Dimer-dimer zero crossing in a one-dimensional mixture

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LPTMS

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Motivation

- Bosonic mixtures of two different components with competing attractive inter and repulsive intra species interaction caught recently a great deal of attention.
- Ability to form self trapped liquid droplets.
- Finite piece of liquid in equilibrium with vacuum without any external potential !
- Many possible applications of these droplets. (cooling, ...)
- Direct manifestation of Beyond-Mean-Field (BMF) effects.

Mixture & Stability

- Mixture : 2 $\neq$ particles $(\uparrow, \downarrow)$ of equal masses and same densities.
- Interactions $(g_{\uparrow\downarrow}, g_{\uparrow\uparrow}, g_{\downarrow\downarrow})$ with attractive inter and repulsive intra species. What about stability (i.e. no collapse) ?

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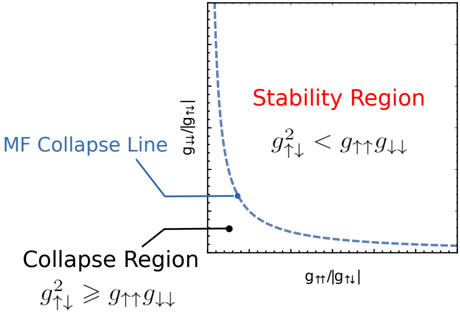
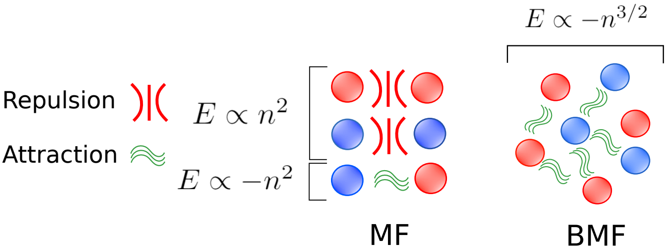
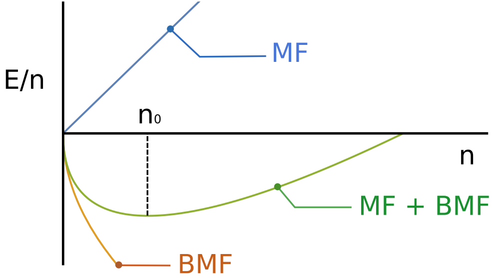


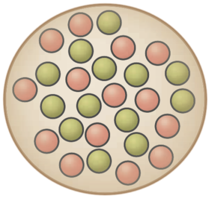
... In 1D, close to this MF collapse line, on the MF repulsive side the mixture liquefy !

$\rightarrow$ Comes from an effectively attractive BMF term.

A Brief Insight

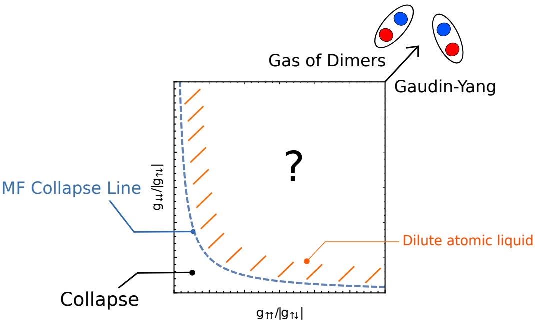






Overview of one dimensional Bose-Bose mixture at this stage

- Increasing $g_{\uparrow\uparrow}g_{\downarrow\downarrow}/g_{\uparrow\downarrow}^2$ makes the system more dilute which in one dimension leads to stronger correlations.
- For $g_{\sigma\sigma} \gg |g_{\uparrow\downarrow}|$, ,the system becomes a gas of $\uparrow\downarrow$ dimers.(Mapping with Gaudin-Yang model which has no other BS than $\uparrow\downarrow$ dimers.)



$$\Psi_B, \quad g_{\uparrow\uparrow} = g_{\downarrow\downarrow} = +\infty \quad \Leftrightarrow \quad \Psi_F, \quad g_{\uparrow\uparrow} = g_{\downarrow\downarrow} = 0$$

Goal

By decreasing the ratio $g_{\uparrow\uparrow}g_{\downarrow\downarrow}/g_{\uparrow\downarrow}^2$:
$\rightarrow$ Find the line curve in the plane $\{g_{\uparrow\uparrow}/|g_{\uparrow\downarrow}|, g_{\downarrow\downarrow}/|g_{\uparrow\downarrow}|\}$ where the dimer-dimer interaction vanishes ($a_{dd} = \infty$).

