Chapter 13

Interacting bosons

Contents

In chapter 7 we discussed superfluidity in a dilute Bose gas in the framework of the Bogoliubov theory. We found a low-energy mode with linear dispersion (the Bogoliubov sound mode) and computed the equation of state. In this chapter we show that perturbation theory beyond the Bogoliubov theory is plagued with infrared divergences (Sec. 13.1.2). Even though it allows us to derive the equation of state in the low-density limit, the Bogoliubov theory violates some exact identities satisfied by the self-energy and does not enable us to obtain the infrared behavior of the one-particle propagator (Sec. 13.1.3). In particular, it misses the divergence of the longitudinal propagator. In this chapter, we discuss two different approaches which do not suffer from the shortcomings of the Bogoliubov theory:

Popov's hydrodynamic theory (Sec. 13.2) and the non-perturbative renormalization group (NPRG) (Sec. 13.3).[1] In section 13.4, we use the NPRG to compute the superfluid transition temperature to leading order in the interactions. Except in this last section, and unless otherwise specified, we focus on the zero-temperature superfluid phase.

13.1 Breakdown of Bogoliubov's theory

In section 7.4, we computed the equation of state of a Bose gas and other thermodynamic quantities to one-loop order (Bogoliubov theory). To do so, it was sufficient to compute the self-energy to leading (zero-loop) order. In this section, we focus on the self-energy and show that in the superfluid state perturbation theory breaks down at one-loop order. Predictions of the Bogoliubov theory regarding the infrared behavior of the one-particle Green function are therefore not correct.[2]

Interacting bosons are described by the (Euclidean) action

$$S = \int_0^\beta d\tau \int d^d r \left[\psi^* \left(\partial_\tau - \mu - \frac{\boldsymbol{\nabla}^2}{2m} \right) \psi + \frac{g}{2} (\psi^* \psi)^2 \right], \tag{13.1}$$

where $\psi(x)$ is a bosonic (complex) field. The interaction is assumed to be local in space and the model is regularized by a momentum cutoff Λ. We consider a space dimension $d > 1$ and restrict ourselves to zero temperature ($T = 1/\beta \to 0$).

It is convenient to introduce the two-component field

$$\Psi(x) = \begin{pmatrix} \psi(x) \\ \psi^*(x) \end{pmatrix}, \quad \Psi^\dagger(x) = \big(\psi^*(x), \psi(x)\big), \tag{13.2}$$

where $x = (\mathbf{r}, \tau)$. The one-particle (connected) propagator becomes a 2×2 matrix whose inverse in Fourier space is given by

$$\mathcal{G}^{-1}(p) = \begin{pmatrix} i\omega + \mu - \epsilon_{\mathbf{p}} - \Sigma_{\rm n}(p) & -\Sigma_{\rm an}(p) \\ -\Sigma_{\rm an}^*(p) & -i\omega + \mu - \epsilon_{\mathbf{p}} - \Sigma_{\rm n}(-p) \end{pmatrix} \tag{13.3}$$

(with $p = (\mathbf{p}, i\omega)$ and ω a bosonic Matsubara frequency), where $\Sigma_{\rm n}$ and $\Sigma_{\rm an}$ are the normal and anomalous self-energies, respectively, and $\epsilon_{\mathbf{p}} = \mathbf{p}^2/2m$.[3]

Alternatively, we can write the boson field

$$\psi(x) = \frac{1}{\sqrt{2}} [\psi_1(x) + i\psi_2(x)] \tag{13.4}$$

in terms of two real fields ψ_1 and ψ_2 and consider the (connected) propagator $G_{ij}(x, x') = -\langle \psi_i(x) \psi_j(x') \rangle_c$. The inverse propagator $G_{ij}^{-1}(p)$ then reads

$$G^{-1}(p) = \begin{pmatrix} -\epsilon_{\mathbf{p}} + \mu - \Sigma_{11}(p) & -\omega - \Sigma_{12}(p) \\ \omega - \Sigma_{21}(p) & -\epsilon_{\mathbf{p}} + \mu - \Sigma_{22}(p) \end{pmatrix}. \tag{13.5}$$

[1]The NPRG approach is used in the following chapter to study strongly correlated bosons and the superfluid–Mott-insulator transition in the framework of the Bose-Hubbard model.

[2]More precisely, we shall see that the predictions of the Bogoliubov theory are essentially correct for the transverse propagator but not for the longitudinal one.

[3]See Sec. 1.7.2.

Figure 13.1: Normal and anomalous self-energies to zero-loop order [Eq. (13.11)]. The zigzag line stands for ψ_0 or ψ_0^* and the dot for the interaction vertex g.

The self-energies in (13.3) and (13.5) are related by

$$\begin{aligned}
\Sigma_{11}(p) &= \frac{1}{2}[\Sigma_{\rm n}(p) + \Sigma_{\rm n}(-p)] + \Re[\Sigma_{\rm an}(p)], \\
\Sigma_{22}(p) &= \frac{1}{2}[\Sigma_{\rm n}(p) + \Sigma_{\rm n}(-p)] - \Re[\Sigma_{\rm an}(p)], \\
\Sigma_{12}(p) &= \frac{i}{2}[\Sigma_{\rm n}(p) - \Sigma_{\rm n}(-p)] + \Im[\Sigma_{\rm an}(p)], \\
\Sigma_{21}(p) &= -\frac{i}{2}[\Sigma_{\rm n}(p) - \Sigma_{\rm n}(-p)] + \Im[\Sigma_{\rm an}(p)].
\end{aligned} \tag{13.6}$$

Note that $\Sigma_{\rm an}(p)$ is real when the superfluid order parameter $\langle\psi(x)\rangle$ is real.

13.1.1 Bogoliubov's theory

The Bogoliubov theory is a Gaussian fluctuation theory about the mean-field solution (Sec. 7.4). The latter is obtained from a saddle-point approximation of the partition function. For a uniform and time-independent solution $\psi(x) = \psi_0$,

$$S_{\rm MF} = \beta V \left(-\mu n_0 + \frac{g}{2} n_0^2 \right), \tag{13.7}$$

where $n_0 = |\psi_0|^2$ is the condensate density (at the mean-field level). Without loss of generality, we can take ψ_0 real. Requiring $S_{\rm MF}$ to be stationary, $\partial S_{\rm MF}/\partial n_0 = 0$, we obtain

$$\mu = g n_0. \tag{13.8}$$

To include Gaussian fluctuations about the mean-field solution, we expand the action to quadratic order in the fluctuation field $\psi'(x) = \psi(x) - \psi_0$,

$$\begin{aligned}
S[\psi'^*, \psi'] = S_{\rm MF} + \sum_p \Big\{ & \psi'^*(p) \left(-i\omega - \mu + \epsilon_{\mathbf p} + 2gn_0\right) \psi'(p) \\
& + \frac{gn_0}{2} [\psi'(-p)\psi'(-p) + \text{c.c.}] \Big\}.
\end{aligned} \tag{13.9}$$

The action can be put in the form

$$S[\psi'^*, \psi'] = S_{\rm MF} - \frac{1}{2} \sum_p (\psi'^*(p), \psi'(-p)) \mathcal{G}^{-1}(p) \begin{pmatrix} \psi'(p) \\ \psi'^*(-p) \end{pmatrix}, \tag{13.10}$$

where $\mathcal{G}^{-1}(p)$ is the inverse propagator (13.3) with the zero-loop self-energies (Fig. 13.1)

$$\Sigma_{\rm n}^{(0)}(p) = 2gn_0 = 2\mu, \qquad \Sigma_{\rm an}^{(0)}(p) = gn_0 = \mu, \tag{13.11}$$

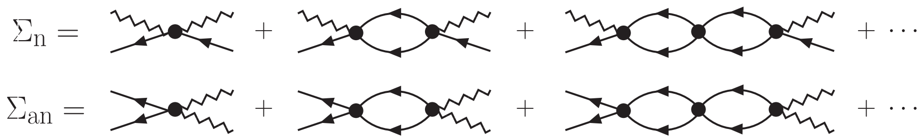

Figure 13.2: Normal and anomalous self-energies obtained from figure 13.1 by replacing the bare interaction vertex g by a renormalized vertex g_R obtained by summing bubble diagrams.

or, equivalently,

$$\Sigma^{(0)}_{11}(p) = 3\mu, \qquad \Sigma^{(0)}_{22}(p) = \mu, \qquad \Sigma^{(0)}_{12}(p) = 0. \tag{13.12}$$

Equations (13.11) were previously obtained in section 7.4. They yield the propagators

$$\begin{aligned} G^{(0)}_{\rm n}(p) &= -\langle \psi(p)\psi^*(p)\rangle_c = \frac{-i\omega - \epsilon_{\mathbf p} - \mu}{\omega^2 + E^2_{\mathbf p}}, \\ G^{(0)}_{\rm an}(p) &= -\langle \psi(p)\psi(-p)\rangle_c = \frac{\mu}{\omega^2 + E^2_{\mathbf p}}, \end{aligned} \tag{13.13}$$

where $E_{\mathbf p} = [\epsilon_{\mathbf p}(\epsilon_{\mathbf p} + 2\mu)]^{1/2}$ is the Bogoliubov quasi-particle excitation energy. When $|\mathbf p|$ is larger than the healing momentum $p_h = (2m\mu)^{1/2}$, the spectrum $E_{\mathbf p} \simeq \epsilon_{\mathbf p} + \mu$ is particle-like, whereas it becomes sound-like for $|\mathbf p| \ll p_h$ with a velocity $c = \sqrt{\mu/m}$ ($p_h = \sqrt{2}mc$).[4] In the weak-coupling limit, we can neglect (to leading order) the condensate depletion, i.e. $n_0 \simeq \bar n$ and $\mu \simeq g\bar n$ ($\bar n$ is the mean boson density), so that p_h can equivalently be defined as $p_h = (2gm\bar n)^{1/2}$. In the hydrodynamic regime $|\mathbf p| \ll p_h$,

$$G^{(0)}_{11}(p) = -\frac{\epsilon_{\mathbf p}}{\omega^2 + c^2\mathbf p^2}, \quad G^{(0)}_{22}(p) = -\frac{2\mu}{\omega^2 + c^2\mathbf p^2}, \quad G^{(0)}_{12}(p) = \frac{\omega}{\omega^2 + c^2\mathbf p^2}. \tag{13.14}$$

Note that in the Bogoliubov approximation, the occurrence of a linear spectrum at low energy (which implies superfluidity according to Landau's criterion, see Sec. 7.1), is due to $\Sigma_{\rm an}(0)$ being nonzero.

In section 7.4, all thermodynamic quantities were expressed in terms of the s-wave scattering length a with no reference to the (bare) interaction g. This was achieved by eliminating all dependencies on the ultraviolet cutoff Λ and the interaction g in favor of the scattering length $a \equiv a(g, \Lambda)$. A similar procedure can be followed here. We may replace g by a renormalized interaction g_R defined by the sum of the bubble diagrams (Fig. 13.2). In the low-density limit, these diagrams can be evaluated in the vacuum limit ($\mu = 0$), which gives

$$\frac{1}{g_R} = \frac{1}{g} + \int_p \frac{1}{(i\omega - \epsilon_{\mathbf p})(-i\omega - \epsilon_{\mathbf p})} = \frac{1}{g} + \int_{\mathbf p} \frac{1}{2\epsilon_{\mathbf p}}. \tag{13.15}$$

In a three-dimensional system $g_R = 4\pi a/m$ is simply related to the scattering length a, and we reproduce the results of the Bogoliubov theory to leading order in $ma^2\mu$: $\mu = g_R\bar n = 4\pi a\bar n/m$, $c = \sqrt{g_R\bar n/m} = \sqrt{4\pi a\bar n}/m$, etc. In a two-dimensional system, g_R vanishes logarithmically in vacuum. At finite density, the vanishing is cut off by the nonzero chemical potential. We will see that $g_R(\mu) \simeq (2\pi/m)/|\ln\sqrt{2ma^2\mu}|$ (Sec. 13.3.3) which again gives the results of Bogoliubov's theory.

[4] p_h^{-1} is nothing but the healing length ξ_h introduced in Sec. 7.3.1.

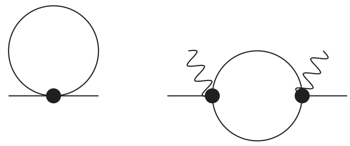

Figure 13.3: One-loop correction $\Sigma^{(1)}$ to the self-energy.

13.1.2 Infrared divergences and the Ginzburg scale

Let us now consider the one-loop correction $\Sigma^{(1)}_{ij}(p)$ to the Bogoliubov result $\Sigma^{(0)}_{ij}(p)$ (we use the representation (13.4) of the boson field) (Fig. 13.3). The first diagram in figure 13.3,[5]

$$-\frac{g}{2}\int_q \sum_{i_1,i_2} G_{i_1i_2}(q)(\delta_{i,j}\delta_{i_1,i_2}+\delta_{i,i_1}\delta_{j,i_2}+\delta_{i,i_2}\delta_{j,i_1}), \tag{13.16}$$

is always finite. For $d\leq 3$ the second one,

$$-\frac{1}{2}g^2n_0\int_q \sum_{i_1\cdots i_4} G_{i_1i_2}(q)G_{i_3i_4}(p+q)(\delta_{i,1}\delta_{i_2,i_3}+\delta_{i_2,1}\delta_{i,i_3}+\delta_{i_3,1}\delta_{i,i_2}) \\ \times(\delta_{j,1}\delta_{i_4,i_1}+\delta_{i_4,1}\delta_{j,i_1}+\delta_{i_1,1}\delta_{j,i_4}), \tag{13.17}$$

gives a divergent contribution when the two internal lines correspond to transverse fluctuations ($i_1=i_2=2$ and $i_3=i_4=2$). This is possible only for Σ_{11}, so that Σ_{22} is finite at the one-loop level. Thus the normal and anomalous self-energies exhibit the same divergence,

$$\Sigma^{(1)}_{\rm n}(p)\simeq\Sigma^{(1)}_{\rm an}(p)\simeq-\frac{1}{2}g^2n_0\int_q G^{(0)}_{22}(q)G^{(0)}_{22}(p+q), \tag{13.18}$$

where we use the notation $q=(\mathbf{q},i\omega')$ and $\int_q=\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\int_{\mathbf{q}}$. For small p, the main contribution to the integral in (13.18) comes from momenta $|\mathbf{q}|\lesssim p_h$ and frequencies $|\omega'|\lesssim cp_h$, so that we can use (13.14) to obtain

$$\Sigma^{(1)}_{\rm n}(p)\simeq\Sigma^{(1)}_{\rm an}(p)\simeq-2\frac{g^4n_0^3}{c^3}\int_{\mathbf{Q}}\frac{1}{\mathbf{Q}^2(\mathbf{Q}+\mathbf{P})^2}, \tag{13.19}$$

where $\mathbf{Q}=(\mathbf{q},\omega'/c)$ and $\mathbf{P}=(\mathbf{p},\omega/c)$ are $(d+1)$-dimensional vectors. The momentum integral in (13.19) is restricted by $|\mathbf{Q}|\lesssim p_h$ and is given by (for $p\to 0$)

$$\int_{\mathbf{Q}}\frac{1}{\mathbf{Q}^2(\mathbf{Q}+\mathbf{P})^2}=\begin{cases} A_{d+1}(|\mathbf{p}|^2+\omega^2/c^2)^{(d-3)/2} & \text{if} \quad d<3, \\ A_4\ln\dfrac{p_h}{\sqrt{\mathbf{p}^2+\omega^2/c^2}} & \text{if} \quad d=3, \end{cases} \tag{13.20}$$

where

$$A_d=\begin{cases} -\dfrac{2^{1-d}\pi^{1-d/2}}{\sin(\pi d/2)}\dfrac{\Gamma(d/2)}{\Gamma(d-1)} & \text{if} \quad d<4, \\ \dfrac{1}{8\pi^2} & \text{if} \quad d=4, \end{cases} \tag{13.21}$$

[5]We do not write explicitly the convergence factors $e^{\pm i\omega 0^+}$ in Eq. (13.16) (see chapter 1).

(see Eq. (10.324)).

The one-loop self-energy correction (13.19) diverges for $(\mathbf{p},\omega)\to 0$ when $d\le 3$ so that the perturbation expansion about the Bogoliubov approximation breaks down. By comparing the one-loop result to the zero-loop one, $|\Sigma_{\rm n}^{(1)}(p)|\sim\Sigma_{\rm n}^{(0)}(p)$ or $|\Sigma_{\rm an}^{(1)}(p)|\sim\Sigma_{\rm an}^{(0)}(p)$ for $|\mathbf{p}|=p_G$ and $|\omega|=cp_G$, one can define a characteristic (Ginzburg) momentum scale p_G above which the Bogoliubov approximation remains valid,

$$p_G\sim\begin{cases}(A_{d+1}gmp_h)^{1/(3-d)} & \text{if} \quad d<3,\\ p_h\exp\left(-\dfrac{1}{\sqrt{2}A_4gmp_h}\right) & \text{if} \quad d=3.\end{cases} \tag{13.22}$$

This result can be rewritten as

$$p_G\sim\begin{cases}p_h(A_{d+1}\tilde g^{d/2})^{1/(3-d)} & \text{if} \quad d<3,\\ p_h\exp\left(-\dfrac{1}{A_4\sqrt{2}\tilde g^{3/2}}\right) & \text{if} \quad d=3,\end{cases} \tag{13.23}$$

where

$$\tilde g=gm\bar n^{1-2/d}\sim\left(\frac{p_h}{\bar n^{1/d}}\right)^2\sim\begin{cases}(p_G/p_h)^{6/d-2} & \text{if} \quad d<3,\\ [\ln(p_h/p_G)]^{-2/3} & \text{if} \quad d=3,\end{cases} \tag{13.24}$$

is the dimensionless coupling constant obtained by comparing the mean interaction energy per particle $g\bar n$ to the typical kinetic energy $1/m\bar r^2$ where $\bar r\sim\bar n^{-1/d}$ is the mean distance between particles. A superfluid is weakly correlated if $\tilde g\ll 1$, i.e. $p_G\ll p_h\ll\bar n^{1/d}$. In this case, the Bogoliubov theory applies to a large part of the spectrum where the dispersion is linear (i.e. $|\mathbf{p}|\lesssim p_h$) and breaks down only at very small momenta $|\mathbf{p}|\lesssim p_G\ll p_h$.[6] When the dimensionless coupling $\tilde g$ becomes of order unity, the three characteristic momentum scales $p_G\sim p_h\sim\bar n^{1/d}$ become of the same order. The momentum range $[p_G,p_h]$ where the linear spectrum can be described by the Bogoliubov theory is then suppressed. We expect the strong-coupling regime $\tilde g\gtrsim 1$ to be governed by a single characteristic momentum scale, namely $\bar n^{1/d}$.[7]

13.1.3 Hugenholtz-Pines theorem and infrared limit of the self-energy

Although the one-loop correction $\Sigma^{(1)}$ diverges when $p\to 0$ for $d\le 3$, it is nevertheless possible to obtain the exact value of $\Sigma(p=0)$ using the U(1) symmetry of the action, i.e. the invariance under the field transformation

$$\psi(x)\to e^{i\theta}\psi(x) \quad \text{and} \quad \psi^*(x)\to e^{-i\theta}\psi^*(x). \tag{13.25}$$

Let us consider the effective action

$$\Gamma[\phi]=-\ln Z[J_1,J_2]+\int_0^\beta d\tau\int d^dr(J_1\phi_1+J_2\phi_2), \tag{13.26}$$

where J_i is an external (real) source which couples linearly to the boson field ψ_i and $\phi_i(x)=\langle\psi_i(x)\rangle$ the superfluid order parameter (see Secs. 1.6.2 and 1.7.2). The U(1) symmetry

[6] In the next sections, we shall see that the weakly correlated superfluid bears many similarities with the ordered phase of the classical $O(N)$ model away from the critical regime (Secs. 10.7.3 and 11.3.3). The healing scale p_h plays the role of the hydrodynamic scale p_c introduced in Sec. 10.7.3.

