Infrared behavior in systems with a broken continuous symmetry: Classical O(N) model versus interacting bosons

N. Dupuis
Laboratoire de Physique Théorique de la Matière Condensée, CNRS–UMR 7600, Université Pierre et Marie Curie, 4 Place Jussieu, F-75252 Paris Cedex 05, France

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In systems with a spontaneously broken continuous symmetry, the perturbative loop expansion is plagued by infrared divergences due to the coupling between transverse and longitudinal fluctuations. As a result, the longitudinal susceptibility diverges and the self-energy becomes singular at low energy. We study the crossover from the high-energy Gaussian regime, where perturbation theory remains valid, to the low-energy Goldstone regime characterized by a diverging longitudinal susceptibility. We consider both the classical linear O (N) model and interacting bosons at zero temperature, using a variety of techniques: perturbation theory, hydrodynamic approach (i.e., for bosons, Popov's theory), large-N limit, and nonperturbative renormalization group. We emphasize the essential role of the Ginzburg momentum scale p_G, below which the perturbative approach breaks down. Even though the action of (nonrelativistic) bosons includes a first-order time derivative term, we find remarkable similarities in the weak-coupling limit between the classical O(N) model and interacting bosons at zero temperature.

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I. INTRODUCTION

In the context of critical phenomena, it is well known that the Gaussian approximation breaks down in the vicinity of a second-order phase transition (below the upper critical dimension). When the Ginzburg criterion $|T - T_c|/T_c \gg t_G$ is violated (T_c denotes the critical temperature and $|T - T_c|/T_c \sim t_G$ defines the Ginzburg temperature T_G), the long-distance behavior of the correlation functions cannot be described by a Gaussian fluctuation theory, and more involved techniques, such as the renormalization group, are required (see e.g., Ref. [1]). At the critical point ($T = T_c$), one can nevertheless distinguish two regimes in momentum space: a high-energy Gaussian regime, where the Gaussian approximation remains essentially correct, and a low-energy critical regime, where the correlation function of the order parameter field shows a critical behavior characterized by a nonzero anomalous dimension η. These two regimes are separated by a characteristic momentum scale p_G, which defines the Ginzburg length $\xi_G = p_G^{-1}$ (see, e.g., Ref. [2]).

In systems with a broken continuous symmetry, the physics remains nontrivial in the whole low-temperature phase due to the presence of Goldstone modes, which implies that correlations decay algebraically. The coupling between transverse and longitudinal order parameter fluctuations leads to a divergence of the longitudinal susceptibility [3–5]. Away from the critical regime (i.e., at sufficiently low temperatures), one can distinguish a high-energy Gaussian regime ($|\mathbf{p}| \gg p_G$), where the Gaussian approximation remains correct, and a low-energy Goldstone regime ($|\mathbf{p}| \ll p_G$) dominated by the Goldstone modes and characterized by a divergence of the longitudinal susceptibility. Note that the Ginzburg momentum scale p_G defined here is the same as the one signaling the onset of the critical regime (in momentum space) when the system is near the phase transition. For instance, for the $(\boldsymbol{\varphi}^2)^2$ theory with O(N) symmetry [classical O(N) model], one finds a transverse susceptibility $\chi_\perp(\mathbf{p}) \sim 1/\mathbf{p}^2$ for $\mathbf{p} \to \mathbf{0}$, while the longitudinal susceptibility $\chi_\parallel(\mathbf{p}) \sim 1/|\mathbf{p}|^{4-d}$ is also singular in dimensions $2 < d \leqslant 4$ (the divergence is logarithmic for $d = 4$). At and below the lower critical dimension $d_c^- = 2$, transverse fluctuations lead to a suppression of long-range order (Mermin-Wagner theorem). There is an analog phenomenon in zero-temperature quantum systems with broken continuous symmetry. When the Goldstone mode frequency $\omega = c|\mathbf{p}|$ vanishes linearly with momentum, the longitudinal susceptibility $\chi_\parallel(\mathbf{p},\omega) \sim 1/(\omega^2 - c^2\mathbf{p}^2)^{(3-d)/2}$ has no pole-like structure but a branch cut for $d \leqslant 3$, and the dynamical structure factor exhibits a critical continuum above the usual δ peak $\delta(\omega - c|\mathbf{p}|)$ due to the Goldstone mode [6–8].

Historically, the divergence of the longitudinal susceptibility was encountered (although not recognized as such) early on in interacting boson systems. The first attempts to improve the Bogoliubov theory of superfluidity [9] were made difficult by a singular perturbation theory plagued by infrared divergences [10–13]. As realized later [14–16], the singular perturbation theory is a direct consequence of the coupling between transverse and longitudinal fluctuations.

In this paper, we study the crossover from the high-energy Gaussian regime to the low-energy Goldstone regime in the ordered phase, both for the classical O(N) model and for interacting bosons at zero temperature. Even though the action of (nonrelativistic) bosons includes a first-order time derivative term, which prevents a straightforward description in terms of a classical O(2) model, we find remarkable similarities in the weak-coupling limit between these two models. In contrast, the strong-coupling limit of the O(N) model, that is, the critical regime near the phase transition, has no direct analog in zero-temperature interacting boson systems.

The classical O(N) model is studied in Sec. II, while superfluid systems are discussed in Sec. III. First, we show that the loop expansion about the mean-field solution is plagued by infrared divergences and deduce a perturbative estimate of the Ginzburg momentum scale p_G (Secs. II A and III A). Then we use symmetry arguments to derive the exact value of the self-energies at vanishing momentum (and frequency) (Secs. II A 3 and III A 3). In the case of bosons, we obtain Nepomnyashchii and Nepomnyashchii's result about the vanishing of the anomalous self-energy [14]. In Secs. II B and III B, we show that the difficulties of perturbation theory can be circumvented within a hydrodynamic approach (i.e., for bosons, Popov's theory [17–19]) based on an amplitude-direction representation of the order parameter field. This yields the correlation functions in the hydrodynamic regime defined by a characteristic momentum scale $p_c \gg p_G$. The O(N) model is solved in the large-N limit in Sec. II C. This allows us to obtain the longitudinal correlation function in the whole low-temperature phase, including the critical regime in the vicinity of the phase transition. Finally, we show how the nonperturbative renormalization group (NPRG) provides a natural framework to understand the ordered phase of the O(N) model and the superfluid phase of interacting bosons (Secs. II D and III C).

II. THE $(\boldsymbol{\varphi}^2)^2$ THEORY AT LOW TEMPERATURES

We consider the $(\boldsymbol{\varphi}^2)^2$ theory defined by the action

$$S[\boldsymbol{\varphi}] = \int d^d r \left\{ \frac{1}{2}(\nabla\boldsymbol{\varphi})^2 + \frac{r_0}{2}\boldsymbol{\varphi}^2 + \frac{u_0}{4!}(\boldsymbol{\varphi}^2)^2 \right\}, \tag{1}$$

where $\boldsymbol{\varphi}$ is an N-component real field and d is the space dimension. We assume $N \geqslant 2$ and $d > 2$. The model is regularized by a ultraviolet momentum cutoff Λ. The connected propagator

$$G_{ij}(\mathbf{p}) = \langle \varphi_i(\mathbf{p})\varphi_j(-\mathbf{p})\rangle - \langle \varphi_i(\mathbf{p})\rangle\langle \varphi_j(-\mathbf{p})\rangle \tag{2}$$

is related to the self-energy Σ by Dyson's equation $G^{-1} = G_0^{-1} + \Sigma$, where

$$G_{0,ij}(\mathbf{p}) = \frac{\delta_{i,j}}{\mathbf{p}^2 + r_0} \tag{3}$$

is the bare propagator. In the low-temperature phase, if we denote by $\boldsymbol{\varphi}_0 = \langle \boldsymbol{\varphi}(\mathbf{r})\rangle$ the order parameter, the self-energy

$$\begin{aligned}\Sigma_{ij}(\mathbf{p}) &= \hat{\boldsymbol{\varphi}}_{0,i}\hat{\boldsymbol{\varphi}}_{0,j}\Sigma_l(\mathbf{p}) + (\delta_{i,j} - \hat{\boldsymbol{\varphi}}_{0,i}\hat{\boldsymbol{\varphi}}_{0,j})\Sigma_t(\mathbf{p}) \\ &= \delta_{i,j}[\Sigma_{\rm n}(\mathbf{p}) - \Sigma_{\rm an}(\mathbf{p})] + 2\hat{\boldsymbol{\varphi}}_{0,i}\hat{\boldsymbol{\varphi}}_{0,j}\Sigma_{\rm an}(\mathbf{p})\end{aligned} \tag{4}$$

($\hat{\boldsymbol{\varphi}}_0 = \boldsymbol{\varphi}_0/|\boldsymbol{\varphi}_0|$) can be written in terms of its longitudinal (Σ_l) and transverse (Σ_t) parts. In the second line of Eq. (4), we have introduced the "normal" ($\Sigma_{\rm n}$) and "anomalous" ($\Sigma_{\rm an}$) self-energies. In the following, we assume that the order parameter $\boldsymbol{\varphi}_0$ is along the direction $(1,0,\dots,0)$ so that

$$\Sigma_{ii}(\mathbf{p}) = \begin{cases} \Sigma_{\rm n}(\mathbf{p}) + \Sigma_{\rm an}(\mathbf{p}) & \text{if} \quad i = 1, \\ \Sigma_{\rm n}(\mathbf{p}) - \Sigma_{\rm an}(\mathbf{p}) & \text{if} \quad i \neq 1. \end{cases} \tag{5}$$

