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ALGEBRAIC VARIETIES FOR THE CHIRAL POTTS MODEL*

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We describe the symmetries of the chiral checkerboard Potts model (duality, inversion relation, . . .) and write down the algebraic variety corresponding to the integrable case advocated by Baxter, Perk, Au-Yang. We examine some of its subvarieties, in different limits and for various lattices, with a special emphasis on $q = 3$. This yields for $q = 3$, a new algebraic variety where the standard scalar checkerboard Potts model is solvable. By a comparative analysis of the parametrization of the integrable four-state chiral Potts model and the one of the symmetric Ashkin-Teller model, we bring to light algebraic subvarieties for the q-state chiral Potts model which are invariant under the symmetries of the model. We recover in this manner the Fateev-Zamolodchikov points.

1. Introduction

In the early 70's a breakthrough was made in the field of two-dimensional statistical mechanics on lattices when one realized (see for instance Ref. 1) that it could be easier to solve at once an infinite number of models rather than an isolated one. The step taken was to handle a family of models such that for any two members of the family, their transfer matrices commute. This property (related to the integrability of the model) is meaningless for one isolated model. Subsequent works have shown how these conditions are deeply rooted in very rich algebraic structures (algebraic Bethe Ansatz,[2]. . .). The fruitful step taken was actually to consider the whole parameter space of the model and the symmetries acting on this space.

These symmetries may reflect the invariance of the partition function by all relabellings of the (dummy) summation variables or symmetries of the lattice, or by transformations in the parameter space (inversion relation, duality transformation, . . .). The art is to find appropriate families of models which are at the same time sufficiently large to permit a non-trivial action of the symmetry group and sufficiently small to be handled.

Considering the exact results obtained on the chiral Potts model[3–5] and recalling the exact results on the checkerboard standard scalar Potts model (see for instance Ref. 6), as well as the Ashkin-Teller model,[7] it is natural to analyze the checkerboard chiral Potts model, its symmetry group, and the algebraic

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varieties of the parameter space which are invariant under these symmetries. Indeed in this model, there seems to be a very good ratio of the number of variables in the parameter space to the number of constraints coming from the symmetries.

The solutions given in Ref. 4 for the q-state chiral Potts model are very interesting from various points of view:

First they exhibit remarkable structures (especially curves of genus greater than one) renewing completely the way one can think of an exactly solvable model.

Next, the chiral Potts model is not an academical model: the presence of chirality leads to a very rich phase diagram (incommensurate phases, Lifschitz points, . . . , see for instance Ref. 8).

It is a good place to understand the role played by the different symmetries (relabelling symmetries of the summation variables, duality which is no longer an involution but a transformation of order four, etc. . . , see for instance Ref. 9), to study the relations between these symmetries and the exact properties of the model (automorphisms of the algebraic curves parametrizing the models,[9] . . .), and to clarify the relations between *criticality, integrability, and self-duality.* Finally the checkerboard model enables one to recover different lattices as limits (triangular honeycomb, anisotropic square).

In this paper we propose for the three-state checkerboard chiral Potts model a codimension 1 algebraic variety which is invariant by all the symmetries of the model: it encompasses the known integrable subcases of the model (checkerboard standard scalar Potts model, and chiral anisotropic square Potts model) as well as a conjectured variety for the critical manifold of the $q = 3$ triangular standard scalar Potts model.[10]

We exhibit, for generic q, interesting algebraic varieties of codimension greater than one, which are invariant under the symmetries. These results rely on the careful analysis of the parametrization of the four-state chiral Potts model and the one of the symmetric Ashkin-Teller model.[7]

In the first section we recall some notations and results on the q-state chiral Potts model, the checkerboard standard scalar Potts model and the symmetric Ashkin-Teller model. In the second section we describe the symmetries of the model, write down, for the checkerboard chiral Potts model, the algebraic variety corresponding to the integrable case advocated in Ref. 4. We examine some of its subvarieties, in different limits and for various lattices, with a special emphasis on $q = 3$. In the third section we examine the symmetric Ashkin-Teller model seen as a subcase of the four-state chiral Potts model[4] and see how the corresponding parametrizations match together. We recover very naturally the Fateev-Zamolodchikov points.[11]

2. Some Notations

2.1. *The chiral q-state checkerboard Potts model*

We describe here the checkerboard q-state chiral Potts model and recall different results and notations.[12] The lattice is a square lattice, which we consider as a chessboard: to each black face of the board we associate a Boltzmann weight which is the product around the face of elementary weights w_i $(i = 1, \ldots, 4)$ associated to the bonds.

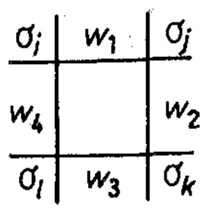

Fig. 1. Spin configuration around a face.

The weight of this configuration is

$$W(\text{face}) = w_1(\sigma_i - \sigma_j) \cdot w_2(\sigma_j - \sigma_k) \cdot w_3(\sigma_l - \sigma_k) \cdot w_4(\sigma_i - \sigma_l) \ .$$

The parameter space of the model is given by four sets of q homogeneous variables $w_i(0), \ldots, w_i(q - 1)$.

The standard scalar limit of the model is obtained if one takes:

$$w_i(1) = w_i(2) = \ldots = w_i(q - 1).$$

In this limit there is no chirality.

