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Fermi liquid theory: a renormalization group approach

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Abstract. We show how Fermi liquid theory results can be systematically recovered using a renormalization group (RG) approach. Considering a two-dimensional system with a circular Fermi surface, we derive RG equations at one-loop order for the two-particle vertex function Γ in the limit of small momentum ($\mathbf{Q}$) and energy (Ω) transfer and obtain the equation which determines the collective modes of a Fermi liquid. The density-density response function is also calculated. The Landau function (or, equivalently, the Landau parameters F_l^s and F_l^a) is determined by the fixed point value of the Ω-limit of the two-particle vertex function (Γ^{Ω^*}). We show how the results obtained at one-loop order can be extended to all orders in a loop expansion. Calculating the quasi-particle life-time and renormalization factor at two-loop order, we reproduce the results obtained from two-dimensional bosonization or Ward Identities. We discuss the zero-temperature limit of the RG equations and the difference between the Field Theory and the Kadanoff-Wilson formulations of the RG. We point out the importance of n-body ($n \geq 3$) interactions in the latter.

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1 Introduction

Since the original work of Landau [1–4], the Fermi liquid theory (FLT) is one of the main basis of our understanding of interacting fermions. The discovery of new materials showing strong deviations with respect to FLT, like high-T_c superconductors, has revived interest in the microscopic derivation of Landau's theory. In particular, this has motivated the application of RG methods to interacting fermions in dimension $d \geq 2$ [5–13].

RG methods are well-known for one-dimensional interacting fermions [14,15]. In these systems, the low-order perturbation theory is characterized by two logarithmically singular and interfering channels of correlation, namely, the particle-particle (Cooper) and $2k_F$ particle-hole (Peierls) channels. This invalidates any RPA-like approach which implicitly assumes the independence of the channels to lowest order. The RG approach allows one to sum up the leading, next-to-leading... logarithmic singularities in a consistent way. It has been also successfully applied to quasi-one-dimensional conductors (*i.e.* weakly coupled chains systems) where the interchain coupling can lead either to a Fermi liquid behavior or to a state of broken symmetry [15,16]. RG methods have also been used to study the instabilities of isotropic two-dimensional systems with respect to superconductivity or charge (spin) density wave [17]. While the independence of various channels of correlation is in general expected to be a good approximation (therefore allowing an RPA approach), for particular Fermi surfaces the situation can become more complicated due to the presence of nesting, Van Hove singularities... In such cases, the RG can be a useful tool to study the instabilities of the system and to investigate in detail phenomena like superconductivity induced by exchange of spin fluctuations.

In the above mentioned examples, one focuses on "highly quantum" degrees of freedom corresponding to energies larger than the temperature. The perturbation theory is characterized in general by logarithmic singularities of the type $\ln(E_0/T)$ (E_0 being an ultra-violet cut-off), which clearly shows that the temperature plays the role of an infrared cut-off. This should be contrasted with the standard diagrammatic derivation of FLT [2–4] (which in the following is referred to as the microscopic FLT) where these "quantum" degrees of freedom are in general not considered explicitly but simply included in the definition of some regular low-energy effective interactions. FLT concentrates on the Landau (or zero-sound (ZS)) channel (particle-hole pairs at small total momentum and energy) where the important degrees of freedom are known to be within the thermal broadening of the Fermi surface. This latter property can be seen from the polarization diagram (or particle-hole bubble) which is proportional to $\partial n_F/\partial \epsilon$ to lowest order in perturbation theory ($n_F(\epsilon)$ is the Fermi occupation factor). Thus the role of temperature is to fix the typical energy scale rather than to provide an infrared

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cut-off. This makes the application of RG methods in that case somewhat different from the above mentioned cases.

The first attempt to recover (in detail) Landau's theory from a RG method is due to Shankar [5,7]. However, Shankar's discussion of FLT rather relies on usual perturbative theory than on RG approach. Although RG arguments were used to identify the relevant couplings, the low-energy effective degrees of freedom were indeed explicitly integrated out by means of standard diagrammatic calculations. Using a finite temperature formalism, Chitov and Sénéchal have shown how FLT can be understood from a RG approach. In particular, they have correctly analyzed (within the RG framework) the singularities of the Landau channel which are at the heart of the microscopic FLT. Chitov and Sénéchal's analysis has recently been further developed and a detailed connection between the microscopic FLT and the RG approach has emerged [10].

The aim of this paper is to show how FLT results can be systematically derived in a RG approach. On the one hand we derive in detail the results reported in reference [10]. On the other hand, we present new results and discuss at length some particular points concerning the application of RG methods to interacting fermions. In the next section, we recall some aspects of FLT. Our aim is not to give an exhaustive summary of FLT, but to mention the main ideas underlying the microscopic FLT while emphasizing some points which will turn out to be crucial in the RG approach, such as the singularity of the two-particle vertex function at small momentum and energy transfer or its symmetry properties. In Section 3, we derive the RG equations at one-loop order for the Q-limit (Γ^Q) and Ω-limit (Γ^Ω) of the forward scattering vertex. In order to satisfy the Fermi statistics, the forward scattering zero-sound (ZS) and exchange (ZS') are both taken into account. As a result, we find that both the flows of Γ^Q and Γ^Ω are non zero. We show that the antisymmetry of Γ^Q under exchange of the two incoming or outgoing particles is conserved under RG, while the antisymmetry of Γ^Ω is lost. We solve (approximately) the RG equations to obtain a relation between the fixed point values Γ^{Q*} and $\Gamma^{\Omega*}$. We then extend the RG equations to the case of finite momentum and energy transfers and obtain the equation determining the collective modes of a Fermi liquid. The standard results of FLT are recovered if one identifies the Landau parameters F_l^s, F_l^a with $\Gamma^{\Omega*}$. We calculate the density-density correlation function and discuss the zero temperature limit of the RG equations. In Section 4, we discuss in detail some subtle points concerning the implementation of RG methods to interacting fermions. We point out the importance of three-, four-, ... -body interactions in the Kadanoff-Wilson (KW) formulation of the RG and discuss the differences between this approach and the Field Theory (FT) approach. In Section 5, we show how the results obtained at one-loop order can be extended to all orders. The quasi-particle life-time and renormalization factor are explicitly calculated at two-loop order in Section 6. The scattering rate is found to be $\tau^{-1} \sim T^2 \ln T$, a result previously known for a two-dimensional Fermi liquid. The expression for the renormalization factor agrees with the one obtained from two-dimensional bosonization or Ward Identities. We only consider in this paper the case of a neutral system with short range interactions.

2 Some aspects of Fermi liquid theory

2.1 Phenomenological approach

Landau's first approach to Fermi liquids is phenomenological [1,3,4]. The main assumption is the existence of low-energy elementary excitations (quasi-particles) which can be put in a one-to-one correspondence with the elementary excitations ("particles" and "holes") of the non-interacting fermion gas. Landau further postulated that a weak perturbation applied to the system in its ground state induces a change of the total energy given by (from now on we consider spin one-half fermions)

$$\delta E = \sum_{\mathbf{K},\sigma} \epsilon^0_{\mathbf{K}} \delta n_{\mathbf{K}\sigma} + \frac{1}{2\nu} \sum_{\mathbf{K},\mathbf{K}',\sigma,\sigma'} f_{\sigma,\sigma'}(\mathbf{K},\mathbf{K}') \delta n_{\mathbf{K}\sigma} \delta n_{\mathbf{K}'\sigma'} + O(\delta n^3) , \tag{1}$$

where $\delta n_{\mathbf{K}\sigma}$ is the change in the occupation number of the quasi-particles of momentum $\mathbf{K}$ and spin σ, and ν the volume of the system. $\epsilon^0_{\mathbf{K}}$ is the energy of a quasi-particle in the absence of other excited quasi-particles. For states near the Fermi surface and in an isotropic liquid (the only case we shall consider in this paper), it can be written as $\epsilon^0_{\mathbf{K}} = v_F(K - K_F) + \mu$ where K_F is the Fermi wave-vector, v_F the Fermi velocity of the quasi-particles and μ the chemical potential (we set $\hbar = k_B = 1$ throughout the paper). The effective mass is defined by $m = K_F/v_F$. The second term of the rhs of (1) comes from the interaction between quasi-particles. For states very close to the Fermi surface, $K \simeq K_F$ and $K' \simeq K_F$, the Landau function $f_{\sigma,\sigma'}$ is a function of the angle between $\mathbf{K}$ and $\mathbf{K}'$. If the spin dependent part of the quasi-particle interaction is due purely to exchange, $f_{\sigma,\sigma'}$ can be written as

$$f_{\sigma,\sigma'}(\theta) = f^s(\theta) + f^a(\theta)\sigma\sigma' = \frac{1}{2N(0)} \sum_{l=0}^{\infty} (F_l^s + F_l^a \sigma\sigma') P_l(\cos\theta) , \tag{2}$$

where $N(0) = K_F^2/2\pi^2 v_F$ is the density of states per spin at the Fermi level and $\sigma = 1$ (-1) for up (down) spins. In the last line of (2), we have expanded $f^s(\theta)$ and $f^a(\theta)$ on the basis of Legendre polynomials $P_l(\cos\theta)$. Using (1) and a transport equation for the distribution function of the quasi-particles, one can relate the physical quantities to the Landau parameters F_l^s and F_l^a [18]. For example, one obtains for the specific heat and the Pauli susceptibility, $C = C^0(1 + \frac{F_1^s}{3})$ and $\chi_P = \chi_P^0/(1 + F_0^a)$, where C^0 and χ_P^0 denote the corresponding quantities in a free

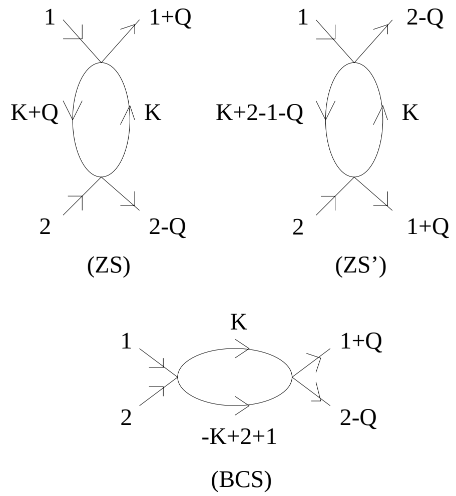

Fig. 1. Lowest order (one-loop) corrections to the two particle vertex function Γ. We use the simplified notation $1 = \tilde{K}_1$, $1+Q = \tilde{K}_1 + \tilde{Q}$...

fermion gas. Equation (1) holds only when the spin projection of the quasi-particles along a given axis is a good quantum number. In a more general case, ϵ^0 and δn should be considered as matrices in the spin variables. The Landau function then becomes a function $f_{\sigma_i} \equiv f_{\sigma_1\sigma_2,\sigma_3\sigma_4}$ of four spin variables. In the following, we shall use f_{σ_i} rather than $f_{\sigma,\sigma'}$.

2.2 Microscopic approach

2.2.1 Bethe-Salpeter equation in the ZS channel

The foundations of FLT were rapidly established using field theoretical methods [2–4]. The key microscopic quantity is the two-particle vertex function, which, together with the one-particle propagator G, plays an essential role in the theory of the Fermi liquid. We denote this quantity by $\Gamma_{\sigma_i}(\tilde{K}_1,\tilde{K}_2;\tilde{K}_2-\tilde{Q},\tilde{K}_1+\tilde{Q}) \equiv \Gamma_{\sigma_i}(\tilde{K}_1,\tilde{K}_2;\tilde{Q})$ with $\Gamma_{\sigma_i} \equiv \Gamma_{\sigma_1\sigma_2,\sigma_3\sigma_4}$. Here $\tilde{K} = (\mathbf{K},\omega)$ where $\mathbf{K}$ is a three-component vector and $\omega = \pi T(2n+1)$ (n integer) a fermionic Matsubara frequency. $\mathbf{Q}$ is the momentum transfer between the two particles and the bosonic Matsubara frequency $\Omega = 2\pi T m$ (m integer) the energy transfer. Landau noted that among the three lowest order corrections for Γ shown in Figure 1 (one-loop corrections), the ZS graph plays a special role when $\tilde{Q} \to 0$ (and $T \to 0$) since the product $G(\tilde{K})G(\tilde{K}+\tilde{Q})$ becomes singular in this limit if one assumes a quasi-particle form for the one-particle propagator, *i.e.* $G(\tilde{K}) = z[i\omega - v_F(K-K_F)]^{-1}$ where z is the quasi-particle renormalization factor. (We do not consider the incoherent part $G_{inc}(\tilde{K})$ which can easily be taken into account.) This motivated Landau to organize the perturbation expansion of Γ as follows [2–4]. One first introduces the quantity $\tilde{\Gamma}$ defined as the sum of all diagrams which do not contain the singular product $G(\tilde{K})G(\tilde{K}+\tilde{Q})$. The exact two-particle vertex function is then determined by the following Bethe-Salpeter equation:

$$\begin{aligned}\Gamma_{\sigma_i}(\tilde{K}_1,\tilde{K}_2;\tilde{Q}) &= \tilde{\Gamma}_{\sigma_i}(\tilde{K}_1,\tilde{K}_2;\tilde{Q}) \\ &+ \frac{T}{\nu}\sum_{\sigma,\sigma',\tilde{K}} \tilde{\Gamma}_{\sigma_1\sigma',\sigma\sigma_4}(\tilde{K}_1,\tilde{K};\tilde{Q}) \\ &\times G(\tilde{K})G(\tilde{K}+\tilde{Q})\Gamma_{\sigma\sigma_2,\sigma_3\sigma'}(\tilde{K},\tilde{K}_2;\tilde{Q}) . \end{aligned} \tag{3}$$

Equation (3) determines Γ as a function of the irreducible (with respect to the ZS channel) two-particle vertex function $\tilde{\Gamma}$. In order to simplify it, we use the two following properties: i) $\tilde{\Gamma}$ is a non-singular function of $\tilde{Q}$ (this is explicitly verified at one-loop order). This allows us to neglect its $\tilde{Q}$-dependence at small $\tilde{Q}$. ii) The singularity of $G(\tilde{K})G(\tilde{K}+\tilde{Q})$ comes from $\tilde{K}$ in the vicinity of the Fermi surface. In this area, the K,ω-dependence of the Γ's in (3) can be neglected so that the energy and radial momentum integral can be done [19]. Note that this amounts to decoupling the ZS channel from the other channels. Using (i) and (ii), (3) transforms into (for $T \to 0$)

$$\begin{aligned}\Gamma_{\sigma_i}(\tilde{K}_1,\tilde{K}_2;\tilde{Q}) &= \tilde{\Gamma}_{\sigma_i}(\tilde{K}_1,\tilde{K}_2) \\ &+ z^2 N(0)\sum_{\sigma,\sigma'}\int\frac{d\Omega_{\hat{\mathbf{K}}}}{4\pi}\tilde{\Gamma}_{\sigma_1\sigma',\sigma\sigma_4}(\tilde{K}_1,\tilde{K}) \\ &\times \Gamma_{\sigma\sigma_2,\sigma_3\sigma'}(\tilde{K},\tilde{K}_2;\tilde{Q})\frac{v_F\hat{\mathbf{K}}\cdot\mathbf{Q}}{i\Omega - v_F\hat{\mathbf{K}}\cdot\mathbf{Q}} ,\end{aligned} \tag{4}$$

where $\hat{\mathbf{K}} = \mathbf{K}/K$ is a unit vector and $d\Omega_{\hat{\mathbf{K}}}$ the corresponding angular integration. Equation (4) is the basis of the microscopic derivation of FLT. Before solving it (Sect. 2.2.3), we discuss in the next section the importance of the ZS' channel with respect to the Fermi statistics.

