Chapter 5

The electron gas

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The aim of this chapter is twofold. On the one hand, we wish to give a short (and incomplete) description of the three-dimensional electron gas at zero temperature. The analysis is based on the jellium model where the electrons interact via the Coulomb interaction and the neutrality of the system is maintained by a uniform background of positive charge with infinite inertia. We discuss the results obtained within the RPA: plasmon mode, screening, Fermi-liquid behavior, etc. In this respect, this chapter is complementary to our discussion of the neutral Fermi liquid in Chap. 4 as well as the general considerations of Sec. 3.4. On the other hand, the electron gas provides an opportunity to introduce standard many-body approaches (RPA, (time-dependent) Hartree-Fock theory) that come up in different contexts, and discuss in particular how they are formulated within diagrammatic or functional integral techniques. Except in Sec. 5.6, where we briefly discuss the Wigner crystallization of the electron gas, we deal only with a three-dimensional system.
5.1 The jellium model

In the jellium model, one assumes that the global neutrality of the system is maintained by the presence of a uniform background of positive charge. This can be seen as a first approximation to a metallic system. In a real crystal the positive charge (ions) is not uniform and can also have its own dynamics (lattice vibrations).

The Hamiltonian reads $\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}} + \hat{H}_{e-b} + \hat{H}_{b-b}$, where

$$\hat{H}_0 = \sum_{k,\sigma} \xi_k \hat{\psi}_k^\dagger(\mathbf{k})\hat{\psi}_\sigma(\mathbf{k})$$

and

$$\hat{H}_{\text{int}} = \frac{1}{2} \sum_{\substack{k, k', \sigma, \sigma'}} \frac{e^2}{4\pi\varepsilon_0 |\mathbf{r}_i - \mathbf{r}_j|},$$

are the kinetic and interaction Hamiltonians of the electrons. In second-quantized form, one obtains

$$\hat{H}_0 = \sum_{k,\sigma} \xi_k \hat{\psi}_k^\dagger(\mathbf{k})\hat{\psi}_\sigma(\mathbf{k})$$

(5.2)

and

$$\hat{H}_{\text{int}} = \frac{1}{2V} \sum_{k, k', \sigma, \sigma'} v(q) \hat{\psi}_\sigma^\dagger(\mathbf{k} + q)\hat{\psi}_{\sigma'}(\mathbf{k}' - q)\hat{\psi}_{\sigma'}(\mathbf{k}')\hat{\psi}_\sigma(\mathbf{k})$$

(5.3)

$$= \frac{1}{2} \sum_{\mathbf{q}} v(\mathbf{q}) \left[ \hat{n}(\mathbf{-q})\hat{n}(\mathbf{q}) - \frac{\hat{N}}{V} \right],$$

(5.4)

where $\hat{n}(\mathbf{q}) = \frac{1}{\sqrt{V}} \int d^3 r e^{-i\mathbf{q}\cdot\mathbf{r}} \hat{n}(\mathbf{r}) = \frac{1}{\sqrt{V}} \sum_{k,\sigma} \hat{\psi}_\sigma^\dagger(\mathbf{k})\hat{\psi}_\sigma(\mathbf{k})$ is the Fourier transform of the density operator $\hat{n}(\mathbf{r}) = \sum_\sigma \hat{\psi}_\sigma^\dagger(\mathbf{r})\hat{\psi}_\sigma(\mathbf{r})$. $v(\mathbf{q}) = e^2/\varepsilon_0 q^2$ is the Fourier transform of the Coulomb potential $\epsilon^2/4\pi\varepsilon_0 |\mathbf{r}|$. We now use the grand canonical ensemble and include the term $-\mu N$ into the Hamiltonian $\hat{H}_0$ ($\xi_k = k^2/2m - \mu$). The background of positive charge gives rise to the two additional terms

$$\hat{H}_{e-b} = -e^2 \int d^3 r d^3 r' \frac{n\hat{n}(\mathbf{r})}{4\pi\varepsilon_0 |\mathbf{r} - \mathbf{r'}|} = -v(0)n\hat{N},$$

$$\hat{H}_{b-b} = \frac{e^2}{2} \int d^3 r d^3 r' \frac{n^2}{4\pi\varepsilon_0 |\mathbf{r} - \mathbf{r'}|} = \frac{1}{2} n^2 \nabla v(0),$$

(5.5)

where $v(0) \equiv v(\mathbf{q} = 0)$. $\hat{H}_{b-b}$ gives the electrostatic energy of the uniform background of charge density $-en$ ($n = N/V$ is the electron density), and $\hat{H}_{e-b}$ the interaction energy between the positive background and the electrons.

Because of the long-range nature of the Coulomb interaction, the integrals appearing in (5.5) are not convergent and $v(0)$ is not defined. Since the system is neutral, the various infinites should cancel and the energy of the system be finite. To deal with this difficulty, one regularizes the Coulomb interaction by choosing

$$v(\mathbf{r}) = \frac{e^2}{4\pi\varepsilon_0} \frac{e^{-|\mathbf{r}|}}{|\mathbf{r}|},$$

i.e., $v(\mathbf{q}) = \frac{e^2}{\varepsilon_0 (q^2 + \kappa^2)}$.

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All terms in (5.5) are now well defined. Physical quantities are obtained by taking first the
thermodynamic limit $V \to \infty$ and then $\kappa \to 0$. This procedure can be justified \textit{a posteriori}
by noting that the final results are independent of $\kappa$.

The part of the Hamiltonian $\hat{H}$ involving $v(0) = v(q = 0)$ reads

$$
\frac{1}{2V} v(0)(\hat{N}^2 - \hat{N}) - v(0)n\hat{N} + \frac{1}{2}Vv(0)n^2 \simeq - \frac{1}{2} v(0)n,
$$

(5.7)
since the fluctuations of the operator $\hat{N}$ becomes negligible in the thermodynamic limit.\footnote{Note that $\Pi_{nn}$ is the right quantity to consider in order to obtain satisfying expressions for $\chi_{nn}$ and $\epsilon_{\parallel}$. Any approximate $\Pi_{nn}$ will give a result for $\chi_{nn}$ and $\epsilon_{\parallel}$ to infinite order in the Coulomb interaction. By contrast, a finite order approximation to $\chi_{nn}$ is likely to violate basic properties such as causality. This is the case for instance if one retains only the first two diagrams in Fig. 5.2. Note that a similar observation was made when discussing the one-particle Green function $G$ (Sec. 3.5): The self-energy $\Sigma$ (and not $G$) is the right quantity to consider in perturbation theory.}

The energy $\langle \hat{H} \rangle$ being an extensive quantity, the part involving $v(0)$ is of relative order

$$
\frac{v(0)n}{V} = \frac{ne^2}{V\epsilon_0 \kappa^2}
$$

(5.8)
and vanishes in the thermodynamic limit ($V \to \infty$ with $\kappa$ fixed). Thus we see that $\hat{H}_{e-b} + \hat{H}_{b-e}$ cancel the $q = 0$ term in $\hat{H}_{int}$. The Hamiltonian in the jellium model can therefore be written as $\hat{H}_0 + \hat{H}_{int}$ without the $q = 0$ term in $\hat{H}_{int}$. It is convenient to define $v(q)$ as $(1 - \delta_{q,0})e^2/\epsilon_0 q^2$ so that $v(0) = 0$.

\section{The random-phase approximation}

The RPA is one of the simplest approximations to the density-density response function and the dynamic longitudinal dielectric function of the electron gas. In this section we derive the RPA from several points of view. In particular we show that the RPA arises very naturally in the functional integral formalism.

\subsection{Dielectric function and density-density response function}

In Sec. 3.4.1 we have shown that the longitudinal dielectric function $\epsilon_{\parallel}(q)$ is related to the density-density response function $\chi_{nn}(q)$,

$$
\frac{1}{\epsilon_{\parallel}(q)} = 1 - v(q)\chi_{nn}(q) \quad \text{or} \quad \epsilon_{\parallel}(q) = 1 + v(q)\Pi_{nn}(q),
$$

(5.9)
where $\Pi_{nn}(q)$ is the “irreducible” part of $\chi_{nn}(q)$, i.e. the part that cannot be split into two disconnected pieces by cutting a single Coulomb line $v(q)$. The general properties of $\chi_{nn}$ and $\epsilon_{\parallel}$ are discussed in Sec. 3.4.1 and the sum rules satisfied by $\epsilon_{\parallel}(q, \omega)$ are summarized in Table 5.1. These sum rules, together with the fact that the particle-hole pair excitations do not contribute to the spectral function $\Im[1/\epsilon_{\parallel}(q, \omega)]$ for $q \to 0$, imply that the electron gas possesses a (collective) plasmon mode with energy $\omega_p = (ne^2/\epsilon_0 m)^{1/2}$ in the long wavelength limit.

An explicit expression of $\epsilon_{\parallel}(q)$ and $\chi_{nn}(q)$ requires to compute, at least approximately, the irreducible part $\Pi_{nn}(q)$.\footnote{Note that $\Pi_{nn}$ is the right quantity to consider in order to obtain satisfying expressions for $\chi_{nn}$ and $\epsilon_{\parallel}$. Any approximate $\Pi_{nn}$ will give a result for $\chi_{nn}$ and $\epsilon_{\parallel}$ to infinite order in the Coulomb interaction. By contrast, a finite order approximation to $\chi_{nn}$ is likely to violate basic properties such as causality. This is the case for instance if one retains only the first two diagrams in Fig. 5.2. Note that a similar observation was made when discussing the one-particle Green function $G$ (Sec. 3.5): The self-energy $\Sigma$ (and not $G$) is the right quantity to consider in perturbation theory.} In the following, we consider the zeroth-order approximation –
Table 5.1: Sum rules satisfied by the longitudinal dielectric function \( \epsilon_{\parallel}^R(q, \omega) \) (see Sec. 3.4.1).