The anisotropic triangular, honeycomb, or square limits of the checkerboard chiral model are obtained respectively by setting for one value of the index i

$$w_i(1) = w_i(2) = \ldots = w_i(q - 1) = 0 \ ,$$

$$w_i(0) = w_i(1) = w_i(2) = \ldots = w_i(q - 1) \ ,$$

or

$$w_1(n) = w_3(n) = w(n) \ , \quad \text{and} \quad w_2(n) = w_4(n) = \overline{w}(n) \ .$$

In their study of the integrability of the anisotropic square q-state chiral Potts model, Baxter, Perk, and Au-Yang[4] have introduced a very adequate set of variables (two sets of rapidity vectors (a_p, b_p, c_p, d_p) and (a_q, b_q, c_q, d_q)). These four component vectors occur only through particular combinations

$$x_1 = b_q d_p \ , \quad x_2 = a_p c_q \ , \quad x_3 = b_p d_q \ , \quad x_4 = c_p a_q \ ,$$
$$x_5 = a_p d_q \ , \quad x_6 = d_p a_q \ , \quad x_7 = c_p b_q \ , \quad x_8 = b_p c_q \ . \tag{1}$$

They are given by the overdetermined system

$$w(n)x_1 - w(n)\omega^{n+1}x_2 - w(n+1)x_3 + w(n+1)\omega^{n+1}x_4 = 0 \tag{2}$$

$$\overline{w}(n)\omega x_5 - \overline{w}(n)\omega^{n+1}x_6 - \overline{w}(n+1)x_7 + \overline{w}(n+1)\omega^{n+1}x_8 = 0 \ , \tag{3}$$

where ω is a qth root of unity. As soon as $q > 3$, the system (2), (respectively (3)) is compatible only when the w's (and respectively the $\overline{w}$'s) are constrained to a certain subvariety V_x (resp. $\overline{V}_x$), obtained by the vanishing of determinants. Notice that the periodicity of w and $\overline{w}$, i.e. $w(n+q) = w(n)$ yields

$$x_1^q + x_4^q = x_2^q + x_3^q \quad \text{and} \quad x_5^q + x_8^q = x_6^q + x_7^q \ . \tag{4}$$

Moreover the form of the x_i's (Eq. (1)) yields two conditions on the homogeneous variables x_i:

$$x_2x_3 = x_5x_8 \quad \text{and} \quad x_1x_4 = x_6x_7$$

that is to say one equation

$$\frac{x_1x_4}{x_2x_3} \cdot \frac{x_5x_8}{x_6x_7} = 1 \ . \tag{5}$$

It is possible to write this equation as

$$F(w(0), w(1), \ldots, w(q-1)) \cdot F(\overline{w}(0), \overline{w}(1), \ldots, \overline{w}(q-1)) = \omega \tag{6}$$

where

$$F = \frac{x_1x_4}{x_2x_3} \ .$$

For the checkerboard model one may introduce linear systems similar to (2, 3), obtained by replacing w by w_1 (resp. w_3), and $\overline{w}$ by w_2 (resp. w_4). These systems are compatible only if each of the w_i's verify determinantal conditions i.e. belong to some subvariety which we denote by $V_x(i)$. The cases $q = 3$ and $q > 3$ are essentially different and should be distinguished.

For $q = 3$, the systems (2), (3) are always compatible, and has the following solution

$$x_1 = w(2)w(0)^2 + \omega w(1)w(2)^2 + \omega^2 w(0)w(1)^2 \ ,$$

$$x_2 = w(0)w(1)^2 + \omega w(1)w(2)^2 + \omega^2 w(2)w(0)^2 ,$$
$$x_3 = w(2)w(1)^2 + \omega w(1)w(0)^2 + \omega^2 w(0)w(2)^2 , \tag{7}$$
$$x_4 = w(0)w(2)^2 + \omega w(1)w(0)^2 + \omega^2 w(2)w(1)^2 .$$

Taking into account the value of the x_i's and the expression of F, Eq. (6) reads

$$P(w(0), w(1), w(2)) \cdot Q(\overline{w}(0), \overline{w}(1), \overline{w}(2)) + Q(w(0), w(1), w(2)) \cdot P(\overline{w}(0), \overline{w}(1), w(2)) = 0 \tag{8}$$

where

$$P(w(0), w(1), w(2)) = \left(\sum_{i=0}^{2} w(i)^3\right) w(0)w(1)w(2) - 3w(0)^2 w(1)^2 w(2)^2 \tag{9}$$

and

$$Q(w(0), w(1), w(2)) = \left(\sum_{i=0}^{2} w(i)^3\right) w(0)w(1)w(2) + 3w(0)^2 w(1)^2 w(2)^2 - 2\sum_{i\neq j} w(i)^3 w(j)^3 . \tag{10}$$

One has the relations

$$2x_1x_4 = Q - (\omega - \omega^2)P$$

and

$$2x_2x_3 = \omega Q + (\omega^2 - \omega^3)P .$$

In the limit of the standard scalar Potts model, condition (6) splits into two equations which are respectively the ferromagnetic[16] and antiferromagnetic criticality (and integrability) conditions[14]:

$$A \cdot B + A + B = 0 \tag{11}$$

$$A \cdot B - A - B = 2 \tag{12}$$

with $A = w(0)/w(1)$ and $B = \overline{w}(0)/\overline{w}(1)$.

For $q = 4$, the homogeneous system (2) is overdetermined. One has non-trivial solutions when the determinant of the system vanishes, namely:

$$[w(0)^2 + w(2)^2]w(1)w(3) + [w(3)^2 + w(1)^2]w(0)w(2) - 2w(0)^2w(2)^2 - 2w(1)^2w(3)^2 = 0 \,. \qquad (13)$$

Remark. An important subcase of the solvable chiral Potts model is the 'superintegrable' chiral Potts model.[17–19] It corresponds to the conditions

$$x_1 = x_7 \,, \quad x_2 = x_8 \,, \quad x_3 = x_5 \,, \quad x_4 = x_6 \,.$$

Equation (5) is automatically verified.