The anomalous self-energy $\Sigma_{\rm an}$ is related to the spontaneously broken O(N) symmetry and vanishes in the high-temperature phase. $\Sigma_{\rm n}$ and $\Sigma_{\rm an}$ are analogous to the normal and anomalous self-energies that are usually introduced in the theory of superfluidity [20,21]. For $N = 2$, we can introduce the complex field

$$\psi(\mathbf{r}) = \frac{1}{\sqrt{2}}[\varphi_1(\mathbf{r}) + i\varphi_2(\mathbf{r})]. \tag{6}$$

Making use of the two-component field

$$\Psi(\mathbf{r}) = \begin{pmatrix} \psi(\mathbf{r}) \\ \psi^*(\mathbf{r}) \end{pmatrix}, \quad \Psi^\dagger(\mathbf{r}) = [\psi^*(\mathbf{r}), \psi(\mathbf{r})], \tag{7}$$

the two-point propagator becomes a 2×2 matrix in Fourier space, whose inverse is given by

$$\begin{pmatrix} \mathbf{p}^2 + r_0 + \Sigma_{\rm n}(\mathbf{p}) & \Sigma_{\rm an}(\mathbf{p}) \\ \Sigma_{\rm an}(\mathbf{p}) & \mathbf{p}^2 + r_0 + \Sigma_{\rm n}(\mathbf{p}) \end{pmatrix}, \tag{8}$$

and it bears some similarities to the single-particle propagator in a superfluid (Sec. III).

A. Gaussian approximation and breakdown of perturbation theory

Let us begin with a dimensional analysis of the action (1). If we assign the scaling dimension 1 to momenta (i.e., $[\mathbf{p}] = 1$), the field has engineering dimension $[\boldsymbol{\varphi}] = \frac{d-2}{2}$, $[r_0] = 2$, and $[u_0] = 4 - d$. We can then define two characteristic length scales:

$$\begin{aligned}\xi &\sim |r_0|^{-1/2}, \\ \xi_G &\sim u_0^{1/(d-4)}.\end{aligned} \tag{9}$$

In the critical regime of the low-temperature phase ($\xi \gg \xi_G$), ξ_G is the characteristic length scale associated with the onset of critical fluctuations, while $\xi \equiv \xi_J$ is the Josephson length separating the critical regime from a regime dominated by Goldstone modes [22]. When critical fluctuations are taken into account, one finds that ξ_J diverges with a critical exponent ν that differs from the mean-field value 1/2. At low temperatures away from the critical regime ($\xi \ll \xi_G$), $\xi \equiv \xi_c$ corresponds to a correlation length for the gapped amplitude fluctuations, while direction fluctuations are gapless due to Goldstone's theorem. The physical meaning of the Ginzburg length ξ_G in this temperature range will become clear below.

1. Gaussian approximation

Within the mean-field (or saddle-point) approximation, one finds $\varphi_0 = |\boldsymbol{\varphi}_0| = (-6r_0/u_0)^{1/2}$ in the low-temperature phase ($r_0 < 0$). In the Gaussian approximation, one expands the action to quadratic order in the fluctuations $\boldsymbol{\varphi} - \boldsymbol{\varphi}_0$ [1]. This yields the (zero-loop) self-energy

$$\Sigma_{ii}^{(0)}(\mathbf{p}) = \begin{cases} -3r_0 & \text{if} \quad i = 1, \\ -r_0 & \text{if} \quad i \neq 1, \end{cases} \tag{10}$$

from which we obtain the longitudinal and transverse propagators:

$$\begin{aligned} G_l^{(0)}(\mathbf{p}) &= G_{11}^{(0)}(\mathbf{p}) = \frac{1}{\mathbf{p}^2 + 2|r_0|}, \\ G_t^{(0)}(\mathbf{p}) &= G_{22}^{(0)}(\mathbf{p}) = \frac{1}{\mathbf{p}^2}. \end{aligned} \tag{11}$$

In agreement with Goldstone's theorem, the transverse propagator is gapless, whereas the longitudinal susceptibility $G_l(\mathbf{p} = 0) = 1/|2r_0|$ is finite. We shall see below that this last property is an artifact of the Gaussian approximation.

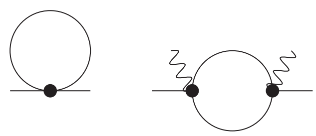

FIG. 1. (Color online) One-loop correction $\Sigma^{(1)}$ to the self-energy. Filled circles represent the bare interaction; zigzag lines, the order parameter φ_0; and solid lines, the connected propagator $G^{(0)}$.

2. One-loop correction and the Ginzburg momentum scale

The one-loop correction $\Sigma^{(1)}$ to the self-energy is shown diagrammatically in Fig. 1. While the first diagram is finite, the second one gives a diverging contribution to Σ_{11} in the infrared limit $\mathbf{p} \to 0$ when $d \leqslant 4$. The divergence arises when both internal lines correspond to transverse fluctuations, which is possible only for Σ_{11}. Thus Σ_{22} is finite at the one-loop level and the normal and anomalous self-energies exhibit the same divergence,

$$\Sigma_{\rm n}^{(1)}(\mathbf{p}) \simeq \Sigma_{\rm an}^{(1)}(\mathbf{p}) \simeq -\frac{N-1}{36} u_0^2 \varphi_0^2 \int_{\mathbf{q}} \frac{1}{\mathbf{q}^2 (\mathbf{p}+\mathbf{q})^2}, \qquad (12)$$

where we use the notation $\int_{\mathbf{q}} = \int \frac{d^d q}{(2\pi)^d}$. The momentum integration in (12) gives [23]

$$\int_{\mathbf{q}} \frac{1}{\mathbf{q}^2(\mathbf{p}+\mathbf{q})^2} = \begin{cases} A_d |\mathbf{p}|^{d-4} & \text{if} \quad d < 4, \\ A_4 \ln(\Lambda/|\mathbf{p}|) & \text{if} \quad d = 4, \end{cases} \qquad (13)$$

for $|\mathbf{p}| \ll \Lambda$, where

$$A_d = \begin{cases} -\frac{2^{1-d}\pi^{1-d/2}}{\sin(\pi d/2)} \frac{\Gamma(d/2)}{\Gamma(d-1)} & \text{if} \quad d < 4, \\ \frac{1}{8\pi^2} & \text{if} \quad d = 4. \end{cases} \qquad (14)$$

The one-loop correction (12) diverges for $\mathbf{p} \to 0$ and the perturbation expansion about the Gaussian approximation breaks down. By comparing the one-loop correction to the zero-loop result, that is, $|\Sigma_{\rm n}^{(1)}(\mathbf{p})| \sim \Sigma_{\rm n}^{(0)}(\mathbf{p})$ or $|\Sigma_{\rm an}^{(1)}(\mathbf{p})| \sim \Sigma_{\rm an}^{(0)}(\mathbf{p})$, one can, nevertheless, extract a characteristic (Ginzburg) momentum scale,

$$p_G \sim \begin{cases} [A_d(N-1)u_0]^{1/(4-d)} & \text{if} \quad d < 4, \\ \Lambda \exp\left(\frac{-1}{A_4(N-1)u_0}\right) & \text{if} \quad d = 4, \end{cases} \qquad (15)$$

which was obtained previously from dimensional analysis [Eq. (9)]. While the Gaussian or perturbative approach remains valid for $|\mathbf{p}| \gg p_G$, the limit $|\mathbf{p}| \ll p_G$ cannot be studied perturbatively. We shall see in Sec. II B that the breakdown of perturbation theory is due to the coupling between transverse and longitudinal fluctuations.

3. Exact results for $\Sigma_{\rm n}(\mathbf{p}=0)$ and $\Sigma_{\rm an}(\mathbf{p}=0)$

Although the one-loop correction $\Sigma^{(1)}(\mathbf{p})$ diverges when $\mathbf{p} \to 0$ for $d \leqslant 4$, it is nevertheless possible to obtain the exact value of $\Sigma(\mathbf{p}=0)$ using the O(N) symmetry of the model.

