

INFRARED BEHAVIOR OF INTERACTING BOSONS AT ZERO TEMPERATURE

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We review the infrared behavior of interacting bosons at zero temperature. After a brief discussion of the Bogoliubov approximation and the breakdown of perturbation theory due to infrared divergences, we show how the non-perturbative renormalization group enables to obtain the exact infrared behavior of the correlation functions.

Keywords: Boson systems; renormalization group methods.

1. Introduction

Many of the predictions of the Bogoliubov theory of superfluidity¹ have been confirmed experimentally, in particular in ultracold atomic gases.^{2,3} Nevertheless, a clear understanding of the infrared behavior of interacting bosons at zero temperature has remained a challenging theoretical issue for a long time. Early attempts to go beyond the Bogoliubov theory have revealed a singular perturbation theory plagued by infrared divergences due to the presence of the Bose–Einstein condensate and the Goldstone mode.^{4–6} In the 1970s, Nepomnyashchii and Nepomnyashchii proved that the anomalous self-energy vanishes at zero frequency and momentum in dimension $d \leq 3$.⁷ This exact result shows that the Bogoliubov approximation, where the linear spectrum and the superfluidity rely on a finite value of the anomalous self-energy, breaks down at low energy. As realized later on,^{8,9} the singular perturbation theory is a direct consequence of the coupling between transverse and longitudinal fluctuations and reflects the divergence of the longitudinal susceptibility — a general phenomenon in systems with a continuous broken symmetry.¹⁰

In this paper, we review the infrared behavior of interacting bosons. A more detailed discussion together with a comparison to the classical $O(N)$ model can be found in Ref. 11. In Sec. 2, we briefly review the Bogoliubov theory and the appearance of infrared divergences in perturbation theory. We introduce the Ginzburg momentum scale p_G signaling the breakdown of the Bogoliubov approximation. In Sec. 3, we discuss the non-perturbative renormalization group (NPRG) and show how it enables to obtain the exact infrared behavior of the

normal and anomalous single-particle propagators without encountering infrared divergences.^{12–20}

2. Perturbation Theory and Breakdown of the Bogoliubov Approximation

We consider interacting bosons at zero temperature with the (Euclidean) action

$$S = \int dx \left[\psi^* \left(\partial_\tau - \mu - \frac{\nabla^2}{2m} \right) \psi + \frac{g}{2} (\psi^* \psi)^2 \right], \quad (1)$$

where $\psi(x)$ is a bosonic (complex) field, $x = (\mathbf{r}, \tau)$, and $\int dx = \int_0^\beta d\tau \int d^d r$. $\tau \in [0, \beta]$ is an imaginary time, $\beta \rightarrow \infty$ the inverse temperature, and μ denotes the chemical potential. The interaction is assumed to be local in space and the model is regularized by a momentum cut-off Λ . We consider a space dimension $d > 1$.

Introducing the two-component field

$$\Psi(p) = \begin{pmatrix} \psi(p) \\ \psi^*(-p) \end{pmatrix}, \quad \Psi^\dagger(p) = (\psi^*(p), \psi(-p)), \quad (2)$$

(with $p = (\mathbf{p}, i\omega)$ and ω a Matsubara frequency), the one-particle (connected) propagator becomes a 2×2 matrix whose inverse in Fourier space is given by

$$\begin{pmatrix} i\omega + \mu - \epsilon_{\mathbf{p}} - \Sigma_n(p) & -\Sigma_{\text{an}}(p) \\ -\Sigma_{\text{an}}^*(p) & -i\omega + \mu - \epsilon_{\mathbf{p}} - \Sigma_n(-p) \end{pmatrix}, \quad (3)$$

where Σ_n and Σ_{an} are the normal and anomalous self-energies, respectively, and $\epsilon_{\mathbf{p}} = \mathbf{p}^2/2m$. If we choose the order parameter $\langle \psi(x) \rangle = \sqrt{n_0}$ to be real (with n_0 the condensate density), then the anomalous self-energy $\Sigma_{\text{an}}(p)$ is real.

