M2 Quantum Physics – Academic year 2013-2014
Broken symmetries and quantum phase transitions

Tutorial class 1: Quantization of the harmonic chain[1]

We consider a one-dimensional "crystal" made of pointlike masses $m = \rho a$ separated by a distance a at rest. Neighboring masses are linked by springs with constant $k_s = \kappa/a$. We want to study the quantum dynamics of this system first by following the canonical quantization rules and then by writing the partition function as a functional integral over a displacement field.

1) Classical analysis

1.1) Write the Lagrangian L and determine the eigenmodes of the system from the Euler-Lagrange equations. Give the expression of the sound velocity c in the crystal.

1.2) In the low-energy (or long-distance) limit, we can approximate the crystal by a continuous elastic chain. We note $\phi(x,t)$ the displacement of the infinitesimal mass ρdx located between x and $x+dx$ at rest. The Lagrangian

$$L[\phi] = \int dx\, \mathcal{L}(\partial_x \phi, \dot{\phi}) \tag{1}$$

becomes a functional of the field $\phi(x,t)$. What is the Lagrangian density $\mathcal{L}$? What is the equation of motion satisfied by ϕ? Compare with the result of question (1.1).

2) Canonical quantization

2.1) What is the momentum $\Pi(x,t)$ conjugated to the displacement field? Give the expression of the classical Hamiltonian of the system.

2.2) In the quantum description, the classical fields $\phi(x,t)$ and $\Pi(x,t)$ become operators $\hat{\phi}(x)$ and $\hat{\Pi}(x)$ (time independent in the Schrödinger picture). What are the commutation rules satisfied by these operators? Using the operators

$$\begin{aligned} \hat{\phi}(k) &= \frac{1}{\sqrt{L}} \int_0^L dx\, e^{-ikx} \hat{\phi}(x) = \hat{\phi}^\dagger(-k), \\ \hat{\Pi}(k) &= \frac{1}{\sqrt{L}} \int_0^L dx\, e^{-ikx} \hat{\Pi}(x) = \hat{\Pi}^\dagger(-k), \end{aligned} \tag{2}$$

show that the quantum Hamiltonian $\hat{H}$ corresponds to a set of decoupled harmonic oscillators with frequencies $\omega_k = c|k|$.

2.3) Show that the Hamiltonian can be diagonalized by introducing the ladder operators

$$\begin{aligned} \hat{a}(k) &= \sqrt{\frac{\rho\omega_k}{2}} \Big[\hat{\phi}(k) + \frac{i}{\rho\omega_k} \hat{\Pi}(k)\Big], \\ \hat{a}^\dagger(k) &= \sqrt{\frac{\rho\omega_k}{2}} \Big[\hat{\phi}^\dagger(k) - \frac{i}{\rho\omega_k} \Pi^\dagger(k)\Big]. \end{aligned} \tag{3}$$

What is the physical meaning of the operators $\hat{a}_k$ and $\hat{a}_k^\dagger$? What are the eigenstates of $\hat{H}$?

[1]Students are encouraged to answer questions 1 and 2 before the tutorial class, only question 3 will be discussed in detail.

3) Functional integral

3.1) We introduce the states $|\phi\rangle$ and $|\Pi\rangle$ defined by $\hat{\phi}(x)|\phi\rangle = \phi(x)|\phi\rangle$ and $\hat{\Pi}(x)|\Pi\rangle = \Pi(x)|\Pi\rangle$. Show that these states satisfy the closure relations

$$\begin{aligned} \mathcal{N} \lim_{a\to 0} \int \prod_{l=0}^{L/a} d\phi(la)\, |\phi\rangle\langle\phi| = \hat{I}, \\ \mathcal{N}' \lim_{a\to 0} \int \prod_{l=0}^{L/a} d\Pi(la)\, |\Pi\rangle\langle\Pi| = \hat{I}, \end{aligned} \tag{4}$$

Where $\hat{I}$ is the identity operator. In order to correctly define the closure relations, we have discretized the chain (the continuous variable x becoming a discrete variable la); the continuous chain is obtained in the limit $a \to 0$. $\mathcal{N}$ and $\mathcal{N}'$ are normalization constants that will be ignored in the following.

3.2) Show that the partition function

$$Z = \int_{\phi(x,\beta)=\phi(x,0)} \mathcal{D}[\phi]\, e^{-S[\phi]} \tag{5}$$

can be written as a functional integral over a real field $\phi(x,\tau)$ ($\tau \in [0,\beta]$), with the (Euclidean) action

$$S[\phi] = \frac{1}{2}\int_0^\beta d\tau \int_0^L dx \left[\rho\dot{\phi}^2 + \kappa(\partial_x\phi)^2\right]. \tag{6}$$

3.3) Write the action $S[\phi]$ as a function of the Fourier transformed field

$$\phi(k, i\omega_n) = \frac{1}{\sqrt{\beta L}} \int_0^\beta d\tau \int_0^L dx\, e^{-i(kx-\omega_n\tau)}\phi(x,\tau). \tag{7}$$

What is the expression of the frequency ω_n.

3.4) Using properties of Gaussian integrals, compute the partition function Z from equation (5).

3.5) What can we deduce from the equation of motion $\delta S/\delta\phi(x,\tau) = 0$? Show that at sufficiently high temperature, only the field $\phi(k, i\omega_{n=0})$ contribute to the functional integral. What is then the expression of the action? Show that at zero temperature we recover the action of a classical two-dimensional system.

