Scattering in Euclidean space: from Bethe-Salpeter to LQCD

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"Hadrons and Nuclear Physics meet Ultra-Cold Atoms: a french-japanese workshop"
Institut Poincaré, January 29, 2018
Despite that physical space-time has a Minkowski structure, the interest in formulating a relativistic theory (QFT) in an Euclidean metric was already present in the very early days. It was first applied by G. Wick to solve the Bethe-Salpeter equation: “Wick rotation” $k_0 = i k_4$

\[ k^2 = k_0^2 - \bar{k}^2 = -(k_4^2 + \bar{k}^2) \equiv -k_E^2 \]

**G. Wick**, Properties of the Bethe-Salpeter wave functions, Phys. Rev. 96 (1954) 1124 and formulated in a more general QFT framework soon later

**J. Schwinger**

- Euclidean Quantum Electrodynamics, Phys Rev (1959) 721

**K. Symanzik**

- International School of Physics Enrico Fermi, Course XLV, Ed. by R. Jost (1968)

This interest was enhanced by the success of **Lattice** calculations, where it is mandatory! The main theoretical problem is to prove the equivalence between both formulations


Gave conditions for a QFT to ensure that the “analytic continuation” of Green functions can be safely done (no singularities) …but they are not yet proved for theories of interest
Prelude

The **practical benefit** of Minkowski -> Euclidean is clear:

Everything becomes smooth (allowing standard methods for solving integral equations)
Euclidean actions $S_E$ becomes positive definite (allowing path integrals)

......

**Much less clear is what is lost** when using an Euclidean formulation...

I Form factors $F(Q^2)$ in time-like region are not allowed

$$Q^2 = Q_0^2 - q^2 = -(Q_4^2 + q^2) = -Q_E^2$$

It is even not clear that the space-like are correctly computed

II Scattering is lost:

**Maiani-Testa** “no go theorem“ PLB 245 (1990) 585: no scattering in **infinite** euclidean space

However circunvented in LQCD finite volume by “Luscher method”

III Problems for introducing chemical potential (at T>0)

....
Luscher method (87)

Energy levels $\varepsilon_n(L)$ of 2-particle in a periodic box (L) provide phase-shifts

In the simplest case: ground state $\varepsilon_0(L)$ provides the scattering length $A_0$

$\varepsilon_0(L) = \frac{4\pi A_0}{(aL)^3} \left\{ 1 + c_1 \left( \frac{A_0}{aL} \right) + c_2 \left( \frac{A_0}{aL} \right)^2 + \ldots \right\}$

$c_1$ and $c_2$ are universal known coefficients

More involved expressions provide the phase shifts

$k \cot \delta_0(k) = \frac{1}{\pi a L} S(\eta)$

It works well! …but has problems with open channels and for large lattices
Prelude

General results are sometimes too deep to be clear....

Bethe-Salpeter (BS) framework is an ideal landscape to have some light on this problem

I

Formulated in terms of BS amplitude which - contrary to wave functions – have a clear definition in terms of QFT and is accessible to LQCD

II

It fulfills a 4D integral equation which - in its (simplified) version – can be solved both in Euclidean and Minkowski space

Aim of this talk:

I Present a method for obtaining BS scattering solutions in Minkowski space
II Discuss the problem with Euclidean BS scattering solutions
III Obtain a purely Euclidean equation for zero energy (scattering length) Alternative to Luscher method in Lattice calculations
IV Present possible extension to non zero energies (Effective Range approximation)
Introduction

Bethe Salpeter equation deals with a - pre-existing - QFT object (Gell-Mann Low)

\[ \Phi(x_1, x_2, P) = <0 | T\{\phi(x_1)\phi(x_2)\} | P> \]

Its Fourier transform \( \Phi(k,P) \)

\[ \Phi(x_1, x_2, P) = \int \frac{dp_1}{(2\pi)^4} \frac{dp_2}{(2\pi)^4} \Phi(p_1, p_2) \ e^{-iPx} \ e^{-ikx} = e^{-iPx} \int \frac{dk}{(2\pi)^4} \Phi(k, P) \ e^{-ikx} \]

satisfies a 4D equation. For bound state case:

\[ \Phi(k, P) = S_1(k, P) S_2(k, P) \int \frac{d^4k'}{(2\pi)^4} \ iK(k, k'; P) \ \Phi(k', P) \]  \hspace{1cm} (1)