2.2. *The standard scalar checkerboard Potts model and its limits*

The standard scalar limit of the model is obtained when

$$w_i(1) = w_i(2) = \ldots = w_i(q - 1) = w_i \,.$$

If one denotes A, B, C, D the ratios $w_i(0)/w_i$ for $i = 1, 2, 3, 4$ respectively, the criticality and integrability algebraic variety reads:

$$ABCD - (AB + AC + AD + BC + BD + CD) - (q - 2)(A + B + C + D) - (q - 1)(q - 3) = 0 \,. \qquad (14)$$

In the limit of the anisotropic square Potts model, i.e. $A = C$, $B = D$, Eq. (14) also splits and we recover Eqs. (11), (12). The anisotropic square model may also be obtained in the limits $C = \infty$, $D = 1$, or $C = 1$, $D = \infty$, but then one recovers only Eq. (12).

In the triangular limit, i.e. $D = \infty$ two varieties emerge[10,16]:

$$\frac{(q - 2)ABC + AB + BC + CA - 1}{AB + BC + CA + (q - 2)(A + B + C) + (q - 2)^2 - 1} = \pm 1 \,. \qquad (15)$$

The case with the plus sign corresponds to the well-known criticality (and integrability) variety of the anisotropic triangular model,[13] and may be obtained as limit of Eq. (14)

The case with the minus sign was introduced by Martin and Maillard.[10] For the *isotropic* three-state model it gives three points remarkably well fitted by numerical studies of the zeroes of the partition function of the model.[10] For $q > 3$ no interesting property has yet been found for this case.

There is a parametrization of the standard scalar checkerboard model (or its limits) permitting the analysis of the symmetries of the parameter space, and the search for criticality or integrability[6,15]:

$$x_A = \frac{w_1(0) - q_+ w(1)}{w_1(0) - q_- w(1)} = \frac{A - q_+}{A - q_-} \qquad (16)$$

$q_\pm$ being the two roots of the equation

$$t^2 + (q - 2)t + 1 = 0 \ ,$$

and similarly for B, C, D.

Note that in the standard scalar limit, for $q = 4$, the determinantal condition (13) reduces to

$$w_i(0) - w_i = 0 \ .$$

For $q = 4$, the parametrization (16) becomes singular. In an appropriate limit one finds an additive parametrization instead of a multiplicative one, by setting

$$X_A = \frac{1}{1 + A} \ , \quad X_B = \ldots$$

Equation (14) for $q = 4$ then reads:

$$X_A + X_B + X_C + X_D = 1 \ .$$

2.3. *The symmetric Ashkin-Teller model*

The symmetric Ashkin-Teller model is an interesting subcase of the four-state chiral Potts model. To make this explicit, let us consider the four-state spin of the Potts model as two Ising spins σ_i and τ_i. The Boltzmann weight associated to the horizontal bond $\langle ij \rangle$ reads then, up to a multiplicative factor:

$$\left[\left(\frac{w(0)}{w(2)}\right)^{1/4}\right]^{\sigma_i \cdot \sigma_j \cdot \tau_i \cdot \tau_j + \tau_i \cdot \tau_j} \left[\left(\frac{w(0)w(2)}{w(1)w(3)}\right)^{1/4}\right]^{\sigma_i \cdot \sigma_j} \left[\left(\frac{w(3)}{w(1)}\right)^{1/4}\right]^{(\sigma_i - \sigma_j) \cdot \tau_i \cdot \tau_j} \tag{17}$$

This Boltzmann weight identifies with the one of the symmetric Ashkin-Teller model when $w(1) = w(3)$ (or $w(0) = w(2)$).

There is a nice parametrization[7] of the (anisotropic) model which makes transparent the action of the symmetries in the parameter space (see Sec. 2.4). It is of course very appropriate for the description of the exact solvability. Introduce the foliation of the parameter space of the anisotropic model:

$$\Delta = \frac{w(1)^2 - w(0)w(2)}{w(1)(w(0) - w(2))} \ , \qquad \overline{\Delta} = \frac{\overline{w}(1)^2 - \overline{w}(0)\overline{w}(2)}{\overline{w}(1)(\overline{w}(0) - w(2))} \ .$$

The good variables of the symmetric Ashkin-Teller model are x and $\overline{x}$ given by:

$$x = \frac{w(0) - w(2) - 2w(1)\,\tilde{q}_+}{w(0) - w(2) - 2w(1)\,\tilde{q}_-} \tag{18}$$

where $\tilde{q}_{\pm}$ are the two roots of

$$t^2 - 2\Delta t + 1 = 0 ,$$

and the same for $\overline{w}$. An analysis based on an extensive use of these variables permits to recover the algebraic varieties where the model is solvable.[7]

In the anisotropic case such varieties have equations:

$$\begin{aligned} \Delta &= \overline{\Delta} , \\ x^2\overline{x}^2 &= \tilde{q}_+^4 . \end{aligned} \tag{19}$$

We recognize in (19) both the known integrability condition $x\overline{x} = \tilde{q}_+^2$ and the candidate conjectured by Truong $x\overline{x} = -\tilde{q}_+^2$.[20]