3.6) Using properties of Gaussian integrals, calculate the propagator of the field ϕ

$$G(k, i\omega_n) = \langle\phi(k, i\omega_n)\phi(-k, -i\omega_n)\rangle = \frac{1}{Z}\int \mathcal{D}[\phi]\, \phi(k, i\omega_n)\phi(-k, -i\omega_n)\, e^{-S[\phi]}. \tag{8}$$

Show that $G(k, \tau-\tau') = \langle\phi(k,\tau)\phi(-k,\tau')\rangle$ coincides with the correlation function

$$\langle T_\tau \hat{\phi}(k,\tau)\hat{\phi}(-k,\tau')\rangle \equiv \theta(\tau-\tau')\langle\hat{\phi}(k,\tau)\hat{\phi}(-k,\tau')\rangle + \theta(-\tau+\tau')\langle\hat{\phi}(-k,\tau')\hat{\phi}(k,\tau)\rangle, \tag{9}$$

where

$$\hat{\phi}(k,\tau) = e^{\hat{H}\tau}\hat{\phi}(k)e^{-\hat{H}\tau} \tag{10}$$

is an operator in the Heisenberg picture (in imaginary time).

3.7) Which information do we obtain from the poles of the "retarded" propagator $G^R(k,\omega) = G(k, i\omega_n \to \omega + i\eta)$ ($\eta \to 0^+$)? Show that the spectral function $A(k,\omega) = \Im[G^R(k,\omega)]$ contains information about the excitation spectrum.

3.8) We consider a more general case where the Lagrangian $L[\phi]$ includes the term $\int dx\, V(\phi)$ (with V a polynomial)? What is then the Euclidean action $S[\phi]$? What is the relation with the quantization of a relativistic scalar field in field theory?

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Tutorial class 2 : $\left(\boldsymbol{\varphi}^2\right)^2$ theory with O(N) symmetry

1) From Ising model to φ^4 theory.

We consider the Ising model,

$$\beta H = -\frac{1}{2}\sum_{i,j}\sigma_i K_{ij}\sigma_j \tag{1}$$

defined on a hypercubic lattice in d dimensions ($\sigma_i = \pm$). K_{ij} is equal to $\beta J = J/T$ for two neighboring sites and vanishes otherwise.

1.1) Calculate the critical temperature T_{c0} in the mean-field approximation.

1.2) Using the Hubbard-Stratonovich transformation

$$e^{\frac{1}{2}\sum_{i,j}\sigma_i K_{ij}\sigma_j} \propto \int_{-\infty}^{\infty}\prod_i d\varphi_i \, e^{-\frac{1}{2}\sum_{i,j}\varphi_i K_{ij}^{-1}\varphi_j + \sum_i \varphi_i\sigma_i}, \tag{2}$$

rewrite the partition function as an integral over real fields φ_i varying between $-\infty$ and ∞. Show that the corresponding action (or Hamiltonian) can be written as

$$S[\varphi] = \int d^d r \left\{ \frac{1}{2}(\boldsymbol{\nabla}\varphi)^2 + \frac{r_0}{2}\varphi^2 + \frac{u_0}{4!}\varphi^4 \right\} \tag{3}$$

in the continuum limit if terms of order φ^6, as well as higher-order derivative terms, are neglected. What is the expression of r_0 and u_0 ? Can we generalize the derivation of Eq. (3) to the Heisenberg model (or to a N-component spin model) ?

2) O(N) model.

We consider the O(N) model, defined by the action

$$S[\boldsymbol{\varphi}] = \int d^d r \left\{ \frac{1}{2}(\boldsymbol{\nabla}\boldsymbol{\varphi})^2 + \frac{r_0}{2}\boldsymbol{\varphi}^2 + \frac{u_0}{4!}\left(\boldsymbol{\varphi}^2\right)^2 \right\}, \tag{4}$$

on a hypercubic lattice in d dimensions. $\boldsymbol{\varphi} = (\varphi_1, \cdots, \varphi_N)^T$ is a N-component vector, $\boldsymbol{\varphi}^2 = \sum_{i=1}^N \varphi_i^2$ and $(\boldsymbol{\nabla}\boldsymbol{\varphi})^2 \equiv \sum_{i=1}^N (\boldsymbol{\nabla}\varphi_i)^2$. We note $r_0 = \bar{r}_0(T - T_{c0})$, and assume $\bar{r}_0$ and u_0 to be temperature independent. The model is regularized in the large momentum limit by a cutoff Λ.

2.1) Calculate $\mathbf{m} = \langle \boldsymbol{\varphi}(\mathbf{r}) \rangle$ in the mean-field approximation. Which symmetry is spontaneously broken in the low-temperature phase ?

2.2) Expand the action $S[\boldsymbol{\varphi}]$ about its mean-field value to quadratic order in the fluctuations $\boldsymbol{\varphi} - \mathbf{m}$ (Gaussian approximation). Deduce the expression of the propagator

$$G_i(\mathbf{q}) = \langle \varphi_i(\mathbf{q})\varphi_i(-\mathbf{q}) \rangle - \langle \varphi_i(\mathbf{q}) \rangle \langle \varphi_i(-\mathbf{q}) \rangle \tag{5}$$

(consider separately the high- and low-temperature phases). Do the results in the ordered phase agree with Goldstone's theorem ?

2.3) What are the values of the critical exponents ν, β, γ and η in the Gaussian approximation ?

2.4) Calculate the mean value $\langle (\varphi_i(\mathbf{r}) - m_i)^2 \rangle$ in the ordered phase. What can we conclude regarding the existence of an ordered phase at nonzero temperatures in dimension $d \leq 2$ (Mermin-Wagner theorem) ?