\( P^2=M^2 \) with \( M \) the total mass of the two-body system

\( S_1 = \) free propagators

\[ S_1(k, P) = \frac{i}{(P/2 + k)^2 - m^2 + i\epsilon} \]

\[ S_2(k, P) = \frac{i}{(P/2 - k)^2 - m^2 + i\epsilon} \]

\( iK=\)Interaction kernel

- if \( K \) would contain all the IR graphs, solving (1) would be equivalent to solve the full QFT
- This is however a wishful thinking. In practice one uses a very poor restriction: ladder+simple kernels
It is usually written in terms of the “vertex function”

\[ F(k, p) = \frac{\Phi(k, p)}{S_1(k, p) S_2(k, p)} \]

That is

\[ F(k; p) = \int \frac{d^4k'}{(2\pi)^4} \frac{iK(k, k') F(k'; p)}{\left( \frac{p}{2} + k' \right)^2 - m^2 + i\epsilon \left[ \left( \frac{p}{2} - k' \right)^2 - m^2 + i\epsilon \right]} \]

As one can see the equation has singularities in the free propagators
As well as in the simplest (one boson exchange ladder) kernel

\[ K(k, k) = -\frac{g^2}{(k - k')^2 - \mu^2 + i\epsilon} \]
**Bound-state Solutions in Euclidean space**

Bound states are easily solved with the "Wick rotation" $k_0 = \text{i}k_4$

Minkowski $(k_0, k) \rightarrow$ Euclidean $(k_4, k)$

$$k^2 = k_0^2 - \vec{k}^2 = -(k_4^2 + \vec{k}^2) = -k_E^2$$

It leads to a smooth integral equation for the euclidean amplitude

$$\Phi_E(k_4, \vec{k}) \equiv \Phi_M(k_0 = \text{i}k_4, \vec{k})$$

soluble by standard methods.

Until very recently, most of the existing solutions were found in this way

Rm: Not a simple « variable change » but a straightforward application of Cauchy theorem:

$$\int_{\gamma_1} f(z) dz = \int_{\gamma_2} f(z) dz$$

If a continous deformation between two paths do not cross a singularity of $f(z)$

Ex: free propagators
There are however some problems:

- The validity of the Wick rotation requires a careful analysis of kernel singularities. It is done only in some cases (ladder WC).

- The total mass $M$ is invariant but the BS amplitude is not: impossible to recover $\Phi_M$ from $\Phi_E$.

- What about other observables: scattering amplitudes, form factors, … only accessible in Minkowski space!

All these reasons motivated a series of works for obtaining the Minkowski BS solutions:

I. Compute scattering observable  
II. Compute form factors

We developed two totally independent methods:

- Light-Front projection of the BS equation and Nakanishi representation of the amplitude

- A « direct » approach (*)

Minkowski space solutions: direct method

In terms of “Vertex function” \( F(k, p) = \frac{\Phi(k, p)}{S_1(k, p)S_2(k, p)} \)

and for the scattering process \( k_{1s} + k_{2s} \rightarrow k_1 + k_2 \) the BS equation reads

\[
F(k; k_s) = K(k, k_s) - i \int \frac{d^4k'}{(2\pi)^4} \frac{K(k, k')F(k'; k_s)}{\left[\left(\frac{p}{2} + k'\right)^2 - m^2 + i\epsilon\right]\left[\left(\frac{p}{2} - k'\right)^2 - m^2 + i\epsilon\right]}
\]

(2)

In Minkowski space, this equation is plagued with singularities from 4 different sources:

1. The singular inhomogeneous Born term
2. The 2+2 poles of the constituent propagators
3. The kernel singularities
4. The singular behaviour of \( F \) itself (due to \( K \), but not only)

A careful analysis of all these singularities allows (*) to obtain the full BS (off-mass shell) scattering amplitude, even above the inelastic thresholds \( (N_1 + N_2 \rightarrow N_1 + N_2 + m + ..) \).

For the first time since its formulation … in 54 !