[7] This situation is realized in the Bose-Hubbard model near a quantum multicritical point where the transition occurs at fixed density (chapter 14).

of the action implies that $\Gamma[\phi]$ is invariant under a uniform rotation of the vector field $(\phi_1(x), \phi_2(x))^T$ [Eq. (13.25)]. For an infinitesimal rotation angle θ, this yields

$$\int_0^\beta d\tau \int d^d r \sum_{i,j} \frac{\delta \Gamma[\phi]}{\delta \phi_i(x)} \epsilon_{ij} \phi_j(x) = 0, \tag{13.27}$$

where ϵ_{ij} is the totally antisymmetric tensor.

Taking the functional derivative $\delta/\delta\phi_l(y)$ and setting $\phi_i(x) = \delta_{i,1}\sqrt{2n_0}$ (which corresponds to $\langle \psi(x) \rangle = \sqrt{n_0}$ real) leads to

$$\Gamma^{(2)}_{2l}(p=0) = 0, \tag{13.28}$$

where $\Gamma^{(2)}_{ij}$ denotes the two-point vertex (Sec. 1.6.2). $\Gamma^{(2)}$ is related to the one-particle propagator by the (matrix) equation $\Gamma^{(2)} = -G^{-1}$. For $l = 2$, equation (13.28) yields the Hugenholtz-Pines theorem

$$\Gamma^{(2)}_{22}(p=0) = \Sigma_{22}(p=0) - \mu = \Sigma_{\rm n}(p=0) - \Sigma_{\rm an}(p=0) - \mu = 0, \tag{13.29}$$

which is nothing but the Goldstone theorem associated with the spontaneously broken U(1) symmetry in the superfluid state.

If we now take the second-order functional derivative $\delta^{(2)}/\delta\phi_l(y)\delta\phi_m(z)$ of (13.27) and set $\phi_i(x) = \delta_{i,1}\sqrt{2n_0}$, we obtain the Ward identity

$$\sum_i \Gamma^{(2)}_{im}(y,z)\epsilon_{il} + \sum_i \Gamma^{(2)}_{il}(z,y)\epsilon_{im} - \sqrt{2n_0}\int_0^\beta d\tau \int d^d r\, \Gamma^{(3)}_{2lm}(x,y,z) = 0. \tag{13.30}$$

Integrating over y and z and setting $l = 2$ and $m = 1$, we deduce

$$\Gamma^{(3)}_{122}(0,0,0) = \frac{1}{\sqrt{\beta V}} \frac{\Gamma^{(2)}_{11}(0,0)}{\sqrt{2n_0}} \tag{13.31}$$

in Fourier space,[8] making use of (13.29).

Let us now consider the exact diagrammatic representation of the self-energy shown in figure 13.4 (see Sec. 1.6.2). We know from perturbation theory that the third diagram is potentially dangerous when the two internal lines correspond to transverse fluctuations. We therefore write the self-energy Σ_{11} as

$$\Sigma_{11}(p) = \tilde{\Sigma}_{11}(p) - g\sqrt{\frac{n_0}{2\beta V}} \sum_q G_{22}(q) G_{22}(p+q) \Gamma^{(3)}_{122}(-p,-q,p+q), \tag{13.32}$$

where $\tilde{\Sigma}_{11}(p)$ denotes the regular part of the self-energy (i.e. the part that does not contain pairs of lines corresponding to $G_{22}G_{22}$). If we assume that the transverse propagator $G_{22}(q) \sim -1/(\omega^2 + c^2\mathbf{q}^2)$ at low energies (this result will be shown in the following sections), the integral $\int_q G_{22}(q)^2$ is infrared divergent for $d \leq 3$. To obtain a finite self-energy $\Sigma_{11}(p=0)$, one must therefore require $\Gamma^{(3)}_{122}(0,0,0)$ to vanish. The Ward identity (13.31) then implies $\Gamma^{(2)}_{11}(p=0) = 0$ and in turn

$$\begin{aligned} \Sigma_{\rm n}(p=0) &= \mu + \frac{1}{2}\left[\Gamma^{(2)}_{11}(p=0) + \Gamma^{(2)}_{22}(p=0)\right] = \mu, \\ \Sigma_{\rm an}(p=0) &= \frac{1}{2}\left[\Gamma^{(2)}_{11}(p=0) - \Gamma^{(2)}_{22}(p=0)\right] = 0, \end{aligned} \tag{13.33}$$

[8] In Eq. (13.31), $\Gamma^{(2)}_{11}(0,0) \equiv \Gamma^{(2)}_{11}(p_1=0, p_2=0)$, etc.

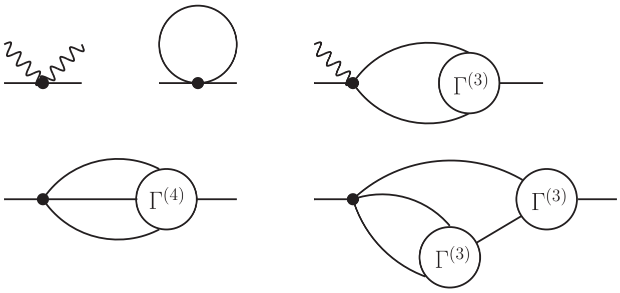

Figure 13.4: Exact diagrammatic representation of the self-energy in terms of the three- and four-leg vertices $\Gamma^{(3)}$ and $\Gamma^{(4)}$. Dots represent the bare interaction, zigzag lines the order parameter, and solid lines the exact propagator.

using the Hugenholtz-Pines theorem (13.29). In marked contrast with Bogoliubov's approximation, we find that the anomalous self-energy $\Sigma_{\rm an}(p)$ vanishes for $p \to 0$ (Nepomnyashchii identity). We shall see in the next section that this result is not incompatible with the existence of a sound mode with linear dispersion (Sec. 13.2.3).

13.2 Popov's hydrodynamic theory

Popov's hydrodynamic theory is based on the phase-density representation of the boson field, $\psi = \sqrt{n} e^{i\theta}$, and bears some similarities with the analysis of the $(\boldsymbol{\varphi}^2)^2$ theory using the amplitude-direction representation (Sec. 10.7.3). It is free of infrared divergence and yields a simple derivation of the infrared behavior of the normal and anomalous propagators of the boson field.

13.2.1 Perturbative approach

In terms of the density and phase fields, the action (13.1) reads

$$S[n,\theta] = \int_0^\beta d\tau \int d^dr \left\{ in\dot\theta + \frac{n}{2m}(\boldsymbol{\nabla}\theta)^2 + \frac{(\boldsymbol{\nabla}n)^2}{8mn} - \mu n + \frac{g}{2}n^2 \right\}. \tag{13.34}$$

At the saddle-point level, $n(x) = \bar n = \mu/g$ and $\theta = \text{const}$. Expanding the action to second order in $\delta n = n - \bar n$, $\dot\theta$ and $\boldsymbol{\nabla}\theta$, we obtain

$$S[\delta n,\theta] = \int_0^\beta d\tau \int d^dr \left\{ i\delta n\dot\theta + \frac{\bar n}{2m}(\boldsymbol{\nabla}\theta)^2 + \frac{(\boldsymbol{\nabla}\delta n)^2}{8m\bar n} + \frac{g}{2}(\delta n)^2 \right\}. \tag{13.35}$$

The equations of motion deduced from the action (13.35) have been discussed in section 7.3.1. The dynamics of the system can also be deduced from the correlation functions

$$\begin{aligned} G^0_{nn}(p) &= \langle \delta n(p)\delta n(-p)\rangle_0 = \frac{\bar n}{m}\frac{\mathbf{p}^2}{\omega^2 + E^2_{\mathbf{p}}}, \\ G^0_{n\theta}(p) &= \langle \delta n(p)\theta(-p)\rangle_0 = -\frac{\omega}{\omega^2 + E^2_{\mathbf{p}}}, \\ G^0_{\theta\theta}(p) &= \langle \theta(p)\theta(-p)\rangle_0 = \frac{\frac{\mathbf{p}^2}{4m\bar n} + g}{\omega^2 + E^2_{\mathbf{p}}}, \end{aligned} \tag{13.36}$$

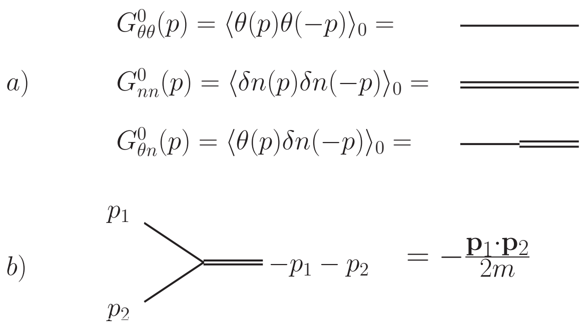

Figure 13.5: (a) Diagrammatic representation of the propagators obtained from the Gaussian action (13.35). (b) Interaction vertex defined by equation (13.39).

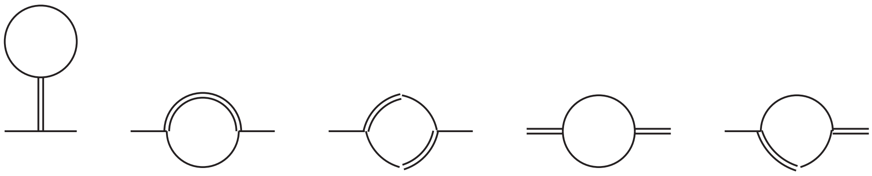

Figure 13.6: Self-energy corrections to the propagators $G^0_{\theta\theta}$, G^0_{nn} and $G^0_{\theta n}$.

where $E_{\mathbf{p}}$ is the Bogoliubov quasi-particle excitation energy defined in section 13.1.1. The average $\langle\cdots\rangle_0$ are taken with the Gaussian action (13.35). In the hydrodynamic regime $|\mathbf{p}| \ll p_h = \sqrt{2gm\bar n}$,

$$G^0_{nn}(p) = \frac{\bar n}{m}\frac{\mathbf{p}^2}{\omega^2 + c^2\mathbf{p}^2}, \quad G^0_{n\theta}(p) = -\frac{\omega}{\omega^2 + c^2\mathbf{p}^2}, \quad G^0_{\theta\theta}(p) = \frac{mc^2}{\bar n}\frac{1}{\omega^2 + c^2\mathbf{p}^2}, \tag{13.37}$$

where $c = \sqrt{\mu/m} = \sqrt{g\bar n/m}$ is the Bogoliubov sound mode velocity ($p_h = \sqrt{2}mc$).

Contrary to the perturbation theory discussed in section 13.1, the perturbation theory based on the phase-density representation is free of infrared divergences. Let us consider the terms not included in the Gaussian action (13.35),

$$\begin{aligned} S_{\rm int}[\delta n, \theta] &= \int_0^\beta d\tau \int d^d r \left\{ \frac{\delta n}{2m}(\boldsymbol{\nabla}\theta)^2 + \frac{(\boldsymbol{\nabla}\delta n)^2}{8m}\left(\frac{1}{\delta n + \bar n} - \frac{1}{\bar n}\right)\right\} \\ &= \int_0^\beta d\tau \int d^d r \left\{ \frac{\delta n}{2m}(\boldsymbol{\nabla}\theta)^2 - \frac{\delta n(\boldsymbol{\nabla}\delta n)^2}{8m\bar n^2} + \cdots \right\}. \end{aligned} \tag{13.38}$$

The most important term in $S_{\rm int}$ is the first one since it involves the (Goldstone) phase variable. Thus, in Fourier space, we obtain

$$S_{\rm int}[\delta n, \theta] = -\frac{1}{\sqrt{\beta V}} \sum_{p_1,p_2} \frac{\mathbf{p}_1\cdot\mathbf{p}_2}{2m}\theta(p_1)\theta(p_2)\delta n(-p_1-p_2) + \cdots \tag{13.39}$$

The basic ingredients (propagators and interaction vertex) appearing in the perturbation theory are shown diagrammatically in figure 13.5. The one-loop self-energy corrections to $G^0_{\theta\theta}$, G^0_{nn} and $G^0_{\theta n}$ are shown in figure 13.6. It is easy to show that the perturbation theory has no infrared divergences. The singularities of the propagators are canceled out by the factor $\mathbf{p}_1\cdot\mathbf{p}_2/2m$ coming from the vertex. Let us consider for example the one-loop correction to $G^0_{nn}(p)$ (which is the most dangerous one since it involves $G^0_{\theta\theta}G^0_{\theta\theta}$). It is proportional to

$$\frac{1}{m^2\beta V}\sum_{p'}[\mathbf{p}'\cdot(\mathbf{p}+\mathbf{p}')]^2 G^0_{\theta\theta}(p')G^0_{\theta\theta}(p+p') \tag{13.40}$$

and is manifestly convergent.

Phase-only action and Beliaev's damping?

13.2.2 Exact hydrodynamic description

Equations (13.37) are in fact exact in the low-energy limit $|\mathbf{p}|,|\omega|/c \ll p_h$ provided that c is the exact sound mode velocity and $\bar n$ the actual mean density (which may differ from μ/g). Let us consider the effective action

$$\Gamma[n,\theta] = -\ln Z[J_n,J_\theta] + \int_0^\beta d\tau \int d^dr (J_n n + J_\theta\theta) \tag{13.41}$$

defined as the Legendre transform of the free energy $-\ln Z[J_n,J_\theta]$. J_n and J_θ are external sources that couple to n and θ.[9] Since the microscopic action (13.34) is invariant in the (semilocal) transformation

$$\psi(x)\to\psi(x)e^{i\alpha(\tau)}, \quad \psi^*(x)\to\psi^*(x)e^{-i\alpha(\tau)}, \quad \mu\to\mu+i\partial_\tau\alpha(\tau) \tag{13.42}$$

($\alpha(\tau)$ is an arbitrary time-dependent phase), the effective action $\Gamma[n,\theta]$ must be invariant in the transformation

$$\theta(x)\to\theta(x)+\alpha(\tau), \qquad \mu\to\mu+i\partial_\tau\alpha(\tau) \tag{13.43}$$

and the chemical potential must appear in the combination $i\partial_\tau\theta-\mu$. Moreover, at zero temperature, Galilean invariance implies the invariance of the effective action in the transformation

$$n'(x') = n(x), \qquad \theta'(x') = \theta(x) - \frac{i}{2}m\mathbf{v}^2\tau - m\mathbf{v}\cdot\mathbf{r} \tag{13.44}$$

(see Sec. 2.2.5), where $\mathbf{r}'=\mathbf{r}+i\mathbf{v}\tau$ and $\tau'=\tau$. $n(x)$ and its space derivatives are invariant (but $\partial_\tau n$ is not) as well as $i\partial_\tau\theta+\frac{1}{2m}(\boldsymbol{\nabla}\theta)^2$.

Thus, including all second-order derivatives, the most general effective action compatible with symmetries reads[10]

$$\Gamma[n,\theta] = \int_0^\beta d\tau \int d^dr \biggl\{ \frac{Y(n)}{8m}(\boldsymbol{\nabla} n)^2 + U(n) + \sum_{p=1}^2 c_p(n)\Bigl[i\partial_\tau\theta-\mu+\frac{1}{2m}(\boldsymbol{\nabla}\theta)^2\Bigr]^p \biggr\}, \tag{13.45}$$

up to an additive (field-independent) term. $Y(n)$, $U(n)$ and $c_p(n)$ are arbitrary functions of n. By considering the mean boson density[11]

$$\bar n = \frac{1}{\beta V}\frac{d\ln[J_n,J_\theta]}{d\mu}\bigg|_{J_n=J_\theta=0} = -\frac{1}{\beta V}\frac{d\Gamma[n,\theta]}{d\mu}\bigg|_{n(x)=\bar n,\theta(x)=\text{const}}, \tag{13.46}$$

[9] To alleviate the notations, we use the same symbols (n and θ) for the density and phase fields in $S[n,\theta]$ and their average values in $\Gamma[n,\theta]$.

[10] $\boldsymbol{\nabla}^2\theta$ is also invariant in the (semilocal) U(1) and Galilean transformations but is odd under time-reversal symmetry.

[11] Since $\delta\Gamma[n,\theta]/\delta n(x)$ vanishes for $n(x)=\bar n$ and $\theta(x)=\text{const}$, $d\Gamma[\bar n,\theta]/d\mu = \partial\Gamma[\bar n,\theta]/\partial\mu$ where ∂_μ does not act on $\bar n\equiv\bar n(\mu)$.

we obtain $c_1(n) = n$ and $c_2(n) = 0$. We conclude that

$$\Gamma[n,\theta] = \int_0^\beta d\tau \int d^dr \biggl\{ \frac{Y(n)}{8m}(\boldsymbol{\nabla} n)^2 + U(n) + n \left[i\partial_\tau \theta - \mu + \frac{(\boldsymbol{\nabla}\theta)^2}{2m} \right] \biggr\} \tag{13.47}$$

to second order in derivatives.