2.5) Using dimensional analysis, determine the dimension $[\varphi]$ of the field $\boldsymbol{\varphi}(\mathbf{r})$ as well as $[r_0]$ and $[u_0]$.[1] We recall that X has dimension $x = [X]$ if it is expressed in physical units of L^{-x} (with L a length scale). Show that we can rewrite the action in the dimensionless form

$$S[\tilde{\boldsymbol{\varphi}}] = \int d^d\tilde{r} \left\{ \frac{1}{2}(\boldsymbol{\nabla}_{\tilde{\mathbf{r}}}\tilde{\boldsymbol{\varphi}})^2 + \frac{1}{2}\tilde{\boldsymbol{\varphi}}^2 + \frac{\tilde{u}_0}{4!}\left(\tilde{\boldsymbol{\varphi}}^2\right)^2 \right\}. \tag{6}$$

Deduce that the Gaussian approximation studied in (2.2), and more generally the perturbative approach in u_0, is correct for $d > 4$ but fails for $d < 4$. Show that in the latter case, one can nevertheless define a temperature T_G (the Ginzburg temperature) above which the Gaussian approximation remains valid.

1. $[\boldsymbol{\varphi}]$ – as defined in the text – is called the naive scaling dimension (or engineering dimension) of the field $\boldsymbol{\varphi}$.

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Tutorial class 3: Bogoliubov's theory of superfluidity

We consider a three-dimensional boson gas at zero temperature ($\beta \to \infty$) described by the action

$$S[\psi^*, \psi] = \int_0^\beta d\tau \int d^3r \left\{ \psi^*(\mathbf{r},\tau) \left(\partial_\tau - \mu - \frac{\boldsymbol{\nabla}^2}{2m} \right) \psi(\mathbf{r},\tau) + \frac{g}{2} \psi^*(\mathbf{r},\tau)\psi^*(\mathbf{r},\tau)\psi(\mathbf{r},\tau)\psi(\mathbf{r},\tau) \right\}, \qquad (1)$$

with a ultraviolet momentum cutoff Λ ($|\mathbf{k}| \leq \Lambda$).

1) Excitation spectrum.

1.1) What is the expression of the action in a saddle-point approximation where the field $\psi(\mathbf{r},\tau) = \psi_0$ is assumed uniform and time independent. Under which condition do we have $\psi_0 \neq 0$? What is the symmetry which is spontaneously broken? Why can we choose ψ_0 real with no loss of generality?

1.2) We now consider fluctuations $\psi'(\mathbf{r},\tau) = \psi(\mathbf{r},\tau) - \psi_0$ of the field about its saddle-point value ψ_0. What is the action $S[\psi'^*, \psi']$ to quadratic order in ψ'?

1.3) We introduce the two-component field

$$\Psi(\mathbf{k}, i\omega_n) = \begin{pmatrix} \psi'(\mathbf{k}, i\omega_n) \\ \psi'^*(-\mathbf{k}, -i\omega_n) \end{pmatrix}, \qquad \Psi^\dagger(\mathbf{k}, i\omega_n) = \big(\psi'^*(\mathbf{k}, i\omega_n), \psi'(-\mathbf{k}, -i\omega_n) \big), \qquad (2)$$

where

$$\begin{aligned} \psi'(\mathbf{k}, i\omega_n) &= \frac{1}{\sqrt{\beta V}} \int_0^\beta d\tau \int d^3r \, e^{-i(\mathbf{k}\cdot\mathbf{r} - \omega_n\tau)} \psi'(\mathbf{r},\tau), \\ \psi'^*(\mathbf{k}, i\omega_n) &= \frac{1}{\sqrt{\beta V}} \int_0^\beta d\tau \int d^3r \, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega_n\tau)} \psi'^*(\mathbf{r},\tau). \end{aligned} \qquad (3)$$

Show that we can write the action in the form

$$S = S_0 + \frac{1}{2} \sum_{\mathbf{k},\omega_n} \Psi^\dagger(\mathbf{k}, i\omega_n) \mathcal{D}(\mathbf{k}, i\omega_n) \Psi(\mathbf{k}, i\omega_n), \qquad (4)$$

where S_0 is the saddle-point value and $\mathcal{D}(\mathbf{k}, i\omega_n)$ a 2×2 matrix. What is the excitation spectrum in the superfluid phase ($\psi_0 \neq 0$)? Does it agree with Goldstone theorem?

2) Thermodynamic potential and equation of state.

2.1) Using equation (10), show that the thermodynamic potential reads

$$\Omega = \Omega_0 + \frac{1}{2\beta} \sum_{\mathbf{k},\omega_n} \ln \det \mathcal{D}(\mathbf{k}, i\omega_n), \qquad (5)$$

where Ω_0 is the saddle-point contribution. The Matsubara frequency sum is divergent and the correct expression is

$$\Omega = \Omega_0 + \frac{1}{2\beta} \sum_{\mathbf{k},\omega_n} \left[\ln \mathcal{D}_{11}(\mathbf{k}, i\omega_n) \, e^{i\omega_n 0^+} + \ln \mathcal{D}_{22}(\mathbf{k}, i\omega_n) \, e^{-i\omega_n 0^+} + \ln \left(1 - \frac{\mathcal{D}_{12}(\mathbf{k}, i\omega_n)^2}{\mathcal{D}_{11}(\mathbf{k}, i\omega_n)\mathcal{D}_{22}(\mathbf{k}, i\omega_n)} \right) \right]. \qquad (6)$$

2.2) Using the results (11), express the thermodynamic potential Ω as a function of μ (do not carry out the sum over $\mathbf{k}$). Deduce the particle density n.