Let us consider the effective action

$$\Gamma[\boldsymbol{\phi}] = -\ln Z[\mathbf{h}] + \int d^d r \, \mathbf{h} \cdot \boldsymbol{\phi}, \qquad (16)$$

defined as the Legendre transform of the free energy $-\ln Z[\mathbf{h}]$, where $\mathbf{h}$ is an external field that couples linearly to the $\boldsymbol{\varphi}$ field and

$$\phi_i(\mathbf{r}) = \frac{\delta \ln Z[\mathbf{h}]}{\delta h_i(\mathbf{r})} = \langle \varphi_i(\mathbf{r}) \rangle_{\mathbf{h}}. \qquad (17)$$

The notation $\langle \cdots \rangle_{\mathbf{h}}$ means that the average value is computed in the presence of the external field $\mathbf{h}$. $\Gamma[\boldsymbol{\phi}]$ satisfies the equation of state,

$$\frac{\delta \Gamma[\boldsymbol{\phi}]}{\delta \phi_i(\mathbf{r})} = h_i(\mathbf{r}). \qquad (18)$$

At equilibrium and in the absence of external field, the order parameter $\boldsymbol{\varphi}_0 = \langle \boldsymbol{\varphi}(\mathbf{r}) \rangle$ is obtained from the stationary condition of the effective action,

$$\left. \frac{\delta \Gamma[\boldsymbol{\phi}]}{\delta \phi_i(\mathbf{r})} \right|_{\boldsymbol{\phi}(\mathbf{r}) = \boldsymbol{\varphi}_0} = 0. \qquad (19)$$

$\Gamma[\boldsymbol{\phi}]$ is the generating functional of the one-particle irreducible vertices,

$$\Gamma^{(n)}_{i_1 \cdots i_n}(\mathbf{r}_1, \ldots, \mathbf{r}_n) = \left. \frac{\delta^{(n)} \Gamma[\boldsymbol{\phi}]}{\delta \phi_{i_1}(\mathbf{r}_1) \cdots \delta \phi_{i_n}(\mathbf{r}_n)} \right|_{\boldsymbol{\phi}(\mathbf{r}) = \boldsymbol{\varphi}_0}. \qquad (20)$$

The latter fully determine the correlation functions. In particular, the two-point vertex $\Gamma^{(2)}$ is related to the propagator by $\Gamma^{(2)} = G^{-1} = G_0^{-1} + \Sigma$.

The O(N) invariance of action (1) implies that the effective action $\Gamma[\boldsymbol{\phi}]$ is invariant under a rotation of the field $\boldsymbol{\phi}$. Let us consider the case $N=3$ for simplicity (the following results are easily extended to arbitrary N). For an infinitesimal rotation $\boldsymbol{\phi} \to \boldsymbol{\phi} + \theta \mathbf{n} \times \boldsymbol{\phi}$ about the axis $\mathbf{n}$ ($\mathbf{n}^2 = 1$ and $\theta \to 0$), the invariance of the effective action yields

$$\int d^d r \sum_{ijk} \frac{\delta \Gamma[\boldsymbol{\phi}]}{\delta \phi_i(\mathbf{r})} \epsilon_{ijk} n_j \phi_k(\mathbf{r}) = 0, \qquad (21)$$

where ϵ_{ijk} is the totally antisymmetric tensor. Taking the first-order functional derivative $\delta/\delta\phi_l(\mathbf{r}')$ and setting $\phi_i(\mathbf{r}) = \delta_{i,1}\varphi_0$, we obtain

$$\int d^d r \sum_{i,j} \Gamma^{(2)}_{il}(\mathbf{r},\mathbf{r}') \epsilon_{ij1} n_j = 0. \qquad (22)$$

With $\mathbf{n} = (0,0,1)$, this gives

$$\Gamma^{(2)}_{22}(\mathbf{p}=0) = r_0 + \Sigma_{22}(\mathbf{p}=0) = 0, \qquad (23)$$

where $\Gamma^{(2)}(\mathbf{p})$ denotes $\Gamma^{(2)}(\mathbf{p},-\mathbf{p})$. Equation (23) is a direct consequence of Goldstone's theorem. If we now take the second-order functional derivative $\delta^{(2)}/\delta\phi_l(\mathbf{r}')\delta\phi_m(\mathbf{r}'')$ of (21) and set $\phi_i(\mathbf{r}) = \delta_{i,1}\varphi_0$, we obtain the Ward identity

$$\sum_{i,j} \left[\Gamma^{(2)}_{im}(\mathbf{r}',\mathbf{r}'')\epsilon_{ijl} + \Gamma^{(2)}_{il}(\mathbf{r}'',\mathbf{r}')\epsilon_{ijm} \right] n_j + \int d^d r \sum_{i,j} \Gamma^{(3)}_{ilm}(\mathbf{r},\mathbf{r}',\mathbf{r}'')\epsilon_{ij1} n_j \varphi_0 = 0. \qquad (24)$$

Choosing $l=2$, $m=1$, and $j=3$, this gives

$$\Gamma^{(2)}_{11}(\mathbf{r}',\mathbf{r}'') - \Gamma^{(2)}_{22}(\mathbf{r}'',\mathbf{r}') - \varphi_0 \int d^d r \, \Gamma^{(3)}_{221}(\mathbf{r},\mathbf{r}',\mathbf{r}'') = 0. \qquad (25)$$

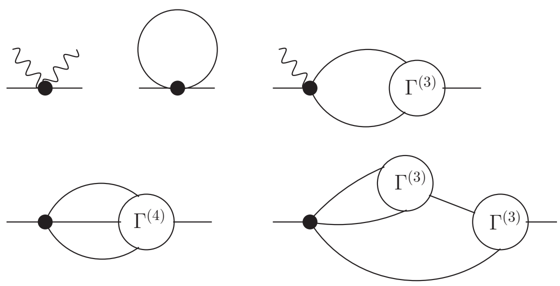

FIG. 2. Exact diagrammatic representation of the self-energy in terms of the three- and four-leg vertices $\Gamma^{(3)}$ and $\Gamma^{(4)}$. Filled circles represent the bare interaction; zigzag lines, the order parameter; and solid lines, the (exact) connected propagator.

Integrating over $\mathbf{r}'$ and $\mathbf{r}''$ and using (23), we deduce (in Fourier space)

$$\Gamma^{(3)}_{122}(0,0,0) = \frac{\Gamma^{(2)}_{11}(0,0)}{\sqrt{V}\varphi_0}, \tag{26}$$

where V is the volume of the system.

Let us now consider the exact diagrammatic representation of the self-energy shown in Fig. 2. We know from perturbation theory that the third diagram in Fig. 2 is potentially dangerous when the two internal lines correspond to transverse fluctuations. We therefore write the self-energy $\Sigma_{11}(\mathbf{p})$ as

$$\Sigma_{11}(\mathbf{p}) = \tilde{\Sigma}_{11}(\mathbf{p}) - \frac{N-1}{6\sqrt{V}} u_0\varphi_0 \sum_{\mathbf{q}} G_{22}(\mathbf{q})G_{22}(\mathbf{p}+\mathbf{q}) \times\Gamma^{(3)}_{122}(-\mathbf{p},-\mathbf{q},\mathbf{p}+\mathbf{q}), \tag{27}$$

where $\tilde{\Sigma}_{11}(\mathbf{p})$ denotes the part of the self-energy that is regular in perturbation theory (i.e., the part that does not contain pairs of lines corresponding to $G_{22}G_{22}$). If we assume that the transverse propagator $G_{22}(\mathbf{q})$ is proportional to $1/\mathbf{q}^2$ for $\mathbf{q}\to 0$ (this result is shown in the following sections), the integral $\int_{\mathbf{q}} G_{22}(\mathbf{q})^2$ is infrared divergent for $d \leqslant 4$. To obtain a finite self-energy $\Sigma_{11}(\mathbf{p}=0)$, one must require that

$$\lim_{\mathbf{q}\to 0} \Gamma^{(3)}_{122}(0,-\mathbf{q},\mathbf{q}) = \Gamma^{(3)}_{122}(0,0,0) = 0. \tag{28}$$

The Ward identity (26), then implies $\Gamma^{(2)}_{11}(\mathbf{p}=0)=0$, so that we finally obtain

$$\begin{aligned} \Sigma_{\mathrm{n}}(\mathbf{p}=0) &= -r_0 + \tfrac{1}{2}\left[\Gamma_{11}(\mathbf{p}=0) + \Gamma_{22}(\mathbf{p}=0)\right] = -r_0, \\ \Sigma_{\mathrm{an}}(\mathbf{p}=0) &= \tfrac{1}{2}\left[\Gamma_{11}(\mathbf{p}=0) - \Gamma_{22}(\mathbf{p}=0)\right] = 0. \end{aligned} \tag{29}$$

It may appear surprising that the anomalous self-energy, which is related to the spontaneously broken O(N) symmetry, vanishes for $\mathbf{p}=0$. The equivalent property in interacting boson systems is a fundamental result of the theory of superfluidity (Sec. III).

B. Amplitude-direction representation

The difficulties of the perturbation theory in Sec. II A can be avoided if one uses the "good" hydrodynamic variables in the low-temperature phase, namely, the amplitude and the direction of the $\boldsymbol{\varphi}$ field. We thus write

$$\boldsymbol{\varphi}(\mathbf{r}) = \rho(\mathbf{r})\mathbf{n}(\mathbf{r}), \tag{30}$$

where $\mathbf{n}(\mathbf{r})^2 = 1$, and obtain the action

$$S[\rho,\mathbf{n}] = \int d^d r \left\{ \frac{1}{2}(\nabla\rho)^2 + \frac{\rho^2}{2}(\nabla\mathbf{n})^2 + \frac{r_0}{2}\rho^2 + \frac{u_0}{4!}\rho^4 \right\}. \tag{31}$$

At the mean-field level, the amplitude takes the value $\rho_0 = (-6r_0/u_0)^{1/2}$ in the low-temperature phase ($r_0 < 0$). For low-amplitude fluctuations $\rho' = \rho - \rho_0$ (which is expected to be the case at sufficiently low temperatures), we obtain the action

$$S[\rho',\mathbf{n}] = \int d^d r \left\{ \frac{1}{2}(\nabla\rho')^2 + |r_0|\rho'^2 + \frac{\rho_0^2}{2}(\nabla\mathbf{n})^2 \right\} \tag{32}$$

and deduce that the amplitude fluctuations are gapped:

$$\langle \rho'(\mathbf{p})\rho'(-\mathbf{p})\rangle = \frac{1}{\mathbf{p}^2 + p_c^2}. \tag{33}$$

If we are interested only in momenta $|\mathbf{p}| \ll p_c = \sqrt{2|r_0|}$, to first approximation we can ignore the higher-order terms in ρ' that were neglected in (32), since they would only lead to a finite renormalization of the coefficients of the action $S[\rho',\mathbf{n}]$ [23].

