Chapter 10

Renormalization group and critical phenomena

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In the preceding chapters, we have seen examples of interacting systems where mean-field theory is a good starting point. In many cases however, fluctuations about the mean-field approximation are important and cannot be neglected. In low dimensions, they often play a crucial role and tend to suppress long-range order. In a more subtle way, they can also affect the correlation functions near a second-order phase transition and invalidate the mean-field predictions regarding their long-distance and long-time behavior.

In this chapter, we give a general introduction to the theory of second-order phase transitions (i.e. transitions with a continuous order parameter and a diverging correlation length). After a brief introduction to critical phenomena (Sec. 10.1), we review Landau's mean-field theory for a (classical) $(\boldsymbol{\varphi}^2)^2$ theory with $\mathrm{O}(N)$ symmetry (Sec. 10.2). By studying fluctuations about the mean-field approximation, we find that Landau's theory becomes inapplicable near the transition for dimensions below the upper critical dimension $d_c^+ = 4$, while long-range order is suppressed at and below the lower critical dimension $d_c^- = 2$ (Sec. 10.3). We then discuss second-order phase transitions in the framework of the scaling hypothesis and derive relations between critical exponents (scaling laws). The RG - with the important notions of RG flows, fixed points and critical exponents - is discussed in Sec. 10.5. We show how the RG naturally leads to universality and scaling. The critical exponents are computed perturbatively near the upper critical dimension (Sec. 10.6), and near the lower critical dimension in the framework of the NLσM (Sec. 10.7). The Berezinskii-Kosterlitz-Thouless transition is discussed in section. 10.8. In section 10.9, we give a brief introduction to the functional RG, the Wilson-Polchinski equation and its solution in the local potential approximation.[1] Details about the perturbative calculation of critical exponents and the large-N limit of the $(\boldsymbol{\varphi}^2)^2$ theory or NLσM can be found in the appendices. We consider only classical (thermal) phase transtions; (zero-temperature) quantum phase will be discussed in chapter 12.

[1]The functional RG will be discussed at length in chapter 11.

10.1 Introduction to critical phenomena

Let us consider a system with the partition function

$$Z(K) = \mathrm{Tr}\, e^{-\beta \hat{H}}, \tag{10.1}$$

where $K = \{K_i\}$ denotes a set of parameters or "coupling constants" (external fields, microscopic parameters, etc.) of the Hamiltonian $\hat{H}$, as well as temperature. We assume that the thermodynamic limit exists, i.e. that the limit

$$f(K) = \lim_{V\to\infty} \frac{F(K)}{V} = -\lim_{V\to\infty} \frac{1}{\beta V} \ln Z(K) \tag{10.2}$$

is defined.[2] When this is not the case, surface effects remain important in the limit $V \to \infty$ and it is not possible to define a bulk free energy density $f(K)$.

A region in the $K = \{K_i\}$ space where $f(K)$ is analytic defines a phase of the system. Phase transitions correspond to non-analyticities of $f(K)$.[3] In general, the free energy of a finite-size system is analytic. We are interested in the case where non-analyticities arise from the thermodynamic limit $V \to \infty$ in which the number of degrees of freedom becomes infinite, and focus on thermal phase transitions (i.e. transitions driven by thermal fluctuations). Quantum fluctuations do not play an important role in the low-energy behavior of a system near a finite-temperature phase transition and we therefore only consider classical models. Quantum phase transitions, i.e. zero-temperature phase transitions driven by quantum fluctuations, are discussed in chapter 12.

Commenter davantage? Pourquoi pas toujours?

10.1.1 Spontaneous symmetry breaking

There are different kinds of phase transitions. Some, such as the liquid-gas transition or the Mott transition in solids,[4] do not break any symmetry. Others, e.g. the ferromagnetic transition in a magnetic system (chapter 8) or the superfluid transition in a Bose gas (chapter 7), are associated with spontaneous symmetry breaking. We are interested in the case where one of the phases has the full symmetry of the Hamiltonian, while the other one has a reduced symmetry. It is then possible to introduce an order parameter which vanishes in the "disordered" (symmetric) phase and takes a nonzero value in the "ordered" phase.[5] Since the state of the system is determined by the minimum of the free energy $F = E - TS$, the disordered phase usually corresponds to the high-temperature phase (where entropy effects dominate) and the ordered phase to the low-temperature phase.

In the following, we mostly use the language of the ferromagnetic transition and, unless otherwise specified, consider an easy-axis ferromagnet (the easy axis determines the direction of the magnetization). While the system is paramagnetic at high temperatures, below the transition temperature T_c there appears a spontaneous magnetization M in zero magnetic field (Fig. 10.1). To understand the nature of the broken symmetry, we assume that the

[2] A necessary condition for the thermodynamic limit to exist is that the interactions are sufficiently short range.

[3] This definition is in fact ambiguous since it is sometimes possible to go from one phase to the other without crossing a phase boundary. This is possible when, as in the liquid-gas transition, both phases have the same symmetry.

[4] The Mott transition is a metal-insulator transition induced by the Coulomb repulsion.

[5] In the liquid-gas transition, although there is no spontaneous symmetry breaking, it is possible to define an order parameter, namely the density of the fluid (or the difference between the density of the fluid and that of the gas).

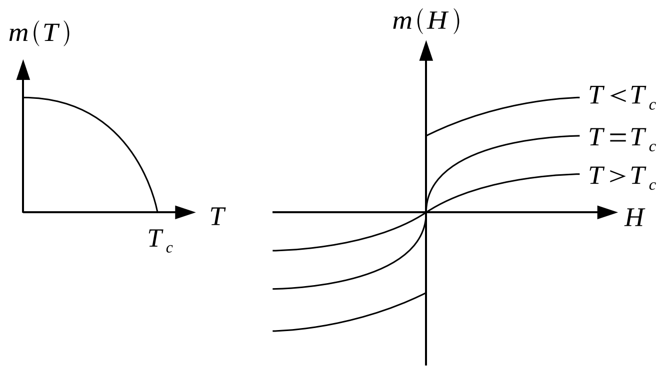

Figure 10.1: Magnetization density of an easy-axis ferromagnet *vs* temperature in zero field (left panel), and *vs* magnetic field below, above and at the transition temperature (right panel).

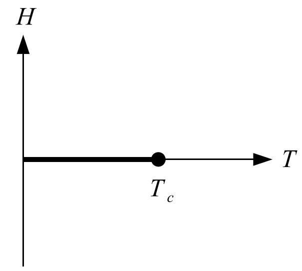

Figure 10.2: Phase diagram of an easy-axis ferromagnet. The discontinuous transition line (thick line) between states with magnetization density $m(T)$ and $-m(T)$ terminates at the critical point $(T_c, H = 0)$.

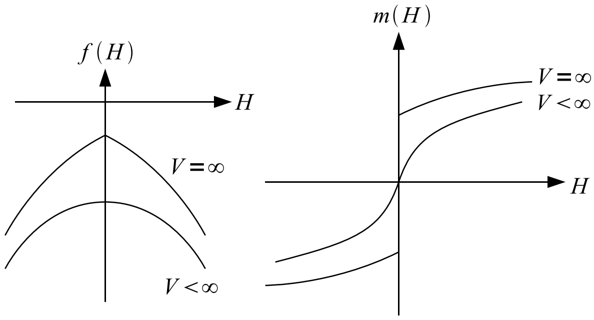

Figure 10.3: Free energy density $f(H) = F(H)/V$ and magnetization density $m(H) = M(H)/V$ of an easy-axis ferromagnet as a function of the magnetic field for finite and infinite systems.

ferromagnet consists of a collection of spins $S_{\mathbf{r}}$ located at the sites $\mathbf{r}$ of a lattice and parallel to the easy axis. Because of time reversal symmetry in the absence of an external field, the Hamiltonian $\mathcal{H}(S_{\mathbf{r}})$ is invariant under spin inversion $S_{\mathbf{r}} \to -S_{\mathbf{r}}$. In the ferromagnetic phase, spin inversion is spontaneously broken. On the other hand, if the ferromagnet has no easy axis but is isotropic, then the Hamiltonian is invariant under a simultaneous rotation of all the spins, $\mathcal{H}(\mathbf{S}_{\mathbf{r}}) = \mathcal{H}(R(\mathbf{S}_{\mathbf{r}}))$, where R is an arbitrary rotation matrix acting in spin space and the spin variables are now three-dimensional vectors. In the ferromagnetic phase, spin rotation invariance is spontaneously broken: $\mathbf{m} = \langle \mathbf{S}_{\mathbf{r}} \rangle \neq 0$. In this case, the broken symmetry is continuous. We shall see that the nature (discrete or continuous) of the spontaneously broken symmetry plays a crucial role in the low-energy properties of the system.

The phase diagram of a ferromagnet is shown in figure 10.2. There is a discontinuity in the magnetization density m as the magnetic field H goes trough zero for $T < T_c$ (the magnetic field is assumed to be along the easy axis of the ferromagnet). This discontinuity terminates at the "critical" point $(T_c, H = 0)$.[6]

Because of time reversal invariance, the free energy does not change if we reverse the direction of the field, $F(H) = F(-H)$. This implies

$$m(H) = -\frac{\partial f(H)}{\partial H} = -\frac{\partial f(-H)}{\partial H} = \frac{\partial f(-H)}{\partial(-H)} = -m(-H), \tag{10.3}$$

so that it seems that the zero-field magnetization density $m(H = 0) = M(H = 0)/V$ must vanish. This argument however requires $f(H) = F(H)/V$ to be analytic at $H = 0$, i.e. $\partial f(H)/\partial H$ to be smooth at $H = 0$. While this is true if the volume is finite, this is violated in the infinite volume limit when $T < T_c$:

$$\lim_{V\to\infty} \lim_{H\to 0} \frac{1}{V} \frac{\partial F(H)}{\partial H} = 0, \tag{10.4}$$

but

$$\lim_{H\to 0} \lim_{V\to\infty} \frac{1}{V} \frac{\partial F(H)}{\partial H} \neq 0, \tag{10.5}$$

[6]See footnote 11 page 609.

as illustrated in figure 10.3. Spontaneous broken symmetry is possible only in the thermodynamic limit where the free energy density $f(H)$ becomes non-analytic at $H=0$.

To illustrate this point, let us consider spins $S_\mathbf{r}$ located at the sites of a lattice (with N the number of sites). The probability to find the system in the state $\{S_\mathbf{r}\}$ is given by the Boltzmann distribution

$$P(\{S_\mathbf{r}\}) = \frac{e^{-\beta\mathcal{H}(\{S_\mathbf{r}\})}}{Z}. \tag{10.6}$$

If $P(\{S_\mathbf{r}\})$ is invariant under $S_\mathbf{r}\to -S_\mathbf{r}$ (time-reversal invariance), then $\langle S_\mathbf{r}\rangle = \mathrm{Tr}P(\{S_\mathbf{r}\})S_\mathbf{r} = 0$ and it seems that spontaneous symmetry breaking is impossible. At low temperature, for a ferromagnetic coupling between spins, the latter are either "up" ($\langle S_\mathbf{r}\rangle = +m$) or "down" ($\langle S_\mathbf{r}\rangle = -m$). These two configurations are related by time reversal symmetry and their probabilities are equal: $P_\oplus = P_\ominus$. Now apply an external positive field H ($H>0$). Due to the coupling term $-H\sum_\mathbf{r} S_\mathbf{r}$,

$$\frac{P_\ominus}{P_\oplus} = e^{-2\beta NHm} \tag{10.7}$$

and

$$\lim_{N\to\infty} \frac{P_\ominus}{P_\oplus} = 0 \tag{10.8}$$

Thus the presence of an infinitesimal field $H\to 0^+$, together with the thermodynamic limit $N\to\infty$, is sufficient to select the configuration $\oplus$. The configuration $\ominus$ is inaccessible. Equivalently, we could set $H=0$ and use a restricted ensemble where the configuration $\ominus$, and more generally all microstates with a negative magnetization, is not allowed. The fact that some part of the phase space is forbidden is known as ergodicity breaking.[7,8]

10.1.1.1 Gibbs free energy

The stability of the system requires the isothermal susceptibility

$$\chi = \frac{\partial m}{\partial H} = -\frac{1}{V}\frac{\partial^2 F(H)}{\partial H^2} \tag{10.9}$$

to be positive. Thus $F''(H)\leq 0$ and the free energy is a convex function of the magnetic field. This allows us to invert the relation $m = -\frac{1}{V}\frac{\partial F}{\partial H}$ and introduce the Gibbs free energy

$$G(m) = F(H) + VHm, \tag{10.10}$$

defined as the Legendre transform of $F(H)$. $G(m)$ satisfies the equation of state

$$\frac{\partial G(m)}{\partial m} = VH. \tag{10.11}$$

From figure 10.3, one easily deduces the general form of the Gibbs free energy (Fig. 10.4). The convexity of $F(H)$ ensures that $G(m)$ is also a convex function. In the ordered phase, the (absolute value of the) magnetization is always larger than the zero-field result $m_0 = -\partial f/\partial H|_{H=0^+}$. The region $]-m_0, m_0[$ is not physically accessible.

[7] A detailed discussion of ergodicity breaking may be found in Ref. [19].
[8] Spontaneous symmetry breaking is further discussed in Sec. 3.6.2.

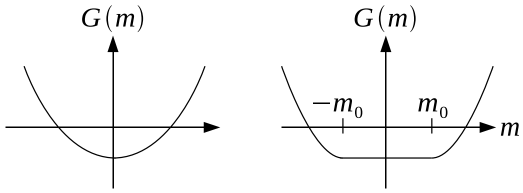

Figure 10.4: Gibbs free energy $G(m)$: high-temperature (left) and low-temperature (right) phase.

10.1.2 Landau's classification of phase transitions

Landau distinguishes between discontinuous (or first-order) and continuous (or second-order) phase transitions.[9] In a first-order transition, the order parameter is discontinuous at the transition (Fig. 10.5). The correlation length ξ is finite,[10] and the two phases (ordered and disordered) coexist at the transition temperature T_c. In a second-order phase transition, the order parameter is continuous (Fig. 10.5) and the correlation length diverges. Fluctuations become correlated over all distances, which forces the whole system to be in a unique phase. The two phases of either side of the transition must therefore become identical at T_c; as the correlation length diverges, the order parameter in the ordered phase goes smoothly to zero. The ferromagnetic transition we have discussed above is an example of a second-order phase transition.

When the correlation length is finite, we can view the system as a collection of subsystems of size ξ^d (with d the space dimension) with no mutual interaction. By the central limit theorem, we expect the fluctuations at large distances ($\gg \xi$) to have a Gaussian probability distribution. By contrast, all degrees of freedom become correlated at a second-order phase transition where ξ diverges (Sec. 10.1.4). We will see that standard perturbation theories break down in the vicinity of a second-order phase transition unless the dimension is high enough or the interactions sufficiently long range (Secs. 10.2.1 and 10.3). The temperature regime where the mean-field or Gaussian theories (Secs. 10.2 and 10.3) are not valid any more is called the critical regime.[11]

On the other hand, the divergence of the correlation length at a second-order phase transition renders microscopic details irrelevant for the long-distance properties. As a consequence, near the critical point, the singular part of the free energy and the asymptotic behavior of the correlation functions depend only on general properties such as the space dimension, the dimension of the order parameter or the symmetry and range of the interactions. This essential property of second-order phase transitions is called universality. We shall see that universal properties of a system near a second-order phase transition can be accurately described within an effective theory involving only long-distance fluctuations.

Unless otherwise specified, we will only consider second-order (continuous) phase transitions in the following.

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[9] Landau's classification differs from Ehrenfest's classification where a transition is said to be nth order if all $(n-1)$th-order derivatives of the free energy are continuous while there is at least one nth-order derivative which is discontinuous. (Ehrenfest did not realize that some thermodynamic quantities (e.g. the specific heat) can diverge rather than exhibiting a simple discontinuity.)

[10] The correlation length ξ is a measure of the distance over which correlations are important; it will be precisely defined in Sec. 10.1.4.

[11] Second-order phase transitions are often called critical phenomena; the transition temperature is then referred to as the critical temperature/point.

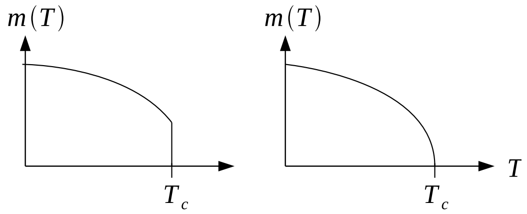


Figure 10.5: Temperature dependence of the order parameter $m(T)$ in a first-order (left) and second-order (right) phase transition.

10.1.3 Critical behavior

The singular behavior at the critical point is characterized by a set of critical exponents (Table 10.1). Below T_c, the magnetization density $m(T,H)$ varies as[12]

$$m(T,0) \sim (-t)^\beta \qquad (T \to T_c^-), \tag{10.12}$$

where

$$t = \frac{T-T_c}{T_c} \tag{10.13}$$

is the reduced temperature. At the transition temperature,

$$m(T_c,H) \sim H^{1/\delta} \qquad (H \to 0), \tag{10.14}$$

which defines the exponent δ. Near the critical point, when the system is about to spontaneously order, the susceptibility (i.e. the response to an external magnetic) becomes very large and diverges at T_c,

$$\chi = \frac{\partial m}{\partial H}\bigg|_{H=0} \sim \begin{cases} t^{-\gamma} & (T \to T_c^+), \\ (-t)^{-\gamma'} & (T \to T_c^-), \end{cases} \tag{10.15}$$

where the two exponents γ and γ' refer to the high- and low-temperature phases, respectively. The critical behavior is also characterized by a divergence of the specific heat

$$C_V \sim \begin{cases} t^{-\alpha} & (T \to T_c^+), \\ (-t)^{-\alpha'} & (T \to T_c^-), \end{cases} \tag{10.16}$$

with an exponent α or α'. In most cases, the exponents on both sides of the transition coincide: $\gamma = \gamma'$ and $\alpha = \alpha'$ (Sec. 10.4.2). The divergence of the correlation length and the singular behavior of the correlation function are discussed in the next section.

10.1.4 Long-range order

A nonzero order parameter implies not only spontaneous broken symmetry but also long-range order. In most cases of interest, the order parameter is the mean value of an observable $\varphi(\mathbf{r})$. Besides $m = \langle \varphi(\mathbf{r}) \rangle$, one can also consider the correlation function of the φ field. In

[12] Note that we use the same notation for the critical exponent defined in (10.12) and the inverse temperature $\beta = 1/T$.

Order parameter	$m(T,0) \sim (-t)^\beta$	$(T \to T_c^-)$
	$m(T_c,H) \sim H^{1/\delta}$	$(H \to 0)$
Susceptibility	$\chi = \frac{\partial m}{\partial H}\Big\vert_{H=0} \sim \begin{cases} t^{-\gamma} & (T \to T_c^+) \\ (-t)^{-\gamma'} & (T \to T_c^-) \end{cases}$	
Specific heat	$C_V \sim \begin{cases} t^{-\alpha} & (T \to T_c^+) \\ (-t)^{-\alpha'} & (T \to T_c^-) \end{cases}$	
Correlation length	$\xi \sim \begin{cases} t^{-\nu} & (T \to T_c^+) \\ (-t)^{-\nu'} & (T \to T_c^-) \end{cases}$	
Correlation function	$G(\mathbf{r}) \sim \frac{1}{\vert\mathbf{r}\vert^{d-2+\eta}}$	$(T = T_c)$
	$G(\mathbf{p}) \sim \frac{1}{\vert\mathbf{p}\vert^{2-\eta}}$	$(T = T_c)$

Table 10.1: Critical exponents at a second-order (continuous) phase transition.

the disordered phase, the order parameter vanishes and the correlation function decays exponentially at large distances,

$$C(\mathbf{r}-\mathbf{r}') = \langle \varphi(\mathbf{r})\varphi(\mathbf{r}')\rangle \sim \exp\left(-\frac{|\mathbf{r}-\mathbf{r}'|}{\xi}\right). \tag{10.17}$$

Equation (10.17) defines the correlation length ξ. For $|\mathbf{r}-\mathbf{r}'| \to \infty$, there should be no correlation between the magnetization densities at point $\mathbf{r}$ and $\mathbf{r}'$, so that

$$\lim_{|\mathbf{r}-\mathbf{r}'|\to\infty} C(\mathbf{r}-\mathbf{r}') = \langle \varphi(\mathbf{r})\rangle\langle\varphi(\mathbf{r}')\rangle = 0, \tag{10.18}$$

in agreement with (10.17). In the ordered phase, one finds instead

$$\lim_{|\mathbf{r}-\mathbf{r}'|\to\infty} C(\mathbf{r}-\mathbf{r}') = \langle \varphi(\mathbf{r})\rangle\langle\varphi(\mathbf{r}')\rangle = m^2. \tag{10.19}$$

Equation (10.19) is the mathematical definition of long-range order; it suggests that ξ diverges as $T \to T_c^+$. This can be shown to be a consequence of the (classical) fluctuation-dissipation theorem (3.60),[13]

There can be LRO without SSB!

$$\chi = \beta \int d^dr \langle\varphi(\mathbf{r})\varphi(0)\rangle \leq \beta\int d^dr\, e^{-|\mathbf{r}|/\xi} \sim \beta\xi^d. \tag{10.20}$$

Since the susceptibility χ diverges at the transition [Eq. (10.15)], $\xi \sim t^{-\nu}$ must also diverge when $T \to T_c^+$, which defines the exponent ν.

In the ordered phase, it is convenient to work with the connected correlation function

$$G(\mathbf{r}-\mathbf{r}') = \langle(\varphi(\mathbf{r})-m)(\varphi(\mathbf{r}')-m)\rangle = C(\mathbf{r}-\mathbf{r}') - m^2 \sim e^{-|\mathbf{r}-\mathbf{r}'|/\xi}, \tag{10.21}$$

which defines the correlation length ξ for $T < T_c$. Both $\chi = \beta G(\mathbf{p}=0) \sim (-t)^{-\gamma'}$ and $\xi \sim (-t)^{-\nu'}$ diverge when $T \to T_c^-$.[14] In almost all cases, the correlation length critical exponents ν and ν' are equal (and $\gamma' = \gamma$) (Sec. 10.4.2).

[13] The correlation function $C(\mathbf{r}) \sim e^{-|\mathbf{r}|/\xi}/|\mathbf{r}|^p$ generally decays faster then $e^{-|\mathbf{r}|/\xi}$.

[14] The divergence of ξ in the ordered phase as $T \to T_c^-$ is obtained from the same argument as that used for the ordered phase with the correlation function C replaced by its connected part G.

***N*-component order parameter.** For a N-component order parameter $\mathbf{m} = \langle \boldsymbol{\varphi}(\mathbf{r}) \rangle$ (e.g. $N = 3$ for a Heisenberg ferromagnet), the connected correlation function is defined by

$$G_{ij}(\mathbf{r} - \mathbf{r}') = \langle (\varphi_i(\mathbf{r}) - m_i)(\varphi_j(\mathbf{r}') - m_j) \rangle = C_{ij}(\mathbf{r} - \mathbf{r}') - m_i m_j. \tag{10.22}$$

If the system is isotropic (O(N) symmetry), the Fourier transformed correlation function takes the form

$$G_{ij}(\mathbf{p}) = \frac{m_i m_j}{m^2} G_\parallel(\mathbf{p}) + \left(\delta_{i,j} - \frac{m_i m_j}{m^2}\right) G_\perp(\mathbf{p}), \tag{10.23}$$

where the longitudinal and transverse components, $G_\parallel$ and $G_\perp$, are functions of $|\mathbf{p}|$. We shall see later on that when $N \geq 2$ and $d \leq 4$ neither $G_\parallel$ nor $G_\perp$ decays exponentially in space below T_c,[15] so that the susceptibility χ diverges in the whole low-temperature phase and it is not possible to define a correlation length. It is nevertheless possible to define a characteristic length, the Josephson length ξ_J, which diverges as $T \to T_c^-$ with an exponent $\nu' = \nu$ (Sec. 10.7.2).

10.1.4.1 Scale invariance

At the critical point ($T = T_c$), the correlation length ξ is infinite and the correlation function decays as a power law,

$$G(\mathbf{r}) \sim \frac{1}{|\mathbf{r}|^{d-2+\eta}}, \tag{10.24}$$

where η is called the anomalous dimension (this terminology is explained in Sec. 10.4).[16] Equation (10.24) will be derived in section 10.4. Under a change of scale, $G(\mathbf{r})$ behaves as

$$G(\mathbf{r}/s) = s^{d-2+\eta} G(\mathbf{r}) \tag{10.25}$$

and is therefore invariant (apart from a multiplication by a factor $s^{d-2+\eta}$). A critical system exhibits scale invariance or self-similarity. The concept of scale invariance at the critical point is central in the renormalization-group approach (Sec. 10.5).

10.2 Landau's theory of phase transitions

10.2.1 Landau's theory as a mean-field theory

10.2.1.1 Microscopic Landau's theory

Faire le lien avec Γ dans FRG?

In a microscopic description, one would like to start from the partition function $Z = \operatorname{Tr} e^{-\beta \hat{H}}$. While the (quantum) Hamiltonian $\hat{H}$ is certainly necessary to understand the microscopic origin of a phase transition, it is not crucial to understand the role of thermal or quantum fluctuations in the vicinity of the phase transition. It appears more appropriate to have a simplified description which emphasizes the role of the order parameter field $\boldsymbol{\varphi}(\mathbf{r})$ associated with the order parameter $\mathbf{m}(\mathbf{r}) = \langle \boldsymbol{\varphi}(\mathbf{r}) \rangle$. $\boldsymbol{\varphi}(\mathbf{r})$ is a quantum field or a function of the classical dynamical variables. The order parameter field is usually defined

[15] Mean-field and Gaussian fluctuation theories predict $G_\parallel(\mathbf{r})$ to decay exponentially below T_c for any value of N. For $N \geq 2$ and $d \leq 4$, this result is an artifact coming from neglecting the coupling between transverse and longitudinal fluctuations (see Sec. 10.7.3).

[16] In Fourier space, Eq. (10.24) gives $G(\mathbf{p}) \sim 1/|\mathbf{p}|^{2-\eta}$.

at a mesoscopic scale Λ^{-1} which is much larger than the lattice spacing (assuming that the microscopic model is defined on a lattice) but small wrt macroscopic scales.[17]

The process of obtaining a low-energy effective description by averaging over many unit cells is called "coarse graining". The averaging procedure ensures that $\boldsymbol{\varphi}(\mathbf{r})$ is a continuous variable, and the partition function can be expressed as a functional integral,[18,19]

$$Z = \int \mathcal{D}[\boldsymbol{\varphi}]\, e^{-S[\boldsymbol{\varphi}]}, \tag{10.26}$$

where the momentum of the Fourier transformed field $\boldsymbol{\varphi}(\mathbf{p})$ satisfied $|\mathbf{p}| \leq \Lambda$. Formally, we can obtain the low-energy effective action $S[\boldsymbol{\varphi}]$ from a partial trace over all microscopic configurations compatible with a given configuration of the coarse grained field $\boldsymbol{\varphi}(\mathbf{r})$. In general the partial integration involves a finite number of degrees of freedom and $S[\boldsymbol{\varphi}]$ is an analytic function of $\boldsymbol{\varphi}$. In some cases, it is possible to explicitly derive $S[\boldsymbol{\varphi}]$ from a microscopic action as in the example of classical spin models discussed in section 10.2.2. In many cases however, we simply include in $S[\boldsymbol{\varphi}]$ all terms allowed by symmetry within a derivative expansion. For example, for a N-component field, the simplest action with $\mathrm{O}(N)$ symmetry reads

$$S[\boldsymbol{\varphi}] = \int d^d r \biggl\{ \frac{1}{2} (\boldsymbol{\nabla}\boldsymbol{\varphi})^2 + \frac{r_0}{2} \boldsymbol{\varphi}^2 + \frac{u_0}{4!} \bigl(\boldsymbol{\varphi}^2\bigr)^2 \biggr\}, \tag{10.27}$$

corresponding to the so-called $(\boldsymbol{\varphi}^2)^2$ theory (or linear $\mathrm{O}(N)$ model).

Such an approach is clearly inappropriate to compute the transition temperature (which depends on microscopic parameters of the system) but is sufficient to understand the behavior of the system near the critical point and in particular the universal properties (Sec. 10.5).

We can compute the partition function (10.26) using the perturbative methods introduced in chapter 1. The simplest approach is to perform a mean-field (or saddle-point) approximation,

$$Z_{\rm MF} \simeq e^{-S[\boldsymbol{\varphi}]}, \tag{10.28}$$

where the field $\boldsymbol{\varphi}$ is determined from the saddle-point equation $\delta S/\delta\boldsymbol{\varphi} = 0$. The free energy is simply given by

$$F = -\frac{1}{\beta} \ln Z_{\rm MF} = \frac{1}{\beta} S[\mathbf{m}], \tag{10.29}$$

where the actual value of the order parameter $\mathbf{m} = \langle \boldsymbol{\varphi}(\mathbf{r}) \rangle$ is obtained by minimizing the action.

10.2.1.2 Phenomenological Landau's theory

It is possible to rephrase the preceding discussion from a more phenomenological point of view with no reference to any microscopic model. In the phenomenological Landau theory one postulates that the free energy density $f = F/V$ is an analytic function of the order parameter, whose absolute minimum specifies the state of the system. Near the critical point of a second-order phase transition, the order parameter is small and f can be expanded in

[17] In practice, Λ^{-1} is often identified with the lattice spacing.

[18] The action $S[\boldsymbol{\varphi}]$ is sometimes written as $\beta H[\boldsymbol{\varphi}]$, or merely $H[\boldsymbol{\varphi}]$, where $H[\boldsymbol{\varphi}]$ is the effective Hamiltonian. It is often (somewhat improperly) referred to as the "microscopic" action.

[19] In a quantum description, the field $\boldsymbol{\varphi}$ would also depend on an imaginary time $\tau \in [0, \beta]$. A classical description is nevertheless sufficient to study the critical behavior at a finite-temperature phase transition (Sec. 12).

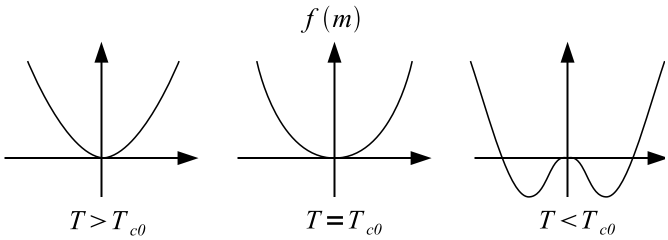


Figure 10.6: Free energy density $f(m)$ in zero field ($\mathbf{h} = 0$) near a second-order phase transition [Eq. (10.30)].

a power series. The form of f and its expansion must be consistent with the symmetries of the system.

Let us consider the case of a N-component real order parameter $\mathbf{m}(\mathbf{r})$ and assume the system to be O(N) symmetric: the free energy density is invariant if we uniformly rotate the order parameter. For a uniform order parameter and in the absence of magnetic field, f must be a function of the O(N) invariant $\mathbf{m}^2$. Near the phase transition, assuming that the free energy density is an analytic function of $\mathbf{m}$, we write

$$\beta f = \beta f_0 + \frac{r_0}{2}\mathbf{m}^2 + \frac{u_0}{4!}\left(\mathbf{m}^2\right)^2 - \mathbf{h}\cdot\mathbf{m}, \tag{10.30}$$

where the last term, which breaks the O(N) symmetry, is due to the coupling to an external field $\mathbf{H} = \mathbf{h}/\beta$ (i.e. the external magnetic field in the case of a ferromagnet). When the order parameter is inhomogeneous, we must include in f a term corresponding to the energy coast due to deviations from spatial uniformity. For a slowly varying order parameter, this leads to the Ginzburg-Landau free energy

$$F[\mathbf{m}] = \int d^d r\, f(\mathbf{m}, \boldsymbol{\nabla}\mathbf{m}), \tag{10.31}$$

where

$$\beta f = \beta f_0 + \frac{1}{2}(\boldsymbol{\nabla}\mathbf{m})^2 + \frac{r_0}{2}\mathbf{m}^2 + \frac{u_0}{4!}\left(\mathbf{m}^2\right)^2 - \mathbf{h}\cdot\mathbf{m}. \tag{10.32}$$

We will see below that higher-order terms such as $(\mathbf{m}^2)^3$ or $(\mathbf{m}\cdot\boldsymbol{\nabla}\mathbf{m})^2$ are negligible near the phase transition. It is always possible to rescale the order parameter to set the coefficient of $(\boldsymbol{\nabla}\mathbf{m})^2$ equal to 1/2. The free energy functional (10.31) is identical to that obtained from a saddle-point approximation of the $(\boldsymbol{\varphi}^2)^2$ theory [Eqs. (10.27,10.29)] and is to be identified with the coarse-grained action $S[\mathbf{m}]$ of the order parameter field.

f_0 is the free energy density in the disordered phase in the absence of magnetic field. Since it is expected to vary smoothly with temperature, it is usually omitted. We assume

$$r_0 = \bar{r}_0(T - T_{c0}) \tag{10.33}$$

and neglect the temperature dependence of u_0 which, as we shall see, is unimportant near the phase transition. The stability of the system requires $u_0 > 0$ (otherwise the free energy is minimized for $|\mathbf{m}| \to \infty$).

The minimization of the free energy is straightforward (Fig. 10.6). In the presence of a uniform field along the $\mathbf{e}_1$ axis ($\mathbf{h} = h\mathbf{e}_1$), the magnetization density $\mathbf{m} = m\mathbf{e}_1$ satisfies

$$\frac{\partial \beta f}{\partial m} = r_0 m + \frac{u_0}{6}m^3 - h = 0. \tag{10.34}$$

For a vanishing field, one finds

$$m = \begin{cases} 0 & \text{if} \quad T \geq T_{c0}, \\ \sqrt{\dfrac{-6r_0}{u_0}} & \text{if} \quad T \leq T_{c0}, \end{cases} \tag{10.35}$$

i.e. a phase transition at the temperature T_{c0} (referred to as the mean-field transition temperature). It usually differs from the actual transition temperature T_c. Right at the transition and in the presence of an external field, the order parameter takes the value

$$m(T_{c0}, h) = \left(\frac{6h}{u_0} \right)^{1/3}. \tag{10.36}$$

From (10.32) and (10.35), we deduce the singular part of the free energy,

$$f = \begin{cases} -\dfrac{3}{2} \dfrac{T\bar{r}_0^2}{u_0} (T - T_{c0})^2 & \text{if} \quad T \leq T_{c0}, \\ 0 & \text{if} \quad T \geq T_{c0}. \end{cases} \tag{10.37}$$

The regular part of the free energy comes from the term f_0 in (10.32), which we have omitted in our discussion.

Equations (10.35) and (10.36) yield the critical exponents $\beta = 1/2$ and $\delta = 3$. Differentiating (10.34) with respect to $H = \beta^{-1} h$, one obtains the uniform susceptibility

$$\chi = \beta \frac{\partial m}{\partial h} = \frac{\beta}{r_0 + \frac{u_0}{2} m^2} = \begin{cases} \dfrac{\beta}{r_0} & \text{if} \quad T > T_{c0}, \\ \dfrac{\beta}{2|r_0|} & \text{if} \quad T < T_{c0}, \end{cases} \tag{10.38}$$

and a critical exponent $\gamma = \gamma' = 1$. The singular part of the specific heat per unit volume,[20]

$$c_V = -T \frac{\partial^2 f}{\partial T^2} = \begin{cases} 0 & \text{if} \quad T > T_{c0}, \\ 3\dfrac{\bar{r}_0^2}{u_0} T^2 & \text{if} \quad T < T_{c0}, \end{cases} \tag{10.39}$$

is discontinuous at the transition.

Let us now we consider the equation

$$0 = \frac{\delta \beta F[\mathbf{m}]}{\delta m_i(\mathbf{r})} = r_0 m_i(\mathbf{r}) - \boldsymbol{\nabla}^2 m_i(\mathbf{r}) + \frac{u_0}{6} \mathbf{m}(\mathbf{r})^2 m_i(\mathbf{r}) - h_i(\mathbf{r}) \tag{10.40}$$

in an arbitrary field $\mathbf{h}(\mathbf{r})$. From (10.40), we see that for r_0 and h small,

$$|\mathbf{m}| \sim |r_0|^{1/2}, \quad |h| \sim |r_0|^{3/2}, \quad \frac{|\nabla \mathbf{m}|}{|\mathbf{m}|} \sim |r_0|^{1/2}, \tag{10.41}$$

so that all terms in (10.32) are of order r_0^2. Terms not included are of higher order and can be neglected: $(\mathbf{m}^2)^3, (\mathbf{m} \cdot \boldsymbol{\nabla} \mathbf{m})^2 \sim r_0^3$, $(\mathbf{m}^2)^4 \sim r_0^4$, etc.

To compute the susceptibility

$$\chi_{ij}(\mathbf{r} - \mathbf{r}') = \frac{\delta m_i(\mathbf{r})}{\delta H_j(\mathbf{r}')}\bigg|_{\mathbf{H}=0} = \beta \frac{\delta m_i(\mathbf{r})}{\delta h_j(\mathbf{r}')}\bigg|_{\mathbf{h}=0}, \tag{10.42}$$

[20] The regular part of the specific heat comes from the term f_0 in (10.32).

	Landau approximation	Gaussian model
$\nu = \nu'$	$1/2$	$1/2$
β	$1/2$	$1/2$
$\gamma = \gamma'$	1	1
δ	3	3
$\alpha = \alpha'$	c_V discontinuous	$2 - d/2$
η	0	0

Table 10.2: Critical exponents of the $(\boldsymbol{\varphi}^2)^2$ theory with O(N) symmetry [Eq. (10.27)] in the Landau approximation (Sec. 10.2) and in the Gaussian approximation for $d \leq 4$ (Sec 10.3). (For $d > 4$, the Gaussian approximation predicts the specific heat to be discontinuous.)

we take the functional derivative $\delta/\delta h_j(\mathbf{r}')$ in (10.40) and set $\mathbf{h}(\mathbf{r}) = 0$ (i.e. $\mathbf{m}(\mathbf{r}) = m\mathbf{e}_1$),

$$\left(r_0 - \boldsymbol{\nabla}^2 + \frac{u_0}{6} m^2 + \delta_{i,1} \frac{u_0}{3} m^2 \right) \chi_{ij}(\mathbf{r} - \mathbf{r}') = \beta \delta_{i,j} \delta(\mathbf{r} - \mathbf{r}'). \tag{10.43}$$

In Fourier space, we finally obtain

$$\chi_\parallel(\mathbf{p}) = \chi_\perp(\mathbf{p}) = \frac{\beta}{\mathbf{p}^2 + r_0} \quad \text{if} \quad T > T_{c0} \tag{10.44}$$

and

$$\left\{ \begin{array}{l} \chi_\parallel(\mathbf{p}) = \dfrac{\beta}{\mathbf{p}^2 + 2|r_0|} \\ \chi_\perp(\mathbf{p}) = \dfrac{\beta}{\mathbf{p}^2} \end{array} \right. \quad \text{if} \quad T < T_{c0}, \tag{10.45}$$

where $\chi_\parallel$ and $\chi_\perp$ are the longitudinal and transverse components of the susceptibility (see Eq. (10.23)).[21] Equations (10.44) and (10.45) imply $\chi_\parallel(\mathbf{r}) \sim e^{-|\mathbf{r}|/\xi}$ with[22]

$$\xi = \left\{ \begin{array}{ll} r_0^{-1/2} & \text{if} \quad T > T_{c0}, \\ |2r_0|^{-1/2} & \text{if} \quad T < T_{c0}, \end{array} \right. \tag{10.46}$$

and therefore a critical exponent $\nu = \nu' = 1/2$. The connected correlation functions $G_\parallel(\mathbf{p})$ and $G_\perp(\mathbf{p})$ of the order parameter field $\boldsymbol{\varphi}$ can be deduced from (10.44) and (10.45) by using the (classical) fluctuation-dissipation theorem $\chi_{ij}(\mathbf{p}) = \beta G_{ij}(\mathbf{p})$ [Eq. (3.60)]. We deduce that the anomalous dimension η vanishes: $G(\mathbf{p}, T_c) = 1/\mathbf{p}^2$. The critical exponents obtained within the Landau approximation are sometimes referred to as "classical" exponents (Table 10.2). The $1/\mathbf{p}^2$ divergence of the transverse susceptibility for $\mathbf{p} \to 0$ is a manifestation of Goldstone theorem; it will be further discussed in section 10.3.2.

[21] Since the magnetization is along the $\mathbf{e}_1$ axis, $\chi_\parallel = \chi_{11}$ and $\chi_\perp = \chi_{22}$.
[22] See footnote 15 page 612.

10.2.1.3 Effective action $\Gamma[\mathbf{m}]$ within the Landau approximation

It is sometimes convenient to work with the effective action $\Gamma[\mathbf{m}]$ (or the Gibbs free energy $G[\mathbf{m}] = \beta^{-1}\Gamma[\mathbf{m}]$) rather than the Helmholtz free energy $F[\mathbf{h}]$ or the microscopic action $S[\boldsymbol{\varphi}]$.[23] When the system is coupled to an external field, the partition function reads

$$Z[\mathbf{h}] = \int \mathcal{D}[\boldsymbol{\varphi}]\, e^{-S[\boldsymbol{\varphi}]+\int d^d r\, \mathbf{h}\cdot\boldsymbol{\varphi}} \tag{10.47}$$

and the order parameter is given by

$$m_i(\mathbf{r}) = \langle \varphi_i(\mathbf{r})\rangle = \frac{\delta \ln Z[\mathbf{h}]}{\delta h_i(\mathbf{r})}. \tag{10.48}$$

The effective action is defined as the Legendre transform

$$\Gamma[\mathbf{m}] = -\ln Z[\mathbf{h}] + \int d^d r\, \mathbf{h}\cdot\mathbf{m}, \tag{10.49}$$

where $\mathbf{h} \equiv \mathbf{h}[\mathbf{m}]$ is obtained by inverting (10.48). It satisfies the equation of state

$$\frac{\delta\Gamma[\mathbf{m}]}{\delta m_i(\mathbf{r})} = h_i(\mathbf{r}). \tag{10.50}$$

In the mean-field approximation, $\ln Z[\mathbf{h}] = -S[\boldsymbol{\varphi}] + \int d^d r\, \mathbf{h}\cdot\boldsymbol{\varphi}$, where $\boldsymbol{\varphi}$ satisfies the saddle-point equation

$$\frac{\delta S[\boldsymbol{\varphi}]}{\delta \varphi_i(\mathbf{r})} - h_i(\mathbf{r}) = 0. \tag{10.51}$$

We deduce

$$\begin{aligned} m_i(\mathbf{r}) &= \frac{\delta \ln Z[\mathbf{h}]}{\delta h_i(\mathbf{r})} \\ &= -\int d^d r' \sum_j \frac{\delta S[\boldsymbol{\varphi}]}{\delta\varphi_j(\mathbf{r}')}\frac{\delta\varphi_j(\mathbf{r}')}{\delta h_i(\mathbf{r})} + \int d^d r' \sum_j h_j(\mathbf{r}')\frac{\delta\varphi_j(\mathbf{r}')}{\delta h_i(\mathbf{r})} + \varphi_i(\mathbf{r}) \\ &= \varphi_i(\mathbf{r}), \end{aligned} \tag{10.52}$$

so that the effective action

$$\Gamma[\mathbf{m}] = S[\mathbf{m}] \tag{10.53}$$

reduces to the microscopic action within the Landau (mean-field) approximation, in agreement with the general discussion of section 1.7.2. The zero-field (connected) correlation function $G_{ij}(\mathbf{r}-\mathbf{r}') = \langle\varphi_i(\mathbf{r})\varphi_j(\mathbf{r}')\rangle - \langle\varphi_i(\mathbf{r})\rangle\langle\varphi_j(\mathbf{r}')\rangle$ [Eq. (10.21)] is given by the inverse of the two-point vertex

$$\Gamma^{(2)}_{ij}(\mathbf{r}-\mathbf{r}') = \frac{\delta^{(2)}\Gamma[\mathbf{m}]}{\delta m_i(\mathbf{r})\delta m_j(\mathbf{r}')}\bigg|_{\mathbf{m}(\mathbf{r})=\mathbf{m}} = \frac{\delta^{(2)}S[\mathbf{m}]}{\delta m_i(\mathbf{r})\delta m_j(\mathbf{r}')}\bigg|_{\mathbf{m}(\mathbf{r})=\mathbf{m}} \tag{10.54}$$

(see Sec. 1.6.2), i.e.

$$\begin{aligned} \Gamma^{(2)}_{\parallel}(\mathbf{p}) &= \mathbf{p}^2 + r_0 + \frac{u_0}{2}\mathbf{m}^2, \\ \Gamma^{(2)}_{\perp}(\mathbf{p}) &= \mathbf{p}^2 + r_0 + \frac{u_0}{6}\mathbf{m}^2, \end{aligned} \tag{10.55}$$

where $\mathbf{m}$ is determined from the stationarity condition $\delta\Gamma[\mathbf{m}]/\delta\mathbf{m} = 0$ [Eq. (10.50)]. We thus recover the mean-field propagator $G_{ij} = T\chi_{ij}$ [Eqs. (10.44) and (10.45)].

[23] The effective action $\Gamma[\mathbf{m}]$ is introduced in Sec. 1.6.2.

10.2.1.4 Universality

Within mean-field theory, the critical exponents of the $(\boldsymbol{\varphi}^2)^2$ theory (10.27) are independent of the values of the coupling constants $\bar{r}_0$ and u_0. Two apparently different systems share the same set of critical exponents: this is called universality. Two systems with the same critical exponents are said to be in the same universality class. We shall see that the universality predicted by the Landau theory is too strong. The critical exponents take their mean-field ("classical") value only above the upper critical dimension d_c^+. When $d < d_c^+$, they generally depend on the dimension d, the number N of components of the order parameter as well as the symmetries and range of the interactions. The explanation of universality and the computation of the critical exponents is one of the great successes of the renormalization group (Sec. 10.5).

10.2.1.5 Breakdown of mean-field theory

Mean-field theory is a good approximation if fluctuations of the order parameter about its mean value are small. Since we have been able, using the fluctuation-dissipation theorem, to obtain the correlation functions from the mean-field theory, we can check the internal consistency of the theory. Let us consider a coherence volume $V \sim \xi^d \sim |r_0|^{-d/2}$ in which the fluctuations are correlated. The average magnetization is

$$M \sim Vm \sim \xi^d \sqrt{\frac{6|r_0|}{u_0}} \sim \xi^{d-1} \tag{10.56}$$

and the fluctuation

$$\Delta M^2 \sim \int d^d r\, d^d r' \langle (\varphi(\mathbf{r}) - m)(\varphi(\mathbf{r}') - m)\rangle \sim V \int d^d r\, G(\mathbf{r}) \sim \xi^{d+2} \tag{10.57}$$

(for simplicity we assume a scalar order parameter, i.e. $N = 1$) can be related to the correlation function[24]

$$G(\mathbf{r}) = \int \frac{d^d p}{(2\pi)^d} \frac{e^{i\mathbf{p}\cdot\mathbf{r}}}{\mathbf{p}^2 + \xi^{-2}} \sim \frac{e^{-|\mathbf{r}|/\xi}}{|\mathbf{r}|^{d-2}} \qquad (|\mathbf{r}| \lesssim \xi). \tag{10.58}$$

We therefore obtain

$$\frac{\Delta M^2}{M^2} \sim \xi^{4-d}. \tag{10.59}$$

When $d > 4$, the rhs of (10.59) vanishes as $T \to T_{c0}^-$ and $\xi \to \infty$, and the mean-field theory appears internally consistent. When $d < 4$, the mean-field theory is clearly not reliable in the vicinity of the critical point. The dimension $d_c^+ = 4$ where the mean-field theory breaks down is called the upper critical dimension. The criterion $\Delta M^2 \sim M^2$ giving the size of the critical region is known as the Ginzburg criterion; it will be discussed in more detail in section 10.3.4.

10.2.2 $(\varphi^2)^2$ theory for classical spin models.

Let us first consider the Hamiltonian

$$\beta H = -\frac{1}{2} \sum_{i,j} \sigma_i K_{ij} \sigma_j - \sum_i h_i \sigma_i \tag{10.60}$$

[24] Correlation functions in real space are discussed in Sec. 10.3.1.

of the Ising model, where $\sigma_i = \pm 1$ is a classical spin variable and i denotes a site of a d-dimensional cubic lattice. h_i/β is an external magnetic field and $K_{ij} = \beta J_{ij}$. We assume the exchange coupling constant J_{ij} to be equal to J for nearest-neighbor spins and to vanish otherwise. The mean-field Hamiltonian is obtained by writing $\sigma_i = (\sigma_i - m_i) + m_i$, with $m_i = \langle \sigma_i \rangle$, and linearizing wrt the fluctuation term $\sigma_i - m_i$,

$$\beta H_{\rm MF} = -\sum_i \Big(h_i + \sum_j K_{ij} m_j \Big) \sigma_i + \frac{1}{2} \sum_{i,j} m_i K_{ij} m_j. \tag{10.61}$$

A self-consistent equation for m_i is obtained by computing the mean value $\langle \sigma_i \rangle$ with (10.61),

$$m_i = \frac{\mathrm{Tr}\big(\sigma_i e^{-\beta H_{\rm MF}}\big)}{\mathrm{Tr}\big(e^{-\beta H_{\rm MF}}\big)} = \tanh\Big(h_i + \sum_j K_{ij} m_j \Big). \tag{10.62}$$

In the absence of external field ($h_i = 0$) and for a uniform order parameter $m_i = m$, we obtain

$$m = \tanh(2dKm), \tag{10.63}$$

where $K = \beta J$. Equation (10.63) admits a nonzero solution when $2dK > 1$, i.e. below the mean-field transition temperature $T_{c0} = 2dJ$.

To rewrite the Ising model as a functional integral over a continuous field φ_i, we use the identity

$$e^{\frac{1}{2}\sum_{i,j} \sigma_i K_{ij} \sigma_j} = \int_{-\infty}^{\infty} \prod_i d\varphi_i \, e^{-\frac{1}{2}\sum_{i,j} \varphi_i K^{-1}_{ij} \varphi_j + \sum_i \varphi_i \sigma_i} \tag{10.64}$$

(Hubbard-Stratonovich transformation) and rewrite the partition function as

$$Z = \sum_{\{\sigma_i\}} \int_{-\infty}^{\infty} \prod_i d\varphi_i \, e^{-\frac{1}{2}\sum_{i,j} (\varphi_i - h_i) K^{-1}_{ij} (\varphi_j - h_j) + \sum_i \varphi_i \sigma_i} \tag{10.65}$$

(we have shifted the field $\varphi_i \to \varphi_i - h_i$). The sum over the discrete variables σ_i can be done,

$$\sum_{\{\sigma_i\}} e^{\sum_i \varphi_i \sigma_i} = \prod_i 2\cosh(\varphi_i), \tag{10.66}$$

which leads to

$$Z = \int_{-\infty}^{\infty} \prod_i d\varphi_i \, e^{-\frac{1}{2}\sum_{i,j} (\varphi_i - h_i) K^{-1}_{ij} (\varphi_j - h_j) + \sum_i \ln(2\cosh\varphi_i)}. \tag{10.67}$$

The Fourier transform of K_{ij} is given by

$$K(\mathbf{p}) = 2K \sum_{\nu=1}^{d} \cos(q_\nu) = 2K \left(d - \frac{\mathbf{p}^2}{2} \right) + \mathcal{O}(p_\nu^4), \tag{10.68}$$

so that

$$K^{-1}(\mathbf{p}) = (2Kd)^{-1} \left(1 + \frac{\mathbf{p}^2}{2d} \right) + \mathcal{O}(p_\nu^4), \tag{10.69}$$

where we have taken the lattice spacing as the unit length. Assuming the field φ to be small (which allows us to expand $\ln(2\cosh\varphi_i)$) and slowly varying (which justifies the continuum limit), we obtain the action

$$S[\varphi] = \int d^d r \left[\left(\frac{1}{4Kd} - \frac{1}{2} \right) \varphi^2 + \frac{1}{8Kd^2} (\boldsymbol{\nabla}\varphi)^2 + \frac{1}{12}\varphi^4 \right] \tag{10.70}$$

in zero magnetic field. With an appropriate rescaling of the field, equation (10.70) takes the form (10.27) with a momentum cutoff $\Lambda \sim \pi$, $N = 1$, and

$$\frac{r_0}{2} \simeq \frac{d}{T_{c0}}(T - T_{c0}), \qquad \frac{u_0}{4!} \simeq \frac{4J^2 d^4}{3T_{c0}^2} = \frac{d^2}{3} \tag{10.71}$$

for T near the mean-field transition temperature T_{c0}. It should be noted that the derivation of (10.70) is questionable since the matrix K_{ij} has no inverse. This difficulty is circumvented in (10.68) and (10.69) by considering the small $\mathbf{p}$ limit of $K(\mathbf{p})$.