2.3) For a contact interaction, the s-wave scattering length a is defined by the equation (see page 3)

$$\frac{m}{4\pi a} = \frac{1}{g} + \frac{1}{V}\sum_{\mathbf{k}} \frac{1}{2\epsilon_{\mathbf{k}}}. \tag{7}$$

Using the results (12), express n as a function of μ and a (take the limit $\Lambda \to \infty$). Deduce the equation of state $\mu(n)$ in the limit $na^3 \ll 1$.

3) Condensate density. The particle density can be written as

$$n = n_0 + \frac{1}{V}\sum_{\mathbf{k}} \langle \psi'^*_{\mathbf{k}} \psi'_{\mathbf{k}} \rangle, \tag{8}$$

where n_0 is the particle density in the condensate.

3.1) Express $n'_{\mathbf{k}} = \langle \psi'^*_{\mathbf{k}} \psi'_{\mathbf{k}} \rangle$ as a function of $\mathcal{D}(\mathbf{k}, i\omega_n)$ and deduce

$$n'_{\mathbf{k}} = \frac{\epsilon_{\mathbf{k}} + \mu}{E_{\mathbf{k}}} - \frac{1}{2}. \tag{9}$$

3.2) Using (12), determine the condensate density n_0 in the limit $na^3 \ll 1$.

4) Derive equations (11) and (12).

Supplementary material

- $$\int \mathcal{D}[\psi^*,\psi]\, \exp\left\{-\frac{1}{2}\int dxdy\, (\psi^*(x),\psi(x))A(x,y)\begin{pmatrix}\psi(y)\\ \psi^*(y)\end{pmatrix}\right\} = (\det A)^{-1/2}, \tag{10}$$

  where $x=(\mathbf{r},\tau)$, $\int dx = \int_0^\beta d\tau \int d^3r$, etc.

- $$\begin{aligned} \frac{1}{\beta}\sum_{\omega_n} \ln(-i\omega_n + a)\, e^{i\omega_n 0^+} &= 0 \quad \text{pour} \quad a>0,\ T=0, \\ \frac{1}{\beta}\sum_{\omega_n} \ln\left(\frac{\omega_n^2+a^2}{\omega_n^2+b^2}\right) &= a-b \quad \text{pour} \quad 0<a<b,\ T=0. \end{aligned} \tag{11}$$

- $$\begin{aligned} \int_{\mathbf{k}} \left(\frac{1}{E_{\mathbf{k}}} - \frac{1}{\epsilon_{\mathbf{k}}}\right) &= -\frac{2}{\pi^2} m^{3/2}\mu^{1/2}, \\ \int_{\mathbf{k}} \left(1 - \frac{\epsilon_{\mathbf{k}}+\mu}{E_{\mathbf{k}}}\right) &= -\frac{2}{3\pi^2}(m\mu)^{3/2}, \end{aligned} \tag{12}$$

  where $\int_{\mathbf{k}} = \int \frac{d^3k}{(2\pi)^3}$, $\epsilon_{\mathbf{k}} = \mathbf{k}^2/2m$ and $E_{\mathbf{k}} = [\epsilon_{\mathbf{k}}(\epsilon_{\mathbf{k}}+\mu)]^{1/2}$.

s-wave scattering length

When the interaction potential $g(\mathbf{r}) = g(\mathbf{r}_1 - \mathbf{r}_2)$ between particles is rotation invariant, collisions take place only in the s-wave channel at low energy. In the center-of-mass frame, a scattering state reads

$$\psi_{\mathbf{k}}(\mathbf{r}) \sim e^{i\mathbf{k}\cdot\mathbf{r}} + f_{\mathbf{k},\mathbf{k}'}\frac{e^{ikr}}{r}, \tag{13}$$

where $\mathbf{k}' = |\mathbf{k}|\hat{\mathbf{r}}$ and

$$f_{\mathbf{k},\mathbf{k}'} \to \frac{-a}{1+ika} \quad (k\to 0). \tag{14}$$

Low-energy interactions interactions are therefore entirely parameterized by the scattering length a. The latter can be obtained from the T matrix,

$$T_{\mathbf{k},\mathbf{k}'}(\mathbf{k}^2/m) = -\frac{4\pi}{m} f_{\mathbf{k},\mathbf{k}'}, \tag{15}$$

and the equation

$$T(\epsilon) = g + gG_0^+(\epsilon)T(\epsilon), \tag{16}$$

where $G_0^+(\epsilon) = (\epsilon + i0^+ - H_0)^{-1}$ (H_0 is the kinetic energy of the two particles in the center-of-mass frame). For a contact potential $g(\mathbf{r}) = g\delta(\mathbf{r})$, $T_{\mathbf{k},\mathbf{k}'}(\epsilon)$ depends only on ϵ and we obtain

$$T(\epsilon) = g + g\int_{\mathbf{k}} \frac{1}{\epsilon + i0^+ - \mathbf{k}^2/m} T(\epsilon) \tag{17}$$

and in turn equation (7).

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Tutorial class 4: BCS theory of superconductivity

1) Free electron gas. We consider non-interacting electrons. Write the Hamiltonian $\hat{H}_0$ in second-quantized form. What is the ground state of the system? Give the expression of the (Euclidean) action $S_0[\psi^*,\psi]$ at finite temperatures, the one-particle propagator $G(\mathbf{k},i\omega_n)$ and the spectral function $A(\mathbf{k},\omega)=-\frac{1}{\pi}\Im[G^R(\mathbf{k},\omega)]$. What is the physical meaning of $A(\mathbf{k},\omega)$?