2.1. Bogoliubov approximation

The Bogoliubov approximation is a Gaussian fluctuation theory about the saddle point solution $\psi(x) = \sqrt{n_0} = \sqrt{\mu/g}$. It is equivalent to a zero-loop calculation of the self-energies,^{21,22}

$$\Sigma_n^{(0)}(p) = 2gn_0, \quad \Sigma_{\text{an}}^{(0)}(p) = gn_0. \quad (4)$$

This yields the (connected) propagators

$$\begin{aligned} G_n^{(0)}(p) &= -\langle \psi(p) \psi^*(p) \rangle_c = \frac{-i\omega - \epsilon_{\mathbf{p}} - gn_0}{\omega^2 + E_{\mathbf{p}}^2}, \\ G_{\text{an}}^{(0)}(p) &= -\langle \psi(p) \psi(-p) \rangle_c = \frac{gn_0}{\omega^2 + E_{\mathbf{p}}^2}, \end{aligned} \quad (5)$$

where $E_{\mathbf{p}} = [\epsilon_{\mathbf{p}} + 2gn_0]^{1/2}$ is the Bogoliubov quasi-particle excitation energy. When $|\mathbf{p}|$ is larger than the healing momentum $p_c = (2gmn_0)^{1/2}$, the spectrum $E_{\mathbf{p}} \simeq \epsilon_{\mathbf{p}} + gn_0$ is particle-like, whereas it becomes sound-like for $|\mathbf{p}| \ll p_c$ with a

velocity $c = \sqrt{gn_0/m}$. In the weak-coupling limit, $n_0 \simeq \bar{n}$ (\bar{n} is the mean boson density) and p_c can equivalently be defined as $p_c = (2gm\bar{n})^{1/2}$. In the Bogoliubov approximation, the occurrence of a linear spectrum at low energy (which implies superfluidity according to Landau's criterion) is due to $\Sigma_{\text{an}}(p = 0)$ being non-zero.

2.2. Infrared divergences and the Ginzburg scale

The lowest-order (one-loop) correction $\Sigma^{(1)}$ to the Bogoliubov result $\Sigma^{(0)}$ is divergent for $d \leq 3$. Retaining only the divergent part, we obtain

$$\Sigma_n^{(1)}(p) \simeq \Sigma_{\text{an}}^{(1)}(p) \simeq -2 \frac{g^4 n_0^3}{c^3} A_{d+1} \left(\mathbf{p}^2 + \frac{\omega^2}{c^2} \right)^{(d-3)/2}, \tag{6}$$

if $d < 3$ and

$$\Sigma_n^{(1)}(p) \simeq \Sigma_{\text{an}}^{(1)}(p) \simeq -\frac{g^4 n_0^3}{c^3} A_4 \ln \left(\frac{p_c^2}{\mathbf{p}^2 + \omega^2/c^2} \right), \tag{7}$$

if $d = 3$, where

$$A_d = \begin{cases} -\frac{2^{1-d} \pi^{1-d/2}}{\sin(\pi d/2)} \frac{\Gamma(d/2)}{\Gamma(d-1)} & \text{if } d < 4, \\ \frac{1}{8\pi^2} & \text{if } d = 4. \end{cases} \tag{8}$$

We can estimate the characteristic (Ginzburg) momentum scale p_G below which the Bogoliubov approximation breaks down from the condition $|\Sigma_n^{(1)}(p)| \sim \Sigma_n^{(0)}(p)$ or $|\Sigma_{\text{an}}^{(1)}(p)| \sim \Sigma_{\text{an}}^{(0)}(p)$ for $|\mathbf{p}| = p_G$ and $|\omega| = cp_G$,

$$p_G \sim \begin{cases} (A_{d+1} g m p_c)^{1/(3-d)} & \text{if } d < 3, \\ p_c \exp \left(-\frac{1}{A_4 g m p_c} \right) & \text{if } d = 3. \end{cases} \tag{9}$$

