chosen among the quadratic polynomials (4.30) and the one deduced by permutations of directions 1, 2 and 3.

Considering the subgroup of Γ_{3D} generated by two involutions among these four, or equivalently considering the iteration of K only[11], one can show that the orbits of this transformation are *algebraic surfaces* given by intersection of quadrics [1, 9, 11, 31].

These additional quadrics have been written explicitly [31][12] :

$$ab_1 - b_2 b_3 - c_1^2 - d_1^2, \qquad (a + b_1)\, c_1 - d_2 d_3 - c_2 c_3, \qquad (b_2 + b_3)\, d_1 - d_2\, c_3 - d_3\, c_2 \tag{4.32}$$

On figure (1a), figure (1b) and figure (1c), given at the end of the paper, it is clear that the orbits of K are (algebraic) *surfaces*. These orbits strongly suggest an interpretation in terms of curves winding round a two-dimensional torus in the generic "incommensurate" situation.

In contrast with the situation encountered in section (3.1) (see relation (3.8)) the successive iterates of M_0 live in the *whole nine dimensional* affine matrix space (4.28):

$$K^{2n}(M_0) \;=\; a_0^{(n)} \cdot M_0 \,+\, a_1^{(n)} \cdot M_2 \cdots + a_8^{(n)} \cdot M_{16} \,+\, a_9^{(n)} \cdot M_{18} \tag{4.33}$$

The visualization of the orbits of K, has also been performed if *one relaxes the matrix symmetry condition* (4.27). *One does not yield surfaces anymore.* Figures (2a), (2b) and (2c) illustrate such a situation. Figure (2a) corresponds to an orbit for an initial matrix "almost" symmetric (symmetric up to 10^{-6}). This first figure, which corresponds to a very small asymmetry of the initial matrix, may look similar (at least for the first 10^5 iterations) to figures (1a), (1b) and (1c). In fact one can see on figure (2a) that the density of points is more "fuzzy" compared to figures (1a), (1b) and (1c) which suggest a curve moving on a two-dimensional torus. The density of points of figures (2b) and (2c) clearly corresponds to the projection of points living in algebraic varieties of dimension greater than two.

These results have to be compared with the one given by Korepanov [32], or the one described in a forthcoming section (6). The fact that a polynomial growth occurs when some of the " arrows" in the vertex models take 2 colors and that exponential growth (generically) occurs when the number of colors of the " arrows" is no longer 2 (see section (3.2)) deserves some comment: it will be seen that this *polynomial growth* is related to the fact that transformation K *can be represented as a shift on a Jacobian variety* naturally associated with K. We previously recalled that algebraic varieties having an *infinite set of automorphisms* cannot be of the so-called *general type* [28]. The fact that one can associate with the algebraic surface given by the intersection of quadrics (4.32) and (4.30), some Jacobian variety should help to characterize in more details these surfaces which are not of the general type[13].

4.4 An integrable subcase of the three-dimensional generalization of the Baxter model

In order to shed some light on the relations between the polynomial growth and the occurrence of *algebraic varieties which are not of the so-called "general" type* [28] (abelian varieties, products of elliptic curves, ...), let us consider a situation for which *elliptic curves occur*. At this point, it is worth recalling that, for particular patterns of the three-dimensional generalization of the Baxter model considered in section (4.3) (conditions (4.16), (4.15) together with *the additional condition* (4.27)), the iteration of K (or $\widehat{K}$) can actually *yield elliptic curves* (see [8, 9]). These particular patterns amount to imposing that the initial matrix (and therefore the successive matrices M_n) is *invariant under the permutation of the two directions* "1" and "2". In fact we will see in this section, that there is no need to impose the matrix symmetry condition (4.27) to get integrable subcases of (4.18).

[11]Since I commutes with the matrix transposition $t_1 \cdot t_2 \cdot t_3$, the product of two inversions, for instance $I_1 \cdot I = t_1 \cdot I \cdot t_1 \cdot t_2 \cdot t_3 \cdot t_1 \cdot I = K^2 \cdot t_1 \cdot t_2 \cdot t_3$ is equivalent, up to the matrix transposition $t_1 \cdot t_2 \cdot t_3$, to K^2.

[12]One should note a misprint in [31]: one should read $ab_1 - b_2 b_3 - c_1^2 - d_1^2$ instead of $ab_1 - b_2 b_3 - c_1^2 + d_1^2$.

[13]Note that the space where this Jacobian variety lives is, in general, not the same as the parameter space $\mathbb{C}P_{q^2-1}$ where these algebraic varieties generated by K live.

With the previous order (4.17) for the rows and columns of the 8×8 matrix, these additional conditions read:

$$R[1,6] = R[1,7]\,,\ R[2,8] = R[3,8]\,,\ R[2,5] = R[3,5]\,,\ R[4,6] = R[4,7]\,,\ R[2,2] = R[3,3]\,,\ R[2,3] = R[3,2]$$

Recalling matrix (4.18) and its notations, this symmetry between direction 2 and 3 yields the additional equalities among the entries: $m = l$, $j = i$, $g = d$, $e = c$, $f = b$ and $p = n$. The corresponding 8×8 matrix, thus depends on *ten* homogeneous parameters:

$$R^{3d} = \begin{pmatrix} a & 0 & 0 & k & 0 & l & l & 0 \\ 0 & b & c & 0 & n & 0 & 0 & d \\ 0 & c & b & 0 & n & 0 & 0 & d \\ q & 0 & 0 & h & 0 & i & i & 0 \\ 0 & i & i & 0 & h & 0 & 0 & q \\ d & 0 & 0 & n & 0 & b & c & 0 \\ d & 0 & 0 & n & 0 & c & b & 0 \\ 0 & l & l & 0 & k & 0 & 0 & a \end{pmatrix} \qquad (4.34)$$

or, on the two identical 4×4 block matrices (4.19):

$$B^{3d} = \begin{pmatrix} a & k & l & l \\ q & h & i & i \\ d & n & b & c \\ d & n & c & b \end{pmatrix} \qquad (4.35)$$

This 4×4 matrix is invariant under the permutation of direction 2 and 3 which amounts to permuting the last two rows and columns of the two four dimensional subspaces $\Big[(+1,+1,+1)\ (+1,-1,-1)\ (-1,+1,-1)$ $(-1,-1,+1)\Big]$ and $\Big[\ (-1,-1,-1)\ (-1,+1,+1)\ (+1,-1,+1)\ (+1,+1,-1)\ \Big]$.

In this subcase (4.34), the factorization relations (4.20) are slightly modified. Imposing these additional constraints (4.34), one remarks that factorizations (4.21) and (4.22) are slightly, but definitely, modified as follows:

$$M_1 = K(M_0), \ f_1 = \det(M_0), \ f_2 = \frac{\det(M_1)}{f_1^5}, \ M_2 = \frac{K(M_1)}{f_1^4}, \ f_3 = \frac{\det(M_2)}{f_1^3 \cdot f_2^5},$$
$$M_3 = \frac{K(M_2)}{f_1^2 \cdot f_2^4}, \ f_4 = \frac{\det(M_3)}{f_1^7 \cdot f_2^3 \cdot f_3^5}, \ M_4 = \frac{K(M_3)}{f_1^6 \cdot f_2^2 \cdot f_3^4}, \ f_5 = \frac{\det(M_4)}{f_2^7 \cdot f_3^3 \cdot f_4^5} \ \cdots \qquad (4.36)$$

and for arbitrary n:

$$K(M_n) = M_{n+1} \cdot f_n^4 \cdot f_{n-1}^2 \cdot f_{n-2}^6, \qquad det(M_n) = f_{n+1} \cdot f_n^5 \cdot f_{n-1}^3 \cdot f_{n-2}^7 \qquad (4.37)$$

yielding:

$$\widehat{K}(M_n) = \frac{K(M_n)}{det(M_n)} = \frac{M_{n+1}}{f_{n-2} \cdot f_{n-1} \cdot f_n \cdot f_{n+1}} \qquad (4.38)$$

Note that the "universal" relation (4.38) is actually modified for subcase (4.34). The new polynomials, defined in this restricted (integrable) subcase (4.34), can actually be shown to verify non-linear recurrences. Since the f_n's are *perfect squares*, one can introduce their square-roots $f'_n = \sqrt{f_n}$. Remarkably, these polynomials f'_n's *do verify the same hierarchy of recurrences as for the sixteen vertex model* (see section (3.1) and [1]):

$$\frac{f'_n f'^2_{n+3} - f'_{n+4} f'^2_{n+1}}{f'_{n-1} f'_{n+3} f'_{n+4} - f'_n f'_{n+1} f'_{n+5}} = \frac{f'_{n-1} f'^2_{n+2} - f'_{n+3} f'^2_n}{f'_{n-2} f'_{n+2} f'_{n+3} - f'_{n-1} f'_n f'_{n+4}} \qquad (4.39)$$

or:

$$\frac{f'_{n+1} f'^2_{n+4} f'_{n+5} - f'_{n+2} f'^2_{n+3} f'_{n+6}}{f'^2_{n+2} f'_{n+3} f'_{n+7} - f'_n f'_{n+4} f'^2_{n+5}} = \frac{f'_{n+2} f'^2_{n+5} f'_{n+6} - f'_{n+3} f'^2_{n+4} f'_{n+7}}{f'^2_{n+3} f'_{n+4} f'_{n+8} - f'_{n+1} f'_{n+5} f'^2_{n+6}} \qquad (4.40)$$

These recurrences are known to yield *elliptic curves* [1, 2, 3]. The generating functions $\alpha(x)$, $\beta(x)$, $\mu(x)$ and $\nu(x)$ read:

$$\alpha(x) = \frac{8\,(1 + 5\,x + 3\,x^2 + 7\,x^3)}{(1+x)\,(1-x)^3}, \quad \beta(x) = \frac{8\,x}{(1+x)\,(1-x)^3} \qquad (4.41)$$

$$\mu(x) = \frac{x\,(5 + 2\,x^2 - x^3)}{(1+x)\,(1-x)^3}, \quad \nu(x) = \frac{2\,x\,(2 + x + 3\,x^2)}{(1+x)\,(1-x)^3} \qquad (4.42)$$

The integrability of this subcase (4.34) is thus associated with the occurrence of one more singularity (compare with (4.7) or (4.24)).

In contrast with the situation we had in section (3.1) (see relation (3.8)) the successive iterates of M_0 belong, for this subcase (4.34), to a *seven* dimensional affine subspace of the *nine* dimensional affine matrix space ((4.34) depends on *ten homogeneous parameters*):

$$K^{2\,n}(M_0) = a_0^{(n)} \cdot M_0 + a_1^{(n)} \cdot M_2 \cdots + a_7^{(n)} \cdot M_{14} \qquad (4.43)$$

that is a codimension-two subspace of the space where the matrices M_0 live.