\( \kappa \) denotes the compressibility and \( \omega_p \) the plasma frequency.

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<th>Expression</th>
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<td>( f )-sum rule</td>
<td>( \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \omega^3 \left[ \frac{1}{\epsilon_{\parallel}^R(q, \omega)} \right] = -\omega_p^2 )</td>
</tr>
<tr>
<td>perfect-screening sum rule</td>
<td>( \lim_{q \to 0} \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \frac{1}{\omega^3} \left[ \epsilon_{\parallel}^R(q, \omega) \right] = -1 )</td>
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<tr>
<td>longitudinal-conductivity sum rule</td>
<td>( \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \omega^2 \left[ \epsilon_{\parallel}^R(q, \omega) \right] = \omega_p^2 )</td>
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<tr>
<td>compressibility sum rule</td>
<td>( \lim_{q \to 0} \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \frac{1}{\omega} \left[ \epsilon_{\parallel}^R(q, \omega) \right] = \frac{e^2 n^2 \kappa}{\epsilon_0 q^2} )</td>
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known as the RPA – where \( \Pi_{nn} \) is identified with the non-interacting susceptibility (Fig. 5.1),

\[
\Pi_{nn}^{\text{RPA}}(q) = \chi_{nn}^0(q) = -\frac{2}{\beta V} \sum_k G_0(k)G_0(k + q).
\]  

This gives

\[
\chi_{nn}^{\text{RPA}}(q) = \frac{\chi_{nn}^0(q)}{1 + v(q)\chi_{nn}^0(q)} \quad \text{and} \quad \epsilon_{\parallel}^{\text{RPA}}(q) = 1 + v(q)\chi_{nn}^0(q).
\]  

The density-density response function \( \chi_{nn}^{\text{RPA}} \) can be represented by the sum of bubble diagrams shown if Fig. 5.2.

To estimate the domain of validity of a perturbative treatment of the Coulomb interaction, we consider the ratio

\[
\frac{\text{Potential energy}}{\text{Kinetic energy}} \sim \frac{e^2}{\epsilon_0 k_F^2} \sim \frac{e^2 m}{\epsilon_0 k_F} \sim \frac{1}{n^{1/3} a_0},
\]  

(5.12)

where \( a_0 = \hbar^2 \epsilon_0 / \pi m e^2 \) is the Bohr radius. It is customary to write the volume per particle as

\[
\frac{1}{n} = \frac{3 \pi^2}{k_F^3} = \frac{4}{3} \pi (r_s a_0)^3
\]  

(5.13)

so that

\[
\frac{\text{Potential energy}}{\text{Kinetic energy}} \sim r_s.
\]  

(5.14)

Thus the RPA is expected to be justified at high density when \( r_s \ll 1 \). Since typical metals correspond to a range \( r_s \sim 2 - 5 \), the RPA results have limited applicability even for conventional metals. In Sec. 5.5 we briefly discuss how one can go beyond the RPA.

In the opposite limit \( r_s \gg 1 \), the electrons are expected to localize and form a Wigner crystal since at very low density the zero-point kinetic energy associated with localizing the electrons eventually becomes negligible in comparison with the electrostatic energy of a classical lattice (Sec. 5.6).

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5.2 The random-phase approximation

\[
\Gamma_{nn} = + + + + \cdots
\]

Figure 5.1: Lowest-order contributions to \(\Pi_{nn}\). The RPA retains only the zeroth-order term.

\[
\chi_{nn}^{\text{RPA}} = + + \cdots
\]

Figure 5.2: Diagrammatic representation of the density-density response function \(\chi_{nn}\) in the RPA.

5.2.2 Mean-field character of the RPA

To see the mean-field character of the RPA, let us rederive it within an elementary approach. We consider the system in the presence of an external potential \(\phi_{\text{ext}}(\mathbf{r}, t) = \frac{1}{\sqrt{V}} \sum \phi_{\text{ext}}(\mathbf{q}, \omega) e^{i[q \cdot \mathbf{r} - (\omega + i\eta)t]} + c.c. \quad (\eta \to 0^+)\). (5.15)

Keeping only the part that oscillates with \(e^{i[q \cdot \mathbf{r} - (\omega + i\eta)t]}\) (we are interested only in the linear response to \(\phi_{\text{ext}}\)), the Hamiltonian reads

\[
\hat{H} = \hat{H}_0 + \frac{1}{2} \sum_{\mathbf{q}} v(\mathbf{q}) \left[ \hat{n}(\mathbf{-q})\hat{n}(\mathbf{q}) - \frac{\hat{N}}{V} \right] + \sum_{\mathbf{q}} \phi_{\text{ext}}(\mathbf{q}, \omega) e^{-i(\omega + i\eta)t} \hat{n}(\mathbf{-q}). \quad (5.16)
\]

The term \(\hat{N}/V\) can be ignored since it does not play any role in the following argument. In a mean-field approximation, the interacting Hamiltonian \(\hat{H}_{\text{int}}\) is linearized with respect to the fluctuations \(\delta \hat{n}(\mathbf{q}) = \hat{n}(\mathbf{q}) - \langle \hat{n}(\mathbf{q}) \rangle\) of the density operator,

\[
\begin{align*}
\hat{n}(\mathbf{-q})\hat{n}(\mathbf{q}) = & \left[ \delta \hat{n}(\mathbf{-q}) + \langle \hat{n}(\mathbf{-q}) \rangle \right] \left[ \delta \hat{n}(\mathbf{q}) + \langle \hat{n}(\mathbf{q}) \rangle \right] \\
= & \delta \hat{n}(\mathbf{-q}) \langle \hat{n}(\mathbf{q}) \rangle + \langle \hat{n}(\mathbf{-q}) \rangle \delta \hat{n}(\mathbf{q}) + \langle \hat{n}(\mathbf{-q}) \rangle \langle \hat{n}(\mathbf{q}) \rangle + O \left( \delta \hat{n}^2 \right) \\
= & \langle \hat{n}(\mathbf{q}) \rangle \langle \hat{n}(\mathbf{-q}) \rangle + \langle \hat{n}(\mathbf{-q}) \rangle \langle \hat{n}(\mathbf{q}) \rangle - \langle \hat{n}(\mathbf{-q}) \rangle \langle \hat{n}(\mathbf{q}) \rangle + O \left( \delta \hat{n}^2 \right),
\end{align*}
\] (5.17)
so that
\[ \hat{H}_{\text{MF}} = \hat{H}_0 + \sum_q [\phi_{\text{ext}}(q, \omega)e^{-i(\omega + i\eta)t} + v(q)\langle \hat{n}(q) \rangle] \hat{n}(-q) \] (5.18)

(up to an irrelevant constant), where the expectation value \( \langle \hat{n}(q) \rangle \equiv \langle \hat{n}(q) \rangle(t) \) is time dependent. We are therefore left with a system of free electrons subject to the effective potential
\[ \phi_{\text{eff}}(q, t) = \phi_{\text{ext}}(q, \omega)e^{-i(\omega + i\eta)t} + v(q)\langle \hat{n}(q) \rangle. \] (5.19)

The normalization constant \( N \)
\[ S \] We rewrite
\[ \text{we rewrite} \]
so that \( \chi_{nn}(q, \omega) = \frac{\chi^{0R}_{nn}(q, \omega)}{1 + v(q)\chi^{0R}_{nn}(q, \omega)}, \] (5.21)
which is the RPA result for the retarded density-density correlation function.

### 5.2.3 The RPA in the functional integral formalism

#### 5.2.3.1 Hubbard-Stratonovich transformation

We consider the action \( S = S_0 + S_{\text{int}} + S_J \) in the presence of external sources that couple to the density,
\[ S_{\text{int}} = \frac{1}{2\beta V} \sum_{k, k', q} v(q)\psi^{\ast}_{\sigma}(k + q)\psi^{\ast}_{\sigma'}(k' - q)\psi_{\sigma'}(k')\psi_{\sigma}(k) \]
\[ = \frac{1}{2} \sum_q v(q)n(-q)n(q), \] (5.22)
\[ S_J = -\sum_q J(-q)n(q), \]

where
\[ n(q) = \frac{1}{\sqrt{\beta V}} \sum_{k, \sigma} \psi^{\ast}_{\sigma}(k)\psi_{\sigma}(k + q). \] (5.23)

We rewrite \( S_{\text{int}} \) by means of a Gaussian functional integral over an auxiliary real bosonic field \( \phi(x) \),
\[ e^{-S_{\text{int}}} = \mathcal{N} \int \mathcal{D}[\phi] e^{-\frac{1}{2} \sum_q \phi(-q)\nu(q)^{-1}\phi(q) + \sum_q \phi(-q)n(q)} \] (5.24)

The normalization constant \( \mathcal{N} = \det(v^{-1/2}) \) will be ignored in the following. This kind of transformation – known as a Hubbard-Stratonovich transformation – will be used repeatedly in this book.\(^3\) It transforms a system of interacting particles into a system of “free” particles interacting with a bosonic field. The action now reads
\[ S[\psi^{\ast}, \psi, \phi] = S_0[\psi^{\ast}, \psi] + \frac{1}{2} \sum_q \left\{ \phi(-q)v(q)^{-1}\phi(q) - 2i\phi(-q) + J(-q)n(q) \right\}, \] (5.25)