2) Superconducting metal. We now assume the electrons to interact *via* an attractive interaction. The Hamiltonian of the superconductor writes $\hat{H}_0+\hat{H}_{\rm int}$, where

$$\hat{H}_{\rm int}=-\lambda\int d^3r\,\hat{\psi}^\dagger_\uparrow(\mathbf{r})\hat{\psi}^\dagger_\downarrow(\mathbf{r})\hat{\psi}_\downarrow(\mathbf{r})\hat{\psi}_\uparrow(\mathbf{r}) \tag{1}$$

and $\lambda>0$. In conventional superconductors, the attractive interaction is due to the exchange of phonons between electrons and affects only particles in the vicinity of the Fermi surface, $|\xi_\mathbf{k}|=|\epsilon_\mathbf{k}-\mu|\leq\omega_D$ ($\omega_D\ll\epsilon_F$ is the phonon Debye frequency and ϵ_F the Fermi energy). This last condition can be implemented explicitly by rewriting equation (1) in terms of the operators $\hat{\psi}^{(\dagger)}_\sigma(\mathbf{k})$.

2.1) What is the action $S_{\rm int}[\psi^*,\psi]$ corresponding to the Hamiltonian $\hat{H}_{\rm int}$?

2.2) Using the Hubbard-Stratonovich transformation

$$e^{-S_{\rm int}[\psi^*,\psi]}=\int\mathcal{D}[\Delta^*,\Delta]\,e^{-\frac{1}{\lambda}\int_0^\beta d\tau\int d^3r|\Delta(\mathbf{r},\tau)|^2+\int_0^\beta d\tau\int d^3r[\Delta(\mathbf{r},\tau)^*\psi_\downarrow(\mathbf{r},\tau)\psi_\uparrow(\mathbf{r},\tau)+{\rm c.c.}]} \tag{2}$$

($\psi_\sigma(\mathbf{r},\tau)$ is a Grassmann variable and $\Delta(\mathbf{r},\tau)$ a complex field), one can rewrite the partion function

$$Z=\int\mathcal{D}[\psi^*,\psi,\Delta^*,\Delta]\,e^{-S[\psi^*,\psi,\Delta^*,\Delta]} \tag{3}$$

as a functional integral over $\psi^{(*)}$ and $\Delta^{(*)}$. What is the expression of the action $S[\psi^*,\psi,\Delta^*,\Delta]$?

3) Mean-field approximation. We consider the action $S[\psi^*,\psi,\Delta^*,\Delta]$ within a saddle-point approximation (or mean-field approximation) where the fluctuations of the field $\Delta(\mathbf{r},\tau)$ about its mean value Δ are neglected. We assume Δ to be real.

3.1) Show that the action of the electrons in the presence of the mean field Δ writes

$$S_{\rm MF}[\psi^*,\psi]=\beta V\frac{\Delta^2}{\lambda}-\sum_{\mathbf{k},\omega_n}\Psi^\dagger(\mathbf{k},i\omega_n)\mathcal{G}^{-1}(\mathbf{k},i\omega_n)\Psi(\mathbf{k},i\omega_n) \tag{4}$$

as a function of the Nambu spinor

$$\Psi(\mathbf{k},i\omega_n)=\begin{pmatrix}\psi_\uparrow(\mathbf{k},i\omega_n)\\ \psi^*_\downarrow(-\mathbf{k},-i\omega_n)\end{pmatrix},\quad \Psi^\dagger(\mathbf{k},i\omega_n)=(\psi^*_\uparrow(\mathbf{k},i\omega_n),\psi_\downarrow(-\mathbf{k},-i\omega_n)). \tag{5}$$

What is the expression of the 2×2 matrix $\mathcal{G}(\mathbf{k},i\omega_n)$? We shall introduce the energy $E_\mathbf{k}=\sqrt{\xi_\mathbf{k}^2+\Delta^2}$. Show that the elements of $\mathcal{G}(\mathbf{k},i\omega_n)$ can be expressed as a function of

$$\begin{aligned}G(\mathbf{k},i\omega_n)&=-\langle\psi_\sigma(\mathbf{k},i\omega_n)\psi^*_\sigma(\mathbf{k},i\omega_n)\rangle\quad\text{(normal Green function)},\\ F(\mathbf{k},i\omega_n)&=-\langle\psi_\uparrow(\mathbf{k},i\omega_n)\psi_\downarrow(-\mathbf{k},-i\omega_n)\rangle\\ F^\dagger(\mathbf{k},i\omega_n)&=-\langle\psi^*_\downarrow(-\mathbf{k},-i\omega_n)\psi^*_\uparrow(\mathbf{k},i\omega_n)\rangle\end{aligned}\quad\text{(anomalous Green function)}. \tag{6}$$

Give the expressions of $G(\mathbf{k}, i\omega_n)$ and $F(\mathbf{k}, i\omega_n)$.