For the isotropic symmetric Ashkin-Teller model, one recovers in particular the self-dual lines

$$w(0) = w(1) + w(2) + w(3) \quad \text{and} \quad w(1) = w(3) \tag{20}$$

that is to say

$$x = \overline{x} = -\tilde{q}_+ ,$$

and

$$-3w(0) = w(1) + w(2) + w(3) \quad \text{and} \quad w(1) = w(3) , \tag{21}$$

that is to say

$$x = \overline{x} = \tilde{q}_+ .$$

The symmetric Ashkin-Teller model can be mapped onto a staggered six-vertex model.[21] On the self-dual line (20) the staggering vanishes and the model is exactly solvable and equivalent to a six-vertex model. In this case there are remarkable values of Δ, for which the exponents become rational ($\Delta = \cos(k\pi/m)$, with k and m integers). Among these values of Δ, we emphasize $\Delta = -1$, $\Delta = -\frac{1}{2}$, and especially $\Delta = 0$ and $\Delta = -\frac{1}{\sqrt{2}}$, for which the model corresponds to a free fermion model.[22]

2.4. *The symmetries of the q-state chiral Potts model*

We recall the results of Ref. 12 and use the same notations. We denote by $\mathcal{G}$ the group of symmetries of the model. Generators of $\mathcal{G}$ are

- A cycle of order q, C: $w(n) \rightarrow w(n+1)$

- The spin reversal R: $w(n) \to w(q-n)$
- The duality transformation D: $w(n) \to \hat{w}(n) = \sum_{m=0}^{q-1} \omega^{nm} w(m)$
- The matrix inverse I: inversion of the (cyclic) matrix W of entries $W_{i,j} = w(i-j)$
- The dyadic inverse J: $w(n) \to 1/w(n)$
- The horizontal-vertical permutation S: $w(n) \leftrightarrow \overline{w}(n)$
- Another cycle of order q, Γ: $w(n) \to \omega^n w(n)$.

Notice that these generators are not independent and verify constraints:

$$I = DJD^{-1} , \qquad \Gamma = D^{-1}CD ,$$

$$R = D^2 , \qquad R^2 = 1 , \qquad JR = RJ .$$

Transformations I and J correspond to the inversion relation[23] acting on the horizontal and vertical weights W and $\overline{W}$ respectively. The group generated is an infinite discrete group of birational transformations in the parameter space, isomorphic to a semi-direct product of $\mathbb{Z}$ by a finite group.[12]

This symmetry group can be generalized straightforwardly to the checkerboard model.

This group of course admits various representations, obtained by restricting its action to subvarieties of the space of parameters.

If we restrict ourselves to the varieties $V_x \cap \overline{V}_x$ where the x_i's exist, the previous transformations read on the x_i's:

- C: $x_1 \to x_1$, $\quad x_2 \to \omega x_2$, $\quad x_3 \to x_3$, $\quad x_4 \to \omega x_4$
- D: $x_1 \to \omega x_2$, $\quad x_2 \to x_4$, $\quad x_3 \to x_1$, $\quad x_4 \to x_3$
- J: $x_1 \to x_3$, $\quad x_2 \to x_4$, $\quad x_3 \to x_1$, $\quad x_4 \to x_2$
- S: $x_1 \leftrightarrow \omega x_5$, $\quad x_2 \leftrightarrow x_6$, $\quad x_3 \leftrightarrow x_7$, $\quad x_4 \leftrightarrow x_8$.

Up to the multiplicative factors, the representation of the group $\mathcal{G}$ is by the group of symmetries of the square. This representation of the group of symmetry is *finite, and thus not faithful.*

If we do not restrict ourselves to $V_x \cap \overline{V}_x$ any longer, there are still two interesting limiting cases: the standard scalar Potts model and for $q=4$ the symmetric Ashkin-Teller model. In terms of the variables x_A and x previously introduced, the action of the group becomes multiplicative:

$$x_A \to q_+^{2n} x_A , \qquad x \to \tilde{q}_+^{4n} x , \ldots$$

and the equations of the solvability varieties take an especially nice form. For example Eq. (14) becomes[24]

$$x_A x_B x_C x_D = q_+^2$$

for the standard scalar checkerboard model or Eq. (19) for the symmetric Ashkin-Teller model. Of course Eq. (19) can be straightforwardly generalized to the checkerboard symmetric Ashkin-Teller model. One should notice that the relation between the homogeneous variables w and the various sets of 'good variables' x_i or x or x_A is in all cases a bilinear system.

Recall however the exception of the standard scalar Potts model at $q = 4$ (and at $q = 0$) for which the parametrization (16) becomes additive. In terms of the previously introduced variables X_A, the matrix inverse and the dyadic inverse read respectively:

$$I: X \to -X , \qquad J: X \to 1 - X .$$

For the symmetric Ashkin-Teller model this switch from multiplicative to additive parametrization corresponds to $\Delta = \pm 1$, for which the model reduces to the $q = 4$ standard scalar Potts model.

In the $q = 3$ case of the standard scalar Potts model, we even have an exact identification of the parametrizations x_i and x_A. Equation (7) yields the following values for the x_i's (up to a multiplicative factor):

$$\begin{aligned} x_1 &= (A-1)(A-\omega) \\ x_2 &= \omega^2(A-1)(A-\omega^2) \\ -x_3 &= \omega(A-1)(A-\omega^2) \\ -x_4 &= \omega(A-1)(A-\omega) , \end{aligned}$$

and we recover

$$x_A = \frac{x_4}{x_3} .$$

Note that:

$$q_+ = \omega \qquad \text{with} \qquad \omega^3 = 1 .$$

In the case $q > 3$ the relation between the different parametrizations is not so clear, because of the determinantal constraints like (13), and the infinite nature of the representation of the symmetry group. This question will be addressed in the next section.

Remark. Another transformation, which is known to play a role for the exactly solvable models has a nice representation in terms of the x_i's. It is the so-called shift operator (see Eq. (15) of Ref. 28, and Eq. (16) of Ref. 19).