\(^3\)See Sec. 6.3.3 for a discussion of the limitations of Hubbard-Stratonovich transformations.
and the partition function involves a functional integral over both the fermionic field $\psi$ and the bosonic field $\phi$. Note that the Hubbard-Stratonovich field $\phi$ is nothing but the scalar potential of the electromagnetic field (Sec. 1.9.3). Its propagator can therefore be expressed in terms of the dielectric function,

$$D_{00}(q) = \langle \phi(q)\phi(-q) \rangle = \frac{v(q)}{\epsilon_\parallel(q)}.$$  \hspace{1cm} (5.26)

### 5.2.3.2 Saddle-point approximation: Hartree theory

The Hubbard-Stratonovich transformation suggests various approximations to treat the electron-electron interactions (now encoded in the dynamics of the bosonic field $\phi$). The simplest one amounts to performing a saddle-point (or mean-field) approximation for the functional integral over $\phi$ whereby

$$Z = \int D[\psi^*, \psi, \phi] e^{-S[\psi^*, \psi, \phi]} \to Z_{\text{MF}} = \int D[\psi^*, \psi] e^{-S_{\text{MF}}[\psi^*, \psi]}.$$  \hspace{1cm} (5.27)

The mean field $\phi(q) = \sqrt{\beta V} \delta_{q,0} \phi_0$ (i.e. $\phi(r, \tau) = \phi_0$) is chosen static and uniform. For $J = 0$, the mean-field action then reads

$$S_{\text{MF}}[\psi^*, \psi] = \frac{\beta V}{2} v(0)^{-1} \phi_0^2 - \sum_{k, \sigma} \psi^*_\sigma(k)(i\omega_n - \xi_k + i\phi_0)\psi_\sigma(k).$$  \hspace{1cm} (5.28)

The value of $\phi_0$ is determined by requiring the thermodynamic potential to be extremum: $\partial \Omega_{\text{MF}} / \partial \phi_0 = 0$ or, equivalently, $\partial Z_{\text{MF}} / \partial \phi_0 = 0$. This gives

$$\int D[\psi^*, \psi] \left[ \beta V v(0)^{-1} \phi_0 - i \sum_{k, \sigma} \psi^*_\sigma(k)\psi_\sigma(k) \right] e^{-S_{\text{MF}}[\psi^*, \psi]} = 0,$$

i.e.

$$i\phi_0 = -v(0) \frac{1}{\beta V} \sum_{k, \sigma} \langle \psi^*_\sigma(k)\psi_\sigma(k) \rangle e^{i\omega_n \eta} = -v(0)n.$$  \hspace{1cm} (5.30)

(we have added the usual convergence factor $e^{i\omega_n \eta}$). Note that $\phi_0$ is purely imaginary, which ensures that the mean-field action is real. \footnote{Recall that mathematically there is nothing that prevents the saddle point $\phi_0$ to be imaginary even though the auxiliary bosonic field $\phi(x)$ is real. One can deform the integration contour into the complex plane when looking for a saddle point; see Sec. 1.7.1.}

The result (5.30) has a simple interpretation in terms of Feynman diagrams. It corresponds to the first-order self-consistent self-energy shown in Fig. 5.3,

$$\Sigma_H(k) = v(0) \frac{1}{\beta V} \sum_{k, \sigma} G_\sigma(k) e^{i\omega_n \eta} = v(0)n.$$  \hspace{1cm} (5.31)

In the jellium model, $v(0) = 0$ and the Hartree self-energy (5.31) vanishes ($\phi_0 = 0$). In short-range interaction models however, $\Sigma_H$ is in general finite. Being independent of $k$, it merely renormalizes the chemical potential.
Gaussian fluctuations: RPA

To proceed beyond the saddle-point approximation, we derive the effective action of the bosonic field to second order in \( \phi(x) - \phi_0 \equiv \phi(x) \). To do so, one has to integrate out the \( \psi \) field retaining only the second-order cumulant,

\[
Z[J] = \int \mathcal{D}[\psi^*, \psi, \phi] e^{-S_0[\psi^*, \psi] - \frac{1}{2} \sum_q [\phi(-q)v(q)^{-1}\phi(q) - 2[i\phi(-q) + J(-q)]n(q)]}
\]

where

\[
S[\phi] = \frac{1}{2} \sum_q \left\{ \phi(-q)v(q)^{-1}\phi(q) - [i\phi(-q) + J(-q)]\chi_{nn}^0(q)[i\phi(q) + J(q)] \right\}, \tag{5.33}
\]

For \( J = 0 \), one obtains the propagator of the auxiliary field,

\[
D_{00}(q) = \langle \phi(q)\phi(-q) \rangle = \frac{1}{v(q)^{-1} + \chi_{nn}^0(q)} = \frac{v(q)}{1 + v(q)\chi_{nn}^0(q)},
\tag{5.34}
\]

which reproduces the RPA result (5.11) for the longitudinal dielectric function (compare (5.26) and (5.34)). From the action (5.33), one can also obtain the partition function by integrating out the \( \phi \) field,

\[
Z[J] = Z_0 \exp \left\{ \frac{1}{2} \sum_q J(-q) \left[ \chi_{nn}^0(q) - \frac{v(q)\chi_{nn}^0(q)}{1 + v(q)\chi_{nn}^0(q)} J(q) \right] \right\}, \tag{5.35}
\]

and deduce the density-density correlation function

\[
\chi_{nn}(q) = \left. \frac{\delta^2 \ln Z[J]}{\delta J(-q)\delta J(q)} \right|_{J=0} = \frac{\chi_{nn}^0(q)}{1 + v(q)\chi_{nn}^0(q)}. \tag{5.36}
\]

Thus we see that by taking into account the Gaussian fluctuations of \( \phi \) about the saddle point \( \phi_0 \), we reproduce the RPA.

\footnote{The first-order cumulant gives a trivial contribution to the action: \( \sum_q [i\phi(-q) + J(-q)]\langle n(q) \rangle = J(q = 0) \sqrt{\beta V} n \) since \( \langle n(q) \rangle = \delta_{q,0} \sqrt{\beta V} n \).}
5.3 Physical properties of the RPA

In this section, we discuss some basic properties of the electron gas in the RPA: density fluctuations spectrum and plasmon mode, screening, ground-state energy and compressibility. We start with a calculation of the Lindhard function $\chi_{mn}^{R}$.

5.3.1 The Lindhard function

After summation over the Matsubara frequency in (5.10) and the analytic continuation $q = (q, i\omega_n) \to (q, \omega + i\eta)$, one obtains the retarded response function (or Lindhard function)

$$\chi_{mn}^{0R}(q, \omega) = -2 \int_{k} \frac{n_F(\xi_k) - n_F(\xi_{k+q})}{\omega + i\eta + \xi_k - \xi_{k+q}}$$  (5.37)

An elementary change of variables gives

$$\chi_{mn}^{0R}(q, \omega) = -2 \int_{k} \frac{n_F(\xi_k)}{\omega + i\eta + \xi_k - \xi_{k+q}} + (\omega + i\eta \to -\omega - i\eta),$$  (5.38)

with $n_F(\xi_k) = \Theta(k_F - |k|)$ at zero temperature. Introducing the dimensionless variables $x = |k|/k_F$, $\tilde{q} = |q|/k_F$ and $\tilde{\omega}^+ = (\omega + i\eta)/v_F|q|$, we have

$$\frac{\Theta(k_F - |k|)}{2q} = \frac{mk_F^2}{4\pi^2|q|} \int_{0}^{1} dx \int_{0}^{\pi} d\theta \frac{k^2 dk}{\omega + i\eta - \frac{|k|}{m} \cos \theta - \frac{q^2}{2m}}$$

$$= \frac{4\pi^2 N(0)}{m k_F^2} \int_{0}^{1} dx \ln \frac{\tilde{\omega}^+ + x - \tilde{q}/2}{\tilde{\omega}^+ - x - \tilde{q}/2}$$

$$= \frac{N(0)}{2\tilde{q}} \left[ a - \frac{a^2 - 1}{2} \ln \frac{a + 1}{a - 1} \right],$$  (5.39)

where $a = \tilde{\omega}^+ - \tilde{q}/2$. $N(0) \equiv N(\xi = 0) = mk_F/2\pi^2$ is the non-interacting density of states per spin at the Fermi level. From (5.38) and (5.39), we finally obtain

$$\chi_{mn}^{0R}(q, \omega) = N(0) \left\{ 1 - \frac{1}{2\tilde{q}} \left[ 1 - \left( \tilde{\omega} - \frac{\tilde{q}}{2} \right)^2 \right] \ln \left( \frac{\tilde{\omega} + 1 - \tilde{q}/2}{\tilde{\omega} - 1 - \tilde{q}/2} \right) \right.$$

$$- \frac{1}{2\tilde{q}} \left[ 1 - \left( \tilde{\omega} + \frac{\tilde{q}}{2} \right)^2 \right] \ln \left( \frac{\tilde{\omega} - 1 + \tilde{q}/2}{\tilde{\omega} + 1 + \tilde{q}/2} \right) \right\}.$$  (5.40)

The real and imaginary parts of the Lindhard function read

$$\Re \left[ \chi_{mn}^{0R}(q, \omega) \right] = N(0) \left\{ 1 - \frac{1}{2\tilde{q}} \left[ 1 - \left( \tilde{\omega} - \frac{\tilde{q}}{2} \right)^2 \right] \ln \left| \frac{\tilde{\omega} + 1 - \tilde{q}/2}{\tilde{\omega} - 1 - \tilde{q}/2} \right| \right.$$

$$- \frac{1}{2\tilde{q}} \left[ 1 - \left( \tilde{\omega} + \frac{\tilde{q}}{2} \right)^2 \right] \ln \left| \frac{\tilde{\omega} - 1 + \tilde{q}/2}{\tilde{\omega} + 1 + \tilde{q}/2} \right| \right\}.$$  (5.41)

The Matsubara sum is done explicitly in Appendix 1.F.
and

$$\Im [\chi^{0R}_{nn}(q, \omega)] = N(0) \frac{\pi}{2q} \left[ 1 - \left( \frac{\tilde{\omega} - q}{2} \right)^2 \right] \Theta \left[ 1 - \left( \frac{\tilde{\omega} - q}{2} \right)^2 \right]$$

$$- \left[ 1 - \left( \frac{\tilde{\omega} + q}{2} \right)^2 \right] \Theta \left[ 1 - \left( \frac{\tilde{\omega} + q}{2} \right)^2 \right],$$

(5.42)

where \( \tilde{\omega} = \omega/v_F |q| \).