Minkowski espace solutions : direct method

After many steps (*) the S-wave BS equation in Minwoski space takes the form

\[
F_0(k_0, k) = F_0^B(k_0, k) + \frac{i\pi^2 k_s}{8\varepsilon_{k_s}} W_0^S(k_0, k, 0, k_s) F_0(0, k_s) \\
+ \frac{\pi}{2M} \int_0^\infty \frac{dk'}{\varepsilon_{k'}(2\varepsilon_{k'} - M)} \left[ k'^2 W_0^S(k_0, k, a_-, k') F_0(|a_-|, k') - \frac{2k_s^2\varepsilon_{k'}}{\varepsilon_{k'} + \varepsilon_{k_s}} W_0^S(k_0, k, 0, k_s) F_0(0, k_s) \right] \\
- \frac{\pi}{2M} \int_0^\infty \frac{k'^2 dk'}{\varepsilon_{k'}(2\varepsilon_{k'} + M)} W_0^S(k_0, k, a_+, k') F_0(a_+, k') \\
+ \frac{i}{2M} \int_0^\infty \frac{k'^2 dk'}{\varepsilon_{k'}} \int_0^\infty dk'_0 \left[ \frac{W_0^S(k_0, k, k'_0, k') F_0(k'_0, k') - W_0^S(k_0, k, a_-, k') F_0(|a_-|, k')}{k'^2_0 - a_+^2} \right] \\
- \frac{i}{2M} \int_0^\infty \frac{k'^2 dk'}{\varepsilon_{k'}} \int_0^\infty dk'_0 \left[ \frac{W_0^S(k_0, k, k'_0, k') F_0(k'_0, k') - W_0^S(k_0, k, a_+, k') F_0(a_+, k')}{k'^2_0 - a_-^2} \right]
\]

\[a_\pm(k', k_s) = \varepsilon_{k'} \pm \varepsilon_{k_s}\]

In this form all the PV coming from propagator poles have been absorbed (by sustraction).
It remains « only » to treat the singularities of the kernel \(W_0^S\), of the Born term \(F^B\) … and solve it !

Quite a nasty equation for an S-wave… compared to the NR Lipmann-Schwinger one
« c’est la vie » in Minkowski space !

This is however the only way to compute in the whole kinematical domain:
- the off-mass shell $F(k,k_s;p)$
- the half off-mass shell $F(k,k_s)$ (with $p$ related to $k_s$)

and solve the BS scattering problem in its full complexity (including inelastic thresholds)

The observables are obtained from the on-shell value $F_{on}^{on}(k_0=0,k=k_s)$

\[ \delta_l = \frac{1}{2i} \log \left( 1 + \frac{2ip_s}{\varepsilon_{ps}} F_{l}^{on} \right) \]
Phase shifts and inelasticities

The amplitude thus obtained can be further used to calculate the transition form factor. In its full off-shell form, it can be used as input in the here were limited to S-wave in the spinless case and the ladder kernel but they can be extended to any partial wave. Coming on mass of the original BS equation are properly treated. A regular equation is obtained and solved by standard methods. The results presented

References

4. Conclusion

Table 2 (dashed).

We have presented the first results of the BS off-shell scattering amplitude in Minkowski space. The different kinds of singularities computing this quantity, and related on-shell observables, is the main result of this work. Together with the bound state solution in a two-dimensional domain.
The problem with the Euclidean space solutions

Assume we want to obtain a scattering BS equation for the euclidean amplitude \( F_E(k_4, k) = F_M(k_0=ik_4, k) \) by properly applying the Wick rotation, i.e. taking into account the singularities

\[
\begin{align*}
    k'_0^{(1)} (k, k_s) &= \varepsilon_{k_s} + \varepsilon_{k'} - i\epsilon = +a_+ - i\epsilon \\
    k'_0^{(2)} (k, k_s) &= \varepsilon_{k_s} - \varepsilon_{k'} + i\epsilon = -a_- + i\epsilon \\
    k'_0^{(3)} (k, k_s) &= -\varepsilon_{k_s} + \varepsilon_{k'} - i\epsilon = +a_- - i\epsilon \\
    k'_0^{(4)} (k, k_s) &= -\varepsilon_{k_s} - \varepsilon_{k'} + i\epsilon = -a_+ + i\epsilon \\
\end{align*}
\]

The initial BS equation (2) becomes

\[
F^E(k_4, \vec{k}; \vec{k}_s) = V^B(k_4, \vec{k}; \vec{k}_s) + \int \frac{d^4k'}{(2\pi)^4} \frac{V(k_4, \vec{k}'_4; k'_4, \vec{k}'_s)F^E(k'_4, \vec{k}'_4; \vec{k}'_s)}{(k'_4^2 + a_-^2)(k'_4^2 + a_+^2)} + S(k_4, k, k_s) \tag{E1}
\]

\[
V(k_4, \vec{k}; k'_4, \vec{k}') = \frac{16\pi m^2\alpha}{(k_4 - k'_4)^2 + (\vec{k} - \vec{k}')^2 + \mu^2},
\]

with «rotated» kernels

\[
V^B(k_4, \vec{k}; \vec{k}_s) = V(k_4, \vec{k}; k'_4 = 0, \vec{k}' = \vec{k}_s)
\]

Pole contribution results into a term (S) mixing \( F_E \) and \( F_M \): equation for \( F_E \) alone is impossible !!!
The (integral) term $S$ contains the Minkowski amplitude at the particular value $F_M(k_0 = \epsilon_{k_s} - \epsilon_k, k)$

In addition to (E1), a second integral equation is needed to solve the problem!