Equation (32) shows that in the "hydrodynamic" regime $|\mathbf{p}| \ll p_c$ direction fluctuations are described by a nonlinear σ model. It is convenient to use the standard parametrization $\mathbf{n} = (\sigma,\boldsymbol{\pi})$, where σ is the component of $\mathbf{n}$ along the direction of order and $\boldsymbol{\pi}$ is an $(N-1)$–component field ($\mathbf{n}^2 = \sigma^2 + \boldsymbol{\pi}^2 = 1$). Integrating over σ, one obtains

$$S[\rho',\boldsymbol{\pi}] = \int d^d r \left\{ \frac{1}{2}(\nabla\rho')^2 + |r_0|\rho'^2 + \frac{\rho_0^2}{2}(\nabla\boldsymbol{\pi})^2 \right\} \tag{34}$$

for small transverse fluctuations $\boldsymbol{\pi}$ [24]. In this limit, we can treat $\pi_i(\mathbf{r})$ as a variable varying between $-\infty$ and ∞. From (34), we deduce the propagator of the $\boldsymbol{\pi}$ field,

$$\langle \pi_i(\mathbf{p})\pi_j(-\mathbf{p})\rangle = \frac{\delta_{i,j}}{\rho_0^2\mathbf{p}^2}. \tag{35}$$

Again, we note that the terms neglected in (34) would only lead to a finite renormalization of the (bare) stiffness ρ_0^2 of the nonlinear σ model at sufficiently low temperatures. In fact, Eq. (34) gives an exact description of the low-energy behavior $|\mathbf{p}| \ll p_c$ if one replaces ρ_0^2 by the exact stiffness and $p_c^{-1} = (2|r_0|)^{-1/2}$ by the exact correlation length of the ρ' field.

We are now in a position to compute the longitudinal and transverse propagators using

$$\begin{aligned} \varphi_l &= \rho\sigma = \rho\sqrt{1-\boldsymbol{\pi}^2} \simeq \rho_0 + \rho' - \tfrac{1}{2}\rho_0\boldsymbol{\pi}^2, \\ \boldsymbol{\varphi}_t &= \rho\boldsymbol{\pi} \simeq \rho_0\boldsymbol{\pi}. \end{aligned} \tag{36}$$

Since the long-distance physics is governed by transverse fluctuations, we have retained in (36) the leading contributions in $\boldsymbol{\pi}$. Making use of (35), one readily obtains

$$G_t(\mathbf{p}) \simeq \rho_0^2\langle \pi_i(\mathbf{p})\pi_i(-\mathbf{p})\rangle = \frac{1}{\mathbf{p}^2}. \tag{37}$$

The longitudinal propagator is given by

$$\begin{aligned} G_l(\mathbf{r}) &= \langle \rho'(\mathbf{r})\rho'(0)\rangle + \frac{1}{4}\rho_0^2\langle \boldsymbol{\pi}(\mathbf{r})^2\boldsymbol{\pi}(0)^2\rangle_c \\ &= \langle \rho'(\mathbf{r})\rho'(0)\rangle + \frac{N-1}{2\rho_0^2}G_t(\mathbf{r})^2, \end{aligned} \tag{38}$$

where $\langle\cdots\rangle_c$ stands for the connected part of $\langle\cdots\rangle$. The second line is obtained using Wick's theorem. In Fourier space, this gives

$$G_l(\mathbf{p}) = \frac{1}{\mathbf{p}^2+p_c^2} + \frac{N-1}{2\rho_0^2}\int_{\mathbf{q}} \frac{1}{\mathbf{q}^2(\mathbf{p}+\mathbf{q})^2}, \quad (39)$$

where the momentum integral is given by (13) for $|\mathbf{p}| \ll \Lambda$ and $d \leqslant 4$. By comparing the two terms in the rhs of (39), we recover the Ginzburg momentum scale (15). For $|\mathbf{p}| \gg p_G$, the longitudinal propagator $G_l(\mathbf{p}) \simeq 1/(\mathbf{p}^2+p_c^2)$ is dominated by amplitude fluctuations and we reproduce the result of the Gaussian approximation. In contrast, for $|\mathbf{p}| \ll p_G$, $G_l(\mathbf{p}) \sim 1/|\mathbf{p}|^{4-d}$ is dominated by direction fluctuations and diverges for $\mathbf{p} \to 0$.

The divergence of the longitudinal propagator is a direct consequence of the coupling between longitudinal and transverse fluctuations [3]. In the long-distance limit, amplitude fluctuations become frozen so that $|\boldsymbol{\varphi}| = \rho \simeq \rho_0$. This implies that the longitudinal and transverse components φ_l and $\boldsymbol{\varphi}_t$ cannot be considered independently as in the Gaussian approximation (Sec. II A) but satisfy the constraint $\varphi_l^2 + \boldsymbol{\varphi}_t^2 \simeq \rho_0^2$. To leading order, $\varphi_l \simeq \rho_0(1-\frac{\pi^2}{2})^{1/2}$ and $G_l(\mathbf{r}) \sim G_t(\mathbf{r})^2$ [Eq. (38)], that is, $G_l(\mathbf{p}) \sim 1/|\mathbf{p}|^{4-d}$ for $d \leqslant 4$ (the divergence is logarithmic for $d=4$).

Equations (37) and (39) imply that the self-energies must satisfy

$$\begin{aligned} \Sigma_{11}(\mathbf{p}) &= -r_0 - \mathbf{p}^2 + C_1|\mathbf{p}|^{4-d}, \\ \Sigma_{22}(\mathbf{p}) &= -r_0 + C_2\mathbf{p}^2 \end{aligned} \quad (40)$$

for $\mathbf{p} \to 0$ and $d < 4$, that is,

$$\begin{aligned} \Sigma_{\mathrm{n}}(\mathbf{p}) &= -r_0 + \frac{C_1}{2}|\mathbf{p}|^{4-d} + \mathcal{O}(\mathbf{p}^2), \\ \Sigma_{\mathrm{an}}(\mathbf{p}) &= \frac{C_1}{2}|\mathbf{p}|^{4-d} + \mathcal{O}(\mathbf{p}^2). \end{aligned} \quad (41)$$

For $d=4$, one finds

$$\begin{aligned} \Sigma_{\mathrm{n}}(\mathbf{p}) &= -r_0 + \frac{C_1}{\ln(\Lambda/|\mathbf{p}|)} + \mathcal{O}(\mathbf{p}^2), \\ \Sigma_{\mathrm{an}}(\mathbf{p}) &= \frac{C_1}{\ln(\Lambda/|\mathbf{p}|)} + \mathcal{O}(\mathbf{p}^2). \end{aligned} \quad (42)$$

For $\mathbf{p}=0$, we reproduce the exact results of Sec. II A 3. Equations (41) and (42) show that $\Sigma_{\mathrm{n}}(\mathbf{p})$ and $\Sigma_{\mathrm{an}}(\mathbf{p})$ contain nonanalytic terms that are dominant for $\mathbf{p} \to 0$.

C. Large-N limit

In this section, we show that the previous results for the longitudinal propagator are fully consistent with the large-N limit of the $(\boldsymbol{\varphi}^2)^2$ theory. Furthermore, the large-N limit enables computation of the longitudinal propagator not only at low temperatures but also in the critical regime near the transition to the high-temperature (disordered) phase.