2.2.2 Symmetry considerations

As pointed out by Mermin [20,21] (see also Ref. [12]), the ZS' graph also becomes singular in the limit $\mathbf{K}_1 - \mathbf{K}_2 \to 0$ since it contains the product $G(\tilde{K})G(\tilde{K}+\tilde{K}_2-\tilde{K}_1-\tilde{Q}) \to G(\tilde{K})G(\tilde{K}-\tilde{Q})$. It is therefore necessary in this case (at least in principle) to consider the ZS and ZS' channels on the same footing and to add to the rhs of (4) the term

$$\begin{aligned} &-\frac{T}{\nu}\sum_{\sigma,\sigma'}\sum_{\tilde{K}}\tilde{\Gamma}_{\sigma_1\sigma',\sigma\sigma_3}(\tilde{K}_1,\tilde{K}+\tilde{K}_2-\tilde{K}_1;\tilde{K}_2-\tilde{K}_1-\tilde{Q}) \\ &\quad\times G(\tilde{K})G(\tilde{K}+\tilde{K}_2-\tilde{K}_1-\tilde{Q}) \\ &\quad\times \Gamma_{\sigma\sigma_2,\sigma_4\sigma'}(\tilde{K},\tilde{K}_2;\tilde{K}_2-\tilde{K}_1-\tilde{Q}) \\ &\simeq -z^2N(0)\sum_{\sigma,\sigma'}\int\frac{d\Omega_{\hat{\mathbf{K}}}}{4\pi}\tilde{\Gamma}_{\sigma_1\sigma',\sigma\sigma_3}(\tilde{K}_1,\tilde{K}) \\ &\quad\times \Gamma_{\sigma\sigma_2,\sigma_4\sigma'}(\tilde{K},\tilde{K}_2;\tilde{K}_2-\tilde{K}_1-\tilde{Q}) \\ &\quad\times \frac{v_F\hat{\mathbf{K}}\cdot(\mathbf{K}_2-\mathbf{K}_1-\mathbf{Q})}{-i\Omega - v_F\hat{\mathbf{K}}\cdot(\mathbf{K}_2-\mathbf{K}_1-\mathbf{Q})} ,\end{aligned} \tag{5}$$

where the second line is obtained in the limit $Q \to 0$ and $|\mathbf{K}_2 - \mathbf{K}_1| \ll T/v_F$ (and $T \to 0$). Note that the ZS

graph alone does not satisfy the Fermi statistics [12,20, 21]. Indeed, if one exchanges the two incoming or outgoing lines, the ZS graph transforms into the ZS' graph and *vice versa*. The consideration of the ZS' graph when $\mathbf{K}_1-\mathbf{K}_2 \to 0$ ensures that the antisymmetry properties of the vertex function are satisfied. From (4, 5), we obtain

$$\Gamma_{\sigma_1\sigma_1,\sigma_2\sigma_3}(\tilde{K}_1,\tilde{K}_1;\tilde{Q}) = 0 \;\text{ for }\; Q \neq 0 \;\text{ and }\; \Omega \neq 0\,. \quad (6)$$

Since the value of $\Gamma_{\sigma_i}(\tilde{K}_1,\tilde{K}_2;\tilde{Q})$ for $\tilde{Q} \to 0$ depends in an essential way on the ratio Q/Ω (as can be seen from (4, 5)), we introduce the Q- and Ω-limits of the two-particle vertex function:

$$\Gamma^Q_{\sigma_i}(\tilde{K}_1,\tilde{K}_2) = \lim_{Q\to 0}\left[\Gamma_{\sigma_i}(\tilde{K}_1,\tilde{K}_2;\tilde{Q})\Big|_{\Omega=0}\right],$$
$$\Gamma^\Omega_{\sigma_i}(\tilde{K}_1,\tilde{K}_2) = \lim_{\Omega\to 0}\left[\Gamma_{\sigma_i}(\tilde{K}_1,\tilde{K}_2;\tilde{Q})\Big|_{Q=0}\right]. \quad (7)$$

The limit $\tilde{Q} \to 0$ in (6) is still ill-defined since the limits $\mathbf{K}_1-\mathbf{K}_2, \tilde{Q} \to 0$ do not commute in (5). Following Mermin [20,21], we first take the limit $\tilde{Q} \to 0$ (with either $Q/\Omega \to 0$ or $\Omega/Q \to 0$) and then $\mathbf{K}_1 - \mathbf{K}_2 \to 0$: this ensures that Γ^Q and Γ^Ω are continuous functions in the forward direction ($\mathbf{K}_1 = \mathbf{K}_2$). (Since Γ^Q and Γ^Ω are ultimately connected to the (physical) forward scattering amplitude of two particles on the Fermi surface and to the Landau function f_{σ_i}, respectively, this requirement of continuity in the forward direction is very natural.) From (4, 5) we then conclude that the Q-limit of the forward scattering vertex function (Γ^Q) satisfies the Pauli principle while the Ω-limit (Γ^Ω) does not, *i.e.*

$$\Gamma^Q_{\sigma_1\sigma_1,\sigma_2\sigma_3}(\tilde{K}_1,\tilde{K}_1) = 0\,,$$
$$\Gamma^\Omega_{\sigma_1\sigma_1,\sigma_2\sigma_3}(\tilde{K}_1,\tilde{K}_1) \neq 0\,. \quad (8)$$

2.2.3 Landau's solution

We first consider the functions $\Gamma^{Q,\Omega}_{\sigma_i}(\tilde{K}_1,\tilde{K}_2)$ and restrict ourselves to states on the Fermi surface ($\omega = 0$ and $K = K_F$). $\Gamma^{Q,\Omega}_{\sigma_i}(\theta)$ become functions of the angle θ between $\mathbf{K}_1$ and $\mathbf{K}_2$, and can be expanded on the basis of Legendre polynomials (with coefficients $\Gamma^{Q,\Omega}_{\sigma_i}(l)$). The usual diagrammatic derivation of FLT does not take into account the singularity which appears for $\tilde{Q} \to 0$ in the ZS' channel when $|\mathbf{K}_1 - \mathbf{K}_2| \ll T/v_F$. This can be justified as follows. In general, physical quantities probe all the possible values of the angle θ. For example, the compressibility and the Pauli susceptibility are entirely determined by $\Gamma^Q_{\sigma_i}(l=0)$. The singularity in the ZS' channel affects only small angles $|\theta| \ll T/E_F$ (where $E_F \sim v_F K_F$ is the Fermi energy), while the singularity in the ZS channel affects all the angles. $\Gamma^Q_{\sigma_i}(l)$ is therefore determined by (4) with an accuracy of order T/E_F for any reasonable value of l ($l \ll E_F/T$).

When (5) is not taken into account, $\tilde{\Gamma} = \Gamma^\Omega$ and (4) becomes

$$\Gamma^Q_{\sigma_i}(l) = \Gamma^\Omega_{\sigma_i}(l) - \frac{z^2N(0)}{2l+1}\sum_{\sigma,\sigma'}\Gamma^\Omega_{\sigma_1\sigma',\sigma\sigma_4}(l)\Gamma^Q_{\sigma\sigma_2,\sigma_3\sigma'}(l)\,. \quad (9)$$

If the spin dependent part of the interaction is due purely to exchange, one can write $\Gamma^{Q,\Omega}_{\sigma_i}$ as a function of a spin symmetric ($A^{Q,\Omega}$) and antisymmetric ($B^{Q,\Omega}$) part:

$$2N(0)z^2\Gamma^{Q,\Omega}_{\sigma_i}(l) = A^{Q,\Omega}(l)\delta_{\sigma_1,\sigma_4}\delta_{\sigma_2,\sigma_3} + B^{Q,\Omega}(l)\boldsymbol{\tau}_{\sigma_1\sigma_4}\cdot\boldsymbol{\tau}_{\sigma_2\sigma_3}\,, \quad (10)$$

where $\boldsymbol{\tau}$ denotes the Pauli matrices. We then obtain from (9)

$$A^Q_l = \frac{A^\Omega_l}{1+\frac{A^\Omega_l}{2l+1}}\,; \quad B^Q_l = \frac{B^\Omega_l}{1+\frac{B^\Omega_l}{2l+1}}\,\cdot \quad (11)$$

Equation (4) can also be used to obtain the collective modes (which correspond to poles in the retarded vertex function) and any response function at finite $\tilde{Q}$. One then recovers the results of the phenomenological approach if one defines the Landau parameters by $F^s_l = A^\Omega_l$ and $F^a_l = B^\Omega_l$, or equivalently

$$f_{\sigma_i}(\theta) = z^2\Gamma^\Omega_{\sigma_i}(\theta)\,. \quad (12)$$

Thus the microscopic FLT not only justifies the results obtained in the phenomenological approach but also provides a microscopic definition of the Landau parameters. It is important to note that the Landau parameters do not correspond to a quantity entering the microscopic action or some low-energy effective action: it is necessary to integrate all the degrees of freedom to obtain $\Gamma^\Omega_{\sigma_i}$ and therefore the Landau parameters. Generally, one does not try to calculate $\Gamma^\Omega_{\sigma_i}$ as a function of the microscopic parameters but only establishes the relation between this quantity and physical quantities which can be measured experimentally.

3 RG equations at one-loop order

From now on we restrict ourselves to a two-dimensional system since our discussion can be straightforwardly extended to the three-dimensional case. We consider interacting spin one-half fermions with a circular Fermi surface. We write the partition function Z as a functional integral over Grassmann variables,

$$Z = \int \mathcal{D}\psi^*\mathcal{D}\psi e^{-S}\,, \quad (13)$$

where, assuming that the high-energy degrees of freedom have been integrated out (in a functional sense), S is a low-energy effective action describing the fermionic degrees of freedom with $|K-K_F| < \Lambda_0 \ll K_F$. We write the effective action as

$$\begin{aligned} S = &-\sum_{\tilde{K},\sigma} \psi^*_\sigma(\tilde{K})(i\omega - \epsilon(\mathbf{K}))\psi_\sigma(\tilde{K}) \\ &+ \frac{1}{4\beta\nu} \sum_{\tilde{K}_1\ldots\tilde{K}_4} \sum_{\sigma_1\ldots\sigma_4} U_{\sigma_1\sigma_2,\sigma_3\sigma_4}(\mathbf{K}_1,\mathbf{K}_2,\mathbf{K}_3,\mathbf{K}_4) \\ &\times \psi^*_{\sigma_4}(\tilde{K}_4)\psi^*_{\sigma_3}(\tilde{K}_3)\psi_{\sigma_2}(\tilde{K}_2)\psi_{\sigma_1}(\tilde{K}_1) \\ &\times \delta_{\mathbf{K}_1+\mathbf{K}_2,\mathbf{K}_3+\mathbf{K}_4}\delta_{\omega_1+\omega_2,\omega_3+\omega_4} , \end{aligned} \tag{14}$$

where the wave-vectors $\mathbf{K}$ satisfy $|K-K_F| < \Lambda_0$. $\beta = 1/T$ is the inverse temperature and ν and $\tilde{K}$ have the same meaning as in Section 2. Ignoring irrelevant terms, we write the single particle energy as $\epsilon(\mathbf{K}) = v_F(K - K_F) \equiv v_F k$ (choosing the chemical potential as the origin of the energies). The summation over the wave-vectors is defined by

$$\frac{1}{\nu}\sum_{\mathbf{K}} = \int \frac{d^2\mathbf{K}}{(2\pi)^2} \equiv K_F \int_{-\Lambda_0}^{\Lambda_0} \frac{dk}{2\pi} \int_0^{2\pi} \frac{d\theta}{2\pi} , \tag{15}$$

ignoring irrelevant terms at tree-level. The coupling function $U_{\sigma_1\sigma_2,\sigma_3\sigma_4}(\mathbf{K}_1,\mathbf{K}_2,\mathbf{K}_3,\mathbf{K}_4)$ is antisymmetric with respect to exchange of the two incoming or outgoing particles,

$$\begin{aligned} &U_{\sigma_1\sigma_2,\sigma_3\sigma_4}(\mathbf{K}_1,\mathbf{K}_2,\mathbf{K}_3,\mathbf{K}_4) \\ &\quad = -U_{\sigma_2\sigma_1,\sigma_3\sigma_4}(\mathbf{K}_2,\mathbf{K}_1,\mathbf{K}_3,\mathbf{K}_4) \\ &\quad = -U_{\sigma_1\sigma_2,\sigma_4\sigma_3}(\mathbf{K}_1,\mathbf{K}_2,\mathbf{K}_4,\mathbf{K}_3) , \end{aligned} \tag{16}$$

and is assumed to be a non-singular function of its arguments.