The equations of these elliptic curves can be simply written down as the intersection of the quadrics and of the hyperplanes preserved by K^2 (see (4.34)). For model (4.28), analyzed in [8, 9] (see the previous section (4.3)), which amounts to imposing the Boltzmann matrix to be symmetric (condition (4.27)) these quadrics are (4.30) and (4.32), and these hyperplanes read, with notations (4.28):

$$b_2 = b_3\,, \qquad c_2 = c_3\,, \qquad d_2 = d_3 \qquad (4.44)$$

4.5 A three-dimensional generalization of the six-vertex model

Another example of "restricted factorization" corresponds to the vertex-model defined in [8, 9, 31] which can be seen as a three-dimensional generalization of the *six-vertex model* [31]. This model corresponds to the K-compatible conditions (4.2) together with the additional conditions:

$$R^{i_1 i_2 i_3}_{+1+1+1} = 0 \ \text{ if } \ (i_1, i_2, i_3) \neq (+1,+1,+1) \ \text{ and } \ R^{i_1 i_2 i_3}_{-1-1-1} = 0 \ \text{ if } \ (i_1, i_2, i_3) \neq (-1,-1,-1) \ \text{ and}$$
$$R^{+1,+1,+1}_{j_1 j_2 j_3} = 0 \ \text{ if } \ (j_1, j_2, j_3) \neq (+1,+1,+1) \ \text{ and } \ R^{-1,-1,-1}_{j_1 j_2 j_3} = 0 \ \text{ if } \ (j_1, j_2, j_3) \neq (-1,-1,-1) \quad (4.45)$$

One should note that this particular form for the 8×8 matrix (4.45) is not stable by transformation K (basically because transposition t_1 does not preserve the form (4.45)) but it is *actually preserved under the action* of K^2 [31][14]. Taking into account the simplicity of this model one can relax the matrix symmetry condition (4.27). This gives (with notations (4.18)) the following 8×8 matrix:

$$R^{3d} = \begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b & c & 0 & n & 0 & 0 & 0 \\ 0 & e & f & 0 & p & 0 & 0 & 0 \\ 0 & 0 & 0 & h & 0 & i & j & 0 \\ 0 & j & i & 0 & h & 0 & 0 & 0 \\ 0 & 0 & 0 & p & 0 & f & e & 0 \\ 0 & 0 & 0 & n & 0 & c & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a \end{pmatrix} \qquad (4.46)$$

[14] This situation generalizes the one encountered in two dimensions with six vertex models.

or, recalling the relabelling previously introduced ($(1,2,3,4,5,6,7,8) \to (1,4,6,7,8,5,3,2) = \Big[(+1,+1,+1)\,(+1,-1,-1)\,(-1,+1,-1)\,(-1,-1,+1)\Big]$ and $\Big[\,(-1,-1,-1)\,(-1,+1,+1)\,(+1,-1,+1)\,(+1,+1,-1)\,\Big]$), the two identical 4×4 matrices (4.19) can be written:

$$B^{3d} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & h & i & j \\ 0 & p & f & e \\ 0 & n & c & b \end{pmatrix} \tag{4.47}$$

This model depends on *ten homogeneous parameters*. For this three-dimensional generalization of the six-vertex model [9, 31], one can introduce the same f_n's as the ones given in section (4.2). The corresponding matrices can be seen to be product of 3×3 and 1×1 matrices. One straight consequence is that all the determinants one calculates are perfect squares which factorize into determinants of 3×3 matrices and terms corresponding to the 1×1 blocks. This enables to introduce variables which are the determinants of these 3×3 matrices which will be denoted g_n in the following, instead of the variables f_n related to the determinant of the whole 8×8 matrix. Introducing two variables w_0 and w_1 related to two particular entries of the matrix M_0 and its transformed by K, one gets factorizations:

$$w_0 = (M_0)_{11}, \quad g_1 = \frac{(\det(M_0))^{1/2}}{w_0}, \quad M_1 = \frac{K(M_0)}{g_1\, w_0},$$
$$w_1 = \frac{(M_1)_{44}}{w_0}, \quad g_2 = \frac{(\det(M_1))^{1/2}}{w_0^3\, w_1}, \quad M_2 = \frac{K(M_1)}{g_2\, w_0^5\, w_1},$$
$$g_3 = \frac{(\det(M_2))^{1/2}}{w_0\, w_1^3\, g_1^3}, \quad M_3 = \frac{K(M_2)}{w_0\, w_1^5\, g_1^5\, g_3}, \quad g_4 = \frac{(\det(M_3))^{1/2}}{w_0^3\, w_1\, g_2^3}, \quad M_4 = \frac{K(M_3)}{w_0^5\, w_1\, g_2^5\, g_4} \quad \cdots \tag{4.48}$$

and, for arbitrary n:

$$\begin{aligned} K(M_n) &= M_{n+1} \cdot w_0 \cdot w_1^5 \cdot g_{n-1}^5 \cdot g_{n+1}, &&\text{for n even} \\ K(M_n) &= M_{n+1} \cdot w_0^5 \cdot w_1 \cdot g_{n-1}^5 \cdot g_{n+1}, &&\text{for n odd} \end{aligned} \tag{4.49}$$

together with:

$$\begin{aligned} (\det(M_n))^{1/2} &= g_{n+1} \cdot w_0 \cdot w_1^3 \cdot g_{n-1}^3, &&\text{for n even} \\ (\det(M_n))^{1/2} &= g_{n+1} \cdot w_0^3 \cdot w_1 \cdot g_{n-1}^3, &&\text{for n odd} \end{aligned} \tag{4.50}$$

yielding for $n \geq 2$:

$$\widehat{K}(M_n) = \frac{K(M_n)}{det(M_n)} = \frac{M_{n+1}}{g_{n-1} \cdot g_{n+1} \cdot w_0 \cdot w_1} \tag{4.51}$$

Similarly, one can introduce the degrees of the determinants of these matrices M_n and the degrees of the successive polynomials g_n's , namely α_n and $\widehat{\beta}_n$ and their corresponding generating functions $\alpha(x)$ and $\widehat{\beta}(x)$. These generating functions read:

$$\alpha(x) = \frac{8\,(1+x+x^2-x^3)}{(1+x)\,(1-x)^3}, \qquad \widehat{\beta}(x) = \frac{x\,(3-2\,x)}{(1-x)^3} \tag{4.52}$$

The exponents α_n and $\widehat{\beta}_n$ read:

$$\alpha_n = 4\,n^2 + 16\,n + 6 + 2\,(-1)^n, \qquad \widehat{\beta}_n = \frac{n^2 + 5\,n}{2} \tag{4.53}$$

Again, one can study the "right action" of K on matrices M_n (equations (3.3)). However, since the form of matrix (4.46) is only preserved by K^2, the right action is a little bit more involved, namely:

$$\begin{aligned} 2\,(g_1)_{K^2} &= g_3\, g_2^3\, g_1^{23}\, w_0^{36}\, w_1^5, \qquad 8\,(g_2)_{K^2} = g_4\, g_2^9\, g_1^{53}\, w_0^{84}\, w_1^{11}, \\ 64\,(g_3)_{K^2} &= g_5\, g_2^{17}\, g_1^{90}\, w_0^{144}\, w_1^{18} \quad \cdots \end{aligned} \tag{4.54}$$

and for arbitrary n:

$$2^{z_n^{(1)}}\,(g_n)_{K^2} = g_{n+2}\, g_2^{z_n^{(2)}}\, g_1^{z_n^{(3)}}\, w_0^{z_n^{(4)}}\, w_1^{z_n^{(5)}} \tag{4.55}$$

where the $z_n^{(i)}$'s are quadratic integers:

$$z_n^{(1)} = \frac{n\,(n+1)}{2}, \quad z_n^{(2)} = n^2 + 3\,n - 1, \quad z_n^{(3)} = \frac{(7\,n+39)\,n}{2}, \quad z_n^{(4)} = 6\,n\,(n+5), \quad z_n^{(5)} = \frac{(n+9)\,n}{2}$$

Imposing the matrix symmetry condition (4.27) does not modify these factorizations.

The successive iterates of M_0 belong, for subcase (4.46), to an *eight* dimensional affine subspace of the *nine* dimensional affine matrix space (matrix (4.46) depends on *ten homogeneous parameters*):

$$K^{2\,n}(M_0) = a_0^{(n)} \cdot M_0 + a_1^{(n)} \cdot M_2 \cdots + a_8^{(n)} \cdot M_{16} \tag{4.56}$$

that is a codimension-one subspace of the space (4.46) where the matrices M_0 live.

If one imposes for this three-dimensional generalization of the six vertex model (4.46), the matrix symmetry condition (4.27), the model now depends on seven homogeneous parameters and the successive iterates belong to the *six* dimensional affine space (4.46) together with condition (4.27). The factorization scheme ((4.49), (4.50)) is not modified when condition (4.27) takes place. In contrast if one relaxes the spin reversal condition (4.15) (twenty homogeneous parameters) the factorization scheme (4.49) is reminiscent of the "string-like" factorizations (4.3). The affine subspace (4.56) is then of dimension greater than nine.

A more detailed analysis, with a particular emphasis on the "pre-Bethe ansatz" conditions (see [10] and (6.2) in the following) of this three-dimensional generalization of the six vertex model has been performed in [31].

5 Generalization to d-dimensional vertex models and monodromy matrices

One will here consider matrix (2.3) for an arbitrary m-dimensional space when there are only two spin states on direction (1), that is q=2.

Transposition t_1 amounts to permuting two off-diagonal $m \times m$ submatrices of this $2\,m \times 2\,m$ R-matrix [15]:

$$t_1 : \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \longrightarrow \begin{pmatrix} A & C \\ B & D \end{pmatrix} \tag{5.1}$$

where A, B, C and D are $m \times m$ matrices.

Such formalism can represent many different situations encountered in lattice statistical mechanics for vertex models. Namely, it can describe d-dimensional vertex models as well as the corresponding *monodromy matrices* (see section (6.1)) [33].

Let us remark that for d-dimensional vertex models, m is equal to 2^{d-1}. In this case one can consider transpositions t_2, t_3, ... t_{d-1} [8, 9] like t_1 associated with the $d-1$ other directions, and of course one obtains similar results for all the t_i's.