This result can be rewritten as

$$p_G \sim \begin{cases} p_c (A_{d+1} \tilde{g}^{d/2})^{1/(3-d)} & \text{if } d < 3, \\ p_c \exp \left(-\frac{1}{A_4 \sqrt{2} \tilde{g}^{3/2}} \right) & \text{if } d = 3, \end{cases} \tag{10}$$

where

$$\tilde{g} = gm\bar{n}^{1-2/d} \sim \left(\frac{p_c}{\bar{n}^{1/d}} \right)^2, \tag{11}$$

is the dimensionless coupling constant obtained by comparing the mean interaction energy per particle $g\bar{n}$ to the typical kinetic energy $1/m\bar{r}^2$ where $\bar{r} \sim \bar{n}^{-1/d}$ is the mean distance between particles.²³ A superfluid is weakly correlated if $\tilde{g} \ll 1$, i.e. $p_G \ll p_c \ll \bar{n}^{1/d}$ (the characteristic momentum scale $\bar{n}^{1/d}$ does not however play any role in the weak-coupling limit). In this case, the Bogoliubov theory applies

to a large part of the spectrum where the dispersion is linear (i.e. $|\mathbf{p}| \lesssim p_c$) and breaks down only at very small momenta $|\mathbf{p}| \lesssim p_G \ll p_c$. When the dimensionless coupling \tilde{g} becomes of order unity, the three characteristic momentum scales $p_G \sim p_c \sim \bar{n}^{1/d}$ become of the same order. The momentum range $[p_G, p_c]$ where the linear spectrum can be described by the Bogoliubov theory is then suppressed. We expect the strong-coupling regime $\tilde{g} \gg 1$ to be governed by a single characteristic momentum scale, namely $\bar{n}^{1/d}$.^a

2.3. Vanishing of the anomalous self-energy

The exact values of $\Sigma_n(p = 0)$ and $\Sigma_{\text{an}}(p = 0)$ can be obtained using the U(1) symmetry of the action, i.e. the invariance under the field transformation $\psi(x) \rightarrow e^{i\theta}\psi(x)$ and $\psi^*(x) \rightarrow e^{-i\theta}\psi^*(x)$. On the one hand, the self-energies satisfy the Hugenholtz–Pines theorem,⁴

$$\Sigma_n(p = 0) - \Sigma_{\text{an}}(p = 0) = \mu. \tag{12}$$

On the other hand, the anomalous self-energy vanishes,

$$\Sigma_{\text{an}}(p = 0) = 0. \tag{13}$$

The last result was first proven by Nepomnyashchii and Nepomnyashchii.^{7,11,20} It shows that the Bogoliubov theory, where the linear spectrum and the superfluidity rely on a finite value of the anomalous self-energy, breaks down at low energy in agreement with the conclusions drawn from perturbation theory (Sec. 2.2).

3. The non-perturbative RG

The NPRG enables us to circumvent the difficulties of perturbation theory and derive the correlation functions in the low-energy limit.^{12–20} The strategy of the NPRG is to build a family of theories indexed by a momentum scale k such that fluctuations are smoothly taken into account as k is lowered from the microscopic scale Λ down to 0.^{25,26} This is achieved by adding to the action (1) an infrared regulator term

$$\Delta S_k[\psi^*, \psi] = \sum_p \psi^*(p) R_k(p) \psi(p). \tag{14}$$

The main quantity of interest is the so-called average effective action

$$\Gamma_k[\phi^*, \phi] = -\ln Z_k[J^*, J] + \sum_p [J^*(p)\phi(p) + \text{c.c.}] - \Delta S_k[\phi^*, \phi], \tag{15}$$

^aNote however that in the strong-coupling limit (i.e. $\bar{n}a^3 \gg 1$ in three dimensions, with a the s -wave scattering length), the bound states of the interaction potential cannot be ignored. As a result, stability is not guaranteed and the system is at best in a quasistable state. The strong-coupling limit of interacting bosons can nevertheless be reached in the Bose–Hubbard model.²⁹ See Ref. 24.