From (13.47), we obtain the two-point vertex in constant fields $n(x) = \bar n$ and $\theta(x) =$ const (with $\bar n$ the actual boson density),

$$\Gamma^{(2)}(p) = \begin{pmatrix} \Gamma^{(2)}_{nn}(p) & \Gamma^{(2)}_{n\theta}(p) \\ \Gamma^{(2)}_{\theta n}(p) & \Gamma^{(2)}_{\theta\theta}(p) \end{pmatrix} = \begin{pmatrix} \frac{Y(\bar n)}{4m}\mathbf{p}^2 + U''(\bar n) & \omega \\ -\omega & \frac{\bar n}{m}\mathbf{p}^2 \end{pmatrix}. \tag{13.48}$$

By inverting $\Gamma^{(2)}(p)$, we recover the propagators (13.37) in the small momentum limit $|\mathbf{p}| \ll p_h = [2mU''(\bar n)/Y(\bar n)]^{1/2}$ but with a sound mode velocity c given by

$$c = \sqrt{\frac{\bar n U''(\bar n)}{m}}. \tag{13.49}$$

Noting that the compressibility $\kappa = \bar n^{-2} d\bar n/d\mu$ can also be expressed as[12]

$$\kappa = \frac{1}{\bar n^2 U''(\bar n)}, \tag{13.50}$$

we conclude that the Bogoliubov sound mode velocity c is equal to the macroscopic sound velocity $(m\bar n\kappa)^{-1/2}$.

The superfluid density can be defined as the rigidity of the system wrt a twist of the order parameter (see Eq. (7.41) in Sec. 7.2.2), i.e.

$$\Gamma^{(2)}_{\theta\theta}(\mathbf{p},0) = \frac{n_s}{m}\mathbf{p}^2 \qquad (\mathbf{p}\to 0). \tag{13.51}$$

From (13.48) we then deduce that the superfluid density $n_s = \bar n$ is given by the fluid density at zero temperature (see also Sec. 7.1).

13.2.3 Normal and anomalous propagators

To compute the propagator of the ψ field, we write

$$\psi(x) = \sqrt{n_0 + \delta n(x)}\, e^{i\theta(x)}, \tag{13.52}$$

where $n_0 = |\langle\psi(x)\rangle|^2 = |\langle\sqrt{n(x)}e^{i\theta(x)}\rangle|^2$ is the condensate density. For a weakly interacting superfluid at zero temperature, $n_0 \simeq \bar n$, and we expect the fluctuations δn to be small. Let us assume that the superfluid order parameter $\langle\psi(x)\rangle = \sqrt{n_0}$ is real. Transverse and longitudinal fluctuations are then expressed as

$$\begin{aligned} \delta\psi_2 &= \sqrt{2n_0}\,\theta + \cdots \\ \delta\psi_1 &= \frac{\delta n}{\sqrt{2n_0}} - \sqrt{\frac{n_0}{2}}\theta^2 + \cdots \end{aligned} \tag{13.53}$$

[12] Equation (13.50) is the usual expression of the compressibility of a system with free energy density $U(\bar n)$ in the canonical ensemble [Eq. (3.108)].

where the ellipses stand for subleading contributions to the low-energy behavior of the correlation functions. ψ_1 and ψ_2 refer to the real and imaginary parts of the boson field [Eq. (13.4)]. For the transverse propagator, we obtain

$$G_{22}(p) \simeq -2n_0 G_{\theta\theta}(p) = -\frac{2n_0 mc^2}{\bar{n}} \frac{1}{\omega^2 + c^2 \mathbf{p}^2} \tag{13.54}$$

to leading order in the hydrodynamic regime, while

$$G_{12}(p) \simeq -G_{n\theta}(p) = \frac{\omega}{\omega^2 + c^2 \mathbf{p}^2}. \tag{13.55}$$

The longitudinal propagator is given by

$$\begin{aligned} G_{11}(x) &= -\frac{1}{2n_0} G_{nn}(x) - \frac{n_0}{2} \langle \theta(x)^2 \theta(0)^2 \rangle_c \\ &= -\frac{1}{2n_0} G_{nn}(x) - n_0 G_{\theta\theta}(x)^2, \end{aligned} \tag{13.56}$$

where the second line is obtained using Wick's theorem (which is justified since the Goldstone (phase) mode is effectively non-interacting in the hydrodynamic limit). In Fourier space,

$$G_{11}(p) = -\frac{\bar{n}}{2mn_0} \frac{\mathbf{p}^2}{\omega^2 + c^2 \mathbf{p}^2} - n_0 G_{\theta\theta} \star G_{\theta\theta}(p), \tag{13.57}$$

where

$$G_{\theta\theta} \star G_{\theta\theta}(p) = \int_q G_{\theta\theta}(q) G_{\theta\theta}(p+q) \tag{13.58}$$

with the dominant contribution to the integral coming from momenta $|\mathbf{q}| \lesssim p_h$ and frequencies $|\omega'|/c \lesssim p_h$. Using (13.21), we find

$$G_{\theta\theta} \star G_{\theta\theta}(p) = \begin{cases} A_{d+1} c \left(\frac{m}{\bar{n}}\right)^2 \left(\mathbf{p}^2 + \frac{\omega^2}{c^2}\right)^{(d-3)/2} & \text{if} \quad d < 3, \\ \frac{A_4}{2} c \left(\frac{m}{\bar{n}}\right)^2 \ln\left(\frac{p_h^2}{\mathbf{p}^2 + \frac{\omega^2}{c^2}}\right) & \text{if} \quad d = 3. \end{cases} \tag{13.59}$$

By comparing the two terms in the rhs of (13.57), we recover the Ginzburg scale (13.22). For $|\mathbf{p}| \gg p_G$ or $|\omega|/c \gg p_G$, the last term in the rhs of (13.57) can be neglected and we reproduce the result of the Bogoliubov theory (noting that $\bar{n} \simeq n_0$),[13] while

$$G_{11}(p) \sim -\frac{1}{(\omega^2 + c^2 \mathbf{p}^2)^{(3-d)/2}} \quad \text{for} \quad |\mathbf{p}|, |\omega|/c \ll p_G \tag{13.60}$$

is dominated by phase fluctuations. The longitudinal susceptibility $G_{11}(\mathbf{p}, i\omega = 0) \sim -1/|\mathbf{p}|^{3-d}$ diverges for $\mathbf{p} \to 0$ in contrast to the Bogoliubov approximation where $G_{11}(\mathbf{p}, i\omega = 0) = -1/2mc^2$ for $|\mathbf{p}| \ll p_h$. The divergence of G_{11} is similar to that of the longitudinal propagator in the broken-symmetry phase of the classical O(N) model [Eq. (10.337)].[14]

[13] The Bogoliubov theory predicts $G_{11}(p) = -\epsilon_{\mathbf{p}}/(\omega^2 + c^2 \mathbf{p}^2)$ for $|\mathbf{p}| \ll p_h$.

[14] Equation (13.60) follows from $G_\parallel(\mathbf{p} \to 0) \sim -1/|\mathbf{p}|^{4-d}$ (longitudinal propagator in the ordered phase of the classical O(N) model) by replacing d by $d+1$ (which accounts for the imaginary-time dimension) and $\mathbf{p}^2$ by $\mathbf{p}^2 + \omega^2/c^2$. Note that the definition of G_{ij} in chapter 10 differs from that used in this chapter by a minus sign.

From these results, we deduce the hydrodynamic behavior of the normal propagator $G_{\rm n}(p) = -\langle \psi(p)\psi^*(p)\rangle$,

$$\begin{aligned} G_{\rm n}(p) &= \frac{1}{2}\left[G_{11}(p) - 2iG_{12}(p) + G_{22}(p)\right] \\ &= -\frac{n_0 mc^2}{\bar n}\frac{1}{\omega^2+c^2\mathbf{p}^2} - \frac{i\omega}{\omega^2+c^2\mathbf{p}^2} + \frac{1}{2}G_{11}(p), \end{aligned} \tag{13.61}$$

as well as that of the anomalous propagator $G_{\rm an}(p) = -\langle \psi(p)\psi(-p)\rangle$,

$$G_{\rm an}(p) = \frac{1}{2}\left[G_{11}(p) - G_{22}(p)\right] = \frac{n_0 mc^2}{\bar n}\frac{1}{\omega^2+c^2\mathbf{p}^2} + \frac{1}{2}G_{11}(p), \tag{13.62}$$

where $G_{11}(p)$ is given by (13.57). As will be shown in section 13.3 the leading order terms in (13.61) and (13.62) (coming from G_{22}) are exact. They coincide with the predictions of the Bogoliubov theory if n_0 is identified with $\bar n$. The $i\omega$ term is also obtained from the Bogoliubov theory but a factor $(mc^2/\bar n)dn_0/d\mu$ is missing in (13.61) (see Sec. 13.3.1). On the other hand, the Bogoliubov theory misses the divergence of the longitudinal propagator.

13.2.3.1 Normal and anomalous self-energies

To compute the self-energies $\Sigma_{\rm n}(p)$ and $\Sigma_{\rm an}(p)$, we use the relations

$$\begin{aligned} \Sigma_{\rm n}(p) &= G_0^{-1}(p) - \frac{G_{\rm n}(-p)}{G_{\rm n}(p)G_{\rm n}(-p) - G_{\rm an}(p)^2}, \\ \Sigma_{\rm an}(p) &= \frac{G_{\rm an}(p)}{G_{\rm n}(p)G_{\rm n}(-p) - G_{\rm an}(p)^2}, \end{aligned} \tag{13.63}$$

with $G_0^{-1}(p) = i\omega - \epsilon_{\mathbf{p}} + \mu$ and

$$\begin{aligned} G_{\rm n}(p)G_{\rm n}(-p) - G_{\rm an}(p)^2 &= G_{11}(p)G_{22}(p) + G_{12}(p)^2 \\ &= G_{22}(p)\left[n_0 G_{\theta\theta}\star G_{\theta\theta}(p) + \frac{\bar n}{2n_0 mc^2}\right]. \end{aligned} \tag{13.64}$$

Setting

$$G_{\rm n}(p) \simeq \frac{1}{2}G_{22}(p), \qquad G_{\rm an}(p) \simeq -\frac{1}{2}G_{22}(p) \tag{13.65}$$

in the numerator of equations (13.63), we obtain

$$\begin{aligned} \Sigma_{\rm an}(p) &= \Sigma_{\rm n}(p) - G_0^{-1}(p) \\ &= \begin{cases} \dfrac{\bar n^2}{2A_{d+1}c^{4-d}n_0 m^2}(\omega^2+c^2\mathbf{p}^2)^{(3-d)/2} & \text{if} \quad d<3, \\ \dfrac{\bar n^2}{A_4 c n_0 m^2}\left[\ln\left(\frac{c^2p_h^2}{\omega^2+c^2\mathbf{p}^2}\right)\right]^{-1} & \text{if} \quad d=3, \end{cases} \end{aligned} \tag{13.66}$$

in the infrared limit $|\mathbf{p}|, |\omega|/c \ll p_G$. Equations (13.66) agree with the exact results (13.33), and show that $\Sigma_{\rm n}(p) - G_0^{-1}(p)$ and $\Sigma_{\rm an}(p)$ are dominated by non-analytic terms for $p\to 0$.

Let us now show that these results for the self-energies yield a sound mode with linear dispersion in the low-energy limit. In the low-energy limit, the self-energies (13.63) can be written as

$$\begin{aligned} \Sigma_{\rm n}(p) - G_0^{-1}(p) &= \Delta\Sigma(p) + \tilde\Sigma_{\rm n}(p), \\ \Sigma_{\rm an}(p) &= \Delta\Sigma(p) + \tilde\Sigma_{\rm an}(p), \end{aligned} \tag{13.67}$$

where $\Delta\Sigma(p)$ denotes the singular part (13.66) while $\tilde\Sigma_{\rm n}(p)$ and $\tilde\Sigma_{\rm an}(p)$ are regular contributions of order $\mathbf{p}^2,\omega^2$. Using $\Delta\Sigma(p) \gg \tilde\Sigma_{\rm n}(p) - G_0^{-1}(p), \tilde\Sigma_{\rm an}(p)$ for $p\to 0$ and inverting (13.3), we obtain

$$\begin{aligned} G_{\rm n}(p) &= -\frac{\Delta\Sigma(p)+\tilde\Sigma_{\rm n}(p)}{2\Delta\Sigma(p)[\tilde\Sigma_{\rm n}(p)-\tilde\Sigma_{\rm an}(p)]+\tilde\Sigma_{\rm n}(p)^2-\tilde\Sigma_{\rm an}(p)^2} \\ &\simeq -\frac{1}{2[\tilde\Sigma_{\rm n}(p)-\tilde\Sigma_{\rm an}(p)]} \end{aligned} \tag{13.68}$$

and

$$\begin{aligned} G_{\rm an}(p) &= \frac{\Delta\Sigma(p)+\tilde\Sigma_{\rm an}(p)}{2\Delta\Sigma(p)[\tilde\Sigma_{\rm n}(p)-\tilde\Sigma_{\rm an}(p)]+\tilde\Sigma_{\rm n}(p)^2-\tilde\Sigma_{\rm an}(p)^2} \\ &\simeq \frac{1}{2[\tilde\Sigma_{\rm n}(p)-\tilde\Sigma_{\rm an}(p)]} \end{aligned} \tag{13.69}$$

to leading order. Since both $\tilde\Sigma_{\rm n}(p)$ and $\tilde\Sigma_{\rm an}(p)$ can be expanded to order $\mathbf{p}^2,\omega^2$, we conclude that equations (13.68) and (13.69) predict the existence of a sound mode with linear dispersion. Of course, these equations are nothing but (13.54) and (13.65). As for the longitudinal propagator, we obtain

$$G_{11}(p) = \frac{1}{2}[G_{\rm n}(p)+G_{\rm n}(-p)] + G_{\rm an}(p) \simeq -\frac{1}{2\Sigma_{\rm an}(p)} \tag{13.70}$$

for $p\to 0$. Thus we conclude that the singularity of the longitudinal propagator is directly related to the vanishing of the anomalous self-energy. A similar result has been obtained in the ordered phase of the classical O(N) model (Sec. 10.7.3).

A crucial point to obtain a sound mode despite the vanishing of $\Sigma_{\rm an}(p=0)$ is the cancellation of the $i\omega$ term of $G_0^{-1}(p)$ by the self-energy $\Sigma_{\rm n}(p)$. In the following section, we shall see that indeed the superfluid phase exhibits a relativistic (Lorentz) invariance at low energies.

> In deriving the low-energy expression (13.66) of the self-energies, we have assumed that the hydrodynamic description holds up to the momentum scale p_h and ignored the contribution of the non-hydrodynamic modes. In Popov's original approach [13], one introduces a momentum cutoff p_0 satisfying $p_G \ll p_0 \ll p_h$. Since $p_0 \gg p_G$, modes with momenta $|\mathbf{p}| \geq p_0$ can be taken into account within standard perturbation theory (see Sec. 13.1). On the other hand, low-momentum modes $|\mathbf{p}| \leq p_0 \ll p_h$ are naturally treated in the hydrodynamic approach discussed in this section. The final results are independent of p_0. The only difference with our results (13.66) is that p_h in the expression of the self-energy for $d=3$ [Eq. (13.66)] is replaced by a smaller momentum scale.

13.2.4 One-dimensional interacting bosons

A faire!

13.3 Non-perturbative RG approach

In this section, we show how the NPRG enables us to overcome the difficulties of the perturbative approach (Sec. 13.1) and determine the exact infrared behavior of the one-particle propagator. We closely follow the NPRG approach to the classical O(N) model (chapter 11).[15]

[15] As in the preceding sections, we focus on the $T=0$ superfluid phase (unless otherwise specified).

To implement the NPRG approach, we add the action (13.1) the infrared regulator term

$$\Delta S_k[\psi^*,\psi] = \int_0^\beta d\tau d\tau' \int d^d r d^d r' \psi^*(x) R_k(x-x')\psi(x') \\ = \sum_p \psi^*(p) R_k(p)\psi(p) \tag{13.71}$$

and consider the k-dependent effective action

$$\Gamma_k[\phi^*,\phi] = -\ln Z_k[J^*,J] + \int_0^\beta d\tau \int d^d r (J^*\phi + \text{c.c.}) - \Delta S_k[\phi^*,\phi], \tag{13.72}$$

where

$$\phi(x) = \langle \psi(x) \rangle = \frac{\delta \ln Z[J^*,J]}{\delta J(x)} \tag{13.73}$$

is the superfluid order parameter. J denotes a complex external source that couples linearly to the boson field ψ. Γ_k satisfies the RG equation[16]

$$\partial_k \Gamma[\phi^*,\phi] = \frac{1}{2}\mathrm{Tr}\left[\partial_k R_k \left(\Gamma_k^{(2)}[\phi^*,\phi] + R_k\right)^{-1}\right], \tag{13.74}$$

where $\Gamma_k^{(2)}$ is the second-order functional derivative of Γ_k. In Fourier space, the trace in (13.74) involves a sum over frequencies and momenta as well as a trace over the two components (real and imaginary parts) of the complex field ϕ.

As in chapter 11, we choose the cutoff function R_k such that all fluctuations are suppressed for $k=\Lambda$ (so that $\Gamma_\Lambda[\phi^*,\phi] = S[\phi^*,\phi]$) and $R_{k=0}(p) = 0$. $R_k(p)$ can act only on momenta or on both momenta and frequencies, i.e.

$$R_k(\mathbf{p}) = Z_{A,k}\epsilon_{\mathbf{p}} r\left(\frac{\mathbf{p}^2}{k^2}\right) \tag{13.75}$$

or

$$R_k(p) = \frac{Z_{A,k}}{2m}\left(\mathbf{p}^2 + \frac{\omega^2}{c_\Lambda^2}\right) r\left(\frac{\mathbf{p}^2}{k^2} + \frac{\omega^2}{c_\Lambda^2 k^2}\right). \tag{13.76}$$

The k-dependent variable $Z_{A,k}$ is defined below. A natural choice for the velocity c_Λ would be the actual (k-dependent) Goldstone mode velocity c_k as in the quantum O(N) model (Sec. 12.4). In the weak coupling limit, however, c_k renormalizes only weakly and is well approximated by its initial value $c_\Lambda = \sqrt{\mu/m}$. In general it is preferable to use a frequency-independent cutoff function which not does violate the causality of the propagator in the vacuum ($\mu \leq 0$ and $T=0$).[17] This is true in particular when one studies the thermodynamics of the dilute Bose gas, whose universality follows from the existence of a quantum critical point between the vacuum ($\mu \leq 0$) and the superfluid phase ($\mu \geq 0$) (see Sec. 7.4.4). On the other hand, we will see that the infrared behavior of the propagator in the superfluid phase is best understood using a "relativistic" cutoff such as (13.76).

[16] The derivation of Eq. (13.74) is similar to that of (11.30).
[17] In vacuum, the propagator $G(\mathbf{p},\tau) \propto \Theta(\tau)$ is retarded.