Outline

Reminders about 1D systems

Two-body scattering theory in 1D

$$[-\frac{\hbar^2}{2\mu}\frac{d^2}{dx^2} + V(|x|)]\Psi(x) = E\Psi(x) \quad (1)$$

- $|x| \gg R_e \to \Psi(x) = A\sin(k|x| + \delta(k))$
- δ is the so-called *phase shift*
- k is the relative momentum and verify $E = \hbar^2 k^2/2\mu$.

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Scattering length a

- $\lim_{x\rightarrow 0} \Psi(x) \propto |x| - a(k)$ where $a(k) = -\frac{\sin(\delta)}{k\cos(\delta)}$
- Small momenta $kR_e \ll 1$, we define the scattering length a :
  $a = \lim_{k\rightarrow 0} a(k) = \lim_{k\rightarrow 0} -\frac{\tan\delta}{k}$
- Expansion : $k\cot\delta(k) = -1/a + (r_e/2)k^2 + ...$

Two-body contact interaction

- Short range interactions between particle modeled by a δ-potential such that : $V_{ij}(x) = g_{ij}\delta(x_{ij})$.
- It's the so-called *Zero-Range Approximation*, valid when $|\lambda_{dB}| \gg R_e$
- One can show with the logarithm derivative of $\Psi_{|0}$ that the constant interaction g_{ij} is related to the scattering length.

$$\boxed{g_{ij} = -\frac{\hbar^2}{\mu a_{ij}}}$$

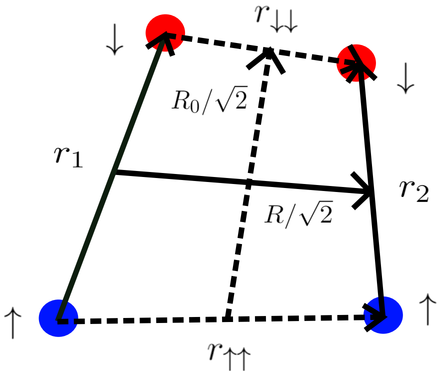
- If $g_{ij} > 0$ one deals with a repulsive potential, whereas for $g_{ij} < 0$ we deal with an attractive potential.
- If $g_{ij} < 0$ a bound state (BS) exists.

$$\boxed{\Psi_{BS} \propto e^{-|x|/a_{ij}} \qquad E_{BS}(\{m_1, m_2\}, g_{ij}) = -\frac{\hbar^2}{2\mu a_{ij}^2}}$$

$\rightarrow$ a_{ij} can be seen as the *size* of the bound state.

The Four-Body system

Schrödinger's equation



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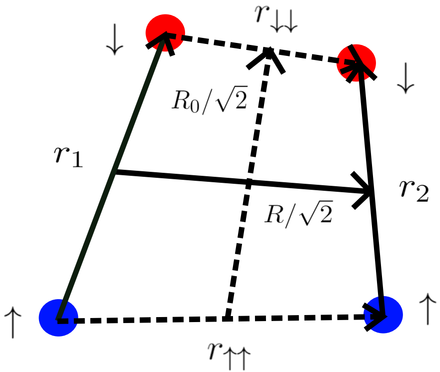

Symmetries ($e_{\sigma\sigma} = \pm 1$)

$$
\begin{aligned}
\Psi(r_1, r_2, R) &= e_{\downarrow\downarrow}\Psi(r_+, r_-, \frac{r_1 - r_2}{\sqrt{2}}) \\
&= e_{\uparrow\uparrow}\Psi(r_-, r_+, \frac{r_2 - r_1}{\sqrt{2}}) \\
&= e_{\uparrow\uparrow}e_{\downarrow\downarrow}\Psi(r_2, r_1, -R)
\end{aligned}
$$

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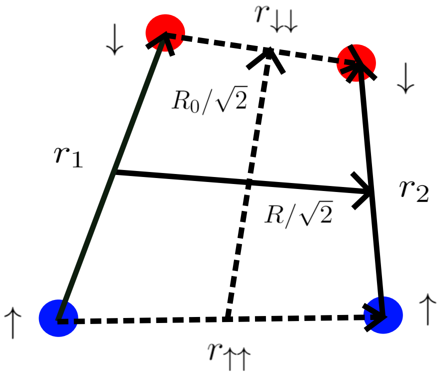