To obtain a meaningful large-N limit, we write the coefficient of the $(\boldsymbol{\varphi}^2)^2$ term in Eq. (1) as u_0/N and take the limit $N \to \infty$ with u_0 fixed. Following Ref. [23], we express the partition function as

$$Z = \int \mathcal{D}[\boldsymbol{\varphi},\rho,\lambda] e^{-\int d^d r[\frac{1}{2}(\nabla\boldsymbol{\varphi})^2 + \frac{r_0}{2}\rho + \frac{u_0}{4!N}\rho^2 + i\frac{\lambda}{2}(\boldsymbol{\varphi}^2-\rho)]}. \quad (43)$$

It can be easily verified that by integrating out λ and then ρ, one recovers the original action $S[\boldsymbol{\varphi}]$. If, instead, one first integrates out ρ, one obtains

$$Z = \int \mathcal{D}[\boldsymbol{\varphi},\lambda] e^{-\int d^d r[\frac{1}{2}(\nabla\boldsymbol{\varphi})^2 + i\frac{\lambda}{2}\boldsymbol{\varphi}^2] + \frac{3N}{2u_0}\int d^d r(i\lambda - r_0)^2}. \quad (44)$$

As in Sec. II B, it is convenient to split the $\boldsymbol{\varphi}$ field into a σ field and an $(N-1)$–component field $\boldsymbol{\pi}$. The integration over the $\boldsymbol{\pi}$ field gives

$$\int \mathcal{D}[\boldsymbol{\pi}] e^{-\int d^d r[\frac{1}{2}(\nabla\boldsymbol{\pi})^2 + i\frac{\lambda}{2}\boldsymbol{\pi}^2]} = (\det g)^{(N-1)/2}, \quad (45)$$

where

$$g^{-1}(\mathbf{r},\mathbf{r}') = -\nabla^2\delta(\mathbf{r}-\mathbf{r}') + i\lambda(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}') \quad (46)$$

is the inverse propagator of the π_i field in the fluctuating λ field. We thus obtain the action

$$\begin{aligned} S[\sigma,\lambda] = {} & \frac{1}{2}\int d^d r[(\nabla\sigma)^2 + i\lambda\sigma^2] \\ & - \frac{3N}{2u_0}\int d^d r(i\lambda - r_0)^2 + \frac{N-1}{2}\mathrm{Tr}\ln g^{-1}. \end{aligned} \quad (47)$$

In the limit $N \to \infty$, the action becomes proportional to N (this is easily seen by rescaling the σ field, $\sigma \to \sqrt{N}\sigma$) and the saddle-point approximation becomes exact. For uniform fields $\sigma(\mathbf{r}) = \sigma$ and $\lambda(\mathbf{r}) = \lambda$, the action is given by

$$\frac{1}{V}S[\sigma,\lambda] = \frac{i}{2}\lambda\sigma^2 - \frac{3N}{2u_0}(i\lambda - r_0)^2 + \frac{N}{2V}\mathrm{Tr}\ln g^{-1} \quad (48)$$

(we use $N-1 \simeq N$ for large N), with $g^{-1}(\mathbf{p}) = \mathbf{p}^2 + i\lambda$ in Fourier space. From (48), we deduce the saddle-point equations

$$\begin{aligned} \sigma m^2 &= 0, \\ \sigma^2 &= \frac{6N}{u_0}(m^2 - r_0) - N\int_{\mathbf{p}} \frac{1}{\mathbf{p}^2+m^2}, \end{aligned} \quad (49)$$

where we use the notation $m^2 = i\lambda$ ($i\lambda$ is real at the saddle point). These equations show that the component σ of the $\boldsymbol{\varphi}$ field that was singled out plays the role of an order parameter.

In the low-temperature phase, σ is nonzero and $m=0$. The propagator $g(\mathbf{p}) = 1/\mathbf{p}^2$ is gapless, thus identifying the π_i fields as the $N-1$ Goldstone modes associated with the spontaneously broken O(N) symmetry. From Eq. (49), we deduce

$$\sigma^2 = -\frac{6N}{u_0}(r_0 - r_{0c}), \quad (50)$$

where

$$r_{0c} = -\frac{u_0}{6}\int_{\mathbf{p}} \frac{1}{\mathbf{p}^2} = -\frac{u_0}{6}\frac{K_d}{d-2}\Lambda^{d-2} \quad (51)$$

[with $K_d = 2^{1-d}\pi^{-d/2}/\Gamma(d/2)$] is the critical value of r_0. Since the saddle-point approximation is exact in the large-N limit, the effective action $\Gamma[\sigma,\lambda]$ is simply given by the action $S[\sigma,\lambda]$ [Eq. (47)] [25]. We deduce

$$\Gamma^{(2)}(\mathbf{r}-\mathbf{r}') = \begin{pmatrix} \Gamma^{(2)}_{\sigma\sigma}(\mathbf{r}-\mathbf{r}') & \Gamma^{(2)}_{\sigma\lambda}(\mathbf{r}-\mathbf{r}') \\ \Gamma^{(2)}_{\lambda\sigma}(\mathbf{r}-\mathbf{r}') & \Gamma^{(2)}_{\lambda\lambda}(\mathbf{r}-\mathbf{r}') \end{pmatrix}$$

$$= \begin{pmatrix} -\nabla^2\delta(\mathbf{r}-\mathbf{r}') & i\sigma\delta(\mathbf{r}-\mathbf{r}') \\ i\sigma\delta(\mathbf{r}-\mathbf{r}') & \frac{N}{2}\Pi(\mathbf{r}-\mathbf{r}')+\frac{3N}{u_0}\delta(\mathbf{r}-\mathbf{r}') \end{pmatrix}, \quad (52)$$

where

$$\Pi(\mathbf{r}-\mathbf{r}') = g(\mathbf{r}-\mathbf{r}')g(\mathbf{r}'-\mathbf{r}), \quad (53)$$

and we use the notation $\Gamma^{(2)}_{\sigma\sigma}(\mathbf{r}-\mathbf{r}') = \delta^{(2)}\Gamma/\delta\sigma(\mathbf{r})\delta\sigma(\mathbf{r}')$, etc. The two-point vertex $\Gamma^{(2)}$ is computed for the saddle-point values of the fields σ and λ. In Fourier space, we obtain

$$\Gamma^{(2)}(\mathbf{p}) = \begin{pmatrix} \mathbf{p}^2 & i\sigma \\ i\sigma & \frac{N}{2}\Pi(\mathbf{p})+\frac{3N}{u_0} \end{pmatrix} \quad (54)$$

and the propagator $G = \Gamma^{(2)-1}$ takes the form

$$G(\mathbf{p}) = \frac{1}{\det\Gamma^{(2)}(\mathbf{p})} \begin{pmatrix} \frac{N}{2}\Pi(\mathbf{p})+\frac{3N}{u_0} & -i\sigma \\ -i\sigma & \mathbf{p}^2 \end{pmatrix}, \quad (55)$$

with

$$\det\Gamma^{(2)}(\mathbf{p}) = \mathbf{p}^2\left[\frac{N}{2}\Pi(\mathbf{p})+\frac{3N}{u_0}\right]+\sigma^2 \quad (56)$$

and $\Pi(\mathbf{p}) = \int_{\mathbf{q}} g(\mathbf{q})g(\mathbf{p}+\mathbf{q})$. Equation (56), together with the small-$\mathbf{p}$ behavior of $\Pi(\mathbf{p})$ [Eq. (13)], leads us to introduce three characteristic momentum scales:

$$p_G = \left(\frac{u_0A_d}{6}\right)^{1/(4-d)},$$
$$p_J = \left(\frac{2\sigma^2}{NA_d}\right)^{1/(d-2)} = \left[\frac{12}{u_0A_d}(r_{0c}-r_0)\right]^{1/(d-2)}, \quad (57)$$
$$p_c = \left(\frac{u_0\sigma^2}{3N}\right)^{1/2} = [2(r_{0c}-r_0)]^{1/2}.$$

For simplicity, we discuss only the case $d<4$; equivalent results for $d=4$ are easily deduced. The Josephson length $\xi_J = p_J^{-1}$—which separates the critical regime from the Golstone regime (see below) [22]—diverges at the transition with the critical exponent $\nu = 1/(d-2)$, which also characterizes the divergence of the correlation length in the high-temperature phase [23]. The momentum scales (57) are not independent since

$$p_c^2 = p_G^2\left(\frac{p_J}{p_G}\right)^{d-2}. \quad (58)$$

If we vary r_0 with u_0 fixed, we find that the three characteristic scales (57) are equal when $T=T_G$, where T_G is defined by

$$\bar{r}_0(T_c-T_G) = \frac{1}{2}\left(\frac{u_0A_d}{6}\right)^{2/(4-d)} \quad (59)$$

(see Fig. 3). We have assumed that $r_0 = \bar{r}_0(T-T_0)$ (with T_0 the mean-field transition temperature) and used $r_{0c} = \bar{r}_0(T_c-T_0)$. We recognize in Eq. (59) the Ginzburg criterion [2] so that we can identify T_G with the Ginzburg temperature separating the critical regime near the transition from the noncritical regime at sufficiently low temperatures.

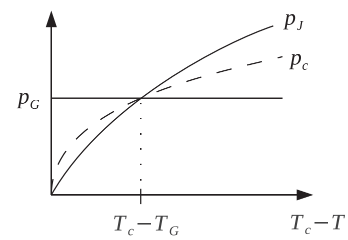

FIG. 3. Characteristic momentum scales p_G, p_J, and p_c vs T_c-T for fixed u_0 [Eqs. (57) with $r_0 = \bar{r}_0(T-T_0)$].