13.3.1 Derivative expansion

We consider the ansatz

$$\Gamma_k[\phi^*,\phi] = \int_0^\beta d\tau \int d^dr \Big[\phi^* \Big(Z_{C,k}\partial_\tau - V_{A,k}\partial_\tau^2 - \frac{Z_{A,k}}{2m}\boldsymbol{\nabla}^2 \Big)\phi + U_k(n)\Big] \tag{13.77}$$

based on a derivative expansion.[18] It is similar to the LPA' discussed for the classical O(N) model in section 11.3. The effective action (13.77) contains a second-order time-derivative term which is not present in the initial condition. We shall see that this term plays a crucial role when $d \leq 3$. Because of the U(1) symmetry of the microscopic action (13.1), the effective potential $U_k(n)$ is a function of the condensate density $n = |\phi|^2$. Its minimum determines the condensate density $n_{0,k}$ and the thermodynamics potential per unit volume $U(n_{0,k})$ in the equilibrium state. The initial condition is given by[19]

$$U_\Lambda(n) = -\mu n + \frac{g}{2}n^2, \qquad Z_{A,\Lambda} = Z_{C,\Lambda} = 1, \qquad V_{A,\Lambda} = 0. \tag{13.78}$$

It is convenient to write the boson field

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \tag{13.79}$$

in terms of two real fields ϕ_1 and ϕ_2. The effective action then reads

$$\begin{aligned} \Gamma_k[\phi] = \int_0^\beta d\tau \int d^dr \Big\{ &\frac{1}{2}\sum_{j,j'} \phi_j \big[i\epsilon_{j,j'} Z_{C,k}\partial_\tau \\ &- \delta_{j,j'} \Big(V_{A,k}\partial_\tau^2 + Z_{A,k}\frac{\boldsymbol{\nabla}^2}{2m} \Big) \big] \phi_{j'} + U_k(n) \Big\}, \end{aligned} \tag{13.80}$$

where $\epsilon_{j,j'}$ is the antisymmetric tensor and $n = \frac{1}{2}(\phi_1^2 + \phi_2^2)$. The two-point vertex is defined as

$$\Gamma^{(2)}_{k,ij}[x;x';\phi] = \frac{\delta^{(2)}\Gamma_k[\phi]}{\delta\phi_i(x)\delta\phi_j(x')} \tag{13.81}$$

and is related to the one-particle propagator by[20]

$$G_k[\phi] = -\big(\Gamma_k^{(2)}[\phi] + R_k\big)^{-1} \tag{13.82}$$

(see Sec. 11.1.1). Because of the U(1) symmetry, the two-point vertex in a constant (i.e. uniform and time-independent) field takes the form[21]

$$\Gamma^{(2)}_{k,ij}(p;\phi) = \delta_{i,j}\Gamma_{A,k}(p;n) + \phi_i\phi_j\Gamma_{B,k}(p;n) + \epsilon_{i,j}\Gamma_{C,k}(p;n), \tag{13.83}$$

[18] For a discussion of the symmetries of the effective action, see Appendix 13.A.1.

[19] *Stricto sensu*, the initial condition of the RG flow is not given by the mean-field solution $\Gamma_\Lambda = S$ since $R_\Lambda < \infty$ (see remark 1 in the discussion page 726). In the end, however, all physical properties will be expressed in terms of "infrared" quantities (boson density, superfluid density, scattering length, etc.) with no reference to the microscopic action.

[20] The minus sign in (13.82) is due to the definition of the one-particle propagator $G_{ij}(x,x') = -\langle\psi_i(x)\psi_j(x')\rangle$ which differs from that used in chapter 11.

[21] $\delta_{i,j}$, $\phi_i\phi_j$ and $\epsilon_{i,j}$ are the three tensors that one can form from the two-dimensional vector $(\phi_1,\phi_2)^T$.

where $\Gamma_{A,k}$, $\Gamma_{B,k}$ and $\Gamma_{C,k}$ are functions of the condensate density n. Parity and time-reversal invariance imply[22]

$$\begin{aligned}\Gamma_{A,k}(p;n) &= \Gamma_{A,k}(-p;n) = \Gamma_{A,k}(\mathbf{p},-i\omega;n), \\ \Gamma_{B,k}(p;n) &= \Gamma_{B,k}(-p;n) = \Gamma_{B,k}(\mathbf{p},-i\omega;n), \\ \Gamma_{C,k}(p;n) &= -\Gamma_{C,k}(-p;n) = -\Gamma_{C,k}(\mathbf{p},-i\omega;n).\end{aligned} \tag{13.84}$$

With the ansatz (13.80), we find

$$\begin{aligned}\Gamma_{A,k}(p;n) &= Z_{A,k}\epsilon_{\mathbf{p}} + V_{A,k}\omega^2 + U_k'(n), \\ \Gamma_{B,k}(p;n) &= U_k''(n), \qquad \Gamma_{C,k}(p;n) = Z_{C,k}\omega,\end{aligned} \tag{13.85}$$

in agreement with the general symmetry properties (13.84).

The functions Γ_A, Γ_B and Γ_C can be related to the normal and anomalous self-energies. For a constant and real field $\phi(x) = \sqrt{n}$, i.e. $\phi_i(x) = \delta_{i,1}\sqrt{2n}$,

$$\begin{aligned}\Sigma_{k,\mathrm{n}}(p;n) - G_0^{-1}(p) &= \Gamma_{A,k}(p;n) + n\Gamma_{B,k}(p;n) - i\Gamma_C(p;n) \\ &= U_k'(n) + nU_k''(n) + Z_{A,k}\epsilon_{\mathbf{p}} - Z_{C,k}i\omega + V_{A,k}\omega^2,\end{aligned} \tag{13.86}$$

and the anomalous self-energy

$$\Sigma_{k,\mathrm{an}}(p;n) = n\Gamma_B(p;n) = nU_k''(n) \tag{13.87}$$

is real.

13.3.1.1 Hugenholtz-Pines theorem, sound mode velocity and superfluid density

From the effective action (13.80), we can extract the main physical properties of the superfluid phase. The condensate density $n_{0,k}$ is defined by $U_k'(n_{0,k}) = 0$. Using equations (13.86) and (13.87), we then deduce

$$\Sigma_{k,\mathrm{n}}(p=0;n_{0,k}) - \Sigma_{k,\mathrm{an}}(p=0;n_{0,k}) = \mu, \tag{13.88}$$

which is nothing but the Hugenholtz-Pines theorem (Sec. 13.1.3).

The excitation spectrum is obtained from the zeros of the determinant of the 2×2 matrix $\Gamma_k^{(2)}(p;n_{0,k})$ (after analytic continuation $i\omega \to \omega + i0^+$),

$$\begin{aligned}\det \Gamma_k^{(2)}(p;n_{0,k}) &= \Gamma_{A,k}(p;n_{0,k})[\Gamma_{A,k}(p;n_{0,k}) + 2n_{0,k}\Gamma_{B,k}(p;n_{0,k})] + \Gamma_{C,k}(p;n_{0,k})^2 \\ &\simeq 2n_{0,k}\Gamma_{A,k}(p;n_{0,k})\Gamma_{B,k}(p;n_{0,k}) + \Gamma_{C,k}(p;n_{0,k})^2,\end{aligned} \tag{13.89}$$

The last result is valid in the low-energy limit $|\mathbf{p}| \ll p_h$ (p_h is the healing momentum scale; see Sec. 13.3.3) where (as we shall see) $\Gamma_{A,k}(p;n_{0,k}) \ll 2n_{0,k}\Gamma_{B,k}(p;n_{0,k})$. Using $U_k'(n_{0,k}) = 0$, we obtain

$$\det \Gamma_k^{(2)}(p;n_{0,k}) = 2n_{0,k}\lambda_k(Z_{A,k}\epsilon_{\mathbf{p}} + V_{A,k}\omega^2) + (Z_{C,k}\omega)^2, \tag{13.90}$$

[22] Time-reversal invariance follows from the invariance of the microscopic action (13.1) under $\psi(\mathbf{r},\tau) \leftrightarrow \psi^*(\mathbf{r},-\tau)$. This implies $\Gamma^{(2)}_{k,ij}(\mathbf{p},i\omega;\phi) = (2\delta_{i,j}-1)\Gamma^{(2)}_{k,ij}(\mathbf{p},-i\omega;\phi^*)$ where $\phi = (\phi_1,\phi_2)$ and $\phi^* = (\phi_1,-\phi_2)$.

where $\lambda_k = U_k''(n_{0,k})$. We deduce the existence of a Goldstone mode (the Bogoliubov sound mode) with velocity

$$c_k = \left(\frac{Z_{A,k}/2m}{V_{A,k} + Z_{C,k}^2/2\lambda_k n_{0,k}} \right)^{1/2}. \tag{13.91}$$

From $\det \Gamma_k^{(2)}(p; n_{0,k}) = 0$, we also obtain a gapped mode, with a gap which is larger than $c_k k$ and therefore outside the domain of validity of the derivative expansion ($|\mathbf{p}|, |\omega|/c_k \ll k$). The existence of two modes in the superfluid phase follows from $\det \Gamma_k^{(2)}(p)$ being of order ω^4. Pushing the derivative expansion to higher order in ω^2 would yield additional modes, beyond the domain of validity of the derivative expansion.

The superfluid density $n_{s,k}$ is defined by the rigidity against a (static) twist of the phase of the order parameter. If $\phi(\mathbf{r}) = \sqrt{2n_{0,k}}(\cos\theta(\mathbf{r}), \sin\theta(\mathbf{r}))$ varies slowly in space, the effective action increases by

$$\begin{aligned} \Delta\Gamma_k &= \frac{1}{2} \sum_p \Gamma_{A,k}(p; n_{0,k}) \phi_2(-p) \phi_2(p) \\ &= n_{0,k} \sum_p \Gamma_{A,k}(\mathbf{p}, i\omega = 0; n_{0,k}) \theta(-\mathbf{p}) \theta(\mathbf{p}) \\ &= \beta \frac{Z_{A,k} n_{0,k}}{2m} \int d^d r (\boldsymbol{\nabla}\theta)^2 \end{aligned} \tag{13.92}$$

to lowest order in $\boldsymbol{\nabla}\theta$. Comparing with (7.41), we deduce

$$n_{s,k} = Z_{A,k} n_{0,k}. \tag{13.93}$$

13.3.1.2 Symmetries and thermodynamic relations

At zero temperature and in the superfluid phase, Galilean invariance and gauge invariance imply the following Ward identities (see Appendix 13.A.2)

$$\begin{aligned} Z_{A,k} &= \frac{\partial}{\partial \epsilon_{\mathbf{p}}} \Gamma_{A,k}(p; n_{0,k}) \Big|_{p=0} = \frac{\bar{n}_k}{n_{0,k}}, \\ V_{A,k} &= \frac{\partial}{\partial \omega^2} \Gamma_{A,k}(p; n_{0,k}) \Big|_{p=0} = -\frac{1}{2n_{0,k}} \frac{\partial^2 U_k}{\partial \mu^2} \Big|_{n_{0,k}}, \\ Z_{C,k} &= \frac{\partial}{\partial \omega} \Gamma_{C,k}(p; n_{0,k}) \Big|_{p=0} = -\frac{\partial^2 U_k}{\partial \mu \partial n} \Big|_{n_{0,k}}, \end{aligned} \tag{13.94}$$

where we consider the effective potential $U_k(\mu, n)$ as a function of the two independent variables μ and n. The condensate density $n_{0,k} \equiv n_{0,k}(\mu)$ is then defined by

$$\frac{\partial U_k}{\partial n} \Big|_{n_{0,k}} = 0, \tag{13.95}$$

while the mean boson density $\bar{n}_k$ is obtained from

$$\bar{n}_k = -\frac{d}{d\mu} U_k(\mu, n_{0,k}) = -\frac{\partial U_k}{\partial \mu} \Big|_{n_{0,k}} - \frac{\partial U_k}{\partial n} \Big|_{n_{0,k}} \frac{dn_{0,k}}{d\mu} = -\frac{\partial U_k}{\partial \mu} \Big|_{n_{0,k}}, \tag{13.96}$$

where $d/d\mu$ is a total derivative. Equation (13.95) being valid for any μ, we deduce

$$0 = \frac{d}{d\mu}\frac{\partial U_k}{\partial n}\bigg|_{n_{0,k}} = \frac{\partial^2 U_k}{\partial\mu\partial n}\bigg|_{n_{0,k}} + \frac{\partial^2 U_k}{\partial n^2}\bigg|_{n_{0,k}} \frac{dn_{0,k}}{d\mu}. \tag{13.97}$$

From equations (13.94) and (13.97), we finally obtain

$$n_{s,k} = Z_{A,k} n_{0,k} = \bar{n}_k, \quad V_{A,k} = -\frac{1}{2n_{0,k}} \frac{\partial^2 U_k}{\partial\mu^2}\bigg|_{n_{0,k}}, \quad Z_{C,k} = \lambda_k \frac{dn_{0,k}}{d\mu}. \tag{13.98}$$

We recover the fact that the superfluid density is given by the full density at zero temperature (see Secs. 7.1 and 13.2.2).

The compressibility $\kappa_k = \bar{n}_k^{-2} d\bar{n}_k/d\mu$ is given by

$$\bar{n}_k^2 \kappa_k = -\frac{\partial^2 U_k}{\partial\mu^2}\bigg|_{n_{0,k}} - \frac{\partial^2 U_k}{\partial\mu\partial n}\bigg|_{n_{0,k}} \frac{dn_{0,k}}{d\mu} = 2n_{0,k} V_{A,k} + \frac{Z_{C,k}^2}{\lambda_k}, \tag{13.99}$$

which implies that the Bogoliubov sound mode velocity (13.91) takes the form

$$c_k = \left(\frac{n_{s,k}}{m\kappa_k \bar{n}_k^2} \right)^{1/2}. \tag{13.100}$$

Since $n_{s,k} = \bar{n}_k$ at zero temperature, the Bogoliubov sound mode velocity coincides with the the macroscopic sound velocity $1/(m\kappa_k\bar{n}_k)^{1/2}$.

13.3.2 Infrared behavior in the superfluid phase

As pointed out in section 11.2, the derivation expansion is correct only for momenta and frequencies satisfying $|\mathbf{p}|, |\omega|/c_k \ll k$. It is nevertheless possible to obtain the infrared behavior of the one-particle Green function by stopping the flow at a finite value k of order $\max(|\mathbf{p}|, |\omega|/c_k)$. In this section, we show that this can be done without actually solving the RG equations (this will be done in Sec. 13.3.3).

Let us consider the anomalous self-energy $\Sigma_{k,\mathrm{an}}(p; n_{0,k}) = \lambda_k n_{0,k}$ in the equilibrium state (Eq. (13.87) with $U_k''(n_{0,k}) = \lambda_k$). Since $\Sigma_{k=0,\mathrm{an}}(p = 0; n_{0,k=0}) = 0$ [Eq. (13.33)], λ_k must vanish when $k \to 0$. Furthermore, assuming that the result obtained from Popov's hydrodynamic theory [Eq. (13.66)] is correct, we expect that

$$\lambda_k \sim \begin{cases} k^{3-d} & \text{for} \quad d < 3, \\ |\ln k|^{-1} & \text{for} \quad d = 3, \end{cases} \tag{13.101}$$

when $k \to 0$. If we assume that the p dependence of the anomalous self-energy can be retrieved by stopping the flow at $k \sim \sqrt{\mathbf{p}^2 + \omega^2/c^2}$, then equation (13.66) follows from (13.101) ($c \equiv c_{k=0}$). The hypothesis (13.101) (which will be confirmed in section 13.3.3) is sufficient, when combined with the thermodynamic relations (13.98), to obtain the exact infrared behavior of the one-particle propagator. Furthermore we will see that this hypothesis is internally consistent.

Since thermodynamic quantities, including the condensate "compressibility" $dn_{0,k}/d\mu$ must remain finite in the limit $k \to 0$, we deduce from (13.98) that $Z_{C,k} \sim \lambda_k \sim k^{3-d}$ vanishes in the infrared limit. It follows that

$$\lim_{k\to 0} c_k = \lim_{k\to 0} \left(\frac{Z_{A,k}}{2mV_{A,k}} \right)^{1/2}. \tag{13.102}$$

Both $Z_{A,k} = \bar{n}_k/n_{0,k}$ and the macroscopic sound velocity c_k being finite at $k = 0$, $V_{A,k}$ (which vanishes in the Bogoliubov approximation) must be nonzero when $k \to 0$. The suppression of $Z_{C,k}$, together with a finite value of $V_{A,k}$, shows that the effective action Γ_k exhibits a "relativistic" (Lorentz) invariance in the infrared limit and therefore becomes equivalent to that of the classical O(2) model in dimension $d+1$. In the ordered phase, the coupling constant of this model vanishes as $\lambda_k \sim k^{4-(d+1)}$ (Sec. 11.3.3), which is nothing but our starting assumption (13.101). Thus, for $k \to 0$, we find that the existence of a linear spectrum is due to the relativistic form of the action rather than a nonzero value of λ_k as in the Bogoliubov theory.