defined as a modified Legendre transform of $-\ln Z_k[J^*, J]$ which includes the subtraction of $\Delta S_k[\phi^*, \phi]$. J denotes a complex external source that couples linearly to the boson field ψ and $\phi(x) = \langle \psi(x) \rangle$ is the superfluid order parameter. The cut-off function R_k is chosen such that at the microscopic scale Λ it suppresses all fluctuations, so that the mean-field approximation $\Gamma_\Lambda[\phi^*, \phi] = S[\phi^*, \phi]$ becomes exact. The effective action of the original model (1) is given by $\Gamma_{k=0}$ provided that $R_{k=0}$ vanishes. For a generic value of k , the cut-off function $R_k(p)$ suppresses fluctuations with momentum $|\mathbf{p}| \lesssim k$ and frequency $|\omega| \lesssim ck$ but leaves those with $|\mathbf{p}|, |\omega|/c \gtrsim k$ unaffected ($c \equiv c_k$ is the velocity of the Goldstone mode). The dependence of the average effective action on k is given by Wetterich's equation²⁷

$$\partial_t \Gamma_k[\phi^*, \phi] = \frac{1}{2} \text{Tr} \{ \dot{R}_k (\Gamma_k^{(2)}[\phi^*, \phi] + R_k)^{-1} \}, \quad (16)$$

where $t = \ln(k/\Lambda)$ and $\dot{R}_k = \partial_t R_k$. $\Gamma_k^{(2)}[\phi^*, \phi]$ denotes the second-order functional derivative of $\Gamma_k[\phi]$. In Fourier space, the trace involves a sum over momenta and frequencies as well as the internal index of the ϕ field.

3.1. Derivative expansion and infrared behavior

Because of the regulator term ΔS_k , the vertices $\Gamma_k^{(n)}(p_1, \dots, p_n)$ are smooth functions of momenta and frequencies and can be expanded in powers of \mathbf{p}_i^2/k^2 and ω_i^2/c^2k^2 . Thus if we are interested only in the low-energy properties, we can use a derivative expansion of the average effective action.^{25,26} In the following, we consider the ansatz^b

$$\Gamma_k[\phi^*, \phi] = \int dx \left[\phi^* \left(Z_{C,k} \partial_\tau - V_{A,k} \partial_\tau^2 - \frac{Z_{A,k}}{2m} \nabla^2 \right) \phi + \frac{\lambda_k}{2} (n - n_{0,k})^2 \right], \quad (17)$$

where $n = |\phi|^2$. $n_{0,k}$ denotes the condensate density in the equilibrium state. We have introduced a second-order time derivative term. Although not present in the initial average effective action Γ_Λ , we shall see that this term plays a crucial role when $d \leq 3$.^{14,16}

In a broken U(1) symmetry state with real order parameter $\phi = \sqrt{n_0}$, the normal and anomalous self-energies are given by

$$\begin{aligned} \Sigma_{k,n}(p) &= \mu + V_{A,k} \omega^2 + (1 - Z_{C,k}) i\omega - (1 - Z_{A,k}) \epsilon_{\mathbf{p}} + \lambda_k n_{0,k}, \\ \Sigma_{k,\text{an}}(p) &= \lambda_k n_{0,k}. \end{aligned} \quad (18)$$

These expressions imply the existence of a sound mode with velocity

$$c_k = \left(\frac{Z_{A,k}/2m}{V_{A,k} + Z_{C,k}^2/2\lambda_k n_{0,k}} \right)^{1/2}. \quad (19)$$

^bOne could consider a more general ansatz where $Z_{A,k}(n)$, $Z_{C,k}(n)$ and $V_{A,k}(n)$ are functions of the condensate density n and the full effective potential $U_k(n)$ is retained rather than expanded to quadratic order about $n_{0,k}$. In the superfluid phase, $U_k(n)$ is convex but otherwise the conclusions regarding the infrared behavior are unchanged.

At the initial stage of the flow, $Z_{A,\Lambda} = Z_{C,\Lambda} = 1$, $V_{A,\Lambda} = 0$, $\lambda_\Lambda = g$ and $n_{0,\Lambda} = \mu/g$, which reproduces the results of the Bogoliubov approximation. A crucial property of the RG flow is that

$$\lambda_k \sim k^{3-d} \tag{20}$$

vanishes with k when $d \leq 3$. Equation (20) follows from the numerical solution of the RG equations, but can also be anticipated from the expected singular behavior of the longitudinal propagator.¹¹