The form of the action (14) is usually justified by arguing that the omitted terms are irrelevant according to tree-level analysis [7,8]. This is not entirely correct. The integration of high-energy modes ($|k| > \Lambda_0$) generates terms of order $n \geq 6$ in the $\psi^{(*)}$ fields (*i.e.* three-, four-... -body interactions) which are marginal although a naive tree-level analysis would predict them to be irrelevant. The RG approach for the low-energy modes ($|k| < \Lambda_0$) also produces such terms. The role of these terms will be mentioned below and discussed in detail in Section 4. Nevertheless, the form (14) of the action is sufficient for the purpose of this section. Moreover, the possibility to assume that U_{σ_i} is a regular function of $\mathbf{K}_1, \mathbf{K}_2$... and to ignore its dependence on the Matsubara frequencies ω_1, ω_2... (which is irrelevant at tree-level) is not obvious. Indeed, according to the results of the microscopic FLT, we expect U_{σ_i} to acquire singularities for small momentum and energy transfers. As will be shown below, these singularities arise in the renormalization process when the momentum cut-off becomes smaller than T/v_F so that they can be ignored in the (bare) effective action (14) if we choose $T \ll v_F\Lambda_0$. Note also that the integration of high-energy degrees of freedom generates a wave-function renormalization factor $z_{\Lambda_0} < 1$ which has been eliminated from (14) *via* a rescaling of the fermion fields. The $\psi^{(*)}$'s in (14) therefore do not correspond to the bare fermions but to quasi-particles with a renormalization factor z_{Λ_0} [4].

As shown in reference [7], the constraint to have all momenta in the shell $|k| < \Lambda_0$ restricts the allowed scatterings to diffusion of particle-hole, or particle-particle, pairs with small total momentum ($Q \lesssim \Lambda_0$). Consequently, only two coupling functions U_{σ_i} have to be considered: the forward scattering coupling function and the BCS coupling function. In the following, we neglect the latter by assuming it is irrelevant so that no BCS instability occurs. As in Section 2, the forward scattering coupling function is denoted by $\Gamma_{\sigma_i}(\tilde{K}_1,\tilde{K}_2;\tilde{Q})$. According to tree-level analysis, this quantity is marginal and its dependence on $k_{1,2}$ and $\omega_{1,2}$ is irrelevant. We therefore introduce the coupling function $\Gamma_{\sigma_i}(\theta_1,\theta_2;\tilde{Q}) = \Gamma_{\sigma_i}(\mathbf{K}^F_1,\mathbf{K}^F_2;\tilde{Q})$ where $\mathbf{K}^F = K_F(\cos\theta,\sin\theta)$ is a wave-vector on the Fermi surface. It is not possible to put $\tilde{Q} = 0$ in Γ (although the dependence on $\tilde{Q}$ is irrelevant at tree-level) because Γ will acquire a singular dependence on $\tilde{Q}$ in the process of renormalization (this point is further discussed at the end of Sect. 3.1). We decompose Γ into a spin triplet amplitude Γ_t and a spin singlet amplitude Γ_s [22,23]:

$$\Gamma_{\sigma_i}(\theta_1,\theta_2;\tilde{Q}) = \Gamma_t(\theta_1,\theta_2;\tilde{Q}) I^{\sigma_1\sigma_2}_{\sigma_3\sigma_4} + \Gamma_s(\theta_1,\theta_2;\tilde{Q}) T^{\sigma_1\sigma_2}_{\sigma_3\sigma_4} , \tag{17}$$

where the functions

$$\begin{aligned} I^{\sigma_1\sigma_2}_{\sigma_3\sigma_4} &= \frac{1}{2}(\delta_{\sigma_1,\sigma_4}\delta_{\sigma_2,\sigma_3} + \delta_{\sigma_1,\sigma_3}\delta_{\sigma_2,\sigma_4}) , \\ T^{\sigma_1\sigma_2}_{\sigma_3\sigma_4} &= \frac{1}{2}(\delta_{\sigma_1,\sigma_4}\delta_{\sigma_2,\sigma_3} - \delta_{\sigma_1,\sigma_3}\delta_{\sigma_2,\sigma_4}) \end{aligned} \tag{18}$$

satisfy the relations

$$\begin{aligned} I^{\sigma_1\sigma_2}_{\sigma_3\sigma_4} &= I^{\sigma_2\sigma_1}_{\sigma_3\sigma_4} = I^{\sigma_1\sigma_2}_{\sigma_4\sigma_3} , \\ T^{\sigma_1\sigma_2}_{\sigma_3\sigma_4} &= -T^{\sigma_2\sigma_1}_{\sigma_3\sigma_4} = -T^{\sigma_1\sigma_2}_{\sigma_4\sigma_3} , \end{aligned}$$

and

$$\begin{aligned} I^{\sigma_1\sigma}_{\sigma_3\sigma'} I^{\sigma'\sigma_2}_{\sigma\sigma_4} &= \frac{5}{4} I^{\sigma_1\sigma_2}_{\sigma_3\sigma_4} - \frac{3}{4} T^{\sigma_1\sigma_2}_{\sigma_3\sigma_4} , \\ I^{\sigma_1\sigma}_{\sigma_3\sigma'} T^{\sigma'\sigma_2}_{\sigma\sigma_4} &= -\frac{1}{4} I^{\sigma_1\sigma_2}_{\sigma_3\sigma_4} + \frac{3}{4} T^{\sigma_1\sigma_2}_{\sigma_3\sigma_4} , \\ T^{\sigma_1\sigma}_{\sigma_3\sigma'} T^{\sigma'\sigma_2}_{\sigma\sigma_4} &= \frac{1}{4} I^{\sigma_1\sigma_2}_{\sigma_3\sigma_4} + \frac{1}{4} T^{\sigma_1\sigma_2}_{\sigma_3\sigma_4} , \end{aligned} \tag{19}$$

where a sum over σ and σ' is implied.

The KW RG procedure consists in successive partial integrations of the fermion field degrees of freedom in the infinitesimal momentum shell $\Lambda_0 e^{-dt} \leq |k| \leq \Lambda_0$ where dt is the RG generator and $\Lambda(t) = \Lambda_0 e^{-t}$ the effective momentum cut-off at step t . Each partial integration is followed by a rescaling of radial momenta, frequencies and fields (*i.e.* $\omega' = s\omega$, $k' = sk$ and $\psi^{(*)'} = \psi^{(*)}$ with $s = e^{dt}$) in order to let the quadratic part of the action (14) invariant and to restore the initial value of the cut-off. (See Refs. [7,15] for a detailed presentation of the KW RG method applied to fermion systems.) The partial integration, which is evaluated perturbatively within the framework of a loop expansion, modifies the parameters of the action which become functions of the flow parameter t. It also generates higher order interactions (three-, four-... body interactions) whose relevance or irrelevance should be controlled (see Sect. 4) [15].

At one-loop order, the three diagrams which have to be considered for the renormalization of $\Gamma_{\sigma_i}(\theta_1,\theta_2;\tilde{Q})$ are shown in Figure 1. In these diagrams, the momenta of the internal lines should be in the infinitesimal shell which is integrated out.

3.1 RG equations for $\Gamma^{\mathbf{Q}}_{\sigma_i}$ and $\Gamma^{\Omega}_{\sigma_i}$

We first consider the RG equations for the Q- and Ω-limits of the forward scattering coupling function:

$$\Gamma^Q_{\sigma_i}(\theta_1-\theta_2) = \lim_{Q\to 0}\left[\Gamma_{\sigma_i}(\theta_1,\theta_2;\tilde{Q})\Big|_{\Omega=0}\right],$$
$$\Gamma^\Omega_{\sigma_i}(\theta_1-\theta_2) = \lim_{\Omega\to 0}\left[\Gamma_{\sigma_i}(\theta_1,\theta_2;\tilde{Q})\Big|_{Q=0}\right], \qquad (20)$$

since these two quantities play an important role in the microscopic FLT. $\Gamma^{Q,\Omega}_{\sigma_i}(\theta)$ are even functions of θ. The only remnant of the antisymmetry of U_{σ_i} (Eq. (16)) is the condition [8]

$$\Gamma^Q_t(\theta=0)|_{\Lambda(t)=\Lambda_0} = \Gamma^\Omega_t(\theta=0)|_{\Lambda(t)=\Lambda_0} = 0, \qquad (21)$$

using the fact that U_{σ_i} is assumed to be a regular function of its arguments.

We ignore the symmetry-preserving contribution of the BCS channel (see however Sect. 4.3) and first discuss the contribution of the ZS graph. This graph involves the quantity

$$T\sum_\omega G(\tilde{K})G(\tilde{K}+\tilde{Q}) = \frac{1}{2}\frac{\tanh\left[\frac{\beta}{2}\epsilon(\mathbf{K}+\mathbf{Q})\right]-\tanh\left[\frac{\beta}{2}\epsilon(\mathbf{K})\right]}{i\Omega+\epsilon(\mathbf{K})-\epsilon(\mathbf{K}+\mathbf{Q})} \simeq \frac{v_F\hat{\mathbf{K}}\cdot\mathbf{Q}}{i\Omega-v_F\hat{\mathbf{K}}\cdot\mathbf{Q}}\frac{\beta}{4\cosh^2(\beta v_F k/2)} \qquad (22)$$

for small Q. Here $G(\tilde{K}) = (i\omega - v_F k)^{-1}$ is the one-particle Green's function and $|k| = \Lambda(t)$. It is clear that the limit of (22) for $\tilde{Q}\to 0$ depends on the ratio Q/Ω. It equals $-(\beta/4)\cosh^{-2}(\beta v_F k/2)$ in the Q-limit while it vanishes in the Ω-limit. We obtain

$$\left.\frac{d\Gamma^Q_{\sigma_i}(\theta_1-\theta_2)}{dt}\right|_{ZS} = -\frac{N(0)\beta_R}{\cosh^2(\beta_R)}\int\frac{d\theta}{2\pi}\sum_{\sigma,\sigma'}\Gamma^Q_{\sigma_1\sigma',\sigma\sigma_4}(\theta_1-\theta) \times\Gamma^Q_{\sigma\sigma_2,\sigma_3\sigma'}(\theta-\theta_2),$$
$$\left.\frac{d\Gamma^\Omega_{\sigma_i}(\theta_1-\theta_2)}{dt}\right|_{ZS} = 0, \qquad (23)$$

a result which was first obtained in reference [8]. Here $\beta_R = v_F\beta\Lambda(t)/2$ is a dimensionless inverse temperature and $N(0) = K_F/2\pi v_F$ the density of states per spin. Note that we have obtained (23) without rescaling the radial momenta, frequencies and fields. However, the same result is obtained if one chooses to do this rescaling. In this case, T should be replaced by $T(t) = Te^t$ (this t-dependence follows from the rescaling of the Matsubara frequencies) and $|k| = \Lambda_0$ in (22), which leads to (23) with $\beta_R = v_F\beta(t)\Lambda_0/2$ (which can be also written $\beta_R = v_F\beta\Lambda(t)/2$ with $\Lambda(t) = \Lambda_0 e^{-t}$). Since the two procedures (with or without rescaling) are equivalent, we will in general use $\beta = 1/T$ and $\Lambda(t) = \Lambda_0 e^{-t}$ which amounts to keeping the original units for the frequencies and momenta. Equations (23) show that we recover two important results of the microscopic FLT: i) the contribution of the ZS graph for $\tilde{Q}\to 0$ strongly depends on the order of the limits $Q\to 0$ and $\Omega\to 0$; ii) this singularity at $\tilde{Q}\to 0$ comes from the integration of states near the Fermi surface ($\Lambda(t)\lesssim T/v_F$) since $\beta_R/\cosh^2(\beta_R)\ll 1$ for $\beta_R\gg 1$. Since $\lim_{\beta\to\infty}(\beta/4)\cosh^{-2}(\beta x/2) = \delta(x)$, the ZS graph gives a singular contribution (with respect to $\Lambda(t)$) to the RG flow of Γ^Q when $T\to 0$.

The ZS' graph involves the quantity

$$T\sum_\omega G(\tilde{K})G(\tilde{K}+\mathbf{K}^F_{21}-\tilde{Q}) = \frac{1}{2}\frac{\tanh\left[\frac{\beta}{2}\epsilon(\mathbf{K}+\mathbf{K}^F_{21}-\mathbf{Q})\right]-\tanh\left[\frac{\beta}{2}\epsilon(\mathbf{K})\right]}{-i\Omega+\epsilon(\mathbf{K})-\epsilon(\mathbf{K}+\mathbf{K}^F_{21}-\mathbf{Q})}, \qquad (24)$$

where $\mathbf{K}^F_{21} = \mathbf{K}^F_2 - \mathbf{K}^F_1$. Since both internal lines should have their momenta in the infinitesimal shell near the cut-off $\Lambda(t)$, the ZS' graph is non zero in the limit $Q\to 0$ only if $\mathbf{K}^F_{21}\to 0$. We therefore obtain a (non-physical) discontinuity in the forward direction ($\theta = 0$). As discussed in Section 4, it is necessary to consider three-body interactions between fermions to restore the continuity in the forward direction. The ZS' graph with $\mathbf{K}^F_{21}\neq 0$ is then generated from the three-particle vertex function. To calculate the ZS' graph for small but finite $|\mathbf{K}_{21}|$, we adopt in this section the following approximate procedure which is justified in Section 4. We impose $\mathbf{K}$ to be at the cut-off but relax the constraint to have $\mathbf{K}+\mathbf{K}^F_{21}-\mathbf{Q}$ at the cut-off. For small $\mathbf{K}^F_{21}$ (*i.e.* for $|\theta_1-\theta_2|\ll T/E_F$) and small Q, (24) becomes

$$\frac{v_F\hat{\mathbf{K}}\cdot(\mathbf{K}^F_{21}-\mathbf{Q})}{-i\Omega-v_F\hat{\mathbf{K}}\cdot(\mathbf{K}^F_{21}-\mathbf{Q})}\frac{\beta}{4\cosh^2(\beta v_F k/2)}. \qquad (25)$$

It is clear from this expression that the limits $\tilde{Q},\mathbf{K}^F_{21}\to 0$ do not commute. The same problem arises in the microscopic FLT as pointed out by Mermin [12,20,21] and briefly discussed in Section 2. In agreement with the procedure followed in the microscopic FLT, we first take the limit $\tilde{Q}\to 0$ (which is well defined for $\mathbf{K}^F_{21}\neq 0$) and then $\theta_1-\theta_2\to 0$. This procedure (together with the consideration of three-body interactions) ensures the continuity of

$\Gamma^{Q,\Omega}(\theta)$ in the forward direction. We have

$$\lim_{\theta_1-\theta_2\to 0}\Big[T\sum_{\omega}G(\tilde K)G(\tilde K+\mathbf{K}^F_{21}-\tilde Q)\Big|_{\tilde Q=0}\Big] = -\frac{\beta}{4\cosh^2(\beta v_F k/2)}\,. \quad (26)$$

At low temperature, the ZS' graph also gives a singular contribution (with respect to $\Lambda(t)$) to the flows of Γ^Q and Γ^Ω when $\theta_1-\theta_2\to 0$. The singularity in the ZS' channel is restricted to the angles $|\theta_1-\theta_2|\ll T/E_F$. For $|\theta_1-\theta_2|\gg T/E_F$, the ZS' channel gives a smooth contribution to the flow of $\Gamma^{Q,\Omega}$. Note also that as in the microscopic FLT the ZS' graph does not differentiate between Γ^Q and Γ^Ω contrary to the ZS graph.