One ends with a system of two-coupled equations involving both $F_E$ and $F_M$ denoted symbolically

\[
F_E(k_A, k) = I_1 [F_E(k_A', k'), F_M(\epsilon_{k_s} - \epsilon_{k'}, k')] \quad \text{(E1)}
\]

\[
F_M(\epsilon_{k_s} - \epsilon_k, k) = I_2 [F_E(k_A', k'), F_M(\epsilon_{k_s} - \epsilon_{k'}, k')] \quad \text{(E2)}
\]

This system of equations was first obtained in M. Levin, J. Wright, and J. Tjon, Phys Rev 254 (1967) 1433

It was derived independently in J.C. and V.A. Kamanov, Phys Rev D90 (2014) 056002

where it was used to check the « direct » solution $F_M$ (for the particular value of $k_0$)

R1: Equations (E1+E2) remain singular (in the Minkowski part)

R2: They do not provide the amplitude $F_M$ in the full $(k_0,k)$ plane but in one-dimensional manifold (string)

R3: On-mass shell, i.e. $k_0=0$, one has $F_M(0,k_s) = F_E(0,k_s)$ and the solution provides elastic phase shifts

The system of equations (E1+E2) coupling $F_E$ and $F_M$ shows the impossibility to solve the BS scattering problem using an eucliden metric

.....Always ???
The case of zero energy scattering \((k_s=0)\)

It can be shown\(^(*)\) that the additional term \(S\) coupling to the Minkowski vanishes in the limit \(k_s=0\).

One obtain this way a **regular purely Euclidean equation for** \(F_E\). For S-waves it reads

\[
F_E(k_4, k) = V^B(k_4, k) + \int_0^{\infty} k'^2 dk' \int_0^{\infty} dk'_4 \frac{V_0(k_4, k; k'_4, k')}{(k'_4^2 + a'_-^2)(k'_4^2 + a'_+^2)} F_E(k'_4, k')
\]

The scattering length is directly given by \(a_0=-F_E(0,0)/m\).

Apart from providing a very stable and cheap scattering length in BS equation (like bound states) it demonstrates, in the BS framework, the possibility to obtain (till now zero energy!) **scattering results from purey Euclidean solutions**.

This suggests an **alternative way to Luscher method** for computing scattering observables in Lattice calculations: one needs only (the Fourier transform of) the Euclidean version of

\[
\Phi(x_1, x_2, P) = \langle 0 \mid T\{\phi(x_1)\phi(x_2)\} \mid P \rangle
\]

This quantity has been computed since many years in LQCD collaborations (see Ikeda san talk) but never used to compute \(a_0\). Could you please try ???

The scattering length value is given by

\[ a_0 = \text{constant} \]

The numerical solutions of Eq. (16) with the kernel (15) have been obtained by spline

The results have been tested by solving the Euclidean Bethe-Salpeter equation

The same amplitude is shown in Figure 3 for

The same than in Fig. 2 with the parameters

The numerical results

\[ \alpha \]

was due to an unadapted choice of the grid parameters for large

The lack of precision

and from those of Ref. [15], both obtained using di

independent methods.

in a scalar one-boson exchange model where the scattering length

The amplitudes are monotonic

Values the two-body system has two bound states with the second

These parameters the two-body system has no bound state and the

We have shown that in the limit of zero incident energy the Bethe-Salpeter scattering equation

The numerical values of the coupling constant

\[ \text{Table 1 - 3 of [15]} \]

In view of the agreement with the previous results (Table 1 of [15]), noticed in Table 1 of [16]

References


Quantum field theory (Springer-Verlag 1994).
Conclusion

We have presented a method for solving Bethe-Salpeter equation in Minkowski space. Based on a **direct solution** with a careful analysis of singularities, we have shown that **Minkowski solutions are mandatory** for the **scattering problem**.

We propose a new method to compute the **scattering length** (zero scattering energy) from a purely **Euclidean BS amplitude** $F_E$ (vertex function). This method provides an **alternative** to Luscher method used in **Lattice** calculations and it is extensible to non-zero energy in the effective range approximation.