The parameters $Z_{A,k}$, $Z_{C,k}$ and $V_{A,k}$ can be related to thermodynamic quantities using Ward identities,^{5,13,18}

$$\begin{aligned} n_{s,k} &= Z_{A,k} n_{0,k} = \bar{n}_k, \\ V_{A,k} &= - \left. \frac{1}{2n_{0,k}} \frac{\partial^2 U_k}{\partial \mu^2} \right|_{n_{0,k}}, \\ Z_{C,k} &= - \left. \frac{\partial^2 U_k}{\partial n \partial \mu} \right|_{n_{0,k}} = \lambda_k \frac{dn_{0,k}}{d\mu}, \end{aligned} \tag{21}$$

where \bar{n}_k is the mean boson density and $n_{s,k}$ the superfluid density. Here we consider the effective potential U_k as a function of the two independent variables n and μ . The first equation in (21) states that in a Galilean invariant superfluid at zero temperature, the superfluid density is given by the full density of the fluid.⁵ The equations in (21) also imply that the Goldstone mode velocity c_k coincides with the macroscopic sound velocity,^{5,13,18} i.e.

$$\frac{d\bar{n}_k}{d\mu} = \frac{\bar{n}_k}{mc_k^2}. \tag{22}$$

Since thermodynamic quantities, including the condensate “compressibility” $dn_{0,k}/d\mu$ should remain finite in the limit $k \rightarrow 0$, we deduce from (21) that $Z_{C,k} \sim \lambda_k \sim k^{3-d}$ vanishes in the infrared limit, and

$$\lim_{k \rightarrow 0} c_k = \lim_{k \rightarrow 0} \left(\frac{Z_{A,k}}{2mV_{A,k}} \right)^{1/2}. \tag{23}$$

Both $Z_{A,k} = \bar{n}_k/n_{0,k}$ and the macroscopic sound velocity c_k being finite at $k = 0$, $V_{A,k}$ (which vanishes in the Bogoliubov approximation) takes a non-zero value when $k \rightarrow 0$.

The suppression of $Z_{C,k}$, together with a finite value of $V_{A,k=0}$ shows that the average effective action (17) exhibits a “relativistic” invariance in the infrared limit and therefore becomes equivalent to that of the classical O(2) model in dimensions $d + 1$. In the ordered phase, the coupling constant of this model vanishes as $\lambda_k \sim k^{4-(d+1)}$,¹¹ which agrees with (20). For $k \rightarrow 0$, the existence of a linear spectrum is due to the relativistic form of the average effective action (rather than a non-zero value of $\lambda_k n_{0,k}$ as in the Bogoliubov approximation).

To obtain the $k = 0$ limit of the propagators (at fixed p), one should in principle stop the flow when $k \sim \sqrt{\mathbf{p}^2 + \omega^2/c^2}$.¹⁸ Since thermodynamic quantities are not expected to flow in the infrared limit, they can be approximated by their $k = 0$ values. Making use of the Ward identities (21), we deduce the exact infrared behavior of the normal and anomalous propagators (at $k = 0$),¹⁸

$$G_n(p) = -\frac{n_0 m c^2}{\bar{n}} \frac{1}{\omega^2 + c^2 \mathbf{p}^2} - \frac{m c^2}{\bar{n}} \frac{d n_0}{d \mu} \frac{i \omega}{\omega^2 + c^2 \mathbf{p}^2} - \frac{1}{2} G_{\parallel}(p),$$

$$G_{\text{an}}(p) = \frac{n_0 m c^2}{\bar{n}} \frac{1}{\omega^2 + c^2 \mathbf{p}^2} - \frac{1}{2} G_{\parallel}(p),$$
(24)

where

$$G_{\parallel}(p) = \frac{1}{2 n_0 C (\omega^2 + c^2 \mathbf{p}^2)^{(3-d)/2}},$$
(25)

is the propagator of the longitudinal fluctuations. The constant C follows from the replacement $\lambda_k \rightarrow C (\omega^2 + c^2 \mathbf{p}^2)^{(3-d)/2}$. The leading terms in (24) agree with the results of Gavoret and Nozières.⁵ The contribution of the diverging longitudinal correlation function was first identified by Nepomnyashchii and Nepomnyashchii.^{8,9}