In the critical regime ($T_c-T \ll T_c-T_G$ or $p_J \ll p_G$), using $p_J \ll p_c \ll p_G$, one finds the longitudinal correlation function

$$G_{\sigma\sigma}(\mathbf{p}) = \begin{cases} \frac{p_J^{2-d}}{|\mathbf{p}|^{4-d}} & \text{if} \quad |\mathbf{p}| \ll p_J, \\ \frac{1}{\mathbf{p}^2} & \text{if} \quad |\mathbf{p}| \gg p_J, \end{cases} \quad (60)$$

while in the noncritical regime ($T_c-T_G \ll T_c-T$ or $p_G \ll p_c$),

$$G_{\sigma\sigma}(\mathbf{p}) = \begin{cases} \frac{1}{p_c^2}\left(\frac{p_G}{|\mathbf{p}|}\right)^{4-d} & \text{if} \quad |\mathbf{p}| \ll p_G, \\ \frac{1}{\mathbf{p}^2+p_c^2} & \text{if} \quad |\mathbf{p}| \gg p_G. \end{cases} \quad (61)$$

In the noncritical regime, we recover the results in Sec. II B. We find two characteristic momentum scales (p_G and p_c) and two regimes for the behavior of $G_{\sigma\sigma}(\mathbf{p})$: (i) a Goldstone regime ($|\mathbf{p}| \ll p_G$), characterized by a diverging longitudinal propagator $G_{\sigma\sigma}(\mathbf{p}) \sim 1/|\mathbf{p}|^{4-d}$, and (ii) a Gaussian (perturbative) regime ($|\mathbf{p}| \gg p_G$) where $G_{\sigma\sigma}(\mathbf{p}) \simeq 1/(\mathbf{p}^2+p_c^2)$. The critical regime is characterized by two momentum scales (p_J and p_G) and three regimes for the behavior of $G_{\sigma\sigma}(\mathbf{p})$: (i) a Goldstone regime ($|\mathbf{p}| \ll p_J$) with a diverging longitudinal propagator; (ii) a critical regime ($p_J \ll |\mathbf{p}| \ll p_G$) where $G_{\sigma\sigma}(\mathbf{p}) \sim 1/|\mathbf{p}|^{2-\eta}$, with a vanishing anomalous dimension η (η is $\mathcal{O}(1/N)$ in the large-N limit [23,26]); and (iii) a Gaussian regime ($p_G \ll |\mathbf{p}|$) where $G_{\sigma\sigma}(\mathbf{p}) \simeq 1/\mathbf{p}^2$. These results are summarized in Fig. 4.

D. The nonperturbative renormalization group

1. *The average effective action*

The strategy of the NPRG is to build a family of theories indexed by a momentum scale k such that fluctuations

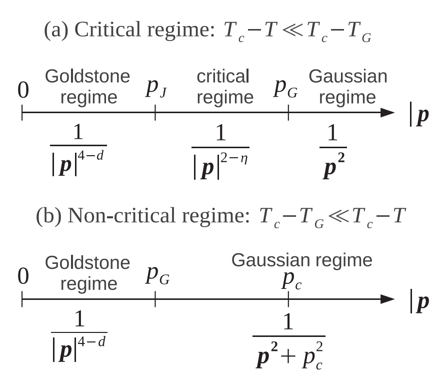

FIG. 4. Momentum dependence of the longitudinal correlation function $G_{\sigma\sigma}(\mathbf{p})$ in the critical and noncritical regimes of the low-temperature phase as obtained from the large-N limit ($2<d<4$).

are smoothly taken into account as k is lowered from the microscopic scale Λ down to 0 [27,28]. This is achieved by adding to action (1) the infrared regulator

$$\Delta S_k[\boldsymbol{\varphi}] = \frac{1}{2}\sum_{\mathbf{p},i}\varphi_i(-\mathbf{p})R_k(\mathbf{p})\varphi_i(\mathbf{p}). \tag{62}$$

The average effective action

$$\Gamma_k[\boldsymbol{\phi}] = -\ln Z_k[J] + \int d^d r \sum_i J_i\phi_i - \Delta S_k[\boldsymbol{\phi}] \tag{63}$$

is defined as a modified Legendre transform of $-\ln Z_k[J]$ that includes the subtraction of $\Delta S_k[\boldsymbol{\phi}]$. Here J_i is an external source that couples linearly to the φ_i field and $\boldsymbol{\phi}(\mathbf{r}) = \langle\boldsymbol{\varphi}(\mathbf{r})\rangle_J$. The cutoff function R_k is chosen such that, at the microscopic scale Λ, it suppresses all fluctuations, so that the mean-field approximation $\Gamma_\Lambda[\boldsymbol{\phi}] = S[\boldsymbol{\phi}]$ becomes exact. The effective action of the original model (1) is given by $\Gamma_{k=0}$ provided that $R_{k=0}$ vanishes. For a generic value of k, the cutoff function $R_k(\mathbf{p})$ suppresses fluctuations with momentum $|\mathbf{p}| \lesssim k$ but leaves unaffected those with $|\mathbf{p}| \gtrsim k$. The variation of the average effective action with k is governed by Wetterich's equation [29]

$$\partial_t\Gamma_k[\boldsymbol{\phi}] = \tfrac{1}{2}\mathrm{Tr}\big\{\dot{R}_k\big(\Gamma_k^{(2)}[\boldsymbol{\phi}] + R_k\big)^{-1}\big\}, \tag{64}$$

where $t = \ln(k/\Lambda)$ and $\dot{R}_k = \partial_t R_k$. $\Gamma_k^{(2)}[\boldsymbol{\phi}]$ denotes the second-order functional derivative of $\Gamma_k[\boldsymbol{\phi}]$. In Fourier space, the trace involves a sum over momenta as well as the internal index of the $\boldsymbol{\phi}$ field.

Because of the regulator term ΔS_k, the vertices $\Gamma^{(n)}_{k,i_1\cdots i_n}(\mathbf{p}_1,\ldots,\mathbf{p}_n)$ are smooth functions of momenta and can be expanded in powers of $\mathbf{p}_i^2/k^2$. Thus if we are interested only in the long-distance physics, we can use a derivative expansion of the average effective action [27,28]. In the following, we consider the ansatz

$$\Gamma_k[\boldsymbol{\phi}] = \int d^d r\left\{\frac{Z_k}{2}(\nabla\boldsymbol{\phi})^2 + U_k(\rho)\right\}. \tag{65}$$

Because of the O(N) symmetry, the effective potential $U_k(\rho)$ must be a function of the O(N) invariant $\rho = \boldsymbol{\phi}^2/2$. To further simplify the analysis, we expand $U_k(\rho)$ about its minimum $\rho_{0,k}$,

$$U_k(\rho) = U_k(\rho_{0,k}) + \frac{\lambda_k}{2}(\rho - \rho_{0,k})^2. \tag{66}$$

We consider only the ordered phase where $\rho_{0,k} > 0$. In a broken symmetry state with order parameter $\boldsymbol{\phi} = (\sqrt{2\rho_{0,k}},0,\ldots,0)$, the two-point vertex is given by

$$\Gamma^{(2)}_{k,ii}(\mathbf{p}) = \begin{cases} Z_k\mathbf{p}^2 + 2\lambda_k\rho_{0,k} & \text{if} \quad i = 1,\\ Z_k\mathbf{p}^2 & \text{if} \quad i \neq 1.\end{cases} \tag{67}$$

By inverting $\Gamma_k^{(2)}$, we obtain the longitudinal and transverse parts of the propagator,

$$G_{k,l}(\mathbf{p}) = \frac{1}{Z_k\mathbf{p}^2 + 2\lambda_k\rho_{0,k}}, \qquad G_{k,t}(\mathbf{p}) = \frac{1}{Z_k\mathbf{p}^2}. \tag{68}$$

Since these expressions are obtained from a derivative expansion of the average effective action, they are valid only in the limit $|\mathbf{p}| \ll k$. In practice, however, one can retrieve the momentum dependence of $G_{k=0}(\mathbf{p})$ at finite $\mathbf{p}$ by stopping the renormalization group (RG) flow at $k \sim |\mathbf{p}|$; that is, $G_{k=0}(\mathbf{p}) \simeq G_{k\sim|\mathbf{p}|}(\mathbf{p})$, where $G_{k\sim|\mathbf{p}|}(\mathbf{p})$ can be approximated by the result of the derivative expansion. It is possible to obtain the full momentum dependence of the correlation functions in a more rigorous and precise way, within the so-called Blaizot–Mendez-Galain–Weschbor scheme [30–32], but this requires a much more involved numerical analysis of the RG equations.

The transverse propagator $G_{k,t}(\mathbf{p})$ is gapless [Eq. (68)], in agreement with Goldstone's theorem, which is a mere consequence of the O(N) symmetry of the average effective action (65). In contrast, the divergence of the longitudinal susceptibility obtained in the previous sections suggests that $\lambda_k \to 0$ for $k \to 0$ ($\lim_{k\to 0}\rho_{0,k} > 0$ in the ordered phase). We will see that this is indeed the result obtained from the RG equations.