For arbitrary m (equal to 2^{d-1} or not), the analysis of the factorizations of the iterations of transformation K yields:

$$M_1 = K(M_0), \; f_1 = \det(M_0), \; f_2 = \frac{\det(M_1)}{f_1^{2\,(m-2)}}, \; M_2 = \frac{K(M_1)}{f_1^{2\,m-5}}, \; f_3 = \frac{\det(M_2)}{f_1^7 \cdot f_2^{2\,(m-2)}}, \; M_3 = \frac{K(M_2)}{f_1^5 \cdot f_2^{2\,m-5}},$$
$$f_4 = \frac{\det(M_3)}{f_1^{4\,(m-2)} \cdot f_2^7 \cdot f_3^{2\,(m-2)}}, \; M_4 = \frac{K(M_3)}{f_1^{2\,(2\,m-5)} \cdot f_2^5 \cdot f_3^{2\,m-5}}, \; f_5 = \frac{\det(M_4)}{f_1^8 \cdot f_2^{4\,(m-2)} \cdot f_3^7 \cdot f_4^{2\,(m-2)}} \cdots \tag{5.2}$$

[15] This problem exactly corresponds to the one considered by Korepanov in [32].

and, for arbitrary n, the following "string-like" factorizations:

$$K(M_n) \;=\; M_{n+1} \cdot f_n^{2m-5} \cdot f_{n-1}^5 \cdot f_{n-2}^{2(2m-5)} \cdot f_{n-3}^8 \cdot f_{n-4}^{2(2m-5)} \cdot f_{n-5}^6 \, \cdots \tag{5.3}$$

$$det(M_n) \;=\; f_{n+1} \cdot f_n^{2(m-2)} \cdot f_{n-1}^7 \cdot f_{n-2}^{4(m-2)} \cdot f_{n-3}^8 \cdot f_{n-4}^{4(m-2)} \cdot f_{n-5}^8 \cdot f_{n-6}^{4(m-2)} \, \cdots \tag{5.4}$$

Equations (5.3) and (5.4) yield the following relation *independent of m* :

$$\widehat{K}(M_n) \;=\; \frac{K(M_n)}{det(M_n)} \;=\; \frac{M_{n+1}}{\left(f_1 \cdot f_2 \cdots f_{n-1}\right)^2 \cdot f_n \cdot f_{n+1}} \tag{5.5}$$

Equation (5.5) gives again a generalization of equation (4.6) for arbitrary m:

$$(1+x) \cdot \alpha(x) \;-\; \frac{2m\,(1+x^2)}{1-x} \cdot \beta(x) \;-\; 2m \;=\; 0 \tag{5.6}$$

From (5.3) and (5.4) and also from (3.3) which is indeed valid, one gets:

$$\alpha(x) \;=\; 2m\, \frac{(1-x)^4 + 2m\,x\,(1+x^2)}{(1+x)\,(1-x)^4}, \qquad \beta(x) \;=\; \frac{2m\,x}{(1-x)^3}, \tag{5.7}$$

$$\mu(x) \;=\; \frac{x\,(2m-4+3x-x^2)}{(1-x)^3}, \qquad \nu(x) \;=\; \frac{x\,\left((2m-5)\,(1+x^2)+5x+x^3\right)}{(1-x)^4\,(1+x)} \tag{5.8}$$

Let us underline that, for $m = 4$, one recovers (4.7) and (4.8) taking the $m = 4$ limit of expressions (5.7). One also recovers factorizations (4.3) and (4.4) taking the $m = 4$ limit of factorizations (5.3) and (5.4).

For this kind of permutations and associated transformations K one can also (similarly to what has been done in section (3.1)) consider, for a given initial matrix M_0, the successive iterates of M_0 under transformation K^2 and see what is the dimension r of the *affine projective space* where these successive matrices live on:

$$K^{2\,n}(M_0) \;=\; a_0^{(n)} \cdot M_0 \;+\; a_1^{(n)} \cdot M_2 \;+ \cdots +\; a_r^{(n)} \cdot M_{2\,r} \tag{5.9}$$

For $m = 3$ (6×6 matrices), performing iterations up to K^{28}, we have been able to show that $r \geq 11$. It will be shown, in the next section, that one can find an upper bound for r considering "gauge-like" symmetries of this problem (see next section (6)).

5.1 Some comments on the generating functions: from vertex to spin models

For all the birational transformations described here, one remarks that one always has the three following factorization relations:

$$\det(M_n) \;=\; f_{n+1} \cdot f_n^{\zeta_1} \cdot f_{n-1}^{\zeta_2} \cdot f_{n-2}^{\zeta_3} \cdot f_{n-3}^{\zeta_4} \cdot f_{n-4}^{\zeta_5} \cdots f_1^{\zeta_n} \tag{5.10}$$

$$K(M_n) \;=\; M_{n+1} \cdot f_n^{\eta_0} \cdot f_{n-1}^{\eta_1} \cdot f_{n-2}^{\eta_2} \cdot f_{n-3}^{\eta_3} \cdot f_{n-4}^{\eta_4} \cdots f_1^{\eta_{n-1}} \tag{5.11}$$

$$\widehat{K}(M_n) \;=\; \frac{K(M_n)}{\det(M_n)} \;=\; \frac{M_{n+1}}{f_{n+1} \cdot f_n^{\rho_1} \cdot f_{n-1}^{\rho_2} \cdot f_{n-2}^{\rho_3} \cdot f_{n-3}^{\rho_4} \cdots f_1^{\rho_n}} \tag{5.12}$$

Let us introduce a new generating function for the ζ_n's:

$$\zeta(x) \;=\; 1 + \zeta_1\,x + \zeta_2\,x^2 + \zeta_3\,x^3 + \cdots \tag{5.13}$$

With this new generating function, $\zeta(x)$, relation (5.10) simply reads:

$$x\,\alpha(x) \;=\; \zeta(x) \cdot \beta(x) \tag{5.14}$$

One can also introduce generating functions for the η_n's and ρ_n's:

$$\eta(x) \;=\; \eta_0 + \eta_1\,x + \eta_2\,x^2 + \eta_3\,x^3 + \cdots \tag{5.15}$$

$$\rho(x) \;=\; 1 + \rho_1\,x + \rho_2\,x^2 + \rho_3\,x^3 + \cdots \tag{5.16}$$

One gets, from relation (5.12), the following relation between $\alpha(x)$, $\beta(x)$ and $\rho(x)$:

$$N \;+\; N\,\rho(x) \cdot \beta(x) \;=\; (1+x) \cdot \alpha(x) \tag{5.17}$$

which generalizes equations (5.6) for an arbitrary $N \times N$ matrix.

Many more relations can be obtained between these various generating functions ($\alpha(x)$, $\beta(x)$, $\mu(x)$ and $\nu(x)$ ...) (see [1, 2]).

Among these more or less involved generating functions, it appears that two generating functions are especially simple namely $\beta(x)$ and particularly $\rho(x)$. Let us give here the explicit expressions of $\rho(x)$ for various vertex models considered in this paper. For t_I for 4×4, as well as $q^2 \times q^2$, matrices (see section (3.1) and section (3.2)) the expression of $\rho(x)$ is:

$$\rho(x) \;=\; 1 + x^2 \tag{5.18}$$

while for t_I for $2^d \times 2^d$ (or $2m \times 2m$) matrices (see sections (4) and (5) and equation (5.6)) $\rho(x)$ is given by:

$$\rho(x) \;=\; \frac{1+x^2}{1-x} \tag{5.19}$$

These two kinds of generalizations of transformation t_I for arbitrary size of the matrices, are of a quite different nature. In particular the size dependence of the generating functions (in particular $\beta(x)$) is quite different. It is simpler for the generalizations described in sections (4) and (5) (compare for instance the expression of $\beta(x)$ in (5.7) and in (3.17)). Note however that $\rho(x)$ is remarkably simple for both kinds of size-generalizations, since it has zeros or poles *only on the unit circle and that it is actually independent of the matrix size.*

The polynomial, or exponential, growth of the calculations of the iterations is made clear on the singularities of the other generating functions $\alpha(x)$, $\beta(x)$, ... or even on the generating functions $\eta(x)$ or $\zeta(x)$. This provides a condition for the *polynomial growth* of the calculations, which can therefore be checked quickly from relations (5.10), (5.12).

Let us finally make the following remark: the *polynomial growth* of the calculations corresponds to *the occurrence of* only *Nth root of unity* in the denominators of the rational functions $(\alpha(x), \beta(x), \cdots)$. In most of the examples we have introduced [1, 2, 3] one gets, most of the time, $x = \pm 1$ singularities. At the present moment we have obtained very few Nth root of unity different from $x = \pm 1$. One example corresponds to an integrable subcase of a birational transformation (denoted class IV in [3]) which yields, when restricted to this integrable subcase [2]:

$$\beta(x) \;=\; \frac{4\,x}{(1-x)\,(1-x^2)\,(1-x^3)} \tag{5.20}$$

Another interesting example correspond to a (*six-state chiral*) edge spin model [5, 6] for which a foliation in terms of elliptic functions exists [5]. The analysis developed here, or in [1, 2, 3] for vertex models, has to be slightly modified [12] when considering edge spin models or IRF models. However it is worth noticing that one gets, for these integrable birational mappings, a generating function for the growth of the calculations where *third and fourth roots of unity occur* [14]:

$$G(x) \;=\; \frac{(1+x+2x^2+x^3+2x^4)\,(1+2x+2x^2+2x^3)}{(1-x)\,(1-x^3)\,(1-x^4)} \tag{5.21}$$

The growth of these coefficients, that is the growth of the degree of the successive iterations, is dominated by the coefficients of the expansion of:

$$G_{dom}(x) \;=\; \frac{49}{12\,(1-x)^3} \tag{5.22}$$

which grows like $49\,(n+1)\,(n+2)/24$. Another example is a *five-state* Potts model [6, 34] (symmetric and cyclic 5×5 matrices) which yields integrable birational mappings (a foliation of $\mathbb{C}P_2$ in algebraic elliptic curves) [6]. For this edge spin model the generating function for the growth of the calculations is given by [14][16]:

$$G_{Potts}(x) \;=\; \frac{(1+x+2\,x^2)^2}{(1-x)^2\,(1-x^3)} \tag{5.23}$$

The similarity with expression (5.21) is striking. The growth of these coefficients is dominated by the coefficients of the expansion of:

$$G^{Potts}_{dom}(x) \;=\; \frac{16}{3\,(1-x)^3} \tag{5.24}$$

Another interesting example of spin model is the q-state standard scalar Potts model on a triangular lattice with two and three site interactions introduced by Baxter, Temperley and Ashley [17]. Because of the three site interactions on the up-pointing triangles, this model is not an edge spin model. It can also be represented as a vertex model on a triangular lattice [17]. It has been shown that the symmetry group generated by the inversion relations yields birational representations of hyperbolic Coxeter groups [35]. Some of the generators of this group have been shown to yield algebraic elliptic curves and even rational curves [35]. Let us consider the factorizations of corresponding to the iteration of one of these generators which yields curves. The analysis of the polynomial growth of the degree of these iterations is sketched in Appendix B and leads to a quite simple generating function:

$$G_{BTA}(x) \;=\; \frac{1+2\,x^3}{(1-x)^3\,(1+x)} \tag{5.25}$$

This greater complexity of the generating function one encounters with edge spin models comes from the fact that the involution which plays the role of the transpositions t_1, t_2 ... for vertex models is a non-linear transformation (namely the Hadamard inverse [5, 6]) which amounts to taking the inverse of each entries of the matrix $R[i,j] \rightarrow 1/R[i,j]$). One cannot find, as simply as for vertex models, "Plücker-like" variables [1] of a reasonable degree that "linearize" the action of the matrix inversion I and of the other involution: the algebraic expressions covariant under the action of the action of the matrix inversion I and of the Hadamard inverse are of a higher degree [5, 6]. In fact it is always possible, after Kadanoff and Wegner [36], to map a spin edge model for which the edge Boltzmann weight interaction depends on the *difference* between nearest neighbor spins, onto a vertex model [36]. Introducing the edge Boltzmann weight interaction $W(\sigma_i - \sigma_j)$ (associated with the horizontal bonds) between two neighboring spins σ_i, σ_j and $\bar{W}(\sigma_k, \sigma_l)$ another edge Boltzmann weight (associated with the vertical bonds between two neighboring spins σ_k, σ_l), the two bonds $[\sigma_i$-$\sigma_j]$ $[\sigma_k$-$\sigma_l]$ being dual bonds, one can easily associate a vertex Boltzmann weight given by:

$$W_{vert}(i,j,k,l) \;=\; W(\sigma_i - \sigma_j)\cdot \bar{W}(\sigma_k - \sigma_l) \tag{5.26}$$

$$\text{with}\quad i = \sigma_i - \sigma_k\,,\; j = \sigma_k - \sigma_j\,,\; k = \sigma_j - \sigma_l\,,\; l = \sigma_l - \sigma_i \qquad \text{and therefore} \qquad i+j+k+l = 0$$

This transformation maps the edge spin model onto a vertex model, thus allowing to introduce linear involutions like t_1, t_2 ... However this "linearization" of the problem, multiplies by two the degree of all the algebraic expressions encountered.