This derivation can easily be generalized to a spin Hamiltonian $H = \frac{1}{2}\sum_{i,j} J_{i,j}\mathbf{S}_i \cdot \mathbf{S}_j$ where $\mathbf{S}_i$ is a N-component spin with $\mathbf{S}_i^2 = 1$. $N = 2$ ($N = 3$) corresponds to the classical XY (Heisenberg) model. The long-distance physics of this model is described by the action (10.27) of the $(\boldsymbol{\varphi}^2)^2$ theory with $\boldsymbol{\varphi}$ a N-component field.

The $(\boldsymbol{\varphi}^2)^2$ theory is not strictly equivalent to the original spin model since we have neglected $(\boldsymbol{\varphi}^2)^3$, $(\boldsymbol{\varphi}^2)^4$, etc., as well as higher-order derivative terms. However these terms can be ignored at the mean-field level in the close vicinity of the phase transition (Sec. 10.2.1). Furthermore, they do not affect the long-distance (universal) physics. In particular, the value of the critical exponents is the same in the spin model and in the $(\boldsymbol{\varphi}^2)^2$ theory. In the renormalization group sense, both models belong to the same universality class (Sec. 10.5). The utility of the $(\boldsymbol{\varphi}^2)^2$ theory is that it allows us to use powerful field theoretical methods to study the critical behavior (Secs. 10.6 and 10.B).[25]

10.2.3 First-order phase transitions

The Landau theory can also be used to study first-order phase transitions although the order parameter is not necessarily small at the transition (unless the transition is weakly first order). Let us consider the free energy density

$$\beta f = \beta f_0 + \frac{r_0}{2} m^2 - u_3 m^3 + \frac{u_0}{4!} m^4 \tag{10.72}$$

for a scalar order parameter, where $r_0 = \bar{r}_0(T - T_0)$ and $u_0, u_3 > 0$ are independent of temperature.[26] The cubic term in (10.72) makes the transition first order: the order parameter is discontinuous at the transition (Fig. 10.7). The phase of the system is determined by requiring the free energy to be minimum. In the high-temperature phase, $m = 0$ and $f = f_0$. The first-order transition temperature T_c and the value of the order parameter m_c at T_c are obtained by requiring that $f(m_c)$ is a minimum of $f(m)$ and that the ordered and disordered phases have the same free energy,

$$\begin{aligned} \beta f'(m) &= m\left(r_0 - 3u_3 m + \frac{u_0}{6} m^2 \right) = 0, \\ \beta f(m) &= m^2 \left(\frac{r_0}{2} - u_3 m + \frac{u_0}{4!} m^2 \right) = 0, \end{aligned} \tag{10.73}$$

[25] See also chapter 11.

[26] Note that if the order parameter vanishes in the disordered phase, then $\partial f/\partial m|_{m=0} = 0$ and there is no linear term in the free energy density $f(m)$.

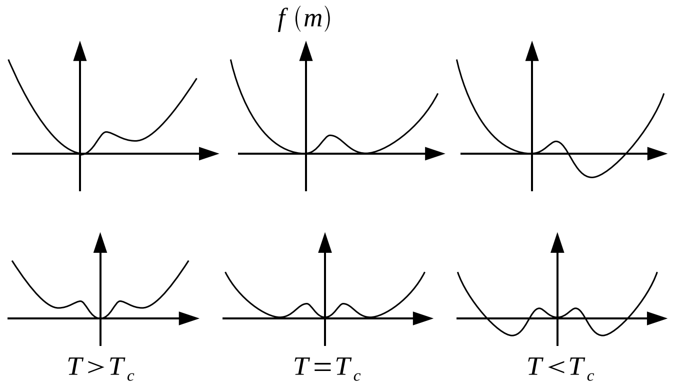


Figure 10.7: Free energy density $f(m)$ near a first-order phase transition. $f(m)$ is defined by (10.72) (top) and (10.77) (bottom).

where we have assumed that $f_0(T_c) = 0$. This yields

$$\begin{aligned} r_{0c} &= \bar{r}_0(T_c - T_0) = 12\frac{u_3^2}{u_0}, \\ m_c &= 12\frac{u_3}{u_0}. \end{aligned} \tag{10.74}$$

In the disordered phase, the entropy density $s = S/V = -\partial f/\partial T$ is simply given by $s = -\partial f_0/\partial T$. In the ordered phase, an elementary calculation gives

$$s = -\frac{\partial f_0}{\partial T} - \frac{\bar{r}_0}{2}m^2T. \tag{10.75}$$

The entropy density is therefore discontinuous at the transition from the low- to the high-temperature phase,

$$\Delta s = 2T_c\bar{r}_0\left(\frac{6u_3}{u_0}\right)^2, \tag{10.76}$$

which corresponds to a latent heat $Q/V = T_c\Delta s$ per unit volume.

Another exemple of first-order phase transition is provided by the free energy density

$$\beta f = \beta f_0 + \frac{r_0}{2}\mathbf{m}^2 + \frac{u_0}{4!}\left(\mathbf{m}^2\right)^2 + u_6\left(\mathbf{m}^2\right)^3. \tag{10.77}$$

When $u_0 > 0$, the phase transition is second order and the sixth-order term can be neglected when the order parameter is small. If $u_0 < 0$, the sixth-order term is necessary to maintain stability. In this case, the transition is first order (Fig. 10.7). The transition temperature T_c and the value of the order parameter at T_c can be calculated as in the preceding example,

$$r_{0c} = \bar{r}_0(T_c - T_0) = \begin{cases} 0 & \text{if} \quad u_0 \geq 0, \\ \dfrac{1}{2u_6}\left(\dfrac{u_0}{4!}\right)^2 & \text{if} \quad u_0 \leq 0, \end{cases} \tag{10.78}$$

and

$$\mathbf{m}_c^2 = \begin{cases} 0 & \text{if} \quad u_0 \geq 0, \\ \dfrac{|u_0|}{48u_6} & \text{if} \quad u_0 \leq 0. \end{cases} \tag{10.79}$$

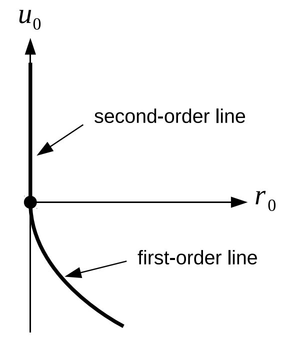


Figure 10.8: Phase diagram obtained from the free energy density (10.77).

The phase diagram in the plane (r_0, u_0) is shown in figure 10.8. The line of second-order transitions ($u_0 > 0$) and that of first-order transitions ($u_0 < 0$) meet at the tricritical point $r_0 = u_0 = 0$.

10.3 Gaussian model

The simplest improvement of Landau's theory consists in taking into account Gaussian fluctuations of the field $\boldsymbol{\varphi}$ about its saddle-point value. For $T > T_{c0}$, this amounts to neglecting the interacting part of the action. For $T < T_{c0}$, we write the field as

$$\boldsymbol{\varphi}(\mathbf{r}) = \left(\sqrt{\frac{-6r_0}{u_0}} + \varphi_1'(\mathbf{r}) \right) \mathbf{e}_1 + \sum_{i=2}^{N} \varphi_i'(\mathbf{r}) \mathbf{e}_i \tag{10.80}$$

(with $\mathbf{e}_1$ the direction of the order parameter) and expand the action to quadratic order in $\boldsymbol{\varphi}'$. This gives

$$S[\boldsymbol{\varphi}] = \begin{cases} \dfrac{1}{2} \displaystyle\sum_{\mathbf{p},i} |\varphi_i(\mathbf{p})|^2 (\mathbf{p}^2 + r_0) & \text{if} \quad T > T_{c0}, \\ S_{\rm MF} + \dfrac{1}{2} \displaystyle\sum_{\mathbf{p}} \left[|\varphi_1'(\mathbf{p})|^2 (\mathbf{p}^2 - 2r_0) + \sum_{i=2}^{N} |\varphi_i'(\mathbf{p})|^2 \mathbf{p}^2 \right] & \text{if} \quad T < T_{c0}, \end{cases} \tag{10.81}$$

for the $(\boldsymbol{\varphi}^2)^2$ theory (10.27).

10.3.1 Correlation functions

The propagator $G_{ij} = T\chi_{ij}$ deduced from the action (10.81) agrees with (10.44) and (10.45). The longitudinal propagator reads[27]

$$G_\parallel(\mathbf{p}) = \frac{1}{\mathbf{p}^2 + \xi^{-2}}, \tag{10.82}$$

with $\xi = r_0^{-1/2}$ ($|2r_0|^{-1/2}$) if $T > T_{c0}$ ($T < T_{c0}$). In the disordered phase, $G_\perp = G_\parallel$, while

$$G_\perp(\mathbf{p}) = \frac{1}{\mathbf{p}^2} \tag{10.83}$$

[27] See footnote 15 page 612.

in the ordered phase. The Gaussian fluctuations do not change the value of the order parameter $\mathbf{m} = \langle \boldsymbol{\varphi}(\mathbf{r}) \rangle$ and the transition temperature is still given by T_{c0}. The critical exponents $\beta, \delta, \gamma, \nu$ and the anomalous dimension η keep their mean-field value (Table 10.1).

Correlation functions in direct space ($d \geq 2$).[28] To obtain $G_\parallel(\mathbf{r})$ and $G_\perp(\mathbf{r})$, we need to Fourier transform $1/\mathbf{p}^2$ and $1/(\mathbf{p}^2 + \xi^{-2})$. The Fourier transform $G(\mathbf{r})$ of $1/\mathbf{p}^2$ satisfies

$$-\boldsymbol{\nabla}^2 G(\mathbf{r}) = \delta(\mathbf{r}). \tag{10.84}$$

Since $G(\mathbf{r}) = G(|\mathbf{r}|)$, $\boldsymbol{\nabla} G(\mathbf{r}) = G'(|\mathbf{r}|)\frac{\mathbf{r}}{|\mathbf{r}|}$ and equation (10.84) can be solved by considering[29]

$$\int_V d^d r\, \delta(\mathbf{r}) = -\int_V d^d r\, \boldsymbol{\nabla}^2 G(\mathbf{r}) = -\int_S d\mathbf{S} \cdot \boldsymbol{\nabla} G(\mathbf{r}) = -|\mathbf{r}|^{d-1} S_d G'(|\mathbf{r}|), \tag{10.85}$$

where V and S are the volume and the surface of the sphere of radius $|\mathbf{r}|$ centered at the origin. $S_d = 2\pi^{d/2}/\Gamma(d/2)$ is the surface of the unit sphere in d dimensions. From (10.85), we deduce $G'(|\mathbf{r}|) = -|\mathbf{r}|^{1-d}/S_d$ and

$$G(\mathbf{r}) = \int \frac{d^d p}{(2\pi)^d} \frac{e^{i\mathbf{p}\cdot\mathbf{r}}}{\mathbf{p}^2} = \begin{cases} \dfrac{1}{(d-2)S_d |\mathbf{r}|^{d-2}} & \text{if} \quad d > 2, \\ -\dfrac{1}{2\pi} \ln|\mathbf{r}| + \text{const} & \text{if} \quad d = 2, \end{cases} \tag{10.86}$$

where we have used $\lim_{|\mathbf{r}|\to\infty} G(\mathbf{r}) = 0$ when $d > 2$. When $d = 2$, the constant can be fixed if one knows the long- or short-distance behavior of $G(\mathbf{r})$.

We now consider the Fourier transform $G(\mathbf{r}) = G(|\mathbf{r}|)$ of $1/(\mathbf{p}^2 + \xi^{-2})$. It satisfies

$$\left(-\boldsymbol{\nabla}^2 + \xi^{-2}\right) G(\mathbf{r}) = \left(-\frac{\partial^2}{\partial|\mathbf{r}|^2} - \frac{d-1}{|\mathbf{r}|}\frac{\partial}{\partial|\mathbf{r}|} + \xi^{-2}\right) G(|\mathbf{r}|) = \delta(\mathbf{r}). \tag{10.87}$$

Let us try a solution $G(|\mathbf{r}|) \propto e^{-|\mathbf{r}|/\xi}/|\mathbf{r}|^p$ which decays exponentially at large distance. For $|\mathbf{r}| \neq 0$, equation (10.87) is satisfied if

$$\frac{p(p+1)}{|\mathbf{r}|^2} + \frac{2p}{|\mathbf{r}|\xi} + \frac{1}{\xi^2} - \frac{d-1}{|\mathbf{r}|}\left(\frac{p}{|\mathbf{r}|} + \frac{1}{\xi}\right) - \frac{1}{\xi^2} = 0. \tag{10.88}$$

The choice of ξ as the decay length ensures that the constant term vanishes. For $|\mathbf{r}| \ll \xi$, the $1/|\mathbf{r}|^2$ terms are the most important and we must have $p(p+1) - p(d-1) = 0$, i.e. $p = d-2$ or $p = 0$. Since for $|\mathbf{r}| \ll \xi$ we must reproduce the result (10.86), we deduce $p = d-2$ and

$$\int \frac{d^d p}{(2\pi)^d} \frac{e^{i\mathbf{p}\cdot\mathbf{r}}}{\mathbf{p}^2 + \xi^{-2}} \simeq \frac{e^{-|\mathbf{r}|/\xi}}{(d-2)S_d|\mathbf{r}|^{d-2}} \quad \text{if} \quad |\mathbf{r}| \ll \xi \quad \text{and} \quad d > 2. \tag{10.89}$$

The $|\mathbf{r}|$ dependence is logarithmic when $d = 2$ [Eq. (10.86)] and the assumption $G(|\mathbf{r}|) \propto e^{-|\mathbf{r}|/\xi}/|\mathbf{r}|^p$ is not correct. For $|\mathbf{r}| \gg \xi$, the $1/|\mathbf{r}|\xi$ terms dominate in (10.88) and therefore $2p - (d-1) = 0$. We thus obtain

$$\int \frac{d^d p}{(2\pi)^d} \frac{e^{i\mathbf{p}\cdot\mathbf{r}}}{\mathbf{p}^2 + \xi^{-2}} \sim \frac{\xi^{(3-d)/2}}{(d-2)S_d} \frac{e^{-|\mathbf{r}|/\xi}}{|\mathbf{r}|^{(d-1)/2}} \quad \text{if} \quad |\mathbf{r}| \gg \xi, \tag{10.90}$$

where the prefactor is fixed from the condition that (10.89) and (10.90) match for $|\mathbf{r}| \sim \xi$.

---

[28] In one dimensional, a straightforward calculation gives $\int_{-\infty}^{\infty} \frac{dq}{2\pi} \frac{e^{iqx}}{q^2+\xi^{-2}} = \frac{\xi}{2} e^{-|x|/\xi}$.

[29] Eq. (10.84) is Poisson's equation satisfied by the potential $G(|\mathbf{r}|)$ created by a charge located at $\mathbf{r} = 0$ and (10.85) is nothing but Gauss' theorem.

These results can also be obtained from the exact solution

$$G(\mathbf{r}) = (2\pi)^{-d/2} \frac{\xi^{(2-d)/2}}{|\mathbf{r}|^{(d-2)/2}} K_{(d-2)/2}(|\mathbf{r}|/\xi), \tag{10.91}$$

where K_n is the modified Bessel function of the second kind. Equations (10.89) and (10.90) are then simply obtained from the following asymptotic properties,

$$\begin{aligned} K_n(x) &\simeq \left(\frac{\pi}{2x}\right)^{1/2} e^{-x} && (x \gg |n^2 - 1/4|), \\ K_n(x) &\simeq \frac{\Gamma(n)}{2} \left(\frac{x}{2}\right)^{-n} && (x \ll \sqrt{n+1} \text{ and } n > 0), \\ K_0(x) &\simeq -\ln\left(\frac{x}{2}\right) - \gamma && (x \ll 1), \end{aligned} \tag{10.92}$$

where $\gamma \simeq 0.5772$ is the Euler constant and $\Gamma(z)$ the Gamma function. The last equation gives the behavior of the two-dimensional correlation function in the limit $|\mathbf{r}|/\xi \ll 1$ in agreement with (10.86). For $d = 3$, one deduces from (10.92) with $n = 1/2$ that

$$G(\mathbf{r}) = \frac{e^{-|\mathbf{r}|/\xi}}{4\pi|\mathbf{r}|}. \tag{10.93}$$

10.3.2 Goldstone's theorem

In the broken-symmetry phase, $G_\perp(\mathbf{p}) = 1/\mathbf{p}^2$ and the uniform transverse susceptibility $\chi_\perp = \beta G_\perp(\mathbf{p} = 0)$ is infinite: it requires an infinitesimal field to rotate the direction of the magnetization. This result is due to the existence of soft modes, i.e. field configurations $\boldsymbol{\varphi}(\mathbf{p})$ whose action $S[\boldsymbol{\varphi}]$ vanishes in the long-wavelength limit $\mathbf{p} \to 0$. It can also be seen as a manifestation of Goldstone's theorem for quantum systems discussed in section 3.6.3: a spontaneously broken continuous symmetry implies the existence of a low-energy mode whose energy $\omega_{\mathbf{p}}$ vanishes for $\mathbf{p} \to 0$.[30] For the $(\boldsymbol{\varphi}^2)^2$ theory with O(N) symmetry, there are $N - 1$ Goldstone modes.[31] These modes play a crucial role since they often give the main contribution to the observables of the system. When $N = 1$, the broken symmetry is discrete and there are no Goldstone modes (as expected since it is not possible to produce slowly varying rotations from one state to an equivalent one).

Faire démonstration de la footnote?

Another important consequence of a spontaneously broken continuous symmetry is the emergence of rigidity. Let us write the field as $\boldsymbol{\varphi}(\mathbf{r}) = m(\mathbf{e}_1 + \delta\tilde{\varphi}_\parallel(\mathbf{r})\mathbf{e}_1 + \delta\tilde{\boldsymbol{\varphi}}_\perp(\mathbf{r}))$, where $\delta\tilde{\varphi}_\parallel$ denotes a longitudinal fluctuation and $\delta\tilde{\boldsymbol{\varphi}}_\perp \perp \mathbf{e}_1$ a transverse fluctuation, i.e., in the limit $\delta\tilde{\boldsymbol{\varphi}}_\perp \to 0$, a fluctuation of the direction of the magnetization. According to (10.81), the action corresponding to transverse fluctuations reads

$$S[\delta\tilde{\boldsymbol{\varphi}}_\perp] = \frac{\rho_s}{2} \sum_{\mathbf{p}} \delta\tilde{\boldsymbol{\varphi}}_\perp(-\mathbf{p}) G_\perp^{-1}(\mathbf{p}) \delta\tilde{\boldsymbol{\varphi}}_\perp(\mathbf{p}) = \frac{\rho_s}{2} \int d^d r \, (\boldsymbol{\nabla}\delta\tilde{\boldsymbol{\varphi}}_\perp)^2 \tag{10.94}$$

(with $\rho_s = m^2$), which implies that the transverse correlation function is given by

$$G_\perp(\mathbf{p}) = \frac{m^2}{\rho_s \mathbf{p}^2}, \tag{10.95}$$

[30] Goldstone's theorem requires the interactions to be short range (Sec. 3.6.3).

[31] More generally, let us consider a model with a symmetry group $\mathcal{G}$, and call $\mathcal{H}$ the subgroup of $\mathcal{G}$ which leaves the order parameter $\mathbf{m} = \langle\boldsymbol{\varphi}(\mathbf{r})\rangle$ invariant. Then there are $g - h$ Goldstone modes, where g and h are the number of generators of the Lie algebras of $\mathcal{G}$ and $\mathcal{H}$. For the $(\boldsymbol{\varphi}^2)^2$ theory, $\mathcal{G} = O(N)$, $\mathcal{H} = O(N-1)$, $g = \frac{1}{2}N(N-1)$ and $h = \frac{1}{2}(N-1)(N-2)$, so that there are $g - h = N - 1$ Goldstone modes.

in agreement with the general discussion of section 3.6.3 on spontaneous symmetry breaking and Goldstone's theorem (see Eq. (3.322)). Equation (10.94) shows that any spatial variation of the order parameter in the direction perpendicular to the ordering raises the energy of the system. In the mean-field (Landau) approximation, the stiffness $\rho_s = m^2$ is simply given by the square of the order parameter. The stiffness plays an essential role in superfluid systems where it determines the superfluid density (chapter 7). We shall see in section 10.8 that in certain two-dimensional systems, the stiffness can be finite although there is no broken symmetry.

10.3.3 Mermin-Wagner theorem – Lower critical dimension

For the ordered phase to be stable, the fluctuations of the $\boldsymbol{\varphi}$ field must be finite. For $N \geq 2$, we can study the stability by looking at the most dangerous modes, namely the transverse fluctuations $\delta\boldsymbol{\varphi}_\perp = m\delta\tilde{\boldsymbol{\varphi}}_\perp$,

$$\langle \delta\boldsymbol{\varphi}_\perp(\mathbf{r})^2 \rangle = (N-1) \int \frac{d^d p}{(2\pi)^d} G_\perp(\mathbf{p}) = (N-1) \frac{S_d}{(2\pi)^d} \int_0^\Lambda \frac{d|\mathbf{p}|}{|\mathbf{p}|^{3-d}}. \tag{10.96}$$

When $d \leq 2$, the integral is infrared divergent and the assumption of spontaneous symmetry breaking cannot be maintained: thermally excited transverse fluctuations destroy long-range order. We recover the Mermin-Wagner theorem discussed in section 3.6.4: at finite temperature, a continuous symmetry cannot be broken in dimension $d \leq 2$ in systems with sufficiently short-range interactions. The dimension at and below which fluctuations prevent long-range order is called the lower critical dimension d_c^-. For the $(\boldsymbol{\varphi}^2)^2$ theory with $N \geq 2$, $d_c^- = 2$. The Mermin-Wagner theorem does not, however, preclude a phase transition (without long-range order) in two-dimensional systems with a continuous symmetry. The most famous example of such a transition is the Berezinskii-Kosterlitz-Thouless transition in the two-dimensional XY model (Sec. 10.8).

Dire qu'on parle ici de droplets: most dangerous excitations

When $N = 1$, the broken symmetry is discrete and there are no Goldstone modes. It is nevertheless possible to determine the lower critical dimension from a simple argument based on the Ising model. We assume a d-dimensional hypercubic lattice, a coupling constant J between nearest-neighbor spins (Sec. 10.2.2), and periodic boundary conditions. In the ground state, all the spins are up. Let us now consider a configuration with an island of down spins with a linear size L equal to a fraction of the linear size N of the system. The energy of such a configuration is $E \sim 2JL^{d-1}$, while the entropy (related to the number of ways to locate the island in the system) $S \sim \ln(N-L) \sim \ln L$. In one dimension, the entropy term always dominates in the thermodynamic limit $L, N \to \infty$, and we expect the formation of islands of down spins to lower the free energy, which makes the magnetization vanish. In contrast, for $d > 1$, the energy term dominates and the presence of islands increases the free energy.[32] We then expect the magnetization to remain finite at sufficiently low temperatures.[33] We conclude that the lower critical dimension of the Ising model (and the $(\boldsymbol{\varphi}^2)^2$ theory with $N = 1$) is $d_c^- = 1$.

Commentaires sur O(1) model: kinks restore the symmetry, analog to instantons for double well.

[32] This argument does not say anything about islands with typical sizes $L \ll N$ (e.g. a single down spin). For $d > 1$, a small density of such defects does not destroy long-range order.

[33] The existence of a finite temperature phase transition in the two-dimensional Ising model is confirmed by Onsager's exact solution [48].

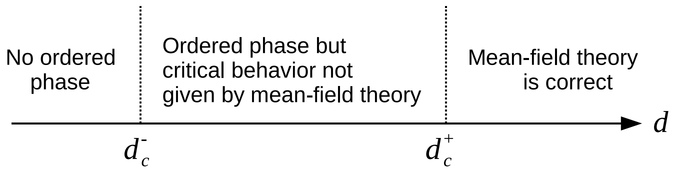


Figure 10.9: Long-range order and critical behavior *vs* lower (d_c^-) and upper (d_c^+) critical dimensions.

10.3.4 Breakdown of mean-field theory – Upper critical dimension

In the preceding section, we have seen that the mean-field theory breaks down at the lower critical dimension d_c^- since it erroneously predicts long-range order for $d \leq d_c^-$. In this section, we show that mean-field theory also breaks down below the upper critical dimension d_c^+ ($d_c^+ > d_c^-$) in a slightly more subtle way. Although long-range order is not suppressed for $d_c^- < d < d_c^+$, the critical behavior in the vicinity of a second-order phase transition is not given by mean-field theory (as already anticipated in section 10.2.1) (Fig. 10.9).

10.3.4.1 Fluctuation corrections to the specific heat – Ginzburg criterion

In the Gaussian approximation, we can integrate out the φ field and compute the free energy. In the disordered phase, one finds

$$Z = \prod_{\mathbf{p}} \left(r_0 + \mathbf{p}^2\right)^{-N/2} \tag{10.97}$$

(see equation (10.81)), where we have neglected an unimportant multiplicative constant. The (most) singular part of the specific heat (per unit volume) $c_V = -T\frac{\partial^2 f}{\partial T^2}$ reads

$$c_V^{\rm sing} = \frac{NT^2}{2V} \sum_{\mathbf{p}} \frac{\bar r_0^2}{(\mathbf{p}^2 + r_0)^2} = \frac{N}{2} T^2 \bar r_0^2 K_d \int_0^\Lambda d|\mathbf{p}| \frac{|\mathbf{p}|^{d-1}}{(\mathbf{p}^2 + r_0)^2}, \tag{10.98}$$

where $K_d = S_d/(2\pi)^d = 1/2^{d-1}\pi^{d/2}\Gamma(d/2)$. Setting $x = |\mathbf{p}|\xi = |\mathbf{p}| r_0^{-1/2}$, this result can be rewritten as

$$c_V^{\rm sing} = \frac{N}{2} T^2 \bar r_0^2 K_d \xi^{4-d} I(\Lambda\xi), \tag{10.99}$$

where

$$I(\Lambda\xi) = \int_0^{\Lambda\xi} dx \frac{x^{d-1}}{(1+x^2)^2}. \tag{10.100}$$

If $d > 4$, the integral I is dominated by the upper cutoff, $I \simeq (\Lambda\xi)^{d-4}/(d-4)$, while $I \sim \ln(\Lambda\xi)$ for $d = 4$. If $d < 4$, the integral is independent of $\Lambda\xi$. We therefore obtain

$$c_V^{\rm sing} \simeq \frac{N}{2} (T\bar r_0)^2 K_d \times \begin{cases} \dfrac{\Lambda^{d-4}}{d-4} & \text{if} \quad d > 4, \\ \ln(\Lambda\xi) & \text{if} \quad d = 4, \\ I\xi^{4-d} & \text{if} \quad d < 4. \end{cases} \tag{10.101}$$

When $d \leq 4$, the specific heat diverges as $T \to T_{c0}^+$,

$$c_V^{\rm sing} \sim \xi^{4-d} \sim (T - T_{c0})^{-\alpha}, \tag{10.102}$$

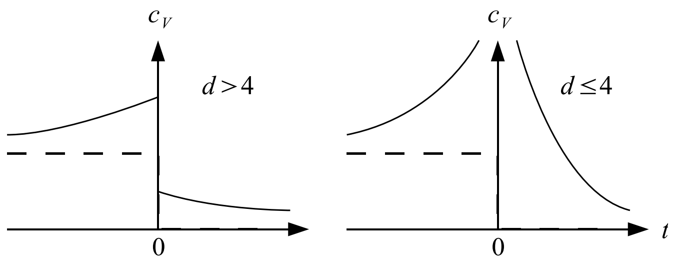


Figure 10.10: Specific heat below and above $d = 4$ in the Gaussian model. The dashed line shows the result of the mean-field approximation. The smooth contribution coming from f_0 in (10.30) is not shown.

with a critical exponent $\alpha = 2 - d/2$ (the divergence is logarithmic for $d = 4$).

A similar analysis can be made below the transition temperature T_{c0}. From (10.81), we obtain

$$Z = Z_{\rm MF} \prod_{\mathbf{p}} \left(\mathbf{p}^2 - 2r_0\right)^{-1/2} \left(\mathbf{p}^2\right)^{-(N-1)/2} . \tag{10.103}$$

We deduce the singular part of the specific heat

$$\begin{aligned} c_V^{\rm sing} &= c_{V,\rm MF}^{\rm sing} + 2(T\bar r_0)^2 K_d \xi^{4-d} I(\Lambda\xi), \\ &= c_{V,\rm MF}^{\rm sing} + 2(T\bar r_0)^2 K_d \times \begin{cases} \dfrac{\Lambda^{d-4}}{d-4} & \text{if} \quad d>4, \\ \ln(\Lambda\xi) & \text{if} \quad d=4, \\ I\xi^{4-d} & \text{if} \quad d<4, \end{cases} \end{aligned} \tag{10.104}$$

where $\xi = |2r_0|^{-1/2}$. While for $d > 4$ the specific heat remains discontinuous at the transition, the fluctuation corrections to the mean-field result yields a divergence when $d \leq 4$ (Fig. 10.10). As far as the critical behavior is concerned, this is the only change wrt mean-field theory, since all other critical exponents are unchanged (Table 10.2). The divergence of the specific heat implies that the mean-field results are not reliable below $d = 4$ in the vicinity of the phase transition: fluctuations dominate the thermodynamics and the predictions of Landau's theory are not valid. We therefore recover our previous result that the upper critical dimension is $d_c^+ = 4$ for the $(\boldsymbol{\varphi}^2)^2$ theory (Sec. 10.2.1).

Below the upper critical dimension, the mean-field theory remains valid sufficiently far away from the transition. One can estimate the temperature at which the mean-field theory breaks down by comparing the mean-field discontinuity to the fluctuation correction, $\Delta c_{V,\rm MF} \sim c_{V,\rm fl}$, i.e.

$$\Delta c_{V,\rm MF} \sim \frac{N}{2} K_d \xi_0^{-d} |t|^{d/2-2} \tag{10.105}$$

(the factor $N/2$ is absent for $t < 0$), where t is defined in (10.13). $\xi_0 \sim (\bar r_0 T_{c0})^{-1/2}$ is a microscopic length, of the order of the correlation length far away from the transition. Thus, for the mean-field theory to be valid below the upper critical dimension, the temperature must satisfy the Ginzburg criterion

$$|t| \gg t_G \sim \left(\frac{N K_d}{2\xi_0^d \Delta c_{V,\rm MF}} \right)^{1/(2-d/2)} . \tag{10.106}$$

$t_G = |T_G - T_c|/T_c$ is related to the Ginzburg temperature T_G, the temperature corresponding to the onset of the critical regime where fluctuations become important.[34]

In practice, the mean-field theory can be valid very close to the transition temperature if ξ_0 is large. This is the case in systems with long-range forces or in conventional superconductors where $\xi_0 \sim 1000$ Angströms corresponds to the BCS coherence length (Sec. 7.5) and is sufficiently large to make the critical regime unobservable. In many systems however, ξ_0 is of the order of a few Angströms, and the crossover from mean-field to critical behavior is observable.

One can also define a Ginzburg length $\xi_G = \xi(t_G) \sim \xi_0 t_G^{-1/2}$, i.e.

$$\xi_G \sim \xi_0 \left(\frac{2\xi_0^d \Delta c_{V,\mathrm{MF}}}{NK_d} \right)^{1/(4-d)} \sim \left(\frac{6}{NK_d u_0} \right)^{1/(4-d)}. \tag{10.107}$$

Below the upper critical dimension d_c^+, ξ becomes larger than ξ_G when $|t| \lesssim t_G$ (critical regime). We shall see in section 10.6 that for a critical system ($T = T_c$), ξ_G separates a high-momentum (perturbative) regime $|\mathbf{p}| \gtrsim \xi_G^{-1}$ where the Gaussian model is essentially correct from a low-momentum (critical) regime where the propagator acquires an anomalous dimension.

The fact that Gaussian fluctuations yield strong corrections to the mean-field results calls into question the validity of the Gaussian model itself when $d < 4$. One could try to perform a systematic loop expansion about the mean-field solution (see Sec. 1.7.2) and see whether it converges or not. Simple dimensional analysis is sufficient to answer this question. In the high-temperature phase, the loop expansion is merely an expansion wrt u_0. The actual expansion parameter can be identified by dimensional analysis. Since $S[\boldsymbol{\varphi}]$ is dimensionless, we must have $[(\boldsymbol{\nabla}\boldsymbol{\varphi})^2] = d$, and therefore $[\boldsymbol{\varphi}] = (d-2)/2$.[35] Similarly, one finds $[r_0] = 2$ and $[u_0] = 4 - d$.[36] In terms of the dimensionless variables

$$\tilde{\mathbf{r}} = \frac{\mathbf{r}}{\xi}, \quad \tilde{\boldsymbol{\varphi}}(\tilde{\mathbf{r}}) = \xi^{(d-2)/2} \boldsymbol{\varphi}(\mathbf{r}), \quad \tilde{u}_0 = \xi^{4-d} u_0 \tag{10.108}$$

(with $\xi = r_0^{-1/2}$), the action becomes

$$S[\tilde{\boldsymbol{\varphi}}] = \int d^d\tilde{r} \left[\frac{1}{2} (\boldsymbol{\nabla}_{\tilde{\mathbf{r}}} \tilde{\boldsymbol{\varphi}})^2 + \frac{1}{2} \tilde{\boldsymbol{\varphi}}^2 + \frac{\tilde{u}_0}{4!} \left(\tilde{\boldsymbol{\varphi}}^2 \right)^2 \right]. \tag{10.109}$$

Il faut aussi vérifier que les intégrales convergent

In dimension $d > 4$, since $\tilde{u}_0 \to 0$ when $T \to T_{c0}$, it is reasonable to expect mean-field theory to become increasingly accurate as the transition is approached. For $d < 4$ on the other hand, $\tilde{u}_0$ diverges as $T \to T_{c0}$ and perturbation theory becomes meaningless. If we estimate that perturbation theory breaks down when $\tilde{u}_0 \sim (\xi/\xi_G)^{4-d}$ becomes of order 1, we recover the Ginzburg criterion (10.106).

[34] *Stricto sensu* there are two Ginzburg temperatures, one (T_G^+) above and one (T_G^-) below T_c. Fluctuations are important in the temperature range $[T_G^-, T_G^+]$. In Sec. 10.7.3, we shall see that for $N \geq 2$ (continuous broken symmetry) the Gaussian approximation breaks down in the whole temperature phase.

[35] The notation $[A] = n$ means that the quantity A is expressed in units of L^{-n} (with L the unit length). For example, $[\mathbf{r}] = -1$ and $[\mathbf{p}] = 1$.

[36] $[r_0] = 2$ and $[u_0] = 4 - d$ are referred to as the naive scaling dimensions of r_0 and u_0 (see Sec. 10.4).

10.3.4.2 One-loop correction to the two-point vertex $\Gamma^{(2)}(\mathbf{p})$

Let us consider the lowest-order (one-loop) correction to the two-point vertex $\Gamma^{(2)}(\mathbf{p}) = G(\mathbf{p})^{-1}$ [Eq. (10.55)]. In the high-temperature phase ($m = 0$), one finds

$$\begin{aligned} \Gamma^{(2)}(\mathbf{p}) &= \mathbf{p}^2 + r_0 + \frac{N+2}{6} u_0 \int \frac{d^d q}{(2\pi)^d} G_0(\mathbf{q}) \\ &= \mathbf{p}^2 + r_0 + \frac{N+2}{6} K_d u_0 \int_0^\Lambda d|\mathbf{q}| \frac{|\mathbf{q}|^{d-1}}{\mathbf{q}^2 + r_0}, \end{aligned} \tag{10.110}$$

where $G_0(\mathbf{q}) = (\mathbf{q}^2 + r_0)^{-1}$ is the bare propagator. We deduce

$$\begin{aligned} \Gamma^{(2)}(\mathbf{p} = 0) = r &= r_0 + \frac{N+2}{6} K_d u_0 \int_0^\Lambda d|\mathbf{q}| \frac{|\mathbf{q}|^{d-1}}{\mathbf{q}^2 + r_0} \\ &\simeq r_0 + \frac{N+2}{6} K_d u_0 \int_0^\Lambda d|\mathbf{q}| \frac{|\mathbf{q}|^{d-1}}{\mathbf{q}^2 + r}, \end{aligned} \tag{10.111}$$

where the replacement of r_0 by r in the last term introduces corrections of order u_0^2 which are beyond the one-loop accuracy. The critical temperature is obtained from the condition

$$0 = r = \bar{r}_0(T_c - T_{c0}) + \frac{N+2}{6} K_d u_0 \int_0^\Lambda d|\mathbf{q}|\, |\mathbf{q}|^{d-3}. \tag{10.112}$$

For $d > 2$, the integral in the rhs is convergent and the perturbative calculation of the shift $T_c - T_{c0}$ of the transition temperature makes sense. Subtracting (10.112) from (10.111) to eliminate T_{c0}, we obtain

$$r = \bar{r}_0(T - T_c) - \frac{N+2}{6} K_d u_0 r \int_0^\Lambda d|\mathbf{q}| \frac{|\mathbf{q}|^{d-1}}{\mathbf{q}^2(\mathbf{q}^2 + r)}. \tag{10.113}$$

If $d > 4$, the integral converges even when $r = 0$. In the limit $r \to 0$, we then obtain

$$r = \bar{r}_0(T - T_c) - Cr = \frac{\bar{r}_0(T - T_c)}{1 + C}, \tag{10.114}$$

with C a constant. Since $\chi^{-1} \sim r \sim T - T_c$, the susceptibility critical exponent keeps its mean-field value $\gamma = 1$. On the other hand, the integral in (10.113) diverges when $d \leq 4$ and $r \to 0$, thus signaling the breakdown of the perturbative expansion. For $d < 4$, we can write

$$r = \bar{r}_0(T - T_c) - \frac{N+2}{6} u_0 \tilde{K}_d r (\sqrt{r})^{d-4}, \tag{10.115}$$

where

$$\tilde{K}_d = K_d \int_0^{\Lambda/\sqrt{r}} dx \frac{x^{d-1}}{x^2(x^2+1)} \simeq K_d \int_0^\infty dx \frac{x^{d-1}}{x^2(x^2+1)} = -\frac{\pi}{2} \frac{K_d}{\sin(\pi d/2)}. \tag{10.116}$$

Since the integral is convergent for $|x| \to \infty$, we have taken the limit $\Lambda \to \infty$. Equation (10.115) is incompatible with $r \sim T - T_c$ and the value of the critical exponent γ cannot be equal to the mean-field prediction $\gamma = 1$. We can recover the Ginzburg criterion (10.106) for the validity of the mean-field approximation by demanding the last term in (10.115) to be a small correction,

$$\frac{N+2}{6} u_0 \tilde{K}_d (\sqrt{r})^{d-4} \simeq \frac{N+2}{6} u_0 \tilde{K}_d [\bar{r}_0(T - T_c)]^{d/2-2} \ll 1. \tag{10.117}$$

10.4 The scaling hypothesis

An essential feature of second-order phase transitions is the divergence of the correlation length $\xi \sim |t|^{-\nu}$. The scaling hypothesis states that this divergence is responsible for the singular dependence on $t = |T-T_c|/T_c$ of physical quantities. It leads to "scaling laws", i.e. relations between critical exponents (Table 10.3). The scaling hypothesis will be justified by the RG approach in section 10.5.3.

If ξ were the only relevant length scale *stricto sensu*, the singular behavior could be simply obtained from (naive) dimensional analysis. For a physical quantity X with naive scaling dimension $d_X^0 = [X]$ (also called engineering dimension),[37] the singular part would indeed be given by $X \sim \xi^{-d_X^0}$. We shall see below that the singular behavior is determined by a scaling dimension d_X which may differ from d_X^0. The difference between d_X and d_X^0 determines the "anomalous" dimension of X. For the latter to be nonzero without violating dimensional analysis, another characteristic length a must necessarily be involved (besides the correlation length ξ) to yield the (dimensionally correct) result $X \sim \xi^{-d_X} a^{d_X - d_X^0}$.[38]

10.4.1 Scaling form of the correlation function

Let us first consider the propagator $G(\mathbf{p}) = \langle \varphi(\mathbf{p})\varphi(-\mathbf{p})\rangle$ in the φ^4 theory with a one-component field. From (naive) dimensional analysis, $[\varphi(\mathbf{r})] = d/2-1$ and $[G(\mathbf{p})] = -2$, one can write the singular part of $G(\mathbf{p})$ in the scaling form

$$G(\mathbf{p}) = \frac{1}{\mathbf{p}^2}\mathcal{G}\left(|\mathbf{p}|\xi, \frac{a}{\xi}\right), \tag{10.118}$$

where a is a characteristic length which does not diverge at the transition.[39] The following discussion can be straightforwardly generalized to the case where the function $\mathcal{G}$ depends on several lengths a_1, a_2... We assume that in the limit $|\mathbf{p}|\xi \to \infty$, $|\mathbf{p}|a \to 0$ and $a/\xi \to 0$, $\mathcal{G}$ behaves as

$$\mathcal{G}\left(|\mathbf{p}|\xi, \frac{a}{\xi}\right) \sim (|\mathbf{p}|\xi)^{x_1}(a/\xi)^{x_2} \qquad (\xi \to \infty). \tag{10.119}$$

For $G(\mathbf{p})$ to be defined and nonzero at the critical point, we must have $x_1 = x_2 \equiv \eta$. We deduce

$$G(\mathbf{p}, T_c) \sim \frac{a^\eta}{|\mathbf{p}|^{2-\eta}}. \tag{10.120}$$

More generally, in the vicinity of the critical point, one can write[40]

$$\begin{aligned} G(\mathbf{p}) &= \frac{1}{|\mathbf{p}|^{2-\eta}}\mathcal{G}\left(|\mathbf{p}|\xi, \frac{a}{\xi}\right) \\ &= \frac{1}{|\mathbf{p}|^{2-\eta}}\left[\mathcal{G}(|\mathbf{p}|\xi, 0) + \text{higher powers of } a/\xi\right], \end{aligned} \tag{10.121}$$

[37] Recall that a quantity X has (naive) scaling dimension $d_X^0 = [X]$ if it is expressed in physical units of $L^{-d_X^0}$ (with L the unit length).

[38] For a thorough discussion of anomalous dimensions, see [8].

[39] a is typically the Ginzburg length ξ_G (Sec. 10.3.4), which in many model is of the order of the lattice spacing (i.e. the inverse of the upper momentum cutoff Λ of the theory).

[40] To avoid a proliferation of symbols, we denote different scaling functions by the same symbol. For instance, in Eqs. (10.118,10.121,10.124) $\mathcal{G}$ denote different functions.

where the scaling function $\mathcal{G}$ has a well-defined limit when $\xi \to \infty$ ($\mathcal{G} \sim a^\eta$). Consider now the change $\mathbf{p} \to s\mathbf{p}$, $\xi \to \xi/s$ and $a \to a/s$. Recalling that $\mathcal{G} \sim a^\eta$, we have $G(s\mathbf{p}\xi/s, a/s) = s^{-2}G(\mathbf{p}, \xi, a)$ in agreement with the naive scaling dimension $[G(\mathbf{p})] = -2$ of the propagator.

If we are interested in the long-distance physics near the phase transition, we can take the limit $\xi/a \to \infty$ while keeping $|\mathbf{p}|\xi$ fixed. From (10.121) we then obtain

$$G(\mathbf{p}, \xi) = \frac{1}{|\mathbf{p}|^{2-\eta}}\mathcal{G}(|\mathbf{p}|\xi), \qquad G(s\mathbf{p}, \xi/s) = s^{-2+\eta}G(\mathbf{p}, \xi), \tag{10.122}$$

and it appears that the field has acquired an anomalous dimension,

$$[\varphi(\mathbf{p})] = -1 + \eta/2, \quad \text{i.e.} \quad d_\varphi = [\varphi(\mathbf{r})] = \frac{d}{2} - 1 + \frac{\eta}{2} = d_\varphi^0 + \frac{\eta}{2}. \tag{10.123}$$

$d_\varphi^0 = d/2 - 1$ is often referred to as the canonical dimension of the φ field and $\eta/2$ as its anomalous dimension.[41] The correlation function can also be written in the form

$$G(\mathbf{p}, \xi) = T\chi\mathcal{G}(|\mathbf{p}|\xi), \tag{10.124}$$

where $\chi = T^{-1}G(\mathbf{p} = 0, \xi)$ is the susceptibility. The function $\mathcal{G}(x)$ is then a universal scaling function (independent of the parameters of the model). In particular, $\mathcal{G}(0) = 1$.

If we consider the propagator to be a function of $(\mathbf{p}, t)$, instead of $(\mathbf{p}, \xi)$, one has

$$G(\mathbf{p}, t) = \frac{1}{|\mathbf{p}|^{2-\eta}}\mathcal{G}_\pm(|\mathbf{p}||t|^{-\nu}), \qquad G(s\mathbf{p}, s^{1/\nu}t) = s^{-2+\eta}G(\mathbf{p}, t), \tag{10.125}$$

or, in real space,

$$G(\mathbf{r}, t) = \frac{1}{|\mathbf{r}|^{d-2+\eta}}\mathcal{G}_\pm(|\mathbf{r}||t|^{\nu}), \qquad G(\mathbf{r}/s, s^{1/\nu}t) = s^{d-2+\eta}G(\mathbf{r}, t). \tag{10.126}$$

In the presence of a magnetic field, equations (10.126) become

$$\begin{aligned} G(\mathbf{r}, t, h) &= s^{-d+2-\eta}G(\mathbf{r}/s, s^{1/\nu}t, s^{d_h}h) \\ &= \frac{1}{|\mathbf{r}|^{d-2+\eta}}\mathcal{G}_\pm\left(|\mathbf{r}||t|^\nu, \frac{h}{|t|^\Delta}\right), \end{aligned} \tag{10.127}$$

where d_h denotes the scaling dimension of the field (see Sec. 10.4.2) and $\Delta = \nu d_h$ is sometimes referred to as the gap exponent. We now allow for different scaling functions, $\mathcal{G}_+$ and $\mathcal{G}_-$, above and below the critical temperature T_c, and assume $\nu = \nu'$ (this assumption will be justified below). Equations (10.125-10.127) show that the correlation function satisfies a generalized homogeneity relation.[42] Using (10.125) with $\mathbf{p} = 0$ and $s = |t|^{-\nu}$, one obtains $\chi \sim G(\mathbf{p} = 0, t) \sim |t|^{\nu(\eta-2)}$, i.e.

$$\gamma = \gamma' = \nu(2 - \eta). \tag{10.128}$$

This result,[43] which relates the critical exponents γ, ν and η, is called a scaling law.

[41] It is customary to refer to η (and not $\eta/2$) as the anomalous dimension.

[42] A function $f(x)$ is said to be homogeneous if it satisfies $f(sx) = s^k f(x)$ for any rescaling factor s. More generally, a function $f(x_1, \cdots, x_n)$ is homogeneous if $f(s^{k_1}x_1, \cdots, s^{k_n}x_n) = s^k f(x_1, \cdots, x_n)$.

[43] Similarly, from (10.125) with $t = 0$ and $s = 1/|\mathbf{p}|$, one obtains $G(\mathbf{p}, t = 0) \sim 1/|\mathbf{p}|^{2-\eta}$ in agreement with (10.120).

Fisher	$\gamma = \nu(2-\eta)$
Rushbrooke	$\alpha + 2\beta + \gamma = 2$
Widom	$\beta(\delta - 1) = \gamma$
Josephson	$\alpha = 2 - \nu d$

Table 10.3: Scaling laws [Eqs. (10.128,10.131,10.136,10.138)]. All exponents are the same above and below T_c.

In practice, we never consider explicitly microscopic lengths, as we did in this section, to derive scaling forms of correlation functions or thermodynamic quantities. We simply consider the scaling dimensions of the quantity of interest and its argument. For instance, in the case of the correlation function $G(\mathbf{p})$, we use $d_\varphi = (d-2+\eta)/2$, $[\mathbf{p}] = 1$, $[\xi] = -1$ and dimensional analysis to directly obtain equation (10.122), where the scaling function $\mathcal{G}$ is universal (in a sense that will be thoroughly explained in section 10.5.3) except for its amplitude.[44] The anomalous dimension η introduced here is *a priori* unknown and could in fact be zero. We shall now show that all other critical exponents defined in section 10.1.3, namely α, β, γ and δ, are fully determined by ν and η.

10.4.1.1 Scaling of the stiffness

The preceding discussion applies with no change to the disordered phase of the O(N)-symmetric $(\boldsymbol{\varphi}^2)^2$ theory where $G_{ij} = \delta_{i,j} G$. The ordered phase is characterized by long-range transverse correlations and a finite stiffness ρ_s [Eq. (10.95)]. ρ_s vanishes as $t \to 0^-$ with an exponent x which can be obtained from dimensional analysis. From (10.94), since the action is dimensionless while $[\delta\tilde{\boldsymbol{\varphi}}_\perp(\mathbf{r})] = 0$, we deduce that

$$[\rho_s] = d - 2, \tag{10.129}$$

i.e.

$$\rho_s \sim \xi^{-(d-2)} \sim (-t)^{\nu(d-2)} \sim (-t)^{2\beta - \nu\eta} \tag{10.130}$$

for $t \to 0^-$, where the last result is obtained using the scaling law (10.131) derived in section 10.4.2.[45,46] When the anomalous dimension vanishes, one obtains $\rho_s \sim (-t)^{2\beta}$ as in Landau's theory ($\rho_s = m^2$).

10.4.2 Scaling form of the free energy density

The magnetization density m being the average value of the field, one expects $m \sim \xi^{-d_\varphi} \sim (-t)^{\nu d_\varphi}$ for $t < 0$,[46] i.e.

$$\beta = \nu d_\varphi = \frac{\nu}{2}(d - 2 + \eta). \tag{10.131}$$

[44] Note that for a scaling function to be universal (possibly up to a nonuniversal normalization) its arguments must have both vanishing scaling dimensions and dimensionless physical units.

[45] In Eq. (10.130) ξ should be understood as the Josephson length; see Secs. 10.1.4 and 10.7.2.

[46] Since ρ_s vanishes in the high-temperature phase, it has no regular part for $t \to 0^-$. The same is true for the order parameter m.

Since $\ln Z$ is dimensionless and $[V] = -d$, the free energy density $f = -T \ln Z/V$ does not carry an anomalous dimension and has scaling dimension d. Its singular part satisfies $f_s \sim \xi^{-d}$.[47] From

$$m = -\frac{\partial f_s}{\partial H} = -\frac{1}{T}\frac{\partial f_s}{\partial h}, \tag{10.132}$$

we deduce that the magnetic field has scaling dimension

$$d_h = d - d_\varphi = \frac{1}{2}(d + 2 - \eta). \tag{10.133}$$

In practice, the anomalous dimension η is small and d_h is positive. Using dimensional analysis, we can therefore write the singular part of the free energy density in the scaling form

$$f_s(t,h) = \xi^{-d}\mathcal{F}_\pm(h\xi^{d_h}). \tag{10.134}$$

In zero field, $f_s = \xi^{-d}\mathcal{F}_\pm(0) \sim |t|^{d\nu}$, and we obtain

$$c_V = T\frac{\partial^2 f_s}{\partial T^2} \sim |t|^{d\nu-2}, \tag{10.135}$$

i.e.

$$\alpha = \alpha' = 2 - \nu d. \tag{10.136}$$

For the magnetization $m \sim \xi^{d_h-d}\mathcal{F}'_\pm(h\xi^{d_h})$ to be defined at T_c, we must have $\mathcal{F}'_\pm(x) \sim x^{(d-d_h)/d_h} = x^{d_\varphi/d_h}$ for $x \to \infty$. This implies

$$m(T_c) \sim h^{d_\varphi/d_h}, \tag{10.137}$$

and a critical exponent

$$\delta = \frac{d_h}{d_\varphi} = \frac{d+2-\eta}{d-2+\eta}. \tag{10.138}$$

The scaling form (10.134) of the free energy density is often written as

$$f_s(t,h) = |t|^{2-\alpha}\mathcal{F}_\pm\left(\frac{h}{|t|^\Delta}\right), \tag{10.139}$$

where $\Delta = \nu d_h = 2 - \alpha - \beta$ is the gap exponent. The equation of state then takes the form[48]

$$m = -\frac{1}{T}\frac{\partial f_s}{\partial h} = -\frac{|t|^\beta}{T}\mathcal{F}'_\pm\left(\frac{h}{|t|^\Delta}\right). \tag{10.140}$$

It seems that we could postulate a more general form,

$$f_s(t,h) = |t|^{2-\alpha_\pm}\mathcal{F}_\pm\left(\frac{h}{|t|^{\Delta_\pm}}\right), \tag{10.141}$$

with different exponents for $t > 0$ and $t < 0$. For $h \neq 0$, i.e. away from the critical point $t = h = 0$, the free energy density must be analytic in t,

$$f_s(t,h) = f_0(h) + tf_1(h) + \mathcal{O}(t^2) \qquad (h \neq 0). \tag{10.142}$$

[47] It would be more correct to write $f \sim T\xi^{-d}$. However, the factor T is not singular and can be omitted.
[48] In Eq. (10.140) the scaling function $\mathcal{F}_\pm$ cannot be "fully" universal for reasons given in footnote 44.