For \( \omega = 0 \), the Lindhard function is real and given by

$$\chi^{0R}_{nn}(q, 0) = N(0) \left[ 1 - \frac{1}{q} \left( 1 - \frac{q^2}{4} \right) \ln \left| \frac{\tilde{q} - 2}{\tilde{q} + 2} \right| \right].$$

(5.43)

In Sec. 5.3.3 we shall see the non-analyticity at \( \tilde{q} = 2 \) has important consequences. For \( |q| \to 0 \) and \( \tilde{\omega} = \omega/v_F |q| \) fixed, \( \chi^{0R}_{nn} \) takes the form

$$\lim_{|q| \to 0} \chi^{0R}_{nn}(q, \omega) = 2N(0) \left[ 1 - \frac{\tilde{\omega}^+}{2} \left( \frac{\tilde{\omega}^+ + 1}{\tilde{\omega}^+ - 1} \right) \right],$$

(5.44)

a result previously obtained in Fermi-liquid theory (Sec. 4.3.4).

### 5.3.1.1 Spectral function

\( \chi^{0\prime\prime}_{nn}(q, \omega) = \Im [\chi^{0R}_{nn}(q, \omega)] \)

For \( \tilde{q} \leq 2 \), the imaginary part of the Lindhard function has the form shown in Fig. 5.4 (left panel). It is linear at low energy and corresponds to an arc of parabola at higher energy,

$$\chi^{0\prime\prime}_{nn}(q, \omega) = \begin{cases} \frac{\pi N(0)}{2q} \frac{\omega}{v_F |q|} & \text{for } 0 \leq \omega \leq |\omega_q|, \\ \frac{\pi N(0)}{2q} \left[ 1 - \left( \tilde{\omega} - \frac{q}{2} \right)^2 \right] & \text{for } |\omega_q| \leq \omega \leq \omega_q^+, \end{cases}$$

(5.45)

where

$$\omega_q^\pm = \pm v_F |q| + \frac{q^2}{2m}.$$  

(5.46)

The negative energy part can be deduced from (5.45) by noting that \( \chi^{0\prime\prime}_{nn}(q, \omega) = -\chi^{0\prime\prime}_{nn}(q, -\omega) \).
5.3 Physical properties of the RPA

Figure 5.5: Plasmon mode dispersion $\omega_q$ in the electron gas. The shaded area shows the particle-hole excitation continuum. In the RPA the latter has the same boundaries as that of the non-interacting electron gas.

For $\tilde{q} \geq 2$, the linear part is not present any more and the Lindhard function reduces to the arc of parabola (Fig. 5.4, right panel),

$$
\chi_{nn}^{0''}(q, \omega) = \frac{\pi N(0)}{2\tilde{q}} \left[ 1 - \left( \frac{\omega - \tilde{q}}{2} \right)^2 \right] \quad \text{for} \quad \omega_q^- \leq \omega \leq \omega_q^+.
$$

(5.47)

5.3.1.2 Particle-hole pair excitation continuum

In the non-interacting system, the particle-hole pair excitation spectrum is defined by $\chi_{nn}^{0''}(q, \omega) \neq 0$. It gives the values of $(q, \omega)$ for which the equation $\omega = \epsilon_{q+k} - \epsilon_k$ possesses at least a solution with $\xi_k < 0$ and $\xi_{q+k} > 0$. For $|q| \leq 2k_F$, the particle-hole continuum extends from $\omega = 0$ up to $\omega = \omega_q^+$. For $|q| > 2k_F$, there is no excitation at low energy and the particle-hole continuum extends from $\omega_q^-$ to $\omega_q^+$ (Fig. 5.5).

5.3.2 Density fluctuation spectrum

The spectrum of the density fluctuations can be obtained from the spectral function

$$
\chi_{nn}''(q, \omega) = \Im \left[ \chi_{nn}^R(q, \omega) \right],
$$

or the structure factor $S_{nn}(q, \omega; T = 0) = 2\Theta(\omega)\chi_{nn}''(q, \omega)$. In the RPA, there are two contributions to

$$
S_{nn}^{RPA}(q, \omega) = 2\Theta(\omega) \frac{\chi_{nn}^{0''}(q, \omega)}{1 + v(q)\Re \left[ \chi_{nn}^R(q, \omega) \right]}\left(1 + v(q)\chi_{nn}^{0''}(q, \omega)\right)^2.
$$

(5.48)

The first one is due to a non-vanishing $\chi_{nn}^{0''}(q, \omega)$ and comes from the particle-hole pair excitations. These are quite similar to the particle-hole pair excitations of the non-interacting electron gas and gives a continuous excitation spectrum (Fig. 5.5). The second contribution to the structure factor comes from the (collective) plasmon mode at the frequency $\omega_q$ defined by

$$
1 + v(q)\Re \left[ \chi_{nn}^R(q, \omega_q) \right] = 0.
$$

(5.49)

Such an excitation is well defined if its lifetime – determined by $1/\chi_{nn}^{0''}(q, \omega_q)$ – is long enough; only in this case does it show up as a sharp peak in $S_{nn}(q, \omega)$ (Fig. 5.6). In the

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The following discussion is similar to that of Sec. 4.3.4 on the structure factor of the neutral Fermi liquid.

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Chapter 5. The electron gas

Figure 5.6: Structure factor $S_{nn}(\mathbf{q}, \omega)/4N(0)$ of the electron gas in the RPA for different values of $\tilde{q} = |\mathbf{q}|/k_F$ and $r_s = 1$ ($\tilde{\omega} = \omega/v_F|\mathbf{q}|$). The dashed line shows the results for the non-interacting electron gas. The plots are obtained with a small but finite value of $\eta$ (hence the nonzero width of the plasmon peak). For small values of $\tilde{q}$ the plasmon peak is clearly visible – and gives the main contribution to the spectral weight $\int_0^{\tilde{\omega}} d\omega S_{nn}(\mathbf{q}, \omega)$ – while for larger values it merges with the continuum of particle-hole pair excitations.

RPA, the plasmon has an infinite lifetime as long as $\chi_{nn}^{0 \prime \prime}(\mathbf{q}, \omega) = 0$, i.e. as long as it does not overlap with the particle-hole pair continuum. When its energy satisfies $\omega = \epsilon_{\mathbf{k}+\mathbf{q}} - \epsilon_{\mathbf{k}}$ (with $|\mathbf{k}| < k_F$ and $|\mathbf{k}+\mathbf{q}| > k_F$), the plasmon becomes strongly damped (Landau damping) and is not a well-defined excitation of the system anymore. The absence of damping at small $\mathbf{q}$ is an artifact of the RPA. Multipair excitations – not taken into account in the RPA – provides the main damping mechanism when Landau damping is ineffective. These issues have been discussed in detail in connection with the zero-sound mode of the neutral Fermi liquid (Secs. 4.3.3 and 4.3.5).

5.3.2.1 Plasmon mode and sum rules

For $\mathbf{q} \to 0$ with $\omega$ fixed, $\chi_{nn}^{0 R}(\mathbf{q}, \omega) \to -nq^2/\omega^2m$, i.e.

$$\lim_{\mathbf{q} \to 0} \epsilon_{\mathbf{k} \parallel}^{R}(\mathbf{q}, \omega) = 1 - \frac{\omega_p^2}{\omega^2}, \quad (5.50)$$

and we recover $\omega_p = (ne^2/\epsilon_0m)^{1/2}$ as the excitation energy of the plasmon mode in the long-wavelength limit. Pushing the expansion of $\chi_{nn}^{0 R}(\mathbf{q}, \omega)$ to higher order in $\mathbf{q}^2$ yields the
long-wavelength plasmon dispersion
\[ \omega_q^2 = \omega_p^2 + \frac{3}{5} v_F^2 q^2 + \mathcal{O}(|q|^4). \] (5.51)

For \( q \to 0 \) and \( \omega \simeq \omega_p \), one has
\[
1 + v(q) \chi_{nn}^0(q, \omega) \simeq (\omega + i\eta - \omega_p)v(q)\partial_\omega \chi_{nn}^0(q, \omega) \bigg|_{\omega = \omega_p} = (\omega + i\eta - \omega_p) \frac{2}{\omega_p},
\]
(5.52)

and therefore
\[
\chi_{nn}^R(q, \omega) = -\frac{\omega_p}{2v(q)} \frac{1}{\omega + i\eta - \omega_p}, \quad (q \to 0 \text{ and } \omega \to \omega_p).
\]
(5.53)

For \( q \to 0 \) the spectral function \( \chi''_{nn}(q, \omega) \) exhibits a delta peak \((\pi\omega_p/2v(q))\delta(\omega - \omega_p)\) and another one at \( \omega = -\omega_p \) with negative weight. Using (5.53), one obtains
\[
\lim_{q \to 0} \pi \int_{-\infty}^{\infty} d\omega \frac{1}{\omega} \Im \left[ \frac{1}{\epsilon_{\parallel}^R(q, \omega)} \right]_{\text{plasmon}} = -\omega_p^2,
\]
\[
\lim_{q \to 0} \pi \int_{-\infty}^{\infty} d\omega \frac{1}{\omega} \Im \left[ \frac{1}{\epsilon_{\parallel}^R(q, \omega)} \right]_{\text{plasmon}} = -1.
\] (5.54)

We conclude that the RPA satisfies the \( f \)-sum rule and the perfect-screening sum rule in the \( q \to 0 \) limit, the plasmon mode exhausting the spectral weight. Equations (5.54) are exact. They are a consequence of the vanishing contribution of particle-hole pair excitations to the spectral function \( \Im [1/\epsilon_{\parallel}^R(q, \omega)] \) when \( q \to 0 \).