[16] The birational transformations corresponding to these two examples of spin edge models [6, 34] can be "q-deformed", this deformation preserving the integrability (namely the foliation in elliptic curves of the parameter space) [14, 34]. It is worth noticing that these q-deformed birational transformations have the same generating function as (5.21) or (5.23).

6 Pre-Bethe-Ansatz and gauge transformations

Let us give here miscellaneous remarks concerning a key "factorization" relation closely related to the action of the birational transformations K, namely the pre-Bethe-Ansatz condition [10]. Let us first recall the results of [10] on the *sixteen vertex model* (which corresponds to $m = 2$ in the previous section). The weak-graph duality [20] symmetries correspond to a "gauge group" $G = sl_2 \times sl_2$ which acts *linearly* on R by similarity transformations (see [20] for details):

$$\text{If}\,, \qquad g = g_1 \times g_2 \qquad\qquad g(R) \;=\; g_1^{-1} g_2^{-1}\cdot R\cdot g_1\, g_2 \tag{6.1}$$

Let us denote B the group of birational transformations generated by I, t_1 and t_2. The action of G and B do not commute. However G and I do commute, and t_1 (resp t_2) sends orbits of G onto orbits of G. Noticeably a group, larger than the gauge group G, has naturally emerged in the analysis of the symmetries of the sixteen-vertex model: a group we have denoted G_{Bethe} [10]. Actually one of the keys to the Bethe Ansatz is the existence (see equations (B.10), (B.11a) in [37]) of vectors which are pure tensor products (of the form $v \otimes w$) and which R maps onto pure tensor product $v' \otimes w'$ (see also [10, 38, 39]). If

$$v = \begin{pmatrix} 1 \\ p \end{pmatrix}, \quad w = \begin{pmatrix} 1 \\ q \end{pmatrix}, \quad v' = \begin{pmatrix} 1 \\ p' \end{pmatrix}, \quad w' = \begin{pmatrix} 1 \\ q' \end{pmatrix}$$

then the solution of the "pre-Bethe-Ansatz" equation [10]:

$$R\,(v \otimes w) \;=\; \mu\, v' \otimes w' \tag{6.2}$$

verifies the two *biquadratic relations* [10]:

$$l_4 + l_{11}\,p - l_{12}\,p' + l_2\,p^2 + l_1\,p'^2 - (l_9 + l_{18})\,p\,p' - l_{13}\,p^2\,p' + l_{10}\,p\,p'^2 + l_3\,p^2\,p'^2 \;=\; 0 \tag{6.3}$$

$$l_7 + l_{16}\,q - l_{15}\,q' + l_8\,q^2 + l_5\,q'^2 - (l_9 - l_{18})\,q\,q' - l_{17}\,q^2\,q' + l_{14}\,q\,q'^2 + l_6\,q^2\,q'^2 \;=\; 0 \tag{6.4}$$

These two biquadratics *are elliptic curves*. Remarkably, when calculating the *modular invariant* [40] of these curves, one can actually see that these two curves actually reduce to *the same Weierstrass canonical form* [10, 41]:

$$y^2 \;=\; 4\,x^3 - g_2\,x \,-\, g_3 \tag{6.5}$$

A group $G_{Bethe} \simeq sl_2 \times sl_2 \times sl_2 \times sl_2$ naturally acts on (6.2): the four copies of sl_2 act respectively on v, w, v', w'. This induces a linear action on R:

$$R \;\longrightarrow\; g_{1L}^{-1}\cdot g_{2L}^{-1}\cdot R\cdot g_{1R}\cdot g_{2R} \tag{6.6}$$

We have also claimed [10] that our infinite order transformations K, or K_{t_2}, can both be represented as a *shift* of the (spectral) parameter enabling to move along these various elliptic curves: the two biquadratics (6.3), (6.4) and the elliptic curves generated by transformations K, or K_{t_2}, in $\mathbb{C}P_{15}$.

This situation can straightforwardly be generalized to $2m \times 2m$ matrices (section (5)), but now, directions 1 and direction 2 are not on the same footing anymore: vectors w and w' have m coordinates instead of two. Their elimination still yields a relation similar to (6.3) but now of a higher degree. The linear action (6.6) is changed into:

$$R \;\longrightarrow\; g_{2L}^{-1}\cdot R\cdot g_{2R} \tag{6.7}$$

Let us represent g_{2R} and g_{2L}^{-1} as $2m \times 2m$ matrices namely:

$$g_{2R} \;=\; \begin{pmatrix} G_{2R} & 0 \\ 0 & G_{2R} \end{pmatrix} \quad \text{and} \quad g_{2L}^{-1} \;=\; \begin{pmatrix} G_{2L}^{-1} & 0 \\ 0 & G_{2L}^{-1} \end{pmatrix} \tag{6.8}$$

where G_{2R} and G_{2L} are two $m \times m$ matrices.

Using notations (5.1) for the Boltzmann weight matrix, one can easily see that this elimination *yields the following determinantal relation between* p and p':

$$\det(A p' - C - D p + p p' B) \; = \; 0 \qquad (6.9)$$

This determinant is a polynomial of degree m *in each variable* p and p'. It is important to note that this determinant is covariant under the "gauge-like" transformations (6.8):

$$\det(A p' - C - D p + p p' B) \;\longrightarrow\; \det(G_{2R})^2 \cdot \det(G_{2L})^{-2} \cdot \det(A p' - C - D p + p p' B) \qquad (6.10)$$

The compatibility condition (6.9) is therefore invariant under the "gauge-like" transformations (6.7).

We have performed an analysis for the $m = 4$ case (more precisely for an 8×8 Boltzmann matrix corresponding to a three dimensional vertex model) getting *biquartic* relations [31]. Generally, for $2m \times 2m$ matrices, one gets relations of degree m *both in* p and p'. Curve (6.9), except for the remarkable $m = 2$ case (the sixteen-vertex model !) for which the *curve identifies with its Jacobian*, is a curve of genus greater than one. Generically it is a curve of genus[17] $g = (2m-2)(2m-1)/2 - 2 \cdot (m-1)m/2 = (m-1)^2$ and Korepanov after Krichever [32, 39] have claimed that the group of birational transformations we study can actually be represented as a *shift* on the Jacobian variety associated with curve (6.9). Transformation K *linearizes on the Jacobian variety*: transformation K corresponds to a constant shift on the torus. Transformation K amounts to adding a fixed element of the Albanese variety C^g/Γ.

Moreover it is straightforward to see that the two transformations g_{2R} and g_{2L}^{-1} are actually *symmetries* of transformation K. With notations (6.8) one gets immediately:

$$K\Big(g_{2L}^{-1} \cdot M \cdot g_{2R}\Big) \;=\; \det(g_{2R}) \cdot \det(g_{2L})^{-1} \cdot g_{2R}^{-1} \cdot K(M) \cdot g_{2L} \qquad (6.11)$$

$$K^2\Big(g_{2L}^{-1} \cdot M \cdot g_{2R}\Big) \;=\; \Big(\det(g_{2R}) \cdot \det(g_{2L})^{-1}\Big)^{2(m-1)} \cdot g_{2L}^{-1} \cdot K^2(M) \cdot g_{2R} \qquad (6.12)$$

and for arbitrary n:

$$K^{2n}\Big(g_{2L}^{-1} \cdot M \cdot g_{2R}\Big) = \Big(\det(g_{2R}) \cdot \det(g_{2L})^{-1}\Big)^{z_{2n}} \cdot g_{2L}^{-1} \cdot K^{2n}(M) \cdot g_{2R} \qquad (6.13)$$

$$= \Big(\det(G_{2R}) \cdot \det(G_{2L})^{-1}\Big)^{2 z_{2n}} \cdot g_{2L}^{-1} \cdot K^{2n}(M) \cdot g_{2R}$$

$$K^{2n+1}\Big(g_{2L}^{-1} \cdot M \cdot g_{2R}\Big) = \Big(\det(g_{2R}) \cdot \det(g_{2L})^{-1}\Big)^{z_{2n+1}} \cdot g_{2R}^{-1} \cdot K^{2n+1}(M) \cdot g_{2L} \qquad (6.14)$$

$$= \Big(\det(G_{2R}) \cdot \det(G_{2L})^{-1}\Big)^{2 z_{2n+1}} \cdot g_{2R}^{-1} \cdot K^{2n+1}(M) \cdot g_{2L}$$

with:

$$z_{2n} = \frac{(2m-1)^{2n} - 1}{2m}, \qquad z_{2n+1} = \frac{(2m-1)^{2n+1} + 1}{2m} \qquad (6.15)$$

For the inhomogeneous transformations $\widehat{K}$ one also gets:

$$\widehat{K}^{2n}\Big(g_{2L}^{-1} \cdot M \cdot g_{2R}\Big) = g_{2L}^{-1} \cdot \widehat{K}^{2n}(M) \cdot g_{2R}\,, \quad \widehat{K}^{2n+1}\Big(g_{2L}^{-1} \cdot M \cdot g_{2R}\Big) = g_{2R}^{-1} \cdot \widehat{K}^{2n+1}(M) \cdot g_{2L} \qquad (6.16)$$

This is a simple consequence of the relations corresponding to the two transformations I and t_1:

$$I\Big(g_{2L}^{-1} \cdot M \cdot g_{2R}\Big) = \det(g_{2R}) \cdot \det(g_{2L})^{-1} \cdot g_{2R}^{-1} \cdot I(M) \cdot g_{2L} = \det(G_{2R})^2 \cdot \det(G_{2L})^{-2} \cdot g_{2R}^{-1} \cdot I(M) \cdot g_{2L}$$

[17] A formula for getting the genus is for example Noether's formula obtained assuming that the curve of degree d has only ordinary multiple points. Since we have only n-uple points this yields: $g = (d-1)(d-2)/2 - N \cdot n(n-1)/2$, where N is the number of n-uple points [42, 43, 44]. We have here *two* n-uple points. To see this one can, for instance, write curve (6.9) in a homogeneous way, as the intersection of equations $\det(A p' - C t - D p + t' B) = 0$ and $p p' = t t'$.

and:

$$t_1\Big(g_{2L}^{-1} \cdot M \cdot g_{2R}\Big) \;=\; g_{2L}^{-1} \cdot t_1(M) \cdot g_{2R} \qquad (6.17)$$

If one imposes $\det(G_{2R}) = \det(G_{2L})$ one gets an invariance under the homogeneous transformations K^{2n}, but in fact one has, in general, a *covariance* property which *is actually a symmetry* closely linked to the homogeneity of the problem [1, 2]. The f_n's and $\det(M_n)$'s transform very simply under (6.7):

$$f_n \;\longrightarrow\; \Big(\det(g_{2R}) \cdot \det(g_{2L})^{-1}\Big)^{\beta_n/2m} \cdot f_n \;=\; \Big(\det(g_{2R}) \cdot \det(g_{2L})^{-1}\Big)^{n \cdot (n+1)/2} \cdot f_n \qquad (6.18)$$

$$\det(M_n) \;\longrightarrow\; \Big(\det(g_{2R}) \cdot \det(g_{2L})^{-1}\Big)^{\alpha_n/2m} \cdot \det(M_n) \qquad (6.19)$$

In the $m = 2$ case (sixteen vertex model see section (3.1)) the f_n's satisfy recurrences, like (3.12), yielding elliptic curves. These relations are actually *invariant* under symmetry (6.18) (see for instance [1, 2]). It is however important to note from (6.16), that *the inhomogeneous variables* $x_n = l_n \cdot l_{n+1}$, product of two consecutive l_n's ($l_n = \det(\widehat{K}^n(M_0))$), (and therefore recurrences (3.12)) *are actually invariant under* (6.7): variables x_n's actually "gauge-away" this quite large symmetry group (6.7).