Taking into account the spin dependence of the coupling functions (using (19)), we obtain that the contributions of the ZS and ZS' graphs to the RG flow of $\Gamma^Q_t(\theta=0)$ cancel each other:

$$\frac{d\Gamma^Q_t(\theta=0)}{dt}\bigg|_{ZS'} = -\frac{d\Gamma^Q_t(\theta=0)}{dt}\bigg|_{ZS}\,. \quad (27)$$

Consequently, we have $\Gamma^Q_t(\theta=0)=0$ for any value of the flow parameter t. The antisymmetry of Γ^Q is therefore conserved under RG. On the other hand, since the contribution of the ZS graph to the flow of $\Gamma^\Omega(\theta=0)$ vanishes, while the contribution of the ZS' graph does not, the antisymmetry of Γ^Ω is not conserved under RG. The symmetry properties of the two-particle vertex function agree with the results of the microscopic FLT (Eqs. (8)). The antisymmetry of Γ^Ω is lost only when $\Lambda(t)\lesssim T/v_F$. For $\Lambda(t)\gg T/v_F$, the RG flow is determined by the ZS' channel only and $\Gamma^Q(\theta)\simeq\Gamma^\Omega(\theta)$.

Taking into account both the contributions of the ZS and ZS' graphs, the RG equations of $\Gamma^{Q,\Omega}$ can be written as

$$\frac{d\Gamma^Q_{\sigma_i}}{dt} = \frac{d\Gamma^Q_{\sigma_i}}{dt}\bigg|_{ZS} + \frac{d\Gamma^Q_{\sigma_i}}{dt}\bigg|_{ZS'}\,,$$
$$\frac{d\Gamma^\Omega_{\sigma_i}}{dt} = \frac{d\Gamma^\Omega_{\sigma_i}}{dt}\bigg|_{ZS'} = \frac{d\Gamma^Q_{\sigma_i}}{dt}\bigg|_{ZS'}\,. \quad (28)$$

The two preceding equations can be combined (using also (23)) to obtain an equation relating Γ^Q and Γ^Ω:

$$\frac{d\Gamma^Q_{\sigma_i}(\theta_1-\theta_2)}{dt} = \frac{d\Gamma^\Omega_{\sigma_i}(\theta_1-\theta_2)}{dt} - \frac{N(0)\beta_R}{\cosh^2(\beta_R)} \times\int\frac{d\theta}{2\pi}\sum_{\sigma,\sigma'}\Gamma^Q_{\sigma_1\sigma',\sigma\sigma_4}(\theta_1-\theta)\Gamma^Q_{\sigma\sigma_2,\sigma_3\sigma'}(\theta-\theta_2)\,. \quad (29)$$

Introducing the Fourier transforms

$$\Gamma^{Q,\Omega}_{\sigma_i}(l) = \int\frac{d\theta}{2\pi}e^{-il\theta}\Gamma^{Q,\Omega}_{\sigma_i}(\theta)\,, \quad (30)$$

and performing the sum over spins using (19), we obtain

$$\frac{d\Gamma^Q_t(l)}{dt} = -\frac{N(0)\beta_R}{\cosh^2(\beta_R)}\left[\frac54\Gamma^Q_t(l)^2 + \frac12\Gamma^Q_t(l)\Gamma^Q_s(l)+\frac14\Gamma^Q_s(l)^2\right]+\frac{d\Gamma^\Omega_t(l)}{dt},$$
$$\frac{d\Gamma^Q_s(l)}{dt} = -\frac{N(0)\beta_R}{\cosh^2(\beta_R)}\left[\frac34\Gamma^Q_t(l)^2 + \frac32\Gamma^Q_t(l)\Gamma^Q_s(l)-\frac14\Gamma^Q_s(l)^2\right]+\frac{d\Gamma^\Omega_s(l)}{dt}\,\cdot \quad (31)$$

These equations agree with those of Chitov and Sénéchal apart from the terms $d\Gamma^\Omega_{t,s}/dt$ coming from the ZS' graph which were omitted in reference [8]. Introducing the spin symmetric ($A^{Q,\Omega}$) and antisymmetric ($B^{Q,\Omega}$) parts as in equation (10), and using

$$2N(0)\Gamma^{Q,\Omega}_t(l) = A^{Q,\Omega}(l)+B^{Q,\Omega}(l)\,,$$
$$2N(0)\Gamma^{Q,\Omega}_s(l) = A^{Q,\Omega}(l)-3B^{Q,\Omega}(l)\,, \quad (32)$$

the RG equations take the simple form

$$\frac{dA^Q_l}{dt} = \frac{dA^\Omega_l}{dt} - \frac{\beta_R}{\cosh^2(\beta_R)}A^{Q\,2}_l\,,$$
$$\frac{dB^Q_l}{dt} = \frac{dB^\Omega_l}{dt} - \frac{\beta_R}{\cosh^2(\beta_R)}B^{Q\,2}_l\,. \quad (33)$$

The two preceding equations are exact at one-loop order. In order to solve them approximately in the low temperature limit, we will take advantage of the singularities which arise in the RG flow. We integrate equations (33) between 0 and t to obtain (writing explicitly the t dependence)

$$A^Q_l(t) = A^\Omega_l(t) - \int_0^t dt'\,\frac{\beta_R}{\cosh^2(\beta_R)}A^Q_l(t')^2 \quad (34)$$

and a similar equation for $B^Q_l(t)$. Here we have used $A^Q_l(t=0)=A^\Omega_l(t=0)$ since $\Gamma_{\sigma_i}(t=0)=U_{\sigma_i}$ is a non-singular function of its arguments as was justified above when $T\ll v_F\Lambda_0$. Iterating (34), we obtain

$$A^Q_l(t) = A^\Omega_l(t) - \int_0^t dt'\,\frac{\beta_R}{\cosh^2(\beta_R)}A^\Omega_l(t')^2+\ldots \quad (35)$$

We have shown above that $\Gamma^\Omega_{\sigma_i}(\theta)$ is a smooth function of $\Lambda(t)$ except for small angles $|\theta|\ll T/v_F$. The Fourier transform $\Gamma^\Omega_{\sigma_i}(l)$ is then a smooth function of $\Lambda(t)$ (for $l\ll E_F/T$). At low temperature, we can therefore make the approximation $A^\Omega_l(t)|_{\Lambda(t)\sim T/v_F}\simeq A^{\Omega *}_l$ where $A^{\Omega *}_l = A^\Omega_l|_{\Lambda(t)=0}$ is the fixed point (FP) value of A^Ω_l. Since the thermal factor $\beta_R/\cosh^2(\beta_R)$ is strongly peaked for $\Lambda(t')\lesssim T/v_F$, we can replace $A^\Omega_l(t')$ in the rhs of (35) by $A^{\Omega *}_l$. For $\Lambda(t)\lesssim T/v_F$, (35) then becomes

$$A^Q_l(t) = A^{\Omega *}_l - \int_0^t dt'\,\frac{\beta_R}{\cosh^2(\beta_R)}A^Q_l(t')^2\,. \quad (36)$$

The preceding equation shows that the ZS channel is now decoupled from the other channels. As in the microscopic FLT, this property follows from the fact that the singularity of the ZS channel is due to particle-hole pairs close to the Fermi surface ($\Lambda(t) \lesssim T/v_F$). Equation (36) is solved by introducing the parameter $\tau = \tanh \beta_R$, which leads to

$$A_l^Q(\tau) = \frac{A_l^{\Omega *}}{1 + (\tau_0 - \tau) A_l^{\Omega *}} \, ,$$
$$B_l^Q(\tau) = \frac{B_l^{\Omega *}}{1 + (\tau_0 - \tau) B_l^{\Omega *}} \, , \tag{37}$$

for $\Lambda(t) \lesssim T/v_F$. $\tau_0 = \tanh(\beta v_F \Lambda_0/2) \simeq 1$ for $T \ll v_F \Lambda_0$. Equations (37) show that the marginality of the coupling functions is lost at one loop-order. A_l^Q and B_l^Q are either relevant or irrelevant depending on their sign. For $A_l^{\Omega *} < -1$ ($B_l^{\Omega *} < -1$), A_l^Q (B_l^Q) diverges for some value of τ which signals an instability of the Fermi liquid. For instance, if $B_0^{\Omega *} < -1$, the Fermi liquid is unstable with respect to a ferromagnetic phase (in (11), one should replace $2l+1$ by 1 for a two-dimensional system). The stability conditions $A_l^{\Omega *} > -1$ and $B_l^{\Omega *} > -1$ are known as the Pomeranchuk's stability conditions [21]. The FP values of A_l^Q and B_l^Q are obtained for $\tau = 0$ ($\Lambda(\tau = 0) = 0$):

$$A_l^{Q*} = \frac{A_l^{\Omega *}}{1 + A_l^{\Omega *}} \, ; \quad B_l^{Q*} = \frac{B_l^{\Omega *}}{1 + B_l^{\Omega *}} \, \cdot \tag{38}$$

Here and in the following we put $\tau_0 = 1$.

The RG at one-loop order therefore agrees with the microscopic FLT (Eq. (11) with $2l+1$ replaced by 1). Equation (36) (together with the analog equation for B_l^Q) is nothing else but a Bethe-Salpeter equation in the ZS channel for the vertex $\Gamma_{\sigma_i}^Q$, with $\Gamma^{\Omega *}$ the irreducible two-particle vertex function. This shows that the integration of the RG equations generates the same diagrams as those considered by Landau [2–4] (as shown in Sect. 5, Eq. (36) holds at all orders in a loop expansion). From this point of view, there is therefore a strict equivalence between the RG approach and the microscopic FLT.

It has been claimed in reference [7] that the coupling function Γ_{σ_i} cannot become singular because the integration of an infinitesimal momentum shell cannot produce any non-analyticity. This argument is not correct since non-analyticity can originate in the infinite sum over the Matsubara frequencies. In this section, we have obtained a non-analyticity summing the product $G(\tilde{K})G(\tilde{K} + \tilde{Q})$ over ω. Because of this non-analyticity, one should keep in the action all coupling functions $\Gamma(\theta_1, \theta_2; \tilde{Q})$ whatever the value of $\tilde{Q}$ (with $Q \lesssim \Lambda(t)$). To illustrate this point, consider a marginal variable (call it g) which is a function of $\tilde{Q}$. Since g is marginal, it is tempting to neglect its dependence on $\tilde{Q}$ arguing it is irrelevant. This latter point is usually proved by considering the Taylor expansion of $g(\tilde{Q})$:

$$g(\tilde{Q}) = g_{00} + g_{10} Q + g_{01} \Omega + g_{11} Q \Omega + \cdots \tag{39}$$

A dimensional analysis shows that g_{00} is marginal and all other coefficients irrelevant, so that only g_{00} has to be kept in the action. This argument is correct only if g is a regular function at $\tilde{Q} = 0$. Otherwise, it has no Taylor expansion around $\tilde{Q} = 0$ and there is no way to control the marginality or irrelevance of the dependence on $\tilde{Q}$.

The importance of keeping the full dependence of $\Gamma(\theta_1, \theta_2; \tilde{Q})$ on $\tilde{Q}$ can be understood from a more physical argument. A neutral Fermi liquid with short range interaction can sustain a collective charge (or spin) density oscillation with an excitation energy vanishing in the limit of long-wave length (as shown by Mermin [20], there exists at least one such mode). This collective mode strongly affects $\Gamma^*(\theta_1, \theta_2; \tilde{Q})$ since it yields a pole at $\Omega = c_0 Q$ (c_0 is the velocity of the zero-sound or spin-waves mode) in the retarded two-particle vertex function obtained by analytical continuation $i\Omega \to \Omega + i0^+$ (see Sect. 3.2). This shows that the dependence on $\tilde{Q}$ cannot be irrelevant.

3.2 RG equations for $\Gamma_{\sigma_i}(\theta_1, \theta_2; \tilde{\mathbf{Q}})$: collective modes

RG equations for $\Gamma_{\sigma_i}(\theta_1, \theta_2; \tilde{Q})$ with $\tilde{Q} \neq 0$ are *a priori* difficult to obtain, since in general the one-loop graphs vanish when $Q \neq 0$. For example, it is not possible to have both $\mathbf{K}$ and $\mathbf{K} + \mathbf{Q}$ in the infinitesimal momentum shell to be integrated out when $\mathbf{Q} \neq 0$ (except for very rare configurations) so that the ZS graph vanishes. The same kind of problem arises in the calculation of the ZS' graph even when $\tilde{Q} \to 0$ (see preceding section). As discussed in Section 4, a one-loop calculation should involve the consideration of three-body interactions. In this section, we restrict ourselves to the case of finite but small $\tilde{Q}$. We adopt the approximate procedure used in Section 3.1 for the calculation of the ZS' graph: we impose on $\mathbf{K}$ to be in the infinitesimal momentum shell to be integrated out but relax the analog condition for $\mathbf{K}+\mathbf{Q}$. As shown below, this approximate procedure is sufficient (and correct) to obtain the long wave-length limit of $\Gamma_{\sigma_i}(\theta_1, \theta_2; \tilde{Q})$.