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$$\boxed{\begin{aligned}(-\nabla^2_{\mathbf{X}} - E)\Psi(r_1, r_2, R) =& [-g_{\uparrow\downarrow}(\delta(r_1) + \delta(r_2) + \delta(r_+) + \delta(r_-)) \\ & - g_{\uparrow\uparrow}\delta(r_{\uparrow\uparrow}) - g_{\downarrow\downarrow}\delta(r_{\downarrow\downarrow})]\Psi(r_1, r_2, R)\end{aligned}} \quad (2)$$

Skorniakov-Ter-Martirosian (STM)

We introduce the function $f_{\uparrow\downarrow}$ which corresponds by definition to the wavefunction Ψ when one pair $\{\uparrow\downarrow\}$ coincide.

$$\lim_{r_1 \to 0} \Psi(r_1, r_2, R) = f_{\uparrow\downarrow}(r_2, R) \tag{3}$$

We do the same for $r_{\uparrow\uparrow} \to 0$ and then $r_{\downarrow\downarrow} \to 0$:

$$\lim_{r_{\uparrow\uparrow} \to 0} \Psi(r_1, r_2, R) = f_{\uparrow\uparrow}(r_{\downarrow\downarrow}, R_0) \tag{4}$$

$$\lim_{r_{\downarrow\downarrow} \to 0} \Psi(r_1, r_2, R) = f_{\downarrow\downarrow}(r_{\uparrow\uparrow}, R_0) \tag{5}$$

STM Equation

$$
\begin{aligned}
(\mathbf{P}^2 - E)\tilde{\Psi}(p_1, p_2, p) = &- g_{\uparrow\downarrow}\tilde{f}_{\uparrow\downarrow}(p_2, p) - e_{\uparrow\uparrow}e_{\downarrow\downarrow}g_{\uparrow\downarrow}\tilde{f}_{\uparrow\downarrow}(p_1, -p) \\
&- e_{\downarrow\downarrow}g_{\uparrow\downarrow}\tilde{f}_{\uparrow\downarrow}\left(\frac{p_1 + p_2 - \sqrt{2}p}{2}, \frac{p_1 - p_2}{\sqrt{2}}\right) \\
&- e_{\uparrow\uparrow}g_{\uparrow\downarrow}\tilde{f}_{\uparrow\downarrow}\left(\frac{p_1 + p_2 + \sqrt{2}p}{2}, \frac{p_2 - p_1}{\sqrt{2}}\right) \\
&- g_{\uparrow\uparrow}\tilde{f}_{\uparrow\uparrow}\left(\frac{p_2 - p_1 + \sqrt{2}p}{2}, \frac{p_1 + p_2}{\sqrt{2}}\right) \\
&- g_{\downarrow\downarrow}\tilde{f}_{\downarrow\downarrow}\left(\frac{p_1 - p_2 + \sqrt{2}p}{2}, \frac{p_1 + p_2}{\sqrt{2}}\right)
\end{aligned}
\quad (6)
$$

Where $\mathbf{P} = \{p_1, p_2, p\}$ corresponds to a 3D vector in momentum space. Idea is then to end up with a system of three coupled integral equations for $\{\tilde{f}_{\uparrow\downarrow}, \tilde{f}_{\uparrow\uparrow}, \tilde{f}_{\downarrow\downarrow}\}$ since we can show that :

$$
\begin{cases}
\tilde{f}_{\uparrow\downarrow}(k, q) = \int \frac{du}{2\pi}\tilde{\Psi}(u, k, q) \\
\tilde{f}_{\uparrow\uparrow}(k, q) = \int \frac{du}{\pi}\tilde{\Psi}(u, \sqrt{2}q - u, \sqrt{2}(k + u) - q) \\
\tilde{f}_{\downarrow\downarrow}(k, q) = \int \frac{du}{\pi}\tilde{\Psi}(u, \sqrt{2}q - u, \sqrt{2}(k - u) + q)
\end{cases}
\quad (7)
$$

Dimer-dimer Scattering

- We fix $g_{\uparrow\downarrow} < 0$ (attractive interspecies).
- The total energy is $E = -2|\epsilon_{\uparrow\downarrow}| + \epsilon_0$, where $\epsilon_{\uparrow\downarrow} = -\hbar^2/ma_{\uparrow\downarrow}^2$ and ϵ_0 is the dimer-dimer collisional energy.
- Starting from gas of dimers $\uparrow\downarrow$ (Yang Gaudin), we decrease the ratio $g_{\uparrow\uparrow}g_{\downarrow\downarrow}/g_{\uparrow\downarrow}^2$ and look at zero-collision d-d energy to extract $a_{dd}/a_{\uparrow\downarrow}$.