3.2) Which condition should we impose on the grand potential Ω (or, equivalently, the partition function Z) to obtain the value of Δ? Deduce that $\Delta = \lambda\langle\psi_\downarrow(\mathbf{r},\tau)\psi_\uparrow(\mathbf{r},\tau)\rangle$ and obtain the equation satisfied by Δ. Show that a non-trivial solution $\Delta \neq 0$ is possible below a temperature T_c. Give the expression of T_c in the weak-coupling limit $\lambda N(0) \ll 1$. We shall assume that the density of states $N(\xi) = \frac{1}{V}\sum_{\mathbf{k}} \delta(\xi - \xi_{\mathbf{k}})$ can be approximated by $N(0)$ when $|\xi| \leq \omega_D$ and use

$$\int_0^{\omega_D} \frac{d\xi}{\xi} \tanh\left(\frac{\beta\xi}{2}\right) \simeq \ln\left(\frac{2\gamma\omega_D}{\pi T}\right) \qquad (T \ll \omega_D), \tag{7}$$

where γ is the exponential of the Euler constant. What is the physical meaning of T_c? What is the broken symmetry in the low-temperature phase $T < T_c$? By analogy with the bosonic superfluid (where the order parameter is $\langle\hat{\psi}(\mathbf{r})\rangle$), explain why we can interpret the low-temperature as a condensate of "Cooper pairs" $(\mathbf{k}, \uparrow; -\mathbf{k}, \downarrow)$.

3.3) Determine the value Δ_0 of Δ at zero temperature in the limit $\lambda N(0) \ll 1$. Deduce that the ratio Δ_0/T_c is "universal".

3.4) Verify that the unitary transformation

$$\begin{pmatrix} \gamma_\uparrow(\mathbf{k}, i\omega_n) \\ \gamma_\downarrow^*(-\mathbf{k}, -i\omega_n) \end{pmatrix} = \begin{pmatrix} u_{\mathbf{k}} & -v_{\mathbf{k}} \\ v_{\mathbf{k}} & u_{\mathbf{k}} \end{pmatrix} \begin{pmatrix} \psi_\uparrow(\mathbf{k}, i\omega_n) \\ \psi_\downarrow^*(-\mathbf{k}, -i\omega_n) \end{pmatrix}, \tag{8}$$

where

$$u_{\mathbf{k}}^2 = \frac{1}{2}\left(1 + \frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}}\right), \qquad v_{\mathbf{k}}^2 = \frac{1}{2}\left(1 - \frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}}\right), \tag{9}$$

makes the action diagonal. What is the corresponding Hamiltonian in terms of the fermionic operators $\hat{\gamma}_\sigma^\dagger(\mathbf{k})$ and $\hat{\gamma}_\sigma(\mathbf{k})$? Show that the ground state satisfies $\hat{\gamma}_\sigma(\mathbf{k})|\Psi_{\rm BCS}\rangle = 0$ (for all annihilation operators $\hat{\gamma}_\sigma(\mathbf{k})$) and is given by the BCS state

$$|\Psi_{\rm BCS}\rangle = \prod_{\mathbf{k}} \left(u_{\mathbf{k}} + v_{\mathbf{k}}\hat{\psi}_\uparrow^\dagger(\mathbf{k})\hat{\psi}_\downarrow^\dagger(-\mathbf{k})\right)|\text{vac}\rangle \tag{10}$$

originally introduced by Bardeen, Cooper and Schrieffer. What is then the physical meaning of $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$? Plot $u_{\mathbf{k}}$, $v_{\mathbf{k}}$ and $E_{\mathbf{k}}/\Delta$ as a function of $\xi_{\mathbf{k}}/\Delta$. How can we interpret the one-particle excitations $\hat{\gamma}_\sigma^\dagger(\mathbf{k})|\Psi_{\rm BCS}\rangle$?

3.5) Calculate the spectral function $A(\mathbf{k}, \omega) = -\frac{1}{\pi}\Im[G^R(\mathbf{k}, \omega)]$ of the superconductor. Plot $A(\mathbf{k}, \omega)$ vs ω for $\xi_{\mathbf{k}} > 0$ and $\xi_{\mathbf{k}} < 0$.

4) Collective modes. The collective modes of the superconductor correspond to the fluctuation eigenmodes of the complex field $\Delta(\mathbf{r}, \tau)$.

4.1) How many collective modes can we expect?

4.2) Show, without any calculation, that (at least) one of these modes must have a vanishing energy in the large wavelength limit. What low-energy effective action can we expect for this mode?

Tutorial class 5: Superfluid–Mott-insulator transition in a Bose gas

We consider the Bose-Hubbard model,

$$\hat{H} = -t \sum_{\langle \mathbf{r},\mathbf{r}' \rangle} \Big(\hat{b}^\dagger_{\mathbf{r}} \hat{b}_{\mathbf{r}'} + \text{h.c.} \Big) + \sum_{\mathbf{r}} \Big(-\mu \hat{b}^\dagger_{\mathbf{r}} \hat{b}_{\mathbf{r}} + \frac{U}{2} \hat{b}^\dagger_{\mathbf{r}} \hat{b}^\dagger_{\mathbf{r}} \hat{b}_{\mathbf{r}} \hat{b}_{\mathbf{r}} \Big), \tag{1}$$

describing bosons moving in a lattice (t denotes the hopping amplitude between nearest-neighbor sites) and subject to a local (on-site) interaction U. The chemical potential μ fixes the mean boson density (i.e. the mean number of bosons per site). We assume an hypercubic lattice in d dimensions and consider only the zero-temperature limit.

1) Superfluid–Mott-insulator transition

1.1) What is the ground state in the absence of interaction ($U = 0$)? How does the local repulsion affect the boson dynamics? Deduce that, depending on the boson density and the strength of the interactions, the ground state is either a superfluid or a "Mott insulator" (consider the limits $t/U \gg 1$ and $t/U \ll 1$).