According to Section 3.1, the ZS' graph does not produce any singularity when $\tilde{Q} \to 0$. We can safely put $\tilde{Q} = 0$ in the contribution of this graph to Γ. The dependence on $\tilde{Q}$ of the ZS graph is given by (22). The RG equation (29) then becomes:

$$\frac{d\Gamma_{\sigma_i}(\theta_1, \theta_2; \tilde{Q})}{dt} = \frac{d\Gamma_{\sigma_i}^{\Omega}(\theta_1 - \theta_2)}{dt} + \frac{N(0)\beta_R}{\cosh^2(\beta_R)} \int \frac{d\theta}{2\pi} g(\theta, \tilde{Q}) \times \sum_{\sigma, \sigma'} \Gamma_{\sigma_1 \sigma', \sigma \sigma_4}(\theta_1, \theta; \tilde{Q}) \Gamma_{\sigma \sigma_2, \sigma_3 \sigma'}(\theta, \theta_2; \tilde{Q}) \, , \tag{40}$$

where $g(\theta, \tilde{Q}) = v_F \hat{\mathbf{K}} \cdot \mathbf{Q}/(i\Omega - v_F \hat{\mathbf{K}} \cdot \mathbf{Q})$ and $\hat{\mathbf{K}} = (\cos\theta, \sin\theta)$. $\Gamma_{\sigma_i}^{\Omega}(\theta_1 - \theta_2)$ is a smooth function of $\Lambda(t)$ except for $|\theta_1 - \theta_2| \ll T/E_F$. If we are interested in quantities (like the collective modes) which involve all the values of $\theta_1 - \theta_2$, we can consider Γ^{Ω} as a smooth function. It is then possible to proceed as in Section 3.1 to solve (40).

We introduce the quantities $A(\theta_1,\theta_2;\tilde{Q})$ and $B(\theta_1,\theta_2;\tilde{Q})$ by analogy with (32). Equation (40) becomes

$$\begin{aligned}\frac{dA(\theta_1,\theta_2;\tilde{Q})}{dt} &= \frac{dA^{\Omega}(\theta_1-\theta_2)}{dt}\\ &+ \frac{\beta_R}{\cosh^2(\beta_R)}\int\frac{d\theta}{2\pi}g(\theta,\tilde{Q})A(\theta_1,\theta;\tilde{Q})A(\theta,\theta_2;\tilde{Q})\,,\\ \frac{dB(\theta_1,\theta_2;\tilde{Q})}{dt} &= \frac{dB^{\Omega}(\theta_1-\theta_2)}{dt}\\ &+ \frac{\beta_R}{\cosh^2(\beta_R)}\int\frac{d\theta}{2\pi}g(\theta,\tilde{Q})B(\theta_1,\theta;\tilde{Q})B(\theta,\theta_2;\tilde{Q})\,. \end{aligned}\tag{41}$$

Integrating this equation, and taking advantage of the singularity of $\beta_R/\cosh^2(\beta_R)$ when $T\to 0$, we obtain for $\Lambda(t)\lesssim T/v_F$

$$A(\theta_1,\theta_2;\tilde{Q}) = A^{\Omega^*}(\theta_1-\theta_2) - \int\frac{d\theta}{2\pi}g(\theta,\tilde{Q}) \times\int_1^\tau d\tau' A(\theta_1,\theta;\tilde{Q})A(\theta,\theta_2;\tilde{Q}), \tag{42}$$

and a similar equation for B. If we iterate this equation, we obtain:

$$\begin{aligned}A(\theta_1,\theta_2;\tilde{Q}) &= A^{\Omega^*}(\theta_1-\theta_2)\\ &+(1-\tau)\int\frac{d\theta}{2\pi}g(\theta,\tilde{Q})A^{\Omega^*}(\theta_1-\theta)A^{\Omega^*}(\theta-\theta_2)\\ &+(1-\tau)^2\int\frac{d\theta}{2\pi}g(\theta,\tilde{Q})\int\frac{d\theta'}{2\pi}g(\theta',\tilde{Q})A^{\Omega^*}(\theta_1-\theta)\\ &\times A^{\Omega^*}(\theta-\theta')A^{\Omega^*}(\theta'-\theta_2)+\cdots\end{aligned}\tag{43}$$

This expansion is clearly equivalent to the integral equation

$$\begin{aligned}A(\theta_1,\theta_2;\tilde{Q}) &= A^{\Omega^*}(\theta_1-\theta_2)\\ &+(1-\tau)\int\frac{d\theta}{2\pi}g(\theta,\tilde{Q})A^{\Omega^*}(\theta_1-\theta)A(\theta,\theta_2;\tilde{Q})\,,\\ B(\theta_1,\theta_2;\tilde{Q}) &= B^{\Omega^*}(\theta_1-\theta_2)\\ &+(1-\tau)\int\frac{d\theta}{2\pi}g(\theta,\tilde{Q})B^{\Omega^*}(\theta_1-\theta)B(\theta,\theta_2;\tilde{Q})\,.\end{aligned}\tag{44}$$

At the fixed point ($\tau=0$), we recover the equations which determine the two-particle vertex function in the microscopic FLT (Eq. (4) evaluated for states on the Fermi surface). In the above calculation, the dependence on $\tilde{Q}$ follows from (22) which is correct only in the limit $Q\to 0$. It is clear that for finite Q, the singularity $(\beta/4)\cosh^{-2}(\beta v_F k/2)|_{T\to 0}=\delta(v_F k)$ is weakened (only the flow of Γ^Q presents a singularity $\sim\delta(\Lambda)$ for $T\to 0$). Consequently, the determination of $\Gamma^*_{\sigma_i}(\theta_1,\theta_2;\tilde{Q})$ is less accurate at finite Q.

The spectrum of the collective modes is given by the poles of A (zero-sound mode) and B (spin-waves mode) after analytical continuation $i\Omega\to\Omega+i0^+$. As in the microscopic FLT, we define the Landau parameters in such a way that the results of the phenomenological approach are reproduced. We therefore identify the Landau parameters with the FP values of A^{Ω}_l and B^{Ω}_l, $F^s_l=A^{\Omega^*}_l$ and $F^a_l=B^{\Omega^*}_l$, or equivalently $f_{\sigma_i}(\theta)=\Gamma^{\Omega^*}_{\sigma_i}(\theta)$. Considering the fact that the $\psi^{(*)}$'s in (14) have been rescaled to eliminate the wave-function renormalization factor z_{Λ_0}, we eventually come to the following definition of the Landau function:

$$f_{\sigma_i}(\theta)=z^2_{\Lambda_0}\Gamma^{\Omega^*}_{\sigma_i}(\theta)\,, \tag{45}$$

where $\Gamma^{\Omega^*}_{\sigma_i}(\theta)$ now refers to the bare fermions.

In the following, we will sometimes consider the simple case where $F^{s,a}_l=0$ if $l\neq 0$. Equations (44) then yield

$$\begin{aligned}A(\tilde{Q}) &= \frac{F^s_0}{1+(1-\tau)F^s_0\Omega_0(\eta)}\,,\\ B(\tilde{Q}) &= \frac{F^a_0}{1+(1-\tau)F^a_0\Omega_0(\eta)}\,,\end{aligned}\tag{46}$$

where $\eta=i\Omega/v_F Q$ and

$$\Omega_0(x)=\int\frac{d\theta}{2\pi}\frac{\cos\theta}{\cos\theta-x}\,\cdot \tag{47}$$

3.3 Density-density response function

We show in this section how the density-density response function can be calculated in the KW RG approach. We proceed as in reference [15]. We introduce a source term in the action,

$$S_h=-\sum_{\tilde{Q}}h(\tilde{Q})\rho(-\tilde{Q})\,, \tag{48}$$

where the external field $h(\tilde{Q})=h^*(-\tilde{Q})$ couples to the particle density $\rho(\tilde{Q})=(\beta\nu)^{-1/2}\sum_{\tilde{K},\sigma}\psi^*_\sigma(\tilde{K})\psi_\sigma(\tilde{K}+\tilde{Q})$. The density-density response function is determined by

$$\chi_{\rho\rho}(\tilde{Q})=\langle\rho(\tilde{Q})\rho(-\tilde{Q})\rangle=\left.\frac{\delta^{(2)}\ln Z[h]}{\delta h^*(\tilde{Q})\delta h(\tilde{Q})}\right|_{h=0}, \tag{49}$$

where $Z[h]$ is the partition function in presence of the source term. The RG process generates correction to the source field h along with higher order terms in the source field. At step t, the total action can be written as

$$S_{\Lambda(t)}-\sum_{\tilde{Q}}z_h(\tilde{Q})h(\tilde{Q})\rho(-\tilde{Q}) -\sum_{\tilde{Q}}h^*(\tilde{Q})h(\tilde{Q})\chi(\tilde{Q})+O(h^3)\,, \tag{50}$$

where $S_{\Lambda(t)}$ is the action at step t without the external field h. $z_h(\tilde{Q})$ and $\chi(\tilde{Q})$ are both t-dependent quantities. From (49), we deduce that $\chi_{\rho\rho}$ is the FP value of χ, *i.e.* $\chi_{\rho\rho}=\chi^*$. We consider the case where only the Landau parameters F^s_0 and F^a_0 are non zero. Γ_{σ_i}, A and B are then functions of $\tilde{Q}$ only and are given by (46).

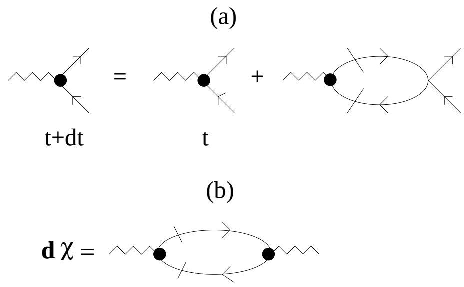

Fig. 2. (a) Diagrammatic representation of the renormalization of the external field h at one-loop order. The wavy line with the black dot represents the renormalized external field $z_h h$. The slashed particle lines indicate momenta in the infinitesimal shell $\Lambda(t)e^{-dt} \leq |k| \leq \Lambda(t)$. (b) Diagrammatic representation of the renormalization of χ at one-loop order.

The renormalization at one-loop order of z_h (see Fig. 2a) is given by

$$dz_h(\tilde{Q}) = z_h(\tilde{Q})\frac{T}{\nu}\sum_{\tilde{K}}{}' G(\tilde{K})G(\tilde{K}+\tilde{Q})\sum_{\sigma'}\Gamma_{\sigma\sigma',\sigma'\sigma}(\tilde{Q}), \tag{51}$$

where $\sum'$ means that the sum is restricted to the degrees of freedom which are in the infinitesimal momentum shell. As in Section 3.2, we impose on $\mathbf{K}$ to be in the infinitesimal shell to be integrated out and let $\mathbf{K}+\mathbf{Q}$ free (and consider only the case of small $\mathbf{Q}$). Using (22) and $\sum_{\sigma'}\Gamma_{\sigma\sigma',\sigma'\sigma}(\tilde{Q}) = A(\tilde{Q})/N(0)$, we obtain

$$\frac{d\ln z_h(\tilde{Q})}{dt} = -\frac{\beta_R}{\cosh^2(\beta_R)}\Omega_0(\eta)A(\tilde{Q}). \tag{52}$$

As expected, the renormalization of z_h involves only the charge part (A) of the interaction. Because of the thermal factor $\beta_R/\cosh^2(\beta_R)$ (which becomes significantly different from zero only when $\Lambda(t) \lesssim T/v_F$), we only need to know the expression of $A(\tilde{Q})$ for $\Lambda(t) \lesssim T/v_F$. Using $z_h(\tilde{Q})|_{\tau=1} = 1$ and (46), we obtain

$$z_h(\tilde{Q}) = \frac{1}{1+(1-\tau)F_0^s\Omega_0(\eta)}\,. \tag{53}$$

The generation of the term of order $O(h^2)$ is shown in Figure 2b. The RG equation for χ is given by

$$d\chi(\tilde{Q}) = -\frac{T}{\nu}\sum_{\tilde{K},\sigma}{}' G(\tilde{K})G(\tilde{K}+\tilde{Q})z_h^2(\tilde{Q}) \tag{54}$$

$$= 2N(0)\frac{\beta_R}{\cosh^2(\beta_R)}\Omega_0(\eta)z_h^2(\tilde{Q})dt\,. \tag{55}$$

Again the factor $\beta_R/\cosh^2(\beta_R)$ allows us to use the expression of $z_h(\tilde{Q})$ for $\Lambda(t) \lesssim T/v_F$ (Eq. (53)). We obtain

$$\chi(\tilde{Q}) = 2N(0)\frac{\Omega_0(\eta)(1-\tau)}{1+(1-\tau)F_0^s\Omega_0(\eta)}\,, \tag{56}$$

using $\chi(\tilde{Q})|_{\tau=1} = 0$, which holds when $T \ll v_F\Lambda_0$, since the external field $h(\tilde{Q})$ couples only to states which are within the thermal broadening of the Fermi surface. At the FP ($\tau = 0$), we recover the standard expression of the density-density response function in the simple case we are considering ($F_l^{s,a} = 0$ if $l \neq 0$). From (56), we deduce the compressibility of the Fermi liquid:

$$\kappa^* = \lim_{Q\to 0}\left[\chi^*(\tilde{Q})\Big|_{\Omega=0}\right] = \frac{2N(0)}{1+F_0^s}\,, \tag{57}$$

a result which holds whatever the values of the Landau parameters (*i.e.* κ^* is determined by F_0^s only). The Pauli susceptibility can be obtained in a similar way by introducing an external magnetic field which couples to the spin density.