To sum up, when considering the iterations of K^2 or $\widehat{K}^2$, one can, without any loose of generality, "gauge-away" the parameters corresponding to these (linear) transformations (6.8): one has two times $m^2 - 1$ inhomogeneous parameters corresponding to G_{2L} and G_{2R}. For $m = 3$ for instance, one can probably "gauge-away" $2(3^2 - 1) = 16$ parameters among the $6^2 - 1 = 35$ inhomogeneous parameters of the 6×6 matrices one considers. This would yield an upper bound for the dimension r of the affine space where the successive iterated matrices under the action of K^2 lie on (5.9). This dimension could not be larger than $35 - 16 = 19$.

Remark.1: Gauge groups like (6.7) can actually be introduced for birational transformations which are not related to vertex models [1, 3], for instance for transformation $K_t = t \cdot I$ associated with transposition t which permutes two entries: $R[1,2] \leftrightarrow R[2,1]$. Let us introduce the following $N \times N$ matrices written in terms of block matrices:

$$g_R = \begin{pmatrix} \mathcal{I} & 0 \\ 0 & G_R \end{pmatrix} \qquad \text{and} \qquad g_L^{-1} = \begin{pmatrix} \mathcal{I} & 0 \\ 0 & G_L^{-1} \end{pmatrix} \qquad (6.20)$$

where $\mathcal{I}$ is the 2×2 identity matrix, G_{2R} and G_{2L}^{-1} are $(N-2) \times (N-2)$ matrices and "0" denote the two rectangular $2 \times (N-2)$ and $(N-2) \times 2$ block matrices with zero entries. It is straightforward to verify that relations (6.16) are still valid for these very transformations, and that one can get relations similar to (6.13), (6.18) or (6.19). This is a simple consequence of the compatibility between (6.20) and symmetry t:

$$t\Big(g_R^{-1} \cdot R \cdot g_L^{-1}\Big) \;=\; g_R^{-1} \cdot t(R) \cdot g_L^{-1} \qquad (6.21)$$

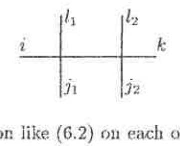
Remark.2: It has been underlined that the analysis of $2m \times 2m$ matrices can be seen as a preliminary study for the $2^d \times 2^d$ matrices corresponding d-dimensional problems (see sections (4) ...). One can similarly write down a d-dimensional "pre-Bethe ansatz" condition [10]:

$$R\,(v_1 \otimes v_2 \otimes \cdots \otimes v_d) \;=\; \mu\, v_1' \otimes v_2' \otimes \cdots \otimes v_d' \qquad (6.22)$$

The elimination of $2(d-1)$ vectors (for instance $v_2 \cdots v_d$ and $v_2' \cdots v_d'$) yields d algebraic curves like (6.9). Among the various d-dimensional R-matrices, the ones for which the genus of the previous d algebraic curves are *all equal* and smaller than $(m-1)^2 = (2^{d-1}-1)^2$ are of particular interest (accidental degeneracy of the genus of *one* of these d curves is easy to find but will yield quite pathological models breaking the symmetry between the d directions of the lattice).

6.1 Representation of the shift doubling

One can also underline that this analysis for $2m \times 2m$ matrices *can also be seen as a complementary study for the* N *sites monodromy matrices*[18] corresponding to two-dimensional vertex models. These monodromy matrices can be written as (5.1) where matrices A, B, C and D are now $2^N \times 2^N$ matrices. Let us just consider the two sites case ($N = 2$):



(6.23)

One can write a "pre-Bethe-Ansatz" equation like (6.2) on each of these two R-matrices and get, after elimination of the "vertical" vectors (w_1,w_2, w'_1,w'_2), two times the same biquadratic relation: namely relation (6.3) and the same relation but where p is replaced by p' and p' is replaced by p''. The resultant of these two biquadratics, which amounts to eliminating the variable p', is a *biquartic* relation between p and p''. Let us now consider a simple (Yang-Baxter integrable) example, namely the symmetric eight vertex Baxter model [45, 46, 47]. For this model the biquadratic (6.3) has a simple form:

$$Q_1 \cdot (1 + p^2 p'^2) \; + 2\, Q_2 \cdot p p' \; - Q_3 \cdot (p^2 + p'^2) \quad = \; 0 \qquad (6.24)$$

where the Q_i's are simple quadratic expressions (with the canonical notations for the Baxter model):

$$Q_1 \; = \; c\, d\,, \qquad Q_2 \; = \; \frac{a^2 + b^2 - c^2 - d^2}{2}\,, \qquad Q_3 \; = \; a\, b \qquad (6.25)$$

The elimination of p' in the two biquadratics of the type (6.24) yields a biquartic relation which *factorizes* as follows:

$$(p - p'')^2 \cdot \Big(Q'_1 \cdot (1 + p^2 p''^2) \; + 2\, Q'_2 \cdot p p'' \; - Q'_3 \cdot (p^2 + p''^2) \Big) \qquad (6.26)$$

This factorization is not surprising if one recalls the elliptic parameterization of (6.24). One has an elliptic parametrization for biquadratic (6.24), as well as for the elliptic curve in $\mathbb{C}P_3$ given by the intersection of the two quadrics (Clebsch's biquadratic) [26, 47, 40]:

$$\frac{Q_1}{Q_3} \; = \; \gamma \; = \; constant\,, \qquad \frac{Q_2}{Q_3} \; = \; \delta \; = \; constant \qquad (6.27)$$

One can actually parameterize p and p' as elliptic sinus [47]: $p = snu$ and $p' = sn(u \pm \lambda)$, where λ is nothing but *a shift on the elliptic curve*. It is straightforward to see that, if one also writes p'' as $sn(u'')$, the argument u'' of the elliptic sinus can be $u \pm 2\lambda$ or $u'' = u \pm \lambda \mp \lambda = u$. The fact that $(p - p'')^2$ factorizes comes from the fact that $u'' = u$ can be obtained in two ways[19] . As a byproduct one gets a representation of the shift doubling, $\lambda \rightarrow 2\lambda$, as a *polynomial transformation on the parameters of the biquadratic* (6.24):

$$(Q_1, Q_2, Q_3) \quad \longrightarrow \quad (Q'_1, Q'_2, Q'_3) \; = \qquad (6.28)$$
$$\Big(-4\, Q_1 \cdot Q_2^2 \cdot Q_3,\; (Q_3^2 - Q_1^2)^2 \; - 2\, Q_2^2 \cdot (Q_1^2 + Q_3^2)\,,\; -(Q_3^2 - Q_1^2)^2 \Big)$$

This polynomial transformation is *not birational*. It maps an elliptic curve of the parameter space $\mathbb{C}P_3$ onto *another* elliptic curve of $\mathbb{C}P_3$. Note that this polynomial representation of the shift doubling is *compatible with the weak-graph duality symmetries, as well as the two inversion relations on the model* or the action of K in the parameter space of the model. This will not be detailed here. It is also

[18] In this framework it is worth recalling the theory of the Quantum Inverse Scattering where such monodromy matrices play a key role, the Yang-Baxter equations yielding an equation relating two monodromy matrices: $RTT = TTR$ [33].

[19] This factorization of $(p - p'')^2$ does not occur for the general sixteen vertex model: in this general case the elimination of p' yields a quite involved (at first sight) biquadratic relation.

important to note that transformation (6.28) preserves the *modular invariant* of the elliptic curves, or more simply, the *modulus* k of the elliptic functions encountered [26, 40]. One has the following relations between k and the Q_i's [40, 47]:

$$k + \frac{1}{k} \; = \; \frac{Q_1^2 + Q_3^2 - Q_2^2}{Q_1 \cdot Q_3} \quad \text{or equivalently} \quad \left(\frac{k+1}{k-1}\right)^2 \; = \; \frac{(Q_1 + Q_3 + Q_2)\cdot(Q_1 + Q_3 - Q_2)}{(Q_1 - Q_3 + Q_2)\cdot(Q_1 - Q_3 - Q_2)} \qquad (6.29)$$

It is straightforward to see that expressions (6.29) *are actually invariant under the polynomial transformation* (6.28). This provides an interesting example of *polynomial, but not birational, transformation the iteration of which yields an algebraic (elliptic) curve* (6.29)[20]. This transformation underlines the existence of new kind of symmetries and transformations in the parameter space of the model. These *isogenies* should not be confused with the birational transformations K [40].

6.2 Further generalizations

Finally let us note that the "pre-Bethe-Ansatz" condition (6.2) can be generalized to $(n \cdot m) \times (n \cdot m)$ matrices. Again vectors w and w' have m coordinates instead of two, but, now, vectors v and v' have n components. Let us just write here the $n = 3$ case. Let us denote the components of vectors v and v' and the $(3 \cdot m) \times (3 \cdot m)$ Boltzmann matrix in terms of $m \times m$ matrices $A_1, \ldots A_9$, as follows:

$$v = \begin{pmatrix} 1 \\ p_1 \\ p_2 \end{pmatrix}, \qquad v' = \begin{pmatrix} 1 \\ p'_1 \\ p'_2 \end{pmatrix}, \qquad g_R \; = \; \begin{pmatrix} A_1 & A_2 & A_3 \\ A_4 & A_5 & A_6 \\ A_7 & A_8 & A_9 \end{pmatrix}$$

The elimination of vectors w and w' yields two determinantal conditions (instead of one for $n = 2$ and, generally, $n - 1$ conditions for arbitrary n):

$$\det((A_1 \; + A_2\, p_1 + \; A_3\, p_2) \cdot p'_2 - (A_7 \; + \; A_8\, p_1 \; + \; A_9\, p_2)) \; = \; 0 \qquad (6.30)$$

$$\det((A_1 \; + \; A_2\, p_1 \; + \; A_3\, p_2) \cdot p'_1 - (A_4 \; + \; A_5\, p_1 \; + \; A_6\, p_2)) \; = \; 0 \qquad (6.31)$$

Curve (6.9) is thus replaced, for $n = 3$, by an *algebraic surface* given by the two conditions (6.30) and (6.31), and, for arbitrary n, by an $(n-1)$-dimensional algebraic variety given by $n - 1$ "determinantal conditions" bearing on $2 \cdot (n-1)$ variables $p_1, \cdots, p_{n-1}$ and $p'_1, \cdots, p'_{n-1}$.