We are now in a position to obtain the $k \to 0$ limit of the propagator in the zero-temperature superfluid phase. The one-particle propagator in a constant field can be written in a form similar to (13.83) or in terms of its longitudinal and transverse components,

$$G_{k,ij}(p;\phi) = \frac{\phi_i\phi_j}{2n} G_{k,\mathrm{ll}}(p;n) + \left(\delta_{i,j} - \frac{\phi_i\phi_j}{2n}\right) G_{k,\mathrm{tt}}(p;n) + \epsilon_{ij} G_{k,\mathrm{lt}}(p;n), \tag{13.103}$$

where[23]

$$\begin{aligned} G_{k,\mathrm{ll}}(p;n) &= -\frac{\Gamma_{A,k}(p;n)}{D_k(p)}, \\ G_{k,\mathrm{tt}}(p;n) &= -\frac{\Gamma_{A,k}(p;n) + 2n\Gamma_{B,k}(p;n)}{D_k(p)} \\ G_{k,\mathrm{lt}}(p;n) &= \frac{\Gamma_{C,k}(p;n)}{D_k(p)}, \end{aligned} \tag{13.104}$$

with $D_k = \Gamma_{A,k}^2 + 2n\Gamma_{A,k}\Gamma_{B,k} + \Gamma_{C,k}^2$. For a real field ($\phi_i = \delta_{i,1}\sqrt{2n}$), G_{ll}, G_{tt} and G_{lt} can be identified with G_{11}, G_{22} and G_{12}, respectively. Using (13.85) and (13.98), we obtain

$$\begin{aligned} G_{k,\mathrm{ll}}(p;n_{0,k}) &= -\frac{1}{2\lambda_k n_{0,k}}, \\ G_{k,\mathrm{tt}}(p;n_{0,k}) &= -\frac{1}{V_{A,k}(\omega^2 + c_k^2\mathbf{p}^2)} = -\frac{2mc_k^2 n_{0,k}}{\bar{n}_k}\frac{1}{\omega^2 + c_k^2\mathbf{p}^2}, \\ G_{k,\mathrm{lt}}(p;n_{0,k}) &= \frac{1}{2\lambda_k n_{0,k}}\frac{Z_{C,k}\omega}{V_{A,k}(\omega^2 + c_k^2\mathbf{p}^2)} = \frac{dn_{0,k}}{d\mu}\frac{mc_k^2}{\bar{n}_k}\frac{\omega}{\omega^2 + c_k^2\mathbf{p}^2}. \end{aligned} \tag{13.105}$$

Since thermodynamic quantities are not expected to flow in the infrared limit (see Sec. 13.3.3), they can be approximated by their $k = 0$ values. As for the longitudinal propagator $G_{k=0,\mathrm{ll}}$, its value is obtained from the replacement $\lambda_k \to C(\omega^2 + c^2\mathbf{p}^2)^{(3-d)/2}$ (with C a constant). We finally obtain

$$\begin{aligned} G_{\mathrm{n}}(p;n_0) &= -\frac{mc^2 n_0}{\bar{n}}\frac{1}{\omega^2 + c^2\mathbf{p}^2} - \frac{dn_0}{d\mu}\frac{mc^2}{\bar{n}}\frac{i\omega}{\omega^2 + c^2\mathbf{p}^2} + \frac{1}{2}G_{\mathrm{ll}}(p;n_0), \\ G_{\mathrm{an}}(p;n_0) &= \frac{mc^2 n_0}{\bar{n}}\frac{1}{\omega^2 + c^2\mathbf{p}^2} + \frac{1}{2}G_{\mathrm{ll}}(p;n_0), \end{aligned} \tag{13.106}$$

for $k = 0$, where

$$G_{\mathrm{ll}}(p;n_0) = -\frac{1}{2n_0 C(\omega^2 + c^2\mathbf{p}^2)^{(3-d)/2}} \tag{13.107}$$

[23] As discussed in chapter 11 (see, e.g., footnote 29 page 749), we define here the physical propagator as $-\Gamma_k^{(2)-1}$ rather than $-(\Gamma_k^{(2)} + R_k)^{-1}$.

and $n_0 \equiv n_{0,k=0}$, $c \equiv c_{k=0}$, etc. The divergence of $G_{\rm ll}(p;n_0)$ for $p\to 0$ is logarithmic when $d=3$.

13.3.3 RG flows

The conclusions of the preceding section can be obtained more rigorously from the RG equation satisfied by the effective action. To simplify the analysis, we truncate the effective potential,

$$U_k(n) = \begin{cases} U_{0,k} + \dfrac{\lambda_k}{2}(n-n_{0,k})^2 & \text{if} \quad n_{0,k}>0, \\ U_{0,k} + \delta_k n + \dfrac{\lambda_k}{2} n^2 & \text{if} \quad n_{0,k}=0, \end{cases} \tag{13.108}$$

where $\delta_k = U_k'(n_{0,k})$ and $\lambda_k = U_k''(n_{0,k})$ (δ_k vanishes in the superfluid phase). The derivation of the flow equations is similar to the case of the (classical) O(N) model and we only give the final results:[24]

$$\begin{aligned} \partial_t n_{0,k} &= \frac{3}{2} I_{k,\rm ll} + \frac{1}{2} I_{k,\rm tt} \quad \text{if} \quad n_{0,k}>0, \\ \partial_t \delta_k &= -2\lambda_k I_{k,\rm ll} \quad \text{if} \quad n_{0,k}=0, \\ \partial_t \lambda_k &= -\lambda_k^2 \bigl[9J_{k;\rm ll,ll}(0) - 6J_{k;\rm lt,lt}(0) + J_{k;\rm tt,tt}(0) \bigr], \end{aligned} \tag{13.109}$$

and

$$\begin{aligned} \partial_t Z_{A,k} &= -2\lambda_k^2 n_{0,k} \frac{\partial}{\partial \epsilon_{\mathbf p}} \bigl[J_{k;\rm ll,tt}(p) + J_{k;\rm tt,ll}(p) + 2J_{k;\rm lt,lt}(p) \bigr]\Bigr|_{p=0}, \\ \partial_t Z_{C,k} &= 2\lambda_k^2 n_{0,k} \frac{\partial}{\partial \omega} \bigl[J_{k;\rm tt,lt}(p) - J_{k;\rm lt,tt}(p) - 3J_{k;\rm ll,lt}(p) + 3J_{k;\rm lt,ll}(p) \bigr]\Bigr|_{p=0}, \\ \partial_t V_{A,k} &= -2\lambda_k^2 n_{0,k} \frac{\partial}{\partial \omega^2} \bigl[J_{k;\rm ll,tt}(p) + J_{k;\rm tt,ll}(p) + 2J_{k;\rm lt,lt}(p) \bigr]\Bigr|_{p=0}, \end{aligned} \tag{13.110}$$

where

$$\begin{aligned} I_{k,\alpha} &= \int_q \tilde\partial_t G_{k,\alpha}(q;n_{0,k}), \\ J_{k,\alpha\beta}(p) &= \int_q [\tilde\partial_t G_{k,\alpha}(q;n_{0,k})] G_\beta(p+q;n_{0,k}), \end{aligned} \tag{13.111}$$

with $\alpha,\beta = \rm ll, tt, lt$ and $\tilde\partial_t = (\partial_t R_k)\partial_{R_k}$. It is sometimes convenient to rewrite these equations with the dimensionless variables

$$\begin{aligned} \tilde n_{0,k} &= k^{-d} Z_{C,k} n_{0,k}, \qquad & \tilde\delta_k &= Z_{A,k}^{-1}\epsilon_k^{-1}\delta_k, \\ \tilde\lambda_k &= k^d \epsilon_k^{-1} Z_{A,k}^{-1} Z_{C,k}^{-1} \lambda_k, \qquad & \tilde V_{A,k} &= \epsilon_k Z_{A,k} Z_{C,k}^{-2} V_{A,k}. \end{aligned} \tag{13.112}$$

We then obtain

$$\begin{aligned} \partial_t \tilde n_{0,k} &= -(d+\eta_{C,k})\tilde n_{0,k} + \frac{3}{2}\tilde I_{k,\rm ll} + \frac{1}{2}\tilde I_{k,\rm tt} \quad \text{if} \quad \tilde n_{0,k}>0, \\ \partial_t \tilde\delta_k &= (\eta_{A,k}-2)\tilde\delta_k - 2\tilde\lambda_k \tilde I_{k,\rm ll} \quad \text{if} \quad \tilde n_{0,k}=0, \\ \partial_t \tilde\lambda_k &= (d-2+\eta_{A,k}+\eta_{C,k})\tilde\lambda_k - \tilde\lambda_k^2 \bigl[9\tilde J_{k;\rm ll,ll}(0) - 6\tilde J_{k;\rm lt,lt}(0) + \tilde J_{k;\rm tt,tt}(0) \bigr], \end{aligned} \tag{13.113}$$

[24] $I_{k,\rm ll} = I_{k,\rm tt}$ when $n_{0,k}=0$.

and

$$\begin{aligned}\eta_{A,k} &= 2\tilde{\lambda}_k^2 \tilde{n}_{0,k} \frac{\partial}{\partial y} \left[\tilde{J}_{k;\text{ll,tt}}(p) + \tilde{J}_{k;\text{tt,ll}}(p) + 2\tilde{J}_{k;\text{lt,lt}}(p) \right] \Big|_{p=0}, \\ \eta_{C,k} &= -2\tilde{\lambda}_k^2 \tilde{n}_{0,k} \frac{\partial}{\partial \tilde{\omega}} \left[\tilde{J}_{k;\text{tt,lt}}(p) - \tilde{J}_{k;\text{lt,tt}}(p) - 3\tilde{J}_{k;\text{ll,lt}}(p) + 3\tilde{J}_{k;\text{lt,ll}}(p) \right] \Big|_{p=0}, \\ \partial_t \tilde{V}_{A,k} &= (2 - \eta_{A,k} + 2\eta_{C,k}) \tilde{V}_{A,k} \\ &\quad - 2\tilde{\lambda}_k^2 \tilde{n}_{0,k} \frac{\partial}{\partial \tilde{\omega}^2} \left[\tilde{J}_{k;\text{ll,tt}}(p) + \tilde{J}_{k;\text{tt,ll}}(p) + 2\tilde{J}_{k;\text{lt,lt}}(p) \right] \Big|_{p=0},\end{aligned} \tag{13.114}$$

where

$$\eta_{A,k} = -\partial_t \ln Z_{A,k}, \quad \eta_{C,k} = -\partial_t \ln Z_{C,k}, \tag{13.115}$$

$y = \mathbf{p}^2/k^2$ and $\tilde{\omega} = \omega Z_{C,k}/Z_{A,k}\epsilon_k$. The expression of the threshold functions $I_{k,\alpha}$, $J_{k,\alpha\beta}(p)$, $\tilde{I}_{k,\alpha}$ and $\tilde{J}_{k,\alpha\beta}(p)$ can be found in Appendix 13.B.

13.3.3.1 RG equations in vacuum

The set of parameters $\tilde{n}_0 = \tilde{\delta} = \tilde{\lambda} = 0$ and $\tilde{V}_A = 0$ corresponds to a fixed point of the flow equations (13.113) and (13.114). We shall see that for $d \geq 2$ this non-interacting (Gaussian) fixed point is the only fixed point in vacuum where both the density and the condensate density vanish. When $n_{0,k} = 0$, $\partial_t Z_{A,k} = \partial_t Z_{C,k} = \partial_t V_{A,k} = 0$, i.e.

$$Z_{A,k} = Z_{C,k} = 1 \quad \text{and} \quad V_{A,k} = 0. \tag{13.116}$$

We then have $\Gamma_{A,k}(p) = \epsilon_{\mathbf{p}} + R_k(p)$, $\Gamma_{B,k}(p) = \lambda_k$, $\Gamma_{C,k}(p) = \omega$ in a vanishing field ($n = 0$), and

$$G_{k,\text{ll}}(p) = G_{k,\text{tt}}(p) = -\frac{\epsilon_{\mathbf{p}} + R_k(p)}{D_k(p)}, \qquad G_{k,\text{lt}}(p) = \frac{\omega}{D_k(p)}, \tag{13.117}$$

where $D_k(p) = [\epsilon_{\mathbf{p}} + R_k(p)]^2 + \omega^2$. From these expressions, we deduce

$$\begin{aligned}\tilde{\partial}_t G_{k,\text{ll}}(p) = \tilde{\partial}_t G_{k,\text{tt}}(p) &= -\frac{\omega^2 - [\epsilon_{\mathbf{p}} + R_k(p)]^2}{D_k(p)^2} \partial_t R_k(p), \\ \tilde{\partial}_t G_{k,\text{lt}}(p) &= \frac{2\omega[\epsilon_{\mathbf{p}} + R_k(p)]}{D_k(p)^2} \partial_t R_k(p)\end{aligned} \tag{13.118}$$

and

$$\partial_t \lambda_k = \lambda_k^2 \int_p \frac{\partial_t R_k(p)}{D_k(p)^3} \{ 10[\epsilon_{\mathbf{p}} + R_k(p)]^3 - 22\omega^2[\epsilon_{\mathbf{p}} + R_k(p)] \}. \tag{13.119}$$

With the frequency-independent cutoff function (13.75), the frequency integral can be done analytically and we obtain

$$\partial_t \lambda_k = \frac{\lambda_k^2}{2} \int_{\mathbf{p}} \frac{\partial_t R_k(\mathbf{p})}{[\epsilon_{\mathbf{p}} + R_k(\mathbf{p})]^2}, \tag{13.120}$$

i.e.

$$\frac{1}{\lambda_k} - \frac{1}{\lambda_\Lambda} = \frac{1}{2} \int_{\mathbf{p}} \left(\frac{1}{\epsilon_{\mathbf{p}} + R_k(\mathbf{p})} - \frac{1}{\epsilon_{\mathbf{p}} + R_\Lambda(\mathbf{p})} \right), \tag{13.121}$$

where $\lambda_\Lambda = g$ is the bare interaction. If we take the function $r(y) = \Theta(1-y)(1-y)/y$ in (13.75), we finally obtain

$$\frac{1}{\lambda_k} - \frac{1}{g} = \frac{1}{2}\int_{\mathbf{p}} \left(\frac{\Theta(|\mathbf{p}|-k)}{\epsilon_{\mathbf{p}}} + \frac{\Theta(k-|\mathbf{p}|)}{\epsilon_k} - \frac{1}{\epsilon_\Lambda} \right). \tag{13.122}$$

In three dimensions, equation (13.122) gives

$$\frac{1}{\lambda_k} - \frac{1}{g} = \frac{m}{2\pi^2}\left(\frac{2\Lambda}{3} - \frac{2k}{3}\right). \tag{13.123}$$

The renormalized value

$$\lambda_{k=0} \equiv \frac{4\pi a}{m} \tag{13.124}$$

of the interaction defines the (s-wave) scattering length

$$a = \frac{m}{4\pi}\frac{g}{1+\frac{mg\Lambda}{3\pi^2}}. \tag{13.125}$$

Thus, the interaction

$$\lambda_k = \frac{4\pi a}{m}\left(1 - \frac{4}{3\pi}ka\right)^{-1} \tag{13.126}$$

is fully determined by the boson mass and the scattering length.

In two dimensions, we obtain

$$\frac{1}{\lambda_k} = \frac{1}{g} + \frac{m}{2\pi}\ln\left(\frac{\Lambda}{k}\right). \tag{13.127}$$

Introducing the two-dimensional scattering length

$$a = \frac{2}{\Lambda}\exp\left(-\frac{2\pi}{mg} - C + \frac{1}{2}\right) \tag{13.128}$$

(C is the Euler constant), we can rewrite (13.127) as

$$\frac{1}{\lambda_k} = -\frac{m}{2\pi}\left[\ln\left(\frac{ka}{2}\right) + C - \frac{1}{2}\right]. \tag{13.129}$$

Both in three and two dimensions, the dimensionless interaction constant $\tilde\lambda_k = k^d\epsilon_k\lambda_k$ vanishes for $k\to 0$. The only fixed point of the RG equations (13.113) and (13.114) in vacuum ($n_0 = 0$) then corresponds to $\tilde\lambda = \tilde\delta = 0$ (the latter condition implies $\mu = 0$). We recover the fact that the upper critical dimension of the vacuum-superfluid transition is $d_c^+ = 2$.[25] Since the one-particle propagator is not renormalized at the quantum critical point ($Z_{A,k} = Z_{C,k} = 1$, $V_{A,k} = 0$ and $\delta_k = 0$) the dynamical critical exponent is $z = 2$. If we linearize the $T = 0$ RG equations about the fixed point $\tilde\delta = \tilde\lambda = 0$, one finds a relevant variable ($\tilde\delta$) in the vacuum phase with scaling dimension $[\tilde\delta] = 2$ so that the correlation-length exponent is $\nu = 1/2$.

[25] One easily verifies that the RG equations in vacuum admit a non-trivial fixed point $\tilde\lambda^* > 0$ for $d < 2$ (see Sec. 7.4.4).

Scattering length in the NPRG approach. In vacuum one can compute the renormalized interaction λ_k directly from the action $S+\Delta S_k$ without solving the RG equation. Summing the ladder diagrams (the only ones that contribute to the interaction in vacuum), one finds

$$\frac{1}{\lambda_k} = \frac{1}{g} + \frac{1}{2}\int_{\mathbf{p}} \frac{1}{\epsilon_{\mathbf{p}} + R_k(\mathbf{p})} \tag{13.130}$$

with a frequency-independent cutoff, which differs from the result (13.121) obtained by integrating the RG equation $\partial_k \lambda_k$ unless $R_\Lambda(\mathbf{p}) = \infty$. When the latter condition is not fulfilled, the initial condition Γ_Λ is not given by the microscopic action (13.1), i.e. $\lambda_\Lambda \neq g$. As noted in chapter 11, this fact does not matter as long as we are interested in the long-distance physics which is here entirely determined by the scattering length a. However, one has to properly define the latter [Eqs. (13.125,13.128)] in order to reproduce the correct expression of $\lambda_k \equiv \lambda_k(a)$ [Eqs. (13.126,13.129)].[26]

13.3.3.2 The healing momentum scale

In the dilute limit, the finite condensate density can be ignored as long as $\epsilon_k \gg 2\lambda_k n_{0,k}$.[27] This defines the characteristic (healing) momentum scale p_h,

$$\epsilon_{p_h} = \lambda_{p_h} n_{0,p_h}. \tag{13.131}$$

The flow is governed by the Gaussian fixed point $\tilde\lambda = \tilde n_0 = 0$ for $k \gg p_h$, and is driven away from that fixed point when $k \ll p_h$ because of the finite boson density. For $k \gg p_h$, we can solve the RG equations to leading order in $p_h^2/k^2 \sim \lambda_{p_h} n_{0,p_h}/\epsilon_k$. The coupling constant λ_k can then be approximated by its value in vacuum. To obtain $\partial_t n_{0,k}$ to leading order in $\lambda_k n_{0,k}$, we use

$$\begin{aligned} G_{k,\mathrm{ll}}(p;n_{0,k}) &= -\frac{\epsilon_p + R_k(p)}{D_k(p)} + \frac{2\lambda_k n_{0,k}}{D_k(p)^2}[\epsilon_p + R_k(p)]^2 + \mathcal{O}(n_{0,k}^2), \\ G_{k,\mathrm{tt}}(p;n_{0,k}) &= -\frac{\epsilon_p + R_k(p)}{D_k(p)} - \frac{2\lambda_k n_{0,k}}{D_k(p)^2}\omega^2 + \mathcal{O}(n_{0,k}^2) \end{aligned} \tag{13.132}$$

($D_k(p)$ is defined after Eq. (13.117)) and[28] $Z_{A,k} = Z_{C,k} = 1$, $V_{A,k} = 0$. This gives

$$\begin{aligned} \partial_t n_{0,k} &= 2\int_p \frac{\partial_t R_k(p)}{D_k(p)^3}\big\{D_k(p)\big([\epsilon_p + R_k(p)]^2 - \omega^2\big) \\ &\quad + \lambda_k n_{0,k}[\epsilon_p + R_k(p)]\big(5\omega^2 - 3[\epsilon_p + R_k(p)]^2\big)\big\} \\ &= -\frac{\lambda_k n_{0,k}}{2}\int_{\mathbf{p}} \frac{\partial_t R_k(p)}{[\epsilon_{\mathbf{p}} + R_k(p)]^2}, \end{aligned} \tag{13.133}$$

where the last result is obtained using the frequency-independent cutoff function (13.75). From (13.120) and (13.133) we deduce that

$$\partial_t(\lambda_k n_{0,k}) = 0 \tag{13.134}$$

[26] Equation (13.129) is obtained by solving (13.130) and using the known expression $a = \frac{2}{\Lambda}e^{-2\pi/mg - C}$ of the two-dimensional scattering length in a system with a ultraviolet violet momentum cutoff Λ (see Sec. 7.4).