- By substituting $\tilde{f}_{\uparrow\downarrow}$ by an appropriate expression, homogeneous STM equation becomes an inhomogeneous equation $MX = Y$
- Leads to a linear problem that we put on the grid to extract $a_{dd}/a_{\uparrow\downarrow}$.

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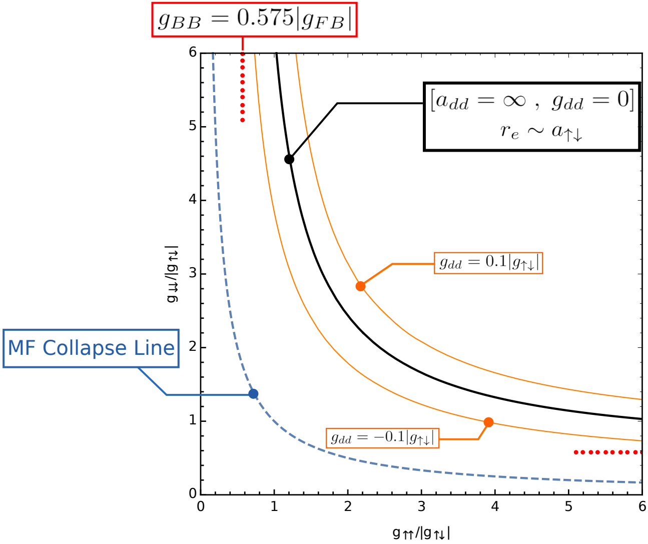
Goal reminder

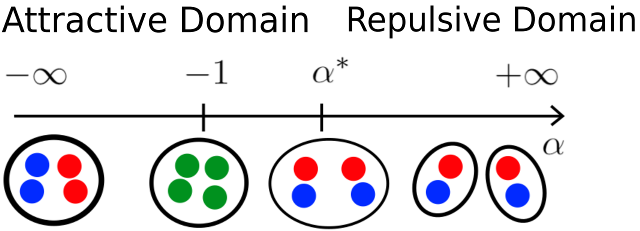
Find the line curve in the plane $\{g_{\uparrow\uparrow}/|g_{\uparrow\downarrow}|, g_{\downarrow\downarrow}/|g_{\uparrow\downarrow}|\}$ where the dimer-dimer interaction vanishes.

Results

Overview of the Bose-Bose mixture in the plane $\{g_{\uparrow\uparrow}/|g_{\uparrow\downarrow}|, g_{\downarrow\downarrow}/|g_{\uparrow\downarrow}|\}$

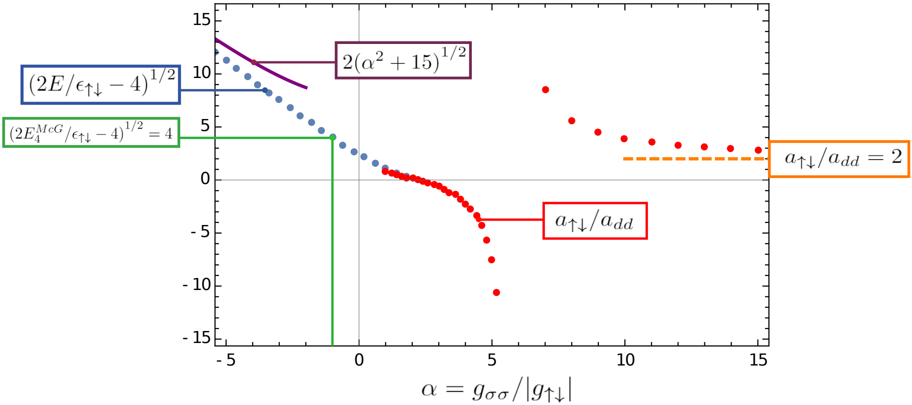


Overview in symmetric case ($g_{\uparrow\uparrow} = g_{\downarrow\downarrow}$) in function of $\alpha = g_{\uparrow\uparrow}/|g_{\uparrow\downarrow}|$