This simple remark enables to better understand *why the number of colors* 2 for the arrows of the vertex models *play such a special role for the occurrence of polynomial growth.*

7 Conclusion

We have used the methods introduced in [1, 2, 3] on various examples of vertex models of lattice statistical mechanics. In particular, we have analyzed the factorization properties of discrete symmetries of the parameter space of these lattice models, represented as birational transformations.

Different features have emerged from this study, namely the *polynomial growth of the complexity* of the iterations of these birational transformations, the existence of recurrences bearing on the factorized polynomials f_n, the existence of determinantal compatibility conditions like (6.9). All these properties have been shown to be related with the integrability or the "quasi-integrability" [10] of these lattice models of statistical mechanics.

For all the examples introduced in this paper, which correspond to matrices of arbitrary size, it has been shown that remarkable factorization relations *independent of the matrix size* occur (see for instance (3.14), (5.5)).

[20] This provides a representation of the (semi-group) $\mathbb{N}$, namely $\lambda \rightarrow 2^n \cdot \lambda$, which should not be confused with the birational representations of the group $\mathbb{Z}$ given in this paper and in [1, 5, 6, 8, 3].

The analysis of the factorizations corresponding to a specific two-dimensional vertex model has shown how the generic exponential growth of the calculations does reduce to a polynomial growth when the model becomes Yang-Baxter integrable. We hope that the search for *polynomial growth* of the complexity of the associated iterations will provide a new way to analyse *three or higher dimensional vertex models* [8, 9, 31], searching systematically for models where a Jacobian variety of an algebraic curve occurs[21]. It has been shown that the (determinantal) compatibility condition associated with the "pre-Bethe-Ansatz" (6.9) naturally yields *curves* of quite high genus *together with their associated Jacobian variety*: one could seek systematically for models (that is specific patterns of matrix Boltzmann weight) for which this genus becomes as small as possible. These compatibility conditions yield as many curves (6.9) as the dimension d of the lattice: one should concentrate on the models for which the d algebraic curves (6.9) are, as much as possible, on the same footing (same genus, same number of coefficients ...).

The examples of birational transformations associated with *vertex* models, detailed here, enable to clarify the occurrence of polynomial growth of the complexity of the iterations: in particular it has been shown that a polynomial growth, not only occur with algebraic elliptic curves [3, 12] but can also occur for transformations yielding *algebraic surfaces or even higher dimensional varieties*. In fact, it will be shown in forthcoming publications, that one can prove the polynomial growth of the calculations [48] when transformation K can be represented as a *shift* on a Jacobian variety C^n/Γ. Let us recall that Jacobian varieties of curves are particular abelian varieties depending only of $3g-3$ moduli among the $g\cdot(g+1)/2$ parameters[22] the abelian varieties depends on.

Conversely it is not clear to see if a polynomial growth necessarily implies the existence of an associated *Jacobian variety* (one can imagine a situation where abelian varieties which are not Jacobian varieties occur together with polynomial growth, or $K3$ surfaces together with polynomial growth ...). We will try in further publications to see if this polynomial growth is necessarily related to *abelian varieties*. We will also try to see to what extend *product* of elliptic curves is a situation favored in lattice statistical mechanics.

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[21] A particular interest may be devoted to the subcase of *hyperelliptic curves*: the (analytical) $3g-3$-dimensional space of moduli (Teichmüller space) has singularities corresponding to the hyperelliptic curves which only depend on $2g-1$ moduli.

[22] The period matrix of the theta functions of g variables has to be symmetric.

Appendix A

The X_n^H, Y_n and Z_n defined in section (3.4), read up to $n=5$:

$$X_1^H \;=\; b-1, \quad Y_1 \;=\; 2b+1, \quad Z_1 \;=\; b+1$$

$$X_2^H \;=\; 2b^2-b-1+bc+c, \quad Y_2 \;=\; 4b^2-2b-2-bc-c, \quad Z_2 \;=\; 2b^2-b-1-bc-c,$$

$$X_3^H \;=\; 4b^3+2b^2c-6b^2-bc^2-bc+2-c^2-c,$$

$$Y_3 \;=\; 4-3bc-12b^2+8b-2c+2b^3c+3b^2c-b^2c^2-3bc^2-2c^2+16b^4-16b^3,$$

$$Z_3 \;=\; 2-bc-6b^2+4b-c+2b^2c-bc^2-c^2+8b^4-8b^3,$$

$$X_4^H \;=\; 16b^5-32b^4+16b^4c+4b^3-12b^3c-18b^2c+20b^2-b^2c^3+b^2c^2-3bc^3-bc^2-4b+8bc-4+6c-2c^3,$$

$$\begin{aligned}Y_4 \;=\; & 16-34bc-96b^2+32b-16c+106b^3c+62b^2c+20b^2c^2-27bc^2-12c^2+272b^4-128b^3\\ &+8c^3+96b^5-320b^6-110b^4c+57b^3c^2+19bc^3+72b^6c-32b^5c^2-80b^5c-8b^4c^3-6b^4c^2\\ &-19b^3c^3+2b^3c^4+8b^2c^4+10bc^4+4c^4+128b^7,\end{aligned}$$

$$\begin{aligned}Z_4 \;=\; & 8-16bc-48b^2+16b-8c+48b^3c+32b^2c+10b^2c^2-14bc^2-6c^2+136b^4\\ &-64b^3+4c^3+48b^5-160b^6-56b^4c+30b^3c^2+9bc^3+32b^6c-16b^5c^2\\ &-32b^5c-4b^4c^3-4b^4c^2-9b^3c^3+b^3c^4+4b^2c^4+5bc^4+2c^4+64b^7,\end{aligned}$$

$$\begin{aligned}X_5^H \;=\; & 48bc-16+128b^2+10bc^5-16b+32c+8b^2c^5+2b^3c^5-40b^5c^3-5b^4c^4+192b^7c\\ &+32b^6c^2+4c^5+128b^8-160b^3c-192b^2c+16b^2c^2-16bc^2-4c^2-400b^4\\ &+32b^3-20c^3+176b^5+416b^6+480b^4c+56b^3c^2+39b^2c^3-38bc^3-448b^6c-40b^5c^2+48b^5c\\ &-17b^4c^3-44b^4c^2+76b^3c^3-12b^3c^4+b^2c^4+12bc^4+4c^4-448b^7,\end{aligned}$$

$$\begin{aligned}Y_5 \;=\; & -137c^6b^2-130c^6b-32c^6+24576b^{10}-128+1032bc+1152b^2+350bc^5-384b+320c+200b^2c^5\\ &-562b^3c^5+1662b^5c^4-6886b^5c^3-794b^4c^4+32448b^8c-26424b^7c-782b^6c^3-696b^4c^5\\ &+5968b^7c^2+1888b^6c^2+96c^5+5120b^9-34560b^8-7368b^3c-2096b^2c+352b^2c^2\\ &-368bc^2-96c^2-5760b^4+3200b^3-272c^3-10368b^5+18816b^6+7776b^4c+2224b^3c^2+602b^2c^3-926bc^3\\ &-20784b^6c-5456b^5c^2+21048b^5c+28b^4c^3-704b^4c^2+4084b^3c^3-1546b^3c^4+27b^2c^4+490bc^4\\ &+128c^4+12672b^7+7744b^9c+5200b^7c^3-19968b^{10}c+488b^8c^3-3104b^8c^2+959b^6c^4+82b^5c^5+44b^3c^6\\ &+6272b^{11}c+1152b^{10}c^2-1856b^9c^2-312b^8c^4-18432b^{11}-16c^7+4096b^{12}-614b^7c^4-1536b^9c^3\\ &+130b^7c^5+400b^6c^5+15b^6c^6+86b^5c^6+154b^4c^6-4b^5c^7-28b^4c^7-76b^3c^7-100b^2c^7-64bc^7,\end{aligned}$$

$$Z_5 = -66\,c^6b^2 - 64\,c^6b - 16\,c^6 + 12288\,b^{10} - 64 + 512\,bc + 576\,b^2 + 176\,bc^5 - 192\,b + 160\,c + 102\,b^2c^5$$
$$-388\,b^6c^3 - 352\,b^4c^5 - 282\,b^3c^5 + 817\,b^5c^4 - 3456\,b^5c^3 - 361\,b^4c^4 + 16128\,b^8c - 12864\,b^7c$$
$$+2688\,b^7c^2 + 1056\,b^6c^2 + 48\,c^5 + 2560\,b^9 - 17280\,b^8 - 14\,b^4c^7 - 38\,b^3c^7 - 50\,b^2c^7 - 32\,bc^7$$
$$-3648\,b^3c - 1056\,b^2c + 192\,b^2c^2 - 176\,bc^2 - 48\,c^2 - 2880\,b^4 + 1600\,b^3 - 136\,c^3 - 5184\,b^5 + 9408\,b^6$$
$$+3936\,b^4c + 1056\,b^3c^2 + 300\,b^2c^3 - 464\,bc^3 - 10464\,b^6c - 2544\,b^5c^2 + 10368\,b^5c + 16\,b^4c^3 - 432\,b^4c^2$$
$$+2048\,b^3c^3 - 759\,b^3c^4 + 2\,b^2c^4 + 240\,bc^4 + 64\,c^4 + 6336\,b^7 + 3584\,b^9c + 2608\,b^7c^3 - 9728\,b^{10}c + 240\,b^8c^3$$
$$-1536\,b^8c^2 + 443\,b^6c^4 + 40\,b^5c^5 + 23\,b^3c^6 + 3072\,b^{11}c + 512\,b^{10}c^2 - 768\,b^9c^2 - 144\,b^8c^4 - 9216\,b^{11}$$
$$-8\,c^7 + 2048\,b^{12} - 302\,b^7c^4 - 768\,b^9c^3 + 66\,b^7c^5 + 202\,b^6c^5 + 7\,b^6c^6 + 41\,b^5c^6 + 75\,b^4c^6 - 2\,b^5c^7$$

Appendix B

Let us sketch here the analysis of the growth of the complexity of the iterations for the two and three site interaction q-state standard scalar Potts model on the triangular lattice [17, 35, 49]. One can introduce a $q \times q$ matrix Boltzmann weight for this model [35, 49]. One inversion relation, transformation I_h, is the (homogeneous) matrix inversion, while other symmetries, playing the role of transformations t_i in this paper, are permutations of the entries of this $q \times q$ Boltzmann matrix [35]. Let us consider an infinite order (homogeneous) birational transformation, we denote K, obtained from I_h and one of these permutations (this transformation is transformation $p_{12}\,I_h$ in [35]). Similarly to the situation encountered with the three-dimensional generalization of the six vertex model (see section (4.5)), the determinant is replaced by two of its factors we denote P_1 and P_2. One has the following factorizations:

$$K(M_{n+4}) \; = \; c_n\, c_{n+1}\, d_{n+1} \cdot M_{n+5} \qquad (7.1)$$

where the c_n's and d_n's are (homogeneous) factorizing polynomials, and:

$$P_1(M_n) \; = \; c_n\, c_{n-3}\, d_{n-3} \qquad (7.2)$$

$$P_2(M_n) \; = \; c_{n-1}\, c_{n-3}\, c_{n-4}^2\, d_n\, d_{n-3} \qquad (7.3)$$

where P_1 and P_2 are polynomials, respectively of degree 1 and 2 in the entries of matrix M_n. These two polynomials are in fact the two prime factors of the determinant of matrix M_n.