[27] For $k \gtrsim p_h$ and with the cutoff function $R_k(\mathbf{p}) = (\epsilon_k - \epsilon_{\mathbf{p}})\Theta(\epsilon_k - \epsilon_{\mathbf{p}})$, $n_{0,k}$ enters the threshold functions in the combination $\epsilon_k + 2\lambda_k n_{0,k}$.

[28] For a justification of this approximation, see Appendix E.4 in Ref. [30].

to leading order in k^2/p_h^2. This equation allows us to relate $\lambda_{p_h} n_{0,p_h}$ to the chemical potential,

$$\lambda_{p_h} n_{0,p_h} \simeq \lambda_\Lambda n_{0,\Lambda} = \mu. \tag{13.135}$$

This yields

$$p_h = \sqrt{2m\lambda_{p_h} n_{0,p_h}} \simeq \sqrt{2m\mu}, \tag{13.136}$$

which is the standard expression of the healing momentum.

For a three-dimensional system, since $\lambda_{p_h} \simeq 4\pi a/m$ for $p_h a \ll 1$, we find

$$p_h = \sqrt{8\pi a \bar{n}}, \tag{13.137}$$

where we have used $n_{0,p_h} \simeq \bar{n}$ in the dilute limit.[29] In two dimensions, the logarithmic vanishing of λ_k in vacuum plays a crucial role. From equation (13.129), we obtain[34]

$$\lambda_{p_h} = -\frac{2\pi/m}{\ln\left(\frac{p_h a}{2}\right) + C - \frac{1}{2}} \simeq \frac{2\pi/m}{\left|\ln\sqrt{ma^2\mu}\right|} \tag{13.138}$$

for $p_h a = \sqrt{2ma^2\mu} \ll 1$, and therefore

$$p_h^2 = 2m\lambda_{p_h} n_{0,p_h} \simeq \frac{4\pi\bar{n}}{\left|\ln\sqrt{ma^2\mu}\right|} \simeq \frac{4\pi\bar{n}}{|\ln\sqrt{\bar{n}a^2}|}, \tag{13.139}$$

using $n_{0,p_h} \simeq \bar{n}$ and $\bar{n}a^2 \ll 1$.

In the weak-coupling limit $\tilde{\lambda}_\Lambda \ll 1$, i.e. $2mg\Lambda \ll 1$ ($d = 3$) or $2mg \ll 1$ ($d = 2$),[30]

$$\lambda_{p_h} \simeq g, \qquad p_h \simeq \sqrt{2mg\bar{n}}, \tag{13.140}$$

except, in two dimensions, when the density is exponentially small.

13.3.3.3 Numerical solution of the RG equations

In section 13.1.2 we pointed out that perturbation theory breaks down below the Ginzburg momentum scale p_G ($p_G \ll p_h$). We therefore expect the RG flow to be non-trivial for $k \ll p_h$ and the Ginzburg scale p_G to manifests itself as a characteristic momentum scale.

We can improve the perturbative estimate of p_G given in section 13.1.2 [Eq. (13.23)] by replacing the bare interaction constant g by λ_{p_h} in order to include fluctuations at momentum scales $k \gtrsim p_h$. This gives

$$p_G \sim \begin{cases} p_h \exp\left(-\mathrm{const}/\sqrt{\bar{n}a^3}\right) & (d = 3), \\ \dfrac{p_h}{|\ln\sqrt{\bar{n}a^2}|} & (d = 2), \end{cases} \tag{13.141}$$

where p_h is defined by equations (13.137) and (13.139).

The numerical solution of the flow equations is shown in figure 13.7 for a two-dimensional system in the weak-coupling limit $mg \ll 1$.[31] We can clearly distinguish two regimes

[29] The result $n_{0,p_h} \simeq \bar{n}$ is correct to leading order in the dimensionless parameter $\tilde{\lambda}_{p_h}$.

[30] The two-dimensional scattering length a is then exponentially small.

[31] For a three-dimensional system in the weak-coupling limit, the Ginzburg scale p_G is exponentially small so that deviations from Bogoliubov theory appear only at extremely small energies.

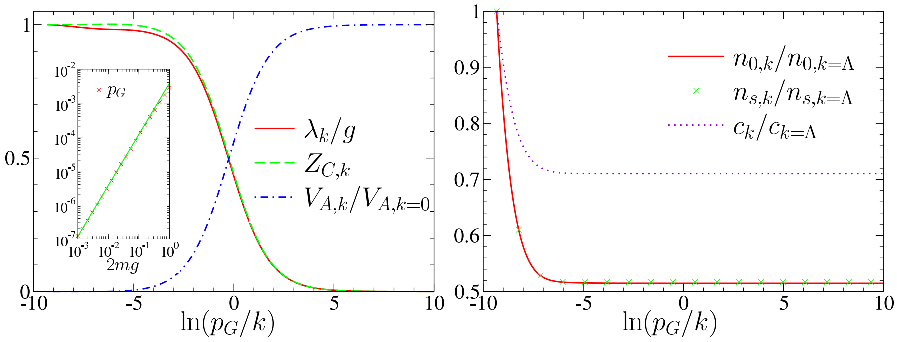

Figure 13.7: RG flow in a two-dimensional system: (left) λ_k, $Z_{C,k}$ and $V_{A,k}$ vs $\ln(p_G/k)$ where $p_G = \sqrt{(gm)^3\bar{n}}/4\pi$ ($\bar{n} = 0.01$, $2mg = 0.1$ and $\ln(p_G/p_h) \simeq -5.87$). The inset shows p_G vs $2mg$ obtained from the criterion $V_{A,p_G} = V_{A,k=0}/2$ (the Green solid line is a fit to $p_G \sim (2mg)^{3/2}$). (right) Condensate density $n_{0,k}$, superfluid density $n_{s,k}$ and Goldstone mode velocity c_k vs $\ln(p_G/k)$.

separated by the Ginzburg momentum scale $p_G \sim \sqrt{(gm)^3\bar{n}}$ (see the inset in the figure). In the (perturbative) Bogoliubov regime $k \gg p_G$, $Z_{A,k} \simeq Z_{C,k} \simeq 1$ and $V_{A,k} \simeq 0$, while λ_k remains nearly equal to its initial value $\lambda_\Lambda = g$.[32] In the (non-perturbative) Goldstone regime $k \ll p_G$, we find that both λ_k and $Z_{C,k}$ vanish linearly with k in agreement with the conclusion of section 13.3.2 [Eq. (13.101)], while $V_{A,k}$ takes a finite value. This regime is dominated by phase fluctuations and characterized by the vanishing of the anomalous self-energy $\Sigma_{k,\mathrm{an}}(p=0) = \lambda_k n_{0,k} \sim k$ and the divergence of the longitudinal propagator (Secs. 13.2.3 and 13.3.2).

The condensate density $n_{0,k}$, the superfluid density $n_{s,k}$ and the Goldstone mode velocity c_k are not sensitive to the Ginzburg scale (Fig. 13.7). This result is particularly remarkable for the velocity c_k, whose expression involves the parameters λ_k, $Z_{C,k}$ and $V_{A,k}$ which all strongly vary when $k \sim p_G$. Thus we see that the coupling between longitudinal and transverse fluctuations does not affect thermodynamic quantities. More generally, all physical quantities related to correlation functions obtained as averages of local gauge-invariant operators (like the density-density correlation function) are expected to be free of infrared divergences and insensitive to the Ginzburg scale.

13.3.3.4 Analytical solution of the RG equations in the infrared limit

In the Goldstone regime it is possible to solve analytically the RG equations. When $k \ll p_G$, the physics is dominated by the Goldstone (phase) mode and longitudinal fluctuations can be ignored. If we take the cutoff function (13.76) with $r(Y) = \Theta(1-Y)(1-Y)/Y$,[33] the threshold functions $\tilde{I}_{k,\alpha}$ and $\tilde{J}_{k,\alpha\beta}(p)$ can be computed exactly and one obtains (see

[32] For the weak-coupling value $2mg = 0.1$, the flow of λ_k for $k \gtrsim p_h$ can be neglected, i.e. $\lambda_{p_h} \simeq g$ [Eq. (13.140)].
[33] In the infrared limit, it is natural to choose a cutoff function $R_k(p)$ acting on both momenta and frequencies and satisfying the Lorentz invariance of the effective action Γ_k.

Appendix 13.B)

$$
\begin{aligned}
\partial_t \tilde n_{0,k} &= -(d+\eta_{C,k})\tilde n_{0,k}, \quad \partial_t \tilde\lambda_k = (d-2+\eta_{C,k})\tilde\lambda_k + 8\frac{v_{d+1}}{d+1}\frac{\tilde\lambda_k^2}{\tilde V_{A,k}^{1/2}}, \\
\eta_{C,k} &= -8\frac{v_{d+1}}{d+1}\frac{\tilde\lambda_k}{\tilde V_{A,k}^{1/2}}, \quad \partial_t \tilde V_{A,k} = (2+2\eta_{C,k})\tilde V_{A,k},
\end{aligned}
\tag{13.142}
$$

while $\eta_{A,k} \simeq 0$. The first and last of these equations can be rewritten as $n_{0,k} = n_{0,k=0}$ and $V_{A,k} = V_{A,k=0}$, respectively. From (13.142), we deduce

$$
\partial_t \tilde\lambda_k = (1-\epsilon)\tilde\lambda_k, \qquad \partial_t \eta_{C,k} = -\epsilon\eta_{C,k} - \eta_{C,k}^2, \tag{13.143}
$$

where $\epsilon = 3-d$. This yields

$$
\begin{cases}
\eta_{C,k} \sim |\ln k|^{-1}, & \tilde\lambda_k \sim k, \quad Z_{C,k},\,\lambda_k \sim |\ln k|^{-1} \quad \text{if} \quad d=3, \\
\eta_{C,k} \to -\epsilon, & \tilde\lambda_k \sim k^{1-\epsilon}, \quad Z_{C,k},\,\lambda_k \sim k^{\epsilon} \quad \text{if} \quad d<3,
\end{cases}
\tag{13.144}
$$

for $k \to 0$ in agreement with conclusions of section 13.3.2 and the numerical results shown in figure 13.7. The anisotropy between time and space in the Goldstone regime $k \ll p_G$ (where the action Γ_k takes a relativistic form) can be eliminated by an appropriate rescaling of frequencies and fields: $\tilde\omega \to \tilde V_{A,k}^{-1/2}\tilde\omega$ and $\tilde\phi \to \tilde V_{A,k}^{-1/4}\tilde\phi$. We thus obtain an isotropic relativistic model with dimensionless condensate density and coupling constant defined by

$$
\tilde n'_{0,k} = \tilde V_{A,k}^{1/2}\tilde n_{0,k}, \qquad \tilde\lambda'_k = \tilde V_{A,k}^{-1/2}\tilde\lambda_k. \tag{13.145}
$$

$\tilde\lambda'_k$ satisfies the RG equation

$$
\partial_t \tilde\lambda'_k = -\epsilon\tilde\lambda'_k + 8\frac{v_{d+1}}{d+1}\tilde\lambda'^{\,2}_k, \tag{13.146}
$$

which is nothing but the RG equation of the coupling constant of the classical O(2) model in dimensions $d+1$ (Eq. (11.114) with $N=2$ and $l_2^d(0,0) = 4/d$ for $r(y) = \Theta(1-y)(1-y)/y$). The corresponding fixed-point value $\tilde\lambda'^{\,*}_k$ is given by (11.117). In the infrared limit, we find

$$
\lambda_k = k^{-d}(Z_{A,k}\epsilon_k)^{3/2} V_{A,k}^{1/2}\tilde\lambda'_k \sim k^{\epsilon}\tilde\lambda'^{\,*}_k \tag{13.147}
$$

if we approximate $Z_{A,k} \simeq Z_{A,k=0}$ and $V_{A,k} \simeq V_{A,k=0}$. The vanishing of $\lambda_k \sim k^{\epsilon}$ is therefore the consequence of the existence of a fixed point $\tilde\lambda'^*$ for the coupling constant of the effective $(d+1)$-dimensional O(2) model that describes the Goldstone regime $k \ll p_G$. (In three dimensions, $\tilde\lambda'^* = 0$ and λ_k vanishes logarithmically.)

13.3.4 Thermodynamics

13.3.4.1 Mean-field theory

The Ginzburg scale is crucial to understand the infrared behavior of the one-particle propagator but is irrelevant for thermodynamic quantities. As a result, the leading-order results for small $ma^2\mu$ (see chapter 7) can be obtained by approximating $\Gamma_{k=0}$ by Γ_{p_h}, i.e. by ignoring any additional renormalization of the effective action as k decreases from p_h to zero. This is equivalent to mean-field theory but with the renormalized interaction constant

λ_{p_h} rather than the bare interaction constant g. We expect this approximation to be valid for thermodynamic quantities if the dimensionless coupling constant $\tilde{\lambda}_{p_h}$ is small,

$$\tilde{\lambda}_{p_h} \simeq 2mp_h^{d-2}\lambda_{p_h} \ll 1, \tag{13.148}$$

where we assume $Z_{A,k} \simeq 1$ and $Z_{C,k} \simeq 1$.

Let us first consider a three-dimensional system. From equations (13.136) and (13.137), we deduce

$$\mu = \frac{4\pi a\bar{n}}{m}, \tag{13.149}$$

so that[34]

$$\tilde{\lambda}_{p_h} = 8\pi p_h a \sim \sqrt{\mu m a^2} \sim \sqrt{\bar{n}a^3}, \tag{13.150}$$

where we have used $n_{0,p_h} \simeq \bar{n}$ in the dilute limit (to leading order in $\tilde{\lambda}_{p_h}$) and $\lambda_{p_h} = 4\pi a/m$. We thus recover the small parameter $\bar{n}a^3 \sim \mu m a^2$ of the three-dimensional dilute Bose gas. The sound mode velocity (13.91) and the superfluid density (13.93) are then given by[35]

$$c = \left(\frac{\lambda_{p_h} n_{0,p_h}}{m}\right)^{1/2} = \frac{\sqrt{4\pi a\bar{n}}}{m}, \qquad n_s = Z_{A,p_h} n_{0,p_h} = \bar{n} \tag{13.151}$$

to leading order in $\tilde{\lambda}_{p_h}$. Equations (13.149) and (13.151) are the standard results for the thermodynamics of a Bose gas in the dilute limit to leading order in $\bar{n}a^3$ (Sec. 7.4).

Verifier dernière expression

For a two-dimensional system, one finds

$$\mu \simeq \frac{2\pi\bar{n}/m}{\left|\ln\left(\sqrt{ma^2\mu}\right)\right|} \simeq \frac{2\pi\bar{n}/m}{|\ln\sqrt{\bar{n}a^2}|}. \tag{13.152}$$

using $n_{0,k} \simeq \bar{n}$ and $\bar{n}a^2 \ll 1$. The sound mode velocity is given by

$$c = \sqrt{\frac{\lambda_{p_h} n_{0,p_h}}{m}} \simeq \frac{1}{m}\sqrt{\frac{2\pi\bar{n}}{|\ln\sqrt{ma^2\mu}|}} \simeq \frac{1}{m}\sqrt{\frac{2\pi\bar{n}}{|\ln\sqrt{\bar{n}a^2}|}}. \tag{13.153}$$

Again we reproduce the mean-field results obtained in section 7.4.[36]

13.3.4.2 Beyond mean-field theory

One can go beyond the mean-field theory by numerically solving the RG equations. All thermodynamic quantities can be derived from the pressure $P(\mu,T) = -U_{k=0}(n_{0,k=0})$, which is obtained by solving the RG equation

$$\partial_t U_k(n_{0,k}) = -\frac{1}{2}\int_p \partial_t R_k(p)\left[\mathcal{G}_{k,11}(p;n_{0,k})e^{i\omega_n 0^+} + \mathcal{G}_{k,22}(p;n_{0,k})e^{-i\omega_n 0^+}\right] \tag{13.154}$$

[34] The dimensionless parameter $\tilde{\lambda}_{p_h}$ can be related to the ratio $\gamma = 2m\lambda_{p_h}\bar{n}^{1-2/d}$ between the mean interaction energy per particle $\lambda_{p_h}\bar{n}$ and the characteristic kinetic energy $\bar{n}^{2/d}/2m$. In three dimensions $\tilde{\lambda}_{p_h} \sim \gamma^{3/2}$, while $\tilde{\lambda}_{p_h} \sim \gamma$ in two dimensions.

[35] The result $n_s = \bar{n}$ is exact (to all order in $\tilde{\lambda}_{p_h}$) due to Galilean invariance (Sec. 13.3.1).

[36] In the weak-coupling limit $mg \ll 1$, where the scattering length (13.128) is exponentially small, $\mu \simeq g\bar{n}$ and $c \simeq \sqrt{g\bar{n}/m}$ except for exponentially small values of the chemical potential or the density.

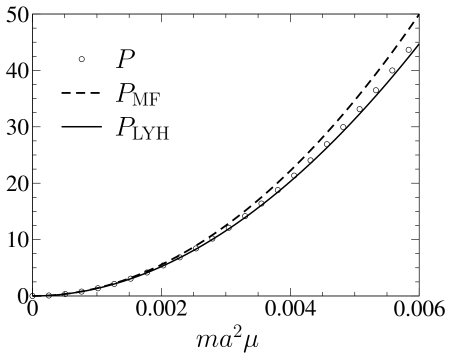

Figure 13.8: Pressure $P(\mu)$ *vs* $ma^2\mu$ in a three-dimensional Bose gas at zero temperature ($2m=1$ and $a \simeq 0.035619$). The circle show the NPRG result obtained from (13.155). The dashed line shows the mean-field expression $m\mu^2/8\pi a$ and the solid one the result including the Lee-Huang-Yang correction [Eq. (7.176)].