2. RG flows

It is convenient to work with the dimensionless variables

$$\tilde{\rho}_{0,k} = Z_k k^{2-d}\rho_{0,k}, \qquad \tilde{\lambda}_k = Z_k^{-2}k^{d-4}\lambda_k. \tag{69}$$

The flow equations for $\tilde{\rho}_{0,k}$, $\tilde{\lambda}_k$ and Z_k are obtained by inserting the ansatz, (65) and (66), into the RG equation, (64). The calculation is standard [27,28] and we only quote the final result:

$$\begin{aligned}\partial_t\tilde{\rho}_{0,k} &= (2 - d - \eta_k)\tilde{\rho}_{0,k} - \frac{3}{2}\tilde{I}_{k,l} - \frac{N-1}{2}\tilde{I}_{k,t},\\ \partial_t\tilde{\lambda}_k &= (d - 4 + 2\eta_k)\tilde{\lambda}_k - \tilde{\lambda}_k^2[9\tilde{J}_{k,ll}(0) + (N-1)\tilde{J}_{k,tt}(0)],\\ \eta_k &= 2\tilde{\lambda}_k^2\tilde{\rho}_{0,k}[\tilde{J}'_{k,lt}(0) + \tilde{J}'_{k,tl}(0)],\end{aligned} \tag{70}$$

where $\eta_k = -\partial_t \ln Z_k$ denotes the running anomalous dimension. With the cutoff function $R_k(\mathbf{p}) = Z_k(\mathbf{p}^2 - k^2)\Theta(\mathbf{p}^2 - k^2)$ [33] ($\Theta(x)$ is the step function), the threshold functions appearing in (70) can be calculated analytically (see Appendix).

In Fig. 5 we show $\tilde{\lambda}_k$, η_k, and $\tilde{\rho}_{0,k}$ vs $-t = \ln(\Lambda/k)$ for $d = 3$ and $N = 3$. We fix $\lambda_{k=0} = u_0/3$ and vary r_0 (i.e., $\rho_{0,k=0} = -3r_0/u_0$). When the system is in the ordered phase away from the critical regime [solid (red) lines in Fig. 5], that is, $p_c \gg p_G$, we see a crossover for $k \sim p_G$ [$t_G = \ln(p_G/\Lambda) \simeq -4$] from the Gaussian regime to the Goldstone regime characterized by $\tilde{\lambda}_k \simeq \tilde{\lambda}^*$, $\eta_k = 0$ and $\tilde{\rho}_{0,k} \sim k^{-1}$ (i.e., $\rho_{0,k} \simeq \rho_0^* = \lim_{k\to 0}\rho_{0,k}$). Since $\tilde{\lambda}_k \simeq \tilde{\lambda}^*$ and $\eta_k = 0$ imply $\lambda_k \sim k$, we find that the longitudinal susceptibility $G_{k,l}(\mathbf{p}) = 1/2\lambda_k\rho_{0,k} \sim 1/k$ diverges when $k \to 0$. Identifying k with $|\mathbf{p}|$ to extract the momentum dependence (as explained above), we recover the singular behavior $G_{k=0,l}(\mathbf{p}) \sim 1/|\mathbf{p}|$ in three dimensions. More generally, for an arbitrary dimension, one finds $\lambda_k \sim k^\epsilon\tilde{\lambda}^*$ and $G_{k,l}(\mathbf{p}) \sim 1/k^\epsilon \equiv 1/|\mathbf{p}|^\epsilon$ with $\epsilon = 4 - d$. Thus in the RG approach, the divergence of the longitudinal susceptibility is a consequence of the existence of a fixed point for the dimensionless coupling constant $\tilde{\lambda}_k$.

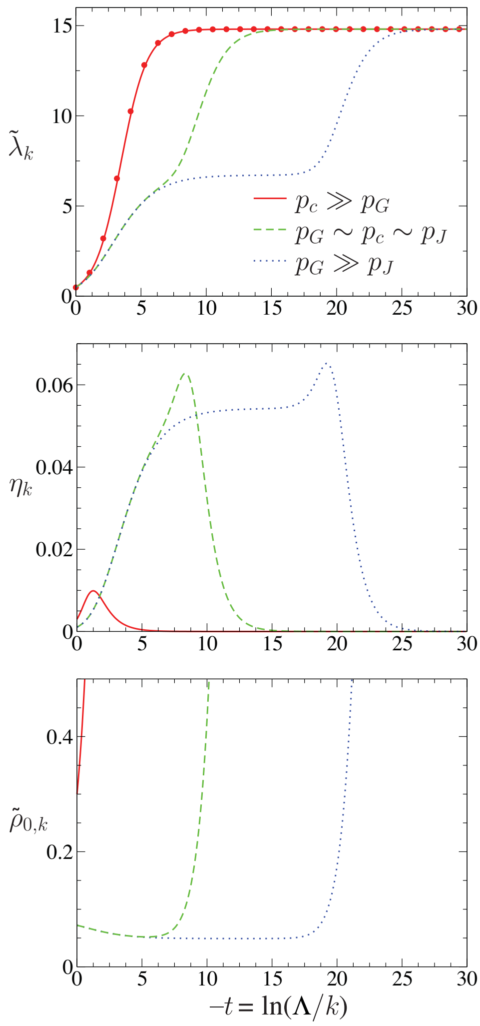

FIG. 5. (Color online) $\tilde\lambda_k$, η_k, and $\tilde\rho_{0,k}$ vs $-t = \ln(\Lambda/k)$ for $d = 3$, $N = 3$, $\Lambda = 1$, and $\lambda_{k=0} = 0.5$. The solid (red) line corresponds to $\rho_{0,k=0} = 0.3$ ($p_c \gg p_G$) and the filled (red) circles are obtained from the analytic solution (73). The dashed (green) line corresponds to $\rho_{0,k=0} = 0.072147$ ($p_c \sim p_G \sim p_J$), and the dotted (blue) line to $\rho_{0,k=0} = 0.072146123$ ($p_G \gg p_J$).

When the system is in the critical regime of the ordered phase [dotted (blue) lines in Fig. 5], that is, $p_G \gg p_J$, there is a first crossover from the Gaussian regime to the critical regime for $k \sim p_G$, followed by a second crossover to the Goldstone regime for $k \sim p_J$. In the critical regime, $p_G \gg k \gg p_J$, $\tilde\lambda_k \simeq \tilde\lambda^*_{\rm cr}$, $\eta_k \simeq \eta^*$, and $\tilde\rho_{0,k} \simeq \tilde\rho_0^*$ are nearly equal to their values at the critical point between the ordered and the disordered phases [34,35]. This gives $G_{k,t}(\mathbf{p}) \simeq G_{k,l}(\mathbf{p}) \sim 1/k^{-\eta^*}\mathbf{p}^2$, that is, $G_{k=0,t}(\mathbf{p}) \simeq G_{k=0,l}(\mathbf{p}) \sim 1/|\mathbf{p}|^{2-\eta^*}$ if we identify k with $|\mathbf{p}|$.

3. Analytical solution in the low-temperature phase

In the low-temperature phase (away from the critical regime, i.e. when $p_c \gg p_G$), it is possible to obtain an analytical solution of the flow equations for $k \ll p_c$. In this limit, the RG flow is dominated by the Goldstone modes and the contribution of the longitudinal mode can be omitted. This amounts to ignoring $\tilde J_{k,ll}(0)$, $\tilde J'_{k,lt}(0)$ and $\tilde J'_{k,tl}(0)$ in Eqs. (70), which is justified by noting that $\tilde\lambda_k\tilde\rho_{0,k}$ becomes very large for $k \ll p_c$ ($\tilde\lambda_k\tilde\rho_{0,k} \sim k^{2-d}$ for $k \to 0$), where the hydrodynamic scale p_c is defined by $2\tilde\lambda_{p_c}\tilde\rho_{0,p_c} \sim 1$. This gives $\eta_k = 0$ and

$$\partial_t\tilde\lambda_k = -\epsilon\tilde\lambda_k + 8\frac{v_d}{d}(N-1)\tilde\lambda_k^2, \qquad (71)$$

where $v_d = [2^{d+1}\pi^{d/2}\Gamma(d/2)]^{-1}$. We have used the expression of the threshold functions given in Appendix. Equation (71) should be solved with the boundary condition $\tilde\lambda_k = \tilde\lambda_c$ for $k = \Lambda_0 \simeq p_c$. For $d < 4$, we then find

$$\begin{aligned}\tilde\lambda_k &= \frac{\epsilon\tilde\lambda_c p_c^\epsilon}{\epsilon k^\epsilon + 8\frac{v_d}{d}(N-1)\tilde\lambda_c(p_c^\epsilon - k^\epsilon)}\\ &\simeq \frac{\epsilon\tilde\lambda_c p_c^\epsilon}{\epsilon k^\epsilon + 8\frac{v_d}{d}(N-1)\tilde\lambda_c p_c^\epsilon}\end{aligned} \qquad (72)$$

for $k \ll p_c$. The last expression can be rewritten in the more insightful form

$$\tilde\lambda_k = \frac{\tilde\lambda^*}{1 + (k/p_G)^\epsilon}, \qquad (73)$$

where

$$\tilde\lambda^* = \lim_{k\to 0}\tilde\lambda_k = \frac{\epsilon d}{8v_d(N-1)} \qquad (74)$$

and

$$\begin{aligned}p_G &= \left[(N-1)\frac{8v_d p_c^\epsilon\tilde\lambda_c}{d\epsilon}\right]^{1/\epsilon}\\ &= \left[(N-1)\frac{8v_d\lambda_c}{d\epsilon Z_{p_c}^2}\right]^{1/\epsilon}.\end{aligned} \qquad (75)$$

Equation (73) is in remarkable agreement with the numerical solution of the flow equations (70) (Fig. 5). In the weak-coupling limit $p_G \ll p_c$, we can ignore the renormalization of Z_k as well as that of λ_k between $k = \Lambda$ and $k = p_c$, and approximate $Z_{p_c} \simeq 1$ and $\lambda_c \simeq \lambda_{k=\Lambda} = u_0/3$. We then recover the expression

$$p_G \simeq \left[(N-1)\frac{8v_d u_0}{3d\epsilon}\right]^{1/\epsilon} \qquad (76)$$

of the Ginzburg momentum scale obtained in previous sections. A similar analysis can be made for the case $d = 4$.