Introducing polynomials f_n's such that (7.1) reads:

$$K(M_n) \; = \; f_n \cdot M_{n+1} \qquad (7.4)$$

with: $f_{n+4} = c_n\, c_{n+1}\, d_{n+1}$.

The product of $P_1(M_n)$ and $P_2(M_n)$ reads:

$$P_1(M_n)\, P_2(M_n) \; = \; f_n^2 \cdot f_{n+3} \qquad (7.5)$$

Introducing α_n the degree of the *entries*[23] of M_n's and β_n the degree of the f_n's one gets from (7.4) and (7.5):

$$2\,\alpha_n \; = \; \beta_n \, + \, \alpha_{n+1}\,, \qquad (7.6)$$

$$3\,\alpha_n \; = \; 2\,\beta_n \, + \, \beta_{n+3} \qquad (7.7)$$

[23] Instead of the degree of the determinant of the M_n's, in most part of this paper.

The elimination of the β_n's yields the recurrence:

$$\alpha_{n+4} \; - 2\,\alpha_{n+3} + \, 2\,\alpha_{n+1} \; - \; \alpha_n \; = \; 0 \qquad (7.8)$$

or equivalently on the corresponding generating function $\alpha(x)$, the relation:

$$\alpha(x) \cdot (1 \, - \, 2\,x + \, 2\,x^3 \; - \; x^4) \; = \; \alpha(x) \cdot (1-x)^3 \cdot (1+x) \; = \; P(x) \qquad (7.9)$$

where $P(x)$ is a polynomial of degree 3. The first coefficients of $\alpha(x)$ read:

$$\alpha(x) \; = \; 1 \, + \, 2\,x + 4\,x^2 \, + 8\,x^3 \, + \cdots \qquad (7.10)$$

From these first coefficients one gets the expression of $P(x)$:

$$P(x) \; = \; 1 \, + 2\,x^3 \qquad (7.11)$$

Relations (7.11) and (7.9) yield the exact expression of the generating function $\alpha(x) = G_{DTA}(x)$ (5.25).