- Interaction between dimers become attractive when $\alpha < \alpha*$
- 3 known integrable cases : $\alpha \to +\infty$, $\alpha = -1$, $\alpha \to -\infty$

Overview of the dimerized symmetric Bose-Bose mixture in function of α

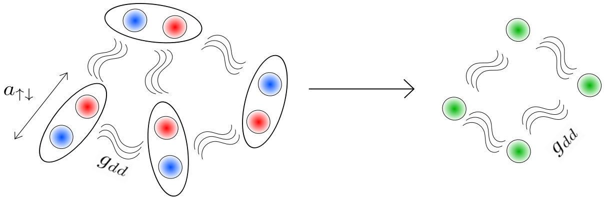


$g_{dd} = 0$ for $\alpha^* = 2.2$

Discussions

Soliton ?

- Consider $N_d > 2$ dimers close to the dimer-dimer zero crossing line in the attractive regime where $a_{dd} \gg a_{\uparrow\downarrow} \sim r_e$.



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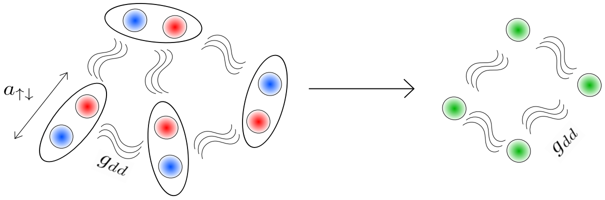
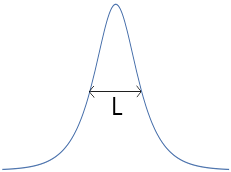


<u>Soliton</u>

$$\boxed{E_{N_d} = \frac{-g_{dd}^2 N_d (N_d^2 - 1)}{12}}$$

$\rightarrow$ Ground State ?



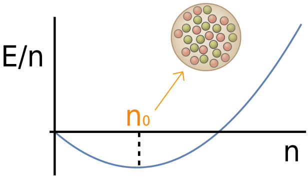
$L \sim a_{dd}/N_d$

Breaks down for $N_d \rightarrow +\infty$ (cf. above the collapse line)

MF & 3-Body Repulsive Interaction

- Idea : A liquid state which is a result of a competition between two- and three- dimer forces ? ($g_{dd} < 0$ and assume $g_3 > 0$)
- MF (for dimers) treatment (cf. Bulgac) :

$$\epsilon := E_{N_d}/N_d = g_{dd} n_d/2 + g_3 n_d^2/6 \qquad (8)$$



Minimum : $n_d^0 = -3g_{dd}/2g_3$

- Applicability : Interaction energy shift much smaller than the energy scale $E \sim {n_d}^2 \rightarrow \{a_{dd} n_d \gg 1$ and $g_3 \ll 1\}$
- Both of these conditions (at n_d^0) lead to $g_3 \ll 1$

About 3-Body Interaction

- What is this g_3 ?

$$\boxed{\text{3-Body in 1D} \rightarrow \quad \text{2-Body in 2D} \quad , \quad \Psi_3 \propto \ln(\rho/a_3) \ , \quad a_3 > 0}$$

- 3-dimer effective potential taken as :

$$g_3 = \frac{\sqrt{3}\pi}{2\ln(2e^{-\gamma}/a_3\kappa)} \tag{9}$$

- κ is the typical momentum of the system
- In the leading order of $g_3 \ll 1$, by assuming that $a_3 \sim a_{\uparrow\downarrow}$, we have in the leading order of g_3 :

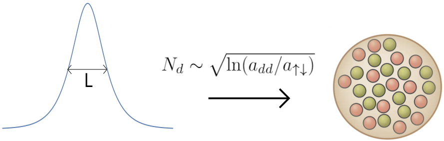
$$g_3 = \sqrt{3}\pi/2\ln(a_{dd}/a_{\uparrow\downarrow}) \ll 1$$

$$\boxed{n_d^0 = (\sqrt{3}/\pi a_{dd})\ln(a_{dd}/a_{\uparrow\downarrow}) \quad , \quad \mu = \epsilon = (\sqrt{3}/4\pi a_{dd}^2)\ln(a_{dd}/a_{\uparrow\downarrow})}$$

Different regimes

- In the region $a_3 \sim a_{\uparrow\downarrow} \ll n_d^{-1}$, precisely $1/\ln(a_{\uparrow\downarrow} n_d) \sim 1/\ln(a_3 n_d) \ll 1$

Crossover : Soliton to Liquid Droplet when increasing N_d



- Dimer-dimer effective range correction (per dimer) ?