(here we consider an arbitrary temperature), where $\mathcal{G}_{11}(p) = -\langle\psi(p)\psi^*(p)\rangle$ and $\mathcal{G}_{22}(p) = -\langle\psi^*(-p)\psi(-p)\rangle$ [Eq. (13.3)]. Since $\mathcal{G}_{11}(p) = \mathcal{G}_{22}(-p) \sim 1/i\omega_n$ for $|\omega_n| \to \infty$, it is necessary to introduce convergence factors $e^{\pm i\omega_n 0^+}$ to make the Matsubara sums well defined.[37] Equation (13.154) can be rewritten as

$$\partial_t U_k(n_{0,k}) = -\frac{1}{2}\int_p \partial_t R_k(p)\,[G_{k,\rm ll}(p;n_{0,k}) + G_{k,\rm tt}(p;n_{0,k}) - 2iG_{k,\rm lt}(p;n_{0,k})]\,e^{i\omega_n 0^+}. \tag{13.155}$$

The Matsubara sums for $G_{k,\rm ll}$ and $G_{k,\rm tt}$ are convergent and do not require the convergence factor. This is not the case for $G_{k,\rm lt}$, which decays as $1/i\omega_n$ for large $|\omega_n|$. Since $G_{k,\rm lt}$ is odd in ω_n, the Matsubara sum would vanish without the convergence factor. This means that the sum is due to large frequencies for which $G_{k,\rm lt}$ is equal to its noninteracting value, i.e.

$$\frac{1}{\beta}\sum_{\omega_n} iG_{k,\rm lt}(p;n_{0,k})e^{i\omega_n 0^+} = \frac{1}{\beta}\sum_{\omega_n}\frac{i\omega_n\, e^{i\omega_n 0^+}}{\omega_n^2 + (\epsilon_{\mathbf p}-\mu)^2} = -\frac{1}{2}. \tag{13.156}$$

Here we assume a frequency-independent cutoff function. As for the contributions of $G_{k,\rm ll}$ and $G_{k,\rm tt}$, they can be computed using the LPA' approximation. At zero temperature, the pressure can also be obtained from $d^2P/d\mu^2 = d\bar n/d\mu$ and the thermodynamic relation (13.99), without solving the flow equation $\partial_t U_k$.

Figure 13.8 shows the pressure as a function of $ma^2\mu$ in a three-dimensional system at zero temperature. In the dilute limit $ma^2\mu \ll 1$, deviations from the mean-field result $m\mu^2/8\pi a$ are very well described by the Lee-Huang-Wang correction (see Sec. 7.4). More generally, the pressure takes the form

$$P(\mu,T) = \left(\frac{m}{2\pi}\right)^{d/2}\mu^{d/2+1}\mathcal{G}^{(d)}_{\rm DBG}\left(\frac{T}{\mu},\tilde\lambda^{(\rm vac)}_{p_h}\right) \tag{13.157}$$

or

$$P(\mu,T) = \left(\frac{m}{2\pi}\right)^{d/2}T^{d/2+1}\mathcal{F}^{(d)}_{\rm DBG}\left(\frac{\mu}{T},\tilde\lambda^{(\rm vac)}_{p_T}\right) \tag{13.158}$$

[37] See the discussion page 70 in chapter 1.

($p_T = \sqrt{2mT}$) in the universal regime $ma^2|\mu|, ma^2T \ll 1$ (Sec. 7.4.4). $\tilde\lambda_k^{(\rm vac)}$ is the dimensionless interaction in vacuum [Eqs. (13.126,13.129)]. The universal scaling functions $\mathcal{G}^{(d)}_{\rm DBG}$ and $\mathcal{F}^{(d)}_{\rm DBG}$, characteristic of the d-dimensional dilute-Bose-gas universality class, are deduced from the numerical solution of the flow equations.[38]

13.4 Superfluid transition in a dilute Bose gas

Appendix 13.A Ward identities in a superfluid Bose gas

In this Appendix, we discuss the symmetry properties of the effective action Γ_k (Sec. 13.A.1). We then derive the Ward identities that follow from the gauge invariance and the Galilean invariance of the microscopic action (13.1) (Secs. 13.A.2.1).

13.A.1 Symmetries of the effective action Γ_k

The microscopic action (13.1) is invariant in the (semilocal) U(1) transformation

$$\psi(x) \to \psi(x)e^{i\alpha(\tau)}, \quad \psi^*(x) \to \psi^*(x)e^{-i\alpha(\tau)}, \quad \mu \to \mu + i\partial_\tau\alpha(\tau), \tag{13.A.1}$$

where $\alpha(\tau)$ is an arbitrary time-dependent phase. The invariance holds because the the combination $\partial_\tau - \mu$ in the action (13.1) acts as a covariant derivative. At zero temperature, the microscopic action is also invariant in the Galilean transformation

$$\psi'(\mathbf{r}',\tau') = e^{\frac{1}{2}m\mathbf{v}^2\tau - im\mathbf{v}\cdot\mathbf{r}}\psi(\mathbf{r},\tau), \quad \psi'^*(\mathbf{r}',\tau') = e^{-\frac{1}{2}m\mathbf{v}^2\tau + im\mathbf{v}\cdot\mathbf{r}}\psi^*(\mathbf{r},\tau), \tag{13.A.2}$$

where $\mathbf{r}' = \mathbf{r} + i\mathbf{v}\tau$ and $\tau' = \tau$ (Sec. 2.2.5). The invariance is ensured by the fact that the derivative terms appear in the kinetic term in the combination $\partial_\tau - \boldsymbol{\nabla}^2/2m$. The interaction term is clearly Galilean invariant for a local interaction.

Since the transformations (13.A.1) and (13.A.2) act linearly on the field, they also leave the effective action $\Gamma[\phi^*,\phi]$ invariant (Sec. 2.3.3). For Γ to be invariant, the derivative operators ∂_τ, $\boldsymbol{\nabla}$ and the chemical potential must appear only in the combination

$$D_\pm = \pm\partial_\tau - \mu - \frac{\boldsymbol{\nabla}^2}{2m}, \tag{13.A.3}$$

with D_+ acting on ϕ and D_- on ϕ^*. Thus we can write Γ as an expansion in the operator $D_\pm$,

$$\begin{aligned}\Gamma[\phi^*,\phi] = \int_0^\beta d\tau \int d^dr \Big\{ & U(n) + \frac{1}{2}Z_1(n)(\phi^* D_+\phi + \phi D_-\phi^*) \\ & + \frac{1}{2}Z_2(n)(\phi^* D_+^2\phi + \phi D_-^2\phi^*) + \cdots \Big\}.\end{aligned} \tag{13.A.4}$$

The action may also contain terms of the type $n^p(\boldsymbol{\nabla}^q n)^r$ (p,q,r integers) since both n and $\boldsymbol{\nabla}n$ are Galilean invariant (with qr even to ensure parity invariance). Note that at finite temperature, Galilean invariance does not hold since it is broken by the thermal

Vérifier avec Floerchinger-Wetterich

[38] In three dimensions, the scaling functions $\mathcal{G}^{(3)}_{\rm DBG}$ and $\mathcal{F}^{(3)}_{\rm DBG}$ can be computed from a loop expansion in some regimes (Sec. 7.4) but not near the superfluid transition temperature (Sec. 13.4). The calculation is much harder in two dimensions where the loop expansion breaks down in the finite-temperature superfluid phase (the superfluid phase at $T > 0$ has a vanishing condensate density ($n_0 = 0$) and algebraic order) [29].

bath.[39] The U(1) invariance (13.A.1) still holds but does not lead to a useful Ward identity (Sec. 13.A.2.1).

When the regulator term ΔS_k is invariant in the transformations (13.A.1) and (13.A.2), the scale-dependent action $\Gamma_k[\phi^*, \phi]$ is also invariant. This is not the case in the approach discussed in section 13.3. On the one hand, the cutoff functions (13.75) and (13.76) are not Galilean invariant (the function (13.76) also breaks the semilocal U(1) invariance). On the other hand, the ansatz (13.77) is not of the form (13.A.4). Nevertheless, the numerical solution of the flow equations shows that the Ward identities associated with the (semilocal) U(1) and Galilean invariances (Sec. 13.A.2) remain very well satisfied.

Vrai uniquement à couplage faible?

13.A.2 Ward identities

13.A.2.1 Gauge invariance

Let us consider the microscopic action

$$S = \int dx \Big[\psi^*(x) \Big(\partial_\tau - \mu(x) - \frac{1}{2m} [\boldsymbol{\nabla} - i\mathbf{A}(x)]^2 \Big) \psi(x) + \frac{g}{2} |\psi(x)|^4 \Big] \tag{13.A.5}$$

in the presence of external sources $\mu(x)$ and $\mathbf{A}(x)$ and at zero temperature ($\beta \to \infty$). S is invariant in the gauge transformation

$$\begin{aligned} &\psi(x) \to \psi(x) e^{i\alpha(x)}, \qquad \psi^*(x) \to \psi^*(x) e^{-i\alpha(x)}, \\ &\mu(x) \to \mu(x) + i\partial_\tau \alpha(x), \qquad \mathbf{A}(x) \to \mathbf{A}(x) + \boldsymbol{\nabla}\alpha(x), \end{aligned} \tag{13.A.6}$$

where $\alpha(x)$ is an arbitrary real function. This implies that the effective action satisfies

$$\Gamma[R(\alpha)\phi; \mu + i\partial_\tau \alpha, A_\nu + \partial_\nu \alpha] = \Gamma[\phi; \mu, A_\nu], \tag{13.A.7}$$

where $\phi = (\phi_1, \phi_2)$ is a two-dimensional vector and

$$R(\alpha) = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix} \tag{13.A.8}$$

is a rotation matrix. Differentiating (13.A.7) with respect to $\alpha(x)$ and setting $\alpha(x) = 0$, we obtain

$$\sum_{i,j} \frac{\delta \Gamma}{\delta \phi_i(x)} \epsilon_{ij} \phi_j(x) + i\partial_\tau \frac{\delta \Gamma}{\delta \mu(x)} + \sum_\nu \partial_\nu \frac{\delta \Gamma}{\delta A_\nu(x)} = 0. \tag{13.A.9}$$

Differentiating now with respect to $\phi_l(x_2)$ and $\mu(x_2)$ and setting $\phi = (\sqrt{2n_0}, 0)$, $\mu(x) = \mu$ and $\mathbf{A}(x) = 0$, we deduce

$$\begin{aligned} &-\sqrt{2n_0}\Gamma^{(2)}_{l2}(x_2, x_1; n_0) + i\partial_{\tau_1} \Gamma^{(2)}_{l;0}(x_2, x_1; n_0) + \sum_{\nu_1} \partial_{\nu_1} \Gamma^{(2)}_{l;\nu_1}(x_2, x_1; n_0) = 0, \\ &-\sqrt{2n_0}\Gamma^{(2)}_{2;0}(x_1, x_2; n_0) + i\partial_{\tau_1} \Gamma^{(2)}_{;00}(x_1, x_2; n_0) + \sum_{\nu_1} \partial_{\nu_1} \Gamma^{(2)}_{;\nu_1 0}(x_1, x_2; n_0) = 0, \end{aligned} \tag{13.A.10}$$

[39] At nonzero temperature ($\beta < \infty$), the transformation (13.A.2) is not allowed since it does not conserve the periodicity of the field: $\psi(\mathbf{r}, \tau + \beta) = \psi(\mathbf{r}, \tau)$.

where we have introduced

$$\begin{aligned}\Gamma^{(2)}_{l;0}(x_2,x_1;n_0) &= \frac{\delta^{(2)}\Gamma}{\delta\phi_l(x_2)\delta\mu(x_1)}\Big|_{n=n_0,\mu(x)=\mu,\mathbf{A}=0}, \\ \Gamma^{(2)}_{;00}(x_2,x_1;n_0) &= \frac{\delta^{(2)}\Gamma}{\delta\mu(x_2)\delta\mu(x_1)}\Big|_{n=n_0,\mu(x)=\mu,\mathbf{A}=0},\end{aligned} \tag{13.A.11}$$

and similar definitions for $\Gamma^{(2)}_{l;\nu}(x_2,x_1;n_0)$ and $\Gamma^{(2)}_{;\nu 0}(x_2,x_1;n_0)$. In Fourier space, equations (13.A.10) lead to the Ward identities

$$\sqrt{2n_0}\Gamma^{(2)}_{12}(p;n_0) + \omega\Gamma^{(2)}_{1;0}(p;n_0) + \sum_\nu ip_\nu\Gamma^{(2)}_{1;\nu}(p;n_0) = 0, \tag{13.A.12a}$$

$$\sqrt{2n_0}\Gamma^{(2)}_{22}(p;n_0) + \omega\Gamma^{(2)}_{2;0}(p;n_0) + \sum_\nu ip_\nu\Gamma^{(2)}_{2;\nu}(p;n_0) = 0, \tag{13.A.12b}$$

$$\sqrt{2n_0}\Gamma^{(2)}_{2;0}(p;n_0) - \omega\Gamma^{(2)}_{;00}(p;n_0) - \sum_\nu ip_\nu\Gamma^{(2)}_{;\nu 0}(p;n_0) = 0. \tag{13.A.12c}$$

From (13.A.12a), we deduce[40]

$$\begin{aligned}\frac{\partial}{\partial\omega}\Gamma^{(2)}_{12}(p;n_0)\Big|_{p=0} &= -\frac{1}{\sqrt{2n_0}}\Gamma^{(2)}_{1;0}(p=0;n_0) \\ &= -\frac{1}{\sqrt{2n_0}}\frac{\partial^2 U}{\partial\phi_1\partial\mu}\Big|_{n_0} = -\frac{\partial^2 U}{\partial n\partial\mu}\Big|_{n_0},\end{aligned} \tag{13.A.13}$$

where the effective potential $U(\mu,n)$ is considered as a function of both μ and n. From (13.A.12b) and (13.A.12c), we obtain

$$\begin{aligned}\frac{\partial}{\partial\omega^2}\Gamma^{(2)}_{22}(p;n_0)\Big|_{p=0} &= -\frac{1}{\sqrt{2n_0}}\frac{\partial}{\partial\omega}\Gamma^{(2)}_{2;0}(p;n_0)\Big|_{p=0}, \\ \frac{\partial}{\partial\omega}\Gamma^{(2)}_{2;0}(p;n_0)\Big|_{p=0} &= \frac{1}{\sqrt{2n_0}}\Gamma^{(2)}_{;00}(0;n_0) = \frac{1}{\sqrt{2n_0}}\frac{\partial^2 U}{\partial\mu^2}\Big|_{n_0}\end{aligned} \tag{13.A.14}$$

and therefore[40]

$$\frac{\partial}{\partial\omega^2}\Gamma^{(2)}_{22}(p;n_0)\Big|_{p=0} = -\frac{1}{2n_0}\frac{\partial^2 U}{\partial\mu^2}\Big|_{n_0}. \tag{13.A.15}$$

13.A.2.2 Galilean invariance

Revoir cette section! Cf. Nozieres ou Pistolesi?

Another Ward identity can be obtained from the Galilean invariance of the microscopic action. The latter is invariant in the transformation $\psi(x)\to\psi(x)e^{i\mathbf{q}\cdot\mathbf{r}}$, $\psi^*(x)\to\psi(x)e^{-i\mathbf{q}\cdot\mathbf{r}}$ if we shift the chemical potential μ by $\mathbf{q}^2/2m$, which implies

$$\Gamma[R(\alpha)\phi,\mu+\mathbf{q}^2/2m] = \Gamma[\phi,\mu], \tag{13.A.16}$$

where $\alpha(x)=\mathbf{q}\cdot\mathbf{r}$ and the chemical potential μ is taken uniform and time independent. To order $\mathbf{q}^2$, equation (13.A.16) gives

$$\begin{aligned}0 &= \frac{\mathbf{q}^2}{2m}\frac{\partial\Gamma[\bar\phi]}{\partial\mu} + n_0\int dxdx'\Gamma^{(2)}_{22}(x-x';n_0)\alpha(x)\alpha(x') \\ &= \frac{\mathbf{q}^2}{2m}\frac{\partial\Gamma[\bar\phi]}{\partial\mu} + \beta V n_0\mathbf{q}^2\frac{\partial}{\partial\mathbf{p}^2}\Gamma^{(2)}_{22}(p;n_0)\Big|_{p=0},\end{aligned} \tag{13.A.17}$$

[40] For $\phi=(\sqrt{2n_0},0)$, we can identify $\Gamma^{(2)}_{12}$ to Γ_C and $\Gamma^{(2)}_{22}$ to Γ_A.

where we have set $\phi = \bar\phi \equiv (\sqrt{2n_0}, 0)$. Since

$$\frac{\partial \Gamma[\bar\phi]}{\partial \mu} = \beta V \frac{\partial U}{\partial \mu}\Big|_{n_0} = -\beta V \bar n \tag{13.A.18}$$

(see Eq. (13.96)), we finally obtain[40]

$$\frac{\partial}{\partial \mathbf{p}^2} \Gamma^{(2)}_{22}(p; n_0)\Big|_{p=0} = \frac{\bar n}{2mn_0}, \tag{13.A.19}$$

where $\bar n$ is the mean boson density.

Appendix 13.B Threshold functions

In this section, we give explicit expressions for the threshold functions I_α and $J_{\alpha\beta}(p)$ involved in the NPRG flow equations. We assume a cutoff function R_k acting both on momenta and frequencies [Eq. (13.76)]. Similar results can be obtained for a cutoff function acting only on momenta.[41]

13.B.1 Dimensionful threshold functions

With the notations

$$\begin{aligned} &A_k(p) = Z_{A,k}\epsilon_{\mathbf{p}} + V_{A,k}\omega^2 + R_k(p), \qquad B_k(p) = A_k(p) + 2n_{0,k}\lambda_k, \\ &C_k(p) = Z_{C,k}\omega, \qquad D_k(p) = C_k(p)^2 + A_k(p)B_k(p), \end{aligned} \tag{13.B.1}$$

we have

$$G_{\mathrm{k,ll}} = -\frac{A_k}{D_k}, \quad G_{\mathrm{k,tt}} = -\frac{B_k}{D_k}, \quad G_{\mathrm{k,lt}} = \frac{C_k}{D_k}, \tag{13.B.2}$$

and

$$\begin{aligned} \tilde\partial_t G_{\mathrm{k,ll}} = -\dot R_k \frac{C_k^2 - A_k^2}{D_k^2}, \quad \tilde\partial_t G_{\mathrm{k,tt}} = -\dot R_k \frac{C_k^2 - B_k^2}{D_k^2}, \\ \tilde\partial_t G_{\mathrm{k,lt}} = -\dot R_k \frac{C_k(A_k + B_k)}{D_k^2}, \end{aligned} \tag{13.B.3}$$

where

$$\dot R_k = \partial_t R_k = -Z_{A,k}\epsilon_k Y(\eta_{A,k} r + 2Yr'), \qquad Y = \frac{\mathbf{p}^2}{k^2} + \frac{\omega^2}{c_\Lambda^2 k^2} \equiv y + \tilde\omega^2, \tag{13.B.4}$$

with $r \equiv r(Y)$ and $r' = \partial r/\partial Y$. All propagators are evaluated in the equilibrium state $n = n_{0,k}$. Equations (13.B.2) and (13.B.3) can be used to compute $I_{k,\alpha}$ and $J_{k,\alpha\beta}(p)$, as well as $\partial_\omega J_{k,\alpha\beta}(p)|_{\omega=0}$ and $\partial_{\omega^2} J_{k,\alpha\beta}(p)|_{\omega=0}$.