Expanding (10.141) for $t \to 0$, we obtain

$$f_s(t,h) = |t|^{2-\alpha_\pm} \left[A_\pm \left(\frac{h}{|t|^{\Delta_\pm}} \right)^{p_\pm} + B_\pm \left(\frac{h}{|t|^{\Delta_\pm}} \right)^{q_\pm} + \cdots \right], \tag{10.143}$$

where $p_\pm, q_\pm$ are the leading powers in the expansion of $g_\pm$ for large arguments ($\Delta_\pm$ is positive). Equations (10.142) and (10.143) require $p_\pm \Delta_\pm = 2 - \alpha_\pm$ and $q_\pm \Delta_\pm = 1 - \alpha_\pm$, so that

$$f_s(t,h) = A_\pm h^{(2-\alpha_\pm)/\Delta_\pm} + B_\pm h^{(1-\alpha_\pm)/\Delta_\pm} |t| + \mathcal{O}(t^2). \tag{10.144}$$

Continuity at $t = 0$ forces $(2-\alpha_+)/\Delta_+ = (2-\alpha_-)/\Delta_-$ and $(1-\alpha_+)/\Delta_+ = (1-\alpha_-)/\Delta_-$, which implies $\alpha_+ = \alpha_-$ and $\Delta_+ = \Delta_-$. We conclude that the critical exponents are necessary the same for $t > 0$ and $t < 0$, which justifies the assumption $\nu = \nu'$ used earlier.

Scaling laws, summarized in Table 10.3, imply that there are only two independent critical exponents, e.g. ν and η. Relations involving d are called hyperscaling relations. They apply only to transitions that are fluctuation dominated, i.e. at or below the upper critical dimension d_c^+. Hyperscaling is satisfied by the mean-field exponents only at the upper critical dimension. Above d_c^+, it breaks down because of the existence of a dangerously irrelevant variable in the renormalization-group sense (see Sec. 10.6.1).

Scaling functions in mean-field theory. For $N = 1$, we can rewrite the correlation function obtained within Landau's theory [Eqs. (10.44,10.45)] in the scaling form

$$G(\mathbf{p}) = T\chi \mathcal{G}(|\mathbf{p}|\xi), \tag{10.145}$$

with $\chi = T^{-1}\xi^2$ and $\mathcal{G}(x) = (1+x^2)^{-1}$. This Lorentzian form of the correlation function was first proposed by Ornstein and Zernicke. The free energy density

$$f(r_0,h) = \begin{cases} -\dfrac{3}{2}\dfrac{Tr_0^2}{u_0} & \text{if} \quad h = 0 \text{ and } r_0 \leq 0, \\ -3\dfrac{6^{1/3}}{4}\dfrac{Th^{4/3}}{u_0^{1/3}} & \text{if} \quad r_0 = 0, \end{cases} \tag{10.146}$$

can be written in the scaling form

$$f(r_0,h) = r_0^2 \mathcal{F}\left(\frac{h}{|r_0|^{3/2}} \right), \tag{10.147}$$

where the scaling function $\mathcal{F}$ satisfies

$$\begin{aligned} \lim_{x\to 0} \mathcal{F}(x) &= -\frac{3T}{2u_0}, \\ \lim_{x\to\infty} \mathcal{F}(x) &= -3\frac{6^{1/3}}{4}\frac{Tx^{4/3}}{u_0^{1/3}}. \end{aligned} \tag{10.148}$$

10.4.3 Finite-size scaling

Experiments, as well as many numerical calculations, use finite systems.[49] How does the finite size of a system affect the various scaling forms discussed above and suppress a phase transition that would exist in the infinite-size limit? To answer this question one simply

[49] Finite-size scaling plays also an important role in the study of quantum phase transitions (Sec. 12.2).

considers the size L of the system as an additional relevant length scale. Since $[L]=-1$ the scaling form of the free energy, in the absence of a magnetic field, becomes

$$f_s(t,L)=\frac{1}{\xi_\infty}\mathcal{F}_\pm\left(\frac{\xi_\infty}{L}\right) \quad \text{or, equivalently,} \quad f_s(t,L)=|t|^{2-\alpha}\mathcal{F}_\pm\left(\frac{|t|^{-\nu}}{L}\right), \tag{10.149}$$

where $\xi_\infty(t)\sim|t|^{-\nu}$ is the correlation length of the infinite system. We expect $\mathcal{F}_\pm(x\ll 1)\simeq \text{const}$ so that when $L\gg\xi_\infty$ the thermodynamic properties are those of the infinite-size system. By contrast, when $L\ll\xi_\infty$, the system is no longer governed by the critical point. The actual correlation length $\xi(t,L)$ cannot grow beyond L and the phase transition appears rounded.

One can write the correlation length of the finite-size system in the scaling form

$$\xi(t,L)=L\,\Xi\left(\frac{\xi_\infty}{L}\right) \quad \text{or} \quad \xi(t,L)=L\,\Xi\left(\frac{|t|^{-\nu}}{L}\right). \tag{10.150}$$

We expect $\Xi(x\ll 1)\simeq x$ and $\Xi(x\gg 1)\simeq \text{const}$ so that $\xi(t,L)\simeq\xi_\infty$ for $L\gg\xi_\infty$ and $\xi(t,L)\sim L$ for $L\ll\xi_\infty$. In the latter case, $\xi(t,L)$ should be an analytic function of t, i.e.

$$\frac{L}{\xi(t,L)}=A+BtL^{1/\nu}+\mathcal{O}(t^2). \tag{10.151}$$

Thus if we plot $L/\xi(t,L)$ vs t (i.e., in practice, vs a coupling constant K) for various values of L, all the curves will pass through the same point when $t=0$ (i.e. $K=K_c$). This allows us to determine K_c and the critical exponent ν using

$$\ln\frac{\partial}{\partial K}\frac{L}{\xi(t,L)}\bigg|_{K=K_c}=\text{const}+\frac{1}{\nu}\ln L. \tag{10.152}$$

10.5 The renormalization group

In the previous sections, we have seen that standard perturbation theory and mean-field approaches break down below the upper critical dimension. In this section, we discuss an alternative method, Wilson's renormalization group (RG). Instead of considering all degrees of freedom on the same footing, one first integrates out short-distance (or high-energy) degrees of freedom. This leads to an effective theory for the long-distance (low-energy) degrees of freedom. This approach is interesting in particular (but not only) for the study of critical phenomena where (at least for universal quantities) an effective description based on the low-energy degrees of freedom should be sufficient.

10.5.1 Renormalization-group transformations

A transformation whereby a subset of (short-distance) degrees of freedom is integrated out is called a RG transformation. For definiteness we consider a field theory with a N-component real field $\boldsymbol{\varphi}(\mathbf{r})$ as in the $(\boldsymbol{\varphi}^2)^2$ theory discussed in previous sections and assume an ultraviolet momentum cutoff Λ. We denote by $S[\boldsymbol{\varphi};K]$ the action with $K=\{K_i\}$ a set of coupling constants. A RG transformation consists in two steps:

- *Mode elimination.*[50] In the first step, one eliminates the short-distance (or high-energy) degrees of freedom. This is achieved by writing the field as

$$\boldsymbol{\varphi}(\mathbf{r})=\boldsymbol{\varphi}_<(\mathbf{r})+\boldsymbol{\varphi}_>(\mathbf{r}), \tag{10.153}$$

[50] Also called decimation or coarse graining.

where $\boldsymbol{\varphi}_<(\mathbf{r})$ has Fourier components in the range $0 \leq |\mathbf{p}| \leq \Lambda/s$ and $\boldsymbol{\varphi}_>(\mathbf{r})$ in $\Lambda/s \leq |\mathbf{p}| \leq \Lambda$ (with $s > 1$). One then integrates out the "fast" modes $\boldsymbol{\varphi}_>$ to obtain an effective action for the "slow" modes $\boldsymbol{\varphi}_<$. The action $S[\boldsymbol{\varphi}_<; K_<]$ governing the dynamics of the slow modes is defined by a new set $K_<$ of coupling constants. In general, the functional form of the action is not preserved, and the set $K_<$ is larger than K.[51]

- *Rescaling.* In the second step of the RG transformation, one rescales momenta and coordinates,
$$\mathbf{p}' = s\mathbf{p} \quad \text{and} \quad \mathbf{r}' = \mathbf{r}/s, \tag{10.154}$$
thus restoring the momentum cutoff to its original value ($0 \leq |\mathbf{p}'| \leq \Lambda$). One also defines a rescaled field,
$$\boldsymbol{\varphi}'(\mathbf{r}') = \lambda_s(K)\boldsymbol{\varphi}_<(\mathbf{r}), \tag{10.155}$$
or
$$\boldsymbol{\varphi}'(\mathbf{p}') = \frac{1}{\sqrt{V'}} \int d^d r' e^{-i\mathbf{p}'\cdot\mathbf{r}'} \boldsymbol{\varphi}'(\mathbf{r}') = s^{-d/2}\lambda_s(K)\boldsymbol{\varphi}_<(\mathbf{p}). \tag{10.156}$$
Equation (10.156) defines a linear RG transformation, since the new field depends linearly on the old one. We will see below how to determine the value of the rescaling parameter $\lambda_s(K)$, which is conveniently written as
$$\lambda_s(K) = s^{d_\varphi^0}\sqrt{Z_s(K)}, \tag{10.157}$$
where d_φ^0 is the canonical dimension of the field $\boldsymbol{\varphi}(\mathbf{r})$ (Sec. 10.4). $Z_s(K)$ determines the anomalous dimension η when the system is critical (Sec. 10.5.2).[52] The rescaling (10.154) and (10.156) transforms $K_<$ into a new set K' of coupling constants.

These two steps can be summarized by[53]
$$e^{-S[\boldsymbol{\varphi}';K']} = \left\{ \int_{\Lambda/s\leq|\mathbf{p}|\leq\Lambda} \mathcal{D}[\boldsymbol{\varphi}]\, e^{-S[\boldsymbol{\varphi};K]} \right\}_{\boldsymbol{\varphi}(\mathbf{p})\to s^{d/2}\lambda_s(K)^{-1}\boldsymbol{\varphi}'(\mathbf{p}')} . \tag{10.158}$$
The coupling constants K'_i are naturally associated with the momentum scale Λ/s and are often referred to as the coupling constants at the scale Λ/s.

The momentum-shell RG we have described so far is not the only possible RG procedure. In particular, for some models such as classical spin models, it is possible to implement a real-space RG following Kadanoff's idea of block spins. This approach has played a very important role in the genesis of Wilson's RG. Quite generally, we can therefore view a RG transformation as a transformation
$$K(s) = R_s(K) \tag{10.159}$$
($s > 1$) acting in the space of possible actions $\{S[\boldsymbol{\varphi}; K]\}$ (or Hamiltonians $\{H(K)\}$). Since two successive transformations R_{s_1} and R_{s_2} should be equivalent to $R_{s_1 s_2}$, the actions

[51] For instance, if the initial action corresponds to a $(\boldsymbol{\varphi}^2)^2$ theory, the renormalized action is likely to include a $(\boldsymbol{\varphi}^2)^3$ term, etc.

[52] Z_s is the inverse of the so-called wave-function renormalization factor.

[53] We ignore any additive contributions to the action $S[\boldsymbol{\varphi}'; K']$ coming from the mode elimination or the Jacobian due to the change of variables $\boldsymbol{\varphi} \to \boldsymbol{\varphi}'$. These matter only when considering the free energy and will be discussed at the end of section 10.5.3.

$S[\boldsymbol{\varphi}; R_{s_1}(R_{s_2}(K))]$ and $S[\boldsymbol{\varphi}; R_{s_1 s_2}(K)]$ should agree up to a global rescaling of the fields. One can always choose the rescaling parameter $\lambda_s(K)$ so that

$$R_{s_1 s_2} = R_{s_1} R_{s_2}. \tag{10.160}$$

The RG transformations $\{R_s\}$ then form a semi-group. This term refers to the action of the RG transformations in the space of field configurations. As some short-scale information is lost in the mode elimination, the procedure cannot be inverted. There is however no problem in inverting the transformation $K \mapsto K(s) = R_s(K)$ in the space of the parameters of the action.

10.5.1.1 Infinitesimal RG transformations

In practice, one often chooses $s = e^{dl}$ with $dl \to 0$ and integrates out fields with momenta in the infinitesimal shell $\Lambda(1 - dl) \le |\mathbf{p}| \le \Lambda$. After l/dl infinitesimal RG transformations, momenta have been rescaled by a factor $s = \lim_{dl \to 0}(1 + dl)^{l/dl} = e^l$. With $s_1 = 1 + \epsilon$ and $s_2 = s$ ($\epsilon \to 0$), equation (10.160) gives the differential RG transformation

$$s \frac{\partial K(s)}{\partial s} = \beta\big(K(s)\big), \tag{10.161}$$

where the beta function

$$\beta\big(K(s)\big) = \frac{\partial R_{s'}(K(s))}{\partial s'}\bigg|_{s'=1} \tag{10.162}$$

is a function of $K(s)$ only (and not of both $K(s)$ and s). It is sometimes convenient to consider $K(s)$ as a function $K(l)$ of the variable $l = \ln s$ (and similarly for the field rescaling factor $Z_l \equiv Z_s(K)$). We can define a "running" (l-dependent) anomalous dimension η_l by

$$\eta_l = \partial_l \ln Z_l \tag{10.163}$$

i.e.

$$Z_{l+dl} = Z_l e^{\eta_l dl} \quad \text{or} \quad Z_l = \exp\left(\int_0^l dl' \eta_{l'} \right), \tag{10.164}$$

so that $\lambda_{l+dl} = \lambda_l e^{(d_\varphi^0 + \eta_l/2)dl}$. η_l is a function of the running coupling constants $K_i(l)$. We shall see below how it is related to the actual anomalous dimension η when the system is critical.

10.5.1.2 Advantages of the RG approach

There are several advantages in computing the partition function by means of RG transformations rather than e.g. standard perturbation theory:

- Since a RG transformation involves only a finite number of degrees of freedom, no singularity (divergence) is expected. Singular behavior can arise only after an infinite number of iterations in which all degrees of freedom in the thermodynamic limit have been integrated.
- The RG turns out to be an efficient computational tool which often goes well beyond standard perturbation theory. Suppose for instance that we compute the change $\frac{d}{ds}K_i(s)$ of the coupling constants to a given order in a series expansion wrt $K_j(s)$.

Solving the flow equations $\beta_i(K(s)) = s\frac{d}{ds}K_i(s) = \frac{d}{dl}K_i(l)$ partially resums the perturbation series to infinite order. For this reason, perturbative RG approaches (i.e. based on a perturbative calculation of the beta functions $\beta_i(K)$) are sometimes referred to as RG improved perturbation theories. However, the computation of the beta functions need not be based on perturbation theory, and the RG provides a natural framework to set up non-perturbative approaches. This will be illustrated in section 10.9 and more generally in chapter 11.

- A RG transformation is not a mere scale transformation as the coarse graining (mode elimination) changes the coupling constants of the action. By iterating the RG transformations, one generates a trajectory $K(s)$ in the coupling constant space. The set of all such trajectories, obtained from different initial conditions $K(s=1)$ generates a RG flow. One generally finds that the trajectories flow into fixed points $K^* = R_s(K^*)$ of the RG transformation. We shall see in the following sections that the fixed points govern the long-distance physics and explain scaling and universality observed in the vicinity of a second-order phase transition.

10.5.2 Fixed points

In a RG transformation R_s, the correlation length transforms as $\xi(K(s)) = \xi(K)/s$.[54] At a fixed point $K^* = R_s(K^*)$ of the transformation, we must have $\xi(K^*) = \xi(K^*)/s$ which implies that $\xi(K^*)$ can only be zero or infinity. We refer to a fixed point with $\xi = \infty$ as a critical fixed point, and a fixed point with $\xi = 0$ as a trivial fixed point. Critical fixed points describe the singular behavior at a second-order phase transition (Sec. 10.5.3), whereas trivial fixed points describe the various phases of the system (see below).

The set of initial conditions K which flow to a given fixed point is called the basin of attraction of that fixed point. The basin of attraction of a critical fixed point is often called the critical manifold or critical surface. Let us consider a physical system represented by a point K in the coupling constant space. A point $K(s) = R_s(K)$ of the RG trajectory originating from K has a correlation length $\xi(K(s)) = \xi(K)/s < \xi(K)$. If the point K belongs to the critical surface, $\lim_{s\to\infty} R_s(K) = K^*$ and in turn $\lim_{s\to\infty}\xi(K(s)) = \infty$, which implies that $\xi(K) = \infty$. We conclude that all points on the critical surface have infinite correlation length.

10.5.2.1 Local behavior of RG flows near a fixed point

Near a fixed point K^* the RG transformation $K' = R_s(K)$ can be linearized,

$$\begin{aligned} K'_i &\simeq K^*_i + \sum_j \frac{\partial K'_i}{\partial K_j}\bigg|_{K^*}(K_j - K^*_j) \\ &= K^*_i + \sum_j T_{ij}(s)(K_j - K^*_j), \end{aligned} \tag{10.165}$$

where

$$T_{ij}(s) = \frac{\partial K'_i}{\partial K_j}\bigg|_{K^*}. \tag{10.166}$$

[54] In the low-temperature phase of the O(N) model with $N \geq 2$, ξ should be understood as the Josephson length ξ_J. See Secs. 10.1.4 and 10.7.2.

The matrix $T_{ij}(s)$ is real, but in general not symmetric and therefore not necessary diagonalizable.[55] We nevertheless assume that the right eigenstates $\mathbf{e}^{(\alpha)}$ form a complete basis with real eigenvalues $\lambda_s^{(\alpha)}$,

$$\sum_j T_{ij}(s) e_j^{(\alpha)} = \lambda_s^{(\alpha)} e_i^{(\alpha)}. \tag{10.167}$$

The group property (10.160) implies that the matrices $T(s)$ for different s commute. It is thus possible to diagonalize them simultaneously in a basis $\{\mathbf{e}^{(\alpha)}\}$ which does not depend on s. Equation (10.160) also implies that $\lambda_s^{(\alpha)} = s^{y_\alpha}$. Writing $\delta K_i = \sum_\alpha t_\alpha e_i^{(\alpha)}$, we can express the action near the fixed point as

$$S[\boldsymbol{\varphi}; K] = S[\boldsymbol{\varphi}; K^*] + \sum_i \delta K_i A_i[\boldsymbol{\varphi}] = S[\boldsymbol{\varphi}; K^*] + \sum_\alpha t_\alpha O_\alpha[\boldsymbol{\varphi}], \tag{10.168}$$

where $O_\alpha = \sum_i e_i^{(\alpha)} A_i$. The t_α's are called the scaling fields (or scaling variables) and the O_α's the scaling operators (or scaling directions). In the linearized RG transformation $T(s)$, the scaling field t_α is multiplied by s^{y_α}. We are then led to distinguish three cases:

1. $y_\alpha > 0$: the scaling field increases with s. t_α is called a relevant scaling field.
2. $y_\alpha = 0$: the scaling field does not change as s varies and is called a marginal field. To determine its behavior, one must go beyond the linear approximation. If t_α turns out to be (ir)relevant, it is said to be marginally (ir)relevant. Marginal scaling fields are responsible for logarithmic corrections to scaling and are important at the upper and lower critical dimensions (Secs. 10.6.2 and 10.7.2).
3. $y_\alpha < 0$: the scaling field decreases as s increases and is called an irrelevant field.

It should be remembered that the terms relevant, marginal and irrelevant should always be specified with respect to a particular fixed point. To complete the description of the linearized RG transformation $T(s)$, one should determine the rescaling parameter λ_s [Eq. (10.156)]. Since a field rescaling multiplies K_i by some power of $\lambda_s(K)$,[56] the only possible form compatible with $\lambda_s^{(\alpha)} = s^{y_\alpha}$ is $\lambda_s = s^{d_\varphi}$. With $s = e^l$, we obtain $\lambda_l = e^{l d_\varphi^0}\sqrt{Z_l} = e^{l d_\varphi}$, which implies that the running anomalous dimension $\eta_l \equiv \eta$ is independent of l and $d_\varphi = d_\varphi^0 + \eta/2$. We will see below that d_φ is nothing but the dimension of the field and η the anomalous dimension (Sec. 10.4.1).

Thus if we start with a set of coupling constants near the fixed point K^* but not in the basin of attraction, the flow along the relevant directions $\mathbf{e}^{(\alpha)}$ ($y_\alpha > 0$) will go away from the fixed point. The irrelevant directions $\mathbf{e}^{(\alpha)}$ ($y_\alpha < 0$) correspond to direction of the flow into the fixed point (for an example, see Fig. 10.11 below). If there are N relevant scaling fields $t_1, \cdots, t_N$, we need to fix $t_1 = \cdots = t_N = 0$ to be in the basin of attraction in the linear approximation. This condition defines the plan tangent to the basin of attraction at the fixed point.

10.5.2.2 Global properties of RG flows

The global behavior of the RG flow determines the phase diagram of the system. In general, any point in the coupling constant space flows to some fixed point. The state of the system

[55] If $T_{ij}(s)$ is not symmetric, there is no guaranty that the eigenvalues are real and that the right or left eigenstates form a complete basis.

[56] For instance, if K_i appears in the action as $K_i \varphi_{j_1} \cdots \varphi_{j_n}$, $K_i \to \lambda_s^{-n} K_i$ when $\varphi_j \to \lambda_s^{-1} \varphi_j$.

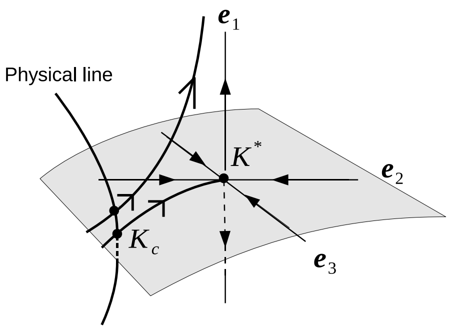


Figure 10.11: RG flow near a critical fixed point (for simplicity, only one of the two relevant directions is shown). Two RG trajectories are shown (thick solid lines): one on the critical surface (gray area) flowing into the fixed point, the other one near the critical surface. The physical line meets the critical surface at the critical point K_c.

described by this fixed point represents the phase at the original point in the coupling constant space. The phase diagram is therefore determined by the global topology of the RG flow.[57]

The distinction between relevant and irrelevant scaling fields, as well as between critical ($\xi = \infty$) and trivial ($\xi = 0$) fixed points, implies a classification of different types of fixed points. Let us briefly list the most important ones:[58]

- *Stable fixed points (or sinks)* have only irrelevant scaling fields and trajectories can only flow into them. Sinks correspond to stable bulk phases ($\xi = 0$), and the nature of the coupling constants at the fixed point characterize the phase. All points in the basin of attraction of the sink correspond to physical systems in the same phase.

- *Unstable fixed points* have only relevant scaling fields and all trajectories flow away from them. These fixed points have no direct physical meaning but play a role in the global topology of the RG flow.

- There are also *fixed points with both relevant and irrelevant scaling fields.* Of particular interest are the fixed points with two relevant scaling fields and an infinite correlation length (critical fixed point). In this case, two variables (e.g. temperature and magnetic field) must be tuned to reach the fixed point. Trajectories which start slightly off the critical surface initially flow towards the fixed point, but are ultimately repelled from the fixed point along the two relevant directions to flow into a stable fixed point (sink) corresponding to the phase of the system (Fig. 10.11). Such a critical fixed point corresponds to a phase transition between two stable phases of matter and the RG flow in its vicinity determines the critical behavior at the phase transition (Sec. 10.5.3). When the temperature changes, K varies along a line in the space of coupling constants, the physical line, which (in zero field) meets the critical surface at the critical point K_c ($T = T_c$).

  Critical fixed points with more than two relevant variables are generically called multicritical fixed points. Tricritical fixed points have three relevant variables; three

[57]This point will be illustrated in Sec. 10.6 and following ones.
[58]The following list is not exhaustive. For a more complete discussion, see Ref. [8].

parameters (e.g. temperature, magnetic field and pressure) must be tuned to hold the system at the critical point.

In some cases, it is possible to obtain a continuum of fixed points in the coupling constant space. A well-known example is the line of fixed points in the RG flow of the two-dimensional XY model (Sec. 10.8). RG flows can also exhibit more exotic behavior such as limit cycles or even chaotic behavior.

10.5.3 Universality and scaling

Let us consider a critical fixed point K^*. We denote by h and t_1 the two relevant scaling fields, and by $t_2, \cdots$, irrelevant scaling fields. We assume that there are no marginal fields.[59] If t_1 and h are initially very small, then K first moves towards K^* and remains a long "time" near K^* before eventually going away along the relevant directions. The critical behavior emerges from the long time where the flow is determined by the vicinity of the fixed point. Now the fixed point $K^* = R_s(K^*)$ and the linear approximation to R_s (i.e. the eigenvalues s^{y_α} and the eigendirections $\mathbf{e}^{(\alpha)}$) are properties of the RG transformation itself. Thus the dynamics near the fixed point is independent of the initial conditions of the RG trajectory. All systems represented by a point K near the critical surface of the fixed point K^* therefore exhibit the same critical behavior (universality). We show below how, by considering a RG transformation near a critical point, we can justify the results obtained from the scaling hypothesis (Sec. 10.4).

Expliquer pourquoi exposants universels: cf. Weinberg article sur gravitation, article a venir d'Adam?, Polchinski sur actions effectives, Wegner

Ici on anticipe le fait que le comportement critique est dicté par le point fixe

Low-energy effective theories and large-river effect

10.5.3.1 Correlation length

Let us first consider the correlation length ξ in the absence of magnetic field, and assume that the relevant field t_1 vanishes linearly with $T - T_c$: $t_1 \sim t = (T - T_c)/T_c$. In a RG transformation, $\xi(K) = s\xi(K')$, so that

$$\xi(t_1, t_2, \cdots) = s\xi(s^{y_1}t_1, s^{y_2}t_2, \cdots) \tag{10.169}$$

in the region near K^* where the transformation can be linearized,[60] with $y_1 > 0$ and $0 > y_2 > y_3 > \cdots$. Setting $s \sim |t_1|^{-1/y_1}$, we obtain[61]

$$\xi(t_1, t_2, \cdots) \sim |t_1|^{-1/y_1}\xi(\pm 1, |t_1|^{-y_2/y_1}t_2, \cdots). \tag{10.170}$$

In the critical region, defined by $|t_1|^{-y_2/y_1}t_2 \ll 1$, all irrelevant scaling fields can be set to zero to leading order and we deduce $\xi \sim |t_1|^{-1/y_1} \sim |T - T_c|^{-1/y_1}$.[62] The correlation length diverges at the transition with the exponent $\nu = 1/y_1$. The correction-to-scaling exponent $\omega = -y_2$, defined as the absolute value of the largest negative eigenvalues y_i, not only determines the size of the critical region but also gives the leading correction to the critical behavior $\xi \sim |t|^{-1/y_1}$. From (10.170), provided that $\xi(\pm 1, t_2', \cdots)$ is analytic in t_2',[62] we deduce

$$\xi \sim |t|^{-1/y_1}\big(1 + A_\pm |t|^{\omega/y_1} + \cdots\big), \tag{10.171}$$

[59] Marginal fields are considered in Secs. 10.6.2 and 10.7.2.

[60] An (approximate) condition is that $|t_i|$ and $s^{y_i}|t_i|$ are at most of order 1.

[61] Here and in the following, we do not write the microscopic length scale that is needed to make dimensional sense of these equations.

[62] We can set $t_2' = |t_1|^{-y_2/y_1}t_2 = 0$ in (10.170) only if ξ is an analytic function of t_2. If not, t_2 is called a dangerously irrelevant variable (see Sec. 10.6.1 for an example).

where $A_\pm$ is a non-universal constant.

If the point K representing the system is too far away from the fixed point K^* for the RG transformation to be linearized, one must first consider a RG transformation $K(\tilde s) = R_{\tilde s}(K)$ which brings $K(\tilde s) = \tilde K$ near the fixed point where its subsequent evolution can be obtained from the linearized RG transformation. The previous argument then gives $\xi(\tilde t_1) = s\xi(s^{1/\nu}\tilde t_1) \sim |\tilde t_1|^{-\nu}$ in the critical region. But $\tilde t_1$, which vanishes for $t=0$, is expected to be proportional to $t = (T-T_c)/T_c$, and therefore $\xi(t_1) = \tilde s\xi(\tilde t_1) \sim |t|^{-\nu}$.

The scaling dimension d_h of the other relevant field (the magnetic field) is easily obtained from the following argument. Since the magnetic field contributes to the action a term $-\int d^dr\,\mathbf{h}\cdot\boldsymbol{\varphi}(\mathbf{r})$, it couples only to the $\mathbf{p}=0$ mode and is not affected by the partial integration of fields with momenta $\Lambda/s \leq |\mathbf{p}| \leq \Lambda$. The renormalization of $\mathbf{h}$ is therefore entirely due to rescaling of momenta (or lengths) and fields,

$$\int d^dr\,\mathbf{h}\cdot\boldsymbol{\varphi}(\mathbf{r}) = s^d\lambda_s(K)^{-1}\int d^dr'\,\mathbf{h}\cdot\boldsymbol{\varphi}'(\mathbf{r}'), \tag{10.172}$$

i.e. $h' = s^{d_h}h$ ($h = |\mathbf{h}|$) with

$$d_h = d - d_\varphi = \frac{1}{2}(d+2-\eta) \tag{10.173}$$

(we use $\lambda_s(K) = s^{d_\varphi}$ in the critical regime). In practice, the smallness of η ensures that d_h is positive and $\mathbf{h}$ a relevant scaling field.

10.5.3.2 Order parameter

The RG transformation $K \mapsto K'$ [Eq. (10.158)] implies

$$\begin{aligned} m_i(K) &= \frac{1}{Z}\int \mathcal{D}[\boldsymbol{\varphi}]\,\varphi_i(\mathbf{r})\,e^{-S[\boldsymbol{\varphi};K]} \\ &= \frac{1}{Z}\int \mathcal{D}[\boldsymbol{\varphi}']\,\lambda_s(K)^{-1}\varphi_i'(\mathbf{r}/s)\,e^{-S[\boldsymbol{\varphi}';K']} \\ &= \lambda_s(K)^{-1}m_i(K'), \end{aligned} \tag{10.174}$$

where $m_i(K')$ is the mean value of $\varphi_i'(\mathbf{r}') = \varphi_i'(\mathbf{r}/s)$ computed with the action $S[\boldsymbol{\varphi}';K']$. Neglecting irrelevant scaling fields (assuming that there is no dangerously irrelevant variable[62]), we obtain

$$m(t,h) = s^{-d_\varphi}m(s^{1/\nu}t, s^{d_h}h) \tag{10.175}$$

in the critical regime. For $t=0$ and $h\neq 0$, we obtain $m(0,h)\sim h^{d_\varphi/d_h}$, i.e. a critical exponent

$$\delta = \frac{d_h}{d_\varphi} = \frac{d+2-\eta}{d-2+\eta}. \tag{10.176}$$

For $h=0$ and $t<0$, $m(t,0)\sim(-t)^{\nu d_\varphi}$ so that

$$\beta = \nu d_\varphi = \frac{\nu}{2}(d-2+\eta). \tag{10.177}$$

10.5.3.3 Correlation function

The two-point correlation function satisfies

$$\begin{aligned} G_{ij}(\mathbf{p};K) &= \frac{1}{Z}\int \mathcal{D}[\boldsymbol{\varphi}]\, \varphi_i(\mathbf{p})\varphi_j(-\mathbf{p})\, e^{-S[\boldsymbol{\varphi};K]} \\ &= \frac{1}{Z}\int \mathcal{D}[\boldsymbol{\varphi}']\, s^d \lambda_s(K)^{-2}\varphi'_i(s\mathbf{p})\varphi'_j(-s\mathbf{p})\, e^{-S[\boldsymbol{\varphi}';K']} \\ &= s^d\lambda_s(K)^{-2}G_{ij}(s\mathbf{p};K') \end{aligned} \tag{10.178}$$

when $|\mathbf{p}| \leq \Lambda/s$, and in turn

$$\begin{aligned} G_{ij}(\mathbf{r}/s;K') &= \int_{|\mathbf{p}'|\leq\Lambda} \frac{d^dp'}{(2\pi)^d} e^{i\mathbf{p}'\cdot\mathbf{r}/s} G_{ij}(\mathbf{p}';K') \\ &= \int_{|\mathbf{p}|\leq\Lambda/s} \frac{d^dp}{(2\pi)^d} e^{i\mathbf{p}\cdot\mathbf{r}} \lambda_s(K)^2 G_{ij}(\mathbf{p};K) \\ &\simeq \lambda_s(K)^2 G_{ij}(\mathbf{r};K) \qquad \text{for} \qquad |\mathbf{r}| \gg \frac{s}{\Lambda}. \end{aligned} \tag{10.179}$$

For a linearized RG transformation, equation (10.179) implies

$$G_{ij}(\mathbf{r};t_1,t_2,\cdots) = s^{-2d_\varphi} G_{ij}(\mathbf{r}/s; s^{y_1}t_1, s^{y_2}t_2,\cdots). \tag{10.180}$$

Thus G_{ij} satisfies a generalized homogeneity relation in agreement with the scaling hypothesis (Sec. 10.4). On the critical surface ($t_1 = 0$), setting $s = |\mathbf{r}|$, we obtain[63]

$$G_{ij}(\mathbf{r};0,t_2,\cdots) = |\mathbf{r}|^{-2d_\varphi} G_{ij}(\mathbf{r}/|\mathbf{r}|; 0, |\mathbf{r}|^{y_2}t_2,\cdots). \tag{10.181}$$

If $|\mathbf{r}|^{y_2}|t_2| \ll 1$ then all irrelevant scaling fields can be neglected and one obtains

$$G_{ij}(\mathbf{r};0,t_2,\cdots) \sim \frac{1}{|\mathbf{r}|^{2d_\varphi}} = \frac{1}{|\mathbf{r}|^{d-2+\eta}}. \tag{10.182}$$

The more negative y_2, the larger the critical region (in coordinate space) $|\mathbf{r}|^{y_2}|t_2| \ll 1$ where (10.182) holds. Away from the critical surface ($t_1 \neq 0$), setting $s = |t_1|^{-1/y_1}$ in (10.180), one finds

$$G_{ij}(\mathbf{r};t_1,t_2,\cdots) = |t|^{2d_\varphi/y_1} G_{ij}(|t_1|^{1/y_1}\mathbf{r}; \pm 1, |t_1|^{-y_2/y_1}t_2,\cdots). \tag{10.183}$$

In the limit where $|t_1|^{-y_2/y_1}t_2$ can be set to zero, we obtain the scaling form

$$G_{ij}(\mathbf{r};t_1,t_2,\cdots) = \frac{1}{|\mathbf{r}|^{d-2+\eta}} f^\pm_{ij}\left(\frac{\mathbf{r}}{\xi}\right), \tag{10.184}$$

where $\xi \sim |t_1|^{-y_1} \sim |t|^{-\nu}$ is the correlation length (in agreement with (10.126)).

A similar analysis can be made for the Fourier transformed correlation function, starting from

$$G_{ij}(\mathbf{p};t_1,t_2,\cdots) = s^{d-2d_\varphi} G_{ij}(s\mathbf{p}; s^{y_1}t_1, s^{y_2}t_1,\cdots), \tag{10.185}$$

[63] Because of space isotropy, $G_{ij}(\mathbf{r}/|\mathbf{r}|; 0, |\mathbf{r}|^{y_2}t_2,\cdots)$ is independent of $\mathbf{r}/|\mathbf{r}|$.

where $d - 2d_\varphi = 2 - \eta$. At the critical point $t_1 = 0$,

$$G_{ij}(\mathbf{p}; 0, t_2, \cdots) = |\mathbf{p}|^{-2+\eta} G_{ij}(\mathbf{p}/|\mathbf{p}|; 0, |\mathbf{p}|^{-y_2} t_2, \cdots). \tag{10.186}$$

In the critical regime (in momentum space) $|\mathbf{p}|^{-y_2}|t_2| \ll 1$, one obtains

$$G_{ij}(\mathbf{p}; 0, t_2, \cdots) \sim \frac{1}{|\mathbf{p}|^{2-\eta}}. \tag{10.187}$$

On the other hand, for $\mathbf{p} = 0$,

$$G_{ij}(0; t_1, t_2, \cdots) = |t_1|^{-(d-2d_\varphi)/y_1} G_{ij}(0; 1, |t_1|^{-y_2/y_1} t_2, \cdots), \tag{10.188}$$

so that the susceptibility $\chi \sim G_{ij}(\mathbf{p} = 0) \sim |t|^{-\gamma}$ diverges with an exponent[64]

$$\gamma = \frac{d - 2d_\varphi}{y_1} = \nu(2 - \eta). \tag{10.189}$$

Finally, in the presence of an external field,

$$G_{ij}(\mathbf{r}, t, h) = s^{-2d_\varphi} G_{ij}(\mathbf{r}/s, s^{1/\nu} t, s^{d_h} h) \tag{10.190}$$

in the critical regime, where we use the scaling variable t rather than t_1. With $s = |\mathbf{r}|$, we obtain

$$G_{ij}(\mathbf{r}, t, h) = \frac{1}{|\mathbf{r}|^{d-2+\eta}} g_{ij}^{\pm}\left(\frac{\mathbf{r}}{\xi}, \frac{h}{|t|^\Delta}\right), \tag{10.191}$$

where $\Delta = \nu d_h$ is the gap exponent introduced in section 10.4.2.

10.5.3.4 Free energy

To obtain the critical exponent α we must consider the free energy. So far we did not keep track of any possible additive contribution to the action produced by the RG transformation. Let us follow the convention that the action vanishes for $\boldsymbol{\varphi} = 0$ and write additive contributions explicitly. The first step in the RG transformation (coarse graining) will in general yield an additive contribution to the action,

$$Z = \int \mathcal{D}[\boldsymbol{\varphi}]\, e^{-S[\boldsymbol{\varphi}; K]} = \int \mathcal{D}[\boldsymbol{\varphi}_<]\, e^{-S[\boldsymbol{\varphi}_<; K_<] - \beta V A(K,s)}. \tag{10.192}$$

When rescaling the momenta and fields, $\boldsymbol{\varphi}'(\mathbf{p}') = s^{-d/2}\lambda_s(K)\boldsymbol{\varphi}_<(\mathbf{p})$, one should take into account the Jacobian,

$$Z = \int \mathcal{D}[\boldsymbol{\varphi}']\, e^{-S[\boldsymbol{\varphi}'; K'] - \beta V A(K,s) - \beta V B(K,s)}, \tag{10.193}$$

where

$$e^{-\beta V B(K,s)} = \prod_{\substack{\mathbf{p} \\ (0 \le |\mathbf{p}| \le \Lambda/s)}} \left[\lambda_s(K)^{-1} s^{d/2}\right]^N. \tag{10.194}$$

[64] If $N \geq 2$, this result holds only in the high-temperature phase, since the susceptibility $\chi = \beta G_\parallel(\mathbf{p} = 0)$ diverges in the whole low-temperature phase (see Secs. 10.1.4 and 10.7.2.

If we denote by $f(K)$ and $f(K')$ the free energy densities associated with $S[\boldsymbol{\varphi};K]$ and $S[\boldsymbol{\varphi}';K']$,

$$\begin{aligned} e^{-\beta V f(K)} &= \int \mathcal{D}[\boldsymbol{\varphi}]\, e^{-S[\boldsymbol{\varphi};K]}, \\ e^{-\beta V' f(K')} &= \int \mathcal{D}[\boldsymbol{\varphi}']\, e^{-S[\boldsymbol{\varphi}';K']} \end{aligned} \tag{10.195}$$

($V'=s^{-d}V$), we then obtain

$$f(K)=s^{-d}f(K')+A(K,s)+B(K,s), \tag{10.196}$$

i.e.

$$f(t,h)=s^{-d}f(s^{1/\nu}t,s^{d_h}h)+A(t,s)+B(t,s) \tag{10.197}$$

if we ignore the irrelevant fields. Note that $A+B$ cannot depend on h, since the latter couples only to the uniform part $\boldsymbol{\varphi}(\mathbf{p}=0)$ of the field. With $s=|t|^{-\nu}\sim\xi$, we deduce

$$f(t,h)=|t|^{d\nu}g_\pm\left(\frac{h}{|t|^\Delta}\right)+A(t,|t|^{-\nu})+B(t,|t|^{-\nu}). \tag{10.198}$$

If we could discard the term $A+B$, we would have derived the scaling form of the free energy. There is however no reason for $A(t,|t|^{-\nu})$ (which represents the free energy density of the modes $|\mathbf{p}|\gtrsim\xi^{-1}$) and $B(t,|t|^{-\nu})$ to be less singular than the free energy density $|t|^{d\nu}g_\pm$ of the modes $|\mathbf{p}|\lesssim\xi^{-1}$. A detailed calculation (see below) shows that

Revoir la phrase precedente! Elle contradit D. Bernard et Mussardo. Considerer modele Gaussien pour comprendre.

$$f(t,h)=|t|^{d\nu}g_\pm\left(\frac{h}{|t|^\Delta}\right)+|t|^{d\nu}\tilde g_\pm, \tag{10.199}$$

from which we deduce the specific heat per unit volume

$$c_V=-T\frac{\partial^2 f}{\partial T^2}\sim|t|^{-\alpha} \tag{10.200}$$

in zero field, with $\alpha=2-d\nu$.

It should be noted that A or B type terms did not appear in the calculation of $m(t,h)$ or $G_{ij}(\mathbf{p},t,h)$. The reason is that $G_{ij}(\mathbf{p},t,h)$ (for $|\mathbf{p}|<\Lambda/s$) and $m_i(t,h)=\langle\varphi_i(\mathbf{r})\rangle=V^{-1/2}\langle\varphi_i(\mathbf{p}=0)\rangle$ do not directly involve the fast modes $|\mathbf{p}|\geq\Lambda/s$ and are affected by the RG transformation only through the renormalization of the coupling constants $K'=R_s(K)$. By contrast, the free energy involves all Fourier components in a direct manner.

To derive (10.199), let us first consider a RG transformation with $s=e^{\Delta l}$,

$$f(K)=e^{-d\Delta l}f\big(R_{\Delta l}(K)\big)+\tilde A(K,\Delta l), \tag{10.201}$$

where $R_{\Delta l}=R_{s=e^{\Delta l}}$ and $\tilde A(K,\Delta l)=A(K,e^{\Delta l})+B(K,e^{\Delta l})$. Using

$$f\big(R_{\Delta l}(K)\big)=e^{-d\Delta l}f\big(R^2_{\Delta l}(K)\big)+\tilde A\big(R_{\Delta l}(K),\Delta l\big), \tag{10.202}$$

we obtain

$$f(K)=e^{-d\Delta l}\Big[e^{-d\Delta l}f\big(R^2_{\Delta l}(K)\big)+\tilde A\big(R_{\Delta l}(K),\Delta l\big)\Big]+\tilde A(K,\Delta l). \tag{10.203}$$

After $l/\Delta l$ iterations, we have

$$f(K) = e^{-dl} f\big(R_l(K)\big) + \sum_{m=0}^{l/\Delta l-1} e^{-md\Delta l} \tilde A\big(R^m_{\Delta l}(K), \Delta l\big). \tag{10.204}$$

Comparing with (10.196), we deduce

$$\begin{aligned} A(K, e^l) + B(K, e^l) &= \sum_{m=0}^{l/\Delta l-1} e^{-md\Delta l} \tilde A\big(R^m_{\Delta l}(K), \Delta l\big) \\ &= \int_0^l dl'\, e^{-l'd} \lim_{\Delta l\to 0} \frac{1}{\Delta l} \tilde A\big(R_{l'}(K), \Delta l\big), \end{aligned} \tag{10.205}$$

where we have taken the limit $\Delta l \to 0$. Since $A(K, 1) + B(K, 1) = 0$,

$$C(K) = \lim_{\Delta l\to 0} \frac{1}{\Delta l} \tilde A(K, \Delta l) = \partial_s [A(K, s) + B(K, s)]\big|_{s=1} \tag{10.206}$$

and equation (10.205) can be rewritten as

$$A(K, s) + B(K, s) = \int_0^{\ln s} dl'\, e^{-l'd} C\big(R_{l'}(K)\big) = \int_0^s \frac{ds'}{s'} s'^{-d} C\big(R_{s'}(K)\big). \tag{10.207}$$

Equation (10.207) implies

$$s\frac{\partial}{\partial s}[A(t, s) + B(t, s)] = s^{-d} C\big(R_s(K)\big) = s^{-d} C(s^{1/\nu} t) \tag{10.208}$$

and, setting $s = |t|^{-\nu} \sim \xi$,

$$\xi \frac{\partial}{\partial \xi}(A + B) = |t|^{d\nu} C(\pm 1) \sim \xi^{-d}. \tag{10.209}$$

We finally obtain

$$\frac{\partial}{\partial t}(A + B) = \frac{1}{\xi}\frac{\partial \xi}{\partial t} \xi \frac{\partial}{\partial \xi}(A + B) \sim |t|^{d\nu-1} \tag{10.210}$$

and

$$A(t, |t|^{-\nu}) + B(t, |t|^{-\nu}) \sim |t|^{d\nu}. \tag{10.211}$$

Together with (10.198), this proves equation (10.199).

Remarque

[Ecrire fonctions d'echelle sous la forme $\mathcal{F}(h/h_0, ...)$ avec h_0 facteur metrique. Les facteurs metriques s'eliminent dans certains rapports, e.g. rapports d'amplitude.]

10.5.3.5 Universal scaling function and two-scale universality

10.6 Perturbative renormalization group

In this section we show how the critical exponents can be computed perturbatively wrt $\epsilon = 4 - d$ near 4 dimensions (see also Appendix 10.A). We start with the RG solution of the Gaussian model before considering the $(\varphi^2)^2$ theory.

10.6.1 RG solution of the Gaussian model

10.6.1.1 RG equation

In the high-temperature phase, the Gaussian model is defined by the action

$$S[\boldsymbol{\varphi}] = \frac{1}{2}\int d^d r \left[(\boldsymbol{\nabla}\boldsymbol{\varphi})^2 + r_0\boldsymbol{\varphi}^2\right] = \frac{1}{2}\sum_{\mathbf{p},i}\varphi_i(-\mathbf{p})(\mathbf{p}^2 + r_0)\varphi_i(\mathbf{p}), \tag{10.212}$$

where $r_0 = \bar{r}_0(T - T_{c0})$. The integration of fields with momenta $\Lambda/s \leq |\mathbf{p}| \leq \Lambda$ yields a constant contribution to the action (i.e. a contribution to the free energy) which we ignore in the following. Rescaling the momenta, $\mathbf{p} \to \mathbf{p}' = s\mathbf{p}$, to restore the original value of the cutoff, we obtain

$$S[\boldsymbol{\varphi}] = \frac{1}{2}\sum_{\mathbf{p},i}\varphi_i(-\mathbf{p})(s^{-2}\mathbf{p}^2 + r_0)\varphi_i(\mathbf{p}). \tag{10.213}$$

We then rescale the field, $\boldsymbol{\varphi}(\mathbf{p}) \to s^{d/2 - d^0_\varphi}\boldsymbol{\varphi}(\mathbf{p})$ with $d^0_\varphi = \frac{d-2}{2}$ the naive scaling dimension of $\boldsymbol{\varphi}$, to restore the coefficient $1/2$ of the $(\boldsymbol{\nabla}\boldsymbol{\varphi})^2$ term,

$$S[\boldsymbol{\varphi}] = \frac{1}{2}\sum_{\mathbf{p},i}\varphi_i(-\mathbf{p})(\mathbf{p}^2 + s^2 r_0)\varphi_i(\mathbf{p}). \tag{10.214}$$

We deduce the RG equation

$$r_0' = s^2 r_0. \tag{10.215}$$

In the presence of an (external) magnetic field $\mathbf{h}$, we must include in the action the term $-\int d^d r\, \mathbf{h}\cdot\boldsymbol{\varphi}(\mathbf{r})$. The RG equation satisfied by the field is given by (10.173),

$$h' = s^{d^0_h} h, \tag{10.216}$$

where $d^0_h = d - d^0_\varphi = d/2 + 1$. Equations (10.215,10.216), together with the vanishing anomalous dimension η ($d_\varphi = d^0_\varphi$), could have been anticipated on purely dimensional ground. The RG transformation for the Gaussian model is a mere scale transformation.

10.6.1.2 Fixed points

For $h = 0$, equation (10.215) admits two fixed points:

- a critical fixed point $r_0^* = 0$ (to be referred to as the Gaussian fixed point) obtained for $T = T_{c0}$ and corresponding to the scale invariant action $S[\boldsymbol{\varphi}] = \frac{1}{2}\int d^d r (\boldsymbol{\nabla}\boldsymbol{\varphi}^2)$ ($\xi = \infty$). r_0 is a relevant scaling field with eigenvalue $y = 2$, which yields $\nu = 1/y = 1/2$.

- a high-temperature fixed point $r_0^* \to \infty$ with the action $S[\boldsymbol{\varphi}] = \frac{r_0^*}{2}\int d^d r \boldsymbol{\varphi}^2$ corresponding to a vanishing correlation length $\xi = 1/\sqrt{r_0^*} \to 0$.

The RG flow is shown in figure 10.12.

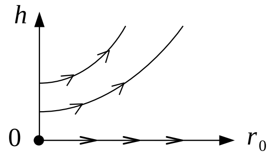


Figure 10.12: RG flow for the Gaussian model. The Gaussian fixed point is located at the origin $r_0 = h = 0$.

10.6.1.3 Critical exponents

One can obtain the correlation length from $\xi(r_0) = s\xi(r_0') = s\xi(s^2 r_0)$. With $s = r_0^{-1/2}$, one obtains $\xi(r_0) = r_0^{-1/2}\xi(1) \propto r_0^{-1/2}$, i.e. $\nu = 1/2$. Similarly, the susceptibility exponent is derived from the scaling law

$$G_{ii}(\mathbf{p}, r_0) = s^{d-2d_\varphi^0} G_{ii}(s\mathbf{p}, r_0') = s^2 G_{ii}(s\mathbf{p}, s^2 r_0) = r_0^{-1} G_{ii}(\mathbf{p} r_0^{-1/2}, 1), \tag{10.217}$$

where the last result is obtained with $s = r_0^{-1/2}$. We deduce that the susceptibility $\chi = \beta G_{ii}(\mathbf{p} = 0, r_0) \sim 1/r_0$ diverges with the exponent $\gamma = 1$ when $r_0 \to 0$.

10.6.1.4 Stability of the Gaussian fixed point

The Gaussian model ignores the $u_0(\boldsymbol{\varphi}^2)^2$ term of the O(N) model (10.27) as well as other terms which would then be generated by the RG procedure, e.g. $u_6(\boldsymbol{\varphi}^2)^3$, $v_0\boldsymbol{\varphi}^2(\boldsymbol{\nabla}\boldsymbol{\varphi})^2$, etc. At the Gaussian fixed point, the field has scaling dimension $d_\varphi^0 = \frac{d-2}{2}$, and we deduce

$$[u_0] = 4 - d, \quad [u_6] = 6 - 2d, \quad [v_0] = 2 - d, \quad \text{etc.} \tag{10.218}$$

In a RG transformation,

$$\begin{aligned} u_0' &= s^{4-d} u_0 + \cdots \\ u_6' &= s^{6-2d} u_6 + \cdots \\ v_0' &= s^{2-d} v_0 + \cdots \end{aligned} \tag{10.219}$$

For small coupling constants, the leading terms in (10.219) come from the rescaling of momenta and fields, and reflect the canonical dimensions (10.218) (i.e. the scaling dimensions at the Gaussian fixed point). The ellipses stand for higher-order (at least quadratic) terms which are generated in the coarse graining step of the RG transformation when the starting action is not quadratic. Since all canonical dimensions, except $[r_0] = 2$, are negative for $d > 4$, the Gaussian fixed point $r_0 = u_0 = u_6 = v_0 = \cdots = 0$ is stable (i.e. has only one relevant direction besides the magnetic field) and the critical exponents take their classical values. Note that this conclusion (which merely follows from a dimensional analysis of the action) has already been reached in section 10.3.4. In section 10.6.2, we shall see that at the upper critical dimension $d_c^+ = 4$, the Gaussian fixed point is still stable but the critical behavior is modified by logarithmic corrections.

Il faut s'assurer que les corrections à une bouche sont convergentes.

10.6.1.5 A dangerously irrelevant variable in Landau's theory

From the analysis of the Gaussian model, we conclude that the hyperscaling law

$$\delta = \frac{d_h^0}{d_\varphi^0} = \frac{d+2}{d-2} \tag{10.220}$$

agrees with the mean-field result $\delta = 3$ only for $d = 4$ whereas one expects it to be also valid for $d > 4$ since the Gaussian fixed point is then stable. This discrepancy is due to the fact that u_0 is a dangerously irrelevant variable for $d > 4$; although irrelevant, it cannot be ignored. Dangerously irrelevant variables lead to a breakdown of hyperscaling relations above the upper critical dimension d_c^+.