It corresponds to the following transformation on one of the four vectors introduced above:

$$(a, b, c, d) \to (b, \omega a, d, c) \ .$$

which translates into simple transformations on the x_i's.

Other automorphisms acting on the rapidities have been considered in Ref. 4:

$$(a, b, c, d) \to (\omega a, b, \omega c, d) \quad \text{or} \quad (\omega a, b, c, d) \quad \text{or} \quad (\omega^{-1/2} c, d, a, \omega^{-1/2} b) \ .$$

Again these automorphisms translate into simple transformations on the x_i's.

2.5. *Miscellaneous results for the q-state checkerboard chiral Potts model*

Among the exact results on the chiral checkerboard Potts model, there exist the so-called disorder solutions[25] for which some dimensional reduction occurs. The different interesting physical quantities can be calculated exactly, when the parameters are restricted to a simple algebraic variety:

$$\hat{w}_1(n)\hat{w}_2(n)\hat{w}_3(n) = \lambda \cdot \sum_{p=0}^{q-1} \omega^{np} \frac{1}{w_4(p)} \tag{22}$$

for every $n = 0, \ldots, q-1$.

Another interesting result[26] gives some insight on the model for small chiralities. If we write

$$w(n) = w + \varepsilon_n \qquad \text{for } n \geq 1$$

one has the following equality for the partition function per site $\mathbb{Z}$:

$$\begin{aligned} &Z(w(0), w + \varepsilon_1, \ldots, w + \varepsilon_{q-1}, \overline{w}(0), \overline{w} + \bar{\varepsilon}_1, \ldots, \overline{w} + \bar{\varepsilon}_{q-1}) \\ &= G \cdot Z\left(w(0)\left[1 - \left(\frac{1}{q-1}\right)\sum_{n=1}^{q-1} \varepsilon_n\right], w, \ldots, w, \right. \\ &\left. \overline{w}(0)\left[1 - \left(\frac{1}{q-1}\right)\sum_{n=1}^{q-1} \bar{\varepsilon}_n\right], \overline{w}, \ldots, \overline{w} \right) \end{aligned} \tag{23}$$

with

$$G = \left[1 - \left(\frac{1}{q-1}\right)\sum_{n=1}^{q-1} \varepsilon_n\right]\left[1 - \left(\frac{1}{q-1}\right)\sum_{n=1}^{q-1} \bar{\varepsilon}_n\right] .$$

The existence of such a relation implies that critical points of the standard scalar Potts model are not isolated: there are critical points in their vicinity in the parameter space (weak chirality neighbourhood).

3. An Algebraic Variety for the Checkerboard Chiral Potts Model

The integrability of the checkerboard q-state chiral Potts model can be understood in a Z-invariant formulation of the model (see p. 141 of Ref. 4). The four Boltzmann weights of the checkerboard model w_1, w_2, w_3, w_4 yield an integrable model if they can be constructed from four sets of rapidities: $(a_{p_i}, b_{p_i}, c_{p_i}, d_{p_i})$ $(i = 1, 2)$ and $(a_{q_i}, b_{q_i}, c_{q_i}, d_{q_i})$ $(i = 1, 2)$. w_1 is then constructed from $p = p_1$, $q = q_1$ with the help of formulas (1) and (2). Then w_2 is constructed from (p_1, q_2), w_3 from (p_2, q_2) and w_4 from (p_2, q_1).

The quantity F of Eq. (6) then equals:

$$F = \frac{a_{q_i} b_{q_i} c_{p_i} d_{p_i}}{a_{p_i} b_{p_i} c_{q_i} d_{q_i}}$$

with $i = 1, 2$ for w_1 and w_3 respectively. For the perpendicular Boltzmann weights w_2 and w_4 we have, because of the extra ω factor occurring in (3), the relation:

$$F(w_4(0), \ldots, w_4(q - 1)) = \omega \frac{a_{p_2} b_{p_2} c_{q_1} d_{q_1}}{a_{q_1} b_{q_1} c_{p_2} d_{p_2}}$$

and the same equation for w_2 where $p_1 \leftrightarrow p_2$, $q_1 \leftrightarrow q_2$. Combining these equations together one gets:

$$F(w_1(0), w_1(1), \ldots, w_1(q - 1)) \cdot F(w_2(0), w_2(1), \ldots, w_2(q - 1)) \\ \times F(w_3(0), w_3(1), \ldots, w_3(q - 1)) \cdot F(w_4(0), w_4(1), \ldots, w_4(q - 1)) = \omega^2 . \tag{24}$$

The integrability subvariety is the intersection of $\cap_i V_x(i)$ with (24).

It is straightforward to check that this subvariety gives back the known integrability varieties previously mentioned for the anisotropic square chiral and the standard scalar checkerboard Potts model. As will be seen in Sec. 4, this is also true for the anisotropic symmetric Ashkin-Teller model (Eqs. (6), (14), (19)). It is also invariant by the symmetry group $\mathcal{G}$.

There are two other non-trivial integrable subcases of the checkerboard chiral Potts model. If w_1, w_2, w_3, w_4 verify:

$$F(w_1(0), w_1(1), \ldots, w_1(q - 1)) \cdot F(w_2(0), w_2(1), \ldots, w_2(q - 1)) = \omega \tag{25}$$

and

$$F(w_3(0), w_3(1), \ldots, w_3(q - 1)) \cdot F(w_4(0), w_4(1), \ldots, w_4(q - 1)) = \omega \tag{26}$$

then the diagonal transfer matrices

$$\mathcal{T}(w_1(0), w_1(1), \ldots, w_1(q - 1), w_2(0), w_2(1), \ldots, w_2(q - 1))$$

and $\mathcal{T}(w_3(0), w_3(1), \ldots, w_3(q-1), w_4(0), w_4(1), \ldots, w_4(q-1))$ commute. Consequently the subvariety of $\cap_i V_x(i)$ defined by (25) and (26) is also an integrability subvariety. Similar results hold for the partition $\{1, 3\}, \{2, 4\}$ of $\{1, 2, 3, 4\}$. These two subcases are clearly compatible with Eq. (24).