1.2) Does the Bogoliubov theory appear appropriate to study the transition between the superfluid state and the Mott insulator?

2) Phase diagram

2.1) We first consider the limit of vanishing hopping amplitude $t = 0$ (local limit). What is the Hamiltonian $\hat{H}_{\text{loc}}$ of a single site? What are the eigenstates? What is the number n_0 of bosons per site as a function of μ/U? Calculate the local Green function $G_{\text{loc}}(\tau) = -\langle T_\tau \hat{b}(\tau) \hat{b}^\dagger(0) \rangle$ and its Fourier transform $G_{\text{loc}}(i\omega_n)$.

2.2) Give the expression of the (Euclidean) action $S[b^*, b]$ of the Bose-Hubbard model. Rewrite the partition function using the Hubbard-Stratonovich transformation

$$e^{\int_0^\beta d\tau \sum_{\mathbf{r},\mathbf{r}'} b^*_{\mathbf{r}} t_{\mathbf{r},\mathbf{r}'} b_{\mathbf{r}'}} = \int \mathcal{D}[\psi^*, \psi]\, e^{-\int_0^\beta d\tau \sum_{\mathbf{r},\mathbf{r}'} \psi^*_{\mathbf{r}} t^{-1}_{\mathbf{r},\mathbf{r}'} \psi_{\mathbf{r}'} + \int_0^\beta d\tau \sum_{\mathbf{r}} (\psi^*_{\mathbf{r}} b_{\mathbf{r}} + \text{c.c.})} \tag{2}$$

(ψ is a complex field). $t_{\mathbf{r},\mathbf{r}'}$ equals t if the sites $\mathbf{r}$ and $\mathbf{r}'$ are nearest neighbors and vanishes otherwise.

2.3) It is possible to integrate out the field b in a cumulant expansion:

$$\int \mathcal{D}[b^*, b]\, e^{-S_{\text{loc}}[b^*,b] - S'[b^*,b]} = Z_{\text{loc}}\, e^{-\langle S' \rangle + \frac{1}{2} \left(\langle S'^2 \rangle - \langle S' \rangle^2 \right) + \cdots} \tag{3}$$

where the mean value $\langle \cdots \rangle$ is to be taken with the local action S_{loc} corresponding to the limit $t = 0$. What is the expression of the action to quadratic order in the auxiliary field ψ?

2.4) Compute the energy $E = -\lim_{T\to 0} T \ln Z$ in a mean-field (or saddle-point) approximation where the fluctuations of the field ψ are neglected (assuming that the saddle-point value of $\psi_{\mathbf{r}}(\tau)$ is uniform and time independent). Deduce the phase diagram in the plane $(2dt, \mu/U)$ (use the notations $D = 2dt$, $\delta\mu = \mu - U(n_0 - 1/2)$ and $x = n_0 + 1/2$).

2.5) We write the energy $E(\psi^*, \psi)$ in the mean-field approximation as

$$E(\psi^*, \psi) = a_0 + a_2 |\psi|^2 + a_4 |\psi|^4 + \mathcal{O}(|\psi|^6), \tag{4}$$

where a_0 and a_2 were computed in question 2.4. What should be the sign of a_4? Calculate the mean boson density n in the Mott insulator and in the superfluid phase. Deduce that the compressibility $\kappa = n^{-2}dn/d\mu$ of the Mott insulator vanishes. Show that there is a special point of the transition line where the superfluid–Mott-insulator transition occurs at constant density.

3) Excitation spectrum in the Mott insulator

3.1) From the action $S[\psi^*, \psi]$ obtained earlier (question 2.3), deduce the propagator $\mathcal{G}(\mathbf{k}, i\omega_n)$ of the field ψ. What is the condition on $\mathcal{G}(\mathbf{k} = 0, i\omega_n = 0)$ for the Mott insulator to be stable? Show that one recovers the mean-field phase diagram (question 2.4).

3.2) Show that the one-particle spectrum exhibits two branches, $E_+(\mathbf{k})$ and $E_-(\mathbf{k})$. Compute $E_\pm(\mathbf{k} = 0)$. How does the transition to the superfluid phase manifest itself? Why is the transition occurring at constant density special?

3.3) What is the behavior of $E_\pm(\mathbf{k})$ in the limit $\mathbf{k} \to 0$? Show that at the Mott transition $E_\pm(\mathbf{k}) \sim |\mathbf{k}|^z$ where the exponent z takes the value 1 or 2.

4) Low-energy effective action

It is possible to write an effective action for the auxiliary field ψ by expanding $\mathcal{G}^{-1}(\mathbf{k}, i\omega_n)$ in powers of $\mathbf{k}$ and ω_n. Show that one obtains

$$S[\psi^*, \psi] = \int_0^\beta d\tau \int d^d r \, \psi^*(\mathbf{r}, \tau) \left(r + K_1 \partial_\tau - K_2 \partial_\tau^2 - K_3 \boldsymbol{\nabla}^2 + \cdots \right) \psi(\mathbf{r}, \tau) + \cdots \tag{5}$$

where

$$r = -\mathcal{G}^{-1}(\mathbf{k} = 0, i\omega_n = 0), \qquad K_1 = -\frac{\partial r}{\partial \mu}. \tag{6}$$

Deduce that K_1 vanishes when the transition occurs at constant density. What can we deduce about the universality class of the superfluid–Mott-insulator transition?