We observe that the partition function with $u_0 = 0$ is not defined for $T < T_{c0}$ so that in this case u_0 has to be included into the analysis one way or the other. Let us try to understand the role of the irrelevant variable u_0 (we now assume $d > 4$) in the RG framework.

The singular part of the free energy satisfies the scaling law

$$f(t, h, u_0) = s^{-d} f(s^{1/\nu} t, s^{d_h^0} h, s^\epsilon u_0) \tag{10.221}$$

(with $\nu = 1/2$), where $t \sim T - T_{c0}$ and $\epsilon = 4 - d$ ($\epsilon < 0$ for $d > 4$). We deduce

$$m(t, h, u_0) = -\frac{1}{T}\frac{\partial f}{\partial h} = s^{-d_\varphi^0} m(s^{1/\nu} t, s^{d_h^0} h, s^\epsilon u_0). \tag{10.222}$$

For $h = 0$ and with $s = |t|^{-\nu}$, we obtain

$$m(t, 0, u_0) = |t|^{\nu d_\varphi^0} m(\pm 1, 0, |t|^{-\nu\epsilon} u_0). \tag{10.223}$$

Since $\epsilon < 0$, it is tempting to set $|t|^{-\nu\epsilon} u_0 = 0$ when $|t| \to 0$. If we do so, we obtain a critical exponent $\beta = \nu d_\varphi^0 = \frac{d-2}{4}$ in disagreement with the exact result $\beta = 1/2$ for $d > 4$. The reason for this disagreement can be understood from Landau's theory. The mean-field result $m = \sqrt{-6r_0/u_0}$ clearly shows that we cannot set $u_0 = 0$ in the scaling law (10.223). The correct result is obtained if we write

$$m(t, 0, u_0) = |t|^{\nu d_\varphi^0} g_\pm(|t|^{-\nu\epsilon} u_0) \tag{10.224}$$

with $g_-(x) \sim x^{-1/2}$ for $x \to 0$. Then

$$m(t, 0, u_0) \sim \frac{|t|^{\nu(d_\varphi^0 + \frac{\epsilon}{2})}}{\sqrt{u_0}} \quad \text{for} \quad t < 0 \tag{10.225}$$

so that

$$\beta = \nu\left(d_\varphi^0 + \frac{\epsilon}{2}\right) = \nu = \frac{1}{2}. \tag{10.226}$$

The fact that the function $g_-(x)$ is not analytic for $x \to 0$ shows that u_0 is a dangerously irrelevant variable; although irrelevant it cannot be set to zero.

The value of δ can be obtained in a similar way. For $t = 0$,

$$\begin{aligned} m(0, h, u_0) &= s^{-d_\varphi^0} m(0, s^{d_h^0} h, s^\epsilon u_0) \\ &= h^{d_\varphi^0/d_h^0} m(0, 1, h^{-\epsilon/d_h^0} u_0) \end{aligned}$$

$$\equiv h^{d_\varphi^0/d_h^0} g(h^{-\epsilon/d_h^0} u_0). \tag{10.227}$$

Again it is not possible to set $h^{-\epsilon/d_h^0} u_0 = 0$ in the scaling function $g(x)$ even though $h^{-\epsilon/d_h^0} u_0 \to 0$ for $h \to 0$. Landau's theory gives $g(x) \sim x^{-1/3}$ for $x \to 0$ so that

$$m(0, h, u_0) \sim h^{(d_\varphi^0 + \frac{\epsilon}{3})/d_h^0}, \tag{10.228}$$

i.e.

$$\delta = \frac{d_h^0}{d_\varphi^0 + \frac{\epsilon}{3}} = 3, \tag{10.229}$$

in agreement with the mean-field result.

10.6.2 The ϵ expansion

We now consider the $(\boldsymbol{\varphi}^2)^2$ theory (10.27). For $d < 4$, u_0 becomes a relevant variable and the Gaussian fixed point $r_0 = u_0 = 0$ does not describe the phase transition. We expect the transition to be described by another fixed point with only one relevant scaling field (besides the magnetic field). We will see that the fixed point value u_0^* is of order $\epsilon = 4 - d$ for d near 4. This enables us to compute the critical exponents below the upper critical dimension $d_c^+ = 4$ within a systematic ϵ expansion. Furthermore, to order ϵ, it is sufficient to compute the RG equations to one-loop order.

10.6.2.1 One-loop RG equations

Following the general RG procedure (Sec. 10.5.1), we split the field $\boldsymbol{\varphi}(\mathbf{r}) = \boldsymbol{\varphi}_<(\mathbf{r}) + \boldsymbol{\varphi}_>(\mathbf{r})$ into slow and fast modes and rewrite the action as

$$S[\boldsymbol{\varphi}_< + \boldsymbol{\varphi}_>] = S_0[\boldsymbol{\varphi}_<] + S_{\rm int}[\boldsymbol{\varphi}_<] + S_0[\boldsymbol{\varphi}_>] + S_{\rm int}[\boldsymbol{\varphi}_<, \boldsymbol{\varphi}_>]. \tag{10.230}$$

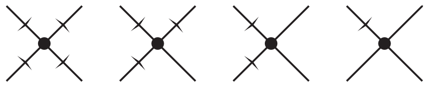
The first two terms in the rhs denote the action in the absence of fast modes ($\boldsymbol{\varphi}_> = 0$), while the last term denotes the interacting part of the action involving fast modes ($S_{\rm int}[\boldsymbol{\varphi}_<, \boldsymbol{\varphi}_> = 0] = 0$). The integration over the fast modes can be done using the linked cluster theorem (Sec. 1.5),

$$\begin{aligned} \int \mathcal{D}[\boldsymbol{\varphi}_>] \exp\{-S_0[\boldsymbol{\varphi}_>] - S_{\rm int}[\boldsymbol{\varphi}_<, \boldsymbol{\varphi}_>]\} \\ &= Z_{0,>} \exp\left\{ \sum_{n=1}^\infty \frac{(-1)^n}{n!} \langle S_{\rm int}[\boldsymbol{\varphi}_<, \boldsymbol{\varphi}_>]^n \rangle_{0,>,c} \right\} \\ &= Z_{0,>} \exp\left\{ \sum \text{connected graphs} \right\}, \end{aligned} \tag{10.231}$$

where

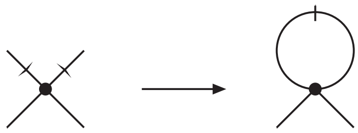
$$\langle \cdots \rangle_{0,>} = \frac{1}{Z_{0,>}} \int \mathcal{D}[\boldsymbol{\varphi}_>] \cdots e^{-S_0[\boldsymbol{\varphi}_>]} \quad \text{and} \quad Z_{0,>} = \int \mathcal{D}[\boldsymbol{\varphi}_>] \, e^{-S_0[\boldsymbol{\varphi}_>]}. \tag{10.232}$$

The notation $\langle \cdots \rangle_{0,>,c}$ means that only the connected graphs are to be considered. The action $S_{\rm int}[\boldsymbol{\varphi}_<, \boldsymbol{\varphi}_>]$ contains several types of vertex depending on the number of legs corresponding to slow and fast modes,



where a slashed line indicates a fast mode. The cumulants $\langle \cdots \rangle_{0,>,c}$ either contribute to the free energy or renormalize the action of the slow modes.

Let us first discuss the case $N = 1$. The one-loop correction to the self-energy of the slow modes is obtained from a vertex with two lines corresponding to fast modes,



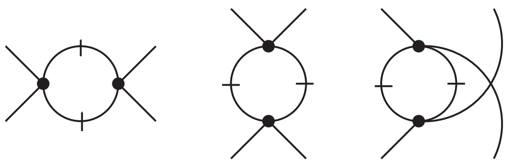
i.e.

$$d\Sigma(\mathbf{p}) = \frac{u_0}{2} \fint_{\mathbf{q}} G_0(\mathbf{q}), \tag{10.233}$$

where $G_0(\mathbf{q}) = (\mathbf{q}^2 + r_0)^{-1}$. We use the notation $\fint_{\mathbf{q}}$ to indicate that the momentum integration is restricted to fast modes $\Lambda/s \leq |\mathbf{q}| \leq \Lambda$. Similarly, the one-loop correction to the (bare) 4-point vertex

$$\Gamma^{(4)}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4) = \frac{u_0}{V} \delta_{\sum_i \mathbf{p}_i, 0} \tag{10.234}$$

of the slow modes is represented diagrammatically by



i.e.

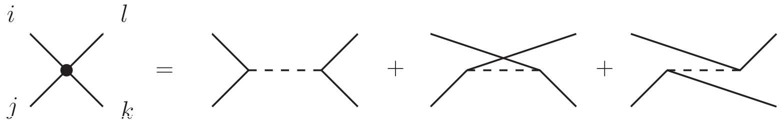
$$d\Gamma^{(4)}(0,0,0,0) = -\frac{3}{2V} u_0^2 \fint_{\mathbf{q}} G_0^2(\mathbf{q}) \tag{10.235}$$

for vanishing momenta $\mathbf{p}_1 = \cdots = \mathbf{p}_4 = 0$. Equations (10.233) and (10.235) follow from standard diagrammatic rules (Sec. 1.5). The momentum dependence of $d\Gamma^{(4)}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4)$ generates new terms in the action, e.g. $\boldsymbol{\varphi}^2(\boldsymbol{\nabla}\boldsymbol{\varphi})^2$, which do not modify the critical exponents to $\mathcal{O}(\epsilon)$ and can therefore be ignored (see the discussion below).

For the $O(N)$ model, the bare 4-point vertex is given by

$$\begin{aligned} \Gamma^{(4)}_{ijkl}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4) &= \frac{\delta^{(4)} S_{\text{int}}[\boldsymbol{\varphi}]}{\delta\varphi_i(-\mathbf{p}_1)\delta\varphi_j(-\mathbf{p}_2)\delta\varphi_k(-\mathbf{p}_3)\delta\varphi_l(-\mathbf{p}_4)} \\ &= \frac{u_0}{3V} \delta_{\sum_i \mathbf{p}_i, 0} \left(\delta_{i,j}\delta_{k,l} + \delta_{i,k}\delta_{j,l} + \delta_{i,l}\delta_{j,k} \right). \end{aligned} \tag{10.236}$$

It convenient to represent this (fully symmetrized) vertex as



where

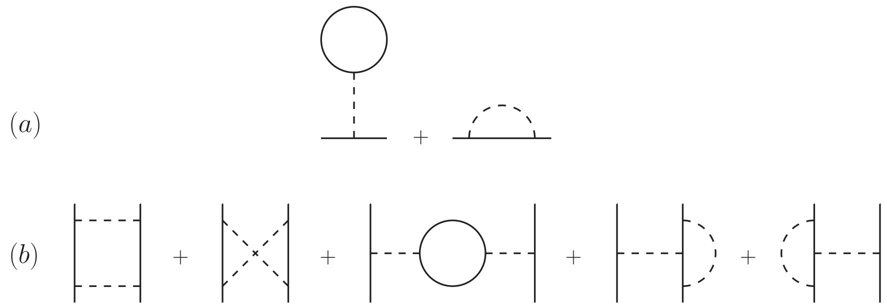
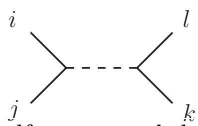


Figure 10.13: One-loop corrections to the self-energy (a) and the 4-point vertex (b).



$$= \frac{u_0}{3}\delta_{i,j}\delta_{k,l}$$

The one-loop corrections to the self-energy and the 4-point vertex are shown in figure 10.13. This leads to

$$d\Sigma(\mathbf{p}) = \frac{N+2}{6} u_0 \fint_{\mathbf{q}} G_0(\mathbf{q}) \tag{10.237}$$

and

$$d\Gamma^{(4)}_{ijkl}(0,0,0,0) = -\left(\delta_{i,j}\delta_{k,l} + \delta_{i,k}\delta_{j,l} + \delta_{i,l}\delta_{j,k}\right) \frac{N+8}{18V} u_0^2 \fint_{\mathbf{q}} G_0^2(\mathbf{q}). \tag{10.238}$$

For $N = 1$, one recovers the previous results (10.233) and (10.235). The one-loop correction due to the fast modes leads to a change of the parameters r_0 and u_0 of the action of the slow modes,

$$\begin{aligned} r_0' &= r_0 + \frac{N+2}{6} u_0 \fint_{\mathbf{q}} G_0(\mathbf{q}), \\ u_0' &= u_0 - \frac{N+8}{6} u_0^2 \fint_{\mathbf{q}} G_0^2(\mathbf{q}). \end{aligned} \tag{10.239}$$

To complete the RG procedure we must rescale momenta ($\mathbf{p} \to \mathbf{p}' = s\mathbf{p}$) and fields. Since the $(\boldsymbol{\nabla}\boldsymbol{\varphi})^2$ term is not renormalized (the self-energy correction $d\Sigma(\mathbf{p})$ is momentum independent), the field rescaling is simply $\boldsymbol{\varphi}(\mathbf{p}) \to \boldsymbol{\varphi}'(\mathbf{p}) s^{-d/2+d^0_\varphi} \varphi(\mathbf{p})$. Finally, it is also convenient to multiply r_0 by Λ^{-2} and u_0 by Λ^{d-4} to obtain coupling constants in dimensionless units. This yields the flow equations

$$\begin{aligned} \tilde r_0' &= s^2 \left[\tilde r_0 + \frac{N+2}{6} \tilde u_0 \Lambda^{2-d} \fint_{\mathbf{q}} G_0(\mathbf{q}) \right], \\ \tilde u_0' &= s^\epsilon \left[\tilde u_0 - \frac{N+8}{6} \tilde u_0^2 \Lambda^\epsilon \fint_{\mathbf{q}} G_0^2(\mathbf{q}) \right] \end{aligned} \tag{10.240}$$

for the dimensionless variables $\tilde r_0$ and $\tilde u_0$.[65] Using

$$\begin{aligned} \int_{\mathbf{q}} G_0(\mathbf{q}) &= K_d \int_{(1-dl)\Lambda}^{\Lambda} d|\mathbf{q}| \frac{|\mathbf{q}|^{d-1}}{\mathbf{q}^2 + r_0} = K_d \frac{\Lambda^{d-2}}{1+\tilde r_0} dl, \\ \int_{\mathbf{q}} G_0^2(\mathbf{q}) &= K_d \int_{(1-dl)\Lambda}^{\Lambda} d|\mathbf{q}| \frac{|\mathbf{q}|^{d-1}}{(\mathbf{q}^2 + r_0)^2} = K_d \frac{\Lambda^{-\epsilon}}{(1+\tilde r_0)^2} dl, \end{aligned} \tag{10.241}$$

for $s = e^{dl}$ (with $dl \to 0$), we finally obtain the equations

$$\begin{aligned} \frac{d\tilde r_0}{dl} &= 2\tilde r_0 + \frac{N+2}{6} K_d \frac{\tilde u_0}{1+\tilde r_0}, \\ \frac{d\tilde u_0}{dl} &= \epsilon \tilde u_0 - \frac{N+8}{6} K_d \frac{\tilde u_0^2}{(1+\tilde r_0)^2} \end{aligned} \tag{10.242}$$

satisfied by $\tilde r_0(l)$ and $\tilde u_0(l)$.

10.6.2.2 Fixed points and critical exponents

The RG equations (10.242) admit two fixed points: the Gaussian fixed point $\tilde r_0^* = \tilde u_0^* = 0$ obtained in section 10.6.1 and the Wilson-Fisher fixed point

$$\begin{aligned} \tilde r_0^* &= -\frac{1}{2}\frac{N+2}{N+8}\epsilon + \mathcal{O}(\epsilon^2), \\ \tilde u_0^* &= \frac{6}{N+8}\frac{\epsilon}{K_4} + \mathcal{O}(\epsilon^2), \end{aligned} \tag{10.243}$$

where $K_4 = 1/8\pi^2$. To obtain the critical exponents associated with these fixed points, we need to find the eigenvalues $e^{y_1 dl}$ and $e^{y_2 dl}$ of the linearized RG transformation $T(dl)$ defined by

$$\begin{pmatrix} \delta\tilde r_0(l+dl) \\ \delta\tilde u_0(l+dl) \end{pmatrix} = T(dl) \begin{pmatrix} \delta\tilde r_0(l) \\ \delta\tilde u_0(l) \end{pmatrix}, \tag{10.244}$$

where $\delta\tilde r_0 = \tilde r_0 - \tilde r_0^*$ and $\delta\tilde u_0 = \tilde u_0 - \tilde u_0^*$. Equation (10.244) can be rewritten as

$$\frac{d}{dl}\begin{pmatrix} \delta\tilde r_0 \\ \delta\tilde u_0 \end{pmatrix} = \frac{T(dl)-1}{dl} \begin{pmatrix} \delta\tilde r_0 \\ \delta\tilde u_0 \end{pmatrix}, \tag{10.245}$$

where the matrix $\frac{T(dl)-1}{dl}$ has eigenvalues y_1 and y_2 for $dl \to 0$. From the linearized RG equations

$$\frac{d}{dl}\begin{pmatrix} \tilde r_0 \\ \tilde u_0 \end{pmatrix} = \begin{pmatrix} 2 & \frac{N+2}{6}K_d \\ 0 & \epsilon \end{pmatrix} \begin{pmatrix} \tilde r_0 \\ \tilde u_0 \end{pmatrix} \tag{10.246}$$

about the Gaussian fixed point, we obtain the eigenvalues $y_1 = 2$ and $y_2 = \epsilon$ and the corresponding eigenvectors

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \mathbf{e}_2 = \begin{pmatrix} -1 \\ \frac{12}{(N+2)K_4} \end{pmatrix}, \tag{10.247}$$

[65] Even without the last rescaling (which implies that $\tilde r_0$ and $\tilde u_0$ are expressed in dimensionless physical units), r_0 and u_0 are "dimensionless" to the extent where they are expressed in units of $(\Lambda'/\Lambda)^{[r_0]}$ and $(\Lambda'/\Lambda)^{[u_0]}$, respectively, i.e. in units of the running cutoff $\Lambda' = \Lambda/s$. This property, which is a consequence of the rescaling of momenta and fields, is crucial for the existence of a fixed point of the RG equations at criticality.

to $\mathcal{O}(\epsilon^0)$. The linearized RG equations about the Wilson-Fisher fixed point read

$$\frac{d}{dl}\begin{pmatrix}\delta\tilde r_0\\ \delta\tilde u_0\end{pmatrix}=\begin{pmatrix}2-\frac{N+2}{N+8}\epsilon & \frac{N+2}{6}\frac{K_d}{1+\tilde r_0^*}\\ 0 & -\epsilon\end{pmatrix}\begin{pmatrix}\delta\tilde r_0\\ \delta\tilde u_0\end{pmatrix}, \tag{10.248}$$

to order ϵ. We thus obtain the eigenvalues $y_1=2-\frac{N+2}{N+8}\epsilon+\mathcal{O}(\epsilon^2)$ and $y_2=-\epsilon+\mathcal{O}(\epsilon^2)$. To $\mathcal{O}(\epsilon^0)$, the eigenvectors are given by (10.247). The Wilson-Fisher fixed point is in the direction $\mathbf{e}_2$ from the origin (Gaussian fixed point) since

$$\begin{pmatrix}\tilde r_0^*\\ \tilde u_0^*\end{pmatrix}_{\rm WF}=\frac{1}{2}\frac{N+2}{N+8}\epsilon\mathbf{e}_2+\mathcal{O}(\epsilon^2). \tag{10.249}$$

For $d>4$, the Gaussian fixed point has one relevant direction ($\mathbf{e}_1$) and governs the critical behavior. The correlation length critical exponent takes the classical value $\nu=1/y_1=1/2$. On the other hand, the Wilson-Fisher fixed point is not physical since $\tilde u_0^*<0$ for $\epsilon<0$, and has two relevant directions (besides the magnetic field). For $d<4$, the Gaussian fixed point has two relevant directions. The critical behavior is governed by the Wilson-Fisher fixed point which has only one relevant direction. The critical exponent ν is given by

$$\nu=\frac{1}{y_1}=\frac{1}{2}+\frac{N+2}{N+8}\frac{\epsilon}{4}+\mathcal{O}(\epsilon^2), \tag{10.250}$$

whereas the correction-to-scaling exponent (Sec. 10.5.3) takes the value

$$\omega=-y_2=\epsilon+\mathcal{O}(\epsilon^2). \tag{10.251}$$

The anomalous dimension η vanishes to $\mathcal{O}(\epsilon)$ since the field has been trivially rescaled in the RG procedure. All other exponents can then be deduced from the scaling laws derived in sections 10.4 and 10.5.3,

$$\begin{aligned}\gamma&=1+\frac{N+2}{N+8}\frac{\epsilon}{2}+\mathcal{O}(\epsilon^2),\\ \beta&=\frac{1}{2}-\frac{3}{2}\frac{\epsilon}{N+8}+\mathcal{O}(\epsilon^2),\\ \delta&=3+\epsilon+\mathcal{O}(\epsilon^2),\\ \alpha&=\frac{4-N}{N+8}\frac{\epsilon}{2}+\mathcal{O}(\epsilon^2).\end{aligned} \tag{10.252}$$

If u_0 vanishes, the phase transition is governed by the Gaussian fixed point even when $d<4$. If u_0 is finite but small and $T=T_c$, the RG trajectory spends a lot of "time" near the Gaussian fixed point before eventually flowing into the Wilson-Fisher fixed point. We can describe this crossover behavior with two relevant scaling fields $\tilde t_1$ and $\tilde t_2$ at the Gaussian fixed point, defined by

$$\begin{pmatrix}\tilde r_0\\ \tilde u_0\end{pmatrix}=\tilde t_1\mathbf{e}_1+\tilde t_2\mathbf{e}_2=\left(\tilde r_0+\frac{(N+2)K_4}{12}\tilde u_0\right)\mathbf{e}_1+\frac{(N+2)K_4}{12}\tilde u_0\mathbf{e}_2. \tag{10.253}$$

The singular part of the free energy satisfies[66]

$$f(t_1,u_0)=s^{-d}f(s^{y_1}t_1,s^{y_2}u_0)=|t_1|^{d/y_1}g_\pm\left(u_0|t_1|^{-\phi}\right) \tag{10.254}$$

[66] Eq. (10.254) can be alternatively written as $f(\tilde t_1(1),\tilde u_0(1))=s^{-d}f(\tilde t_1(s),\tilde u_0(s))$.

(we consider $\tilde{u}_0$ rather than $\tilde{t}_2$), where $y_1 = 2$ and $y_2 = \epsilon$. The crossover exponent ϕ is given by

$$\phi = \frac{y_2}{y_1} > 0. \tag{10.255}$$

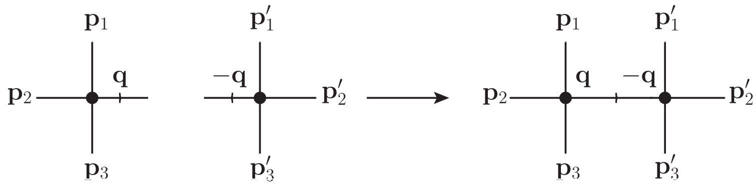
Thus we expect to see effective Gaussian behavior when $u_0|t_1|^{-\phi} \ll 1$ and critical behavior (governed by the Wilson-Fisher fixed point) when $u_0|t_1|^{-\phi} \gg 1$.

To ensure that the preceding discussion is correct, we have to verify that the omitted coupling constants do not modify the critical exponents to $\mathcal{O}(\epsilon)$.[67] Even if we start from the $(\boldsymbol{\varphi}^2)^2$ theory, the RG procedure will generate all coupling constants allowed by symmetry. An arbitrary coupling $\tilde{w}$, different from $\tilde{r}_0$ and $\tilde{u}_0$, satisfies the RG equation

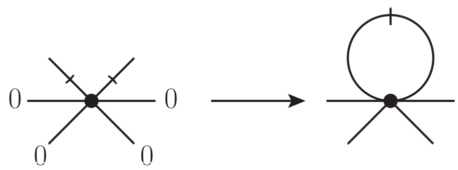
$$\frac{d\tilde{w}}{dl} = \big(d_w^0 + \mathcal{O}(\epsilon^2)\big)\tilde{w} + \mathcal{O}(\tilde{u}_0^2, \tilde{u}_0\tilde{w}, \tilde{w}^2, \cdots), \tag{10.256}$$

where d_w^0 is the engineering dimension of $\tilde{w}$ while the $\mathcal{O}(\epsilon^2)$ term comes from the anomalous dimension η of the $\boldsymbol{\varphi}$ field. There is no term linear in $\tilde{u}_0$ in (10.256) since the only $\mathcal{O}(\tilde{u}_0)$ terms enter the RG equation of $\tilde{r}_0$ and $\tilde{u}_0$.[68] The fixed point value $\tilde{w}^*$ is of order ϵ^2 or higher. Thus the Wilson-Fisher fixed point is characterized by an infinite number of nonzero coupling constants, but only $\tilde{r}_0^*$ and $\tilde{u}_0^*$ are of order ϵ.

We must now ask whether the $\mathcal{O}(\epsilon^2)$ couplings can change the values of the critical exponents to order ϵ. Only terms of order ϵ are important for the equation fixing $\tilde{r}_0^*$, so that $\tilde{w} = \mathcal{O}(\epsilon^2)$ is negligible. By contrast, if a $\mathcal{O}(\epsilon^2)$ term enters the equation $d\tilde{u}_0/dl$ linearly, $\tilde{u}_0^*$ will change to order ϵ and in turn the critical exponents. Thus the $u_6(\boldsymbol{\varphi}^2)^3$ term, which contributes to $d\tilde{u}_0/dl$, is potentially dangerous. The leading contribution to $d\tilde{u}_6/dl$ comes from[69]



where $\mathbf{q} = -\mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3 = \mathbf{p}_1' + \mathbf{p}_2' + \mathbf{p}_3'$, $|\mathbf{q}| \in [\Lambda(1-dl), \Lambda]$ and $|\mathbf{p}_i|, |\mathbf{p}_i'| < \Lambda(1-dl)$ $(i = 1,2,3)$. Now the $\tilde{u}_6$ term contribution to $d\tilde{u}_0/dl$ must have $\mathbf{p}_1 = \mathbf{p}_2 = \mathbf{p}_1' = \mathbf{p}_2' = 0$:



But the condition $\mathbf{p}_1 = \mathbf{p}_2 = \mathbf{p}_1' = \mathbf{p}_2' = 0$ implies that $\mathbf{p}_3 = -\mathbf{p}_3' = -\mathbf{q}$, which is in contradiction with $|\mathbf{p}_3|, |\mathbf{p}_3'| < \Lambda(1-dl)$ and $|\mathbf{q}| \in [\Lambda(1-dl), \Lambda]$. Thus $\tilde{u}_6$ does not contribute an $\mathcal{O}(\epsilon^2)$ term to the equation for $\tilde{u}_0^*$, and the critical exponents to order ϵ are unchanged.

[67] See Ref. [3] for a thorough discussion.

[68] The only diagrams of order $\tilde{u}_0$ are the one-loop self-energy diagram (which contributes to $d\tilde{r}_0/dl$) and the bare vertex $\tilde{u}_0$ (which gives the term linear in $\tilde{u}_0$ in $d\tilde{u}_0/dl$).

[69] There is also a contribution to $\tilde{u}_6$ coming from the $\tilde{u}_8(\boldsymbol{\varphi}^2)^4$ term which turns out to be $\mathcal{O}(\epsilon^3)$ for $\tilde{u}_8 \simeq \tilde{u}_8^*$.

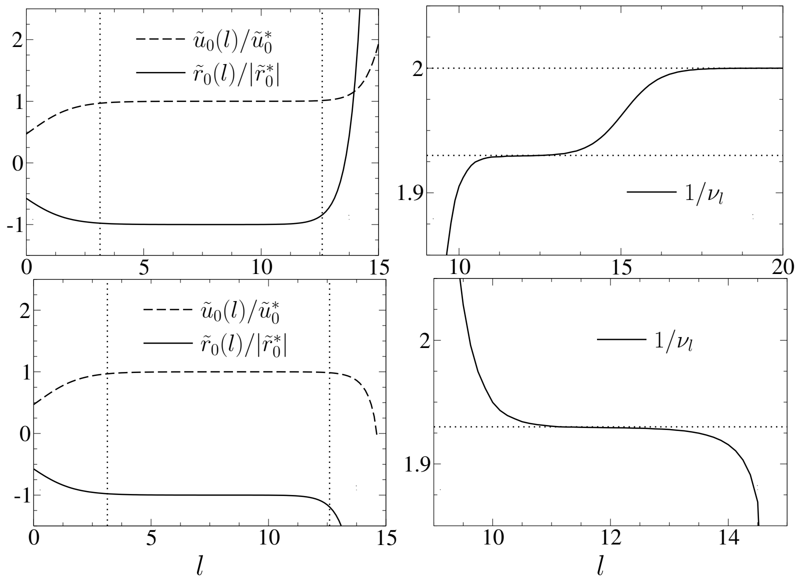


Figure 10.14: Solution of the RG equations (10.242) for initial conditions near the critical surface ($d = 3$, $N = 1$, $\Lambda = 1$, $\tilde{r}_0^* \simeq -0.1$ and $\tilde{u}_0^* \simeq 10.66$): $T > T_c$ (top) and $T < T_c$ (bottom). The vertical dotted lines show the Ginzburg scale $l_G = \ln(\Lambda\xi_G) \simeq 3$ and the correlation length scale $l_\xi = \ln(\Lambda\xi) \simeq 12$.

10.6.2.3 Flow diagrams

The analysis of the RG equations (10.242) is not restricted to the determination of fixed points and critical exponents. In figure 10.14 we show typical solutions for generic initial conditions near the critical surface (i.e. $T \simeq T_c$) for $d = 3$ and $N = 1$ (boldly extrapolating the result of the ϵ-expansion to $\epsilon = 1$). The top plots correspond to $T > T_c$ and the bottom ones to $T < T_c$. In both cases, one can identify a critical regime (in momentum space) where $\tilde{r}_0$ and $\tilde{u}_0$ are nearly equal to their fixed point values $\tilde{r}_0^*$ and $\tilde{u}_0^*$. Note that $\tilde{r}_0$ can take negative values as long as $G_0(\mathbf{p}) = (\mathbf{p}^2 + r_0)^{-1}$ remains positive for $|\mathbf{p}| = \Lambda(l)$, i.e. $1 + \tilde{r}_0 > 0$. The critical regime begins when $\Lambda(l) \sim \xi_G^{-1} \sim u_0^{1/(d-4)}$, where ξ_G is the Ginzburg length introduced in section 10.3.4. It ends when $\Lambda(l) \sim \xi^{-1}$ where ξ is the correlation length. The critical regime is preceded by a perturbative regime $\Lambda(l) \gg \xi_G^{-1}$ where the Gaussian approximation is essentially correct (Sec. 10.3.4).[70]

Since the relevant scaling field $\tilde{t}_1 \propto e^{l/\nu}$ grows exponentially near the fixed point (which implies $\delta\tilde{r}_0 \propto e^{l/\nu}$ and $\delta\tilde{u}_0 \propto e^{l/\nu}$), it is possible to obtain the value of the critical exponent ν from the behavior of $\tilde{r}_0$ and $\tilde{u}_0$ in the critical regime. To this end it is convenient to define a "running" critical exponent ν_l by

$$\frac{1}{\nu_l} = \frac{d}{dl} \ln|\tilde{r}_0 - \tilde{r}_0^*|, \tag{10.257}$$

in the same way as we have defined a running anomalous dimension η_l [Eq. (10.163)]. Near

[70]Strictly speaking, $\tilde{u}_0(l)$ must be small for the perturbative regime to exist. The fact that $\tilde{u}_0^* = \mathcal{O}(\epsilon)$ for $d = 4 - \epsilon$ does not say anything about the value of $\tilde{u}_0(l)$ for $\Lambda(l) \gg \xi_G^{-1}$.

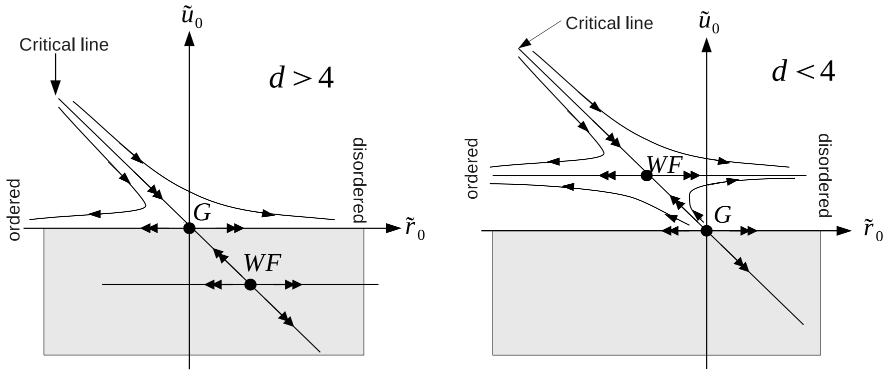


Figure 10.15: Schematic flow diagrams in the $(\tilde{r}_0, \tilde{u}_0)$ plane obtained from the linearized one-loop RG equations for $d > 4$ (left) and $d < 4$ (right). G indicates the Gaussian fixed point and WF the Wilson-Fisher fixed point. The gray areas correspond to the nonphysical region $\tilde{u}_0 \leq 0$.

the end of the critical regime, we observe that ν_l takes a constant value which can be identified with the actual value of the exponent ν; the obtained value clearly differs from the mean-field value $\nu = 1/2$ (Fig. 10.14). In principle, the anomalous dimension η can be obtained by a similar method, since the (running) anomalous dimension η_l coincides with η when $\xi^{-1} \ll \Lambda(l) \ll \xi_G^{-1}$. It is however necessary to go beyond the one-loop RG equations to obtain a nonzero anomalous dimension (see chapter 11).

When $T > T_c$, the flow for $\Lambda(l) \ll \xi^{-1}$ is characteristic of the disordered phase. $\tilde{r}_0$ takes a large value ($\tilde{r}_0 \gg 1$) which suppresses the perturbative corrections to the coupling constants. The flow of $\tilde{r}_0$ and $\tilde{u}_0$ is then purely dimensional, i.e. $\tilde{r}_0 \propto e^{2l}$ and $\tilde{u}_0 \propto e^{\epsilon l}$, while the "dimensionful" variables satisfies $dr_0/dl \simeq du_0/dl \simeq 0$ [Eq. (10.239)]. ν_l takes the mean-field value 1/2. This is a general feature of Wilson's RG: a "gapped" fluctuation mode, with propagator $G(\mathbf{p}) \sim (\mathbf{p}^2 + \xi^{-2})^{-1}$, does not contribute to the RG flow once $\Lambda(l) \ll \xi^{-1}$.

When $T < T_c$, the critical regime is followed by a decrease of $\tilde{r}_0$ until $1 + \tilde{r}_0 = 0$, which corresponds to a pole in the propagator $G_0(\mathbf{q}) = (\mathbf{q}^2 + r_0)^{-1}$ for $|\mathbf{q}| = \Lambda$. One could try to circumvent this difficulty by expanding the action about one of its degenerate minima. For $N \geq 2$ however, one would have to deal with the nontrivial physics of the low-temperature phase due to the (gapless) Goldstone modes associated with the spontaneous broken symmetry, an impossible task within the framework of the perturbative RG we have discussed so far. The ordered phase of the $(\boldsymbol{\varphi}^2)^2$ theory with $N \geq 2$ will be studied in section 10.7 from the NLσM (see also Secs. 10.B and 10.C).

By solving the RG equations (10.242) for various initial conditions, we obtain the flow diagrams shown in figure 10.15 for $d > 4$ and $d < 4$. The critical line (or, more precisely, the line tangent to the critical line at the fixed point) is determined by

$$\tilde{t}_1 = \delta\tilde{r}_0 + \frac{N+2}{12} K_4 \delta\tilde{u}_0 = 0, \tag{10.258}$$

where the relevant scaling field $\tilde{t}_1$ is defined by

$$\begin{pmatrix} \delta\tilde{r}_0 \\ \delta\tilde{u}_0 \end{pmatrix} = \tilde{t}_1 \mathbf{e}_1 + \tilde{t}_2 \mathbf{e}_2. \tag{10.259}$$

We see that fluctuations lead to a reduction of the transition temperature compared to the mean-field result. For a given value of $\tilde{u}_0(0)$, we need to choose $\tilde{r}_0(0) < 0$, i.e. $T_c < T_{c0}$, to be on the critical line (Fig. 10.15).

10.6.2.4 The upper critical dimension

At the upper critical dimension $d_c^+ = 4$, $\tilde{u}_0$ is a marginal variable. To obtain the critical behavior, one must go beyond the linear approximation. We shall see that the Gaussian fixed point governs the critical behavior ($\tilde{u}_0$ is marginally irrelevant at the Gaussian fixed point) but the mean-field predictions are modified by logarithmic corrections.

Expanding the one-loop RG equations (10.242) to quadratic order in $\tilde{r}_0$ and $\tilde{u}_0$, we obtain

$$\begin{aligned} \frac{d\tilde{t}_1}{dl} &= 2\tilde{t}_1 - \frac{N+2}{6} K_4 \tilde{t}_1 \tilde{u}_0 + \mathcal{O}(\tilde{u}_0^2), \\ \frac{d\tilde{u}_0}{dl} &= -\frac{N+8}{6} K_4 \tilde{u}_0^2, \end{aligned} \tag{10.260}$$

where $\tilde{t}_1$ is the relevant scaling field defined in (10.259). The second equation gives

$$\tilde{u}_0(l) = \frac{\tilde{u}_0(0)}{1 + \frac{N+8}{6} K_4 \tilde{u}_0(0) l}. \tag{10.261}$$

$\tilde{u}_0(l)$ vanishes for $l \to \infty$, but only logarithmically wrt the running momentum cutoff $\Lambda(l) = \Lambda e^{-l}$. From

$$\frac{d \ln \tilde{t}_1}{dl} = 2 - \frac{N+2}{6} K_4 \tilde{u}_0, \tag{10.262}$$

we then deduce

$$\tilde{t}_1(l) \propto \tilde{t}_1(0) e^{2l} l^{-\frac{N+2}{N+8}} \tag{10.263}$$

when $\frac{N+8}{6} K_4 \tilde{u}_0(0) l \gg 1$.

To obtain the correlation length (or the Josephson length ξ_J for $N \geq 2$ and $T < T_c$) we use $\xi(\tilde{t}_1(0), \tilde{u}_0(0)) = e^l \xi(\tilde{t}_1(l), \tilde{u}_0(l))$ and choose l such that $|\tilde{t}_1(l)| \sim 1$ and $\tilde{u}_0(l) \ll 1$, i.e.

$$e^{2l} \sim \frac{1}{|\tilde{t}_1(0)|} \big| \ln |\tilde{t}_1(0)| \big|^{\frac{N+2}{N+8}}. \tag{10.264}$$

We then have $\xi(\tilde{t}_1(l), \tilde{u}_0(l)) \sim \xi(1,0) \sim 1$ and therefore

$$\xi \sim \frac{1}{\sqrt{|T - T_c|}} \big| \ln |T - T_c| \big|^{\frac{N+2}{2(N+8)}}, \tag{10.265}$$

where we have used $\tilde{t}_1(0) \propto T - T_c$. Thus the marginal variable $\tilde{u}_0$ leads to a logarithmic correction to the mean-field result $\xi \sim 1/\sqrt{|T - T_c|}$.

Similarly, we can compute the uniform susceptibility[71] $\chi = \beta G_{ii}(\mathbf{p} = 0)$ starting from

$$\chi(\tilde{t}_1(0), \tilde{u}_0(0)) = e^{(d - 2d_\varphi^0) l} \chi(\tilde{t}_1(l), \tilde{u}_0(l)), \tag{10.266}$$

[71] The uniform susceptibility is defined in the ordered phase only for $N = 1$ (see Sec. 10.7.3).

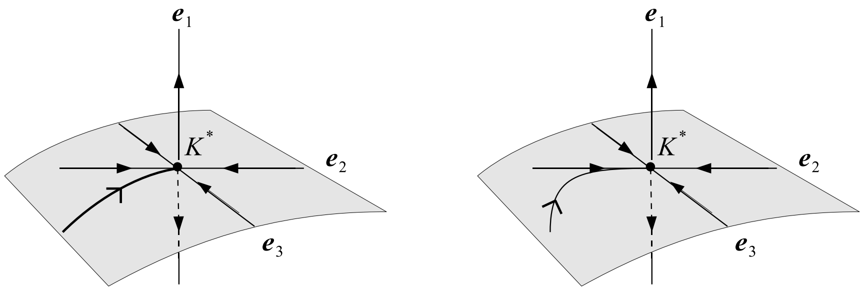


Figure 10.16: RG trajectories on the critical surface ($\mathbf{e}_1$ is a relevant direction): without marginal variable, $y_2, y_3 < 0$ (left), and with a marginally irrelevant variable, $y_2 = 0$ and $y_3 < 0$ (right).

where $d - 2d_\varphi^0 = 2$ ($\eta = 0$). When $|\tilde{t}_1(l)| \sim 1$ and $\tilde{u}_0 \ll 1$, the rhs in (10.266) can be calculated by perturbation theory, which yields

$$\chi \sim \frac{1}{|T - T_c|} \big| \ln |T - T_c| \big|^{\frac{N+2}{N+8}}. \tag{10.267}$$

Again, we obtain a logarithmic correction to the mean-field result $\chi \sim 1/|T - T_c|$. Note that there is no logarithmic ($\ln |\mathbf{p}|$) correction to the critical correlation function $G_{ii}(\mathbf{p}) \sim 1/\mathbf{p}^2$ since the anomalous dimension η vanishes for $d = 4$ (see Appendix 10.A).

To obtain the singular part of the free energy, we use

$$f(\tilde{t}_1(0), \tilde{u}_0(0)) = e^{-4l} f(\tilde{t}_1(l), \tilde{u}_0(l)). \tag{10.268}$$

Let us assume that the system is in the low-temperature phase. For $|\tilde{t}_1(l)| \sim 1$ and $\tilde{u}_0(l) \ll 1$, the rhs in (10.268) can be calculated within mean-field theory,[72]

$$f(\tilde{t}_1(0), \tilde{u}_0(0)) \sim -e^{-4l} \frac{\tilde{r}_0(l)^2}{\tilde{u}_0(l)} \sim -|\tilde{t}_1(0)|^2 l^{1-2\frac{N+2}{N+8}}, \tag{10.269}$$

where we have used $\tilde{r}_0(l) \sim \tilde{t}_1(l) \sim 1$, $\tilde{u}_0(l) \sim 1/l \ll 1$ and (10.263). With l given by (10.264), we finally obtain

$$f(\tilde{t}_1(0), \tilde{u}_0(0)) \sim -|\tilde{t}_1(0)|^2 \big| \ln |\tilde{t}_1(0)| \big|^{\frac{4-N}{N+8}}. \tag{10.270}$$

Since $\tilde{t}_1(0) \propto T - T_c$, we deduce the (most) singular part of the specific heat,[73]

$$C_V \sim \big| \ln |T - T_c| \big|^{\frac{4-N}{N+8}}. \tag{10.271}$$

Thus the singular part of C_V diverges for $N < 4$ but vanishes for $N > 4$. These results significantly differ from the Gaussian model predictions $C_V \sim \big| \ln |T - T_c| \big|$ (Sec. 10.3.4).

Logarithmic corrections are a generic consequence of a marginally irrelevant variable. Figure 10.16 shows two trajectories on the critical surface with and without a marginal

[72] The singular dependence of $f(\tilde{t}_1(l), \tilde{u}_0(l))$ on $\tilde{u}_0(l)$ is due to $\tilde{u}_0(l)$ being a dangerously irrelevant variable in the low-temperature phase (see Sec. 10.6.1).

[73] This result can also be obtained from the high-temperature phase using $f(\tilde{t}_1(l), \tilde{u}_0(l)) \simeq f(\tilde{t}_1(l))$ for $\tilde{u}_0(l) \to 0$ and $s^{(4-d)} \equiv \ln s$ for $d \to 4$ ($s = e^l$).

Exponent	Gaussian model	$\mathcal{O}(\epsilon)$	$\mathcal{O}(\epsilon^5)$	Numerics
ν	1/2	0.583	0.6290(25)	0.6302(1)
β	1/2	0.333	0.3257(25)	
γ	1	1.167	1.2380(50)	
δ	3	4		
α	1/2	0.167		
η	0	0	0.0360(50)	0.0368(2)

Table 10.4: Critical exponents obtained from the ϵ expansion [49] and numerical methods (Monte Carlo and high-temperature series) [50] ($d = 3$ and $N = 1$).

variable. In the absence of a marginal variable, the trajectory rapidly converges to the fixed point. On the other hand, when there is a marginally irrelevant variable, the trajectory first moves closer to the corresponding axis (because the irrelevant variables rapidly decrease to zero) before converging to the fixed point due to the slow vanishing of the marginally irrelevant variable. When the relevant field $\tilde{t}_1$ is nonzero but small, the trajectory eventually runs away from the fixed point. Nevertheless, the marginal irrelevant variable still controls the (slow) approach to the fixed point and leads to logarithmic corrections.

10.6.2.5 Utility of the ϵ expansion

The ϵ expansion to first order does not yield reliable estimates of the critical exponents of three-dimensional systems. Its great virtue is to provide a technically easy way of determining what kind of universality classes one can expect. Although the value of the critical exponents changes when one goes away from the upper critical dimension, the topology of the flow diagram does not. One can therefore investigate the phase transitions and their universality classes in various models. Furthermore, the ϵ expansion can also be applied to the analysis of systems near the lower critical dimension (see Sec. 10.7).

Calculating the critical exponents to order ϵ^2 and beyond in the Wilson approach is quite complicated as one must keep track of additional coupling constants besides $\tilde{r}_0$ and $\tilde{u}_0$. In practice, higher-order calculations are carried out using field theoretical perturbative methods (see Appendix 10.A for an introduction to this type of approach). The expansion has been pushed to $\mathcal{O}(\epsilon^5)$. Although the series in ϵ is only asymptotic, it can be evaluated by the Borel summation method. Results for three-dimensional systems are in very good agreement with numerical approaches. Table 10.4 shows the critical exponents obtained from various methods for $d = 3$ and $N = 1$.[74] Note that the crude estimates obtained from the $\mathcal{O}(\epsilon)$ expansion with $\epsilon = 1$ are closer to the exact results than those of the Gaussian model.

[74] In chapter 11 we shall see that the non-perturbative RG is also a powerful tool to compute the critical exponents with high precision.

10.6.2.6 The large-river effect and the field-theoretical approach

-large-river effect: universality in a broader sense. -continuum limit possible thanks to Gaussian fixed-point. Trajectories parametrized by a single parameter (at criticality): field-theoretical approach -non-critical quantities not accessible in that case, by contrast to the Wilson RG, in particular in its functional version (cf. Ising model in NPRG chapter).

10.7 The nonlinear σ model

In section 10.2.2, we have argued that the critical behavior of the lattice classical spin model $H = J\sum_{\langle\mathbf{r},\mathbf{r}'\rangle}\mathbf{S}_\mathbf{r}\cdot\mathbf{S}_{\mathbf{r}'}$ ($\mathbf{S}_\mathbf{r}^2=1$) with O($N$) symmetry is described by a $(\boldsymbol{\varphi}^2)^2$ field theory. In this section, we study the former model at low temperatures in the ordered phase by expanding about the classical configuration $\mathbf{S}_\mathbf{r}=(1,0,\cdots,0)$. We assume $N\geq 2$. For $T<T_c$, the spontaneous symmetry breaking gives rise to a nontrivial physics due to the (gapless) Goldstone modes. We shall see that, rather surprisingly, the low temperature expansion allows us to study the critical behavior near $d=2$ when $N>2$.

We thus start from the partition function

$$Z=\int\mathcal{D}[\mathbf{n}]\prod_\mathbf{r}\delta(\mathbf{n}_\mathbf{r}^2-1)\exp\biggl\{-\frac{1}{2g}\sum_{\mathbf{r},\mu}(D_\mu\mathbf{n}_\mathbf{r})^2+\frac{\mathbf{h}}{g}\cdot\sum_\mathbf{r}\mathbf{n}_\mathbf{r}\biggr\} \tag{10.272}$$

of the NLσM on a d-dimensional hypercubic lattice, where

$$D_\mu\mathbf{n}_\mathbf{r}=\frac{\mathbf{n}_{\mathbf{r}+\boldsymbol{\mu}}-\mathbf{n}_\mathbf{r}}{a} \tag{10.273}$$

(with a the lattice spacing) denotes a discrete derivative, $g=T/Ja^2$, $\mathbf{h}$ is an external field and $\mu=1,\cdots,d$. In the continuum limit, the partition function of the NLσM is often written as

$$Z=\int\mathcal{D}[\mathbf{n}]\,\delta(\mathbf{n}^2-1)\exp\left\{-\frac{1}{2g}\int d^dr(\boldsymbol{\nabla}\mathbf{n})^2+\frac{\mathbf{h}}{g}\cdot\int d^dr\,\mathbf{n}\right\}. \tag{10.274}$$

However, a proper handling of the measure requires to define the model on a lattice and take the continuum limit only at a later stage (see below).

10.7.1 Perturbative expansion

In the limit of a vanishing coupling constant, $g\to 0$, the dominant field configuration is $\mathbf{n}_\mathbf{r}=(1,0,\cdots,0)$ if the field $\mathbf{h}=(h,0,\cdots,0)$ ($h>0$). To study the fluctuations about this configuration, it is convenient to use the parametrization

$$\mathbf{n}_\mathbf{r}=(\sigma_\mathbf{r},\boldsymbol{\pi}_\mathbf{r}), \tag{10.275}$$

where $\sigma_\mathbf{r}$ denotes the component of $\mathbf{n}_\mathbf{r}$ along $\mathbf{h}$ and $\boldsymbol{\pi}_\mathbf{r}$ is a $(N-1)$-component field perpendicular to $\mathbf{h}$. For small fluctuations, $|\boldsymbol{\pi}_\mathbf{r}|\ll 1$ and $\sigma_\mathbf{r}>0$, we can express $\sigma_\mathbf{r}=(1-\boldsymbol{\pi}_\mathbf{r})^{1/2}$ in terms of $\boldsymbol{\pi}_\mathbf{r}$ using $\mathbf{n}_\mathbf{r}^2=1$. It is then possible to derive an effective action for the $\boldsymbol{\pi}$ field by integrating out $\sigma_\mathbf{r}$,

$$Z=\int\mathcal{D}[\sigma,\boldsymbol{\pi}]\prod_\mathbf{r}\delta(\sigma_r^2+\boldsymbol{\pi}_\mathbf{r}^2-1)\exp\biggl\{-\frac{1}{2g}\sum_{\mathbf{r},\mu}\bigl[(D_\mu\sigma_\mathbf{r})^2+(D_\mu\boldsymbol{\pi}_\mathbf{r})^2\bigr]+\frac{h}{g}\sum_\mathbf{r}\sigma_\mathbf{r}\biggr\}$$

$$= \int \mathcal{D}[\boldsymbol{\pi}] \exp\biggl\{ -\frac{1}{2g} \sum_{\mathbf{r},\mu} \Bigl[(D_\mu \boldsymbol{\pi}_{\mathbf{r}})^2 + (D_\mu \sqrt{1-\boldsymbol{\pi}_{\mathbf{r}}^2})^2 \Bigr] + \frac{h}{g} \sum_{\mathbf{r}} \sqrt{1-\boldsymbol{\pi}_{\mathbf{r}}^2} - \frac{1}{2} \sum_{\mathbf{r}} \ln(1-\boldsymbol{\pi}_{\mathbf{r}}^2) \biggr\}, \tag{10.276}$$

where the integration over $\boldsymbol{\pi}_{\mathbf{r}}$ is restricted to $|\boldsymbol{\pi}_{\mathbf{r}}| \leq 1$. The last term in (10.276) comes from the measure $\mathcal{D}[\mathbf{n}] \prod_{\mathbf{r}} \delta(\mathbf{n}_{\mathbf{r}}^2 - 1)$ in the functional integral (10.272).[75]

We are now in a position to take the continuum limit $a \to 0$ whereby $\mathbf{r}$ becomes a continuous position variable and $D_\mu \boldsymbol{\pi}_{\mathbf{r}} = \partial_\mu \boldsymbol{\pi}$ a standard derivative. The term coming from the measure becomes

$$\frac{\rho}{2} \int d^d r \ln(1-\boldsymbol{\pi}^2) \qquad \text{where} \qquad \rho = \frac{1}{V} \sum_{\mathbf{p}\in \mathrm{BZ}} = a^{-d} \tag{10.277}$$

is the number of degrees of freedom per unit volume. In the continuum limit, the Brillouin zone (BZ) is replaced by a spherical region of radius Λ with the same volume,

$$\rho = a^{-d} \equiv \int \frac{d^d p}{(2\pi)^d} \Theta(\Lambda - |\mathbf{p}|) = \frac{K_d}{d} \Lambda^d. \tag{10.278}$$

The final form of the action therefore reads

$$S[\boldsymbol{\pi}] = \frac{1}{2g} \int d^d r \Bigl[(\boldsymbol{\nabla}\boldsymbol{\pi})^2 + (\boldsymbol{\nabla}\sqrt{1-\boldsymbol{\pi}^2})^2 \Bigr] - \frac{h}{g} \int d^d r \sqrt{1-\boldsymbol{\pi}^2} + \frac{\rho}{2} \int d^d r \ln(1-\boldsymbol{\pi}^2) \tag{10.279}$$

(we set the lattice spacing a equal to unity), where the $\boldsymbol{\pi}$ field satisfies the constraint $|\boldsymbol{\pi}| \leq 1$.