We will now examine the consequences of the existence of this variety for the standard scalar model for $q = 3$ (where we do not have the determinantal constraints like Eq. (13) for $q = 4$).

Denote P_i and Q_i the polynomials in $w_i(0)$, $w_i(1)$, $w_i(2)$ defined as in Eqs. (9) and (10). Equation (24) reads:

$$3(Q_1P_2P_3P_4 + Q_2P_1P_3P_4 + Q_3P_1P_2P_4 + Q_4P_1P_2P_3) \\ - (P_1Q_2Q_3Q_4 + P_2Q_1Q_3Q_4 + P_3Q_1Q_2Q_4 + P_4Q_1Q_2Q_3) = 0 . \qquad (27)$$

In the anisotropic square limit of the three-state chiral Potts model ($w_1(n) = w_3(n)$ and $w_2(n) = w_4(n)$), Eq. (27) becomes:

$$Q_1Q_2 - 3P_1P_2 = 0$$

together with Eq. (8):

$$P_1Q_2 + P_2Q_1 = 0 .$$

The honeycomb and triangular limits cannot be taken by brute force because Eqs. (1, 2) may not be fulfilled in the limit.

The honeycomb limit corresponds to

$$F(w(0), w(1), w(2)) = 1 ,$$

and the triangular limit to

$$F(w(0), w(1), w(2)) = \omega .$$

Equation (24) then becomes:

$$F(w_1(0), w_1(1), \ldots, w_1(q-1)) \cdot F(w_2(0), w_2(1), \ldots, w_2(q-1)) \\ \times F(w_3(0), w_3(1), \ldots, w_3(q-1)) = K \qquad (28)$$

where $K = \omega^2$ and ω for the honeycomb and triangular lattices respectively.

In terms of the polynomial P_i and Q_i, this reads

$$3(Q_1P_2P_3 + Q_2P_1P_3 + Q_3P_1P_2 - P_1P_2P_3) \\ - (P_1Q_2Q_3 + P_2Q_1Q_3 + P_3Q_1Q_2\, m - Q_1Q_2Q_3) = 0 \qquad (29)$$

for the triangular lattice, and

$$3(Q_1P_2P_3 + Q_2P_1P_3 + Q_3P_1P_2 - P_1P_2P_3) + (P_1Q_2Q_3 + P_2Q_1Q_3 + P_3Q_1Q_2 - Q_1Q_2Q_3) = 0 \qquad (30)$$

for the honeycomb lattice.

Notice that for the chiral isotropic three-state Potts model, Eqs. (29) and (30) are invariant under *all permutations of* $w(0)$, $w(1)$, $w(2)$ while for (29) one expects only a symmetry under the exchange of $w(1)$ with $w(2)$. This is the manifestation of an enhanced symmetry at exact solvability.

In the limit of the standard scalar model, Eq. (27) splits into two equations: Eq. (14) for $q = 3$, and a new equation:

$$ABCD + 2(ABC + ABD + ACD + BCD) + (AB + AC + BC + AD + BD + CD) - (A + B + C + D) - 2 = 0 \, . \qquad (31)$$

In the 'good' variables $x_A, \ldots$, this equation reads:

$$x_A x_B x_C x_D = -\omega^2 \qquad (32)$$

to be compared with Eq. (14) in the same variables:

$$x_A x_B x_C x_D = \omega^2 \, . \qquad (33)$$

Note that Eq. (32) represents an algebraic variety of the standard scalar checkerboard Potts model, invariant by the group $\mathcal{G}$.

In the triangular limit ($D \to \infty$), we recover the variety introduced by Martin and Maillard for $q = 3$ (see Eq. (15) with the minus sign):

$$ABC + 2(AB + BC + CA) + A + B + C - 1 = 0 \, . \qquad (34)$$

In the honeycomb limit ($D \to 1$), Eq. (31) gives a new algebraic variety (for $q = 3$):

$$ABC + (AB + AC + BC) - 1 = 0 \, .$$

In the anisotropic square limit ($A = C, B = D$), Eq. (31) gives another new algebraic variety (for $q = 3$):

$$A^2B^2 + 4(A^2B + AB^2) + (A^2 + B^2 + 4AB) - 2(A + B) - 2 = 0 \, .$$

Recalling the equivalence of the standard scalar Potts model with a staggered six-vertex model,[21] one can introduce 'fugacity' variables for these various lattices ($z = x_A x_B x_C x_D / q_+^2$ for the checkerboard lattice, $z = x_A x_B / (-q_+)$ for the anisotropic square lattice, $z = x_A x_B x_C / q_+^2$ for the triangular lattice, $z = x_A x_B x_C / (-q_+)$ for the honeycomb lattice).[29] Condition (14) or (33) corresponds to the case where it is possible to 'wash' the staggering field (i.e. $z = 1$) by gauge transformations. The model then reduces to an integrable six-vertex model. What we have seen for $q = 3$ is that $z = -1$ also corresponds to solvable cases for these various lattices, and even $z = \pm i$ for the anisotropic square lattice. Remark that an analogous phenomenon happens for the symmetric Ashkin-Teller model with the relation $x\bar{x} = -\tilde{q}_+^2$ of Ref. 20.