$\rightarrow$ Scales as $r_e \epsilon n_d \sim \epsilon g_3^{-1} e^{-\sqrt{3}\pi/2g_3}$ smaller than any powers of g_3

- Case $a_{\uparrow\downarrow} \ll 1/n_d \ll a_3$?

$\rightarrow$ Weak 3-body attraction leads to high density phase (cf. Nishida)
$\rightarrow$ Solution breaks down for same reasons than soliton.

Conclusion

Summary

1. We derived STM equations for the 4 body-problem in the case of a mixture with intercomponent dimers.

2. We implemented these equations numerically and verify our numerical method in known integrable cases.

3. We calculated the line where the dimer-dimer interaction vanishes (particularly in the Bose symmetric case $\alpha^* = 2.2$ and in the BF case $g_{BB} = 0.575|g_{FB}|$)

4. For a weak dimer-dimer interaction, we predict a dilute dimerized liquid phase stabilized against collapse by a repulsive three-dimer force.

Open questions

Solve the three-dimer problem / Three dimer zero crossing point ? / Liquid density imbalanced / Pentamer ...

Bose-Fermi Mapping

- In 1D, one can map the case of N impenetrable bosons with an ideal Fermi gas of N particles.
- For fermions, thanks to Pauli principle, the wavefunction vanishes with contact of intraspecies.
- For bosons, if we impose an infinite contact repulsion (impenetrable bosons), we reproduce artificially the Pauli principle.

$$\begin{cases} \Psi_B(x_1, x_2, ..., x_n) = A(x_1, ..., x_n)\Psi_F(x_1, x_2, ..., x_n) \\ A(x_1, ..., x_n) = \prod_{i>j} \text{sgn}(x_i - x_j) \end{cases} \tag{10}$$

$\rightarrow$ Same characteristic such as energy.

- This mapping has been at center of investigations in 1-dimension, in our case, we will resume this by :

$$\boxed{\Psi_B,\ \ g_{\uparrow\uparrow} = g_{\downarrow\downarrow} = +\infty \quad \Leftrightarrow \quad \Psi_F,\ \ g_{\uparrow\uparrow} = g_{\downarrow\downarrow} = 0} \tag{11}$$

Trimer Threshold

- Let us consider the $\uparrow\uparrow\downarrow$ combination (or equivalently, $\downarrow\downarrow\uparrow$) and apply STM.
- In the case $g_{\uparrow\downarrow} < 0$, $\uparrow\uparrow\downarrow$ is always bound except in the limit $g_{\uparrow\uparrow} = +\infty$ where $(\epsilon_{\uparrow\uparrow\downarrow} - \epsilon_{\uparrow\downarrow}) = 0$ and a_{ad} diverges.
- The trimer $\uparrow\uparrow\downarrow$ can be formed if $\epsilon_{\uparrow\uparrow\downarrow} < E = -2|\epsilon_{\uparrow\downarrow}|$ for zero dimer-dimer collision energy.

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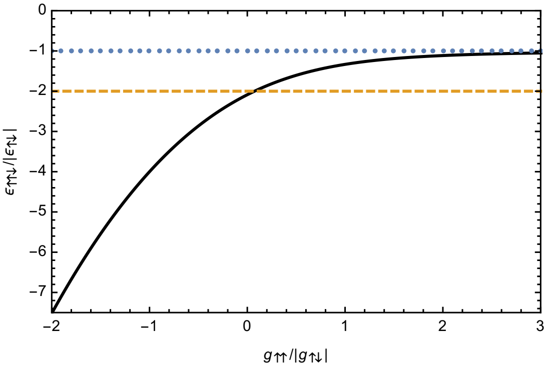