Following the same lines as in the classical O(N) model (see Appendix 11.B), we find

$$\frac{\partial}{\partial \epsilon_{\mathbf{p}}} J_{k,\alpha\beta}(p)\Big|_{p=0} = 4\frac{v_d}{d} k^{d+2} Z_{A,k} \int_\omega \int_0^\infty dy\, y^{d/2} \Big\{ k^2 Y(\eta_{A,k} r + 2Yr') \frac{\partial}{\partial R_k} G'_{k,\alpha}$$

[41] In this case, the expression of R_k and its derivatives should be changed: $R_k(\mathbf{p}) = Z_{A,k}\epsilon_k y r(y)$, $\partial_t R_k(\mathbf{p}) = -Z_{A,k}\epsilon_k y[\eta_{A,k} r(y) + 2yr'(y)]$, etc.

$$+ [\eta_{A,k} r + (\eta_{A,k} + 4)Yr' + 2Y^2 r''] \frac{\partial}{\partial R'_k} G'_{k,\alpha} \Big\} G'_{k,\beta}. \tag{13.B.5}$$

The function $G'_{k,\alpha}(p) = \partial_{\mathbf{p}^2} G_{k,\alpha}(p)$ can be expressed as

$$\begin{gathered} G'_{\mathrm{k,ll}} = -\frac{1}{D_k^2}(C_k^2 A'_k - A_k^2 B'_k), \quad G'_{\mathrm{k,tt}} = -\frac{1}{D_k^2}(C_k^2 B'_k - A'_k B_k^2), \\ G'_{\mathrm{k,lt}} = -\frac{C_k}{D_k^2}(A'_k B_k + A_k B'_k), \end{gathered} \tag{13.B.6}$$

where

$$A'_k = \partial_{\mathbf{p}^2} A_k = \frac{Z_{A,k}}{2m}(1 + r + Yr'), \quad B'_k = \partial_{\mathbf{p}^2} B_k = \frac{Z_{A,k}}{2m}(1 + r + Yr'). \tag{13.B.7}$$

We also have

$$\begin{aligned} \frac{\partial}{\partial R_k} G'_{\mathrm{k,ll}} &= \frac{2}{D_k^3}\left[C_k^2(A_k B'_k + A'_k B_k + A_k A'_k) - A_k^3 B'_k\right], \\ \frac{\partial}{\partial R_k} G'_{\mathrm{k,tt}} &= \frac{2}{D_k^3}\left[C_k^2(A_k B'_k + A'_k B_k + B_k B'_k) - A'_k B_k^3\right], \\ \frac{\partial}{\partial R_k} G'_{\mathrm{k,lt}} &= -\frac{C_k}{D_k^3}\left[(C^2 - A_k B_k)(A'_k + B'_k) - 2A_k^2 B'_k - 2A'_k B_k^2\right], \end{aligned} \tag{13.B.8}$$

and

$$\begin{gathered} \frac{\partial}{\partial R'_k} G'_{\mathrm{k,ll}} = -\frac{1}{D_k^2}(C_k^2 - A_k^2), \qquad \frac{\partial}{\partial R'_k} G'_{\mathrm{k,tt}} = -\frac{1}{D_k^2}(C_k^2 - B_k^2), \\ \frac{\partial}{\partial R'_k} G'_{\mathrm{k,lt}} = -\frac{C_k}{D_k^2}(A_k + B_k). \end{gathered} \tag{13.B.9}$$

Equations (13.B.6), (13.B.8) and (13.B.9) are used to compute $\partial_{\epsilon_{\mathbf{p}}} J_{\alpha\beta}(p)|_{p=0}$ [Eq. (13.B.5)] and in turn $\partial_t Z_{A,k}$.

13.B.2 Dimensionless threshold functions

We introduce the dimensionless propagators

$$\begin{gathered} \tilde G_{\mathrm{k,ll}} = \frac{G_{\mathrm{k,ll}}}{Z_{A,k}\epsilon_k} = -\frac{\tilde A_k}{\tilde D_k}, \qquad \tilde G_{\mathrm{k,tt}} = \frac{G_{\mathrm{k,tt}}}{Z_{A,k}\epsilon_k} = -\frac{\tilde B_k}{\tilde D_k}, \\ \tilde G_{\mathrm{k,lt}} = \frac{G_{\mathrm{k,lt}}}{Z_{A,k}\epsilon_k} = \frac{\tilde C_k}{\tilde D_k}, \end{gathered} \tag{13.B.10}$$

where

$$\begin{aligned} \tilde A_k &= (Z_{A,k}\epsilon_k)^{-1} A_k = \tilde V_{A,k}\tilde\omega^2 + y + Yr, \\ \tilde B_k &= (Z_{A,k}\epsilon_k)^{-1} B_k = \tilde A_k + 2\tilde\lambda_k \tilde n_{0,k}, \\ \tilde C_k &= (Z_{A,k}\epsilon_k)^{-1} C_k = \tilde\omega. \end{aligned} \tag{13.B.11}$$

The dimensionless coefficients $\tilde I_{k,\alpha}$ and $\tilde J_{k,\alpha\beta}(p)$ are then defined by

$$\tilde I_{k,\alpha} = k^{-d} Z_{C,k} I_{k,\alpha} = -2v_d \int_{y,\tilde\omega} y^{d/2-1}(\eta_{A,k} r + 2Yr') \frac{\partial \tilde G_{k,\alpha}}{\partial r} \tag{13.B.12}$$

and[42]

$$\begin{aligned}\tilde J_{k,\alpha\beta}(p) &= \frac{Z_{A,k}Z_{C,k}\epsilon_k}{k^d} J_{k,\alpha\beta}(p) \\ &= -\frac{1}{8\pi^2}\int_0^{(4-d)\pi} d\theta \sin^{d-2}\theta \int_{y,\tilde\omega} y^{d/2-1}(\eta_{A,k}r + 2Yr')\frac{\partial \tilde G_{k,\alpha}(q)}{\partial r}\tilde G_{\beta,k}(p+q), \\ \tilde J_{k,\alpha\beta}(0) &= -2v_d \int_{y,\tilde\omega} y^{d/2-1}(\eta_{A,k}r + 2Yr')\frac{\partial \tilde G_{k,\alpha}}{\partial r}\tilde G_{k,\beta} \end{aligned} \tag{13.B.13}$$

($d=3$ or $d=2$). To evaluate (13.B.13), we use

$$\begin{gathered}\frac{\partial \tilde G_{\rm k,ll}}{\partial r} = \frac{Y}{\tilde D_k^2}(\tilde A_k^2 - \tilde C_k^2), \qquad \frac{\partial \tilde G_{\rm k,tt}}{\partial r} = \frac{Y}{\tilde D_k^2}(\tilde B_k^2 - \tilde C_k^2), \\ \frac{\partial \tilde G_{\rm k,lt}}{\partial r} = -\frac{Y\tilde C_k}{\tilde D_k^2}(\tilde A_k + \tilde B_k).\end{gathered} \tag{13.B.14}$$

In dimensionless form, Eq. (13.B.5) becomes[43]

$$\begin{aligned}\frac{\partial}{\partial y}\tilde J_{k,\alpha\beta}(p)\Big|_{p=0} &= \frac{Z_{A,k}Z_{C,k}\epsilon_k^2}{k^d}\frac{\partial}{\partial \epsilon_{\mathbf p}} J_{k,\alpha\beta}(p)\Big|_{p=0} \\ &= 4\frac{v_d}{d}\int_{y,\tilde\omega} y^{d/2}\bigg\{(\eta_{A,k}r + 2Yr')\frac{\partial \tilde G'_{k,\alpha}}{\partial r}\bigg|_{r+Yr'} \\ &\quad + \left[\eta_{A,k}r + (\eta_{A,k}+4)Yr' + 2Y^2r''\right]Y^{-1}\frac{\partial \tilde G'_{k,\alpha}}{\partial r'}\bigg\}\tilde G'_{k,\beta},\end{aligned} \tag{13.B.15}$$

where ($\tilde G'_{k,\alpha} = \partial_y \tilde G_{k,\alpha}$)

$$\begin{gathered}\tilde G'_{\rm k,ll} = -\frac{1}{\tilde D_k^2}(\tilde C_k^2\tilde A'_k - \tilde A_k^2 \tilde B'_k), \qquad \tilde G'_{\rm k,tt} = -\frac{1}{\tilde D_k^2}(\tilde C_k^2\tilde B'_k - \tilde B_k^2\tilde A'_k), \\ \tilde G'_{\rm k,lt} = -\frac{\tilde C_k}{\tilde D_k^2}(\tilde A'_k\tilde B_k + \tilde A_k\tilde B'_k).\end{gathered} \tag{13.B.16}$$

We also have[43]

$$\begin{aligned}\frac{\partial \tilde G'_{\rm k,ll}}{\partial r}\bigg|_{r+Yr'} &= \frac{2Y}{\tilde D_k^3}\left[\tilde C_k^2(\tilde A_k\tilde B'_k + \tilde A'_k\tilde B_k + \tilde A_k\tilde A'_k) - \tilde A_k^3\tilde B'_k\right], \\ \frac{\partial \tilde G'_{\rm k,tt}}{\partial r}\bigg|_{r+Yr'} &= \frac{2Y}{\tilde D_k^3}\left[\tilde C_k^2(\tilde A_k\tilde B'_k + \tilde A'_k\tilde B_k + \tilde B_k\tilde B'_k) - \tilde A'_k\tilde B_k^3\right], \\ \frac{\partial \tilde G'_{\rm k,lt}}{\partial r}\bigg|_{r+Yr'} &= -\frac{Y\tilde C_k}{\tilde D_k^3}\left[(\tilde C_k^2 - \tilde A_k\tilde B_k)(\tilde A'_k + \tilde B'_k) - 2\tilde A_k^2\tilde B'_k - 2\tilde A'_k\tilde B_k^2\right],\end{aligned} \tag{13.B.17}$$

[42] In Eqs. (13.B.13), $y=\mathbf q^2/k^2$ and $\cos\theta = \mathbf q\cdot\mathbf p/|\mathbf q||\mathbf p|$.

[43] In equations (13.B.15) and (13.B.17), the derivative $\partial/\partial r$ is taken with $r+Yr'$ (i.e. $R'_k = \partial_{\mathbf p^2}R_k$) fixed.

and

$$\frac{\partial \tilde G'_{\mathrm{k,ll}}}{\partial r'} = -\frac{Y}{\tilde D_k^2}(\tilde C_k^2 - \tilde A_k^2), \qquad \frac{\partial \tilde G'_{\mathrm{k,tt}}}{\partial r'} = -\frac{Y}{\tilde D_k^2}(\tilde C_k^2 - \tilde B_k^2),$$
$$\frac{\partial \tilde G'_{\mathrm{k,lt}}}{\partial r'} = -\frac{Y\tilde C_k}{\tilde D_k^2}(\tilde A_k + \tilde B_k). \tag{13.B.18}$$

We have introduced

$$\tilde A'_k = \partial_y \tilde A_k = 1 + r + Yr', \quad \tilde B'_k = \partial_y \tilde B_k = 1 + r + Yr'. \tag{13.B.19}$$

13.B.3 Threshold functions with the theta cutoff

The threshold functions can be computed exactly in the limit $k \ll p_G$ if we take $r(Y) = (1-Y)\Theta(1-Y)/Y$ and $c_k = (Z_{A,k}/2mV_{A,k})^{1/2} \simeq c_0$ instead of c_Λ (c_k is nearly k independent for $k \ll p_G$). For $Y \leq 1$, one then has

$$\tilde A_k = Y + Yr(Y) = 1, \quad \tilde B_k = 1 + 2\tilde\lambda_k \tilde n_{0,k}, \quad \tilde D_k = 1 + 2\tilde\lambda_k \tilde n_{0,k} + \tilde\omega^2. \tag{13.B.20}$$

We also observe that the condition $Y \leq 1$ implies

$$|\tilde\omega| \leq \frac{Z_{C,k}}{Z_{A,k}\epsilon_k} c_0 k \sim k^{2-d}. \tag{13.B.21}$$

On the other hand,

$$\tilde\lambda_k \tilde n_{0,k} = (Z_{A,k}\epsilon_k)^{-1}\lambda_k n_{0,k} \sim k^{1-d}. \tag{13.B.22}$$

In equations (13.B.21) and (13.B.22) we have anticipated that $Z_{C,k}, \lambda_k \sim k^{3-d}$ for $d < 3$. We can therefore neglect $\tilde\omega^2$ with respect to $\tilde B_k$, and

$$\tilde D_k \simeq \tilde B_k \simeq 2\tilde\lambda_k \tilde n_{0,k} \tag{13.B.23}$$

becomes frequency independent. For $d = 3$, $|\tilde\omega| \lesssim 1/|k \ln k|$ and $\tilde\lambda_k \tilde n_{0,k} \sim 1/|k^2 \ln k|$, so that (13.B.23) holds.

We are now in a position to compute the infrared limit of the coefficients $\tilde I_{k,\alpha}$ and $\tilde J_{k,\alpha\beta}$. Since $\eta_{A,k} \to 0$, we have

$$\tilde I_{k,\mathrm{ll}} = -4v_d \int_{y,\tilde\omega} y^{d/2-1} Y^2 r' \frac{\tilde A_k^2 - \tilde\omega^2}{\tilde D_k^2},$$
$$\tilde I_{k,\mathrm{tt}} = -4v_d \int_{y,\tilde\omega} y^{d/2-1} Y^2 r' \frac{\tilde B_k^2 - \tilde\omega^2}{\tilde D_k^2}. \tag{13.B.24}$$

Since $|\tilde\omega|, \tilde A_k \ll \tilde B_k$, we can neglect $\tilde I_{k,\mathrm{ll}}$ with respect to $\tilde I_{k,\mathrm{tt}}$ and approximate

$$\tilde I_{k,\mathrm{tt}} \simeq -4v_d \int_{y,\tilde\omega} y^{d/2-1} Y^2 r' \frac{\tilde B_k^2}{\tilde D_k^2} = 4v_d \int_{y,\tilde\omega} y^{d/2-1}\theta(1-Y). \tag{13.B.25}$$

For any function $f(Y)$,

$$v_d \int_0^\infty dy\, y^{d/2-1} \int_{-\infty}^{\infty} \frac{d\tilde\omega}{2\pi} f(Y) = \tilde V_{A,k}^{-1/2} v_{d+1} \int_0^\infty dY\, Y^{(d-1)/2} f(Y), \tag{13.B.26}$$

so that we finally obtain

$$\tilde{I}_{k,\mathrm{tt}} \simeq 8\frac{v_{d+1}}{d+1}\tilde{V}_{A,k}^{-1/2} \tag{13.B.27}$$

and

$$\begin{aligned}\partial_t \tilde{n}_{0,k} &\simeq -(d+\eta_{C,k})\tilde{n}_{0,k} + \frac{1}{2}\tilde{I}_{k,\mathrm{tt}} \\ &\simeq -(d+\eta_{C,k})\tilde{n}_{0,k} + 4\frac{v_{d+1}}{d+1}\tilde{V}_{A,k}^{-1/2} \\ &\simeq -(d+\eta_{C,k})\tilde{n}_{0,k},\end{aligned} \tag{13.B.28}$$

where we have used the fact that the condensate density $n_{0,k}$ flows to a finite value when $k \to 0$ (so that the flow of $\tilde{n}_{0,k}$ is determined by the purely dimensional contribution).

With a similar reasoning, we find

$$\begin{aligned}\partial_t \tilde{\lambda}_k &\simeq (d-2+\eta_{C,k})\tilde{\lambda}_k - \tilde{\lambda}_k^2 \tilde{J}_{k;\mathrm{tt,tt}}(0) \simeq (d-2+\eta_{C,k})\tilde{\lambda}_k + 8\frac{v_{d+1}}{d+1}\frac{\tilde{\lambda}_k^2}{\tilde{V}_{A,k}^{1/2}}, \\ \eta_{C,k} &\simeq -2\tilde{n}_{0,k}\tilde{\lambda}_k^2 \frac{\partial}{\partial\tilde{\omega}}\tilde{J}_{k;\mathrm{tt,lt}}(p)\Big|_{p=0} \simeq -8\frac{v_{d+1}}{d+1}\frac{\tilde{\lambda}_k}{\tilde{V}_{A,k}^{1/2}}, \\ \partial_t \tilde{V}_{A,k} &\simeq (2+2\eta_{C,k})\tilde{V}_{A,k}.\end{aligned} \tag{13.B.29}$$

All the integrals involved in the derivation of (13.B.28,13.B.29) are $d+1$-dimensional integrals of the type (13.B.26). This is a direct manifestation of the relativistic invariance which emerges in the low-energy limit (Sec. 13.3.2).

Guide to the bibliography

- For references on the Bogoliubov theory, see the bibliography in chapter 7.
- Early attempts to improve the Bogoliubov theory and the occurrence of infrared divergences are discussed in Refs. [1–4].
- The leading infrared behavior of the one-particle Green function [Eqs. (13.106) without the $G_{\rm ll}$ term] was obtained by Gavoret and Nozières [4]. The vanishing of the anomalous self-energy $\Sigma_{\rm an}(p=0)$ was proven by Nepomnyashchii and Nepomnyashchii (NN) [5]. Using diagrammatic techniques, NN then showed that the spectrum remains linear at low energy despite the vanishing of the self-energy [6]. Furthermore, they associated the infrared divergences of the perturbation theory to divergence of the longitudinal propagator [7].
- The necessity to develop a perturbation theory free of infrared divergences was actually realized by Popov before NN [8–10]. Popov's hydrodynamic theory is reviewed in [11, 12]. The infrared behavior of the one-particle propagator obtained by NN was reproduced in Ref. [13].
- The infrared behavior of the one-particle Green function is also discussed in Refs. [14, 15].
- The (non-perturbative) RG approach to interacting boson systems is discussed in Refs. [16–23] (zero temperature) and [24–29] (finite temperature). The spectral function of the one-particle propagator is discussed in [20–23].
- Refs. for the computation of T_c with the BMW approach.

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