4. The Symmetric Ashkin-Teller Limit of the Chiral Potts Model

We have described the model and its parametrization in Sec. 2.3. We describe here how it is related to the one of the four-state chiral Potts model (Sec 2.1).

We may go from the $q = 4$ chiral Potts model to the symmetric Ashkin-Teller model by taking $w(1) = w(3)$.

What is remarkable is that the determinantal condition (13) becomes:

$$\Delta^2 = \frac{1}{2} .$$

For the self-dual symmetric Ashkin-Teller model, this condition is just *a free fermion condition.*

Moreover the overdetermined system (2) gives the following relation on the x_i's:

$$x_1 = \Omega\, x_4 \quad \text{and} \quad x_3 = \Omega\, x_2 \tag{35}$$

$$\text{where} \qquad \Omega^2 = \omega \tag{36}$$

and similar conditions on x_5, x_6, x_7, x_8. Notice that these equations are compatible with Eq. (4) and of course with the group action described in Sec. 2.4. The 'good' variable x of the symmetric Ashkin-Teller model is nothing but the ratio $-(x_1/x_3)^2$, and $\tilde{q}_+$ of Sec. (2.3) is a root of order eight of unity. We also have

$$F = \left(\frac{x_1}{x_3}\right)^2 . \tag{37}$$

The identification comes as follows: relation (35) used in the system (2) for $q = 4$, (keeping in mind $w(1) = w(3)$) yields two equations:

$$[w(0) - w(2) + 2\Omega w(1)]\, x_1 = \Omega[w(0) + w(2)]\, x_3$$

$$[w(0) - w(2) - 2\Omega w(1)]\, x_3 = -\Omega[w(0) + w(2)]\, x_1$$

leading to

$$-\left(\frac{x_1}{x_3}\right)^2 = \frac{[w(0) - w(2) - 2\Omega w(1)]}{[w(0) - w(2) + 2\Omega w(1)]},$$

to be compared with Eq. (18).

Equation (24) identifies with Eq. (19). In the isotropic case it reads:

$$x = \bar{x} = -(\pm\Omega)$$

which are Eqs. (20), (21), and one gets

$$\frac{w(0)}{w(1)} = \frac{\alpha^3 \pm \alpha^{-3}}{\alpha \pm \alpha^{-1}}, \qquad \frac{w(1)}{w(2)} = \frac{\alpha^7 \pm \alpha^{-7}}{\alpha^5 \pm \alpha^{-5}} \tag{38}$$

where

$$\alpha^{32} = 1 .$$

These results (35), (36), (37) hold for arbitrary q. In the anisotropic case, one gets an algebraic curve parametrized by

$$\frac{w(0)}{w(1)} = \frac{1 - \Omega x_1/x_3}{x_1/x_3 - \Omega}, \qquad \frac{w(1)}{w(2)} = \frac{1 - \Omega^3 x_1/x_3}{x_1/x_3 - \Omega^3} \cdots \tag{39}$$

where

$$\Omega^{2q} = 1 .$$

and similar expressions for $\bar{w}$.

In the isotropic case:

$$\left(\frac{x_1}{x_3}\right)^2 = \pm\Omega . \tag{40}$$

Introducing α such that $\alpha^{8q} = 1$, we will have the points

$$\frac{w(0)}{w(1)} = \frac{\alpha^3 \pm \alpha^{-3}}{\alpha \pm \alpha^{-1}}$$

$$\frac{w(1)}{w(2)} = \frac{\alpha^7 \pm \alpha^{-7}}{\alpha^5 \pm \alpha^{-5}}$$

$$\cdots$$

$$\frac{w(n)}{w(n+1)} = \frac{\alpha^{3+4n} \pm \alpha^{-(3+4n)}}{\alpha^{1+4n} \pm \alpha^{-(1+4n)}} . \tag{41}$$

$$\cdots$$

These results are exactly the points found in Ref. 11 for which an enhanced symmetry appears. These self-dual critical points can be described by conformal $\mathbb{Z}_q \times \mathbb{Z}_q$ theories.[11]

For $q = 5$ or $q = 7$ for example, these points are bifurcation (critical) points of the phase diagram.[30] On the other hand, the critical points of the solvable chiral Potts model were given in Ref. 4: they are obtained with

$$c = d \quad \text{and} \quad \frac{a^q + b^q}{c^q + d^q} = \frac{I}{2}$$

for all sets of rapidities. This critical variety should contain for $q \geq 3$ the Lifshitz point where the incommensurate phase begins. In terms of the x_i's it gives: $x_1 = x_7$, $x_2 = x_5$, $x_4 = x_6$, $x_3 = x_8$. In the case $I = 0$, one recovers Eqs. (35), (36).

In the $q = \infty$ limit one recovers from Eq. (39) the $U(1)$ invariant XY model with specific Boltzmann weights (up to multiplicative factors):

$$w_{\varphi,\varphi'} = \left| \sin \frac{(\varphi - \varphi')}{2} \right|^{1 - \theta/\pi}$$

where $x_1/x_3 = \exp(i\theta/q)$ (and similar equations with $\overline{w}$ and $\overline{\theta}$). Equation (6) gives

$$\theta + \overline{\theta} = \pi \,.$$

Remark 1. One sees that, here again, the genus greater than one parametrization degenerates into a rational one.

Remark 2. The Fateev-Zamolodchikov points are self-dual ($\sqrt{q}w(n) = \hat{w}(n)$), and one also has $w(n) = w(q - n)$.