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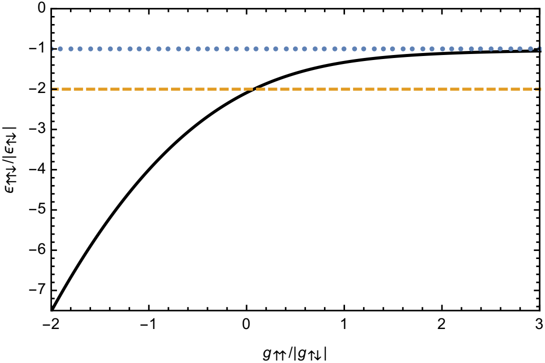


$$\epsilon_{\uparrow\uparrow\downarrow} = -2|\epsilon_{\uparrow\downarrow}| \Leftrightarrow g_{\uparrow\uparrow} = 0.0738|g_{\uparrow\downarrow}|$$

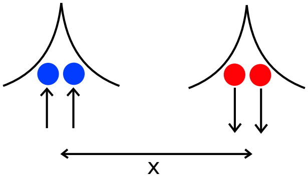
Case $\alpha \to +\infty$

- Thanks to the BFM, the case of infinite repulsion between intracomponents lead to study interacting two species Fermi gas.
- Corresponds equivalently in this study to the fermionic case where $g_{\uparrow\uparrow} = g_{\downarrow\downarrow} = 0 \to$ We end up with 1 Integral equation.
- Four attractively interacting fermions in 1D $\to$ Integrable case (solved by C. Mora) :
- Scattering properties of the two dimers ($\uparrow\downarrow$) system are described by the dimer-dimer scattering length a_{dd}.

$$\boxed{a_{dd} = 0.5 a_{\uparrow\downarrow}} \tag{12}$$

Case $\alpha \to -\infty$

Intraspecies are infinitely attractive : $g_{\uparrow\uparrow} = g_{\downarrow\downarrow} = -\infty$
$\to$ Four-body bound state composed of two intracomponent dimers.



$$[-\frac{\hbar^2}{2\mu}\frac{d^2}{dx^2} + 4g_{\uparrow\downarrow}\delta(x)]\chi_r = E_{BS}\chi_r \tag{13}$$

$$E = -\frac{2}{m{a_{\uparrow\uparrow}}^2} - E_{BS}(\{2m, 2m\}, g = 4g_{\uparrow\downarrow}) \tag{14}$$

$$\boxed{E/|\epsilon_{\uparrow\downarrow}| = -2\alpha^2 - 32} \tag{15}$$

Case $\alpha = -1$

- Known as Lieb-Liniger / Mc Guire model
- Take N as the arbitrary number of particles of equal masses M all interacting via one another via equal strength δ-function potentials.

$$[-\frac{\hbar^2}{2M}\sum_{i=1}^{N}\frac{d^2}{d{x_i}^2} + C\sum_{i>j}\sum_{j=1}^{N}\delta(x_i - x_j)]\Psi = E\Psi \tag{16}$$

- We put $\hbar = M = 1$ and $g = -\sqrt{2}C$ and consider the case of a δ-attractive potential between particles. We end up with the energy of the N-body bound state :

$$E = -\frac{g^2}{48}N(N^2 - 1) \tag{17}$$

In our units for our four-body problem :

$$\boxed{E(N = 4) = -10|\epsilon_{\uparrow\downarrow}|} \tag{18}$$

Case $\alpha \simeq \alpha^*$

- Appearance of a weakly bound four-body bound state :

$$E = -\frac{2}{m{a_{\uparrow\downarrow}}^2} - E_{BS}(\{2m, 2m\}, g_{dd})$$

- Where g_{dd} is the strength of interaction between the two intercomponent dimers and which verify $g_{dd} = -1/a_{dd}$.

$$E/|\epsilon_{\uparrow\downarrow}| = -2 - \frac{{a_{\uparrow\downarrow}}^2}{{a_{dd}}^2} \tag{19}$$

- One can interest to the function A defined by :

$$A(\alpha) = \sqrt{2}\sqrt{\frac{E}{\epsilon_{\uparrow\downarrow}} - 2} \underset{\alpha\simeq\alpha^*}{=} \frac{a_{\uparrow\downarrow}}{a_{dd}} \tag{20}$$

- $A(\alpha^*)$ passes through zero when a_{dd} diverge, namely for the ratio α^* of the gas-liquid transition :

$$A(\alpha^*) = 0 \Leftrightarrow a_{dd} = \infty \Leftrightarrow g_{dd} = 0 \tag{21}$$