Quantum physics: from atom to solid state – Academic year 2012-2013
Broken symmetries and quantum phase transitions

Multidimensional Gaussian integrals

- Real variables $x = (x_1, \cdots, x_n)^T$:

$$Z(J) = \int_{-\infty}^{\infty} \prod_{i=1}^{n} dx_i \, e^{-\frac{1}{2} x^T A x + J^T x} = (2\pi)^{n/2} (\det A)^{-1/2} \, e^{\frac{1}{2} J^T A^{-1} J} \tag{1}$$

  (A definite positive real symmetric matrix, J_i real).

- Complex variables $z = (z_1, \cdots, z_n)^T$:

$$Z(J^*, J) = \int_{-\infty}^{\infty} \prod_{i=1}^{n} \frac{dz_i^* dz_i}{2i\pi} \, e^{-z^\dagger A z + (J^\dagger z + \text{c.c.})} = (\det A)^{-1} \, e^{J^\dagger A^{-1} J}, \tag{2}$$

  where $\frac{dz_i^* dz_i}{2i\pi} = \frac{d\Re(z_i) d\Im(z_i)}{\pi}$ (A complex matrix with positive definite Hermitian part $\frac{1}{2}(A + A^\dagger)$).

- Grassmann variables $\theta = (\theta_1, \cdots, \theta_n)^T$:

$$Z(J^*, J) = \int \prod_{i=1}^{n} d\theta_i^* d\theta_i \, e^{-\theta^\dagger A \theta + (J^\dagger \theta + \text{c.c.})} = \det(A) \, e^{J^\dagger A^{-1} J} \tag{3}$$

  (A complex matrix, J_i, J_i^* Grassmann variables)

Functional Gaussian integrals

- Real field:

$$\begin{aligned} Z[J] &= \int \mathcal{D}[\phi] \, e^{-\frac{1}{2} \int d^d x d^d y \, \phi(\mathbf{x}) A(\mathbf{x}, \mathbf{y}) \phi(\mathbf{y}) + \int d^d x \, J(\mathbf{x}) \phi(\mathbf{x})} \\ &= (\det A)^{-1/2} \, e^{\frac{1}{2} \int d^d x d^d y \, J(\mathbf{x}) A^{-1}(\mathbf{x}, \mathbf{y}) J(\mathbf{y})} \end{aligned} \tag{4}$$

- Complex field:

$$\begin{aligned} Z[J^*, J] &= \int \mathcal{D}[\psi^*, \psi] \, e^{-\int d^d x d^d y \, \psi^*(\mathbf{x}) A(\mathbf{x}, \mathbf{y}) \psi(\mathbf{y}) + \int d^d x \, [J^*(\mathbf{x}) \psi(\mathbf{x}) + \text{c.c.}]} \\ &= (\det A)^{-1} \, e^{\int d^d x d^d y \, J^*(\mathbf{x}) A^{-1}(\mathbf{x}, \mathbf{y}) J(\mathbf{y})} \end{aligned} \tag{5}$$

- Grassmannian field:

$$\begin{aligned} Z[J^*, J] &= \int \mathcal{D}[\psi^*, \psi] \, e^{-\int d^d x d^d y \, \psi^*(\mathbf{x}) A(\mathbf{x}, \mathbf{y}) \psi(\mathbf{y}) + \int d^d x \, [J^*(\mathbf{x}) \psi(\mathbf{x}) + \text{c.c.}]} \\ &= \det(A) \, e^{\int d^d x d^d y \, J^*(\mathbf{x}) A^{-1}(\mathbf{x}, \mathbf{y}) J(\mathbf{y})} \end{aligned} \tag{6}$$

These results hold up to a multiplicative constant which depends on the definition of the measure in the functional integral.