With \( \epsilon_{\parallel}^R(q, \omega) = 1 + v(q) \chi_{nn}^0(q, \omega) \), the longitudinal-conductivity sum rule reduces to the \( f \)-sum rule (3.106) for non-interacting electrons and is therefore trivially satisfied. Similarly if we define the compressibility \( \kappa \) of the charged system by \( n^2 \kappa = \lim_{q \to 0} \Pi_{nn}(q, 0) \), as discussed in Sec. 3.4.1.2, the compressibility sum rule is also trivially satisfied since it follows from that definition. Furthermore \( \kappa \) is independent of the Coulomb interaction since \( \Pi_{nn}^{\text{RPA}} = \chi_{nn}^0 \). The compressibility does depend on the interactions if it is calculated from the density dependence of the ground state energy, as in Sec. 5.3.4.2, but in that case the compressibility sum rule is violated.

5.3.3 Screening

5.3.3.1 Thomas-Fermi theory

Let us start with a simple theory of screening. We consider the response of an electron gas to a static impurity charge density \( n_i(r) \). If the induced charge density varies slowly with

---

9In Sec. 3.4.1, we have shown that the exact result \( \lim_{q \to 0} \omega_q = \omega_p = (ne^2/\epsilon_0 m)^{1/2} \) can be deduced from the sum rules (5.54).

10There is no contribution of the \( q \to 0 \) plasmon to the longitudinal-conductivity and compressibility sum rules since \( \epsilon_{\parallel}^R(q \to 0, \omega) \) is real for \( \omega \simeq \omega_p \).
respect to \( k_F \), we expect a semiclassical description to be justified. The scalar potential \( \phi(\mathbf{r}) \) satisfies Poisson’s equation\(^{\text{11}}\)

\[
\nabla^2 \phi(\mathbf{r}) + \frac{e^2}{\epsilon_0} \left[ \delta n(\mathbf{r}) + n_i(\mathbf{r}) \right] = 0,
\]

(5.55)

where \( \delta n(\mathbf{r}) = \langle \hat{n}(\mathbf{r}) \rangle - n \) is the induced charge density. In a semiclassical picture, \( \mu - \phi(\mathbf{r}) \) acts as a local chemical potential and

\[
\delta n(\mathbf{r}) = -\phi(\mathbf{r}) \frac{\partial n}{\partial \mu} + \mathcal{O}(\phi^2) = -n^2 \kappa \phi(\mathbf{r}) + \mathcal{O}(\phi^2),
\]

(5.56)

where \( \kappa = n^{-2} \partial n / \partial \mu = 3/2 n \epsilon_F \) is the compressibility of the free electron gas. To first order in \( \phi(\mathbf{r}) \), Poisson’s equation then gives

\[
\nabla^2 \phi(\mathbf{r}) - \frac{e^2}{\epsilon_0} n^2 \kappa \phi(\mathbf{r}) = -\frac{e^2}{\epsilon_0} n_i(\mathbf{r}).
\]

(5.57)

In Fourier space, we find

\[
\phi(\mathbf{q}) = \frac{e^2}{\epsilon_0} \frac{n_i(\mathbf{q})}{\mathbf{q}^2 + q_{TF}^2},
\]

(5.58)

where

\[
q_{TF}^2 = \frac{e^2}{\epsilon_0} n^2 \kappa = \frac{3e^2 n}{2\epsilon_0 \epsilon_F} = r_s k_F^2 \frac{4}{\pi} \left( \frac{4}{9\pi} \right)^{1/3}
\]

(5.59)

is the (square of the) Thomas-Fermi screening wavevector. From the standard definition of the longitudinal dielectric function,

\[
\epsilon_{TF}(\mathbf{q}) = 1 + \frac{q_{TF}^2}{\mathbf{q}^2}.
\]

(5.60)

we obtain

\[
\epsilon_{TF}(\mathbf{q}) = 1 + \frac{q_{TF}^2}{\mathbf{q}^2}.
\]

(5.61)

At small distances \(|\mathbf{q}| \gg q_{TF}\), the impurity charge is unscreened since \( \epsilon_{TF}(\mathbf{q}) \simeq 1 \). On the contrary, at large distances the screening is very effective. For a point-like impurity \( n_i(\mathbf{r}) = n_i \delta(\mathbf{r}) \), the screened potential has the form of a Yukawa potential,

\[
\phi(\mathbf{r}) = \frac{n_i e^2}{4\pi \epsilon_0 |\mathbf{r}|} \exp(-q_{TF} |\mathbf{r}|),
\]

(5.62)

and is exponentially suppressed at length scales larger than the screening length \( q_{TF}^{-1} \). Away from the impurity \((\mathbf{r} \neq 0)\), the induced charge density is given by

\[
\delta n(\mathbf{r}) = -\frac{\epsilon_0}{e^2} \nabla^2 \phi(\mathbf{r}) = -n_i q_{TF}^2 \frac{\exp(-q_{TF} |\mathbf{r}|)}{4\pi |\mathbf{r}|}.
\]

(5.63)

A necessary condition for the (semiclassical) Thomas-Fermi theory to be justified is \( q_{TF} \ll k_F \), i.e. \( r_s \ll 1 \). We shall see below that even when this condition is satisfied, the Thomas-Fermi theory fails to provide an accurate description of the response of the electron gas to a static impurity charge density.
Figure 5.7: Static dielectric functions obtained from the Thomas-Fermi theory and the RPA. The non-analyticity of $\epsilon_{\text{RPA}}(q)$ at $|q| = 2k_F$ gives rise to Friedel oscillations and a power-law decrease of the charge induced by a point-like impurity [Eq. (5.66)].

5.3.3.2 Static RPA dielectric function

In the RPA, the screening of a static impurity charge is determined by the dielectric function

$$
\epsilon_{\text{RPA}}(q) = 1 + \frac{q^2}{2q^2} \left\{ 1 - \frac{1}{q} \left( 1 - \frac{q^2}{4} \right) \ln \left| \frac{q - 2}{q + 2} \right| \right\}.
$$

(5.64)

For $q \to 0$, one can approximate $\epsilon_{\text{RPA}}(q)$ by

$$
\epsilon_{\text{RPA}}(q) \simeq 1 + v(q) \lim_{q \to 0} \chi^0_{nn}(q, i\omega_n = 0) = 1 + \frac{e^2 n^2 \kappa}{\epsilon_0 q^2} = \epsilon_{\text{TF}}(q).
$$

(5.65)

It is quite tempting to conclude that at long distances the RPA predicts results similar to those of the Thomas-Fermi theory. This turns out to be wrong because of the non-analyticity of $\epsilon_{\text{RPA}}(q)$ at $|q| = 2k_F$ (Fig. 5.7). This non-analyticity, which originates in the Fermi surface of the electron gas, prevents the mere replacement $\epsilon_{\text{RPA}}(q) \to \epsilon_{\text{TF}}(q)$ when computing the long-distance screening of an external potential. A careful analysis shows that the charge induced by a point-like impurity decays as a power law and exhibits Friedel oscillations [1,3],

$$
\delta n(r) \propto n_i \frac{\cos(2k_F|r|)}{|r|^3},
$$

(5.66)

in marked contrast with the results of the Thomas-Fermi theory [Eq. (5.63)].

5.3.4 Ground state energy and compressibility

5.3.4.1 Ground state energy

The thermodynamic potential can be expressed as a function of the interaction energy by means of a coupling constant integration. Let us write the Hamiltonian as $\hat{H} = \hat{H}_0 + \lambda \hat{H}_\text{int}$ and use (3.277),

$$
\frac{\partial \Omega}{\partial \lambda} = \frac{1}{\lambda} \langle \lambda \hat{H}_\text{int} \rangle_{\lambda},
$$

(5.67)

\textit{See footnote 4 page 345 for the explanation of $e^2$ (and not $e$) appearing in (5.55).}
where the average \((\cdots)_\lambda\) is taken with the Hamiltonian \(\hat{H}_0 + \lambda \hat{H}_{\text{int}}\). The potential energy can be expressed as a function of the density-density response function,

\[
\langle \hat{H}_{\text{int}} \rangle_\lambda = \frac{1}{2} \sum q v(q) \left( \hat{n}(-q) \hat{n}(q) \right)_\lambda - \frac{\hat{N}}{V} \right)
= \frac{1}{2} \sum q v(q) \left[ \frac{1}{\beta} \sum \lambda \chi_{nn}(q) e^{i\omega \eta} - n^\lambda \right],
\]

(5.68)

where both \(\chi_{nn}(q)\) and \(n^\lambda = \langle \hat{N} \rangle_\lambda / V\) \textit{a priori} depend on \(\lambda\).