3.4 Zero-temperature limit

We discuss in this section the zero-temperature limit of the RG equations. For simplicity, we only consider the contribution of the ZS graph to Γ^Q. This contribution can be written as

$$\left.\frac{dA_l^Q}{dt}\right|_{ZS} = -\frac{\beta_R}{\cosh^2(\beta_R)}{A_l^Q}^2\,, \tag{58}$$

where $\beta_R = v_F\beta(t)\Lambda_0/2$ with Λ_0 the cut-off (which is kept fixed through the rescaling procedure) and $\beta(t) = \beta e^{-t}$ the effective temperature at step t. Using $\lim_{\beta\to\infty}\beta_R/\cosh^2(\beta_R) = 2\Lambda_0\delta(\Lambda_0)$, we obtain in the zero temperature limit

$$\left.\frac{dA_l^Q}{dt}\right|_{ZS} = -2\Lambda_0\delta(\Lambda_0){A_l^Q}^2 = 0 \tag{59}$$

for any finite value of the cut-off Λ_0. One can also obtain the preceding equation by taking the limit $T \to 0$ from the very beginning of the calculation (*i.e.* in Eq. (14)). Equation (59) disagrees with the preceding sections where the limit $T \to 0$ was taken only at the end of the calculation. The origin of this disagreement can be understood as follows. The ZS graph describes processes where the two incoming particles exchange a particle-hole pair at zero total momentum and energy. Assume that the particle and the hole of this pair have momenta corresponding to the same (band) energy ϵ. Because of the rescaling of the momenta, $|\epsilon|$ increases ($\epsilon' = e^{dt}\epsilon$ at each step of the renormalization). When it reaches the value $|\epsilon| = \Lambda_0$, we obtain a finite contribution to $dA_l^Q/dt|_{ZS}$. At finite temperature, the particle and the hole of the exchanged pair have energies within the thermal broadening of the Fermi surface ($|\epsilon| \lesssim T$). However, at $T = 0$, the particle-hole pair has to lie exactly on the Fermi surface ($\epsilon = 0$). Under rescaling of the energies ($\epsilon' = e^{dt}\epsilon$), $\epsilon = 0$ remains unchanged and never reaches the value $|\epsilon| = \Lambda_0$. Hence the absence of flow for A_l^Q in the ZS channel. This unphysical result (first pointed out by Chitov and

Sénéchal [8]) follows from the rescaling procedure keeping the cut-off Λ_0 fixed which amounts to integrating all the degrees of freedom except those on the Fermi surface. While this procedure is in general perfectly valid, it fails at $T=0$ due to the somehow pathological behavior of the ZS channel. A natural way to avoid these difficulties is to use a finite temperature formalism taking the limit $T\to 0$ only at the end of the calculations [8]. Alternatively, one can determine $\Gamma(\theta_1,\theta_2;\tilde{Q})$ for small but finite $\tilde{Q}$ taking the limit $\tilde{Q}$ at the end of the calculation (a finite $\mathbf{Q}$ ensures that the particle and the hole are not on the Fermi surface. Under the rescaling procedure, their energies will therefore become of the order of Λ_0). Another possibility would be not to follow the rescaling procedure and to derive RG equations as a function of the effective cut-off Λ. Equation (58) is then replaced by

$$\left.\frac{dA_l^Q}{d\Lambda}\right|_{ZS} = \frac{v_F\beta/2}{\cosh^2(v_F\beta\Lambda/2)}{A_l^Q}^2 \to_{(T\to 0)} 2\delta(\Lambda){A_l^Q}^2\,, \tag{60}$$

which allows to integrate all the degrees of freedom since the cut-off can reach the value $\Lambda=0$.

4 A few remarks

In this section, we discuss in detail some points which were only briefly mentioned in Section 3.

4.1 Content of the low-energy effective action

We first consider the problem which arises in the calculation of the ZS' graph. When $\mathbf{Q}\to 0$, the ZS' graph vanishes unless $\mathbf{K}_2-\mathbf{K}_1\to 0$ since both internal lines should have their momenta in the infinitesimal shell near the cut-off. We therefore obtain a discontinuous contribution in the forward direction ($\mathbf{K}_2=\mathbf{K}_1$). For the same reason, when $\mathbf{K}_2=\mathbf{K}_1$, we obtain a discontinuity at $\mathbf{Q}=\mathbf{0}$ when one varies $\mathbf{Q}$. The same problem arises in the calculation of the ZS graph considered as a function of $\mathbf{Q}$. These discontinuities are clearly unphysical.

Consider the one-loop diagrams of Figure 1. All the internal momenta should be in the infinitesimal shell $\Lambda(t)e^{-dt}\le|k|\le\Lambda(t)$ which has to be integrated out. We therefore consider only the intermediate states where the particle and the hole (or both particles in the case of the BCS graph) have the same energies (in absolute value, *i.e.* $|\epsilon_1|=|\epsilon_2|$). If, in the KW RG method, we consider only these diagrams, we do not take into account processes where the particle and the hole in the intermediate state do not have the same energy. This reduction of the Hilbert space results in unphysical discontinuities. These discontinuities are suppressed if one includes in the action three-body interactions [24].

By iterating "by hand" the RG equations, one can identify the diagrams which are effectively considered *via* the RG approach. For instance, a one-loop RG calculation in the ZS channel (ignoring the ZS' and BCS channels) amounts to suming the series of bubble diagrams in this channel. In the following we explicitly identify some of the diagrams generated by the RG equations to prove the importance of three-body interactions.

Consider the action $S=S_0+S_4+S_6$ where the quadratic and quartic parts, S_0 and S_4, are given by (14) (where we now note $U^{(4)}$ the coupling function of the two-body interaction). S_6 is a three-body interaction given by

$$S_6 = \frac{1}{(3!)^2}\frac{T^2}{\nu^2}\sum_{\tilde{K}_1\ldots\tilde{K}_6} U^{(6)}(\tilde{K}_i)\psi^*(\tilde{K}_6)\psi^*(\tilde{K}_5)\psi^*(\tilde{K}_4) \times\psi(\tilde{K}_3)\psi(\tilde{K}_2)\psi(\tilde{K}_1)\delta_{1+2+3,4+5+6}\,, \tag{61}$$

where we do not consider the spin dependence which is of no importance for our discussion. The function $\delta_{1+2+3,4+5+6}$ ensures the conservation of momentum and energy and $U^{(6)}(\tilde{K}_i)\equiv U^{(6)}(\tilde{K}_1,...,\tilde{K}_6)$. All wave-vectors satisfy $0\le|k|\le\Lambda_0$. If we reduce the cut-off, $\Lambda_0'=\Lambda_0/s$ ($s>1$), and rescale radial momenta, frequencies and fields in the usual way ($k'=sk$, $\omega'=s\omega$, $\psi'=\psi$) to keep the quadratic action S_0 invariant, we obtain ${U^{(4)}}'=U^{(4)}$ and ${U^{(6)}}'=U^{(6)}/s$. One usually concludes that $U^{(4)}$ is marginal and $U^{(6)}$ is irrelevant, so that this latter can be neglected in the small $U^{(6)}$ limit. This conclusion is not correct if $U^{(6)}$ is not an analytic function of its arguments. This is precisely the situation we have to consider. The RG generates a three-body interaction which is a singular function of its arguments and turns out to be marginal. To lowest order, a three-body interaction is generated *via* the process shown in Figure 3a. In this figure, the slashed lines indicate degrees of freedom in the infinitesimal shell $\Lambda(t)e^{-dt}\le|k|\le\Lambda(t)$ which have to be integrated out. The other particle lines are all assumed to be below the infinitesimal momentum shell ($|k|<\Lambda(t)e^{-dt}$). The corresponding contribution to S_6 is of the type (ignoring sign and multiplicative factors)

$$\begin{aligned}&U^{(4)}(\tilde{K}_2,\tilde{K}_3,\tilde{K}_2+\tilde{K}_3-\tilde{K}_6,\tilde{K}_6;t_{23\bar{6}})\\ &\times U^{(4)}(\tilde{K}_1,\tilde{K}_5+\tilde{K}_4-\tilde{K}_1,\tilde{K}_4,\tilde{K}_5;t_{23\bar{6}})\\ &\times G(\tilde{K}_2+\tilde{K}_3-\tilde{K}_6)\\ &\times\psi^*(\tilde{K}_6)\psi^*(\tilde{K}_5)\psi^*(\tilde{K}_4)\psi(\tilde{K}_3)\psi(\tilde{K}_2)\psi(\tilde{K}_1)\,,\end{aligned} \tag{62}$$

where $t_{23\bar{6}}$ is defined by $v_F\Lambda(t_{23\bar{6}})=|\epsilon(\mathbf{K}_2+\mathbf{K}_3-\mathbf{K}_6)|$. Because of the Green's function $G(\tilde{K}_2+\tilde{K}_3-\tilde{K}_6)=(i\omega_2+i\omega_3-i\omega_6-\epsilon(\mathbf{K}_2+\mathbf{K}_3-\mathbf{K}_6))^{-1}$, this contribution is singular when $\epsilon,\omega\to 0$. In a dimensional analysis, $G(\tilde{K}_2+\tilde{K}_3-\tilde{K}_6)$ yields an additional factor s so that the contribution (62) to S_6 is marginal and not irrelevant. Imagine that, once the contribution (62) has been generated, one continues the renormalization process by decreasing the cut-off below $\Lambda(t_{23\bar{6}})$. If two (one incoming and one outgoing) of the six external lines of the six-leg diagram of Figure 3a have the same momentum and energy, for example $\tilde{K}_3=\tilde{K}_4$, then for $v_F\Lambda(t)=|\epsilon(\mathbf{K}_3)|=|\epsilon(\mathbf{K}_4)|$, this diagram generates a four-leg diagram as shown in Figure 3b, assuming that the other four external legs are below the cut-off

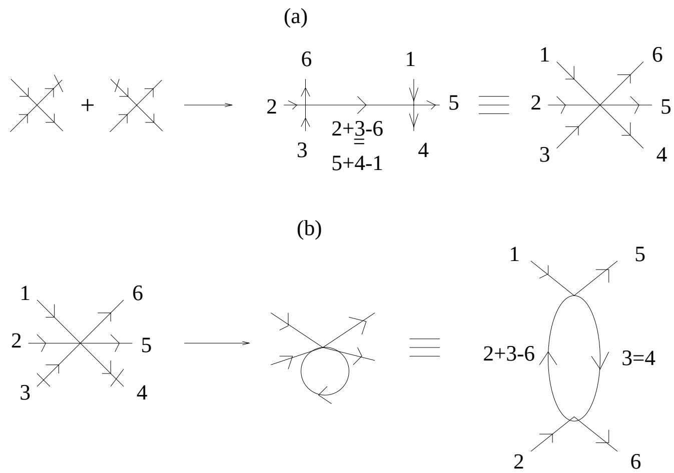

Fig. 3. (a) Generation of a six-leg diagram from four-leg diagrams. (b) Generation of a four-leg diagram from a six-leg diagram. The slashed particle lines indicate momenta in the infinitesimal shell $\Lambda(t)e^{-dt} \leq |k| \leq \Lambda(t)$

(when the cut-off is between $|\epsilon(\mathbf{K}_3)|/v_F = |\epsilon(\mathbf{K}_4)|/v_F$ and $\Lambda(t_{23\bar{6}})$, the contribution (62) to S_6 does not renormalize). We therefore obtain a contribution to S_4 of the type

$$\begin{aligned} &U^{(4)}(\tilde{K}_2, \tilde{K}_3, \tilde{K}_2 + \tilde{K}_3 - \tilde{K}_6, \tilde{K}_6; t_{23\bar{6}}) \\ &\times U^{(4)}(\tilde{K}_1, \tilde{K}_5 + \tilde{K}_4 - \tilde{K}_1, \tilde{K}_4, \tilde{K}_5; t_{23\bar{6}}) \\ &\times G(\tilde{K}_2 + \tilde{K}_3 - \tilde{K}_6) \\ &\times G(\tilde{K}_3)\psi^*(\tilde{K}_6)\psi^*(\tilde{K}_5)\psi(\tilde{K}_2)\psi(\tilde{K}_1). \end{aligned} \quad (63)$$

The important point is that $|\epsilon(\mathbf{K}_3)| \neq |\epsilon(\mathbf{K}_2 + \mathbf{K}_3 - \mathbf{K}_6)|$. Thus, *via* the three-body interaction, we have generated the "missing" processes where the particle and the hole in the intermediate state do not have the same energy. This shows that it is necessary to consider three-body interactions to generate all the one-loop diagrams in the KW RG approach. The calculation of one-loop diagrams generated from six-leg vertices would be in practice very difficult. The reason is that, in the above example (Fig. 3), the contribution to $U^{(4)}$ at $\Lambda(t) = |\epsilon(\mathbf{K}_3)| = |\epsilon(\mathbf{K}_4)|$ involves $U^{(4)}(t_{23\bar{6}})$ where $t_{23\bar{6}} < t$ (*i.e.* $\Lambda(t_{23\bar{6}}) > \Lambda(t)$). Therefore, $dU^{(4)}(t)/dt$ is not a function of t only but depends also on $t' < t$: $U^{(4)}(t)$ is determined by an integro-differential equation. Nevertheless, if $|\epsilon(\mathbf{K}_3)| \simeq |\epsilon(\mathbf{K}_2 + \mathbf{K}_3 - \mathbf{K}_6)|$, we have $\Lambda(t_{23\bar{6}}) \simeq \Lambda(t)$ and we can make the approximation $U^{(4)}(t_{23\bar{6}}) = U^{(4)}(t)$. $dU^{(4)}(t)/dt$ is then entirely determined by $U^{(4)}(t)$. In other words, we have approximated the integro-differential equation which determines $U^{(4)}(t)$ by a differential equation. It is clear that this approximation amounts to calculating $dU^{(4)}(t)/dt$ directly from the one-loop diagram $\propto {U^{(4)}}^2$ (*i.e.* without considering $U^{(6)}$) imposing $\mathbf{K}_3$ to be at the cut-off and relaxing the analog constraint for $\mathbf{K}_2 + \mathbf{K}_3 - \mathbf{K}_6$. This is precisely what we have done in Section 3 to calculate the ZS' graph for $\mathbf{K}_1^F - \mathbf{K}_2^F \neq 0$ or the ZS graph at finite $\tilde{Q}$.