The minimum of the action is reached for $\boldsymbol{\pi} = 0$. For small g, we expect the dominant fluctuations to satisfy $|\boldsymbol{\pi}| \sim \sqrt{g}$. Field configurations with $|\boldsymbol{\pi}| \sim 1$ give exponentially small contributions (of order $\exp(-\mathrm{const}/g)$) to the partition function that can be neglected in the perturbative approach ($g \ll 1$). This allows us to ignore the constraint $|\boldsymbol{\pi}| \leq 1$ and freely integrate over π_μ from $-\infty$ to ∞. The expansion wrt g is then similar to a loop expansion (Sec. 1.7), the only difference being that the term coming from the measure is not multiplied by $1/g$. To leading order (using $|\boldsymbol{\pi}| \sim \sqrt{g}$),

$$S_0[\boldsymbol{\pi}] = \frac{1}{2g} \int d^d r \left[(\boldsymbol{\nabla}\boldsymbol{\pi})^2 + h\boldsymbol{\pi}^2 \right], \tag{10.280}$$

and the propagator of the $\boldsymbol{\pi}$ field reads

$$G_0(\mathbf{p}) = \langle \pi_\mu(\mathbf{p}) \pi_\mu(-\mathbf{p}) \rangle = \frac{g}{\mathbf{p}^2 + h}. \tag{10.281}$$

Alternatively, one can obtain the action (10.280) by introducing a rescaled field $\boldsymbol{\pi}_r = g^{-1/2}\boldsymbol{\pi}$ and then setting $g = 0$. For $h = 0$, the state with $\mathbf{n} = (1, 0, \cdots, 0)$ spontaneously breaks the O(N) symmetry, and $S_0[\boldsymbol{\pi}]$ is nothing but the action of the $N-1$ Goldstone modes (to leading order). A nonzero field h explicitly breaks the O(N) symmetry and gives a "mass"

[75] We have used $\int d\sigma\, \delta(\sigma^2 + \boldsymbol{\pi}^2 - 1) = \int d\sigma \frac{1}{2\sigma} \delta(\sigma - \sqrt{1-\boldsymbol{\pi}^2}) \propto \exp\left\{ -\frac{1}{2} \ln(1-\boldsymbol{\pi}^2) \right\}$.

term to the Goldstone modes. From (10.280) and (10.281), we deduce the (bare) stiffness of the NLσM,[76]

$$\rho_s^0 = \frac{1}{g}. \tag{10.282}$$

In the RG language, the action (10.280) corresponds to the Gaussian fixed point $g = 0$. This fixed point is stable if the O(N) symmetry remains spontaneously broken for $g = 0^+$ and $h \to 0$. We can repeat the argument of section 10.3.3 to show that this should be the case for $d > 2$. The reduction of the order parameter by fluctuations is given by

$$\begin{aligned} \langle \sigma(\mathbf{r}) \rangle &= 1 - \frac{1}{2} \langle \boldsymbol{\pi}(\mathbf{r})^2 \rangle + \mathcal{O}(g^2) \\ &= 1 - \frac{(N-1)g}{2} \int \frac{d^d p}{(2\pi)^d} \frac{1}{\mathbf{p}^2} + \mathcal{O}(g^2) \end{aligned} \tag{10.283}$$

for $h = 0$. The momentum integral in (10.283) is infrared divergent for $d \leq 2$, so that long-range order cannot exist in that case (Mermin-Wagner theorem; Sec. 10.3.3). On the other hand we expect long-range order for $d > 2$ when g is small enough. This suggests that higher-order vertices neglected in (10.280) are irrelevant at the Gaussian fixed point when $d > 2$.

Alternatively, one can check the stability of the Gaussian fixed point using dimensional analysis. By expanding the various terms in (10.279), we find two types of vertices: $(\boldsymbol{\pi} \cdot \boldsymbol{\nabla}\boldsymbol{\pi})^2 (\boldsymbol{\pi}^2)^n$ and $(\boldsymbol{\pi}^2)^n$. At the Gaussian fixed point, the $\boldsymbol{\pi}$ field has scaling dimension[77]

$$[\boldsymbol{\pi}] = \frac{d-2}{2}. \tag{10.284}$$

A vertex with $2n$ fields and $2r$ derivatives has then dimension

$$y_{nr} = d - n(d-2) - 2r. \tag{10.285}$$

The vertices $(\boldsymbol{\pi} \cdot \boldsymbol{\nabla}\boldsymbol{\pi})^2 (\boldsymbol{\pi}^2)^n$ are relevant for $d < 2$ and irrelevant for $d > 2$. The vertices $(\boldsymbol{\pi}^2)^n$ are relevant for $d < 2$. Surprisingly, a finite number of them are relevant also for $d > 2$, which seems to contradict our expectation that the Gaussian fixed point should be stable for $d > 2$. In fact, the vertices coming from the measure maintain the O(N) symmetry of the action and ensure that the propagator of the $\boldsymbol{\pi}$ field remains gapless to all orders in perturbation theory. Their relevance does not, as we shall see, invalidate the conclusion that the Gaussian fixed point is stable for $d > 2$.

10.7.2 RG approach

To implement the RG procedure, we proceed as in the case of the (linear) O(N) model (Sec. 10.6.2). We split the field $\boldsymbol{\pi}(\mathbf{r}) = \boldsymbol{\pi}^<(\mathbf{r}) + \boldsymbol{\pi}^>(\mathbf{r})$ into slow ($|\mathbf{p}| \leq \Lambda/s$) and fast ($\Lambda/s \leq |\mathbf{p}| \leq \Lambda$) modes and integrate out the latter. Since the renormalization procedure maintains the O(N) symmetry for $h = 0$, the action of the slow modes must be of the form $S[\mathbf{n}] = \frac{1}{2g'} \int d^d r (\boldsymbol{\nabla}\mathbf{n})^2 - \frac{\mathbf{h}'}{g'} \int d^d r\, \mathbf{n}$ up to higher-order (irrelevant) terms in the derivative expansion. It therefore depends only on the two renormalized coupling constants g' and h'

[76] Eq. (10.282) follows from $\rho_s^0 = \sigma^2/g$ with $\sigma = 1$ the order parameter in the classical configuration $\mathbf{n_r} = (1, 0, \cdots, 0)$.

[77] To define the scaling dimension of the field, it is convenient to use the rescaled field $\boldsymbol{\pi}_r = g^{-1/2}\boldsymbol{\pi}$. Since $S[\boldsymbol{\pi}_r] = \frac{1}{2} \int d^d r \left[(\boldsymbol{\nabla}\boldsymbol{\pi}_r)^2 + h \boldsymbol{\pi}_r^2 \right]$, $[\boldsymbol{\pi}_r] = d/2 - 1$.

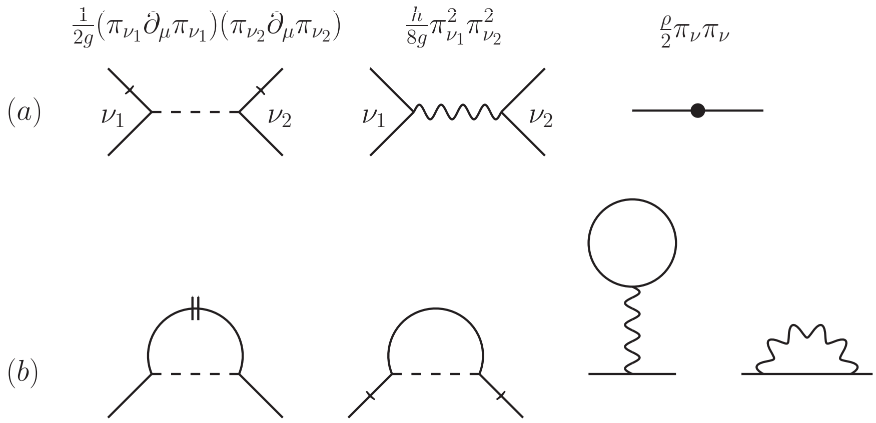


Figure 10.17: (a) Diagrammatic representation of the three terms of $S_1[\boldsymbol{\pi}]$. The slashed lines represent the spatial derivative ∂_μ. (b) One-loop diagrams obtained from the diagrams shown in (a).

as well as on a field renormalization factor necessary to keep the normalization $\mathbf{n}^2 = 1$. This implies that the vertices in the action (10.279) are not independent. For instance, the ratio of the coefficients of $(\boldsymbol{\pi}\cdot\boldsymbol{\nabla}\boldsymbol{\pi})^2$ and $(\boldsymbol{\nabla}\boldsymbol{\pi})^2$ is one. If this ratio were different, the action would not have O(N) symmetry. Thus we need only consider RG equations for the two independent coupling constants g and h/g.

10.7.2.1 One-loop RG equations

To obtain the RG equations to lowest order, it is sufficient to consider the $\mathcal{O}(g)$ correction to S_0,

$$S_1[\boldsymbol{\pi}] = \frac{1}{2g}\int d^dr\Big[(\boldsymbol{\pi}\cdot\boldsymbol{\nabla}\boldsymbol{\pi})^2 + \frac{h}{4}(\boldsymbol{\pi}^2)^2\Big] - \frac{\rho}{2}\int d^dr\,\boldsymbol{\pi}^2. \tag{10.286}$$

The three terms in S_1 are diagrammatically represented in figure 10.17a. Integrating out the fast modes, one finds

$$\begin{aligned} Z &= \int\mathcal{D}[\boldsymbol{\pi}^<]\,e^{-S_0[\boldsymbol{\pi}^<]}\int\mathcal{D}[\boldsymbol{\pi}^>]\,e^{-S_0[\boldsymbol{\pi}^>]-S_1[\boldsymbol{\pi}^<+\boldsymbol{\pi}^>]+\mathcal{O}(g^2)} \\ &= \int\mathcal{D}[\boldsymbol{\pi}^<]\,e^{-S_0[\boldsymbol{\pi}^<]-\langle S_1[\boldsymbol{\pi}^<+\boldsymbol{\pi}^>]\rangle_{0,>}+\mathcal{O}(g^2)}, \end{aligned} \tag{10.287}$$

where

$$\begin{aligned} \langle S_1[\boldsymbol{\pi}^< + \boldsymbol{\pi}^>]\rangle_{0,>} = {}& \frac{1}{2gV}\sum_{\mathbf{p},\nu}\pi^<_\nu(-\mathbf{p})\pi^<_\nu(\mathbf{p})\sum_{\mathbf{p}'}{}' \left[\mathbf{p}^2+\mathbf{p}'^2\right]G_0(\mathbf{p}') \\ &+ \frac{h}{4gV}\sum_{\mathbf{p},\nu}[(N-1)+2]\pi^<_\nu(-\mathbf{p})\pi^<_\nu(\mathbf{p})\sum_{\mathbf{p}'}{}' G_0(\mathbf{p}') \\ &- \frac{1}{2}\rho^>\sum_{\mathbf{p},\nu}\pi^<_\nu(-\mathbf{p})\pi^<_\nu(\mathbf{p}). \end{aligned} \tag{10.288}$$

The notation $\sum'_{\mathbf{p}'}$ means the sum over $\mathbf{p}'$ is restricted to the fast modes $\Lambda/s \leq |\mathbf{p}'| \leq \Lambda$. The first two terms are represented by the diagrams of figure 10.17b. The last term in (10.288) is obtained from $\frac{\rho}{2}\int d^dr\,(\boldsymbol{\pi}^<)^2$ by writing $\rho = \rho^< + \rho^>$ with

$$\rho^> = \frac{1}{V}\sum_{\mathbf{p}'}{}' \equiv \fint_{\mathbf{p}'} \qquad (V \to \infty) \tag{10.289}$$

and including the contribution $-\frac{1}{2}\rho^>\int d^dr\,(\boldsymbol{\pi}^<)^2$ in the action of the slow modes. We therefore obtain the following action of the slow modes to leading order,

$$S[\boldsymbol{\pi}^<] = \frac{1}{2g}(1+dI)\int d^dr(\boldsymbol{\nabla}\boldsymbol{\pi}^<)^2 + \frac{h}{2g}\left(1+\frac{N-1}{2}dI\right)\int d^dr(\boldsymbol{\pi}^<)^2, \tag{10.290}$$

where

$$dI = \fint_{\mathbf{p}} G_0(\mathbf{p}) \tag{10.291}$$

and we have used

$$\fint_{\mathbf{p}} \mathbf{p}^2 G_0(\mathbf{p}) = g\rho^> - h dI. \tag{10.292}$$

Rescaling momenta and field, i.e. $\mathbf{r} \to \mathbf{r}' = \mathbf{r}/s$ and $\boldsymbol{\pi}^< \to \boldsymbol{\pi}^{<\prime} = \lambda\boldsymbol{\pi}^<$, we reproduce the original action $S_0[\boldsymbol{\pi}^<] + \mathcal{O}(g)$ but with renormalized coupling constants,

$$\begin{aligned} \frac{1}{g'} &= \frac{1}{g}(1+dI)\lambda^{-2}s^{d-2}, \\ \frac{h'}{g'} &= \frac{h}{g}\left(1+\frac{N-1}{2}dI\right)\lambda^{-2}s^d. \end{aligned} \tag{10.293}$$

To obtain the field rescaling factor λ, one could compute the renormalized coefficient of $(\boldsymbol{\pi}^< \cdot \boldsymbol{\nabla}\boldsymbol{\pi}^<)^2$. Because of the $\mathrm{O}(N)$ symmetry, this coefficient should be equal to the coefficient of $(\boldsymbol{\nabla}\boldsymbol{\pi}^<)^2$. Alternatively, one can notice that h/g scales trivially. The $\mathrm{O}(N)$ symmetry implies that the renormalization of h/g does not depend of the direction of the field $\mathbf{h}$. If one couples $\mathbf{h}$ to π_μ (rather than to σ), one obtains the following term in the action,

$$-\frac{h}{g}\int d^dr\,\pi_\mu(\mathbf{r}). \tag{10.294}$$

Since h/g couples to $\pi^<_\mu(\mathbf{p}=0)$, it is not affected by the integration of $\boldsymbol{\pi}^>$ and its renormalization is entirely due to the rescaling of momenta and field,

$$\frac{h'}{g'} = \frac{h}{g}s^d\lambda^{-1}. \tag{10.295}$$

Comparing (10.295) with (10.293), we deduce

$$\begin{aligned} \lambda &= 1+\frac{N-1}{2}dI, \\ g' &= g\frac{\left(1+\frac{N-1}{2}dI\right)^2}{1+dI}s^{2-d}. \end{aligned} \tag{10.296}$$

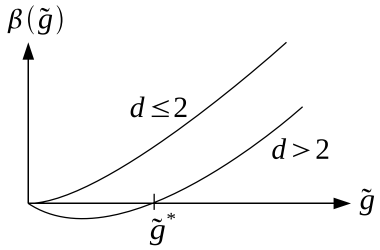


Figure 10.18: β function $\beta(\tilde{g}) = d\tilde{g}/dl$ of the NLσM for $d \leq 2$ and $d > 2$.

Taking $s = e^{dl}$ ($dl \to 0$) and using

$$dI = gK_d\Lambda^{d-2}dl \tag{10.297}$$

for $h = 0$, we can transform the RG equations (10.296) into a differential equation for the coupling constant $g(l)$. Introducing a dimensionless coupling constant $\tilde{g} = \Lambda^{2-d}g$, we finally obtain

$$\frac{d\tilde{g}}{dl} = -\epsilon\tilde{g} + (N-2)K_d\tilde{g}^2 + \mathcal{O}(\tilde{g}^3), \tag{10.298}$$

where $\epsilon = d - 2$.

It is possible to deduce the (running) dimension $d_\pi(l)$ of the field from the rescaling factor λ [Eq. (10.296)]. Because the propagator $G_0(\mathbf{p})$ depends explicitly on the coupling constant g, one should however disentangle the contribution to λ coming from the running of $\tilde{g}$. Let us consider the rescaled fields $\boldsymbol{\pi}_r = g(l)^{-1/2}\boldsymbol{\pi}$ and $\boldsymbol{\pi}'_r = g(l+dl)^{-1/2}\boldsymbol{\pi}'$, where $\boldsymbol{\pi}' = \lambda\boldsymbol{\pi}$. The scaling dimension $d_\pi = [\boldsymbol{\pi}_r]$ can be deduced from the behavior of the $\boldsymbol{\pi}_r$ field in the renormalization process, i.e.

$$\boldsymbol{\pi}'_r = e^{d_\pi(l)dl}\boldsymbol{\pi}_r = \lambda\left(\frac{g(l)}{g(l+dl)}\right)^{1/2}\boldsymbol{\pi}_r \tag{10.299}$$

(see Sec. 10.5.1) with

$$\lambda = 1 + \frac{N-1}{2}\tilde{g}(l)K_d dl + \mathcal{O}(dl^2). \tag{10.300}$$

Equation (10.299) gives

$$d_\pi(l) = \frac{d-2}{2} + \frac{K_d}{2}\tilde{g}(l) \tag{10.301}$$

and in turn the (running) anomalous dimension

$$\eta(l) = K_d\tilde{g}(l). \tag{10.302}$$

10.7.2.2 Fixed points and critical exponents

As expected, $\tilde{g}$ is irrelevant at the Gaussian fixed point $\tilde{g} = 0$ for $d > 2$ and relevant for $d \leq 2$ (Fig. 10.18). In the following, we assume $N > 2$; the case $N = 2$ will be discussed later.

For $d \leq 2$, the growing of the coupling constant invalidates the perturbation approach and indicates that the system is disordered for any finite value of $\tilde{g}$ in agreement with

On utilise $K_2 = 1/2\pi$???

the Mermin-Wagner theorem. Let us consider the two-dimensional case where $\tilde{g} = g$ is marginally relevant,

$$g(l) = \frac{g(0)}{1 - \frac{N-2}{2\pi} g(0) l}. \tag{10.303}$$

We can obtain the correlation length from

$$\xi(g(0)) = \xi(g(l)) e^l = \xi(g(l)) \exp\left\{ \int_{g(0)}^{g(l)} \frac{dg}{\beta(g)} \right\} \tag{10.304}$$

where $\beta(g) = dg/dl$ is β function. For $g(l) \lesssim 1$, we can use (10.298) to obtain

$$\xi(g(0)) = \xi(g(l)) \exp\left\{ \frac{2\pi}{N-2} \left(\frac{1}{g(0)} - \frac{1}{g(l)} \right) \right\} \tag{10.305}$$

For $g(l) \sim 1$, we expect $\xi(g(l)) \sim \Lambda^{-1}$, so that

$$\xi \sim \Lambda^{-1} \exp\left(\frac{2\pi}{(N-2)g} \right) = \Lambda^{-1} \exp\left(\frac{2\pi\rho_s^0}{N-2} \right) \tag{10.306}$$

for $g \equiv g(0) \ll 1$. The correlation length diverges exponentially with $1/g$.

For $d > 2$, there is a critical fixed point $\tilde{g}^*$ located away from the origin corresponding to a second-order phase transition between an ordered phase (described by the Gaussian fixed point $\tilde{g} = 0$) and a disordered phase (Fig. 10.18). Near the lower critical dimension $d_c^- = 2$, this fixed point and the associated critical exponents can be obtained within an ϵ expansion ($\epsilon = d - 2$). To leading order in ϵ, one finds[78]

$$\begin{aligned} \tilde{g}^* &= \frac{2\pi}{N-2}\epsilon + \mathcal{O}(\epsilon^2), \\ \eta &= \frac{\tilde{g}^*}{2\pi} + \mathcal{O}(\epsilon^2) = \frac{\epsilon}{N-2} + \mathcal{O}(\epsilon^2). \end{aligned} \tag{10.307}$$

We can determine a characteristic length ξ from (10.304). If $\tilde{g}(0)$ is close to $\tilde{g}^*$, we linearize the β function,

$$\frac{d\tilde{g}}{dl} = (\tilde{g} - \tilde{g}^*)\beta'(\tilde{g}^*) + \mathcal{O}\left((\tilde{g} - \tilde{g}^*)^2\right), \tag{10.308}$$

to obtain

$$\xi = \xi(l) \left| \frac{\tilde{g}(l) - \tilde{g}^*}{\tilde{g}(0) - \tilde{g}^*} \right|^{1/\beta'(\tilde{g}^*)}, \tag{10.309}$$

which holds provided that $|\tilde{g}(l) - \tilde{g}^*| \lesssim 1$ (thus allowing the linearized form (10.308) to be used). To eliminate the dependence on l in (10.309), we note that we expect $\xi(l) \sim \Lambda^{-1}$ for $\tilde{g}(l) \sim 2\tilde{g}^*$ (the factor 2 is somewhat arbitrary here) in the disordered phase, and $\xi(l) \sim \Lambda^{-1}$ for $\tilde{g}(l) \sim \tilde{g}^*/2$ in the ordered phase. We conclude that

$$\xi \sim |\tilde{g} - \tilde{g}^*|^{-\nu} \tag{10.310}$$

($\tilde{g} \equiv \tilde{g}(0)$) diverges at the transition with an exponent

$$\nu = \frac{1}{\beta'(\tilde{g}^*)} = \frac{1}{\epsilon}. \tag{10.311}$$

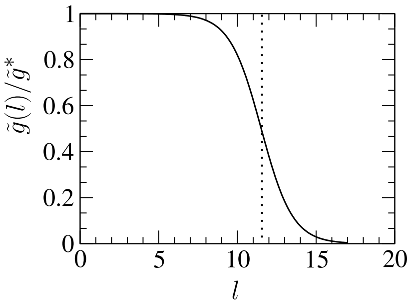


Figure 10.19: Solution of the RG equation (10.298) for $\tilde{g}(0)$ smaller but very close to the critical point $\tilde{g}^*$ ($d = 3$ and $N = 3$). The dotted vertical line indicates the Josephson scale $l_J = \ln(\xi_J \Lambda)$ separating the critical regime $l \ll l_J$ from the Goldstone regime $l \gg l_J$.

10.7.2.3 Josephson length and stiffness

While the characteristic length ξ is naturally identified with the correlation length in the disordered phase, its physical meaning is less clear in the ordered phase. Let us consider the coupling constant $\tilde{g}(l)$ in the ordered phase. When $\tilde{g}(0)$ is close to $\tilde{g}^*$, one can identify a critical regime $l \ll l_J$ where $\tilde{g}(l)$ is very close to the critical value $\tilde{g}^*$ and $\eta_l \simeq \eta$ (Fig. 10.19). For $l \gg l_J$, $\tilde{g}(l)$ goes to zero and the long-distance physics is governed by the Gaussian fixed point $\tilde{g} = 0$. Thus the Josephson length $\xi_J \simeq \Lambda^{-1} e^{l_J}$ separates a critical regime and a "Goldstone" regime where the Goldstone modes are effectively non-interacting.

Since the stiffness has scaling dimension $[\rho_s] = d - 2$ (Sec. 10.4.1), the renormalized stiffness satisfies

$$\rho_s(\tilde{g}(l)) = \rho_s(\tilde{g}(0)) e^{(d-2)l}. \tag{10.312}$$

In the low-temperature phase, the coupling constant $\tilde{g}(l)$ goes to zero for $l \to \infty$ and we can read off the renormalized stiffness directly from the renormalized action,

$$\rho_s(\tilde{g}(l)) \simeq \rho_s^0(\tilde{g}(l)) = \frac{\Lambda^{d-2}}{\tilde{g}(l)} \quad (l \to \infty), \tag{10.313}$$

so that

$$\rho_s \equiv \rho_s(\tilde{g}(0)) = \Lambda^{d-2} \lim_{l \to \infty} \frac{e^{-(d-2)l}}{\tilde{g}(l)}. \tag{10.314}$$

Using the solution

$$\tilde{g}(l) = \frac{\tilde{g}^*}{1 - e^{(d-2)l}\left(1 - \frac{\tilde{g}^*}{\tilde{g}(0)}\right)} \tag{10.315}$$

of the RG equation (10.298), we finally obtain

$$\rho_s = \rho_s^0 \left(1 - \frac{\tilde{g}(0)}{\tilde{g}^*}\right). \tag{10.316}$$

At the transition ($\tilde{g}(0) \to \tilde{g}^*$), the stiffness vanishes with an exponent $\nu(d-2) = 1$.

[78] Note that the one-loop RG equations (giving $\beta(\tilde{g})$ to $\mathcal{O}(\tilde{g}^2)$) are sufficient to obtain the fixed-point value $\tilde{g}^*$ and the anomalous dimension η to $\mathcal{O}(\epsilon)$.

In the Goldstone regime, one can also compute the connected propagator of the longitudinal field σ using $\sigma \simeq 1 - \boldsymbol{\pi}^2/2$ for small transverse fluctuations,

$$\begin{aligned}\langle \sigma(\mathbf{r})\sigma(0)\rangle_c &\simeq \frac{1}{4}\langle \boldsymbol{\pi}(\mathbf{r})^2\boldsymbol{\pi}(0)^2\rangle_c \simeq \frac{1}{4}\sum_{\nu,\nu'}\langle \pi_\nu^2(\mathbf{r})\pi_{\nu'}^2(0)\rangle_c \simeq \frac{N-1}{2}\langle \pi_\nu(\mathbf{r})\pi_\nu(0)\rangle^2 \\ &\sim \frac{1}{|\mathbf{r}|^{2d-4}}, \end{aligned} \tag{10.317}$$

where $\langle \pi_\nu(\mathbf{r})\pi_\nu(0)\rangle \sim 1/|\mathbf{r}|^{d-2}$ is the Goldstone modes propagator in the limit $g \to 0$ obtained from the Fourier transform of $1/\mathbf{p}^2$. The last result in (10.317) is obtained using Wick's theorem (which holds at the Gaussian fixed point). We therefore deduce that the longitudinal propagator

$$\langle \sigma(\mathbf{p})\sigma(-\mathbf{p})\rangle_c \sim \begin{cases} \ln|\mathbf{p}| & \text{for} \quad d=4, \\ \dfrac{1}{|\mathbf{p}|^{4-d}} & \text{for} \quad d<4, \end{cases} \tag{10.318}$$

is singular for $\mathbf{p} \to 0$ below four dimensions. (Equations (10.318) hold for $|\mathbf{p}| \ll \xi_J^{-1}$.) This result should be contrasted with the predictions of the Gaussian approximation (which neglects the coupling between transverse and longitudinal fluctuations) to the $(\boldsymbol{\varphi}^2)^2$ theory, according to which the longitudinal fluctuation mode is gapped and the longitudinal propagator finite in the limit $\mathbf{p} \to 0$ [Eq. (10.82)]. The longitudinal propagator singularity $\sim 1/|\mathbf{p}|^{4-d}$ is weaker than that ($\sim 1/|\mathbf{p}|^2$) of the transverse propagator for $d > d_c^-$. Both singularities would coincide at the lower critical dimension $d_c^- = 2$ if long-range order were not suppressed by fluctuations thus making the propagator of the $\mathbf{n}$ field gapped and O(N) symmetric.

10.7.2.4 Non-linear sigma model *vs* $(\boldsymbol{\varphi}^2)^2$ theory

The $\epsilon = d-2$ expansion of the NLσM near the lower critical dimension bears some similarities with the $\epsilon = 4-d$ expansion of the (linear) O(N) model near the upper critical dimension (section 10.6.2). In both approaches, we find a second-order phase transition and are able to compute the critical exponents to order ϵ using a perturbative (one-loop) RG. In the linear O(N) model near criticality, we found two characteristic lengths, the correlation length (or the Josephson length in the ordered phase) ξ and the Ginzburg length ξ_G. These two characteristic lengths, which determine the critical regime $\xi^{-1} \ll |\mathbf{p}| \ll \xi_G^{-1}$ in momentum space, are related to the two (bare) parameters, r_0 and u_0, of the model. By contrast, there is only one coupling constant in the NLσM and therefore one characteristic length, the correlation length (or the Josephson length) ξ, the critical regime being defined by $\xi^{-1} \ll |\, p| \leq \Lambda$.

Since the $(\boldsymbol{\varphi}^2)^2$ theory and the NLσM derive from the same spin model $H = J\sum_{\langle \mathbf{r},\mathbf{r}'\rangle} \mathbf{S}_\mathbf{r} \cdot \mathbf{S}_{\mathbf{r}'}$, they should belong to the same universality class. There are two limits where this can be shown explicitly. i) In the large N limit: to all orders of the $1/N$ expansion, the correlation functions in the critical regime have the same asymptotic behavior in both models (Appendix 10.B). ii) Near two dimensions: when $d \to d_c^-$, amplitude fluctuations play no role in the critical behavior of the $(\boldsymbol{\varphi}^2)^2$ theory which then reduces to that of the NLσM discussed in this section (chapter 11).

However, in contrast to the $(\boldsymbol{\varphi}^2)^2$ theory, the NLσM does not yield estimates of the critical exponents in three dimensions, the $\epsilon = d-2$ expansion being not Borel summable.

10.7.2.5 The O(2) NLσM

If $N = 2$, the one-loop correction to the β function vanishes and $\beta(\tilde{g}) = d\tilde{g}/dl$ reduces to the purely dimensional term $-\epsilon\tilde{g}$ (coming from the rescaling of momenta and fields) [Eq. (10.298)]. With the parametrization $\mathbf{n} = (\cos\theta, \sin\theta)$, one can write the NL$\sigma$M as a free field theory,

$$S[\theta] = \frac{1}{2g}\int d^d r (\boldsymbol{\nabla}\theta)^2. \tag{10.319}$$

The result $\beta(\tilde{g}) = -\epsilon\tilde{g}$ is therefore exact (to all orders in the loop expansion) for the O(2) model.

For $d > 2$, the continuous symmetry remains broken for any value of g (the fixed point $\tilde{g}^* = 2\pi\epsilon/(N-2) \to \infty$ for $N \to 2^+$), with a mean value of the field given by $|\langle \mathbf{n}(\mathbf{r})\rangle| = \langle e^{i\theta(\mathbf{r})}\rangle$, i.e.

$$|\langle \mathbf{n}(\mathbf{r})\rangle| = \exp\left(-\frac{1}{2}\langle\theta(\mathbf{r})^2\rangle\right) = \exp\left(-\frac{1}{2}\int_{\mathbf{p}}\frac{g}{\mathbf{p}^2}\right). \tag{10.320}$$

Spin-wave excitations alone are not able to disorder the system. There is however no doubt that the original lattice model (10.272) (XY model) is disordered at sufficiently high temperature (i.e. sufficiently large g).[79] This apparent paradox can be explained by the observation that the NLσM ignores the fact that θ is a cyclic variable, which can be justified only at small g (up to exponentially small corrections in $1/g$).

In two dimensions, the β function of the O(2) NLσM vanishes identically. Mermin-Wagner theorem forbids long-range order. However, the fact that the expression $g^* = 2\pi\epsilon/(N-2)$ becomes undetermined when $\epsilon \to 0$ and $N \to 2$ suggests that this case might be special (see Sec. 10.8).

10.7.3 Low-temperature limit of the $(\varphi^2)^2$ theory

In section 10.3, we have studied the $(\boldsymbol{\varphi}^2)^2$ theory with O(N) symmetry in the Gaussian approximation. This approximation breaks down in the vicinity of the phase transition because of critical fluctuations. We shall see below that when $d \leq 4$ the Gaussian approximation breaks down in the whole low-temperature phase for $N \geq 2$ due to the presence of Goldstone modes. It is however possible to circumvent the difficulties of the perturbation theory by considering the "good" hydrodynamic variables, namely the amplitude and the direction of the N-component field $\boldsymbol{\varphi}$. While amplitude fluctuations are gapped, the effective theory describing the low-energy direction fluctuations is a NLσM.

10.7.3.1 Breakdown of perturbation theory

Within the mean-field (or saddle-point) approximation, the order parameter $\boldsymbol{\varphi}_0 = \langle\boldsymbol{\varphi}(\mathbf{r})\rangle$ has an amplitude $\varphi_0 = (-6r_0/u_0)^{1/2}$ in the low-temperature phase ($r_0 < 0$). By expanding the action to quadratic order about the mean-field solution, one finds the (zero-loop) self-energy

$$\Sigma^{(0)}_{ii}(\mathbf{p}) = \begin{cases} -3r_0 & \text{if} \quad i = 1, \\ -r_0 & \text{if} \quad i \neq 1, \end{cases} \tag{10.321}$$

[79] See, e.g., the argument given in Sec. 10.8.

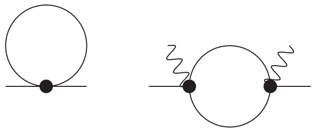


Figure 10.20: One-loop correction $\Sigma^{(1)}$ to the self-energy. The dots represent the bare interaction, the zigzag lines the order parameter φ_0, and the solid lines the connected propagator $G^{(0)}$.

and the longitudinal and transverse propagators

$$\begin{aligned} G_{\parallel}^{(0)}(\mathbf{p}) &= G_{11}^{(0)}(\mathbf{p}) = \frac{1}{\mathbf{p}^2 + 2|r_0|}, \\ G_{\perp}^{(0)}(\mathbf{p}) &= G_{22}^{(0)}(\mathbf{p}) = \frac{1}{\mathbf{p}^2} \end{aligned} \tag{10.322}$$

(see Sec. 10.3.1). We assume the order parameter $\boldsymbol{\varphi}_0$ to be parallel to the direction $(1, 0, \cdots, 0)$. In agreement with Goldstone's theorem, the transverse propagator is gapless whereas the longitudinal susceptibility $G_{\parallel}(\mathbf{p} = 0) = 1/|2r_0|$ is finite.

Let us now consider the one-loop correction $\Sigma^{(1)}$ to the self-energy shown in figure 10.20. While the first diagram is finite, the second one gives a diverging contribution to Σ_{11} in the infrared limit $\mathbf{p} \to 0$ when $d \leq 4$. The divergence arises when both internal lines correspond to transverse fluctuations (which is possible only for Σ_{11}). Retaining only the divergent contribution, we obtain

$$\Sigma_{11}^{(1)}(\mathbf{p}) \simeq -\frac{N-1}{18} u_0^2 \varphi_0^2 \int_{\mathbf{q}} \frac{1}{\mathbf{q}^2 (\mathbf{p} + \mathbf{q})^2}. \tag{10.323}$$

The momentum integration in (10.323) gives[80]

$$\int_{\mathbf{q}} \frac{1}{\mathbf{q}^2 (\mathbf{p} + \mathbf{q})^2} = \begin{cases} A_d |\mathbf{p}|^{d-4} & \text{if} \quad d < 4, \\ A_4 [1 + \ln(\Lambda/|\mathbf{p}|)] & \text{if} \quad d = 4, \end{cases} \tag{10.324}$$

for $|\mathbf{p}| \ll \Lambda$, where [51]

$$A_d = \begin{cases} -\dfrac{2^{1-d} \pi^{1-d/2}}{\sin(\pi d/2)} \dfrac{\Gamma(d/2)}{\Gamma(d-1)} & \text{if} \quad d < 4, \\ \dfrac{1}{8\pi^2} & \text{if} \quad d = 4. \end{cases} \tag{10.325}$$

The one-loop correction (10.323) diverges for $\mathbf{p} \to 0$ and the perturbation expansion about the Gaussian approximation breaks down. By comparing the one-loop correction to the zero-loop result, i.e. $|\Sigma_{11}^{(1)}(\mathbf{p})| \sim \Sigma_{11}^{(0)}(\mathbf{p})$, one can extract the characteristic momentum scale

$$p_G \sim \begin{cases} [A_d (N-1) u_0]^{1/(4-d)} & \text{if} \quad d < 4, \\ \Lambda \exp\left(\frac{-1}{A_4 (N-1) u_0} \right) & \text{if} \quad d = 4. \end{cases} \tag{10.326}$$

[80]For simplicity, in the following we approximate $1 + \ln(\Lambda/|\mathbf{p}|) \simeq \ln(\Lambda/|\mathbf{p}|)$ for $|\mathbf{p}| \ll \Lambda$.

p_G^{-1} is nothing but the Ginzburg length ξ_G introduced in section 10.3.4. In the critical regime, p_G is the momentum scale associated with the breakdown of the Gaussian approximation and the onset of critical fluctuations (Sec. 10.6.2). Here we see that in the whole low-temperature phase, the Ginzburg momentum scale p_G signals the breakdown of the Gaussian approximation: while the Gaussian or perturbative approach remains valid for $|\mathbf{p}| \gg p_G$, the limit $|\mathbf{p}| \ll p_G$ cannot be studied perturbatively. We shall see below that, when the system is away from the critical regime, the breakdown of perturbation theory is due to the coupling between transverse and longitudinal fluctuations.

10.7.3.2 Amplitude-direction representation

To distinguish between amplitude and direction fluctuations in the ordered phase, we use the parametrization

$$\boldsymbol{\varphi}(\mathbf{r}) = \rho(\mathbf{r})\mathbf{n}(\mathbf{r}), \tag{10.327}$$

with $\mathbf{n}(\mathbf{r})^2 = 1$. The partition function becomes

$$Z = \int \mathcal{D}[\rho, \mathbf{n}] \prod_{\mathbf{r}} \rho^{N-1}(\mathbf{r})\, e^{-S[\rho,\mathbf{n}]}, \tag{10.328}$$

where the action is given by

$$S[\rho, \mathbf{n}] = \int d^d r \left[\frac{1}{2}\rho^2(\boldsymbol{\nabla}\mathbf{n})^2 + \frac{1}{2}(\boldsymbol{\nabla}\rho)^2 + \frac{r_0}{2}\rho^2 + \frac{u_0}{4!}\rho^4 \right]. \tag{10.329}$$

In the low-temperature phase ($r_0 < 0$), the mean-field theory yields a finite order parameter $\rho_0 = (-6r_0/u_0)^{1/2}$. To quadratic order in the fluctuations $\rho' = \rho - \rho_0$, we obtain the action

$$S[\rho', \mathbf{n}] = \int d^d r \left[\frac{\rho_0^2}{2}(\boldsymbol{\nabla}\mathbf{n})^2 + \frac{1}{2}(\boldsymbol{\nabla}\rho')^2 + |r_0|\rho'^2 \right] \tag{10.330}$$

and deduce that the amplitude fluctuations are gapped,

$$\langle \rho'(\mathbf{p})\rho'(-\mathbf{p}) \rangle = \frac{1}{\mathbf{p}^2 + p_c^2}. \tag{10.331}$$

If we are interested only in momenta $|\mathbf{p}| \ll p_c = \sqrt{2|r_0|}$, to first approximation we can ignore the higher-order terms in ρ' that were neglected in (10.330), since they would only lead to a finite renormalization of the coefficients of the action $S[\rho', \mathbf{n}]$.

Equation (10.330) shows that in the "hydrodynamic" regime $|\mathbf{p}| \ll p_c$ direction fluctuations are described by a NLσM. It is convenient to use the parametrization $\mathbf{n} = (\sigma, \boldsymbol{\pi})$ introduced in section 10.7.1. Integrating over σ, one then obtains

$$S[\rho', \boldsymbol{\pi}] = \int d^d r \left[\frac{1}{2}(\boldsymbol{\nabla}\rho')^2 + |r_0|\rho'^2 + \frac{1}{2}\rho_0^2(\boldsymbol{\nabla}\boldsymbol{\pi})^2 \right] \tag{10.332}$$

for small transverse fluctuations $\boldsymbol{\pi}$ (i.e. to leading order in the coupling constant $1/\rho_0^2$ of the NLσM). From (10.332), we deduce the propagator of the $\boldsymbol{\pi}$ field,

$$\langle \pi_i(\mathbf{p})\pi_j(-\mathbf{p}) \rangle = \frac{\delta_{i,j}}{\rho_0^2\mathbf{p}^2}. \tag{10.333}$$

Again we note that the terms neglected in (10.332) would only lead to a finite renormalization of the (bare) stiffness ρ_0^2 of the NLσM at sufficiently low temperature. In fact, equation (10.332) gives an exact description of the low-energy behavior $|\mathbf{p}| \ll p_c$ if one replaces ρ_0^2 by the exact stiffness and $p_c^{-1} = (2|r_0|)^{-1/2}$ by the exact correlation length of the ρ' field.

We are now in a position to compute the longitudinal and transverse propagators using

$$\begin{aligned} \varphi_\parallel &= \rho\sigma = \rho\sqrt{1-\boldsymbol{\pi}^2} \simeq \rho_0 + \rho' - \frac{1}{2}\rho_0\boldsymbol{\pi}^2, \\ \boldsymbol{\varphi}_\perp &= \rho\boldsymbol{\pi} \simeq \rho_0\boldsymbol{\pi}. \end{aligned} \tag{10.334}$$

Since the long-distance physics is governed by transverse fluctuations, we have retained in (10.334) the leading contributions in $\boldsymbol{\pi}$. Making use of (10.333), one readily obtains

$$G_\perp(\mathbf{p}) \simeq \rho_0^2 \langle \pi_i(\mathbf{p})\pi_i(-\mathbf{p})\rangle = \frac{1}{\mathbf{p}^2}. \tag{10.335}$$

The longitudinal propagator is given by

$$\begin{aligned} G_\parallel(\mathbf{r}) &= \langle \rho'(\mathbf{r})\rho'(0)\rangle + \frac{1}{4}\rho_0^2 \langle \boldsymbol{\pi}(\mathbf{r})^2 \boldsymbol{\pi}(0)^2\rangle_c \\ &= \langle \rho'(\mathbf{r})\rho'(0)\rangle + \frac{N-1}{2\rho_0^2} G_\perp(\mathbf{r})^2, \end{aligned} \tag{10.336}$$

where $\langle\cdots\rangle_c$ stands for the connected part of $\langle\cdots\rangle$. The second line is obtained using Wick's theorem. In Fourier space, this gives

$$G_\parallel(\mathbf{p}) = \frac{1}{\mathbf{p}^2+p_c^2} + \frac{N-1}{2\rho_0^2}\int_{\mathbf{q}} \frac{1}{\mathbf{q}^2(\mathbf{p}+\mathbf{q})^2}, \tag{10.337}$$

where the momentum integral is given by (10.324) for $|\mathbf{p}| \ll \Lambda$ and $d \leq 4$. By comparing the two terms in the rhs of (10.337), we recover the Ginzburg momentum scale (10.326). For $|\mathbf{p}| \gg p_G$ (Gaussian regime), the longitudinal propagator $G_\parallel(\mathbf{p}) \simeq 1/(\mathbf{p}^2+p_c^2)$ is dominated by amplitude fluctuations and we reproduce the result of the Gaussian approximation. On the other hand, for $|\mathbf{p}| \ll p_G$ (Goldstone regime), $G_\parallel(\mathbf{p}) \sim 1/|\mathbf{p}|^{4-d}$ is dominated by direction fluctuations and diverges for $\mathbf{p} \to 0$:

$$G_\parallel(\mathbf{p}) \sim \begin{cases} \frac{1}{\mathbf{p}^2+p_c^2} & \text{if} \quad |\mathbf{p}| \gg p_G, \\ \frac{1}{|\mathbf{p}|^{4-d}} & \text{if} \quad |\mathbf{p}| \ll p_G, \end{cases} \tag{10.338}$$

for $d \leq 4$ (the divergence is logarithmic for $d=4$).

The divergence of the longitudinal propagator is a direct consequence of the coupling between longitudinal and transverse fluctuations. In the long-distance limit, amplitude-Maria Chamarro ¡edpif.su@edpif.org¿ fluctuations become frozen so that $|\boldsymbol{\varphi}| = \rho \simeq \rho_0$. This implies that the longitudinal and transverse components $\varphi_\parallel$ and $\boldsymbol{\varphi}_\perp$ cannot be considered independently as in the Gaussian approximation (Sec. 10.3) but satisfy the constraint $\varphi_\parallel^2 + \boldsymbol{\varphi}_\perp^2 \simeq \rho_0^2$. To leading order, $\varphi_\parallel \simeq \rho_0(1-\frac{\boldsymbol{\pi}^2}{2})^{1/2}$ and $G_\parallel(\mathbf{r}) \sim G_\perp(\mathbf{r})^2$, i.e. $G_\parallel(\mathbf{p}) \sim 1/|\mathbf{p}|^{4-d}$ for $d \leq 4$ [Eq. (10.338)].

Equations (10.335) and (10.337) imply that in the limit $\mathbf{p} \to 0$ the self-energies are given by

Discuter relations exactes à $\mathbf{p}=0$?

$$\begin{aligned}\Sigma_{11}(\mathbf{p}) &= -r_0 + C_1|\mathbf{p}|^{4-d} + \mathcal{O}(\mathbf{p}^2), \\ \Sigma_{22}(\mathbf{p}) &= -r_0 + \mathcal{O}(\mathbf{p}^2),\end{aligned} \tag{10.339}$$

for $d < 4$, and

$$\begin{aligned}\Sigma_{11}(\mathbf{p}) &= -r_0 + \frac{C_1}{\ln(\Lambda/|\mathbf{p}|)} + \mathcal{O}(\mathbf{p}^2), \\ \Sigma_{22}(\mathbf{p}) &= -r_0 + \mathcal{O}(\mathbf{p}^2),\end{aligned} \tag{10.340}$$

for $d = 4$. $\Sigma_{11}(\mathbf{p})$ contains a non-analytic term that is dominant for $\mathbf{p} \to 0$, in marked contrast with the prediction of the Gaussian approximation [Eq. (10.321)].

10.8 The Berezinskii-Kosterlitz-Thouless phase transition

In two dimensions thermal fluctuations prevent the spontaneous breaking of a continuous symmetry at finite temperature (Mermin-Wagner theorem). In this section we shall see that a phase transition without symmetry breaking, driven by topological defects (vortices), can nevertheless occur in two-dimensional systems with a two-component order parameter and an O(2) symmetry. Although the following discussion holds for all systems in that category, we shall consider the XY model defined on a square lattice (with lattice spacing a) by the Hamiltonian

$$H = -J \sum_{\langle \mathbf{r},\mathbf{r}' \rangle} \mathbf{S}_{\mathbf{r}} \cdot \mathbf{S}_{\mathbf{r}'}, \tag{10.341}$$

where $\mathbf{S}_{\mathbf{r}} = (\cos\theta_{\mathbf{r}}, \sin\theta_{\mathbf{r}})$ is a classical spin of unit length and we assume $J > 0$. We shall first show that the existence of a finite-temperature phase transition can be inferred from the high- and low-temperature expansions and discuss the role of vortices from simple arguments. We shall then turn to the Coulomb-gas and sine-Gordon models for a detailed study of the transition.

10.8.1 Finite-temperature phase transition

10.8.1.1 High-temperature expansion

At high temperatures, the spin-spin correlation function can be determined using the expansion

$$\begin{aligned}\langle \mathbf{S}_{\mathbf{r}} \cdot \mathbf{S}_{\mathbf{r}'} \rangle &= \frac{1}{Z} \left(\prod_{\mathbf{r}''} \int_0^{2\pi} \frac{d\theta_{\mathbf{r}''}}{2\pi} \right) \cos(\theta_{\mathbf{r}} - \theta_{\mathbf{r}'})\, e^{K \sum_{\langle \mathbf{r}_i,\mathbf{r}_j \rangle} \mathbf{S}_{\mathbf{r}_i} \cdot \mathbf{S}_{\mathbf{r}_j}} \\ &= \frac{1}{Z} \left(\prod_{\mathbf{r}''} \int_0^{2\pi} \frac{d\theta_{\mathbf{r}''}}{2\pi} \right) \cos(\theta_{\mathbf{r}} - \theta_{\mathbf{r}'}) \prod_{\langle \mathbf{r}_i,\mathbf{r}_j \rangle} \big[1 + K\cos(\theta_{\mathbf{r}_i} - \theta_{\mathbf{r}_j}) + \mathcal{O}(K^2) \big],\end{aligned} \tag{10.342}$$

where $K = J/T$ and Z is the partition function. Since

$$\begin{aligned}&\int_0^{2\pi} \frac{d\theta_{\mathbf{r}''}}{2\pi} \cos(\theta_{\mathbf{r}_1} - \theta_{\mathbf{r}''}) = 0, \\ &\int_0^{2\pi} \frac{d\theta_{\mathbf{r}''}}{2\pi} \cos(\theta_{\mathbf{r}_1} - \theta_{\mathbf{r}''}) \cos(\theta_{\mathbf{r}''} - \theta_{\mathbf{r}_2}) = \frac{1}{2} \cos(\theta_{\mathbf{r}_1} - \theta_{\mathbf{r}_2}),\end{aligned} \tag{10.343}$$

for the numerator in (10.342) not to vanish one must associate with every factor $\cos(\theta_{\mathbf{r}_1}-\theta_{\mathbf{r}_2})$ another factor $\cos(\theta_{\mathbf{r}_2}-\theta_{\mathbf{r}_3})$. To every nonzero term one can therefore assign a path on the lattice going from site $\mathbf{r}$ to site $\mathbf{r}'$, which gives a contribution $(K/2)^N$ where N is the number of bonds along the path. The dominant contribution in the high-temperature limit is found by choosing the shortest path, i.e. $N \sim |\mathbf{r}-\mathbf{r}'|/a$. We thus obtain

$$\langle \mathbf{S}_\mathbf{r} \cdot \mathbf{S}_{\mathbf{r}'} \rangle \sim \left(\frac{K}{2} \right)^{\frac{|\mathbf{r}-\mathbf{r}'|}{a}} \sim e^{-\frac{|\mathbf{r}-\mathbf{r}'|}{\xi}} \tag{10.344}$$

where $\xi \sim a/\ln(2/K)$.[81] The exponential decay of correlations means that the system is disordered at sufficiently high temperatures.

10.8.1.2 Low-temperature expansion

Since the ground state of the XY model exhibits ferromagnetic order, at low temperatures we expect the dominant fluctuations to be long-wavelength fluctuations (spinwaves) where the phase $\theta_\mathbf{r}$ varies slowly in space. This allows us to approximate the XY model by a continuum model with Hamiltonian

$$H = \frac{\rho_s^0}{2} \int d^2r \, (\boldsymbol{\nabla}\theta)^2, \tag{10.345}$$

where $\rho_s^0 = J$ is the (bare) spinwave stiffness. At sufficiently low temperatures, we can ignore the periodicity of the phase and consider θ in equation (10.345) as a variable varying between $-\infty$ and ∞. We can then use the standard rules of Gaussian integration to obtain

$$\langle \mathbf{S}_\mathbf{r} \cdot \mathbf{S}_{\mathbf{r}'} \rangle = \langle e^{i(\theta_\mathbf{r}-\theta_{\mathbf{r}'})} \rangle = e^{-\frac{1}{2}\langle (\theta_\mathbf{r}-\theta_{\mathbf{r}'})^2 \rangle} = e^{G(\mathbf{r}-\mathbf{r}')-G(0)}, \tag{10.346}$$

where $G(\mathbf{r})$ is the (bare) spinwave propagator,

$$G(\mathbf{r}) = \int \frac{d^2q}{(2\pi)^2} \frac{e^{i\mathbf{q}\cdot\mathbf{r}}}{K\mathbf{q}^2} = \frac{1}{2\pi K} \int_0^\Lambda \frac{dq}{q} J_0(q|\mathbf{r}|). \tag{10.347}$$

J_0 is the zeroth-order Bessel function and we have introduced a UV momentum cutoff $\Lambda \sim 1/a$ since the continuum approximation is valid only at length scales larger than a. We thus obtain

$$G(0) - G(\mathbf{r}) = \frac{1}{2\pi K} \int_0^\Lambda \frac{dq}{q} [1 - J_0(q|\mathbf{r}|)] \simeq \frac{1}{2\pi K} \ln \left| \frac{\mathbf{r}}{a} \right| \tag{10.348}$$

and therefore

$$\langle \mathbf{S}_\mathbf{r} \cdot \mathbf{S}_{\mathbf{r}'} \rangle \sim \left| \frac{a}{\mathbf{r}-\mathbf{r}'} \right|^{\frac{1}{2\pi K}}. \tag{10.349}$$

Since $\langle \mathbf{S}_\mathbf{r} \cdot \mathbf{S}_{\mathbf{r}'} \rangle \to \langle \mathbf{S}_\mathbf{r} \rangle \cdot \langle \mathbf{S}_{\mathbf{r}'} \rangle$ for $|\mathbf{r}-\mathbf{r}'| \to \infty$, we conclude that there is no long-ranger order, $\langle \mathbf{S}_\mathbf{r} \rangle = 0$, in agreement with the Mermin-Wagner theorem. However, the decay of correlations is algebraic rather than exponential as is usually the case in the absence

[81] We assume here that the path of shortest length is unique. Taking into account the multiplicity of shortest lattice paths would modify the correlation length ξ but not the exponential decay of the correlation function.