In the RPA, the one-particle propagator is approximated by the Hartree (non-interacting) propagator. This ensures that the RPA is a conserving approximation that satisfies the conservation laws.\(^{12}\) \(n^\lambda \equiv n\) is thus independent of the interaction \(\lambda v(q)\) and equal to the density of the free gas (for a fixed value of the chemical potential); working at fixed density or fixed chemical potential in the RPA is the same. From (5.67) and (5.68), we then obtain

\[
\Omega = \Omega_0 + \frac{1}{2} \int_0^1 \frac{d\lambda}{\lambda} \sum q \lambda v(q) \left[ \frac{1}{\beta} \sum \lambda \chi_{nn}(q) e^{i\omega \eta} - n \right],
\]

(5.69)

where \(\chi_{nn}(q) = \frac{\chi_{nn}^0(q)}{1 + \lambda v(q) \chi_{nn}^0(q)} = \frac{\chi_{nn}^0(q)}{1 + \lambda v(q) \chi_{nn}^0(q)} - \lambda v(q) \frac{[\chi_{nn}^0(q)]^2}{1 + \lambda v(q) \chi_{nn}^0(q)}\)

(5.70)

and \(\Omega_0 \equiv \Omega(\lambda = 0)\) is the thermodynamic potential of the non-interacting electron gas. Inserting (5.70) into (5.69) and summing over \(\lambda\), one obtains the ground state energy

\[
E_{\text{RPA}} = \Omega(T = 0) + \mu N = E_0 + E_x + E_{\text{corr}},
\]

(5.71)

where \((\beta \to \infty)\)

\[
E_x = \frac{1}{2} \sum q v(q) \left[ \frac{1}{\beta} \sum \lambda \chi_{nn}(q) e^{i\omega \eta} - n \right],
\]

(5.72)

\[
E_{\text{corr}} = \frac{1}{2} \beta \sum q \left\{ -v(q)\chi_{nn}^0(q) + \ln \left[ 1 + v(q)\chi_{nn}^0(q) \right] \right\}.
\]

The energy of the non-interacting gas per electron is given by

\[
\frac{E_0}{N} = \frac{3}{5} \varepsilon_F = \frac{3}{5} \left( \frac{9\pi}{4} \right) \frac{2}{v_s^2} \text{ Ry},
\]

(5.73)

where the Rydberg Ry = \(e^2 / \pi \varepsilon_0 a_0 = \hbar^2 / 2ma_0^2\).

An elementary calculation gives

\[
E_x = -\frac{1}{V} \sum_{k,q} v(q) n_{k+q}^0 (n_k^0 - 1) - \frac{1}{2} \sum q v(q) n = -\frac{1}{V} \sum_{k,q} v(q) n_{k+q}^0 n_k^0,
\]

(5.74)

where \(n_k^0 = \Theta(k_F - |k|)\). \(E_x\) is called the exchange energy. It is nothing but the first-order correction \(\langle \hat{H}_{\text{int}} \rangle\) to the energy \(E_0\) of the non-interacting electron gas and arises from the

---

\(^{12}\)Conserving approximations are discussed in Chap. 9; see in particular the discussion of the electron gas in Sec. 9.1.3.1.
5.3 Physical properties of the RPA

Figure 5.8: Feynman diagrams for the ground state energy $E = \Omega(T = 0) + \mu N$. (a) Exchange energy $E_x$. (b) Correlation energy $E^{\text{RPA}}_{\text{corr}}$ in the RPA. (c) Second-order contributions not included in $E^{\text{RPA}}_{\text{corr}}$.

antisymmetry of the ground state with respect to the exchange of two particles (Fig. 5.8a).\(^{13}\) The exchange energy per electron reads

$$
\frac{E_x}{N} = -\frac{1}{n} \int_{\mathbf{q}, \mathbf{k}} \frac{e^2}{\epsilon_0 |\mathbf{q} - \mathbf{k}|^2} \Theta(k_F - |\mathbf{k}|) \Theta(k_F - |\mathbf{q}|) 
= -\frac{e^2 k_F}{2\pi^2 \epsilon_0 n} \int_k \Theta(k_F - |\mathbf{k}|) \left[ \frac{1}{2} + \frac{1 - \tilde{k}^2}{4\tilde{k}} \ln \frac{1 + \tilde{k}}{1 - \tilde{k}} \right] 
= -\frac{3e^2 k_F}{16\pi^2 \epsilon_0} 
= -\frac{3}{2\pi} \left( \frac{9\pi}{4} \right)^{1/3} \frac{1}{r_s} \text{Ry.} \quad (5.75)
$$

The difference $E - E_0 - E_x$ is referred to as the correlation energy. In the RPA, it is given by (5.72) and corresponds to the bubble (or ring) diagrams shown in Fig. 5.8b. As the full calculation is not possible analytically, we only quote the final result [3],

$$
\frac{E^{\text{RPA}}_{\text{corr}}}{N} = \frac{2}{\pi^2} (1 - \ln 2) \ln r_s - 0.141 \text{ Ry} \simeq 0.0622 \ln r_s - 0.141 \text{ Ry}. \quad (5.76)
$$

The validity of the RPA can be improved by adding the second-order contributions shown in Fig. 5.8c. One then obtains the following expansion of the correlation energy in the high-density limit [3],

$$
\frac{E^{\text{corr}}}{N} = 0.0622 \ln r_s - 0.094 + \mathcal{O}(r_s \ln r_s) \text{ Ry.} \quad (5.77)
$$

5.3.4.2 Compressibility

Let us introduce the notation $\epsilon(r_s) = (E_0 + E_x + E^{\text{corr}})/N$ for the energy per electron,

$$
\epsilon(r_s) = \frac{3}{5} \left( \frac{9\pi}{4} \right)^{2/3} \frac{1}{r_s^2} - \frac{3}{2\pi} \left( \frac{9\pi}{4} \right)^{1/3} \frac{1}{r_s} + 0.0622 \ln r_s - 0.094 + \mathcal{O}(r_s \ln r_s) \text{ Ry.} \quad (5.78)
$$

\(^{13}\)Note that the other first-order correction, the so-called direct term, involves $v(0)$ and vanishes in the jellium model.
We deduce the pressure
\[ P = -\left. \frac{\partial E}{\partial V} \right|_N = n^2 \frac{\partial \epsilon}{\partial n} = -\frac{n r_s}{3} \frac{\partial \epsilon}{\partial r_s} \text{Ry} \]  
(5.79)
and the inverse compressibility
\[ \frac{1}{\kappa} = -\left. V \frac{\partial P}{\partial V} \right|_N = n \frac{\partial P}{\partial n} = \frac{n r_s}{3} \left[ \frac{2}{3} \frac{\partial \epsilon}{\partial r_s} + \frac{r_s}{3} \frac{\partial^2 \epsilon}{\partial r_s^2} \right] \text{Ry}. \]  
(5.80)

The compressibility becomes negative for \( r_s > 4.84 \) (Fig. 5.9). As noted in Sec. 3.4.1, the condition \( \kappa < 0 \) does not necessarily imply an instability of the system as \( \kappa \) does not include the stiffness of the background of positive charge with respect to a change of volume.\(^{14}\)

## 5.4 One-particle properties

The RPA seems to give a reliable description of the density fluctuations: It provides us with a description of screening and the plasmon mode, and satisfies the sum rules of Table 5.1 (except the one related to the compressibility, see Sec. 5.3.2). On the other hand the RPA does not say much about the one-particle properties since it approximates the one-particle propagator by the non-interacting (Hartree) propagator. In particular one would like to know whether the electron gas is a Fermi liquid (in the sense discussed in Chap. 4).

### 5.4.1 Hartree-Fock theory

The simplest, and most natural, way to improve the Hartree theory is to consider the exchange (or Fock) term show in Fig. 5.10,

\[
\Sigma_x(k) = -\frac{1}{\beta V} \sum_{k'} v(k' - k) G(k') e^{i\omega'_n \eta} \\
= -\frac{1}{\beta V} \sum_{k'} v(k' - k) \frac{e^{i\omega'_n \eta}}{i\omega'_n - \xi_{k'} - \Sigma_x(k')} 
\]

\(^{14}\)\( \kappa \) in (5.80) is calculated by assuming that the background of positive charge adjusts itself to maintain neutrality at not energy cost (Sec. 3.4.1).
5.4 One-particle properties

\[ \Sigma_x(k) = -\int_{k'} v(k' - k) n_F(\xi_{k'} + \Sigma_x(k')) \, d\xi_{k'} \]  

At zero temperature one therefore obtains

\[ \Sigma_x(k) = -\int_{k'} v(k' - k) \Theta(k_F - |k'|) \, d\xi_{k'} \]  

where the Fermi wavevector is defined by \( \xi_{k_F} + \Sigma_x(k_F) = 0 \) and

\[ n = \frac{2}{\beta V} \sum_k G(k) e^{i\omega_n \eta} = 2 \int_k n_F(\xi_k + \Sigma_x(k)) = \frac{2}{\beta} \int_k \Theta(k_F - |k|). \]  

Thus \( k_F \) is the same as in the non-interacting electron gas (for a given density). This is nothing but the Luttinger theorem: The interactions do not change the volume of the Fermi surface (Sec. 4.4.6). For a frequency independent self-energy, the theorem is trivially satisfied. Note that \( \Sigma_x(k) \) is actually first order in the Coulomb interaction: Only the first order diagram contributes, all higher-order ones vanishing as can be verified by a direct calculation.