In the same way, the RG generates eight-leg vertices which, for the reason discussed above, are marginal. These vertices generate in turn four-leg vertices. Figure 4 shows

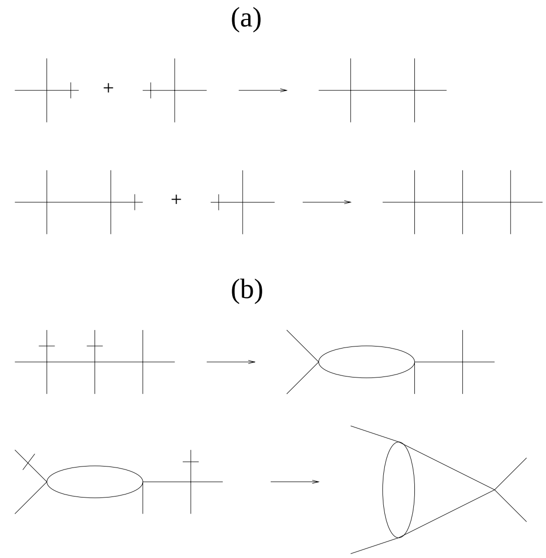

Fig. 4. (a) generation of an eight-leg diagram from four-leg diagrams. (b) generation of a four-leg diagram from an eight-leg diagram.

how a two-loop diagram is generated in this way. This two-loop diagram cannot be generated directly (*i.e.* from the integration of only one infinitesimal momentum shell) because of the constraints imposed by momentum conservation (in the diagram of Fig. 4b, it is not possible to have all the internal momenta in the shell to be integrated out).

n-body ($n > 2$) interactions are also generated by the integration of high-energy degrees of freedom $|\epsilon| > E_0$ ($E_0 = v_F\Lambda_0$ assuming a circular Fermi surface) which is the necessary step to obtain the low-energy effective action. We note $S_{micro}(\psi^*, \psi)$ the exact microscopic action. $\psi^{(*)} \equiv \psi^{(*)}(\mathbf{K}, \omega)$ where $\mathbf{K}$ belongs to the first Brillouin zone (assuming for simplicity a single band). If we note $\psi_>^{(*)}$ ($\psi_<^{(*)}$) the fields with $|\epsilon(\mathbf{K})| > E_0$ ($|\epsilon(\mathbf{K})| < E_0$), then the low-energy effective action is defined by

$$e^{-S(\psi_<^*, \psi_<)} = \int \mathcal{D}\psi_>^* \mathcal{D}\psi_> e^{-S_{micro}(\psi_>^*, \psi_>; \psi_<^*, \psi_<)}. \quad (64)$$

This partial integration generates vertices at all order for the $\psi_<^{(*)}$'s even if the microscopic action S_{micro} contains only a two-body interaction [25]. As discussed above, some of these n-body interactions are marginal due to a non trivial dependence on the external variables. They should therefore be retained in the low-energy effective action. Physically, the role of these n-body interactions is clear. For example, at one-loop order, the three-body interaction is necessary to take into account processes where, in the intermediate state, the particle is below the initial cut-off ($|\epsilon| < E_0$) while the hole (or the other particle) is above the initial cut-off ($|\epsilon| > E_0$).

4.2 Self-energy corrections

Until now, we have not considered the renormalization of the one-particle propagator. The corresponding diagram

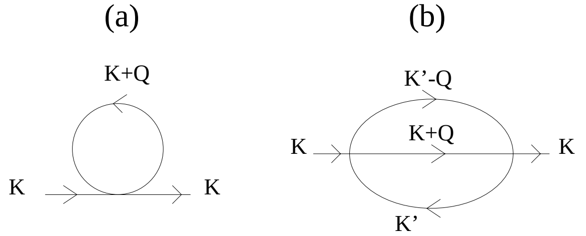

Fig. 5. One-loop (a) and two-loop (b) diagrams for the self-energy.

at one and two-loop orders are shown in Figure 5. The two-loop diagram vanishes because it is not possible to have all internal momenta in the shell to be integrated out. This is also true for higher order diagrams. The only diagram which does not vanish is the one-loop diagram. In general, non-trivial self-energy corrections (finite life-time and wave-function renormalization) originate in the dependence of $U^{(4)}$ on $\tilde{Q}$. The integration of high-energy degrees of freedom also generates a wave-function renormalization. However, it cannot induce a finite life-time for states near the Fermi surface, since this effect comes from real transitions occuring at low-energy. Since the dependence of $U^{(4)}$ on $\tilde{Q}$ arises through the consideration of three-, four-... n-body interactions, the inclusion of $U^{(6)}$, $U^{(8)}$... in the action is also very important for the calculation of the self-energy.

The role of n-body ($n > 2$) is then crucial in the KW RG approach. They should be taken into account according to the order to which the calculation is to be done. For example, for a two-loop order calculation, one should include in the action four-body interactions. In practice, the KW RG method, as described here, would be very difficult to apply. We have shown in Section 3 how a one-loop order RG calculation can be (approximately) done without considering three-body interactions (see also Ref. [15]). Moreover, Bourbonnais and Caron have shown how the KW RG approach can be modified to allow a two-loop calculation [15]. Their method will be used in Section 6 to obtain the quasi-particle life-time and the wave-function renormalization factor.

4.3 Interference between channels

In the KW RG approach, the renormalization of the forward scattering coupling function involves mainly the ZS channel. As discussed above, the interference with the ZS' channel requires the consideration of (at least) three-body interactions. This holds also for the interference between the ZS and BCS channels. This means that the interference between channels involves not only intermediate states at a given energy ($\Lambda(t)$) but also intermediate states at higher energy ($|\epsilon| > \Lambda(t)$). In other words, the interference between channels is "frustrated". This situation is characteristic of a two-dimensional system with a circular Fermi surface and is at the basis of the validity of FLT. It is also what justifies the use of a one-loop RG approach in the BCS channel only [7] (*i.e.* a ladder diagrams summation (or RPA approximation) in the conventional diagrammatic language) to study the BCS instability. For more complicated Fermi surfaces, the interference between channels may become important. An example would be a two-dimensional conductor near half-filling where d-wave superconductivity can be induced by the exchange of spin fluctuations. (This Kohn-Luttinger effect [7,26] always exists but is expected to lead to extremely small critical temperature in the case of a circular Fermi surface.) Another example is given by one-dimensional conductors where the different channels of correlation strongly interfere forbidding any RPA-like calculations [14,15].

The consideration of the ZS' channel in Section 3 was motivated by symmetry considerations. For $|\theta_1 - \theta_2| \ll T/E_F$ ($\theta_1 - \theta_2$ being the angle between the two incoming particles), the interference between the ZS and ZS' channel is not frustrated. This induces a particular behavior of the two-particle vertex function around $\theta_1 - \theta_2 = 0$ and ensures that Γ^Q satisfies the Pauli principle [11].

It turns out that the frustration of the interference between the ZS and BCS channels also disappears for $\theta_1 - \theta_2 \sim \pi$. For these values of $\theta_1 - \theta_2$, $\Gamma(\theta_1, \theta_2; \tilde{Q})$ can be seen both as a forward scattering coupling function or as a BCS coupling function (Shankar's V function [7]). We therefore expect a particular behavior of $\Gamma(\theta_1 - \theta_2 \sim \pi)$. Such a behavior has been found for a dilute Fermi gas (where a well-controlled low-density expansion can be made) in both three-dimensional [21,27] and two-dimensional systems [28].

4.4 Field theory approach

We briefly discuss in this section the differences between the KW approach (which is used in the rest of this paper) and the FT approach. In the latter, one calculates n-point vertices in a cut-off theory as a function of the bare couplings. By requiring the renormalized vertices to be independent of the cut-off, one obtains the evolution of the bare vertices with the cut-off [7,29]. Consider for example the renormalization of Γ^Q. At one-loop order, the renormalized two-particle vertex function can be written as

$$\Gamma^Q\Big|_R = \Gamma^Q + \delta\Gamma^Q\Big|_{ZS,ZS',BCS}, \qquad (65)$$

where $\delta\Gamma^Q$ is the correction calculated with a cut-off $\Lambda(t) = \Lambda_0 e^{-t}$. The dependence of Γ^Q on t is then obtained from the equation:

$$\frac{d}{dt}\Gamma^Q\Big|_R = \frac{d}{dt}\left(\Gamma^Q + \delta\Gamma^Q\Big|_{ZS,ZS',BCS}\right) = 0\,. \qquad (66)$$

Consider now the ZS' graph where the particle and the hole in the intermediate state have energies ϵ_1 and ϵ_2 with $|\epsilon_1| \neq |\epsilon_2|$. In the FT approach, this graph contributes to the RG flow when $\max(|\epsilon_1|, |\epsilon_2|) = \Lambda(t)$. Thus, as pointed out by Shankar [7], $d\Gamma^Q/dt$ is given by the sum of all graphs where one momentum is at the cut-off while the others are below. While the KW and FT approaches are clearly equivalent for the ZS graph for $\mathbf{Q} \to 0$ (since if one momentum of the particle-hole bubble is at the cut-off, then momentum conservation ensures that the other one is also at the cut-off), they in general differ. In the FT approach, it is clear that there is no need to consider three-, four- ... body interactions contrary to the KW approach. This also means that $\Gamma^Q(t)$ does not contain the same information in the KW and FT approaches. In particular, the low-energy effective action (14) (defined for a given cut-off Λ_0) is not the same in both approaches. The two-body interaction U_{σ_i} is different and the action contains n-body interactions ($n > 2$) in the KW approach. Notice that when one defines the low-energy effective action by (64) (*i.e.* by integrating out (in a functional sense) the high-energy degrees of freedom), one always uses (implicitly) the KW approach.

An interesting aspect of the FT approach is that the frustration of the interference between channels appears very naturally since it always involves the small parameter Λ/K_F because of phase space restriction [7]. For instance, the contribution of the ZS' graph to the renormalization of Γ^Q is of order Λ/K_F with respect to the one of the ZS graph. We pointed out in Section 4.3 that the interference between channels in a two-dimensional system with a circular Fermi surface is mainly determined by high-energy states. In the KW approach, its description requires the consideration of (at least) three-body interactions. In the FT approach, processes involving both low and high-energy states are integrated out in the early stage of the renormalization procedure. The interference left at low-energy is suppressed by the small parameter Λ/K_F. As discussed by several authors [7,9], this latter property can be used to control the perturbation expansion in a way similar to the $1/N$ expansion in statistical mechanics.

5 Beyond one-loop

In Section 3, we used the singularity arising at low temperature in the flow of $\Gamma_{\sigma_i}(\theta_1,\theta_2;\tilde{Q})$ for $\Lambda(t)\to 0$ to recover at one-loop order the results of the microscopic FLT. In this section, we show that we can solve in the same way the RG equations at all orders if we assume the existence of well defined quasi-particles (near the Fermi surface) and that the only singular contribution to the RG flow is due to the one-loop ZS graph (as shown in Sect. 3, the contribution of the ZS' graph to the flow of $\Gamma_{\sigma_i}(\theta_1,\theta_2;\tilde{Q})$ can be considered as regular if we are interested in quantities which involve all the values of $\theta_1-\theta_2$). Let us stress again that our aim is not to calculate FP quantities as a function of the bare parameters of the action, but to relate physical quantities with $\Gamma_{\sigma_i}^{\Omega^*}$. In the following, the action is not restricted to a two-body interaction (as in (14)), but contains also three-, four-, ... n-body interactions which are generated either by the RG process or the integration of high-energy degrees of freedom ($|\epsilon| > v_F\Lambda_0$) as discussed in Section 4.

We first consider the renormalization of the one-particle Green's function G. We note $z(t)$ the quasi-particle renormalization factor (and assume $z(t)>0$) and write the Green's function as $G(\tilde{K}) = (i\omega - v_F(t)k)^{-1}$ where the Fermi velocity $v_F(t)$ depends on the flow parameter t. This form, which assumes a proper rescaling of the fields, will be justified below. The integration of the degrees of freedom in the infinitesimal momentum shell modifies the Green's function:

$$G(\tilde{K})^{-1}\Big|_{t+dt} = i\omega - v_F(t)k - d\Sigma(k\omega)\,, \qquad (67)$$

where the self-energy correction $d\Sigma(k\omega)$ depends only on k because of rotational invariance. We analyze the self-energy following reference [7]. We Taylor expand $d\Sigma(k\omega)$ as follows:

$$d\Sigma(k\omega) = d\Sigma(00) + i\omega \frac{\partial d\Sigma(k\omega)}{\partial i\omega}\bigg|_{i\omega=k=0} + k\frac{\partial d\Sigma(k\omega)}{\partial k}\bigg|_{i\omega=k=0} + \cdots, \qquad (68)$$

where the dots denote irrelevant terms at tree-level. In the following, we neglect any effect associated with a finite lifetime of the quasi-particles, *i.e.* we assume that $d\Sigma(00)$, $\partial d\Sigma(k\omega)/\partial i\omega|_{i\omega=k=0}$, and $\partial d\Sigma(k\omega)/\partial k|_{i\omega=k=0}$ are real. This is justified when the scattering rate is much smaller than $\omega\sim T$. In a two-dimensional Fermi liquid, this latter is known to be of order $T^2\ln T$ (see Sect. 6) and can be neglected at low temperature. Ignoring irrelevant terms and $d\Sigma(00)$, which corresponds to a non essential shift of the chemical potential, we write the Green's function as

$$G(\tilde{K})^{-1}\Big|_{t+dt} = i\omega z^{-1}(dt) - v_F(t)kz_m^{-1}(dt)\,, \qquad (69)$$

where

$$\begin{aligned} z^{-1}(dt) &= 1 - \frac{\partial d\Sigma(k\omega)}{\partial i\omega}\bigg|_{i\omega=k=0}, \\ z_m^{-1}(dt) &= 1 + \frac{1}{v_F(t)}\frac{\partial d\Sigma(k\omega)}{\partial k}\bigg|_{i\omega=k=0} \\ &= 1 + \frac{m(t)}{K_F}\frac{\partial d\Sigma(k\omega)}{\partial k}\bigg|_{i\omega=k=0}. \end{aligned} \qquad (70)$$

We have introduced the effective mass $m(t) = K_F/v_F(t)$. Since the coefficients of $i\omega$ and k are modified by different parameters, no rescaling will keep the quadratic part of the action invariant. If one chooses to rescale the fields to keep the coefficient of $i\omega$ fixed at unity, *i.e.* $\psi^{(*)'} = [z(dt)]^{-\frac{1}{2}}\psi^{(*)}$, we obtain

$$G(\tilde{K})^{-1}\Big|_{t+dt} = i\omega - v_F(t)kz_m^{-1}(dt)z(dt)\,. \qquad (71)$$