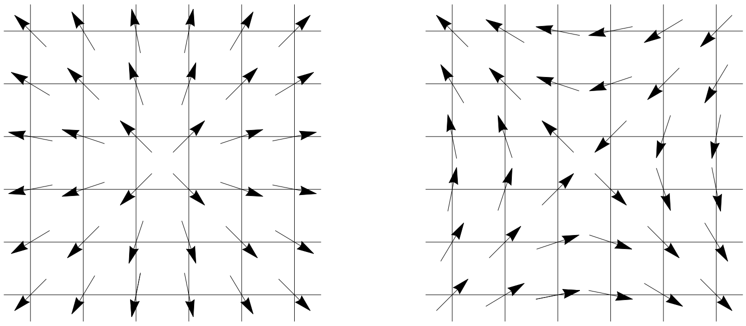


Figure 10.21: A vortex and an antivortex with winding numbers +1 and −1, respectively.

of spontaneous symmetry breaking; the system is said to exhibit (algebraic) quasi-long-range order.[82] In higher dimensions ($d > 2$), an algebraic decay of correlation functions is characteristic of a critical point. Here it arises from the angular growth of angular fluctuations, which is specific to two dimensions, and is expected to hold at sufficiently low temperatures. In other words, rather than a critical point at a given temperature T_c, one finds at low temperatures a critical line characterized by a temperature-dependent anomalous dimension[83]

$$\eta(T) = \frac{1}{2\pi K} = \frac{T}{2\pi\rho_s^0} \tag{10.350}$$

and an infinite correlation length.

This simple analysis suggests the existence of a phase transition at a finite temperature T_c separating the low-temperature phase with quasi-long-range order from the high-temperature phase characterized by a finite correlation length.

10.8.2 The role of vortices

What are the fluctuations responsible for the phase transition? Berezinskii, and Kosterlitz and Thouless, showed that the transition is driven by the unbinding of vortices. A vortex is a topological defect that cannot be obtained from a continuous deformation of the ground state ($\mathbf{S}_\mathbf{r} = \mathbf{n}$ with $\mathbf{n}$ an arbitrary unit vector) and therefore cannot be described by the long-wavelength Hamiltonian (10.345).

Because of the 2π periodicity of the phase θ it is possible to construct spin configurations for which the line integral

$$\oint_\mathcal{C} d\mathbf{l}\cdot\boldsymbol{\nabla}\theta = 2\pi k \qquad (k \in \mathbf{Z}) \tag{10.351}$$

does not vanish when the closed path $\mathcal{C}$ encloses the vortex center (Fig. 10.21). The integer k is known as the topological charge or the winding number of the vortex. Far away from the center, the lattice structure does not matter and we can use a continuum description

[82]The existence of quasi-long-range order at low temperatures follows from the Gaussian Hamiltonian (10.345). Higher-order terms in the derivative expansion, e.g. $[(\boldsymbol{\nabla}\theta)^2]^2$, are irrelevant in the RG sense and would not modify this conclusion.

[83]Recall that in a critical d-dimensional system, the correlation function of the order parameter field decays as $1/|\mathbf{r}|^{d-2+\eta}$.

as in (10.351). If, furthermore, the phase varies slowly in space, one can use the Hamiltonian (10.345) to determine the phase field of the vortex. Minimizing the energy, we obtain

$$0 = \frac{\delta}{\delta\theta(\mathbf{r})}\frac{\rho_s^0}{2}\int d^2r' \big(\boldsymbol{\nabla}_{\mathbf{r}'}\theta(\mathbf{r}')\big)^2 = -\rho_s^0 \boldsymbol{\nabla}^2\theta(\mathbf{r}). \tag{10.352}$$

The phase field $\theta(\mathbf{r}) = k\varphi$, where $\varphi = \arctan(y/x)$, satisfies both the topological constraint (10.351) and the equilibrium condition (10.352). The corresponding velocity field[84]

$$\mathbf{v}(\mathbf{r}) = \boldsymbol{\nabla}\theta(\mathbf{r}) = \frac{k}{\mathbf{r}^2}(-y, x) = -k\boldsymbol{\nabla}\times(\hat{\mathbf{z}}\ln|\mathbf{r}|) \tag{10.353}$$

($\hat{\mathbf{z}}$ is a unit vector in the z direction) is tangential, $\mathbf{v}(\mathbf{r})\cdot\mathbf{r} = 0$, and its Fourier transform

$$\mathbf{v}(\mathbf{q}) = \int d^2r\, e^{-i\mathbf{q}\cdot\mathbf{r}}\mathbf{v}(\mathbf{r}) = 2\pi k\frac{i\mathbf{q}\times\hat{\mathbf{z}}}{\mathbf{q}^2} \tag{10.354}$$

is a transverse vector, $\mathbf{v}(\mathbf{q})\cdot\mathbf{q} = 0$.

The energy of a vortex is given by

$$\begin{aligned} E_{\rm v} &\simeq E_c(k) + \frac{\rho_s^0}{2}\int_{|\mathbf{r}|>a} d^2r (\boldsymbol{\nabla}\theta)^2 \\ &\simeq E_c(k) + \pi\rho_s^0 k^2 \ln\left(\frac{R}{a}\right), \end{aligned} \tag{10.355}$$

where $E_c(k)$ is the core energy (which cannot be obtained from the continuum model (10.345)) and the size of the core is approximated by a. Thus the energy of a vortex diverges with the size of the system (assumed here to be a disk of radius R, the vortex being at the center). The free energy $F_{\rm v} = E_{\rm v} - TS_{\rm v}$ of an isolated vortex is therefore

$$\begin{aligned} F_{\rm v} &= E_c + \pi\rho_s^0 \ln(R/a) - T\ln(R/a)^2 \\ &= E_c + (\pi\rho_s^0 - 2T)\ln(R/a), \end{aligned} \tag{10.356}$$

where the entropy $S_{\rm v} = \ln W$ is obtained by noting that there are $W \sim \pi R^2/\pi a^2$ independent configurations for a vortex whose core has a surface πa^2.[85] In (10.356) we consider only vortices with winding number $k = \pm 1$ since they have the smallest energy. When $T < \pi\rho_s^0/2$, the free energy cost of creating an isolated vortex diverges when $R \to \infty$, and we do not expect isolated vortices to be present in the system at equilibrium. On the contrary, when $T > \pi\rho_s^0/2$, the free energy is smaller when an isolated vortex is present and we do expect vortices to spontaneously appear in the system. Each time a vortex of charge $k = \pm 1$ appears between two points A and B, the corresponding phase difference $\theta_A - \theta_B$ varies by $\pm\pi$. The presence of thermally excited vortices therefore randomizes the phase and destroys the algebraic order. The temperature $T_c = \pi\rho_s^0/2$ can thus be interpreted as the transition temperature between the low-temperature phase with quasi-long-range order and the disordered high-temperature phase.

[84] This terminology comes from boson systems where θ is the phase of the boson field $\psi(\mathbf{r})$ and $\mathbf{v}(\mathbf{r}) \sim \boldsymbol{\nabla}\theta(\mathbf{r})$ the superfluid velocity; see the remark at the end of Sec. 10.8.2 and chapter 7.

[85] The expression (10.355) of the energy assumes the vortex to be at the center of the system. When this is not the case, we expect that it is still possible to use (10.355) since the dependence on R is only logarithmic.

The preceding analysis can easily be extended to the case of a set of vortices with topological charges $k_1, k_2, \cdots$ If the total charge $k_{\rm tot}$ is nonzero, the preceding results are essentially unchanged; far away from the vortices (assumed to be located near $\mathbf{r} = 0$), $|\mathbf{v}| \sim 1/|\mathbf{r}|$, and the energy $E \sim k_{\rm tot}^2 \ln(R/a)$ diverges logarithmically with the size of the system. By contrast the energy remains finite when $k_{\rm tot} = 0$, so that the corresponding vortex configuration has a nonzero probability at any finite temperature. Consider for instance a pair of vortices with opposite winding numbers k and $-k$ and located at $\mathbf{r} \pm \boldsymbol{\ell}/2$ (vortex-antivortex pair). Assuming that far away from the dipole center the velocity field is given by the sum of the fields created independently by the isolated vortices, one finds

$$\mathbf{v}_{\rm pair}(\mathbf{r}) = \boldsymbol{\nabla}\theta_+\left(\mathbf{r} + \frac{\boldsymbol{\ell}}{2}\right) + \boldsymbol{\nabla}\theta_-\left(\mathbf{r} - \frac{\boldsymbol{\ell}}{2}\right) \simeq (\boldsymbol{\ell}\cdot\boldsymbol{\nabla})\boldsymbol{\nabla}\theta_+(\mathbf{r}) = k(\boldsymbol{\ell}\cdot\boldsymbol{\nabla})\frac{\hat{\mathbf{r}}\times\hat{\mathbf{z}}}{|\mathbf{r}|} \tag{10.357}$$

for $|\mathbf{r}| \gg |\boldsymbol{\ell}|$. $\theta_\pm(\mathbf{r} \pm \boldsymbol{\ell}/2)$ denotes the phase field of the vortex with charge $= \pm k$ and is given by (10.353). Thus $\mathbf{v}_{\rm pair}(\mathbf{r})$ decays as $1/\mathbf{r}^2$ far away from the dipole center, which leads to a finite energy,[86]

$$E_{\rm pair}(\boldsymbol{\ell}) = 2E_c(k) + 2\pi\rho_s^0 k^2 \ln\left|\frac{\boldsymbol{\ell}}{a}\right|. \tag{10.358}$$

Since the energy is minimum for $k = \pm 1$, at low temperatures we can ignore vortex-antivortex pairs with larger topological charges (i.e. $|k| > 1$). The probability $P(\boldsymbol{\ell})d^2\ell$ for the vortex and antivortex to be separated by $\boldsymbol{\ell}$, within $d^2\ell$, is determined by

$$P(\boldsymbol{\ell}) = \mathcal{N}\, e^{-2\beta E_c - 2\pi K \ln|\boldsymbol{\ell}/a|} = \mathcal{N} y^2 \left|\frac{a}{\boldsymbol{\ell}}\right|^{2\pi K}, \tag{10.359}$$

where $y = e^{-\beta E_c}$ is the fugacity of vortices of charge ± 1 and $\mathcal{N}$ a normalization constant that is determined from the condition $\int d^2\ell\, P(\boldsymbol{\ell}) = 1$. The mean square distance is

$$\langle \boldsymbol{\ell}^2 \rangle = \int d^2\ell\, P(\boldsymbol{\ell})\boldsymbol{\ell}^2 = \frac{\int_a^\infty d\ell\, \ell^3 (a/\ell)^{2\pi K}}{\int_a^\infty d\ell\, \ell (a/\ell)^{2\pi K}} = a^2 \frac{\pi K - 1}{\pi K - 2} \tag{10.360}$$

for $\pi K > 2$. Thus $\langle \boldsymbol{\ell}^2 \rangle$ is finite at low temperatures and diverges when T approaches $T_c = \pi\rho_s^0/2$.

The low-temperature phase should therefore be visualized as a phase of bound vortex-antivortex pairs with a characteristic size $\langle \boldsymbol{\ell}^2 \rangle^{1/2} < \infty$. These pairs are expected to reduce the spinwave stiffness at short distance, but the long-distance behavior should be determined only by the spinwaves albeit with a renormalized temperature-dependent stiffness $\rho_s \equiv \rho_s(T) < \rho_s^0$, i.e. by the effective Hamiltonian

$$H_{\rm eff} = \frac{\rho_s}{2}\int d^2r\, (\boldsymbol{\nabla}\theta_{\rm sw})^2, \tag{10.361}$$

where the notation $\theta_{\rm sw}$ ("sw" stands for spinwave) emphasizes that the phase should be considered as a slowly varying variable whose 2π periodicity does not matter. This Hamiltonian leads to a power-law decay of the spin-spin correlation function with an anomalous dimension $\eta(T) = T/2\pi\rho_s$ (see Eq. (10.349)). The number and size of the vortex-antivortex pairs increase with temperature. The pairs unbind at the BKT transition temperature $T_c = \pi\rho_s(T_c)/2$, above which individual vortices can move freely in the system leading to

[86] Eq. (10.358) is derived in Sec. 10.8.3; see Eq. (10.373).

a finite correlation length and a vanishing stiffness.[87] In the following section, we shall see how this picture can be confirmed by a more rigorous study.

Let us finally point out that the continuum description of the XY model bears an obvious similarity with the description of a Bose superfluid by a classical field $\psi(\mathbf{r}) = \sqrt{n(\mathbf{r})}e^{i\theta(\mathbf{r})}$. The spinwaves of the XY model correspond to the phonon modes of the superfluid and the stiffness ρ_s can be identified with the superfluid density n_s. The density field $n(\mathbf{r}) = |\psi(\mathbf{r})|^2$ is crucial to describe the core of the vortices (see Sec. 7.3.2) but away from the core $n(\mathbf{r}) \simeq n$ and a description in terms of the phase field θ alone is possible.

Being characterized by the suppression of the topological vortex defects, the low-temperature phase is sometimes said to have topological order, and the BKT transition is identified as a topological phase transition.

10.8.3 The Coulomb gas model

A detailed study of the BKT transition requires to take into account the long-wavelength fluctuations of the phase (spinwaves) as well as the vortices and their interactions. In the continuum description (valid at length scales larger than a) the phase variable can be written as the sum of an analytic part θ_a and a singular θ_s part arising from vortices. The regular field θ_a is generally associated with the spinwave fluctuations (i.e. the field $\theta_{\rm sw}$ in equation (10.361)). Although this is true in the absence of vortices, the field θ_a does not have a clear physical meaning in the presence of vortices and the decomposition $\theta = \theta_a + \theta_s$ should be seen as a mere mathematical trick [39, 40].[88] We shall nevertheless follow the standard terminology and refer to θ_a as the "spinwave" variable.

We thus write the velocity field as the sum of a longitudinal part $\mathbf{v}_\parallel = \boldsymbol{\nabla}\theta_a$ and a transverse part $\mathbf{v}_\perp = \boldsymbol{\nabla}\theta_s$,

$$\mathbf{v} = \mathbf{v}_\parallel + \mathbf{v}_\perp, \qquad \boldsymbol{\nabla}\times\mathbf{v}_\parallel = 0, \qquad \boldsymbol{\nabla}\cdot\mathbf{v}_\perp = 0. \tag{10.362}$$

For a collection of vortices with charges k_i located at $\mathbf{r}_i$, we must have

$$\int_{\mathcal{C}} d\boldsymbol{\ell}\cdot\mathbf{v} = \int_\Sigma d^2r\,\hat{\mathbf{z}}\cdot(\boldsymbol{\nabla}\times\mathbf{v}) = \sum_{i\in\Sigma} 2\pi k_i \tag{10.363}$$

(see Eq. (10.351)) for any surface Σ spanned by the closed path $\mathcal{C}$. Equation (10.363) implies

$$\boldsymbol{\nabla}\times\mathbf{v} = \boldsymbol{\nabla}\times\mathbf{v}_\perp = 2\pi\sum_i k_i\delta(\mathbf{r}-\mathbf{r}_i)\hat{\mathbf{z}}. \tag{10.364}$$

If we write $\mathbf{v}_\perp = -\boldsymbol{\nabla}\times(\hat{\mathbf{z}}\psi)$, by taking the curl of (10.364) we obtain

$$\boldsymbol{\nabla}^2\psi = 2\pi\sum_i k_i\delta(\mathbf{r}-\mathbf{r}_i), \tag{10.365}$$

which is the Poisson equation associated with a set of charges $2\pi k_i$ located at $\mathbf{r}_i$. The solution reads

$$\psi(\mathbf{r}) = -2\pi\sum_i k_i V_{\rm C}(\mathbf{r}-\mathbf{r}_i) \simeq \sum_i k_i \ln\left|\frac{\mathbf{r}-\mathbf{r}_i}{a}\right|, \tag{10.366}$$

[87] The stiffness being the response to twisted boundary conditions (see the following section and Sec. 7.2.2 for a discussion in the context of superfluidity), it should vanish when the correlation length is finite.

[88] This will appear clearly in the following analysis, e.g. by the fact that the "spinwave" variable θ_a does not couple to the vortex excitations [Eq. (10.374)] and the associated stiffness (the coefficient ρ_s^0 in (10.369)) does not renormalize, at odds with the physical picture given at the end of the preceding section. The true spinwave variable $\theta_{\rm sw}$ should therefore be a mixture of θ_a and θ_s [40].

where

$$V_{\rm C}(\mathbf{r}) = \int_{\mathbf{q}} \frac{e^{i\mathbf{q}\cdot\mathbf{r}}}{\mathbf{q}^2} \simeq -\frac{1}{2\pi} \ln\left|\frac{\mathbf{r}}{a}\right| \tag{10.367}$$

is the two-dimensional Coulomb potential. The short-distance cutoff a in (10.367) follows from the UV cutoff $1/a$ that we impose in the momentum integral since the continuum description is not valid at length scales smaller than a. $\psi(\mathbf{r})$ is simply the superposition of the potentials created by isolated vortices (see Eq. (10.353)).

We can now write the energy as

$$H = \frac{\rho_s^0}{2} \int d^2r\, (\mathbf{v}_\parallel + \mathbf{v}_\perp)^2 = \frac{\rho_s^0}{2} \int d^2r\, (\mathbf{v}_\parallel^2 + \mathbf{v}_\perp^2). \tag{10.368}$$

If we impose periodic boundary conditions for θ_a it is readily seen, by an integration by part, that $\int d^2r\, \mathbf{v}_\parallel \cdot \mathbf{v}_\perp = 0$. The Hamiltonian then splits into a spinwave part

$$H_{\rm sw} = \frac{\rho_s^0}{2} \int d^2r\, \mathbf{v}_\parallel^2 \tag{10.369}$$

and a vortex part

$$H_{\rm v} = \frac{\rho_s^0}{2} \int d^2r\, \mathbf{v}_\perp^2 = \frac{\rho_s^0}{2} \int d^2r\, (\boldsymbol{\nabla}\psi)^2. \tag{10.370}$$

By an integration par part, we obtain

$$\int d^2r\, (\boldsymbol{\nabla}\psi)^2 = -\int d^2r\, \psi \boldsymbol{\nabla}^2 \psi + \int_0^L dy\, (\psi\partial_x\psi)\big|_{x=0}^{x=L} + \int_0^L dx\, (\psi\partial_y\psi)\big|_{y=0}^{y=L} \tag{10.371}$$

(with the notation $f(x)|_0^L = f(L) - f(0)$), where L is the linear size of the system. Using (10.366) we see that the surface term in (10.371) gives a contribution to the energy which grows as $\rho_s^0(\sum_i k_i)^2 \ln L$. In a large system, one therefore needs to consider only states with a vanishing total vorticity $k_{\rm tot} = \sum_i k_i = 0$. From (10.365) and (10.366) we then obtain the vortex energy

$$H_{\rm v} = -\pi\rho_s^0 \sum_{i,j} k_i k_j \ln\left|\frac{\mathbf{r}_i - \mathbf{r}_j}{a}\right|, \tag{10.372}$$

where it is understood that the minimal distance between vortices is of order a. The energy is not defined for $i = j$; this is an artefact of the continuum treatment which is not valid at small distances. The self-interaction of the vortex is simply its core energy $E_c(k_i) = k_i^2 E_c$. This finally leads to

$$\begin{aligned} H_{\rm v} &= \sum_i E_c(k_i) - \pi\rho_s^0 \sum_{i\neq j} k_i k_j \ln\left|\frac{\mathbf{r}_i - \mathbf{r}_j}{a}\right| \\ &= \sum_i E_c(k_i) + 2\pi^2\rho_s^0 \int_{|\mathbf{r}-\mathbf{r}'|>a} d^2r\, d^2r'\, n_{\rm v}(\mathbf{r}) V_{\rm C}(\mathbf{r}-\mathbf{r}') n_{\rm v}(\mathbf{r}'), \end{aligned} \tag{10.373}$$

where $n_{\rm v}(\mathbf{r}) = \sum_i k_i \delta(\mathbf{r} - \mathbf{r}_i)$ is the vortex density. One can further simplify the problem by considering only vortices with charges $k_i = \pm 1$, which are the most likely at low temperatures due to their lower core energy $E_c(\pm 1) \equiv E_c$. Apart from the core contribution, $H_{\rm v}$ is identical to the Hamiltonian of a two-dimensional Coulomb gas with point charges $2\pi k_i$.

A state of the XY model is thus specified by the phase field θ_a describing spinwaves and the charge $k_i = \pm 1$ and location $\mathbf{r}_i$ of the vortices, i.e. the vortex density $n_\mathrm{v}(\mathbf{r})$ (with the condition of vanishing total vorticity $\sum_i k_i = 0$). Spinwaves and vortices are uncoupled and the partition function takes the simple form

$$Z = Z_\mathrm{sw} Z_\mathrm{v} \tag{10.374}$$

with

$$\begin{aligned} Z_\mathrm{sw} &= \int \mathcal{D}[\theta_a] \exp\left\{ -\frac{K}{2} \int d^2r\, (\boldsymbol{\nabla}\theta_a)^2 \right\}, \\ Z_\mathrm{v} &= \sum_{N=0}^\infty \frac{y^{2N}}{(N!)^2} \int \prod_{i=1}^N (d^2r_i^+ d^2r_i^-) \exp\left\{ -2\pi^2 K \int_{|\mathbf{r}-\mathbf{r}'|>a} d^2r d^2r'\, n_\mathrm{v}(\mathbf{r}) V_\mathrm{C}(\mathbf{r}-\mathbf{r}') n_\mathrm{v}(\mathbf{r}') \right\} \end{aligned} \tag{10.375}$$

where we denote by $\mathbf{r}_i^\pm$ the positions of the charge $k_i^\pm = \pm 1$ and $n_\mathrm{v}(\mathbf{r}) = \sum_{i,\alpha} k_i^\alpha \delta(\mathbf{r} - \mathbf{r}_i^\alpha)$. The fugacity $y = e^{-\beta E_c}$ controls the density of vortices. The factor $1/(N!)^2$ is necessary to prevent overcounting of states that differ only by a permutation of vortices with the same charge.

10.8.3.1 Spinwave stiffness

A quantity of central importance is the (renormalized) spinwave stiffness ρ_s since, as anticipated at the end of the previous section and shown below, it takes a finite value in the low-temperature phase but vanishes above T_c. ρ_s can be easily computed from the electrostatic analogy. Performing a Hubbard-Stratonovich transformation, we write the partition function of the vortices as

$$Z_\mathrm{v} = \sum_{N=0}^\infty \frac{1}{(N!)^2} \int \mathcal{D}[\varphi] \int \prod_{i=1}^N (d^2r_i^+ d^2r_i^-) \int \mathcal{D}[\varphi]\, e^{-\frac{1}{2K}\int d^2r\, (\boldsymbol{\nabla}\varphi)^2 + 2i\pi \int d^2r\, \varphi(\mathbf{r}) n_\mathrm{v}(\mathbf{r})}, \tag{10.376}$$

which shows that φ is the scalar potential with $\epsilon_0 \equiv 1/K$ the dielectric constant in the absence of charges (see Sec. 1.9.3). In the presence of charges, the renormalized propagator of the scalar potential,

$$\langle \varphi(\mathbf{q}) \varphi(-\mathbf{q}) \rangle = \frac{K_R(\mathbf{q})}{\mathbf{q}^2}, \tag{10.377}$$

defines a renormalized momentum-dependent stiffness $K_R(\mathbf{q})$ or, equivalently, the dielectric constant $\epsilon(\mathbf{q}) = K/K_R(\mathbf{q})$. From the relation (3.126) between $\epsilon(\mathbf{q})$ and the density-density correlation function, one obtains[89]

$$\frac{K_R(\mathbf{q})}{K} = 1 - 4\pi^2 K V_\mathrm{C}(\mathbf{q}) \langle n_\mathrm{v}(\mathbf{q}) n_\mathrm{v}(-\mathbf{q}) \rangle = 1 - \frac{4\pi^2 K}{\mathbf{q}^2} \langle n_\mathrm{v}(\mathbf{q}) n_\mathrm{v}(-\mathbf{q}) \rangle. \tag{10.378}$$

Using $\boldsymbol{\nabla}^2 \mathbf{v}_\perp = -2\pi \boldsymbol{\nabla} \times (n_\mathrm{v} \hat{\mathbf{z}})$, one has

$$\mathbf{v}_\perp(\mathbf{q}) = 2\pi n_\mathrm{v}(\mathbf{q}) \frac{i\mathbf{q} \times \hat{\mathbf{z}}}{|\mathbf{q}|^2}, \qquad \mathbf{v}_\perp(\mathbf{q}) \cdot \mathbf{v}_\perp(-\mathbf{q}) = \frac{4\pi^2}{\mathbf{q}^2} n_\mathrm{v}(\mathbf{q}) n_\mathrm{v}(-\mathbf{q}), \tag{10.379}$$

[89] The factor $4\pi^2$ is due to the fact that the actual charge density is $2\pi n_\mathrm{v}(\mathbf{r})$.

so that

$$K_R(\mathbf{q}) = K - K^2 \langle \mathbf{v}_\perp(\mathbf{q}) \mathbf{v}_\perp(-\mathbf{q}) \rangle. \tag{10.380}$$

If we now introduce the current density $\mathbf{j} = \rho_s^0 \mathbf{v}$ and its correlation function

$$\chi_{il}(\mathbf{q}) = \langle j_i(\mathbf{q}) j_l(-\mathbf{q}) \rangle = \frac{q_i q_l}{|\mathbf{q}|^2} \chi_\parallel(\mathbf{q}) + \left(\delta_{i,l} - \frac{q_i q_l}{|\mathbf{q}|^2} \right) \chi_\perp(\mathbf{q}), \tag{10.381}$$

we obtain

$$K_R(\mathbf{q}) = \frac{1}{T^2} [\chi_\parallel(\mathbf{q}) - \chi_\perp(\mathbf{q})], \tag{10.382}$$

where we have used $\chi_\parallel(\mathbf{q}) = T^2 K$.[90] The renormalized stiffness $\rho_s = T K_R$, defined from the zero-momentum limit of $K_R(\mathbf{q})$, is then given by

$$\rho_s = T \lim_{\mathbf{q}\to 0} K_R(\mathbf{q}) = \frac{1}{T} [\chi_\parallel(0) - \chi_\perp(0)], \tag{10.383}$$

which is analoguous to the expression (7.10) of the superfluid density of a quantum fluid obtained in chapter 7. This confirms the interpretation of ρ_s as the renormalized stiffness of the XY model or, equivalently, the superfluid density of a superfluid at finite temperatures.[91]

Let us show that the renormalized stiffness ρ_s, when it is nonzero, controls the long-distance behavior of the spin-spin correlation function

$$G(\mathbf{r}) = \langle \mathbf{S}(\mathbf{r}) \cdot \mathbf{S}(0) \rangle = \langle e^{i[\theta_a(\mathbf{r}) + \theta_s(\mathbf{r}) - \theta_a(0) - \theta_s(0)]} \rangle = e^{-g_{\rm sw}(\mathbf{r}) - g_{\rm v}(\mathbf{r})}, \tag{10.384}$$

where the spinwave contribution

$$g_{\rm sw}(\mathbf{r}) = \frac{1}{2} \langle [\theta_a(\mathbf{r}) - \theta_a(0)]^2 \rangle = \int_{\mathbf{q}} \frac{1 - e^{i\mathbf{q}\cdot\mathbf{r}}}{K\mathbf{q}^2} \tag{10.385}$$

is given by equations (10.347) and (10.348). We determine $g_{\rm v}$ to lowest nontrivial order in the fugacity.[92] The leading contribution is then given by the second-order cumulant in the functional integral over θ_s,

$$g_{\rm v}(\mathbf{r}) = \frac{1}{2} \langle [\theta_s(\mathbf{r}) - \theta_s(0)]^2 \rangle, \tag{10.386}$$

since $\theta_s = \mathcal{O}(n_{\rm v}) = \mathcal{O}(y)$. Noting that equations (10.378) and (10.379) imply

$$\mathbf{q}^2 \langle \theta_s(\mathbf{q}) \theta_s(-\mathbf{q}) \rangle = \langle \mathbf{v}_\perp(\mathbf{q}) \mathbf{v}_\perp(-\mathbf{q}) \rangle = \frac{1}{K} - \frac{K_R(\mathbf{q})}{K^2} \simeq \frac{1}{K} - \frac{K_R}{K^2} \qquad (\mathbf{q} \to 0), \tag{10.387}$$

we obtain

$$g_{\rm v}(\mathbf{r}) = \int_{\mathbf{q}} \frac{1 - e^{i\mathbf{q}\cdot\mathbf{r}}}{K\mathbf{q}^2} \left(1 - \frac{K_R}{K} \right) \tag{10.388}$$

and

$$g_{\rm sw}(\mathbf{r}) + g_{\rm v}(\mathbf{r}) = \int_{\mathbf{q}} \frac{1 - e^{i\mathbf{q}\cdot\mathbf{r}}}{K\mathbf{q}^2} \left(1 - \frac{K_R - K}{K} \right) \simeq \int_{\mathbf{q}} \frac{1 - e^{i\mathbf{q}\cdot\mathbf{r}}}{K_R\mathbf{q}^2} \tag{10.389}$$

[90] This follows from $\mathbf{j}_\parallel = \rho_s^0 \boldsymbol{\nabla}\theta_a$, $\langle \theta_a(\mathbf{q}) \theta_a(-\mathbf{q}) \rangle = T/\rho_s^0 \mathbf{q}^2$ and $\langle j_{\parallel i}(\mathbf{q}) j_{\parallel l}(-\mathbf{q}) \rangle = T \rho_s^0 q_i q_l / |\mathbf{q}|^2$.

[91] The expression (10.380) of the stiffness can also be obtained from the response of the system to an imposed twist (*via* appropriate boundary conditions) of the phase field θ [7,41].

[92] The RG equations below are also derived to lowest order in the fugacity.

to leading order in y since $K_R/K - 1 = \mathcal{O}(y^2)$. From (10.389) we conclude that

$$G(\mathbf{r}) \sim \left|\frac{a}{\mathbf{r}}\right|^{\frac{1}{2\pi K_R}}, \tag{10.390}$$

which corresponds to a temperature-dependent anomalous dimension,[93]

$$\eta(T) = \frac{1}{2\pi K_R} = \frac{T}{2\pi\rho_s}, \tag{10.391}$$

determined by ρ_s.

A convenient expression of K_R is obtained by expanding the correlation function in (10.378) as follows,

$$\begin{aligned}\langle n_{\rm v}(\mathbf{q}) n_{\rm v}(-\mathbf{q})\rangle &= \int d^2r\, e^{-i\mathbf{q}\cdot\mathbf{r}} \langle n_{\rm v}(\mathbf{r}) n_{\rm v}(0)\rangle \\ &= \int d^2r \left[1 - i\mathbf{q}\cdot\mathbf{r} - \frac{1}{2}(\mathbf{q}\cdot\mathbf{r})^2 + \cdots\right] \langle n_{\rm v}(\mathbf{r}) n_{\rm v}(0)\rangle \\ &= -\frac{1}{2}\sum_{i,j} q_i q_j \int d^2r\, r_i r_j \langle n_{\rm v}(\mathbf{r}) n_{\rm v}(0)\rangle + \mathcal{O}(|\mathbf{q}|^4) \\ &= -\frac{1}{4}\mathbf{q}^2 \int d^2r\, \mathbf{r}^2 \langle n_{\rm v}(\mathbf{r}) n_{\rm v}(0)\rangle + \mathcal{O}(|\mathbf{q}|^4)\end{aligned} \tag{10.392}$$

using parity and charge neutrality $\int d^2r\, n_{\rm v}(\mathbf{r}) = 0$. We can thus rewrite the stiffness as

$$K_R = K - 4\pi^2 K^2 C_2 \quad \text{with} \quad C_2 = -\frac{1}{4}\int d^2r\, \mathbf{r}^2 \langle n_{\rm v}(\mathbf{r}) n_{\rm v}(0)\rangle. \tag{10.393}$$

10.8.3.2 RG equations

In the low-temperature limit, where the density of vortices is small, it makes sense to compute the renormalized stiffness K_R in powers of the fugacity. To obtain $\langle n_{\rm v}(\mathbf{r}) n_{\rm v}(0)\rangle$, one must consider the various vortex configurations and their Boltzmann weight $e^{-\beta E(\mathcal{C})}$ where the energy of the configuration $\mathcal{C}$ is determined by the vortex Hamiltonian (10.373). To lowest nontrivial order in y, it is sufficient to consider the configuration with a vortex located at $\mathbf{r}$ and an antivortex located at the origin (or the reverse). The associated Boltzmann weight is

$$\frac{1}{Z} e^{-\beta(2E_c + 2\pi\rho_s^0 \ln|\mathbf{r}/a|)} = y^2 \left|\frac{a}{\mathbf{r}}\right|^{2\pi K} + \mathcal{O}(y^4) \tag{10.394}$$

since $Z = 1 + \mathcal{O}(y^2)$. This gives[94]

$$\langle n_{\rm v}(\mathbf{r}) n_{\rm v}(0)\rangle \simeq -2\frac{y^2}{a^4}\left|\frac{a}{\mathbf{r}}\right|^{2\pi K} + \mathcal{O}(y^4), \qquad C_2 \simeq \frac{y^2}{2}\int d^2r \frac{|\mathbf{r}|^2}{a^4}\left|\frac{a}{\mathbf{r}}\right|^{2\pi K} + \mathcal{O}(y^4). \tag{10.395}$$

We deduce the renormalized stiffness or, equivalently, its inverse

$$K_R^{-1} = K^{-1} + 4\pi^3 y^2 \int_a^\infty \frac{dr}{a}\left(\frac{r}{a}\right)^{3-2\pi K} + \mathcal{O}(y^4), \tag{10.396}$$

[93] See footnote 83 page 676.

[94] The minus sign in the first of equations (10.395) comes from the opposite charges of the vortex and antivortex and the factor 2 from the two equivalent configurations (obtained by interchanging the vortex and the antivortex). Note also that the product of the vortex densities gives a factor $\sim 1/a^4$.

where $r = |\mathbf{r}|$. The integral on the rhs converges at low temperatures when $2 - \pi K < 0$, i.e. $K = \rho_s^0/T > 2/\pi$, but diverges when $K < 2/\pi$, and perturbation theory in y breaks down. The difficulty associated with the divergence at small K can be overcome with the following renormalization procedure. We split the integral in (10.396) into two parts,

$$\int_a^\infty dr = \int_a^{a\,e^{dl}} dr + \int_{a\,e^{dl}}^\infty dr. \tag{10.397}$$

The nonsingular small-r part is evaluated and incoporated into K^{-1}, i.e.

$$K_R^{-1} = K'^{-1} + 4\pi^3 y^2 \int_{a\,e^{dl}}^\infty \frac{dr}{a} \left(\frac{r}{a}\right)^{3-2\pi K} + \mathcal{O}(y^4), \tag{10.398}$$

with

$$K'^{-1} = K^{-1} + 4\pi^3 y^2 \int_a^{a\,e^{dl}} \frac{dr}{a} \left(\frac{r}{a}\right)^{3-2\pi K}. \tag{10.399}$$

We then rescale the distance $r \to r\,e^{-dl}$ in order to obtain an equation similar to (10.396),

$$K_R^{-1} = K'^{-1} + 4\pi^3 y'^2 \int_a^\infty \frac{dr}{a} \left(\frac{r}{a}\right)^{3-2\pi K'} + \mathcal{O}(y^4), \tag{10.400}$$

but with new parameters

$$\begin{aligned} y' &= y\,e^{(2-\pi K)dl}, \\ K'^{-1} &= K^{-1} + 4\pi^3 y^2 \int_a^{a\,e^{dl}} \frac{dr}{a} \left(\frac{r}{a}\right)^{3-2\pi K}. \end{aligned} \tag{10.401}$$

In (10.400) we have replaced K by K' in the exponent of r, which does not modify the equation to order y^2. For $dl \to 0$, this yields the RG equations

$$\begin{aligned} \frac{d}{dl} y(l) &= [2 - \pi K(l)]y(l) + \mathcal{O}\big(y(l)^3\big), \\ \frac{d}{dl} K^{-1}(l) &= 4\pi^3 y(l)^2 + \mathcal{O}\big(y(l)^4\big) \end{aligned} \tag{10.402}$$

satisfied by $y(l)$ and $K(l)$. Initially a corresponds to the minimal distance between vortices (which is of the order of the lattice spacing in the XY model) and $K(l=0) \equiv K$ and $y(l = 0) \equiv y$ are "coupling constants" at this length scale. Solving the flow equations (10.402) amounts to coarse graining the system by integrating out the short-distance fluctuations. Thus $K(l)$ and $y(l)$ stand for the renormalized stiffness and fugacity once vortices with separation less than $a(l) = a\,e^l$ have been integrated out in the partition function.

Since the renormalized stiffness K_R is invariant in the renormalization procedure, $K_R = K_R(K,y) = K_R(K(l), y(l))$ and

$$K_R = \lim_{l\to\infty} K(l). \tag{10.403}$$

The flow diagram is shown in figure 10.22. $K(l)$ is a decreasing function of l, in agreement with the fact that vortices tend to decrease the stiffness. There is an attractive line of fixed points, defined by $y = 0$ and $K \geq 2/\pi$. All trajectories attracted to this line correspond to the low-temperature phase where vortices exist only in bound pairs, the effective fugacity

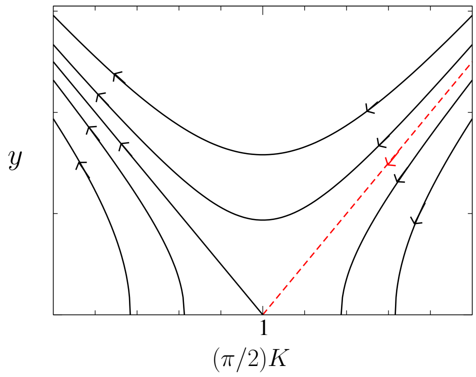


Figure 10.22: Flow diagram of the Coulomb gas model near the BKT point as a function of the stiffness K and fugacity y. The red dashed line shows the critical trajectory.

vanishes at large distance and the renormalized stiffness takes a nonzero value $K_R \geq 2/\pi$. The line of fixed points become repulsive when $K < 2/\pi$ and the fugacity increases with l. The same long-distance behavior is obtained when $K > 2/\pi$ if the initial fugacity is sufficiently large. Although the flow cannot be continued up to arbitrarily large values of l since the RG equations become invalid once $y(l) = \mathcal{O}(1)$, it is clear that this part of the flow diagram describes the high-temperature phase where unbound pairs of vortices can exist (hence the increasing effective fugacity) and the renormalized stiffness K_R vanishes.

To study in more detail the vicinity of the BKT fixed point ($K_c = 2/\pi, y = 0$), i.e. the end point of the attractive line of fixed points corresponding to the low-temperature phase, we introduce the variable $x = 2/\pi K - 1$. To leading order in x and y, the RG equations become

$$\begin{aligned} \frac{dx}{dl} &= 8\pi^2 y^2, \\ \frac{dy}{dl} &= 2xy. \end{aligned} \tag{10.404}$$

Contrary to the flow equations we encountered earlier in this chapter, these equations have no linear terms and, as we shall see, lead to a non-standard critical behavior. From (10.404), we deduce that

$$y^2 = \frac{1}{4\pi^2}(x^2 + C), \tag{10.405}$$

where C is a constant that depends on the initial conditions of the flow at $l = 0$. The RG trajectories in the plane (x, y) are therefore given by hyperbolas. The case $C > 0$ corresponds to the high-temperature phase, $K(l) \to 0$ for $l \to \infty$, while $C < 0$ corresponds to the low-temperature phase: $K(l) \to K_R = (2/\pi)(1 + \sqrt{-C})$. The critical trajectory is defined by $C = 0$ and ends at the BKT point. Thus C measures the distance from the critical point and one can set

$$C = b^2(T - T_c) \tag{10.406}$$

assuming that it vanishes linearly with $T - T_c$, with T_c being the BKT transition temperature. In the low-temperature phase, the renormalized stiffness is then given by

$$K_R = \frac{2}{\pi}\left[1 + b\sqrt{T_c - T}\right] \qquad (T \leq T_c). \tag{10.407}$$

At the transition, where $K_R = 2/\pi$, the anomalous dimension (10.391) takes the universal value $\eta = \eta(T_c) = 1/4$.[95] Since $K_R(T_c^+) = 0$, there is a universal jump of the stiffness at the transition,[96]

$$K(T_c^-) - K(T_c^+) = \frac{\Delta\rho_s}{T_c} = \frac{2}{\pi}. \tag{10.408}$$

In a superfluid, this corresponds to a jump $\Delta n_s/T_c = 2m/\pi$ of the superfluid density n_s (with m the boson mass). From (10.404) and (10.405), one can derive an equation for x alone,

$$\frac{dx}{dl} = 2(x^2 + C). \tag{10.409}$$

In the high-temperature phase ($C > 0$), we deduce

$$\frac{1}{\sqrt{C}}\left[\arctan\left(\frac{x(l)}{\sqrt{C}}\right) - \arctan\left(\frac{x(0)}{\sqrt{C}}\right)\right] = 2l. \tag{10.410}$$

If $C \to 0^+$, i.e. $T \to T_c^+$, $\arctan(x(0)/\sqrt{C}) \to -\pi/2$ since $x(0) < 0$, and

$$\frac{1}{\sqrt{C}}\left[\arctan\left(\frac{x(l)}{\sqrt{C}}\right) + \frac{\pi}{2}\right] = 2l. \tag{10.411}$$

The correlation length $\xi \sim a\, e^{l^*}$ can be estimated from the criterion $x(l^*) \sim 1$ (which signals the end of the critical regime where equations (10.404) are valid),

$$\xi \sim a \exp\left(\frac{\pi}{2\sqrt{C}}\right) \sim a \exp\left(\frac{\pi}{2b\sqrt{T - T_c}}\right). \tag{10.412}$$

Unlike any of the phase transitions discussed so far, the correlation length does not diverge as a power law but exhibits an essential singularity. On approaching the transition from the high-temperature side, the singular part of the free energy density,

$$f_{\text{sing}} \sim \xi^{-2} \sim \exp\left(-\frac{\pi}{b\sqrt{T - T_c}}\right) \tag{10.413}$$

has only an essential singularity. All temperature derivatives are finite at T_c. In particular, the singularity in the specific heat is very weak and essentially unobservable. The usual exponents ν, α, β and γ cannot be defined for the BKT transition.[97] We may however define δ from equation (10.138),

$$\delta = \frac{d + 2 - \eta}{d - 2 + \eta} = 15 \tag{10.414}$$

using $d = 2$ and $\eta = 1/4$.

[95] At the transition temperature T_c, the scaling of the momentum-dependent stiffness $K_R(\mathbf{q})$ implies a logarithmic correction to the spin-spin correlation function: $G(\mathbf{r}) \sim (\ln|\mathbf{r}|)^{1/8}/|\mathbf{r}|^{1/4}$ [7].

[96] To what extent is the result $2/\pi$ truly universal, i.e. independent of the RG procedure? On can try to convince oneself that the exponent $3 - 2\pi K$ in (10.396) is robust and independent of our approximations. A more convincing proof comes from the RG analysis of the sine-Gordon model in the following section.

[97] In the Ehrenfest classification, the BKT phase transition is an infinite-order phase transition.

10.8.4 The sine-Gordon model

In this section we show that the Coulomb gas model can be mapped onto the sine-Gordon model. The advantage of the latter is that it is amenable to standard (momentum-shell) perturbative RG analysis.

10.8.4.1 From the Coulomb gas model to the sine-Gordon model

Let us consider the expression (10.376) of the partition function $Z_{\rm v}$ of the Coulomb gas and compare it with that of the sine-Gordon model,

$$Z_{\rm SG} = \int \mathcal{D}[\varphi]\, e^{-\frac{1}{2K}\int d^2r\,(\boldsymbol{\nabla}\varphi)^2 - 2\frac{y}{a^2}\int d^2r\,\cos(2\pi\varphi)}. \tag{10.415}$$

Expanding $Z_{\rm SG}$ wrt y, we find

$$\begin{aligned} Z_{\rm SG} = &\int \mathcal{D}[\varphi]\, e^{-S_0[\varphi]} \sum_{N_+,N_-=0}^\infty \frac{(-y)^{N_++N_-}}{N_+!N_-!} \\ &\times \int \prod_{i=1}^{N_+} d^2r_i^+ \int \prod_{i=1}^{N_-} d^2r_i^-\, e^{2i\pi\sum_{j=1}^{N_+}\varphi(\mathbf{r}_j^+) - 2i\pi\sum_{j=1}^{N_-}\varphi(\mathbf{r}_j^-)}, \end{aligned} \tag{10.416}$$

where we have used $2\cos(2\pi\varphi) = e^{2i\pi\varphi} + e^{-2i\pi\varphi}$. We must therefore compute

$$\left\langle e^{2i\pi\sum_{j=1}^{N_+}\varphi(\mathbf{r}_i^+) - 2i\pi\sum_{j=1}^{N_-}\varphi(\mathbf{r}_i^-)}\right\rangle_0, \tag{10.417}$$

where the average is taken with the Gaussian part $S_0[\varphi]$ of the sine-Gordon action. Since the latter is invariant in the shift $\varphi(\mathbf{r}) \to \varphi(\mathbf{r})+\alpha$, the average value (10.417) vanishes unless $N_+ = N_-$.[98] With this condition, it is easy to see that $Z_{\rm SG} = Z_{\rm v}$ (up to a multiplicative constant), which shows that the sine-Gordon model has a transition in the same universality class as the Coulomb gas model.

10.8.4.2 RG approach to the sine-Gordon model

The sine-Gordon model is traditionally defined by the action

$$S[\varphi] = \int d^2r \left\{ \frac{1}{2}(\boldsymbol{\nabla}\varphi)^2 - u\cos(\beta\varphi) \right\} \tag{10.418}$$

where $u/\Lambda^2, \beta > 0$ are dimensionless parameters and Λ is a UV momentum cutoff. Introducing the "Luttinger parameter"[99] $K = \beta^2/8\pi$ and rescaling the field, we rewrite the action as

$$S[\varphi] = \int d^2r \left\{ \frac{1}{2\pi K}(\boldsymbol{\nabla}\varphi)^2 - u\cos(\sqrt{8}\varphi) \right\}. \tag{10.419}$$

Note that the parameter $K \equiv K_{\rm SG}$ should be compared with the parameter $(\pi/2)K \equiv (\pi/2)K_{\rm C}$ used in the Coulomb gas model.

[98] If we compute (10.417) from the standard rules of Gaussian integration, it seems that we always find a nonzero result. The reason is that φ is more than a collection of harmonic oscillators; it also possesses a zero mode, which is responsible for the vanishing of (10.417) when $N_+ \neq N_-$.

[99] This terminology comes from one-dimensional quantum fluids, see chapter 15.

For $u = 0$, we obtain a Gaussian field theory with propagator $G(\mathbf{q}) = \pi K/\mathbf{q}^2$. In real space, the propagator

$$G(\mathbf{r}) = \pi K \int_{\mathbf{q}} \frac{e^{i\mathbf{q}\cdot\mathbf{r}}}{\mathbf{q}^2} = -\frac{K}{2} \ln\left|\frac{\mathbf{r}}{a}\right| \tag{10.420}$$

is nothing but the two-dimensional Coulomb potential $V_{\rm C}(\mathbf{r})$ (up to a prefactor). To make contact with the Coulomb gas model, we have taken the cutoff Λ of the order of $1/a$ (a being the lattice spacing in the XY model). Of interest is also the correlation function

$$\langle e^{i\sqrt{8}[p\varphi(\mathbf{r})-p'\varphi(\mathbf{r}')]}\rangle = \delta_{p,p'} e^{-8p^2[G(0)-G(\mathbf{r}-\mathbf{r}')]} \sim \delta_{p,p'} \left|\frac{a}{\mathbf{r}-\mathbf{r}'}\right|^{4p^2K} . \tag{10.421}$$

This power-lay decaying correlation function is characteristic of a critical system. The sine-Gordon model therefore exhibits a line of critical points defined by $u = 0$ and parametrized by K. Equation (10.421) shows that the field $e^{i\sqrt{8}\varphi}$ has scaling dimension $2K$. Since the action is dimensionless, we obtain the scaling dimension

$$[u] = 2 - [e^{i\sqrt{8}\varphi}] = 2 - 2K \tag{10.422}$$

of the perturbation u to the Gaussian theory. We conclude that the line of fixed points at $u = 0$ is stable if $K > 1$ (i.e. $\beta^2 > 8\pi$) and unstable if $K < 1$ ($\beta^2 < 8\pi$). In the former case the line is attractive, in the latter case it is repulsive.

To study the sine-Gordon model away from the line $u = 0$, we use the momentum-shell RG approach following the general principles described in sections 10.5 and 10.6. We split the field $\varphi(\mathbf{q}) = \varphi_<(\mathbf{q}) + \varphi_>(\mathbf{q})$ into slow ($|\mathbf{q}| \leq \Lambda e^{-dl}$) and fast ($\Lambda e^{-dl} \leq |\mathbf{q}| \leq \Lambda$) modes,

$$Z = \int \mathcal{D}[\varphi_<]\, e^{-S_0[\varphi_<]} \int \mathcal{D}[\varphi_>]\, e^{-S_0[\varphi_>]-S_{\rm int}[\varphi_<,\varphi_>]}, \tag{10.423}$$

where

$$S_{\rm int}[\varphi_<,\varphi_>] = -u \int d^2r (\cos\sqrt{8}\varphi_< \cos\sqrt{8}\varphi_> - \sin\sqrt{8}\varphi_< \sin\sqrt{8}\varphi_>). \tag{10.424}$$

The integration over $\varphi_>$ is performed using a cumulant expansion so that the action of the slow modes becomes

$$S_0[\varphi_<] + \langle S_{\rm int}[\varphi_<,\varphi_>]\rangle_{0,>} - \frac{1}{2}\langle S_{\rm int}[\varphi_<,\varphi_>]^2\rangle_{0,>,c} \tag{10.425}$$

to order u^2, with

$$\langle S_{\rm int}\rangle_{0,>} = -u \int d^2r \cos\sqrt{8}\varphi_< \langle \cos\sqrt{8}\varphi_>\rangle_{0,>} \tag{10.426}$$

and[100]

$$\begin{aligned}\langle S_{\rm int}^2\rangle_{0,>,c} = u^2 \int d^2r\, d^2r' \big[&\cos\sqrt{8}\varphi_< \cos\sqrt{8}\varphi'_< \langle \cos\sqrt{8}\varphi_> \cos\sqrt{8}\varphi'_>\rangle_{0,>,c} \\ &+ \sin\sqrt{8}\varphi_< \sin\sqrt{8}\varphi'_< \langle \sin\sqrt{8}\varphi_> \sin\sqrt{8}\varphi'_>\rangle_{0,>}\big]\end{aligned} \tag{10.427}$$

[100] $\langle S_{\rm int}^2\rangle_{0,>,c} = \langle S_{\rm int}^2\rangle_{0,>} - \langle S_{\rm int}\rangle_{0,>}^2$ stands for the connected expectation value obtained with the action $S_0[\varphi_>]$.

using the notation $\varphi \equiv \varphi(\mathbf{r})$, $\varphi' \equiv \varphi(\mathbf{r}')$ and the fact that an expectation value with an odd number of fields vanishes. Introducing the function

$$g_>(\mathbf{r}-\mathbf{r}') = \langle \varphi_>(\mathbf{r})\varphi_>(\mathbf{r}')\rangle_{0,>}, \tag{10.428}$$

one easily finds

$$\begin{aligned} \langle \cos\sqrt{8}\varphi_> \rangle_{0,>} &= e^{-4g_>(0)}, \\ \langle \cos\sqrt{8}\varphi_> \cos\sqrt{8}\varphi'_> \rangle_{0,>} &= e^{-8g_>(0)}\cosh[8g_>(\mathbf{r}-\mathbf{r}')], \\ \langle \sin\sqrt{8}\varphi_> \sin\sqrt{8}\varphi'_> \rangle_{0,>} &= e^{-8g_>(0)}\sinh[8g_>(\mathbf{r}-\mathbf{r}')] \end{aligned} \tag{10.429}$$

and in turn

$$\begin{aligned} S[\varphi_<] = S_0[\varphi_<] &- u\, e^{-4g_>(0)} \int d^2r \cos\sqrt{8}\varphi_< \\ &- \frac{u^2}{4} e^{-8g_>(0)} \int d^2r\, d^2r' \big\{ (e^{8g_>(\mathbf{r}-\mathbf{r}')} - 1)\cos[\sqrt{8}(\varphi_< - \varphi'_<)] \\ &+ (e^{-8g_>(\mathbf{r}-\mathbf{r}')} - 1)\cos[\sqrt{8}(\varphi_< + \varphi'_<)] \big\}. \end{aligned} \tag{10.430}$$

Since $g_>(\mathbf{r}-\mathbf{r}')$ has only Fourier modes near Λ, it is rapidely suppressed when $|\mathbf{r}-\mathbf{r}'| \gg \Lambda^{-1}$. We can therefore approximate the integrand in (10.430) using

$$\begin{aligned} &\cos[\sqrt{8}(\varphi_< - \varphi'_<)] \simeq 1 - 4[(\mathbf{r}-\mathbf{r}')\cdot\boldsymbol{\nabla}\varphi_<]^2, \\ &\cos[\sqrt{8}(\varphi_< + \varphi'_<)] \simeq \cos(2\sqrt{8}\varphi_<) + \sqrt{8}\sin(2\sqrt{8}\varphi_<)[(\mathbf{r}'-\mathbf{r})\cdot\boldsymbol{\nabla}\varphi_<]. \end{aligned} \tag{10.431}$$

The second line in (10.431) corresponds to higher-order harmonics in the action and is irrelevant for small u.[101] We therefore obtain the action

$$\begin{aligned} S[\varphi] = S_0[\varphi] &- u\, e^{-4g_>(0)} \int d^2r \cos\sqrt{8}\varphi \\ &+ \frac{u^2}{2} e^{-8g_>(0)} \int d^2r'\, (e^{8g_>(\mathbf{r}')} - 1)\mathbf{r}'^2 \int d^2r\, (\boldsymbol{\nabla}\varphi)^2, \end{aligned} \tag{10.432}$$

where we now drop the index $<$. The last step of the renormalization procedure is to rescale the coordinates, $\mathbf{r} \to \mathbf{r}\, e^{-dl}$, in order to restore the original value of the cutoff,

$$\begin{aligned} S[\varphi] = S_0[\varphi] &- u\, e^{2dl-4g_>(0)} \int d^2r \cos\sqrt{8}\varphi \\ &+ \frac{u^2}{2} e^{-8g_>(0)} \int d^2r'\, (e^{8g_>(\mathbf{r}')} - 1)\mathbf{r}'^2 \int d^2r\, (\boldsymbol{\nabla}\varphi)^2. \end{aligned} \tag{10.433}$$

Since $g_>$ is $\mathcal{O}(dl)$, it can be written as $g_>(\mathbf{r}) = KI(\mathbf{r})dl$. We also introduce the notation $J = \Lambda^4 \int d^2r\, I(\mathbf{r})\mathbf{r}^2$. Both $I(\mathbf{r})$ and J are dimensionless and independent of K and Λ. We then see that the action (10.433) can be cast in the original form (10.419) but with renormalized parameters,

$$\begin{aligned} \frac{1}{K'} &= \frac{1}{K} + 8\pi\Lambda^{-4}u^2KJdl, \\ u' &= u[1 + (2 - 4KI(0))dl] \end{aligned} \tag{10.434}$$

[101] Since $[\cos(2\sqrt{8}\varphi)] = 4K$, the coupling constants associated with both terms of the second line of (10.431) have scaling dimension $2 - 4K$ and are irrelevant near the BKT point $K = 1$.

in the limit $dl \to 0$. An elementary calculation gives

$$I(0) = \frac{1}{dl} \int_{\Lambda\, e^{-dl} \leq |\mathbf{q}| \leq \Lambda} \frac{\pi}{\mathbf{q}^2} = \frac{1}{2}, \tag{10.435}$$

which leads to the flow equations[102]

$$\begin{aligned} \frac{d}{dl} K(l)^{-1} &= 8\pi J K(l) \tilde{u}(l)^2 + \mathcal{O}(\tilde{u}(l)^4), \\ \frac{d}{dl} \tilde{u}(l) &= [2 - 2K(l)]\tilde{u}(l) + \mathcal{O}(\tilde{u}(l)^3) \end{aligned} \tag{10.436}$$

satisfied by the dimensionless parameters $K(l)$ and $\tilde{u}(l) = u(l)/\Lambda^2$. These equations are identical to the RG equations (10.402) of the Coulomb gas model, once the difference in the definition of the parameter K has been taken into account (see the discussion after (10.419)),[103] with $\tilde{u}$ playing the role of the fugacity. They lead to the same flow diagram (Fig. 10.22) and their physical interpretation is similar.