Remark 3. For the Fateev-Zamolodchikov points, and the three-state standard scalar Potts model, one has

$$\frac{x_1}{x_4} = \frac{x_3}{x_2} = n\text{th root of unity.}$$

Remark 4. We have found algebraic subvarieties of V_x which are invariant by the action of the group $\mathcal{G}$, and compatible with Eqs. (4), (6) for arbitrary q. Conditions (4), (6) are actually quite restrictive, in particular if the subvarieties we look for are given by equalities between the x_i's, up to multiplicative factors.

Remark 5. Equation (14) is known to describe a subvariety where the model is *solvable and critical.* The status of Eq. (27) is less clear. It is a variety where the model is exactly solvable. For $q = 3$, the points given by the isotropic limit of Eq. (34) and (33) could very well be critical points, in view of the numerical studies of the zeroes of the partition function of the isotropic triangular lattice.[10] We will only remark that, because of Kardar's result,[26] all the known or possible critical points can be extended for weak chiralities.

Remark 6. One can also try to relate Eq. (24) and the disorder varieties (22). Actually the disorder solution for the anisotropic triangular lattice, can be seen as a trivializing limit of the star-triangle relation. This is particularly clear on the simple example of the Ising model. The star-triangle relation reads

$$\exp(K_1\sigma_2\sigma_3 + K_2\sigma_1\sigma_3 + K_3\sigma_1\sigma_2) = \Lambda \sum_{\sigma} \exp\left(L_1\sigma_1\sigma + L_2\sigma_2\sigma + L_3\sigma_3\sigma\right) .$$

The disorder solution

$$\mathrm{th}K_3 + \mathrm{th}\, K_1 \cdot \mathrm{th}\, K_2 = 0$$

amounts to say that L_1 or L_2 vanishes. This question will be addressed in further studies.

5. Conclusions

This paper emphasizes the role played by the algebraic subvarieties of the space of parameters, as well as the importance of the choice of variables. Let us go back for example to the star-triangle relation for the chiral Potts model. If we construct three sets of cyclic and diagonal matrices respectively C, C', C'' and D, D', D'', from the weights $w(i)$, $w'(i)$, $w''(i)$, $\overline{w}(i)$, $\overline{w}'(i)$, $\overline{w}''(i)$, with

$$C_{i,j} = w(i - j) \quad \text{for the cyclic matrices, and}$$
$$D_{i,j} = \delta_{i,\, j} \cdot \overline{w}(i) \quad \text{for the diagonal matrices}$$

then the star-triangle relation may be written as the matrix equation:

$$C\,D'\,C'' = D\,C'\,D'' \,. \tag{42}$$

In the limit where $D = D' = D'' =$ *unit matrix* the variables $\hat{w}$ enable to write very simply this matrix equation. The limit $C = C' = C'' =$ *unit matrix* is even more trivial. However when the matrices C and D are non-trivial, the finding of variables appropriate for solving Eq. (42) is not easy. The 'good' variables x used throughout the paper are such a good choice, in which Eq. (42) collapse to extremely simple ones, as, among others:

$$\frac{x'_8}{x'_3} = \frac{x_2}{x_5}\,, \qquad \frac{x'_7}{x'_1} = \frac{x''_8}{x''_3} \quad \cdots \,.$$

On this example, the existence of an (overdetermined) bilinear system is seen as a key structure for the star-triangle relation.

We have seen that these fundamental 'good' variables exist for various models which do not automatically reduce to cyclic matrices (the symmetric Ashkin-Teller model for instance). It is thus natural to try finding models where such structures are available, and provide us with candidates to integrability.

We could envisage generic models with Boltzmann weights $w_{i,j}$, with restrictions of the type

$$w_{i,j} = w_{i',j'} \tag{43}$$

for some pairs of indices $\{i,j\}$, $\{i',j'\}$. Cyclic matrices (i.e. such that $w_{i,j} = w_{i+k,j+k}$), or symmetric matrices (i.e. such that $w_{i,j} = w_{j,i}$) exemplify models of this type. We are interested in families which are (globally) invariant by the two involutions I and J, as described in Sec. 2.4 (I is the matrix inversion, and J is the dyadic inverse $w_{i,j} \to 1/w_{i,j}$.)

We could, in a second step, look for bilinear (overdetermined or not) systems like

$$\sum_{ij;k} E_{ij}^{l,k}\, w_{i,j} \cdot x_k = 0 \,, \quad k = 1, \ldots, r \tag{44}$$

such that not only I and J, but all the generators of $\mathcal{G}$ are simply represented in terms of the x_k's:

$$x_k \to \alpha_k \cdot x_{\sigma(k)}$$

with σ some permutation of r indices, and α_k complex numbers, that is to say $\mathcal{G}$ is represented as a semi-direct product of $\mathbb{Z}$ and of a subgroup of the permutation group of r elements.

One imagines that, because of the bilinear character of Eq. (44), linear transformations on the w's may be simply represented on the x_k's. On the contrary nonlinear transformations like the dyadic inverse J will not be represented simply unless for example the coefficients $E_{ij}^{l,k}$ vanish except for two pairs of indices (i_1, j_1) and (i_2, j_2).

Actually, as far as the first step is concerned, for $q = 3$ the exhaustive list of patterns of the type (43) contains only 17 elements, some being related by trivial relabelling of rows and columns. Most of these patterns define subgroups of $GL(3, R)$. The cyclic matrices form an abelian one. The sets which do not correspond to subgroups of $GL(3, R)$ are the symmetric matrices (symmetric with respect to the diagonal), and the ones which are symmetric with respect to the other diagonal.

A classification of the stable patterns is, in our opinion, a direct way to uncover spin models particularly interesting for further developments in statistical mechanics on lattices. Further exploration in this direction is in progress.

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