The preceding equation shows that the quasi-particle form of the one-particle propagator, $G(\tilde{K}) = (i\omega - v_F(t)k)^{-1}$, is conserved if one ignores irrelevant terms and finite lifetime effects. The condition $z(t)>0$ then ensures the existence of well-defined quasi-particles. The rescaling of the fields modify the wave-function renormalization factor which becomes

$$z(t+dt) = z(dt)z(t)\,. \tag{72}$$

Moreover, from (71) one obtains the following RG equation for the Fermi velocity:

$$v_F(t+dt) = z(dt)z_m^{-1}(dt)v_F(t)\,, \tag{73}$$

or, equivalently,

$$m(t+dt) = z^{-1}(dt)z_m(dt)m(t)\,. \tag{74}$$

We now consider the renormalization of the two-particle vertex function Γ^Q (the renormalization of $\Gamma(\theta_1,\theta_2;\tilde{Q})$ is discussed below). We make the assumption that all graphs, except the one-loop ZS graph, are well-behaved for $\tilde{Q}\to 0$ and give a smooth contribution (with respect to $\Lambda(t)$) to the RG flow of $\Gamma_{\sigma_i}(\theta_1,\theta_2;\tilde{Q})$. Note that the same kind of assumption is made in the microscopic FLT. The RG flows of Γ^Q and Γ^Ω are then determined by (before the rescaling of the fields)

$$\frac{d\Gamma^Q_{\sigma_i}}{dt} = \left.\frac{d\Gamma^Q_{\sigma_i}}{dt}\right|_{ZS} + \left.\frac{d\Gamma^Q_{\sigma_i}}{dt}\right|_{ZS',BCS,2\,loops\ldots} \tag{75}$$

$$\frac{d\Gamma^\Omega_{\sigma_i}}{dt} = \left.\frac{d\Gamma^\Omega_{\sigma_i}}{dt}\right|_{ZS',BCS,2\,loops\ldots} = \left.\frac{d\Gamma^Q_{\sigma_i}}{dt}\right|_{ZS',BCS,2\,loops\ldots} \tag{76}$$

The contribution of the one-loop ZS graph (first term of the rhs of (75)), which gives the only singular contribution to the flow of Γ^Q, has been separated from the non-singular contributions. Equations (75, 76) can be combined to obtain

$$\frac{d\Gamma^Q_{\sigma_i}}{dt} = \frac{d\Gamma^\Omega_{\sigma_i}}{dt} + \left.\frac{d\Gamma^Q_{\sigma_i}}{dt}\right|_{ZS}. \tag{77}$$

According to our assumption, $\Gamma^\Omega_{\sigma_i}$ is a non-singular function of $\Lambda(t)$ since it does not receive any contribution from the one-loop ZS graph (see Sect. 3.1). Using the results of Section 3.1, we have

$$\left.\frac{d\Gamma^Q_{\sigma_i}(l)}{dt}\right|_{ZS} = -\frac{N(0)\beta_R}{\cosh^2(\beta_R)}\sum_{\sigma,\sigma'}\Gamma^Q_{\sigma_1\sigma',\sigma\sigma_4}(l)\Gamma^Q_{\sigma\sigma_2,\sigma_3\sigma'}(l)\,, \tag{78}$$

where $N(0)=K_F/2\pi v_F(t)$ and $\beta_R = v_F(t)\beta\Lambda(t)/2$ since the one-particle Green's function has the form $G(\tilde{K}) = (i\omega - v_F(t)k)^{-1}$. Equations (77) can be written in the compact form

$$\Gamma^Q_{\sigma_i}(l,t+dt) = z^{(\Gamma)}_{\sigma_i}(l,dt)\Gamma^Q_{\sigma_i}(l,t)\,. \tag{79}$$

After the rescaling of the fields, $\psi^{(*)'} = [z(dt)]^{-\frac{1}{2}}\psi^{(*)}$, we obtain the RG equation

$$\Gamma^Q_{\sigma_i}(l,t+dt) = z(dt)^2 z^{(\Gamma)}_{\sigma_i}(l,dt)\Gamma^Q_{\sigma_i}(l,t)\,. \tag{80}$$

The RG equations at all orders are given by (72, 73, 80) which (together with the assumption that Γ^Ω is a smooth function of $\Lambda(t)$) constitute the basis of FLT in the RG language. $z(t)$ and $v_F(t)$ are determined by the dependence of Γ on $\tilde{Q}$ through the self-energy $d\Sigma(k\omega)$. Since the singularity in the flow of Γ is weakened at finite $\tilde{Q}$ (only the flow of Γ^Q presents a singularity $\sim\delta(\Lambda)$ for $T\to 0$), $z(t)$ and $v_F(t)$ are smooth functions of the cut-off $\Lambda(t)$. At low temperature and for $\Lambda(t)\lesssim T/v_F$, they can therefore be approximated by their FP values: $z(t)|_{\Lambda(t)\sim T/v_F}\simeq z^*$ and $v_F(t)|_{\Lambda(t)\sim T/v_F}\simeq v_F^*$. Consequently, we have $z(dt) = z_m(dt)\simeq 1$ for $\Lambda(t)\lesssim T/v_F$. Since the contribution of the ZS graph becomes significantly different from zero only when $\Lambda(t)\lesssim T/v_F$, Equations (72, 73, 80) reduce to (77) with $z(t)$ and $v_F(t)$ equal to their FP values. Equation (77) is similar to (29) and can be solved in the same way which leads again to (38). The expression of Γ^{Q^*} as a function of the Landau parameters is recovered if these latter are defined by $f_{\sigma_i}(\theta) = \Gamma^{\Omega^*}_{\sigma_i}(\theta)$. Since the $\psi^{(*)}$'s have been rescaled at each step of the renormalization, we eventually come to

$$f_{\sigma_i}(\theta) = z^{*2}\Gamma^{\Omega^*}_{\sigma_i}(\theta)\,, \tag{81}$$

where $\Gamma^{\Omega^*}_{\sigma_i}(\theta)$ now refers to the bare fermions. Equation (81) defines the Landau function in the RG approach.

The renormalization of $\Gamma_{\sigma_i}(\theta_1,\theta_2;\tilde{Q})$ can be discussed in the same way. However, for finite Q, the resolution of the RG equations becomes less accurate because the singularity $(\beta/4)\cosh^{-2}(\beta v_F k/2)|_{T\to 0} = \delta(v_F k)$ is weakened (see Sect. 3.2). In particular, since $z(t)$ and $v_F(t)$ depend on $\Gamma_{\sigma_i}(\theta_1,\theta_2;\tilde{Q})$ for finite $\tilde{Q}$, the replacement $z(t)\to z^*$ and $v_F(t)\to v_F^*$ in the RG equation for $\Gamma_{\sigma_i}(\theta_1,\theta_2;\tilde{Q})$ is not exact any more. Equations (72, 73, 80) should be considered together if they were to be solved exactly. This would lead to complicated coupled equations for $\Gamma_{\sigma_i}(\theta_1,\theta_2;\tilde{Q})$, $z(t)$ and $v_F(t)$.

Thus, the relations between Γ^{Ω^*} and physical quantities obtained at one-loop order (Sect. 3) are essentially unchanged by higher order contributions although of course the expression of Γ^{Ω^*} as a function of the microscopic parameters is changed. (The only change is that the "bare" quantities v_F and z_{Λ_0} are replaced by their FP values v_F^* and z^*.) This is a direct consequence of our assumption according to which the only singular contribution to the RG flow comes from the one-loop ZS graph.

6 Quasi-particle properties

In this section, we calculate the quasi-particle life-time and renormalization factor. As discussed in Section 4, the self-energy is obtained from the one-loop diagram (since all higher order diagrams vanish). Thus, one possibility to obtain the self-energy would be to introduce the expression of $\Gamma_{\sigma_i}(\theta_1,\theta_2;\tilde{Q})$ obtained in Section 3.2 in the one-loop diagram. In practice, such a procedure turns out to be very difficult. We follow in this section an alternative method introduced by Bourbonnais and Caron [15].

These authors have shown how the KW RG approach can be modified in order to easily obtain the two-loop corrections and therefore the self-energy at this order. The main idea is to make a distinction between band momenta and transfer momenta. At each step of the renormalization, one integrates the fermion field degrees of freedom corresponding to a band momentum in the infinitesimal shell $\Lambda(t)e^{-dt} \leq |k| \leq \Lambda(t)$ to be integrated out. No cut-off is imposed on the transfer momenta which are let free. In principle, it is necessary to impose an additional constraint on the transfer momenta in order to ensure that every degree of freedom is integrated once and only once (at a given order). In practice, one can ignore this constraint which is automatically taken into account through the Fermi factors. The application of this method to one-dimensional systems has been very successful since it has allowed to recover the results of the multiplicative renormalization group approach [14].

We consider the simple case where only F_0^s and F_0^a are non zero. As discussed at the end of Section 5, the calculation of quantities involving $\Gamma(\theta_1,\theta_2;\tilde{Q})$ with finite Q requires to consider on the same footing the renormalization of $z(t)$ and $v_F(t)$. For simplicity, we assume that all quantities can be calculated with the FP form of the one-particle propagator, $G(\tilde{K}) = (i\omega - v_F^* k)^{-1}$, ignoring any finite life-time as can be justified at low temperature. Moreover, we do not take into account the effect of the rescaling of the fields, $\psi^{(*)'} = [z(dt)]^{-\frac{1}{2}}\psi^{(*)}$, on the RG equation of $\Gamma(\theta_1,\theta_2;\tilde{Q})$. These approximations are expected to be correct as long as one considers only small transfer momentum $\mathbf{Q}$ in the diagram for the self-energy (Fig. 5b). This can be achieved by introducing a cut-off Q_c for the transfer momenta. We can then use the results of Sections 3.2 and 3.3 which hold in the limit of small Q. The cut-off Q_c is also introduced in the microscopic FLT where the self-energy is usually obtained by dressing the particle line with a charge or spin fluctuation propagator with $Q < Q_c$ [30].

The self-energy correction is given by (Fig. 5b)

$$\begin{aligned} d\Sigma(k\omega) &= -\frac{T^2}{2\nu^2}\sum_{\tilde{Q}}\sum_{\tilde{K}'}{}^{'}\sum_{\sigma_1\sigma_2\sigma_3} G(\tilde{K}+\tilde{Q}) \\ &\quad\times G(\tilde{K}')G(\tilde{K}'-\tilde{Q})\Gamma_{\sigma\sigma_1,\sigma_2\sigma_3}(\tilde{Q})\Gamma_{\sigma_3\sigma_2,\sigma_1\sigma}(\tilde{Q}) \\ &= -\frac{1}{4N^2(0)}\frac{T^2}{\nu^2}\sum_{\tilde{Q}}\sum_{\tilde{K}'}{}^{'} G(\tilde{K}+\tilde{Q})G(\tilde{K}') \\ &\quad\times G(\tilde{K}'-\tilde{Q})\Big(A^2(\tilde{Q})+3B^2(\tilde{Q})\Big), \end{aligned} \tag{82}$$

where $\Gamma(\tilde{Q})$, $A(\tilde{Q})$ and $B(\tilde{Q})$ are given by (46). In the above equation, the sum over the transfer momentum $\mathbf{Q}$ is free (with $Q < Q_c$) while the sum over the band momentum $\mathbf{K}'$ is restricted to $\Lambda(t)e^{-dt} \leq |k'| \leq \Lambda(t)$. In the following we consider only the charge part of the interaction (*i.e.* we put $B = 0$). $d\Sigma(k\omega)$ can be expressed as a function of the density-density response function obtained in Section 3.3. Using (46, 53, 54), we have

$$d\chi(\tilde{Q}) = -2\frac{T}{\nu}\sum_{\tilde{K}'}{}^{'} G(\tilde{K}')G(\tilde{K}'+\tilde{Q})\frac{A^2(\tilde{Q})}{F_0^{s\,2}}\,. \tag{83}$$

We therefore obtain

$$d\Sigma(k\omega) = \frac{T}{2\nu}\sum_{\tilde{Q}} G(\tilde{K}+\tilde{Q}) f_0^{s\,2} d\chi(\tilde{Q})\,, \tag{84}$$

where $f_0^s = F_0^s/2N(0)$. Integrating this equation, we obtain the FP value of the self-energy:

$$\Sigma^*(k\omega) = \frac{T}{2\nu}\sum_{\tilde{Q}} G(\tilde{K}+\tilde{Q}) f_0^{s\,2}\chi^*(\tilde{Q})\,. \tag{85}$$

The FP value $\chi^*(\tilde{Q})$ of the density-density response function is determined by (56) [31]. $\Sigma^*(k\omega)$ is the self-energy one would obtain in perturbation theory by dressing the particle line with one density fluctuation propagator $\chi^*(\tilde{Q})$. It is usually in this way that the quasi-particle properties are calculated in the microscopic FLT [30].

The quasi-particle life-time is obtained from the retarded part of the self-energy:

$$\frac{1}{\tau} \sim -Im\left\{\Sigma^*(k\omega)\Big|_{i\omega\to\omega+i0^+}\right\}. \tag{86}$$

For states close to the Fermi surface ($k,\omega \to 0$), this equation yields the standard result for a two-dimensional Fermi liquid: $\tau^{-1} \sim T^2 \ln T$ [32,33,30]. Since $\tau^{-1} \ll T$ at low temperature, the neglect of τ^{-1} in the single-particle propagator $G(\tilde{K})$ during the renormalization is justified.

From (70,72), we obtain

$$d\ln(z) = Re\left[\frac{\partial d\Sigma(k\omega)}{\partial i\omega}\bigg|_{i\omega=k=0}\right], \tag{87}$$

which yields

$$\begin{aligned} z^* &= z_{\Lambda_0}\exp\left\{Re\left[\frac{\partial \Sigma^*(k\omega)}{\partial i\omega}\bigg|_{i\omega=k=0}\right]\right\} \\ &= z_{\Lambda_0}\exp\left\{-\frac{T}{2\nu}\sum_{\tilde{Q}}\frac{f_0^{s\,2}\chi^*(\tilde{Q})}{(i\Omega - v_F^*\hat{\mathbf{K}}\cdot\mathbf{Q})^2}\right\}. \end{aligned} \tag{88}$$

The preceding equation agrees with the result obtained from two-dimensional bosonization [34] or Ward Identities [35]. In the case of short-range interactions, the exponential factor in (88) gives only a small correction to z_{Λ_0} which can be ignored for $\Lambda_0 \ll K_F$ [34,35]. This ensures the existence of quasi-particles ($z^* > 0$) for any non-vanishing value of z_{Λ_0}.

7 Conclusion

We have shown in this paper how FLT results can be derived in a RG approach. While it seems difficult to calculate physical quantities as a function of the bare parameters of the low-energy effective action, it appears possible (and quite natural) to relate them to the FP value of the Ω-limit of the two-particle vertex function which therefore determines the Landau parameters. This result follows from the assumption that the ZS graph is the only singular graph in the limit $\tilde{Q} \to 0$ and also the only one which is dominated by the integration of low-energy states. These assumptions seem reasonable in cases where the "quantum" degrees of freedom ($|\epsilon| \gtrsim T$) do not lead to any instability (such as superconductivity or charge/spin density wave). As we pointed out, these two assumptions also underlie the standard diagrammatic derivation of FLT.

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