Although equations (10.436) are obtained for a sharp cutoff Λ, it is possible to consider an arbitrary cutoff defined by a function $f(\mathbf{q}^2/\Lambda^2)$ satisfying $f(0) = 1$ and $f(\infty) = 0$. All momentum integrals appearing in the perturbation expansion should then be understood as

$$\int_{\mathbf{q}} \equiv \int_{\mathbf{q}} f\left(\frac{\mathbf{q}^2}{\Lambda^2}\right). \tag{10.437}$$

The sharp cutoff corresponds to $f(x) = \Theta(1-x)$; an example of a soft cutoff is $f(x) = e^{-x}$. In the momentum-shell RG analysis, integration over the fast modes is defined by

$$⨍_{\mathbf{q}} \equiv \left[f\left(\frac{\mathbf{q}^2}{\Lambda^2}\right) - f\left(\frac{\mathbf{q}^2}{(\Lambda\, e^{-dl})^2}\right) \right] \equiv -2\frac{dl}{\Lambda^2} \int_{\mathbf{q}} \mathbf{q}^2 f'\left(\frac{\mathbf{q}^2}{\Lambda^2}\right). \tag{10.438}$$

One then finds that

$$I(0) = \frac{1}{dl} ⨍_{\mathbf{q}} \frac{\pi}{\mathbf{q}^2} = -\frac{1}{2} \int_0^\infty dx\, f'(x) = -\frac{1}{2}[f(\infty) - f(0)] = \frac{1}{2} \tag{10.439}$$

is independent of the cutoff function $f(x)$. This confirms the fact that the jump $\Delta K_R = 1$ (or, equivalently, $\Delta K_R = 2/\pi$ in the Coulomb gas model) is universal, i.e. independent of the details of the RG procedure.

The general expression for J is

$$J = -\pi \int_0^\infty dy\, y^3 \int_0^\infty dx\, f'(x) J_0(\sqrt{x} y). \tag{10.440}$$

For a sharp cutoff

$$J = \pi \int_0^\infty dy\, y^3 J_0(y) \tag{10.441}$$

is infinite since $J_0(y) \sim y^{-1/2} \cos(y - \pi/4)$ for $y \to \infty$. It is nevertheless well defined for the soft cutoff $f(x) = e^{-x}$: $J = 8\pi$.

[102] The invariance of the sine-Gordon action (10.419) in the change $u \to -u$ and $\varphi \to \varphi + \pi/\sqrt{8}$ implies that the next-order terms in (10.436) are of order $\tilde{u}^4$ and $\tilde{u}^3$, respectively.

[103] Because of the difference in the definition of K, the BKT point is now located at $K = 1$ while it was defined by $K = 2/\pi$ in the Coulomb gas model.

10.9 Functional renormalization group

In this section, we discuss a RG approach which, contrary to the study of sections 10.6 and 10.7, is not based on perturbation theory. This approach relies on an exact RG equation for the action which can be approximately solved within a derivative expansion. It deals with functions rather than a limited set of coupling constants and is intrinsically non-perturbative (no small parameter is assumed).

10.9.1 Wilson-Polchinski equation

We consider the partition function

$$Z = \int \mathcal{D}[\boldsymbol{\varphi}]\, e^{-\frac{1}{2}\boldsymbol{\varphi}\cdot C_\Lambda^{-1}\cdot\boldsymbol{\varphi} - V_\Lambda[\boldsymbol{\varphi}]}, \tag{10.442}$$

where we use the notation

$$\begin{aligned} \boldsymbol{\varphi}\cdot C_\Lambda^{-1}\cdot\boldsymbol{\varphi} &= \int d^dr \int d^dr' \sum_i \varphi_i(\mathbf{r}) C_\Lambda^{-1}(\mathbf{r}-\mathbf{r}')\varphi_i(\mathbf{r}') \\ &= \sum_{\mathbf{p},i} \varphi_i(-\mathbf{p}) C_\Lambda^{-1}(\mathbf{p})\varphi_i(\mathbf{p}). \end{aligned} \tag{10.443}$$

Λ is an arbitrary momentum cutoff and

$$C_\Lambda(\mathbf{p}) = \frac{1}{\mathbf{p}^2} K\left(\frac{\mathbf{p}^2}{\Lambda^2}\right) \tag{10.444}$$

a cutoff function which ensures that only modes with momenta $|\mathbf{p}| \lesssim \Lambda$ are included in the partition function. In the perturbative RG (section 10.6), we took a sharp cutoff $K(x) = \Theta(1-x)$. In this section, we consider a smooth cutoff as shown in figure 10.23. $K(0) = 1$ and $K(x)$ decays rapidly (typically exponentially) for $x \gg 1$.[104] If we take Λ_0 as the initial value of the cutoff, then

$$S_{\Lambda_0}[\boldsymbol{\varphi}] = \frac{1}{2}\boldsymbol{\varphi}\cdot C_{\Lambda_0}^{-1}\cdot\boldsymbol{\varphi} + V_{\Lambda_0}[\boldsymbol{\varphi}] \tag{10.445}$$

is the (bare) microscopic action.

As Λ is reduced to Λ', fluctuation modes with momenta $\Lambda' \lesssim |\mathbf{p}| \lesssim \Lambda$ are integrated out, which changes the form of the action $S_\Lambda[\boldsymbol{\varphi}]$. Instead of considering a finite number of coupling constants as in section 10.6 (by expanding $S_\Lambda[\boldsymbol{\varphi}]$ in powers of $\boldsymbol{\varphi}$), we want to determine the full action $S_{\Lambda'}$. Let us show that $V_{\Lambda'}$ is related to V_Λ by

$$e^{-V_{\Lambda'}[\boldsymbol{\varphi}]} = \int \mathcal{D}[\boldsymbol{\varphi}']\, e^{-\frac{1}{2}\boldsymbol{\varphi}'\cdot D_{\Lambda,\Lambda'}^{-1}\cdot\boldsymbol{\varphi}' - V_\Lambda[\boldsymbol{\varphi}+\boldsymbol{\varphi}']}, \tag{10.446}$$

where

$$D_{\Lambda,\Lambda'} = C_\Lambda - C_{\Lambda'}. \tag{10.447}$$

Here and in the following we ignore any multiplicative constant which only affects the field-independent part of $V_\Lambda[\boldsymbol{\varphi}]$. From (10.446), we deduce

$$\int \mathcal{D}[\boldsymbol{\varphi}]\, e^{-\frac{1}{2}\boldsymbol{\varphi}\cdot C_{\Lambda'}^{-1}\cdot\boldsymbol{\varphi} - V_{\Lambda'}[\boldsymbol{\varphi}]}$$

[104] The sharp cutoff version of the Wilson-Polchinski equation is known as the Wegner-Houghton equation [44].

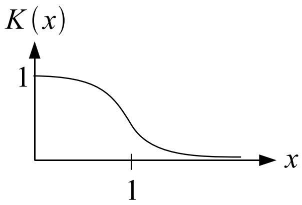


Figure 10.23: Cutoff function $K(x)$ [Eq. (10.444)].

$$= \int \mathcal{D}[\boldsymbol{\varphi}, \boldsymbol{\varphi}'] \, e^{-\frac{1}{2}\boldsymbol{\varphi}\cdot C_{\Lambda'}^{-1}\cdot\boldsymbol{\varphi} - \frac{1}{2}\boldsymbol{\varphi}'\cdot D_{\Lambda,\Lambda'}^{-1}\cdot\boldsymbol{\varphi}' - V_\Lambda[\boldsymbol{\varphi}+\boldsymbol{\varphi}']}$$
$$= \int \mathcal{D}[\boldsymbol{\varphi}, \boldsymbol{\varphi}'] \, e^{-\frac{1}{2}(\boldsymbol{\varphi}-\boldsymbol{\varphi}')\cdot C_{\Lambda'}^{-1}\cdot(\boldsymbol{\varphi}-\boldsymbol{\varphi}') - \frac{1}{2}\boldsymbol{\varphi}'\cdot D_{\Lambda,\Lambda'}^{-1}\cdot\boldsymbol{\varphi}' - V_\Lambda[\boldsymbol{\varphi}]}. \tag{10.448}$$

The integral over $\boldsymbol{\varphi}'$ gives

$$\int \mathcal{D}[\boldsymbol{\varphi}'] \, e^{-\frac{1}{2}\boldsymbol{\varphi}'\cdot(C_{\Lambda'}^{-1}+D_{\Lambda,\Lambda'}^{-1})\cdot\boldsymbol{\varphi}' + \boldsymbol{\varphi}\cdot C_{\Lambda'}^{-1}\cdot\boldsymbol{\varphi}'} = e^{\frac{1}{2}\boldsymbol{\varphi}\cdot C_{\Lambda'}^{-1}(C_{\Lambda'}^{-1}+D_{\Lambda,\Lambda'}^{-1})^{-1}C_{\Lambda'}^{-1}\cdot\boldsymbol{\varphi}}, \tag{10.449}$$

with

$$C_{\Lambda'}^{-1}(C_{\Lambda'}^{-1} + D_{\Lambda,\Lambda'}^{-1})^{-1}C_{\Lambda'}^{-1} = C_{\Lambda'}^{-1} - C_{\Lambda}^{-1}, \tag{10.450}$$

and we recognize in (10.448) the partition function, which proves equation (10.446).

We can write equation (10.446) in differential form by choosing $\Lambda' = \Lambda + d\Lambda$. Then

$$D_{\Lambda,\Lambda'} = -\frac{dC_\Lambda}{d\Lambda} d\Lambda + \mathcal{O}(d\Lambda^2) \tag{10.451}$$

and the fields $\boldsymbol{\varphi}'$ contributing to the functional integral (10.446) are $\mathcal{O}(\sqrt{d\Lambda})$. In the limit $d\Lambda \to 0$, it is sufficient to expand $V_\Lambda[\boldsymbol{\varphi} + \boldsymbol{\varphi}']$ to second order in $\boldsymbol{\varphi}'$,

$$V_\Lambda[\boldsymbol{\varphi} + \boldsymbol{\varphi}'] = V_\Lambda[\boldsymbol{\varphi}] + \int d^d r \sum_i \frac{\delta V_\Lambda[\boldsymbol{\varphi}]}{\delta \varphi_i(\mathbf{r})} \varphi_i'(\mathbf{r})$$
$$+ \frac{1}{2} \int d^d r \int d^d r' \sum_{i,j} \frac{\delta^{(2)} V_\Lambda[\boldsymbol{\varphi}]}{\delta\varphi_i(\mathbf{r})\delta\varphi_j(\mathbf{r}')} \varphi_i'(\mathbf{r})\varphi_j'(\mathbf{r}'). \tag{10.452}$$

Integrating out $\boldsymbol{\varphi}'$ in (10.446) within a cumulant expansion gives

$$\exp\{-V_{\Lambda+d\Lambda}[\boldsymbol{\varphi}]\} = \exp\bigg\{ - V_\Lambda[\boldsymbol{\varphi}] - \frac{1}{2} \int d^d r \int d^d r' \sum_i \bigg(\frac{\delta^{(2)} V_\Lambda[\boldsymbol{\varphi}]}{\delta\varphi_i(\mathbf{r})\delta\varphi_i(\mathbf{r}')}$$
$$- \frac{\delta V_\Lambda[\boldsymbol{\varphi}]}{\delta\varphi_i(\mathbf{r})} \frac{\delta V_\Lambda[\boldsymbol{\varphi}]}{\delta\varphi_i(\mathbf{r}')} \bigg) D_{\Lambda,\Lambda+d\Lambda}(\mathbf{r} - \mathbf{r}') \bigg\}, \tag{10.453}$$

where we have used

$$\langle \varphi_i'(\mathbf{r})\varphi_j'(\mathbf{r}')\rangle = \delta_{i,j} D_{\Lambda,\Lambda+d\Lambda}(\mathbf{r} - \mathbf{r}') = -\delta_{i,j} \frac{dC_\Lambda}{d\Lambda}(\mathbf{r} - \mathbf{r}') d\Lambda. \tag{10.454}$$

$C_\Lambda(\mathbf{r})$ denotes the Fourier transform of $C_\Lambda(\mathbf{p})$. We thus obtain the differential equation

$$\frac{dV_\Lambda[\boldsymbol{\varphi}]}{d\Lambda} = -\frac{1}{2} \int d^d r \int d^d r' \sum_i \frac{dC_\Lambda}{d\Lambda}(\mathbf{r} - \mathbf{r}') \left(\frac{\delta^{(2)} V_\Lambda[\boldsymbol{\varphi}]}{\delta\varphi_i(\mathbf{r})\delta\varphi_i(\mathbf{r}')} - \frac{\delta V_\Lambda[\boldsymbol{\varphi}]}{\delta\varphi_i(\mathbf{r})} \frac{\delta V_\Lambda[\boldsymbol{\varphi}]}{\delta\varphi_i(\mathbf{r}')} \right)$$

$$\equiv -\frac{1}{2}\mathrm{Tr}\frac{dC_\Lambda}{d\Lambda}\left(\frac{\delta^{(2)}V_\Lambda[\boldsymbol{\varphi}]}{\delta\boldsymbol{\varphi}\delta\boldsymbol{\varphi}} - \frac{\delta V_\Lambda[\boldsymbol{\varphi}]}{\delta\boldsymbol{\varphi}}\frac{\delta V_\Lambda[\boldsymbol{\varphi}]}{\delta\boldsymbol{\varphi}}\right). \tag{10.455}$$

With elementary algebra, equation (10.455) can be rewritten as a differential equation for the action (Wilson-Polchinski equation),

$$\begin{aligned}\frac{dS_\Lambda[\boldsymbol{\varphi}]}{d\Lambda} &= -\frac{1}{2}\mathrm{Tr}\frac{dC_\Lambda}{d\Lambda}\left(\frac{\delta^{(2)}S_\Lambda[\boldsymbol{\varphi}]}{\delta\boldsymbol{\varphi}\delta\boldsymbol{\varphi}} - \frac{\delta S_\Lambda[\boldsymbol{\varphi}]}{\delta\boldsymbol{\varphi}}\frac{\delta S_\Lambda[\boldsymbol{\varphi}]}{\delta\boldsymbol{\varphi}}\right) \\ &\quad - \boldsymbol{\varphi}\cdot\frac{d\ln C_\Lambda}{d\Lambda}\cdot\frac{\delta S_\Lambda[\boldsymbol{\varphi}]}{\delta\boldsymbol{\varphi}}.\end{aligned} \tag{10.456}$$

To complete the RG procedure, we still have to rescale momenta and fields. We postpone this step to section 10.9.2.

An alternative derivation of the Wilson-Polchinski equation. Let us consider the functional $W_\Lambda[\mathbf{h}]$ defined by

$$e^{W_\Lambda[\mathbf{h}]} = \int\mathcal{D}[\boldsymbol{\varphi}]\,e^{-\frac{1}{2}\boldsymbol{\varphi}\cdot D^{-1}_{\Lambda_0,\Lambda}\cdot\boldsymbol{\varphi}-V_{\Lambda_0}[\boldsymbol{\varphi}]+\mathbf{h}\cdot\boldsymbol{\varphi}}. \tag{10.457}$$

Using

$$\begin{aligned}\frac{dW_\Lambda[\mathbf{h}]}{d\Lambda}e^{W_\Lambda[\mathbf{h}]} &= -\frac{1}{2}\int\mathcal{D}[\boldsymbol{\varphi}]\,\boldsymbol{\varphi}\cdot\frac{dD^{-1}_{\Lambda_0,\Lambda}}{d\Lambda}\cdot\boldsymbol{\varphi}\,e^{-\frac{1}{2}\boldsymbol{\varphi}D^{-1}_{\Lambda_0,\Lambda}\cdot\boldsymbol{\varphi}-V_{\Lambda_0}[\boldsymbol{\varphi}]+\mathbf{h}\cdot\boldsymbol{\varphi}} \\ &= -\frac{1}{2}\mathrm{Tr}\frac{dD^{-1}_{\Lambda_0,\Lambda}}{d\Lambda}\frac{\delta^{(2)}e^{W_\Lambda[\mathbf{h}]}}{\delta\mathbf{h}\delta\mathbf{h}},\end{aligned} \tag{10.458}$$

we obtain the flow equation

$$\frac{dW_\Lambda[\mathbf{h}]}{d\Lambda} = -\frac{1}{2}\mathrm{Tr}\frac{dD^{-1}_{\Lambda_0,\Lambda}}{d\Lambda}\left(\frac{\delta^{(2)}W_\Lambda[\mathbf{h}]}{\delta\mathbf{h}\delta\mathbf{h}} + \frac{\delta W_\Lambda[\mathbf{h}]}{\delta\mathbf{h}}\frac{\delta W_\Lambda[\mathbf{h}]}{\delta\mathbf{h}}\right). \tag{10.459}$$

One can then relate W_Λ to V_Λ using (10.446),

$$\begin{aligned}e^{-V_\Lambda[\boldsymbol{\varphi}]} &= \int\mathcal{D}[\boldsymbol{\varphi}']\,e^{-\frac{1}{2}\boldsymbol{\varphi}'\cdot D^{-1}_{\Lambda_0,\Lambda}\cdot\boldsymbol{\varphi}'-V_{\Lambda_0}[\boldsymbol{\varphi}+\boldsymbol{\varphi}']} \\ &= \int\mathcal{D}[\boldsymbol{\varphi}']\,e^{-\frac{1}{2}(\boldsymbol{\varphi}'-\boldsymbol{\varphi})\cdot D^{-1}_{\Lambda_0,\Lambda}\cdot(\boldsymbol{\varphi}'-\boldsymbol{\varphi})-V_{\Lambda_0}[\boldsymbol{\varphi}']} \\ &= e^{-\frac{1}{2}\boldsymbol{\varphi}\cdot D^{-1}_{\Lambda_0,\Lambda}\cdot\boldsymbol{\varphi}+W_\Lambda[D^{-1}_{\Lambda_0,\Lambda}\cdot\boldsymbol{\varphi}]}.\end{aligned} \tag{10.460}$$

We can now use (10.459) to derive a flow equation for V_Λ,

$$\begin{aligned}\frac{dV_\Lambda[\boldsymbol{\varphi}]}{d\Lambda} &= \frac{1}{2}\boldsymbol{\varphi}\cdot\frac{dD^{-1}_{\Lambda_0,\Lambda}}{d\Lambda}\cdot\boldsymbol{\varphi} - \frac{\delta W_\Lambda[\mathbf{h}]}{\delta\mathbf{h}}\bigg|_{\mathbf{h}=D^{-1}_{\Lambda_0,\Lambda}\boldsymbol{\varphi}}\cdot\frac{d}{d\Lambda}(D^{-1}_{\Lambda_0,\Lambda}\boldsymbol{\varphi}) \\ &\quad + \frac{1}{2}\mathrm{Tr}\frac{D^{-1}_{\Lambda_0,\Lambda}}{d\Lambda}\left(\frac{\delta^{(2)}W_\Lambda[\mathbf{h}]}{\delta\mathbf{h}\delta\mathbf{h}} + \frac{\delta W_\Lambda[\mathbf{h}]}{\delta\mathbf{h}}\frac{\delta W_\Lambda[\mathbf{h}]}{\delta\mathbf{h}}\right)_{\mathbf{h}=D^{-1}_{\Lambda_0,\Lambda}\boldsymbol{\varphi}}.\end{aligned} \tag{10.461}$$

From the relation (10.460) between V_Λ and W_Λ, we obtain

$$\begin{aligned}\frac{\delta W_\Lambda[\mathbf{h}]}{\delta\mathbf{h}}\bigg|_{\mathbf{h}=D^{-1}_{\Lambda_0,\Lambda}\boldsymbol{\varphi}} &= \boldsymbol{\varphi} - D_{\Lambda_0,\Lambda}\cdot\frac{\delta V_\Lambda[\boldsymbol{\varphi}]}{\delta\boldsymbol{\varphi}}, \\ \frac{\delta^{(2)}W_\Lambda[\mathbf{h}]}{\delta\mathbf{h}\delta\mathbf{h}}\bigg|_{\mathbf{h}=D^{-1}_{\Lambda_0,\Lambda}\boldsymbol{\varphi}} &= D_{\Lambda_0,\Lambda} - D_{\Lambda_0,\Lambda}\frac{\delta^{(2)}V_\Lambda[\boldsymbol{\varphi}]}{\delta\boldsymbol{\varphi}\delta\boldsymbol{\varphi}}D_{\Lambda_0,\Lambda},\end{aligned} \tag{10.462}$$

and in turn

$$\begin{aligned}\frac{dV_\Lambda[\boldsymbol{\varphi}]}{d\Lambda} &= -\frac{1}{2}\mathrm{Tr}\frac{dD^{-1}_{\Lambda_0,\Lambda}}{d\Lambda}D_{\Lambda_0,\Lambda}\frac{\delta^{(2)}V_\Lambda[\boldsymbol{\varphi}]}{\delta\boldsymbol{\varphi}\delta\boldsymbol{\varphi}}D_{\Lambda_0,\Lambda} \\ &\quad+\frac{1}{2}\frac{\delta V_\Lambda[\boldsymbol{\varphi}]}{\delta\boldsymbol{\varphi}}\cdot D_{\Lambda_0,\Lambda}\frac{dD^{-1}_{\Lambda_0,\Lambda}}{d\Lambda}D_{\Lambda_0,\Lambda}\cdot\frac{\delta V_\Lambda[\boldsymbol{\varphi}]}{\delta\boldsymbol{\varphi}} \\ &= \frac{1}{2}\mathrm{Tr}\frac{dD_{\Lambda_0,\Lambda}}{d\Lambda}\left(\frac{\delta^{(2)}V_\Lambda[\boldsymbol{\varphi}]}{\delta\boldsymbol{\varphi}\delta\boldsymbol{\varphi}}-\frac{\delta V_\Lambda[\boldsymbol{\varphi}]}{\delta\boldsymbol{\varphi}}\frac{\delta V_\Lambda[\boldsymbol{\varphi}]}{\delta\boldsymbol{\varphi}}\right), \end{aligned}\tag{10.463}$$

which is nothing but equation (10.455) since $\frac{dD_{\Lambda_0,\Lambda}}{d\Lambda}=-\frac{dC_\Lambda}{d\Lambda}$.

10.9.2 Local potential approximation

The Wilson-Polchinski equation cannot be solved exactly. A possible approximation relies on a field expansion of the functional $V_\Lambda[\varphi]$,

$$V_\Lambda[\varphi]=\sum_{n=0}^\infty\frac{1}{n!}\int d^dr_1\cdots d^dr_n\,V^{(n)}_\Lambda(\mathbf{r}_1,\cdots,\mathbf{r}_n)\varphi(\mathbf{r}_1)\cdots\varphi(\mathbf{r}_n)\tag{10.464}$$

(in this section we consider a scalar field), truncated to a given order. Such an expansion is reminiscent of the perturbative RG studied in section 10.6. To reproduce the $\mathcal{O}(\epsilon)$ results, it is sufficient to retrain $V^{(2)}_\Lambda$, $V^{(4)}_\Lambda$ and $V^{(6)}_\Lambda$ in the expansion (10.464).

Faire calcul à $\mathcal{O}(\epsilon)$?

The functional Wilson-Polchinski equation suggests a different kind of approximation, namely a derivative expansion. To leading order, the action reads

$$S_\Lambda[\varphi]=\frac{1}{2}\varphi\cdot C^{-1}_\Lambda\cdot\varphi+\int d^dr\,U_\Lambda\big(\varphi(\mathbf{r})\big).\tag{10.465}$$

The approximation (10.465) is called is the local potential approximation (Local-potential approximation (LPA)). In the LPA, the action is entirely determined by the function U_Λ, whose RG equation follows from the Wilson-Polchinski equation. Using

$$\begin{aligned}\frac{\delta V_\Lambda[\varphi]}{\delta\varphi(\mathbf{r})}&=\frac{\partial U_\Lambda\big(\varphi(\mathbf{r})\big)}{\partial\varphi(\mathbf{r})}, \\ \frac{\delta^{(2)}V_\Lambda[\varphi]}{\delta\varphi(\mathbf{r})\delta\varphi(\mathbf{r}')}&=\frac{\partial^2U_\Lambda\big(\varphi(\mathbf{r})\big)}{\partial\varphi(\mathbf{r})^2}\delta(\mathbf{r}-\mathbf{r}'),\end{aligned}\tag{10.466}$$

one finds

$$\begin{aligned}\frac{d}{d\Lambda}\int d^dr\,U_\Lambda\big(\varphi(\mathbf{r})\big) &= -\frac{1}{2}\int d^dr\frac{dC_\Lambda}{d\Lambda}(\mathbf{r}=0)U''_\Lambda\big(\varphi(\mathbf{r})\big) \\ &\quad+\frac{1}{2}\int d^drd^dr'\frac{dC_\Lambda}{d\Lambda}(\mathbf{r}-\mathbf{r}')U'_\Lambda\big(\varphi(\mathbf{r})\big)U'_\Lambda\big(\varphi(\mathbf{r}')\big).\end{aligned}\tag{10.467}$$

If we consider this equation for a uniform field $\varphi(\mathbf{r})=\varphi$, one obtains a differential equation for the function $U_\Lambda(\varphi)$,

$$\frac{d}{d\Lambda}U_\Lambda=-\frac{1}{2}\frac{dC_\Lambda}{d\Lambda}(\mathbf{r}=0)U''_\Lambda+\frac{1}{2}\frac{dC_\Lambda}{d\Lambda}(\mathbf{p}=0)U'^2_\Lambda.\tag{10.468}$$

Using (10.444), one finally obtains

$$\frac{d}{d\Lambda}U_\Lambda = -\Lambda^{d-3} I_1 U''_\Lambda + \Lambda^{-3} I_0 {U'_\Lambda}^2, \tag{10.469}$$

where

$$\begin{aligned} I_0 &= \frac{\Lambda^3}{2}\frac{dC_\Lambda}{d\Lambda}(\mathbf{p}=0) = -K'(0), \\ I_1 &= \frac{\Lambda^{3-d}}{2}\frac{dC_\Lambda}{d\Lambda}(\mathbf{r}=0) = -\frac{K_d}{2}\int_0^\infty dx\, x^{d/2-1}K'(x) \end{aligned} \tag{10.470}$$

are Λ-independent parameters depending on the cutoff function K. To obtain a fixed point solution of (10.469), one must first eliminate the explicit dependence on Λ. This corresponds to the momentum and field rescaling step in the perturbative RG of section 10.6. Since $[\varphi] = \frac{d-2}{2}$ and $[U(\varphi)] = d$,[105] one is lead to introduce the dimensionless variables

$$\tilde\varphi = \Lambda^{(2-d)/2}\varphi, \qquad \tilde U_\Lambda(\tilde\varphi) = \Lambda^{-d}U_\Lambda(\varphi), \tag{10.471}$$

which then yields the RG equation

$$\Lambda\frac{d}{d\Lambda}\tilde U_\Lambda = -d\tilde U_\Lambda + \left(\frac{d}{2}-1\right)\tilde\varphi\tilde U'_\Lambda - I_1\tilde U''_\Lambda + I_0{\tilde U'_\Lambda}{}^2. \tag{10.472}$$

The constant I_0 and I_1 can be eliminated by a trivial rescaling, $\tilde\varphi \to \sqrt{I_1}\tilde\varphi$ and $\tilde U_\Lambda \to (I_1/I_0)\tilde U_\Lambda$, leading to

$$\partial_l \tilde U_\Lambda = d\tilde U_\Lambda + \left(1-\frac{d}{2}\right)\tilde\varphi\tilde U'_\Lambda + \tilde U''_\Lambda - {\tilde U'_\Lambda}{}^2, \tag{10.473}$$

with $\Lambda = \Lambda_0 e^{-l}$. Finally, to get rid of the field independent part of $\tilde U_\Lambda$, it is convenient to consider the function $f = \tilde U'_\Lambda$,

$$\dot f = f'' - 2ff' + \left(1+\frac{d}{2}\right)f + \left(1-\frac{d}{2}\right)xf', \tag{10.474}$$

where $\dot f = \partial_l f$, $f' = \partial_x f$ and we denote the dimensionless field $\tilde\varphi$ by x.

We are now in a position to look for the fixed point solutions $\dot f^* = 0$ and the corresponding critical exponents. The RG equation (10.474) admits the trivial fixed point $f^* = 0$ that we first discuss before considering nontrivial fixed points.

10.9.2.1 The Gaussian fixed point in the LPA

The solution $f^* = 0$ corresponds to a vanishing function $\tilde U^*_\Lambda$ and is therefore associated with the Gaussian fixed point.[106] To obtain the critical exponents, we linearize the flow equation about the solution $f^* = 0$,

$$\dot f = f'' + \left(1+\frac{d}{2}\right)f + \left(1-\frac{d}{2}\right)xf'. \tag{10.475}$$

[105] The scaling dimension of φ field is obtained by noting that $[C_\Lambda(\mathbf{p})] = -2$.

[106] The condition $f^* = 0$ implies $\tilde U^*_\Lambda = \text{const}$. Eq. (10.473) with $\partial_l\tilde U_\Lambda = 0$ shows that the only solution is $\tilde U^*_\Lambda = 0$.

To solve this equation, we write

$$f_l(x) = h(\beta x)e^{\lambda l}, \tag{10.476}$$

where $\beta = \sqrt{d-2}/2$ (we assume $d > 2$), which leads to

$$h''(y) - 2yh'(y) + \frac{2}{d-2}(2+d-2\lambda)h(y) = 0 \tag{10.477}$$

with $y = \beta x$. This equation is known to have polynomial solutions,[107] given by the Hermite polynomials $h(y) = \hat{H}_{2k-1}(y) = 2^{k-1/2}H_{2k-1}(y)$ of degree $2k-1$, only for the set of discrete values of λ satisfying

$$2k-1 = \frac{d+2-2\lambda_k}{d-2} \quad \text{i.e.} \quad \lambda_k = d - k(d-2) \qquad (k = 1,2,3,\cdots). \tag{10.478}$$

If one considers symmetric perturbations, even degree Hermite's polynomials are not allowed since in that case the function $f(x)$ must be odd ($\tilde{U}_\Lambda(\tilde{\varphi})$ is even). Note that the λ_k's coincide with the scaling dimension $[v_{2k}]$ of the vertex $v_{2k}\int d^dr\,(\varphi)^{2k}$ about the Gaussian fixed point.

When $d > 4$, all eigenvalues λ_k are negative except $\lambda_1 = 2$, which determines the correlation-length critical exponent $\nu = 1/\lambda_1 = 1/2$. The corresponding relevant eigenvector is given by $\hat{H}_1(y) \propto y$ (which corresponds to a φ^2 term in U_Λ). The less negative eigenvalue $\lambda_2 = 4-d$ determines the correction-to-scaling exponent $\omega = -\lambda_2 = d-4$, i.e. the speed at which $\tilde{U}_k$ approaches the fixed-point solution $\tilde{U}^*$ when the system is critical. For $d < 4$, the eigenvalue $\lambda_2 = 4-d$ becomes relevant[108] and we expect the phase transition to be described by a nontrivial fixed point with a single relevant field. For the φ^4 theory, we shall see below that this fixed point is the Wilson-Fisher fixed point found in the perturbative RG (Sec. 10.6).

10.9.2.2 Non-trivial fixed points in the LPA

Non-Gaussian fixed points cannot be found analytically and one must solve the fixed point equation

$$0 = f^{*\prime\prime} - 2f^*f^{*\prime} + \left(1+\frac{d}{2}\right)f^* + \left(1-\frac{d}{2}\right)xf^{*\prime} \tag{10.479}$$

numerically. Since equation (10.479) is a second-order differential equation, a solution is *a priori* parametrized by two arbitrary constants. But since $U_\Lambda(\varphi)$ is even, $f^*(0) = 0$, and there is only one free parameter, e.g. $\gamma = f^{*\prime}(0)$. However, most solutions are singular at some x_c,

$$f^*(x) \sim \frac{1}{x-x_c} \quad \text{for} \quad x \to x_c. \tag{10.480}$$

By requiring $f^*(x)$ to be defined for all x, the numerical solution of (10.479) shows that only a finite set of values of γ is obtained. In practice, the fixed point solution $f^*(x)$ is therefore determined by fine tuning $\gamma = f^{*\prime}(0)$ until a regular solution is obtained (shooting method). Note that one can easily determine the large x behavior of $f^*(x)$,

$$f^*(x) \simeq x + Cx^{(d-2)/(d+2)} \quad \text{for} \quad x \to \infty, \tag{10.481}$$

where C is a constant.

[107] It can be shown that non-polynomial solutions imply a continuum of eigenvalues and are therefore non physical [43].

[108] To see whether the field associated with the eigenvalue $\lambda_2 = 4-d$ is relevant or irrelevant in four dimensions, one must go beyond the linear approximation (10.475). We do not discuss the case $d = 4$ here.

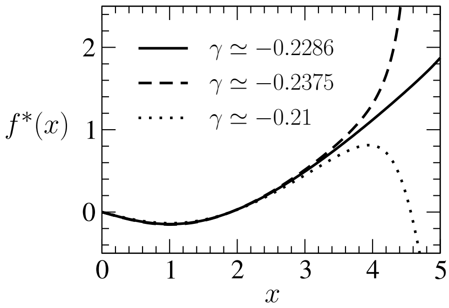


Figure 10.24: Fixed point solution $f^*(x)$ obtained by the shooting method for $d = 3$ vs $\gamma = f^{*\prime}(0)$. The actual (regular) solution is obtained for $\gamma \simeq -0.2286$.

For $d > 4$ only the Gaussian fixed point $f^*(x) = 0$ is found. For $3 \leq d < 4$, a nontrivial fixed point (the Wilson-Fisher fixed point) is found for a nonzero value of γ ($\gamma \simeq -0.2286$ for $d = 3$). Figure 10.24 shows $f^*(x)$ obtained by the shooting method. Note that the vanishing of $f^*(x)$ (i.e. the minimum of $\tilde{U}(x)$) at some $x_0 > 0$ is not in contradiction with the system being critical. Going back to dimensionful variables, one finds that the potential $U^*(\varphi)$ has a minimum at $\varphi_0 \propto \Lambda^{(d-2)/2} x_0 \to 0$ for $\Lambda \to 0$. A new nontrivial fixed point emanates from the Gaussian fixed point each time that one of the eigenvalues λ_k [Eq. (10.478)] vanishes, which occurs at the dimensional thresholds

Signification physique de ces nouveaux points fixes? Discuter par ex. pt fixe (tricritique) apparaissant à $d = 3$.

$$d_k = \frac{2k}{k-1} \quad (k \geq 2). \tag{10.482}$$

Once a fixed point is identified, one can determine the critical exponents by linearizing the flow equation about f^*. Setting

$$f_l(x) = f^*(x) + e^{\lambda l} g(x), \tag{10.483}$$

one finds

$$\lambda g = g'' + \left(1 + \frac{d}{2}\right) g + \left(1 - \frac{d}{2}\right) x g' - 2 f^* g' - 2 f^{*\prime} g \tag{10.484}$$

to linear order in g. Again one expects solutions to be labeled by two parameters. However, one can choose $g(0) = 0$ (since $f_l(x)$ is odd) and $g'(0) = 1$ (arbitrary normalization of the eigenvectors). The solution is then unique for a given λ. Now it is easy to see that for large x,

$$g(x) \sim x^{(d-2-2\lambda)/(d+2)} \quad \text{or} \quad g(x) \sim e^{\frac{d+2}{4} x^2}. \tag{10.485}$$

Discarding the solutions with exponential asymptotic behavior,[107] one finds a regular solution only for a countable set of λ's that can be determined by the shooting method. For $3 \leq d < 4$, the Wilson-Fisher fixed point possesses only one positive eigenvalue $\lambda_1 = 1/\nu$. The less negative eigenvalue λ_2 determines the correction-to-scaling exponent $\omega = -\lambda_2$ (Sec. 10.5.3). For $d = 3$, this gives

$$\begin{aligned} \nu &\simeq 0.6496, \\ \omega &\simeq 0.6557, \end{aligned} \tag{10.486}$$

with $\eta = 0$ in the LPA. These results improve over the $\mathcal{O}(\epsilon)$ perturbative results obtained in section 10.6. To compete with the best estimates obtained from the ϵ expansion (table 10.4), one must however go beyond the LPA.[109]

10.9.2.3 Beyond the LPA

The LPA is the leading order of a derivative expansion. To next order,

$$V_\Lambda[\varphi] = \int d^d r \left\{ U_\Lambda(\varphi(\mathbf{r})) + \frac{1}{2}\left[Z_\Lambda(\varphi(\mathbf{r})) - 1\right] \left(\boldsymbol{\nabla}\varphi(\mathbf{r})\right)^2 \right\} \tag{10.487}$$

and

$$S_\Lambda[\varphi] = \int d^d r \left\{ U_\Lambda(\varphi(\mathbf{r})) + \frac{1}{2} Z_\Lambda(\varphi(\mathbf{r}))\left(\boldsymbol{\nabla}\varphi(\mathbf{r})\right)^2 \right\} + \mathcal{O}\big((\boldsymbol{\nabla}\varphi)^4\big), \tag{10.488}$$

where we have used (10.444) with $K(0) = 1$. The action is now determined by two functions, U_Λ and Z_Λ, whose RG equations can be obtained by inserting (10.487) into the Wilson-Polchinski equation (10.455). Contrary to the LPA, these equations are not independent of the cutoff function K [Eq. (10.444)], but depend on two K-dependent parameters A and B.[110] To eliminate the cutoff dependence, one could try to compute the anomalous dimension as well as other critical exponents for various functions K and then use a principle of minimum sensitivity to choose the "best" function: if K depends on several parameters α_i, then the best value of the anomalous dimension must satisfy $d\eta/d\alpha_i = 0$, i.e. $d\eta/dA = d\eta/dB = 0$. Unfortunately, it appears that such a principle of minimum sensitivity cannot be used since η depends approximately linearly on B. This shortcoming seriously limits the use of the Wilson-Polchinski approach beyond the LPA. It should also be noted that the correlation functions of the "fast" modes (with momenta larger than Λ) cannot be computed from the action S_Λ. To obtain correlation functions with arbitrary momenta, it is necessary to introduce a spatially-varying external field $\mathbf{h}(\mathbf{r})$ in the action S_Λ and compute the resulting flow, which is technically difficult. In chapter 11, we shall see how the non-perturbative RG enables to circumvent the difficulties of the Wilson-Polchinski approach.

Appendix 10.A Perturbative calculation of critical exponents

It is difficult to push the RG calculation of critical exponents beyond order ϵ using the approach of section 10.6. Although the perturbation theory breaks down near the critical point (Sec. 10.3.4), it turns out to be an efficient tool to compute the critical exponents if one admits that the correlation functions take the form predicted by the RG. In this section, we compute the anomalous dimension η to $\mathcal{O}(\epsilon^2)$ and $\mathcal{O}(1/N)$ using the perturbative approach.

10.A.1 ϵ expansion

Let us consider the two-point vertex

$$\Gamma^{(2)}_{ii}(\mathbf{p}) = \mathbf{p}^2 + r_0 + \Sigma(\mathbf{p}) \tag{10.A.1}$$

[109] It is also possible to determine the critical exponents by solving the flow equation (10.474) for a system near criticality. This method will be discussed in section 11.2.2.

[110] More generally, the cutoff dependence can be absorbed into $2k$ parameters at the kth order of the derivative expansion [43].

at the critical temperature T_c (we assume a $(\boldsymbol{\varphi}^2)^2$ theory with O(N) symmetry). The RG predicts

$$\begin{aligned}\Gamma^{(2)}_{ii}(\mathbf{p}) &\propto |\mathbf{p}|^{2-\eta}\left[1+\mathcal{O}(|\mathbf{p}|^{-y_2})\right] \\ &\propto \mathbf{p}^2\left[1-\eta\ln|\mathbf{p}|+\mathcal{O}(|\mathbf{p}|^{-y_2})+\cdots\right]\end{aligned} \tag{10.A.2}$$

(see Eq. (10.186)), where $y_2<0$ refers to the dominant irrelevant field t_2. Thus it seems that we can extract the anomalous dimension η from the coefficient of $\mathbf{p}^2\ln|\mathbf{p}|$ in the small $\mathbf{p}$ expansion of $\Gamma^{(2)}(\mathbf{p})$. But for $d<4$, $y_2=-\epsilon+\mathcal{O}(\epsilon^2)$ (Sec. 10.6) and the $\mathcal{O}(|\mathbf{p}|^{-y_2})$ term will also appear as a series in $\ln|\mathbf{p}|$, i.e. $|\mathbf{p}|^{-y_2}=1+\epsilon\ln|\mathbf{p}|+\cdots$, when expanded in powers of ϵ. If we compute $\Gamma^{(2)}(\mathbf{p})$ in powers of ϵ, we will not only obtain powers of $\ln|\mathbf{p}|$ coming from $|\mathbf{p}|^{2-\eta}$ but also powers of $\ln|\mathbf{p}|$ coming from $|\mathbf{p}|^{-y_2}$. If, however, we are able to set the scaling field t_2 to zero, then we can directly deduce η (as well as other critical exponents) from the perturbative calculation of the two-point vertex. This can be done by choosing a particular value $u_0(\epsilon)$ of the coupling constant u_0.

$u_0(\epsilon)$ can be determined by considering the 4-point vertex $\Gamma^{(4)}$. Near the fixed point,

$$\Gamma^{(4)}_{ijkl}(0,K')=s^{d-4d_\varphi}\Gamma^{(4)}_{ijkl}(0,K), \tag{10.A.3}$$

i.e.

$$\Gamma^{(4)}_{ijkl}(0,t_1',t_2')=s^{d-4d_\varphi}\Gamma^{(4)}_{ijkl}(0,t_1,t_2) \tag{10.A.4}$$

if we retain only the leading irrelevant field t_2. The first argument of $\Gamma^{(4)}_{ijkl}$ indicates that all momenta are set to zero. Equation (10.A.4) can be rewritten as

$$\Gamma^{(4)}_{ijkl}(0,t_1,t_2)=s^{d-4+2\eta}\Gamma^{(4)}_{ijkl}(0,s^{1/\nu}t_1,s^{y_2}t_2). \tag{10.A.5}$$

Instead of t_1, we take the scaling field $r=\chi^{-1}\sim t^\gamma$ and restrict ourselves to the high-temperature phase ($t\geq 0$),

$$\Gamma^{(4)}_{ijkl}(0,r,t_2)=s^{-\epsilon+2\eta}\Gamma^{(4)}_{ijkl}(0,s^{1/\nu}r^{1/\gamma},s^{y_2}t_2). \tag{10.A.6}$$

With $s=r^{-\nu/\gamma}$, equation (10.A.6) gives

$$\Gamma^{(4)}_{ijkl}(0,r,t_2)=r^{(\epsilon-2\eta)\nu/\gamma}\Gamma^{(4)}_{ijkl}(0,1,r^{-y_2\nu/\gamma}t_2). \tag{10.A.7}$$

To $\mathcal{O}(\epsilon)$, $\eta=0$ and $\nu=\gamma/2$, so that

$$\begin{aligned}\Gamma^{(4)}_{ijkl}(0,r,t_2) &= r^{\epsilon/2}\Gamma^{(4)}_{ijkl}(0,1,r^{-y_2/2}t_2) \\ &= A_{ijkl}\left(1+\frac{\epsilon}{2}\ln r+B_{ijkl}t_2\frac{\epsilon}{2}\ln r+\mathcal{O}(\epsilon^2)\right).\end{aligned} \tag{10.A.8}$$

Let us compare this expression with the one-loop result

$$\begin{aligned}\Gamma^{(4)}_{ijkl}(0) &= (\delta_{i,j}\delta_{k,l}+\delta_{i,k}\delta_{j,l}+\delta_{i,l}\delta_{j,k})\frac{1}{V}\left[\frac{u_0}{3}-u_0^2\frac{N+8}{18}\int_{\mathbf{q}}\frac{1}{(\mathbf{q}^2+r_0)^2}\right] \\ &\simeq (\delta_{i,j}\delta_{k,l}+\delta_{i,k}\delta_{j,l}+\delta_{i,l}\delta_{j,k})\frac{u_0}{3V}\left[1-u_0\frac{N+8}{12}K_4\ln\frac{\Lambda^2}{r}\right]\end{aligned} \tag{10.A.9}$$

for $r \ll \Lambda$. Since u_0 will eventually be of order ϵ, we have set $d=4$ and replaced r_0 by r. Equation (10.A.9) is then correct to order ϵ^2. Comparing (10.A.8) and (10.A.9), we see that t_2 vanishes if u_0 takes the value[111]

$$u_0(\epsilon) = \frac{6}{(N+8)K_4}\epsilon + \mathcal{O}(\epsilon^2). \tag{10.A.10}$$

We are now in a position to compute the critical exponents from the perturbation theory. To compute γ to $\mathcal{O}(\epsilon)$ we use equation (10.113),

$$\begin{aligned} r &= \bar r_0(T-T_c) - \frac{N+2}{6}K_4 u_0 r \int_0^\Lambda d|\mathbf{q}| \frac{|\mathbf{q}|}{\mathbf{q}^2+r} + \mathcal{O}(\epsilon^2) \\ &= \bar r_0(T-T_c) - \frac{N+2}{12}K_4 u_0 r \ln\frac{\Lambda^2}{r} + \mathcal{O}(\epsilon^2), \end{aligned} \tag{10.A.11}$$

i.e.

$$t \sim \bar r_0(T-T_c) = r\left(1 - \frac{N+2}{12}K_4 u_0 r \ln\frac{r}{\Lambda^2}\right) + \mathcal{O}(\epsilon^2), \tag{10.A.12}$$

where $u_0 \equiv u_0(\epsilon)$. This expression must be compared with

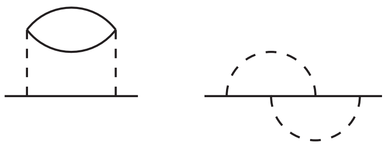
$$t \sim r^{1/\gamma} = r\left[1 + \left(\frac{1}{\gamma}-1\right)\ln r\right] + \mathcal{O}(\epsilon^2). \tag{10.A.13}$$

We deduce

$$\gamma = 1 + \frac{N+2}{N+8}\frac{\epsilon}{2} + \mathcal{O}(\epsilon^2), \tag{10.A.14}$$

in agreement with the result obtained in section 10.6.

To compute the anomalous dimension η to $\mathcal{O}(\epsilon^2)$, we must consider the two-loop contributions



to the self-energy:

$$\Sigma(\mathbf{p}) = -\frac{N+2}{18}u_0^2 \int d^dr\, e^{-i\mathbf{p}\cdot\mathbf{r}} G_0^3(\mathbf{r}) \tag{10.A.15}$$

(other two-loop diagrams, as well as one-loop diagrams, give self-energy corrections independent of the external momentum). This expression cannot be directly used since the bare propagator $G_0(\mathbf{p}) = (\mathbf{p}^2+r_0)^{-1}$ has a finite correlation length $r_0^{-1/2}$ when $T=T_c$ ($T_c \neq T_{c0}$). To circumvent this difficulty, we include $\Sigma(0)$ in the "bare" propagator, i.e. we take $G_0(\mathbf{p}) = [\mathbf{p}^2 + r_0 + \Sigma(0)]^{-1}$.[112] At the critical point, $\Gamma^{(2)}(\mathbf{p}=0) = r_0 + \Sigma(0) = 0$, and the propagator $G_0(\mathbf{p})$ has now an infinite correlation length. The two-point vertex reads

$$\Gamma^{(2)}_{ii}(\mathbf{p}) = \mathbf{p}^2 + \Sigma(\mathbf{p}) - \Sigma(0) \tag{10.A.16}$$

[111] To make dimensional sense of (10.A.8), we must interpret $\ln r$ as $\ln\frac{r}{a\Lambda^2}$. The expression of $u_0(\epsilon)$ is independent of the constant a in the limit $r\to 0$.

[112] For a justification of this procedure (in particular regarding combinatorial factors in the Feynman diagrams), see Sec. 1.6.3.

when $T = T_c$, where $\Sigma(\mathbf{p})$ is given by (10.A.15) to two-loop order.