M2 Quantum Physics – Academic year 2013-2014
Broken symmetries and quantum phase transitions

Tutorial class 6: The two-dimensional quantum O(N) model in the large-N limit

We consider the action

$$S[\boldsymbol{\varphi}] = \int_0^\beta d\tau \int d^2r \left\{ \frac{1}{2}(\boldsymbol{\nabla}\boldsymbol{\varphi})^2 + \frac{1}{2c^2}(\partial_\tau \boldsymbol{\varphi})^2 + \frac{r_0}{2}\boldsymbol{\varphi}^2 + \frac{u_0}{4!N}(\boldsymbol{\varphi}^2)^2 \right\} \tag{1}$$

of the two-dimensional quantum O(N) model, where $\boldsymbol{\varphi}(\mathbf{r},\tau)$ is a N-component real field satisfying $\boldsymbol{\varphi}(\mathbf{r},\tau+\beta) = \boldsymbol{\varphi}(\mathbf{r},\tau)$. r_0 and u_0 are temperature independent and c is a propagation velocity. The model is regularized by a ultraviolet momentum cutoff ($|\mathbf{p}| < \Lambda$).

1) The large N limit.

We use the identity

$$1 = \int_{-\infty}^{\infty} d\rho\, \delta(\rho - \boldsymbol{\varphi}^2) = \mathcal{N} \int_{-\infty}^{\infty} d\rho \int_{-\infty}^{\infty} d\lambda\, e^{-\frac{i}{2}\lambda(\boldsymbol{\varphi}^2 - \rho)} \tag{2}$$

(we ignore the constant $\mathcal{N}$ in the following) to rewrite the partition function as

$$Z = \int \mathcal{D}[\boldsymbol{\varphi},\rho,\lambda] \exp\left\{ - \int_0^\beta d\tau \int d^2r \left[\frac{1}{2}(\boldsymbol{\nabla}\boldsymbol{\varphi})^2 + \frac{1}{2c^2}(\partial_\tau \boldsymbol{\varphi})^2 + \frac{r_0}{2}\rho + \frac{u_0}{4!N}\rho^2 + \frac{i}{2}\lambda(\boldsymbol{\varphi}^2 - \rho) \right] \right\}. \tag{3}$$

1.1) Integrate out the field ρ to obtain the action $S[\boldsymbol{\varphi},\lambda]$.

1.2) We split the field $\boldsymbol{\varphi} = (\sigma, \boldsymbol{\pi})$ into a field σ and a $(N-1)$-component field $\boldsymbol{\pi}$. Integrate out $\boldsymbol{\pi}$ to obtain the action

$$S[\sigma,\lambda] = \int_0^\beta d\tau \int d^2r \left\{ \frac{1}{2}\left[(\nabla\sigma)^2 + \frac{1}{c^2}(\partial_\tau\sigma)^2 + i\lambda\sigma^2 \right] - \frac{3N}{2u_0}(r_0 - i\lambda)^2 \right\} + \frac{N-1}{2}\mathrm{Tr}\ln g^{-1}, \tag{4}$$

where

$$g^{-1}(\mathbf{r},\tau;\mathbf{r}',\tau') = \left[-\left(\boldsymbol{\nabla}_{\mathbf{r}}^2 + c^{-2}\partial_\tau^2\right) + i\lambda(\mathbf{r},\tau) \right] \delta(\mathbf{r}-\mathbf{r}')\delta(\tau-\tau') \tag{5}$$

is the (inverse) propagator of the field π_i in the presence of the fluctuating field λ.

1.3) Show that the saddle-point approximation becomes exact in the limit $N \to \infty$. Give the saddle-point equations for uniform and time-independent fields σ and λ; discuss qualitatively the solutions and their physical meaning.

2) Zero-temperature phase diagram.

2.1) Show that there is a (zero-temperature) quantum phase transition when $r_0 = r_{0c}$. What is the expression of r_{0c}?

2.2) Compute the excitation gap $m_0 = m(T=0)$ ($m^2 = i\lambda c^2$) and the correlation length $\xi = c/m_0$ in the disordered phase when $m_0 \ll ck_G \ll c\Lambda$ (with $k_G \sim cu_0$). Deduce the value of the correlation-length exponent ν, the dynamical exponent z, and the anomalous dimension η.

2.3) Compute the order parameter σ in the ordered phase. Is Goldstone's theorem satisfied? The transverse propagator can be written in the form

$$g(\mathbf{q}, i\omega) = \frac{\sigma^2 c^2}{\rho_s(\omega^2 + c^2\mathbf{q}^2)}, \tag{6}$$

where ρ_s is the "stiffness" or "rigidity". Show that ρ_s has the dimension of an energy.

3) Finite-temperature phase diagram.

3.1) Show that at finite temperature the system is always disordered. Could have this result been anticipated? Show that

$$m = 2T \,\mathrm{argsinh}\left\{ \frac{1}{2} \exp\left[\frac{12\pi}{T u_0} (r_0 - r_{0c}) \right] \right\} \tag{7}$$

in the vicinity of the quantum critical point.

3.2) We consider the case $r_0 \geq r_{0c}$ where the system is disordered at zero temperature. Express m as a function of T and m_0. Show that at low temperatures the system is dominated by quantum fluctuations whereas at high temperatures both classical and quantum fluctuations matter.

3.3) We now consider the case $r_0 \leq r_{0c}$ where the system is ordered at zero temperature. Express m as a function of ρ_s and T and show that we can again distinguish two temperature regimes: a low-temperature regime where the system is disordered by classical fluctuations and the high-temperature regime studied in question 3.2.

3.4) Draw the phase diagram in the plane (r_0, T) with the crossover lines between the various finite-temperature regimes.