From (5.82), we thus obtain

\[ \Sigma_x(k) = -\frac{e^2 k_F}{2\pi^2 \epsilon_0} \left[ \frac{1}{2} + \frac{1 - \tilde{k}^2}{4k} \ln \left| \frac{1 + \tilde{k}}{1 - \tilde{k}} \right| \right] \]  

(Fig. 5.11), where \( \tilde{k} = |k|/k_F \). Since

\[ \frac{\Sigma_x(k_F)}{\epsilon_F} = -\frac{e^2 m}{2\pi^2 \epsilon_0 k_F} = -\frac{2}{\pi k_F a_0} \sim -r_s, \]  

the exchange self-energy is small in the high-density limit \( r_s \ll 1 \) where a perturbative treatment of the Coulomb interaction is expected to be valid. Yet the Hartree-Fock theory is not satisfactory for the electron gas. The reason is that the non-analyticity of \( \Sigma_x(k) \) at \( |k| = k_F \) gives rise to a vanishing effective mass at the Fermi level,

\[ \frac{m}{m^*} = 1 + \frac{m}{k_F^2} \partial |k| \Sigma_x(k) = \lim_{k \to 1} \left[ 1 + \frac{1}{\pi k_F a_0} \ln \left| \frac{2}{1 - k} \right| \right] = \infty. \]  

5.4.2 RPA self-energy

The failure of the Hartree-Fock approximation is not surprising since the Fock self-energy involves the bare, unscreened, Coulomb interaction \( v(q) \). The simplest improvement, consistent with screening, is obtained by replacing \( v(q) \) by \( v(q)/\epsilon_{RPA}(q) \) in \( \Sigma_x \) and neglecting
the self-consistency, i.e. replacing the dressed propagator $G$ by $G_0$. This gives the RPA self-energy

$$\Sigma_{\text{RPA}}(k) = -\frac{1}{\beta V} \sum_q \frac{v(q)}{1 + v(q)\chi_{nn}^0(q)} G_0(k + q),$$

(5.87)

which can be rewritten as

$$\Sigma_{\text{RPA}}(k) = \Sigma_x(k) + \frac{1}{\beta V} \sum_q v(q)^2 \chi_{nn}^{\text{RPA}}(q) G_0(k + q),$$

(5.88)

as shown diagrammatically in Fig. 5.12.

The RPA self-energy can also be justified as follows. The self-energy is related to the density-density response function by (1.439) or (1.440), i.e.

$$\frac{1}{\beta V} \sum_k \Sigma(k) G(k) e^{i\omega_n \eta} = \frac{1}{2} \int_q v(q) \left[ \frac{1}{\beta} \sum_{\omega_n} \chi_{nn}(q) - n \right].$$

(5.89)

One can therefore try to find an expression for the self-energy that satisfies (5.89) when $\chi_{nn}$ is approximated by $\chi_{nn}^{\text{RPA}}$. Given that $\chi_{nn}^{\text{RPA}}$ is expressed in terms of $G_0$, the RPA self-energy can be the solution only if we replace $G$ by $G_0$ in the lhs. In that case both sides of (5.89) can be represented by the sum of the bubble diagrams

To show the validity of (5.89) more rigorously (and the role of $n$ in the rhs), we consider

$$\frac{1}{\beta V} \sum_k \Sigma_{\text{RPA}}(k) G_0(k) e^{i\omega_n \eta} = \frac{1}{\beta V} \sum_k \left[ \Sigma_{\text{RPA}}(k) - \Sigma_x(k) \right] G_0(k)$$
5.4 One-particle properties

\[ \Sigma_{\text{RPA}} = \cdot \cdot \cdot + \cdot \cdot \cdot \]

\[ \Sigma_{\text{RPA}}^{\text{RPA}} = \cdot \cdot \cdot + \cdot \cdot \cdot \]

Since

\[ \Sigma_{x}(k) = \lim\limits_{|\omega_n| \to \infty} \Sigma_{\text{RPA}}(k), \]

the Matsubara sum in the first term in the rhs of (5.90) is convergent and the factor \( e^{i\omega_n \eta} \) is not necessary anymore. One easily finds

\[ \frac{1}{\beta V} \sum_{k} \frac{1}{2} \left[ \Sigma_{\text{RPA}}(k) - \Sigma_{x}(k) \right] G_0(k) = \frac{1}{(\beta V)^2} \sum_{k,q} v(q) \chi_{nn}^\text{RPA}(q) G_0(k+q) \]

\[ = \frac{1}{2\beta V} \sum_{q} v(q) [\chi_{nn}^\text{RPA}(q) - \chi_{nn}^0(q)]. \]

The second term in the rhs of (5.90) gives

\[ \frac{1}{\beta V} \sum_{k} \Sigma_{x}(k) G_0(k) e^{i\omega_n \eta} = - \frac{1}{(\beta V)^2} \sum_{k,q} v(q) G_0(k+q) e^{i(\omega_n + \omega_{\nu}) \eta} G_0(k) e^{i\omega_n \eta} \]

\[ = - \frac{1}{\beta V} \sum_{k,q} v(q) n_F(\xi_{k+q}) n_F(\xi_k). \]

This should be compared with

\[ \frac{1}{\beta V} \sum_{q} v(q) \chi_{nn}^0(q) = - \frac{2}{\beta V^2} \sum_{q,k} v(q) \frac{n_F(\xi_k) - n_F(\xi_{k+q})}{\omega_{\nu} + \xi_k - \xi_{k+q}} e^{i\omega_{\nu} \eta} \]

\[ = \frac{2}{\beta V} \sum_{q,k} v(q) [n_F(\xi_k) - n_F(\xi_{k+q})] n_B(\xi_{k+q} - \xi_k) \]

Using \([n_F(\xi_k) - n_F(\xi_{k+q})] n_B(\xi_{k+q} - \xi_k) = n_F(\xi_{k+q})[1 - n_F(\xi_k)]\), this gives

\[ \frac{1}{\beta V} \sum_{q} v(q) \chi_{nn}^0(q) = - \frac{2}{\beta V^2} \sum_{k,q} v(q) n_F(\xi_k) n_F(\xi_{k+q}) + \frac{1}{\beta V} \sum_{q} v(q) n_0 \]

and therefore

\[ \frac{1}{\beta V} \sum_{k} \Sigma_{x}(k) G_0(k) e^{i\omega_n \eta} = \frac{1}{2\beta V} \sum_{q} v(q) \chi_{nn}^0(q) - \frac{1}{2\beta V} \sum_{q} v(q) n_0, \]

\[15\text{Since the Matsubara sum in (5.94) is convergent, we may add the convergence factor } e^{i\omega_{\nu} \eta}.\]
where \( n_0 \equiv n_0(\mu) = 2 \sum_k n_F(\xi_k) \) is the average density in the non-interacting limit (for a given chemical potential). From (5.92) and (5.95) we finally obtain
\[
\frac{1}{\beta V} \sum_k \Sigma_{\text{RPA}}(k) G_0(k) e^{i\omega n_0} = \frac{1}{2} \int q v(q) \left[ \frac{1}{\beta} \sum_{\omega} \chi_{nn}^{\text{RPA}}(q) - n_0 \right],
\]
which is the expected result except for \( n_0 = (2/\beta V) \sum_k G_0(k) e^{i\omega n_0} \) appearing in the rhs instead of the actual density \( n = (2/\beta V) \sum_k G_{\text{RPA}}(k) e^{i\omega n} \) (with \( G_{\text{RPA}} = (G_0 - \Sigma_{\text{RPA}})^{-1} \)).

### 5.4.2.1 Fermi-liquid results

In order to find out whether the RPA self-energy describes a Fermi liquid, one must calculate the imaginary part \( \Im[\Sigma_{\text{RPA}}(k, \omega)] \) at low energies and temperatures (Sec. 4.4.4). The first term in the rhs of (5.88), \( \Sigma_x \), does not contribute to \( \Im[\Sigma_{\text{RPA}}(k, \omega)] \). The second term is similar to the second-order contribution to the self-energy studied in Sec. 4.4.1 with the bare charge susceptibility \( \chi_{nn}^0 \) replaced by \( \chi_{nn}^{\text{RPA}} \). We therefore deduce from (4.4.9)
\[
\Im[\Sigma_{\text{RPA}}(k, \omega)] = -\frac{m}{4\pi^2 k_F} \int_0^{2k_F} v(q)^2 |q| dq 
\times \int_{-\infty}^{\infty} d\omega' \chi_{nn}^{\text{RPA}}(q, \omega') [n_B(\omega') + n_F(\omega' + \omega)]. \tag{5.98}
\]
Because of the Bose and Fermi factors, the relevant part of the sum over \( \omega' \) corresponds to \( |\omega'| \leq \text{max}(|\omega|, T) \). For \( |\omega|, T \rightarrow 0 \), we can therefore use the low-energy limit of \( \chi_{nn}^{\text{RPA}}(q, \omega') \) (with \( |q| \leq 2k_F \)),
\[
\chi_{nn}^{\text{RPA}}(q, \omega) \approx \frac{\chi_{nn}^0(q, \omega)}{[1 + v(q)\chi_{nn}^0(q, 0)]^2} \approx \frac{\pi N(0)\omega}{[\epsilon_{\text{RPA}}(q, 0)]^2 v_F |q|} \quad (\omega \rightarrow 0). \tag{5.99}
\]
Since \( \chi_{nn}^{\text{RPA}}(q, \omega) \) is linear in \( \omega \) for \( \omega \rightarrow 0 \), we can proceed as in Sec. 4.4.1 to obtain
\[
\Im[\Sigma_{\text{RPA}}(k, \omega)] = -\frac{m^3}{16\pi^3 k_F} (\omega^2 + \pi^2 T^2) \int_0^{2k_F} d|q| \frac{v(q)^2}{[\epsilon_{\text{RPA}}(q, 0)]^2}, \tag{5.100}
\]
which is the standard Fermi-liquid result. Approximating \( \epsilon_{\text{RPA}}(q, 0) \) by the Thomas-Fermi result \( \epsilon_{\text{TF}}(q) = 1 + q^2_{\text{TF}}/q^2 \), we find
\[
\int_0^{2k_F} d|q| \frac{v(q)^2}{[\epsilon_{\text{TF}}(q, 0)]^2} = \frac{2\pi^4}{m^2 k_F} \xi(r_s), \tag{5.101}
\]
where
\[
\xi(r_s) = \int_0^1 dx \left[ 1 + \frac{\pi}{r_s} \left( \frac{9\pi}{4} \right)^{1/3} x^2 \right]^{-2}. \tag{5.102}
\]
We deduce that the quasi-particle lifetime $\tau_k$ is given by

$$\frac{1}{\tau_k} = -2z_k \Im [\Sigma^R(k, \tilde{\xi}_k)] = z_k \frac{\pi^2 T^2}{\epsilon_F} + \frac{\xi^2_k}{\epsilon_F},$$  