To obtain the $\mathbf{p}^2 \ln|\mathbf{p}|$ term in $\Sigma(\mathbf{p})$, it is sufficient to expand

$$e^{-i\mathbf{p}\cdot\mathbf{r}} = 1 - i\mathbf{p}\cdot\mathbf{r} - \frac{1}{2}(\mathbf{p}\cdot\mathbf{r})^2 + \cdots \tag{10.A.17}$$

in (10.A.15). This gives

$$\begin{aligned}\int d^4r\, (e^{-i\mathbf{p}\cdot\mathbf{r}} - 1) G_0^3(\mathbf{r}) &= -\frac{\mathbf{p}^2}{8(2S_4)^3} \int d^4r\, |\mathbf{r}|^{-4} + \mathcal{O}(|\mathbf{p}|^4) \\ &= -\frac{\mathbf{p}^2}{64 S_4^2} \int d|\mathbf{r}|\, |\mathbf{r}|^{-1} + \mathcal{O}(|\mathbf{p}|^4)\end{aligned} \tag{10.A.18}$$

($S_4 = 2\pi^2$), where we have used the expression (10.86) of $G_0(\mathbf{r}) = \int_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{r}} \mathbf{p}^{-2}$. We have also set $d = 4$, since we are interested in the result to $\mathcal{O}(\epsilon^2)$ while $u_0 \equiv u_0(\epsilon) = \mathcal{O}(\epsilon)$ in (10.A.15). The lower and upper limits in the last integral of (10.A.18) are approximately given by Λ^{-1} and $1/|\mathbf{p}|$, so that

$$\Gamma_{ii}^{(2)}(\mathbf{p}) = \mathbf{p}^2 - u_0^2 \frac{N+2}{18} \frac{\mathbf{p}^2}{64 S_4^2} (\ln|\mathbf{p}| + \text{const}) + \mathcal{O}(|\mathbf{p}|^4). \tag{10.A.19}$$

Comparing this result with (10.A.2) (without the $\mathcal{O}(|\mathbf{p}|^{-y_2}$ term) and using (10.A.10), we finally obtain

$$\eta = \frac{1}{2} \frac{N+2}{(N+8)^2} \epsilon^2 + \mathcal{O}(\epsilon^3). \tag{10.A.20}$$

10.A.2 $1/N$ expansion

Perturbation theory can also be used to calculate the critical exponents within a $1/N$ expansion. We start from the action

$$S[\boldsymbol{\varphi}] = \int d^dr \left\{ \frac{1}{2}(\boldsymbol{\nabla}\boldsymbol{\varphi})^2 + \frac{r_0}{2}\boldsymbol{\varphi}^2 + \frac{u_0}{4!N}\left(\boldsymbol{\varphi}^2\right)^2 \right\} \tag{10.A.21}$$

where the factor $1/N$ is introduced to obtain a meaningful limit $N \to \infty$. The $1/N$ expansion is not a mere expansion in u_0 since for each closed loop there is a factor N coming from the sum over internal O(N) index.

10.A.2.1 Leading order

To leading order, the self-energy is given by the "Hartree" approximation (Fig. 10.25),

$$\Sigma = \frac{u_0}{6} \int_{\mathbf{q}} \frac{1}{\mathbf{q}^2 + r_0 + \Sigma}. \tag{10.A.22}$$

The two-point vertex is then given by

$$\Gamma_{ii}^{(2)}(\mathbf{p}) = \mathbf{p}^2 + r_0 + \Sigma = \mathbf{p}^2 + r, \tag{10.A.23}$$

where $r = r_0 + \Sigma$. The critical temperature is determined by $r = 0$,

$$0 = \bar{r}_0(T - T_{c0}) + \frac{u_0}{6} \int_{\mathbf{q}} \frac{1}{\mathbf{q}^2}, \tag{10.A.24}$$

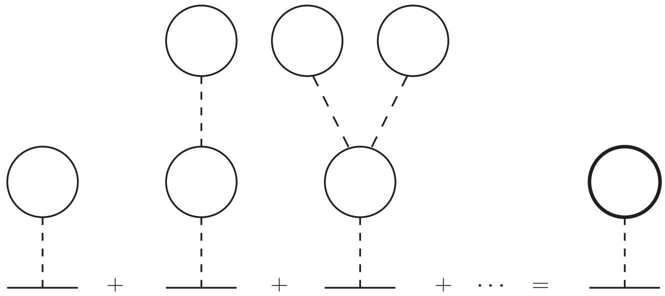


Figure 10.25: Self-energy Σ to leading order in the limit $N \to \infty$. The thick solid line stands for $G = (G_0^{-1} + \Sigma)^{-1}$.

which allows us to express r as

$$\begin{aligned} r &= \bar{r}_0(T - T_c) + \frac{u_0}{6} \int_{\mathbf{q}} \left(\frac{1}{\mathbf{q}^2 + r} - \frac{1}{\mathbf{q}^2} \right) \\ &= \bar{r}_0(T - T_c) + \frac{u_0 r}{6} \int_{\mathbf{q}} \frac{1}{\mathbf{q}^2(\mathbf{q}^2 + r)}. \end{aligned} \tag{10.A.25}$$

For $d > 4$, the integral converges when $r \to 0$ so that $r \sim T - T_c$ when $T \to T_c$, i.e. $\gamma = 1$, in agreement with the mean-field result. For $d = 4$, one finds

$$\int_{\mathbf{q}} \frac{1}{\mathbf{q}^2(\mathbf{q}^2 + r)} = K_4 \int_0^\Lambda d|\mathbf{q}| \frac{|\mathbf{q}|}{\mathbf{q}^2 + r} \simeq \frac{K_4}{2} \ln \frac{\Lambda^2}{r} \tag{10.A.26}$$

for $\Lambda/\sqrt{r} \gg 1$. From (10.A.25), we then deduce

$$\bar{r}_0(T - T_c) \simeq \frac{u_0}{12} K_4 r \ln \frac{\Lambda^2}{r}, \tag{10.A.27}$$

i.e.

$$r \simeq \frac{12}{u_0 K_4} \frac{\bar{r}_0(T - T_c)}{\ln \left(\frac{u_0 K_4 \Lambda^2}{12 \bar{r}_0 (T - T_c)} \right)} \tag{10.A.28}$$

for $T \to T_c$. As expected at the upper critical dimension, there are logarithmic corrections to the mean-field result $r \sim T - T_c$.

For $d < 4$, we use

$$\int_{\mathbf{q}} \frac{1}{\mathbf{q}^2(\mathbf{q}^2 + r)} = K_d \int_0^\Lambda d|\mathbf{q}| \frac{|\mathbf{q}|^{d-3}}{\mathbf{q}^2 + r} = r^{d/2-2} \tilde{K}_d, \tag{10.A.29}$$

where $\tilde{K}_d$ is defined in (10.116). From (10.A.25), we then deduce

$$\bar{r}_0(T - T_c) \simeq \frac{u_0}{6} r^{d/2-1} \tilde{K}_d \tag{10.A.30}$$

for $T \to T_c$ and $d < 4$, i.e.

$$r \sim (T - T_c)^\gamma \tag{10.A.31}$$

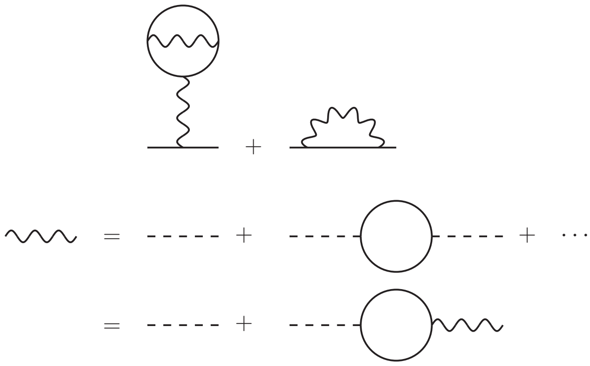


Figure 10.26: Self-energy to order $1/N$.

with the susceptibility critical exponent

$$\gamma = \frac{2}{d-2} + \mathcal{O}(N^{-1}). \tag{10.A.32}$$

Since $\eta = 0$ (the self-energy is momentum independent), we obtain all other critical exponents using the scaling laws (Sec. 10.4),

$$\begin{aligned} \nu &= \frac{1}{d-2} + \mathcal{O}(N^{-1}), \\ \beta &= \frac{1}{2} + \mathcal{O}(N^{-1}), \\ \alpha &= \frac{d-4}{d-2} + \mathcal{O}(N^{-1}), \\ \delta &= \frac{d+2}{d-2} + \mathcal{O}(N^{-1}). \end{aligned} \tag{10.A.33}$$

10.A.2.2 $1/N$ corrections

The self-energy to order $1/N$ is shown in figure 10.26. As in the case of the ϵ expansion, we include $\Sigma(0)$ in the "bare" propagator $G(\mathbf{p}) = (\mathbf{p}^2 + r)^{-1}$. The effective vertex u (the wavy line in Fig. 10.26) is defined by[113]

$$u(\mathbf{p}) = \frac{u_0}{1 + \frac{u_0}{6}\Pi(\mathbf{p})}, \qquad \Pi(\mathbf{p}) = \int_{\mathbf{q}} G(\mathbf{q})G(\mathbf{p}+\mathbf{q}). \tag{10.A.34}$$

Since the first diagram in figure 10.26 is independent of $\mathbf{p}$, we obtain

$$\Sigma(\mathbf{p}) - \Sigma(0) = \frac{1}{3N}\int_{\mathbf{q}} u(\mathbf{q})\left[G(\mathbf{p}+\mathbf{q}) - G(\mathbf{q})\right]. \tag{10.A.35}$$

[113]This result is easily obtained by considering the expansion for the 4-point vertex,

$$\begin{aligned} \Gamma^{(4)}_{ijkl}(\mathbf{p}_1,\cdots,\mathbf{p}_4) &= \delta_{\sum_i \mathbf{p}_i,0}\left(\delta_{i,j}\delta_{k,l} + \delta_{i,k}\delta_{j,l} + \delta_{i,l}\delta_{j,k}\right) \\ &\quad \times \left[\frac{u_0}{3N} - \frac{N}{2}\left(\frac{u_0}{3N}\right)^2 \int_{\mathbf{q}} G(\mathbf{q})G(\mathbf{p}_1+\mathbf{p}_2+\mathbf{q}) + \cdots\right]. \end{aligned}$$

At the critical point ($r=0$), we can use

$$\Pi(\mathbf{p}) = \int_{\mathbf{q}} \frac{1}{\mathbf{q}^2(\mathbf{p}+\mathbf{q})^2} \simeq A_d|\mathbf{p}|^{d-4} \tag{10.A.36}$$

in the limit $|\mathbf{p}| \ll \Lambda$ and for $d<4$ (see Eq. (10.324)). Thus, for $\mathbf{p}\to 0$,

$$u(\mathbf{p}) \simeq \frac{6}{\Pi(\mathbf{p})} \simeq \frac{6}{A_d|\mathbf{p}|^{d-4}} \tag{10.A.37}$$

and

$$\begin{aligned}\Sigma(\mathbf{p}) - \Sigma(0) &\simeq \frac{2}{NA_d}\int_{\mathbf{q}} |\mathbf{q}|^{4-d}\left[\frac{1}{(\mathbf{p}+\mathbf{q})^2} - \frac{1}{\mathbf{q}^2}\right] \\ &\simeq A\mathbf{p}^2 \ln\left(\frac{\Lambda}{|\mathbf{p}|}\right). \end{aligned} \tag{10.A.38}$$

A can be obtained by expanding (10.A.38) for small $\mathbf{p}$ since the $\mathbf{p}^2\ln\Lambda$ term is due to large values of $\mathbf{q}$. Using

$$\frac{1}{(\mathbf{p}+\mathbf{q})^2} - \frac{1}{\mathbf{q}^2} = -2\frac{\mathbf{p}\cdot\mathbf{q}}{|\mathbf{q}|^4} - \frac{\mathbf{p}^2}{|\mathbf{q}|^4} + 4\frac{(\mathbf{p}\cdot\mathbf{q})^2}{|\mathbf{q}|^6} + \mathcal{O}(|\mathbf{p}|^3), \tag{10.A.39}$$

we obtain

$$\int_{\mathbf{q}} |\mathbf{q}|^{4-d}\left[\frac{1}{(\mathbf{p}+\mathbf{q})^2} - \frac{1}{\mathbf{q}^2}\right] \simeq \mathbf{p}^2\frac{4-d}{d}\int_{\mathbf{q}}|\mathbf{q}|^{-d} = \mathbf{p}^2\frac{4-d}{d}K_d\ln\Lambda \tag{10.A.40}$$

and

$$\Gamma^{(2)}_{ii}(\mathbf{p}) = \mathbf{p}^2 + \frac{2}{NA_d}\frac{4-d}{d}K_d\mathbf{p}^2\ln\frac{\Lambda}{|\mathbf{p}|}, \tag{10.A.41}$$

which leads to

$$\eta = \frac{2}{NA_d}\frac{4-d}{d}K_d + \mathcal{O}(N^{-2}). \tag{10.A.42}$$

Appendix 10.B The $(\varphi^2)^2$ theory in the large-N limit

In this section we reconsider the large-N limit of the $(\boldsymbol{\varphi}^2)^2$ theory [Eq. (10.A.21)] and show how we can extend the results of section 10.A.2 to the low-temperature phase.

Using

$$\int_{-\infty}^{\infty} d\rho \int_{-\infty}^{\infty} d\lambda\, e^{-i\frac{\lambda}{2}(\boldsymbol{\varphi}^2-\rho)} = \int_{-\infty}^{\infty} d\rho\, \delta(\boldsymbol{\varphi}^2-\rho) = 1 \tag{10.B.1}$$

(we ignore any multiplicative constant), we rewrite the partition function as

$$\begin{aligned} Z &= \int \mathcal{D}[\boldsymbol{\varphi},\rho,\lambda]\, \exp\left\{-\int d^dr\left[\frac{1}{2}(\boldsymbol{\nabla}\boldsymbol{\varphi})^2 + \frac{r_0}{2}\rho + \frac{u_0}{4!N}\rho^2 + i\frac{\lambda}{2}(\boldsymbol{\varphi}^2-\rho)\right]\right\} \\ &= \int \mathcal{D}[\boldsymbol{\varphi},\lambda]\, \exp\left\{-\frac{1}{2}\int d^dr\left[(\boldsymbol{\nabla}\boldsymbol{\varphi})^2 + i\lambda\boldsymbol{\varphi}^2\right] + \frac{3N}{2u_0}\int d^dr\,(i\lambda - r_0)^2\right\}, \end{aligned} \tag{10.B.2}$$

where the second line is obtained by integrating out the ρ field. We now split $\boldsymbol{\varphi} = (\sigma,\boldsymbol{\pi})$ into a scalar field σ and a $(N-1)$-component field $\boldsymbol{\pi}$. As in the NLσM (Sec. 10.7), this parametrization will allow spontaneous symmetry breaking. The integration over the $\boldsymbol{\pi}$ field gives

$$\int \mathcal{D}[\boldsymbol{\pi}] \exp\left\{-\frac{1}{2}\int d^d r \left[(\boldsymbol{\nabla}\boldsymbol{\pi})^2 + i\lambda\boldsymbol{\pi}^2\right]\right\} = \left(\det g^{-1}\right)^{-(N-1)/2}, \qquad (10.B.3)$$

where

$$g^{-1}(\mathbf{r},\mathbf{r}') = -\boldsymbol{\nabla}^2\delta(\mathbf{r}-\mathbf{r}') + i\lambda(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}') \qquad (10.B.4)$$

is the propagator of the π_i field in the presence of a fluctuating λ field. We thus obtain the partition function

$$\begin{aligned} Z = \int \mathcal{D}[\sigma,\lambda] \exp\biggl\{ &-\frac{1}{2}\int d^d r \left[(\boldsymbol{\nabla}\sigma)^2 + i\lambda\sigma^2\right] \\ &+ \frac{3N}{2u_0}\int d^d r\, (i\lambda - r_0)^2 - \frac{N-1}{2}\mathrm{Tr}\ln g^{-1}\biggr\}. \end{aligned} \qquad (10.B.5)$$

If we rescale the σ field, $\sigma \to \sqrt{N}\sigma$,[114] then the action becomes proportional to N in the limit $N\to\infty$ and the saddle-point approximation becomes exact. For uniform fields $\sigma(\mathbf{r}) = \sigma$ and $\lambda(\mathbf{r}) = \lambda$, the action is given by

$$\frac{1}{V}S[\sigma,\lambda] = \frac{i}{2}\lambda\sigma^2 - \frac{3N}{2u_0}(i\lambda - r_0)^2 + \frac{N}{2V}\mathrm{Tr}\ln g^{-1} \qquad (10.B.6)$$

(we use $N-1 \simeq N$ for large N), with $g^{-1}(\mathbf{p}) = \mathbf{p}^2 + i\lambda$ in Fourier space. Since σ and λ do not fluctuate when $N\to\infty$, $g(\mathbf{p})$ is the propagator of the field π_i.[115] From (10.B.6), we deduce the saddle-point equations

$$\begin{aligned} \sigma m^2 &= 0, \\ \sigma^2 &= \frac{6N}{u_0}(m^2 - r_0) - N\int_{\mathbf{p}} \frac{1}{\mathbf{p}^2 + m^2}, \end{aligned} \qquad (10.B.7)$$

where we use the notation $m^2 = i\lambda$ ($i\lambda$ is real at the saddle point). These equations show that the component σ of the $\boldsymbol{\varphi}$ field which was singled out plays the role of an order parameter. In the disordered phase, $\sigma = 0$ and $m \neq 0$ (the $N-1$ π_i field are gapped). The saddle-point equation for m^2 reproduces our previous result (10.A.22) with $\Sigma = m^2 - r_0$. In the ordered phase, σ is nonzero and the propagator $g(\mathbf{p}) = 1/\mathbf{p}^2$ is gapless, thus identifying the π_i fields as the $N-1$ Goldstone modes associated with spontaneous rotation symmetry breaking.

10.B.1 Correlation functions in the low-temperature phase

In the broken-symmetry phase, $m = 0$ and the saddle-point equation for the order parameter reads

$$\sigma^2 = -\frac{6N}{u_0}(r_0 - r_{0c}), \qquad (10.B.8)$$

[114] In the following, we work with the "unrescaled" field which is therefore $\mathcal{O}(\sqrt{N})$.

[115] If σ and λ were fluctuating, one would have to integrate them out to obtain the propagator of the $\boldsymbol{\pi}$ field.

where

$$r_{0c} = -\frac{u_0}{6}\int_{\mathbf{p}} \frac{1}{\mathbf{p}^2} = -\frac{u_0}{6}\frac{K_d \Lambda^{d-2}}{d-2} \tag{10.B.9}$$

is the critical value of r_0 defining the critical temperature: $r_{0c} = \bar r_0(T_c - T_{c0})$.[116] The correction to the mean-field result $\sigma^2 = -6Nr_0/u_0$ in (10.B.8) is due to the Goldstone modes. Since $\sigma \sim \sqrt{T_c - T}$, the critical exponent β is equal to 1/2 in agreement with (10.A.33).

The action being obtained from a saddle-point approximation, the effective action $\Gamma[\sigma,\lambda]$ is simply given by the action $S[\sigma,\lambda]$ defined by (10.B.5).[117,118] We deduce

$$\begin{aligned}\Gamma^{(2)}(\mathbf{r}-\mathbf{r}') &= \begin{pmatrix} \Gamma^{(2)}_{\sigma\sigma}(\mathbf{r}-\mathbf{r}') & \Gamma^{(2)}_{\sigma\lambda}(\mathbf{r}-\mathbf{r}') \\ \Gamma^{(2)}_{\lambda\sigma}(\mathbf{r}-\mathbf{r}') & \Gamma^{(2)}_{\lambda\lambda}(\mathbf{r}-\mathbf{r}') \end{pmatrix} \\ &= \begin{pmatrix} -\boldsymbol{\nabla}^2\delta(\mathbf{r}-\mathbf{r}') & i\sigma\delta(\mathbf{r}-\mathbf{r}') \\ i\sigma\delta(\mathbf{r}-\mathbf{r}') & \frac{N}{2}\Pi(\mathbf{r}-\mathbf{r}') + \frac{3N}{u_0}\delta(\mathbf{r}-\mathbf{r}') \end{pmatrix}, \end{aligned} \tag{10.B.10}$$

where

$$\Pi(\mathbf{r}-\mathbf{r}') = g(\mathbf{r}-\mathbf{r}')g(\mathbf{r}'-\mathbf{r}) \tag{10.B.11}$$

and we use the notation $\Gamma^{(2)}_{\sigma\sigma}(\mathbf{r}-\mathbf{r}') = \delta^{(2)}\Gamma/\delta\sigma(\mathbf{r})\delta\sigma(\mathbf{r}')$, etc. The two-point vertex $\Gamma^{(2)}$ is computed for the saddle-point values of the σ and λ fields. In Fourier space, we obtain

$$\Gamma^{(2)}(\mathbf{p}) = \begin{pmatrix} \mathbf{p}^2 & i\sigma \\ i\sigma & \frac{N}{2}\Pi(\mathbf{p}) + \frac{3N}{u_0} \end{pmatrix}, \tag{10.B.12}$$

where $\Pi(\mathbf{p}) = \int_{\mathbf{q}} g(\mathbf{q})g(\mathbf{p}+\mathbf{q})$. The propagator $G = \Gamma^{(2)-1}$ takes the form

$$G(\mathbf{p}) = \frac{1}{\det \Gamma^{(2)}(\mathbf{p})} \begin{pmatrix} \frac{N}{2}\Pi(\mathbf{p}) + \frac{3N}{u_0} & -i\sigma \\ -i\sigma & \mathbf{p}^2 \end{pmatrix}, \tag{10.B.13}$$

with

$$\det \Gamma^{(2)}(\mathbf{p}) = \mathbf{p}^2\left[\frac{N}{2}\Pi(\mathbf{p}) + \frac{3N}{u_0}\right] + \sigma^2. \tag{10.B.14}$$

The last equation, together with the small $\mathbf{p}$ behavior (10.A.36) of $\Pi(\mathbf{p})$, leads us to introduce three characteristic momentum scales,

$$\begin{aligned} p_G &= \left(\frac{u_0 A_d}{6}\right)^{1/(4-d)}, \\ p_J &= \left(\frac{2\sigma^2}{NA_d}\right)^{1/(d-2)} = \left[\frac{12}{u_0 A_d}(r_{0c} - r_0)\right]^{1/(d-2)}, \\ p_c &= \left(\frac{u_0\sigma^2}{3N}\right)^{1/2} = [2(r_{0c}-r_0)]^{1/2}, \end{aligned} \tag{10.B.15}$$

which will be referred to as the Ginzburg scale, the Josephson scale and the correlation scale, respectively (p_G and p_c were previously defined in section 10.7.3 while the Josephson

[116] We assume $r_0 = \bar r_0(T - T_{c0})$.

[117] The equality between the effective action Γ and the "microscopic" action S within a saddle-point approximation has been shown in Sec. 10.2.1 when discussing the Landau theory.

[118] To simplify the notations, we note σ and λ the arguments of both the action S and the effective action Γ.

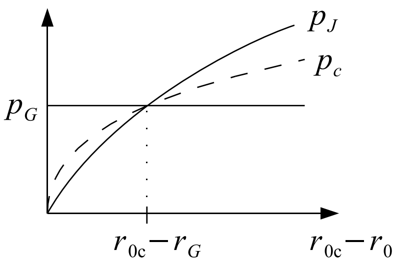


Figure 10.27: Characteristic momentum scales p_G, p_J and p_c [Eq. (10.B.15)] *vs* $r_{0c} - r_0 = \bar r_0(T_c - T)$ for fixed u_0.

length $\xi_J = p_J^{-1}$ was discussed in Sec. 10.7.2). Here and in the following we assume $d < 4$ and postpone the case $d = 4$ to section 10.B.3. These momentum scales are not independent since

$$p_c^2 = p_G^2 \left(\frac{p_J}{p_G}\right)^{d-2} . \tag{10.B.16}$$

If we vary r_0 (i.e. the temperature) with u_0 fixed, we find that the three characteristic scales (10.B.15) are equal for a temperature T_G defined by

$$r_{0c} - r_{0G} = \bar r_0(T_c - T_G) = \frac{1}{2}\left(\frac{u_0 A_d}{6}\right)^{2/(4-d)} \tag{10.B.17}$$

(see Fig. 10.27). Equation (10.B.17) is similar to the Ginzburg criterion (10.117) obtained from the one-loop calculation of the two-point vertex in the disordered phase. As we show below, T_G separates a critical regime from a non-critical regime in the ordered phase.

In the critical regime ($T_c - T \ll T_c - T_G$ or $p_J \ll p_G$), using $p_J \ll p_c \ll p_G$ one finds

$$G_{\sigma\sigma}(\mathbf{p}) = \begin{cases} \dfrac{p_J^{2-d}}{|\mathbf{p}|^{4-d}} & \text{if} \quad |\mathbf{p}| \ll p_J, \\ \dfrac{1}{\mathbf{p}^2} & \text{if} \quad |\mathbf{p}| \gg p_J, \end{cases} \tag{10.B.18}$$

while in the non-critical regime ($T_c - T_G \ll T_c - T$ or $p_G \ll p_c$),

$$G_{\sigma\sigma}(\mathbf{p}) = \begin{cases} \dfrac{1}{p_c^2}\left(\dfrac{p_G}{|\mathbf{p}|}\right)^{4-d} & \text{if} \quad |\mathbf{p}| \ll p_G, \\ \dfrac{1}{\mathbf{p}^2 + p_c^2} & \text{if} \quad |\mathbf{p}| \gg p_G. \end{cases} \tag{10.B.19}$$

In the non-critical regime, we recover the results of section 10.7.3. We find two characteristic momentum scales (p_G and p_c) and two regimes for the behavior of $G_{\sigma\sigma}(\mathbf{p})$: i) a Goldstone regime ($|\mathbf{p}| \lesssim p_G$) characterized by a diverging longitudinal propagator $G_{\sigma\sigma}(\mathbf{p}) \sim 1/|\mathbf{p}|^{4-d}$ in the limit $\mathbf{p} \to 0$, ii) a Gaussian regime ($|\mathbf{p}| \gtrsim p_G$) where the Gaussian approximation (and the perturbative approach) is essentially correct. The critical regime is characterized by two momentum scales (p_J and p_G). The Josephson length diverges at the phase transition with the exponent $\nu = 1/(d-2)$. The same exponent was found for the divergence of the correlation length ξ in the disordered phase [Eq. (10.A.33)]. There are three regimes for

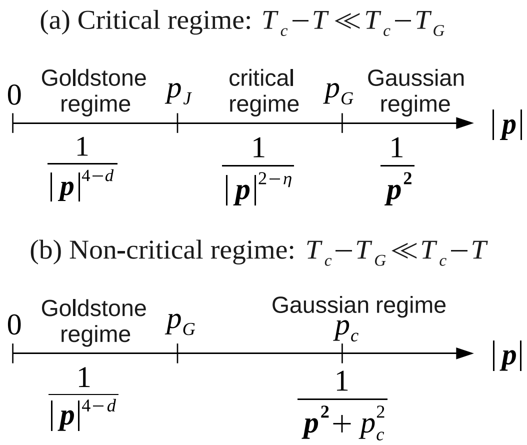


Figure 10.28: Momentum dependence of the longitudinal correlation function $G_{\sigma\sigma}(\mathbf{p})$ in the critical and non-critical regimes of the low-temperature phase ($2 < d < 4$). Note that $\eta = 0$ in the limit $N \to \infty$.

the behavior of $G_{\sigma\sigma}(\mathbf{p})$: i) a Goldstone regime ($|\mathbf{p}| \lesssim p_J$) with a diverging longitudinal propagator, ii) a critical regime ($p_J \lesssim |\mathbf{p}| \lesssim p_G$) with a vanishing anomalous dimension η (η is $\mathcal{O}(1/N)$ in the large-N limit, see Sec. 10.A.2), iii) a Gaussian regime ($p_G \lesssim |\mathbf{p}|$). These results are summarized in figure 10.28.

10.B.1.1 The NLσM

At the critical point in the limit $N \to \infty$, from the preceding results we deduce

$$\begin{aligned} G_{\lambda\lambda}(\mathbf{p}) &= \frac{2}{NA_d|\mathbf{p}|^{d-4}}, \\ G_{\sigma\sigma}(\mathbf{p}) &= \frac{1}{\mathbf{p}^2}, \end{aligned} \tag{10.B.20}$$

so that $[\sigma] = (d-2)/2$ and $[\lambda] = 2$. It follows that the perturbation $\int d^d r\, \lambda^2$ is irrelevant for $d < 4$. Thus, if we shift the λ field by its fixed-point value, $i\lambda \to i\lambda + m^2$, we can omit the λ^2 term in the action,

$$\begin{aligned} S[\sigma,\lambda] = \frac{1}{2}\int d^d r\, \left[(\boldsymbol{\nabla}\sigma)^2 + (m^2 + i\lambda)\sigma^2\right] \\ - \frac{3N}{u_0}\int d^d r\, i\lambda(m^2 - r_0) + \frac{N-1}{2}\mathrm{Tr}\ln g^{-1}. \end{aligned} \tag{10.B.21}$$

We can now reintroduce the $\boldsymbol{\pi}$ field using

$$\exp\left\{-\frac{N-1}{2}\mathrm{Tr}\ln g^{-1}\right\} = \int \mathcal{D}[\boldsymbol{\pi}] \exp\left\{-\frac{1}{2}\int d^d r\, \left[(\boldsymbol{\nabla}\boldsymbol{\pi})^2 + (i\lambda + m^2)\boldsymbol{\pi}^2\right]\right\} \tag{10.B.22}$$

to obtain

$$S[\boldsymbol{\varphi},\lambda] = \frac{1}{2}\int d^d r\, \left[(\boldsymbol{\nabla}\boldsymbol{\varphi})^2 + i\lambda\boldsymbol{\varphi}^2\right] - \frac{3N}{u_0}\int d^d r\, (i\lambda - m^2)(m^2 - r_0), \tag{10.B.23}$$

where we have shifted λ back to its original value. Integrating over λ, we eventually obtain

$$Z = \int \mathcal{D}[\boldsymbol{\varphi}]\, \delta(\boldsymbol{\varphi}^2 - \boldsymbol{\varphi}_0^2) \exp\left\{ -\frac{1}{2} \int d^d r\, (\boldsymbol{\nabla}\boldsymbol{\varphi})^2 \right\} \tag{10.B.24}$$

with $\boldsymbol{\varphi}_0^2 = \frac{6N}{u_0}(m^2 - r_0)$, which is nothing but the action of the NLσM (see Sec. 10.7). We conclude that in the limit $N \to \infty$, the correlation functions of the $(\boldsymbol{\varphi}^2)^2$ theory and the NLσM have the same asymptotic long-distance behavior at the critical point. It can be shown that this equivalence holds to all orders in the $1/N$ expansion [51]. The NLσM in the large-N limit is studied in section 10.C.

10.B.2 Gibbs free energy

Let us consider the system in the presence of an external field,

$$Z[h] = \int \mathcal{D}[\sigma, \lambda]\, e^{-S[\sigma,\lambda] + \int d^d r\, h\sigma}, \tag{10.B.25}$$

where the action $S[\sigma, \lambda]$ is defined by (10.B.5). The Gibbs free energy reads

$$\Gamma[M] = -\ln Z[h] + \int d^d r\, hM, \tag{10.B.26}$$

where h is related to the order parameter $M(\mathbf{r}) = \langle \sigma(\mathbf{r}) \rangle$ by

$$M(\mathbf{r}) = \frac{\delta \ln Z[h]}{\delta h(\mathbf{r})}. \tag{10.B.27}$$

In the large-N limit, the saddle-point approximation in (10.B.25) is exact so that $\Gamma[M] = S[M, \lambda]$. For a uniform order parameter,

$$\frac{1}{V}\Gamma(M) = \frac{1}{2} m^2 M^2 - \frac{3N}{2u_0}(m^2 - r_0)^2 + \frac{N}{2} \int_{\mathbf{p}} \ln(\mathbf{p}^2 + m^2). \tag{10.B.28}$$

The momentum integral in (10.B.28) diverges for small m and should be regularized, e.g. by considering

$$D(m^2) = \int_{\mathbf{p}} [\ln(\mathbf{p}^2 + m^2) - \ln(\mathbf{p}^2)], \tag{10.B.29}$$

which amounts to removing an infinite normalization constant in the partition function. For $m \to 0$ and $d < 4$,[119]

$$D(m^2) = -2\frac{\tilde{K}_d}{d} m^d + K_d \frac{\Lambda^{d-2}}{d-2} m^2 + \frac{K_d}{2} \frac{\Lambda^{d-4}}{4-d} m^4, \tag{10.B.30}$$

where $\tilde{K}_d$ is defined in (10.116). For $d < 4$, we can neglect m^4 wrt m^d for small m and we obtain[120]

$$\frac{1}{NV}\Gamma(M) = -\frac{3r_0^2}{2u_0} + \frac{3}{u_0}(r_0 - r_{0c}) m^2 + \frac{m^2 M^2}{2N} - \frac{\tilde{K}_d}{d} m^d. \tag{10.B.31}$$

[119] Eq. (10.B.30) is obtained from $D(0) = 0$, $D'(m^2) = \frac{\Lambda^d}{dm^2}\, {}_2F_1\left(1, \frac{d}{2}, \frac{d}{2}+1, -\frac{\Lambda^2}{m^2}\right)$, and the expansion of the hypergeometric function ${}_2F_1$ for small m^2.

[120] We use $\frac{1}{2}\frac{K_d}{d-2}\Lambda^{d-2} = -\frac{3r_{0c}}{u_0}$.

The value of m^2 is obtained by requiring that $\Gamma[M]$, as $-\ln Z[h]$, is extremum.[121] The condition $\frac{\partial}{\partial m^2}\Gamma[M]=0$ gives

$$m^2=\left[\frac{1}{\tilde{K}_d}\left(\frac{6\tau}{u_0}+\frac{M^2}{N}\right)\right]^{2/(d-2)} \tag{10.B.32}$$

with $\tau=r_0-r_{0c}=\bar{r}_0(T-T_c)$. This expression makes sense only if $\frac{6\tau}{u_0}+\frac{M^2}{N}>0$. If not, the extremum is reached for $m^2=0$.

In the high-temperature phase, $\tau\geq 0$ and $m^2\neq 0$ for any value of M. This yields

$$\frac{1}{NV}\Gamma(M)=-\frac{3r_0^2}{2u_0}+\frac{d-2}{2d}\tilde{K}_d^{2/(2-d)}\left(\frac{6\tau}{u_0}+\frac{M^2}{N}\right)^{d/(d-2)}. \tag{10.B.33}$$

The minimum of the Gibbs free energy is reached for $M=0$. We deduce the specific heat

$$C_V=-T\frac{\partial^2}{\partial T^2}\beta^{-1}\Gamma(M=0)\sim\tau^{-\alpha}, \tag{10.B.34}$$

with a critical exponent $\alpha=(d-4)/(d-2)$. At the transition ($\tau=0$),

$$\frac{1}{NV}\Gamma(M)\sim M^{2d/(d-2)}+\text{const}\quad\text{and}\quad h=\frac{1}{V}\frac{\partial\Gamma}{\partial M}\sim M^\delta \tag{10.B.35}$$

with a critical exponent $\delta=(d+2)/(d-2)$. The values of α and δ agree with (10.A.33).

In the low-temperature phase, $\tau\leq 0$ and m^2 can be zero or nonzero depending on the value of M. Thus

$$\frac{1}{NV}\Gamma(M)=\begin{cases}-\dfrac{3r_0^2}{2u_0} & \text{if}\quad |M|\leq M_0,\\ -\dfrac{3r_0^2}{2u_0}+\dfrac{d-2}{2d}\tilde{K}_d^{2/(2-d)}\left(\dfrac{M^2-M_0^2}{N}\right)^{d/(d-2)} & \text{if}\quad |M|\geq M_0,\end{cases} \tag{10.B.36}$$

where

$$M_0=\sqrt{\frac{6N|\tau|}{u_0}} \tag{10.B.37}$$

is equal to the saddle point value of the σ field [Eq. (10.B.8)]. Thus the large-N approach gives a convex Gibbs free energy both in the high- and low-temperature phases (see Fig. 10.4 in Sec. 10.1.1).

10.B.3 The upper critical dimension

The results obtained in sections 10.B.1 and 10.B.3 for $2<d<4$ can easily be extended to the case $d=d_c^+=4$. Using the small $\mathbf{p}$ behavior (10.A.36) of $\Pi(\mathbf{p})$ when $d=4$, the three characteristic momentum scales introduced in section 10.B.1 become

$$\begin{aligned}p_G&=\Lambda\exp\left(\frac{-6}{u_0A_4}\right),\\ p_J&=\left[\frac{24(r_{0c}-r_0)}{u_0A_4\ln\left(\frac{u_0A_4\Lambda^2}{24(r_{0c}-r_0)}\right)}\right]^{1/2},\\ p_c&=[2(r_{0c}-r_0)]^{1/2}.\end{aligned} \tag{10.B.38}$$

[121]This follows from the property $\frac{\partial\Gamma[M,m^2]}{\partial m^2}\big|_M=-\frac{\partial\ln Z[h,m^2]}{\partial m^2}\big|_h$, which is a direct consequence of the definition (10.B.26) of the effective action.

As expected, we find a mean-field like divergence of the Josephson length ($\xi_J = p_J^{-1} \sim (T_c - T)^{-1/2}$) with logarithmic corrections. The three momentum scales (10.B.38) satisfy

$$p_J^2 \simeq p_c^2 \frac{\ln(\Lambda/p_G)}{\ln(\Lambda/p_c)}. \tag{10.B.39}$$

and therefore coincide at the Ginzburg temperature T_G defined by

$$r_{0c} - r_{0G} = \bar{r}_0(T_c - T_G) \simeq \frac{\Lambda^2}{2} \exp\left(\frac{-12}{u_0 A_4}\right). \tag{10.B.40}$$

In the critical regime ($T_c - T \ll T_c - T_G$ or $p_J \ll p_G$), one finds

$$G_{\sigma\sigma}(\mathbf{p}) = \begin{cases} \dfrac{\ln(\Lambda/|\mathbf{p}|)}{p_J^2 \ln(\Lambda/p_J)} & \text{if} \quad |\mathbf{p}| \ll p_J, \\ \dfrac{1}{\mathbf{p}^2} & \text{if} \quad |\mathbf{p}| \gg p_J, \end{cases} \tag{10.B.41}$$

while in the non-critical regime ($T_c - T_G \ll T_c - T$ or $p_G \ll p_c$),

$$G_{\sigma\sigma}(\mathbf{p}) = \begin{cases} \dfrac{\ln(\Lambda/|\mathbf{p}|)}{p_c^2 \ln(\Lambda/p_G)} & \text{if} \quad |\mathbf{p}| \ll p_G, \\ \dfrac{1}{\mathbf{p}^2 + p_c^2} & \text{if} \quad |\mathbf{p}| \gg p_G. \end{cases} \tag{10.B.42}$$

The behavior of the longitudinal correlation function is similar to the case $d < 4$ (Fig. 10.28) except that the divergence $\sim 1/|\mathbf{p}|^{4-d}$ in the Goldstone regime is now logarithmic, and the anomalous dimension η vanishes (Sec. 10.A).

To compute the Gibbs free energy $\Gamma(M)$ [Eq. (10.B.28)], we use

$$D(m^2) = \frac{K_4}{2}\left[m^2\Lambda^2 + m^4\left(\ln\frac{m}{\Lambda} - \frac{1}{4}\right)\right] + \mathcal{O}(m^6). \tag{10.B.43}$$

Minimizing $\Gamma(M)$ wrt m^2, we then obtain

$$m^2 \simeq \begin{cases} \dfrac{2X}{K_4 \ln\left(\frac{K_4\Lambda^2}{2X}\right)} & \text{if} \quad X \geq 0, \\ 0 & \text{if} \quad X \leq 0, \end{cases} \tag{10.B.44}$$

where

$$X = \frac{M^2}{N} + \frac{6\tau}{u_0} \tag{10.B.45}$$

and $\tau = r_0 - r_{0c} = \bar{r}_0(T - T_c)$. In the high-temperature phase, this leads to

$$\frac{1}{NV}\Gamma(M) = -\frac{3r_0^2}{2u_0} + \frac{X^2}{2K_4 \ln\left(\frac{K_4\Lambda^2}{2X}\right)}. \tag{10.B.46}$$

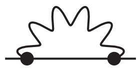
The minimum is reached for $M = 0$ and the singular part of $\Gamma(M = 0)$ is given by $\tau^2/\ln\tau$. The singular part of the specific heat in the large-N limit, $C_V \sim 1/|\ln\tau|$, agrees with the general result (10.271) of section 10.6.2. In the low-temperature phase, we find

$$\frac{1}{NV}\Gamma(M) = \begin{cases} -\dfrac{3r_0^2}{2u_0} & \text{if} \quad |M| \leq M_0, \\ -\dfrac{3r_0^2}{2u_0} + \dfrac{1}{2K_4N^2}\dfrac{(M^2-M_0^2)^2}{\ln\left(\frac{\Lambda^2 K_4 N}{2(M^2-M_0^2)}\right)} & \text{if} \quad |M| \geq M_0, \end{cases} \tag{10.B.47}$$

where M_0 is equal to the saddle-point value of the σ field [Eq. (10.B.37)].

10.B.4 $1/N$ correction

The $\mathcal{O}(1/N)$ correction to the propagator $G_{\sigma\sigma}$ comes from the one-loop self-energy diagram



where the dot stands for the vertex $i\lambda\sigma^2$ (see Eq. (10.B.5)). $G_{\sigma\sigma}$ (solid line) and $G_{\lambda\lambda}$ (wavy line) are the propagators in the limit $N \to \infty$. Thus,

$$\Gamma^{(2)}_{\sigma\sigma}(\mathbf{p}) = \mathbf{p}^2 + \int_{\mathbf{q}} G_{\lambda\lambda}(\mathbf{q})\,[G_{\sigma\sigma}(\mathbf{p}+\mathbf{q}) - G_{\sigma\sigma}(\mathbf{q})] + \mathcal{O}\left(\frac{1}{N^2}\right) \tag{10.B.48}$$

at the critical point ($T = T_c$). As in Sec. 10.A.1, the subtraction of $G_{\sigma\sigma}(\mathbf{q})$ in (10.B.48) ensures that $\Gamma^{(2)}_{\sigma\sigma}(\mathbf{p}=0) = 0$ when $T = T_c$. Using (10.B.20) we finally obtain

$$\Gamma^{(2)}_{\sigma\sigma}(\mathbf{p}) = \mathbf{p}^2 + \frac{2}{NA_d}\int_{\mathbf{q}} \frac{1}{|\mathbf{q}|^{d-4}}\left[\frac{1}{(\mathbf{p}+\mathbf{q})^2} - \frac{1}{\mathbf{q}^2}\right] + \mathcal{O}\left(\frac{1}{N^2}\right), \tag{10.B.49}$$

which agrees with our previous result (10.A.38) and yields the result (10.A.42) for the anomalous dimension η to $\mathcal{O}(1/N)$.

Appendix 10.C The nonlinear σ model in the large-N limit

In section 10.B, we have studied the $(\boldsymbol{\varphi}^2)^2$ theory in the large-N limit and shown that the critical behavior is the same as that of the NLσM. In this section, we directly consider the NLσM in the large-N limit (along similar lines). We start from the partition function

$$Z = \int \mathcal{D}[\mathbf{n}]\delta(\mathbf{n}^2 - 1)\exp\left\{-\frac{N}{2g}\int d^dr\,(\boldsymbol{\nabla}\mathbf{n})^2\right\} \tag{10.C.1}$$

with an implicit ultraviolet cutoff Λ on the momenta. The factor N in (10.C.1) is introduced to yield a meaningful limit $N \to \infty$. As in section 10.7 we write the field $\mathbf{n} = (\sigma, \boldsymbol{\pi})$ in terms of a scalar field σ and a $(N-1)$-component field $\boldsymbol{\pi}$. Introducing a Lagrange multiplier field λ to impose the constraint $\mathbf{n}^2 = 1$, we obtain

$$\begin{aligned} Z &= \int \mathcal{D}[\sigma,\boldsymbol{\pi},\lambda]\exp\left\{-i\int d^dr\frac{\lambda}{2}(\sigma^2 + \boldsymbol{\pi}^2 - 1) - \frac{N}{2g}\int d^dr\ \left[(\boldsymbol{\nabla}\sigma)^2 + (\boldsymbol{\nabla}\boldsymbol{\pi})^2\right]\right\} \\ &= \int \mathcal{D}[\sigma,\lambda]\exp\left\{-i\int d^dr\frac{\lambda}{2}(\sigma^2 - 1) - \frac{N}{2g}\int d^dr\,(\boldsymbol{\nabla}\sigma)^2 - \frac{N-1}{2}\mathrm{Tr}\ln g_\pi^{-1}\right\}, \end{aligned} \tag{10.C.2}$$

where $g_\pi^{-1}(\mathbf{r},\mathbf{r}') = -\frac{N}{g}\boldsymbol{\nabla}^2\delta(\mathbf{r}-\mathbf{r}') + i\lambda(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}')$. The second line in (10.C.2) is obtained by integrating out the $\boldsymbol{\pi}$ field. If we take the limit $N\to\infty$ (and rescale the field $\lambda\to N\lambda$), the action becomes proportional to N and the saddle-point approximation is then exact. For uniform fields σ and λ, the saddle-point equations read

$$\begin{aligned} \sigma m^2 &= 0, \\ \sigma^2 &= 1 - g\int_{\mathbf{p}} \frac{1}{\mathbf{p}^2+m^2}, \end{aligned} \tag{10.C.3}$$

where $m^2 = i\lambda g/N$. The critical coupling constant separating the low- and high-temperature phases is given by

$$g_c = \frac{d-2}{K_d\Lambda^{d-2}}. \tag{10.C.4}$$

g_c vanishes in two dimensions and the system is always in the disordered phase in agreement with the Mermin-Wagner theorem.

10.C.1 The high-temperature phase

In the high-temperature phase, $\sigma=0$ and m is determined by the equation

$$\frac{1}{g_c} - \frac{1}{g} = m^{d-2}K_d\int_0^{\Lambda/m} dx\,\frac{x^{d-3}}{x^2+1}. \tag{10.C.5}$$

For $d<4$, one can take the limit $\Lambda/m\to\infty$ since the integral is convergent. This gives a correlation length

$$\xi = m^{-1} \sim (g-g_c)^{-1/(d-2)} \tag{10.C.6}$$

for $g\to g_c$ so that $\nu = 1/(d-2)$. In two dimensions, the correlation length is determined from

$$\frac{1}{g} = \int_{\mathbf{p}} \frac{1}{\mathbf{p}^2+m^2} \simeq \frac{1}{2\pi}\ln\left(\frac{\Lambda}{m}\right), \tag{10.C.7}$$

which gives

$$\xi \sim \Lambda^{-1}\exp\left(\frac{2\pi}{g}\right). \tag{10.C.8}$$

ξ diverges exponentially for small g in agreement with the RG analysis of section 10.7.2 in the large-N limit.[122]

10.C.1.1 Upper critical dimension

At the upper critical dimension $d_c^+ = 4$,

$$\xi^{-2} = m^2 \sim \frac{2\tau/g}{K_4\ln\left(\frac{\Lambda^2K_4g}{\tau}\right)}, \tag{10.C.9}$$

where $\tau = g/g_c - 1$. The mean-field expectation $\xi\sim 1/\sqrt{\tau}$ is modified by logarithmic corrections.

[122] Without the factor N in (10.C.1), we would obtain $\xi\sim\Lambda^{-1}\exp\left(\frac{2\pi}{Ng}\right)$ in agreement with (10.306) for $N\to\infty$.

10.C.2 The low-temperature phase

In the low-temperature phase, $m = 0$ and the order parameter is determined by

$$\sigma^2 = 1 - g \int_{\mathbf{p}} \frac{1}{\mathbf{p}^2} = 1 - \frac{g}{g_c}, \tag{10.C.10}$$

which gives a critical exponent $\beta = 1/2$.

The propagator of $\boldsymbol{\pi}$ field is given by $g(\mathbf{p}) = g/N\mathbf{p}^2$. To obtain the propagator of the longitudinal field σ, we proceed as in section 10.B.1. In the limit $N \to \infty$, the effective action $\Gamma[\sigma, \lambda] = S[\sigma, \lambda]$ and

$$\Gamma^{(2)}(\mathbf{p}) = \begin{pmatrix} \dfrac{N\mathbf{p}^2}{g} & i\sigma \\ i\sigma & \frac{N}{2}\Pi(\mathbf{p}) \end{pmatrix}, \tag{10.C.11}$$

where $\Pi(\mathbf{p}) = \int_{\mathbf{q}} g_\pi(\mathbf{q}) g_\pi(\mathbf{p}+\mathbf{q})$. We deduce

$$\det \Gamma^{(2)}(\mathbf{p}) = \frac{1}{2} A_d g \left(|\mathbf{p}|^{d-2} + p_J^{d-2} \right) \tag{10.C.12}$$

for $|\mathbf{p}| \ll \Lambda$ and $d < 4$ (A_d is defined by (10.325)). The Josephson momentum scale is defined by

$$p_J = \left(\frac{2\sigma^2}{A_d g} \right)^{1/(d-2)} = \left[\frac{2}{A_d} \left(\frac{1}{g} - \frac{1}{g_c} \right) \right]^{1/(d-2)} \tag{10.C.13}$$

and vanishes with the critical exponent $\nu = 1/(d-2)$. By inverting $\Gamma^{(2)}(\mathbf{p})$, we obtain the propagator

$$G_{\sigma\sigma}(\mathbf{p}) \simeq \frac{g|\mathbf{p}|^{d-4}/N}{|\mathbf{p}|^{d-2} + p_J^{d-2}} = \begin{cases} \dfrac{g}{N\mathbf{p}^2} & \text{if} \quad |\mathbf{p}| \gg p_J, \\ \dfrac{g}{N} \dfrac{p_J^{2-d}}{|\mathbf{p}|^{4-d}} & \text{if} \quad |\mathbf{p}| \ll p_J. \end{cases} \tag{10.C.14}$$

Même rq pour phi4 theory?

If g is sufficiently close to g_c, then $p_J \ll \Lambda$ and the system is in the critical regime. One can then distinguish two regimes for the behavior of $G_{\sigma\sigma}(\mathbf{p})$: i) a Goldstone regime $|\mathbf{p}| \ll p_J$ characterized by a diverging longitudinal susceptibility $G_{\sigma\sigma}(\mathbf{p}) \sim 1/|\mathbf{p}|^{4-d}$, ii) a critical regime $|\mathbf{p}| \gg p_J$ where $G_{\sigma\sigma}(\mathbf{p}) \sim 1/|\mathbf{p}|^{2-\eta}$ (with $\eta = \mathcal{O}(1/N)$). On the other hand, if $p_J \gtrsim \Lambda$, the system is in a non-critical regime and the longitudinal propagator exhibits the behavior $G_{\sigma\sigma}(\mathbf{p}) \sim 1/|\mathbf{p}|^{4-d}$ for any value of the momentum. These results are similar to those obtained from the large-N limit of the $(\boldsymbol{\varphi}^2)^2$ theory. Note however that the Ginzburg momentum scale p_G does not show up in the NLσM. The same conclusion was reached from the RG analysis of section 10.7.2.

10.C.3 Gibbs free energy

The Gibbs free energy can be obtained as in section 10.B.2. In the large-N limit and for a uniform order parameter M,

$$\frac{1}{NV}\Gamma(M) = \frac{m^2}{2g}(M^2 - 1) + \frac{1}{2}\int_{\mathbf{p}} \left[\ln\left(\frac{\mathbf{p}^2 + m^2}{g} \right) - \ln\left(\frac{\mathbf{p}^2}{g} \right) \right], \tag{10.C.15}$$

where the last (M independent) term is introduced to make the Gibbs free energy finite. Using (10.B.30) for $d<4$, one obtains

$$\frac{1}{NV}\Gamma(M) = \frac{m^2}{2g}(M^2-1) + \frac{m^2}{2}\left(\frac{1}{g_c} - \frac{1}{g}\right) - \frac{\tilde K_d}{d} m^d \tag{10.C.16}$$

for small m, where $\tilde K_d$ is defined by (10.116). By requiring $\Gamma(M)$ to be minimum wrt m^2, one deduces

$$m^2 = \begin{cases} \left(\frac{M^2+\tau}{g\tilde K_d}\right)^{2/(d-2)} & \text{if} \quad M^2+\tau \geq 0, \\ 0 & \text{if} \quad M^2+\tau \leq 0. \end{cases} \tag{10.C.17}$$

This yields

$$\frac{1}{NV}\Gamma(M) = \frac{d-2}{2d}\tilde K_d^{2/(d-2)}\left(\frac{M^2+\tau}{g}\right)^{d/(d-2)} \tag{10.C.18}$$

in the high-temperature phase ($\tau>0$) and

$$\frac{1}{NV}\Gamma(M) = \Theta(M^2-M_0^2)\frac{d-2}{2d}\tilde K_d^{2/(d-2)}\left(\frac{M^2-M_0^2}{g}\right)^{d/(d-2)} \tag{10.C.19}$$

in the low-temperature phase ($\tau<0$), where $M_0=\sqrt{-\tau}$ is equal to the saddle-point value of the σ field [Eq. (10.C.10)]. The results (10.C.18) and (10.C.19) are similar to those obtained in the large-N limit of the $(\boldsymbol{\varphi}^2)^2$ theory (Sec. 10.B.2).

Guide to the bibliography

- In addition to Wilson's original papers [1,2], there are many reviews [3–6] and books [7–16] with a detailed presentation of the renormalization group (Ref. [7] contains a general introduction to phase transitions). Many of these references discuss the $\epsilon = 4 - d$ expansion as well as the $1/N$ expansion.
- The field theoretical approach to critical phenomena is discussed in Refs. [11, 17, 18].
- For a discussion of the NLσM, see Refs. [7, 15, 17, 18, 28–32, 51].
- The BKT phase transitions [33–37] is reviewed e.g. in [7, 15, 38].
- The functional renormalization group (FRG) is discussed at length in chapter 11.

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