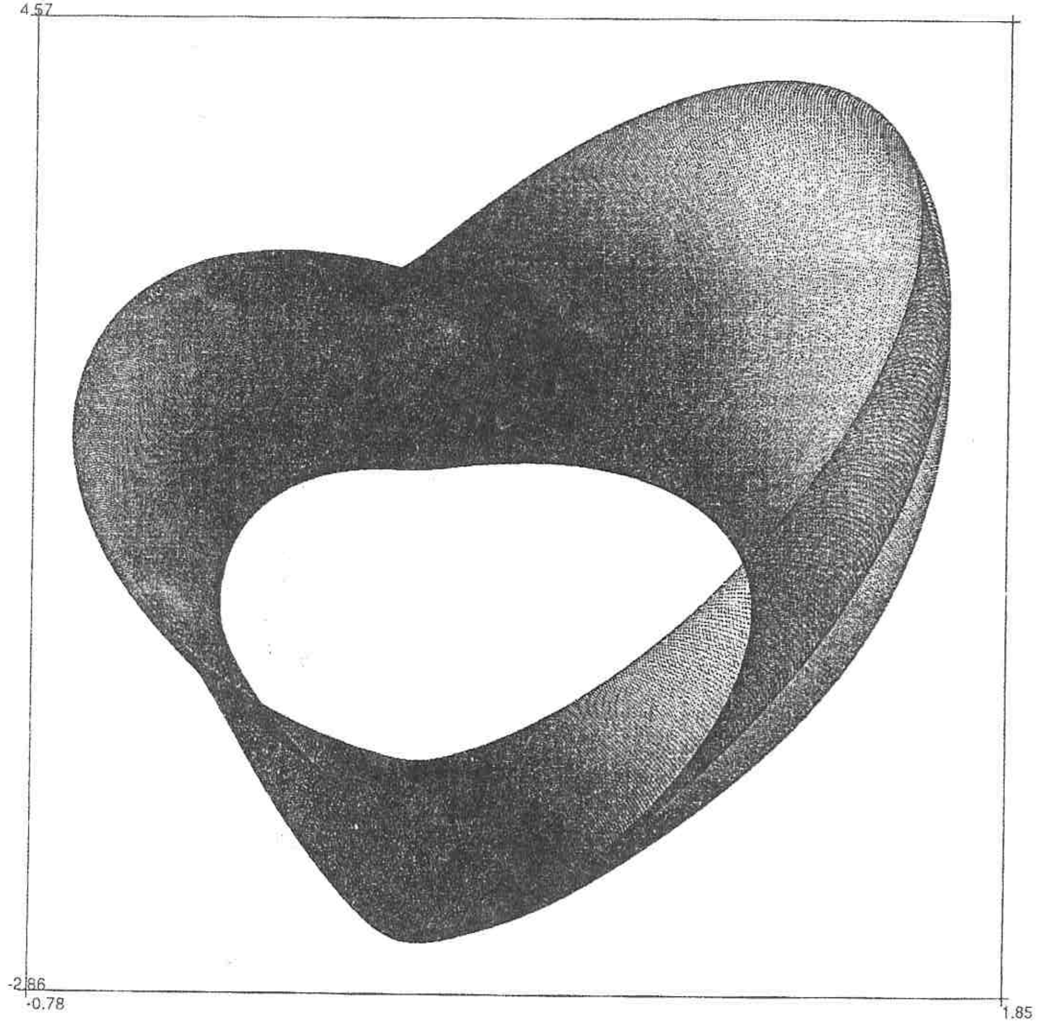

Figure 1a

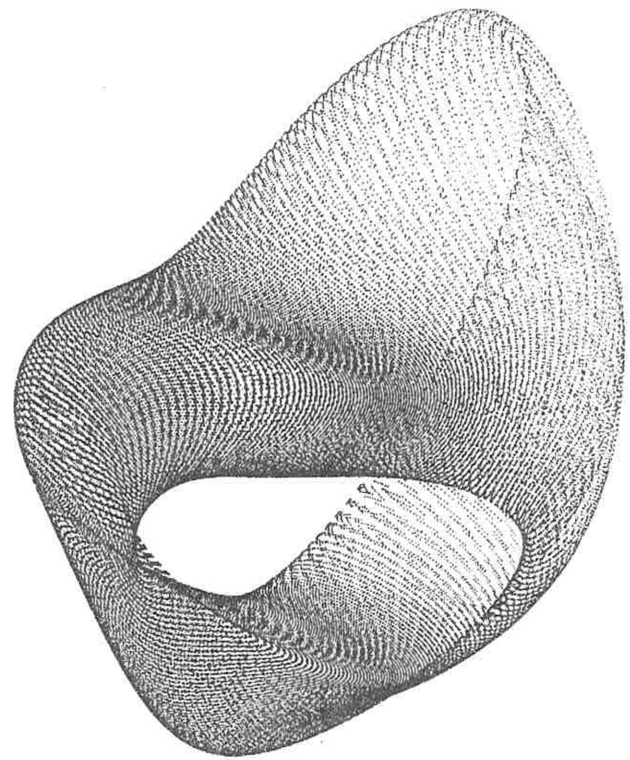

Figure 1b

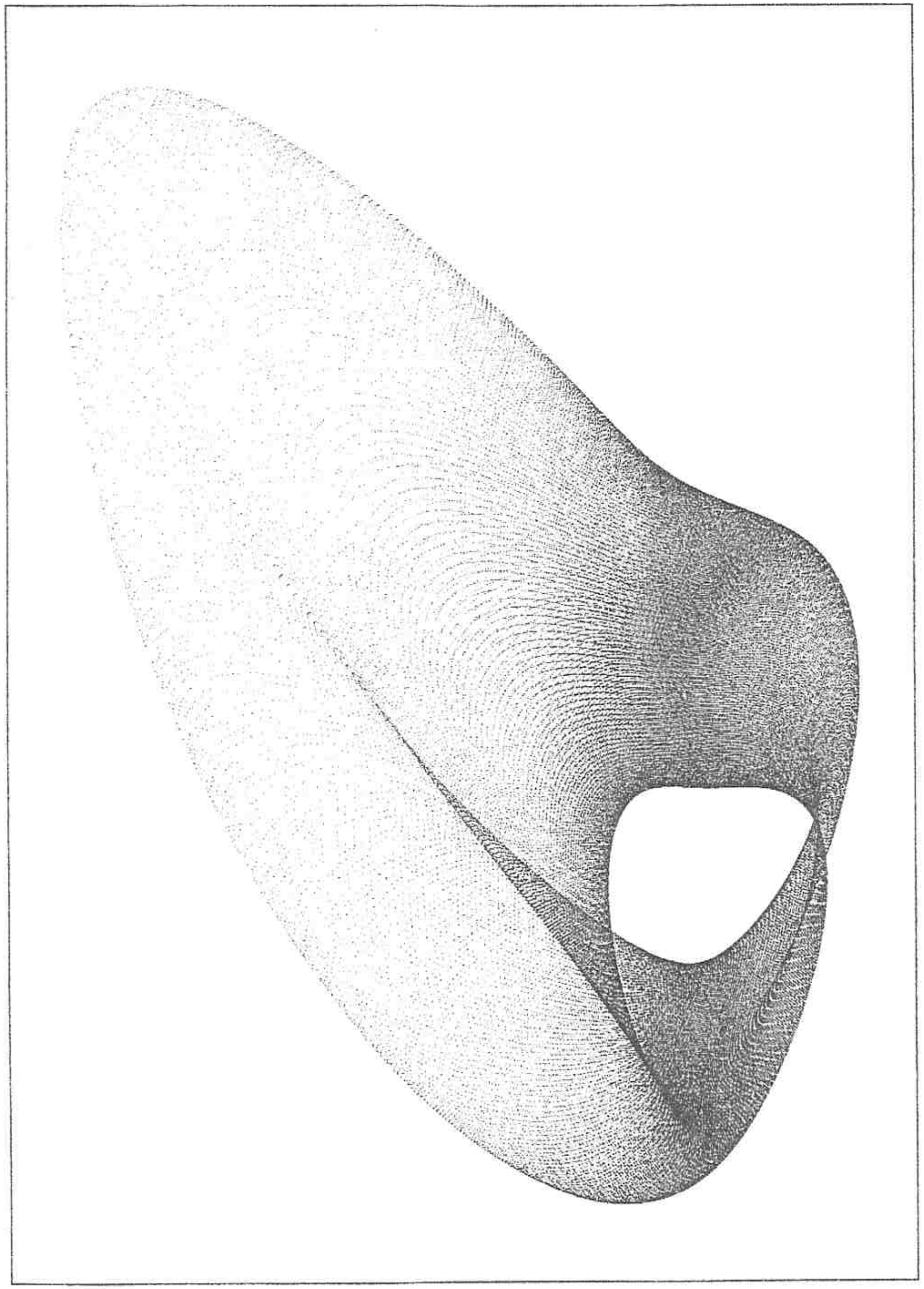

Figure 1c

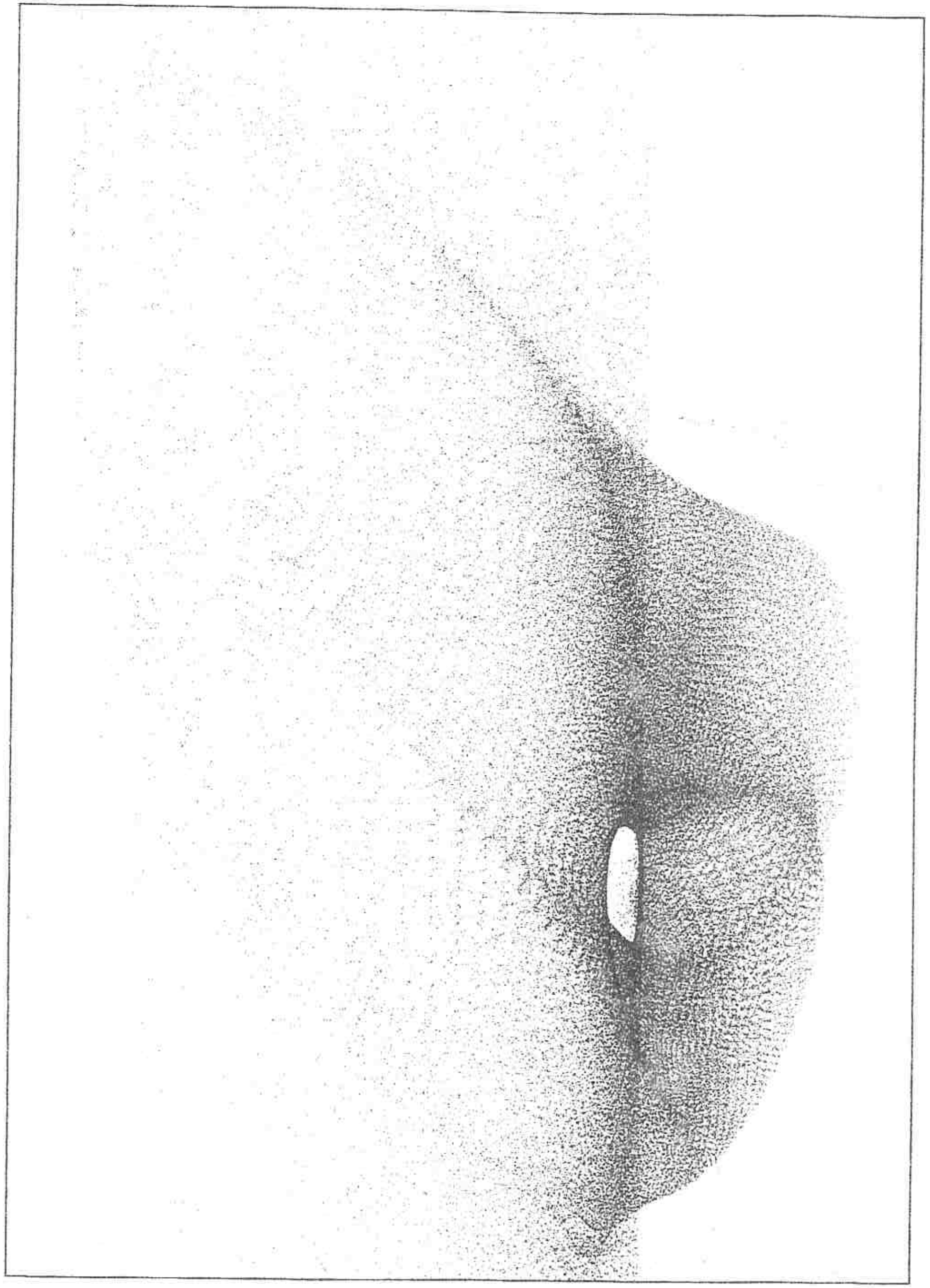

Figure 2a

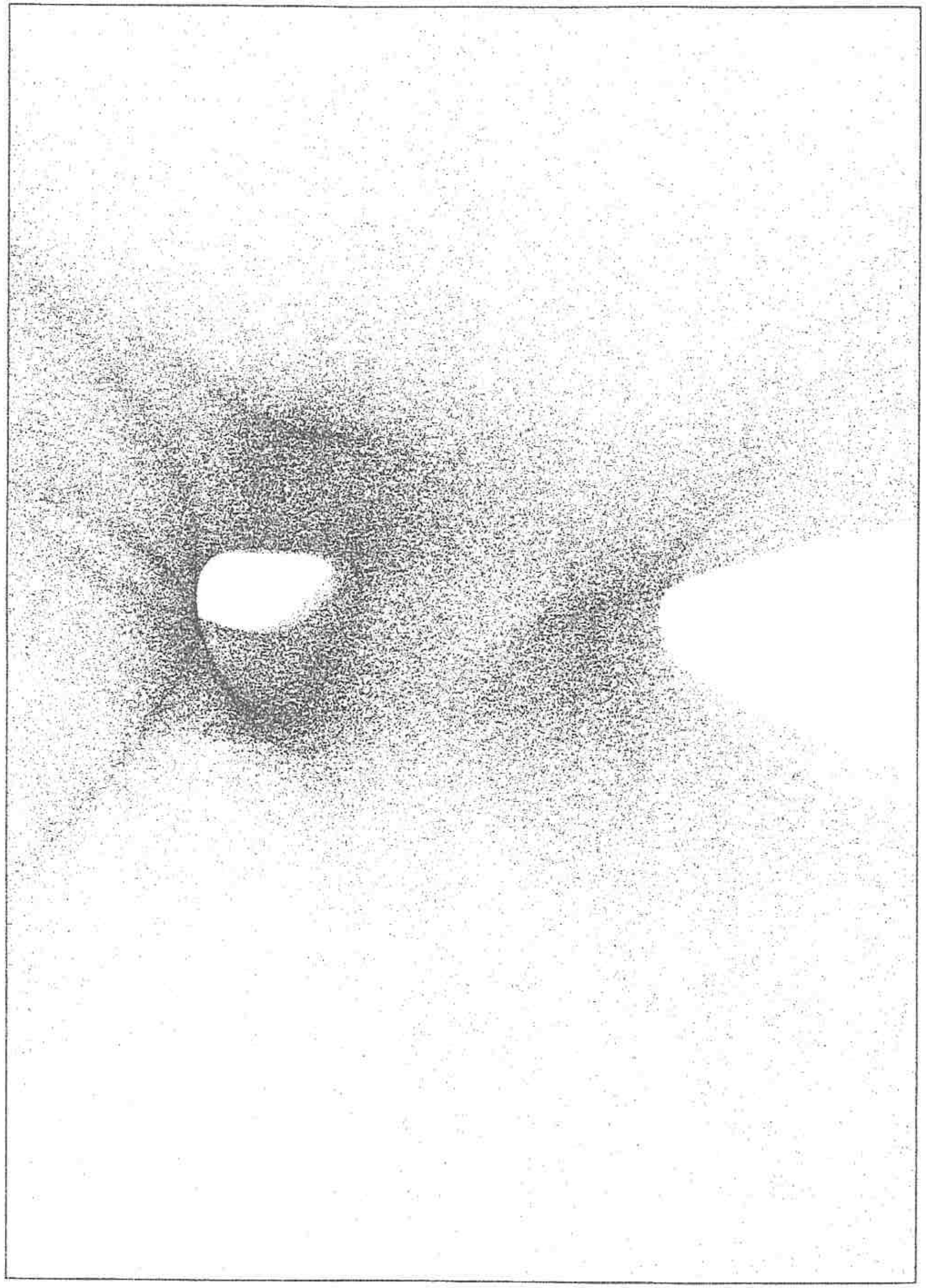

Figure 2b

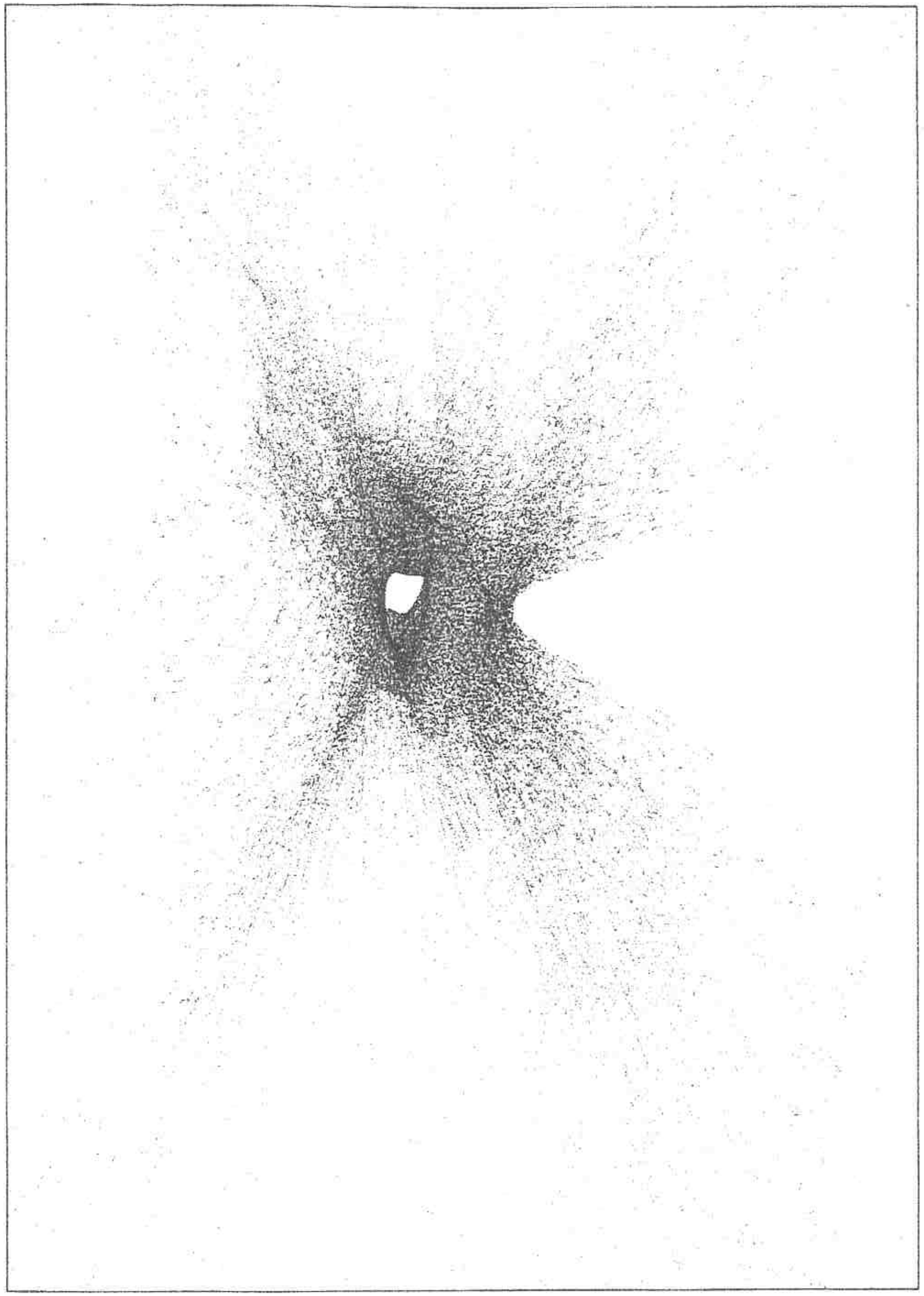

Figure 2c

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