(5.103)

where $z_k$ is the quasi-particle weight and $\tilde{\xi}_k$ the quasi-particle dispersion defined by $\tilde{\xi}_k = \xi_k + \Re [\Sigma^R(k, \tilde{\xi}_k)]$ (Sec. 4.4). In the high-density limit $r_s \to 0$, $\xi(r_s) \sim \sqrt{r_s}$ and (5.103) gives

$$\frac{1}{\tau_k} \sim z_k \sqrt{r_s} \frac{\tilde{\xi}_k^2}{\epsilon_F} \sim z_k \omega_p \frac{\tilde{\xi}_k^2}{\epsilon_F} (T \to 0).$$  

(5.104)

The quasi-particle scattering rate is proportional to the plasma frequency but reduced by the factor $\tilde{\xi}_k^2/\epsilon_F^2$. At metallic densities $r_s \sim 1$, $\xi(r_s) \sim 1$ and $1/\tau_k \sim z_k \tilde{\xi}_k^2/\epsilon_F$. In both cases, $r_s \to 0$ and $r_s \sim 1$, the quasi-particle scattering rate $1/\tau_k \sim (|k| - k_F)^2$ diverges near the Fermi surface as in a Fermi liquid.

The calculation of $z_k$ – as well as the effective mass $m^*$ – is quite cumbersome and we therefore only quote the results in Table 5.2. As expected, $z_{k_F}$ decreases monotonically with $r_s$ but remains finite, confirming the Fermi-liquid behavior of the electron gas up to rather large values of $r_s$. The effective mass is smaller than the bare mass when $r_s \lesssim 2$, but becomes larger at sufficiently low density.\[6\]

Table 5.2: Quasi-particle weight $z_{k_F}$ [7] and effective mass $m^*$ (reproduced from [1]) in the RPA.

<table>
<thead>
<tr>
<th>$r_s$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z_{k_F}^{\text{RPA}}$</td>
<td>1.0859</td>
<td>0.768</td>
<td>0.700</td>
<td>0.646</td>
<td>0.602</td>
<td>0.566</td>
<td></td>
</tr>
<tr>
<td>$(m^*/m)_{\text{RPA}}$</td>
<td>1.07</td>
<td>0.99</td>
<td>1.02</td>
<td>1.04</td>
<td>1.06</td>
<td>1.08</td>
<td></td>
</tr>
</tbody>
</table>

\[6\]This suggests that the Landau parameter $F_1^s$ is negative in the high-density limit of the three-dimensional electron gas ($m^*/m = 1 + F_1^s/3$ in a Galilean invariant system (Sec. 4.1.5)). This agrees with predictions of local field factor approaches (Sec. 5.5) [1].

5.5 Beyond the RPA

The main advantage and appeal of the RPA is its simplicity. Not only does it account for important physical properties of a metal (plasmon mode, screening, etc.) but it also satisfies a number of exact results such as the sum rules discussed in Sec. 5.3.2 or the value of the plasmon energy in the long wavelength limit. However the RPA is based on the one-particle Hartree (non-interacting) propagator and the correlations between a given electron and the other particles are not taken into account. The Pauli principle reduces the probability of finding two electrons with the same spin orientation nearby. This “exchange hole” is not accounted for in the Hartree propagator. The probability of finding two electrons nearby, irrespective of their spin orientations, is further altered by the Coulomb interaction. This “correlation hole” is also neglected.

We have shown in Sec. 5.4.2 how the RPA can nevertheless be used to build a nontrivial approximation for the self-energy ($\Sigma_{\text{RPA}}$). It would be tempting to include the RPA
self-energy into the irreducible density-density response function $\Pi_{nn}$. This would however violate conservation laws, i.e. make the approximation non-conserving.\textsuperscript{17} The simplest conserving approximation beyond the RPA includes both the Hartree and Fock self-energies in the one-particle propagator; it is sometimes referred to as the time-dependent Hartree-Fock approximation (the RPA corresponding to the time-dependent Hartree approximation). In order to satisfy conservation laws and other exact results, one must then include all ladder diagrams in the calculation of $\Pi_{nn}$ (Fig. 5.13). The justification of this statement is given in Chap. 9 where a general method – based on the Luttinger-Ward functional – to generate diagrammatic approximations that satisfy conservation laws is described. The time-dependent Hartree-Fock approximation requires a numerical solution. It does not yield a significant improvement over the RPA and, as discussed in Sec. 5.4.1, the Fock self-energy is itself pathological. Thus the time-dependent Hartree-Fock approximation has its own limitations and difficulties. A detailed discussion of the results is beyond the scope of this chapter; we refer the interested reader to Ref. \cite{1}.

It is possible to treat exchange and correlation effects in a non-diagrammatic approximation that preserves the mathematical structure of the RPA.\textsuperscript{18} In the latter, the electrons are subjected to an effective potential

$$\phi_{\text{ext}}(q, \omega) + v(q)\langle \hat{n}(q) \rangle(\omega)$$

(Sec. 5.2.2). In order to account for the correlations neglected in the RPA, one assumes that the effective potential seen by the spin-$\sigma$ electrons can be written as

$$\phi_{\text{eff}, \sigma}(q, \omega) = \phi_{\text{ext}}(q, \omega) + v(q)\sum_{\sigma'}\langle \hat{n}_{\sigma'}(q) \rangle(\omega) - \sum_{\sigma'} v(q) G_{\sigma\sigma'}(q, \omega)\langle \hat{n}_{\sigma'}(q) \rangle(\omega).$$

The local field factor $G_{\sigma\sigma'}(q, \omega)$ contains the correction to the effective potential stemming from the exchange and correlation effects. The density-density response function then reads

$$\chi_{nn}(q) = \frac{\chi_{nn}^0(q)}{1 + v(q)[1 - G_+(q)]\chi_{nn}^0(q)},$$

(5.107)

where $G_+(q) = \frac{1}{2}[G_{\uparrow\uparrow}(q) + G_{\uparrow\downarrow}(q)]$. Thus $G_{\sigma\sigma'}$ can be seen as an effective vertex correction. The local field factor approach has proven to be a successful method to calculate many quantities of interest (effective mass, compressibility, spin susceptibility, Landau parameters, etc.) while treating exchange and correlation effects beyond the RPA. Again we refer to Ref. \cite{1} for a detailed discussion.

\textsuperscript{17}See also the discussion at the end of Sec. 9.1.3.1.

\textsuperscript{18}The idea to improve the RPA while preserving its simple mathematical structure is also at the heart of the two-particle self-consistent (TPSC) theory of the Hubbard model (Sec. 6.4.5).
5.6 Wigner crystallization

The RPA is valid only in the high-density limit where the kinetic energy dominates. In the opposite limit the interaction energy prevails. When $r_s \to 0$, the system behaves classically and one expects the charged particles to form a crystalline structure, the so-called Wigner crystal. In three dimensions, the most stable structure is body-centered-cubic (bcc). Quantum-mechanical corrections – zero-point motion of the electrons about their classical equilibrium position – to the classical electrostatic energy modify the ground state energy but do not destroy the crystalline order. The lowest-lying excitations of the Wigner crystal are phonons and collective spin excitations corresponding to fluctuations in the relative spin orientations on different lattice sites.

Thus as the density is varied one expects a phase transition between a homogeneous fluid and a crystal. Other phases are actually possible, in particular a spin-polarized fluid. Considering only these three possible ground states, one finds a paramagnetic fluid for $r_s \lesssim 75$, a ferromagnetic fluid for $75 \lesssim r_s \lesssim 100$ and a Wigner crystal for $r_s \gtrsim 100$ [1].

In two dimensions, the ground state is a metallic Fermi liquid at high densities and a Wigner crystal with a hexagonal lattice structure at low densities. The determination of the ground state in the intermediate range $20 \lesssim r_s \lesssim 34$ is very hard and whether or not the fluid is polarized remains an open issue [1].

In one dimension, the concept of Fermi liquid does not apply. A fermion system with short-range repulsive interactions is a Luttinger liquid (Chaps. 4 and 15). The ground state of an electron system with a long-range Coulomb interaction potential $v(x) \sim e^2/|x|$ (i.e. $v(q) \sim -e^2 \ln |q|$) is a Wigner crystal [8] (see Chap. 15).
Guide to the bibliography

The electron gas in the RPA is discussed in Refs. [1–6]. For a discussion which goes beyond the RPA, see Refs. [1,6]. Wigner crystallization is discussed in [1,6].