3.2. RG flows

The conclusions of the preceding section can be obtained more rigorously from the RG equation (16) satisfied by the average effective action. The flow of λ_k , $Z_{C,k}$ and $V_{A,k}$ is shown in Fig. 1 for a two-dimensional system in the weak-coupling limit. We clearly see that the Bogoliubov approximation breaks down at a characteristic momentum scale $p_G \sim \sqrt{(gm)^3 \bar{n}}$. In the Goldstone regime $k \ll p_G$, we find that both λ_k and $Z_{C,k}$ vanish linearly with k in agreement with the conclusions of

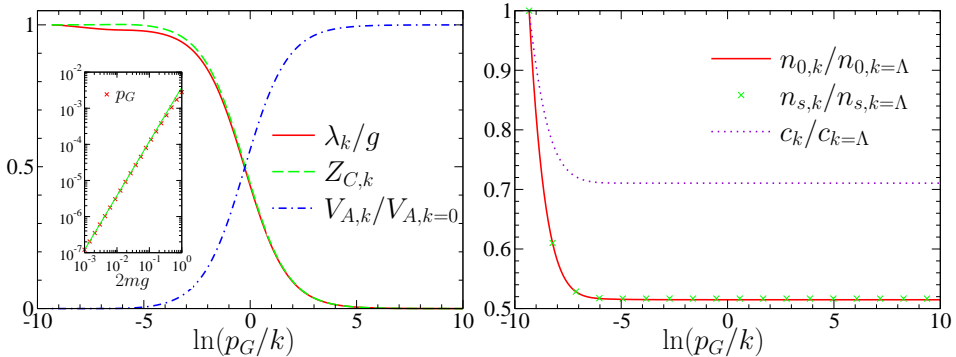


Fig. 1. (Color online) (Left panel) λ_k , $Z_{C,k}$ and $V_{A,k}$ versus $\ln(p_G/k)$ where $p_G = \sqrt{(gm)^3 \bar{n}}/4\pi$ for $\bar{n} = 0.01$, $2mg = 0.1$ and $d = 2$ [$\ln(p_G/p_c) \simeq -5.87$]. The inset shows p_G versus $2mg$ obtained from the criterion $V_{A,p_G} = V_{A,k=0}/2$ [the Green solid line is a fit to $p_G \sim (2mg)^{3/2}$]. (Right panel) Condensate density $n_{0,k}$, superfluid density $n_{s,k}$ and Goldstone mode velocity c_k versus $\ln(p_G/k)$.

Sec. 3.1. Furthermore, $V_{A,k}$ takes a finite value in the limit $k \rightarrow 0$ in agreement with the limiting value (23) of the Goldstone mode velocity. Figure 1 also shows the behavior of the condensate density $n_{0,k}$, the superfluid density $n_{s,k} = Z_{A,k}n_{0,k}$ and the velocity c_k . Since $Z_{A,k=0} \simeq 1.004$, the mean boson density $\bar{n}_k = n_{s,k}$ is nearly equal to the condensate density $n_{0,k}$. Apart from a slight variation at the beginning of the flow, $n_{0,k}$, $n_{s,k} = Z_{A,k}n_{0,k}$ and c_k do not change with k . In particular, they are not sensitive to the Ginzburg scale p_G . This result is quite remarkable for the Goldstone mode velocity c_k , whose expression (19) involves the parameters λ_k , $Z_{C,k}$ and $V_{A,k}$, which all strongly vary when $k \sim p_G$. These findings are a nice illustration of the fact that the divergence of the longitudinal susceptibility does not affect local gauge invariant quantities.^{13,18}

4. Conclusion

Interacting bosons at zero temperature are characterized by two momentum scales: the healing (or hydrodynamic) scale p_c and the Ginzburg scale p_G . p_G sets the scale at which the Bogoliubov approximation breaks down. For momenta $|\mathbf{p}| \ll p_c$, it is possible to use a hydrodynamic description in terms of density and phase variables. This description allows one to compute the correlation functions without encountering infrared divergences.^{11,28} In this paper, we have reviewed another approach, based on the NPRG, which enables one to describe the system at all energy scales and yields the exact infrared behavior of the single-particle propagator. A nice feature of the NPRG is that it can be used to study models of strongly-correlated bosons such as the Bose–Hubbard model.²⁹

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