III. INTERACTING BOSONS

We consider interacting bosons at zero temperature with the (Euclidean) action

$$S = \int dx\left[\psi^*\left(\partial_\tau - \mu - \frac{\nabla^2}{2m}\right)\psi + \frac{g}{2}(\psi^*\psi)^2\right], \qquad (77)$$

where $\psi(x)$ is a bosonic (complex) field, $x = (\mathbf{r},\tau)$, and $\int dx = \int_0^\beta d\tau\int d^dr$. $\tau \in [0,\beta]$ is an imaginary time, $\beta \to \infty$ the inverse temperature, and μ denotes the chemical potential. The interaction is assumed to be local in space and the model is regularized by a momentum cutoff Λ. We consider a space dimension $d > 1$.

Introducing the two-component field

$$\Psi(p) = \begin{pmatrix} \psi(p) \\ \psi^*(-p) \end{pmatrix}, \quad \Psi^\dagger(p) = (\psi^*(p),\psi(-p)), \quad (78)$$

with $p = (\mathbf{p}, i\omega)$ and ω a Matsubara frequency, the one-particle (connected) propagator becomes a 2×2 matrix whose inverse in Fourier space is given by

$$\begin{pmatrix} i\omega + \mu - \epsilon_{\mathbf{p}} - \Sigma_{\rm n}(p) & -\Sigma_{\rm an}(p) \\ -\Sigma^*_{\rm an}(p) & -i\omega + \mu - \epsilon_{\mathbf{p}} - \Sigma_{\rm n}(-p) \end{pmatrix}, \quad (79)$$

where $\Sigma_{\rm n}$ and $\Sigma_{\rm an}$ are the normal and anomalous self-energies, respectively, and $\epsilon_{\mathbf{p}} = \mathbf{p}^2/2m$. If we choose the order parameter $\langle\psi(x)\rangle = \sqrt{n_0}$ to be real (with n_0 the condensate density), then the anomalous self-energy $\Sigma_{\rm an}(p)$ is real.

To make contact with the classical $(\boldsymbol{\varphi}^2)^2$ theory with O(N) symmetry studied in Sec. II, it is convenient to write the boson field

$$\psi(x) = \frac{1}{\sqrt{2}}[\psi_1(x) + i\psi_2(x)] \quad (80)$$

in terms of two real fields ψ_1 and ψ_2 and consider the (connected) propagator $G_{ij}(x,x') = \langle\psi_i(x)\psi_j(x')\rangle_c$. The inverse propagator $G^{-1}_{ij}(p)$ reads

$$\begin{pmatrix} \epsilon_{\mathbf{p}} - \mu + \Sigma_{11}(p) & \omega + \Sigma_{12}(p) \\ -\omega + \Sigma_{21}(p) & \epsilon_{\mathbf{p}} - \mu + \Sigma_{22}(p) \end{pmatrix}, \quad (81)$$

where

$$\begin{aligned} \Sigma_{11}(p) &= \frac{1}{2}[\Sigma_{\rm n}(p) + \Sigma_{\rm n}(-p)] + \Sigma_{\rm an}(p), \\ \Sigma_{22}(p) &= \frac{1}{2}[\Sigma_{\rm n}(p) + \Sigma_{\rm n}(-p)] - \Sigma_{\rm an}(p), \\ \Sigma_{12}(p) &= \frac{i}{2}[\Sigma_{\rm n}(p) - \Sigma_{\rm n}(-p)], \\ \Sigma_{21}(p) &= -\frac{i}{2}[\Sigma_{\rm n}(p) - \Sigma_{\rm n}(-p)], \end{aligned} \quad (82)$$

when $\Sigma_{\rm an}(p)$ is real.

A. Perturbation theory and infrared divergences

1. Bogoliubov's theory

The Bogoliubov approximation is a Gaussian fluctuation theory about the saddle-point solution $\psi(x) = \sqrt{n_0} = \sqrt{\mu/g}$ [i.e., $\psi_1(x) = \sqrt{2n_0}$ and $\psi_2(x) = 0$]. It is equivalent to a zero-loop calculation of the self-energies,

$$\Sigma^{(0)}_{\rm n}(p) = 2gn_0, \quad \Sigma^{(0)}_{\rm an}(p) = gn_0, \quad (83)$$

or, equivalently,

$$\Sigma^{(0)}_{11}(p) = 3gn_0, \quad \Sigma^{(0)}_{22}(p) = gn_0, \quad \Sigma^{(0)}_{12}(p) = 0. \quad (84)$$

This yields the (connected) propagators

$$\begin{aligned} G^{(0)}_{\rm n}(p) &= -\langle\psi(p)\psi^*(p)\rangle_c = \frac{-i\omega - \epsilon_{\mathbf{p}} - gn_0}{\omega^2 + E^2_{\mathbf{p}}}, \\ G^{(0)}_{\rm an}(p) &= -\langle\psi(p)\psi(-p)\rangle_c = \frac{gn_0}{\omega^2 + E^2_{\mathbf{p}}}, \end{aligned} \quad (85)$$

where $E_{\mathbf{p}} = [\epsilon_{\mathbf{p}}(\epsilon_{\mathbf{p}} + 2gn_0)]^{1/2}$ is the Bogoliubov quasiparticle excitation energy. When $|\mathbf{p}|$ is larger than the healing momentum $p_c = (2gmn_0)^{1/2}$, the spectrum $E_{\mathbf{p}} \simeq \epsilon_{\mathbf{p}} + gn_0$ is particle-like, whereas it becomes sound-like for $|\mathbf{p}| \ll p_c = \sqrt{2}mc$, with a velocity $c = \sqrt{gn_0/m}$. In the weak-coupling limit, $n_0 \simeq \bar{n}$ ($\bar{n}$ is the mean boson density) and p_c can equivalently be defined as $p_c = (2gm\bar{n})^{1/2}$. In the hydrodynamic regime $|\mathbf{p}| \ll p_c$,

$$\begin{aligned} G^{(0)}_{11}(p) &= \frac{\epsilon_{\mathbf{p}}}{\omega^2 + c^2\mathbf{p}^2}, \\ G^{(0)}_{22}(p) &= \frac{2gn_0}{\omega^2 + c^2\mathbf{p}^2}, \\ G^{(0)}_{12}(p) &= -\frac{\omega}{\omega^2 + c^2\mathbf{p}^2}. \end{aligned} \quad (86)$$

Note that in the Bogoliubov approximation, the occurrence of a linear spectrum at low energy (which implies superfluidity according to Landau's criterion) is due to $\Sigma_{\rm an}(0)$ being nonzero.

2. Infrared divergences and the Ginzburg scale

Let us now consider the one-loop correction $\Sigma^{(1)}$ to the Bogoliubov result $\Sigma^{(0)}$. For $d \leqslant 3$, the second diagram in Fig. 1 gives a divergent contribution when the two internal lines correspond to transverse fluctuations, which is possible only for Σ_{11}. Thus Σ_{22} is finite at the one-loop level and the normal and anomalous self-energies exhibit the same divergence,

$$\Sigma^{(1)}_{\rm n}(p) \simeq \Sigma^{(1)}_{\rm an}(p) \simeq -\frac{1}{2}g^2 n_0 \int_q G^{(0)}_{22}(q) G^{(0)}_{22}(p+q), \quad (87)$$

where we use the notation $q = (\mathbf{q}, i\omega')$ and $\int_q = \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \int_{\mathbf{q}}$. For small p, the main contribution to the integral in (87) comes from momenta $|\mathbf{q}| \lesssim p_c$ and frequencies $|\omega'| \lesssim cp_c$, so that we can use (86) and obtain

$$\Sigma^{(1)}_{\rm n}(p) \simeq \Sigma^{(1)}_{\rm an}(p) \simeq -2\frac{g^4 n_0^3}{c^3} \int_{\mathbf{Q}} \frac{1}{\mathbf{Q}^2(\mathbf{Q}+\mathbf{P})^2}, \quad (88)$$

where $\mathbf{Q} = (\mathbf{q}, \omega'/c)$ and $\mathbf{P} = (\mathbf{p}, \omega/c)$ are $(d+1)$–dimensional vectors. The momentum integral in (88) is restricted by $|\mathbf{Q}| \lesssim p_c$ and is given by (13), with Λ replaced by p_c, d by $d+1$, and $|\mathbf{p}|$ by $(\mathbf{p}^2 + \omega^2/c^2)^{1/2}$. We can estimate the characteristic (Ginzburg) momentum scale p_G below which the Bogoliubov approximation breaks down from the condition $|\Sigma^{(1)}_{\rm n}(p)| \sim \Sigma^{(0)}_{\rm n}(p)$ or $|\Sigma^{(1)}_{\rm an}(p)| \sim \Sigma^{(0)}_{\rm an}(p)$ for $|\mathbf{p}| = p_G$ and $|\omega| = cp_G$,

$$p_G \sim \begin{cases} (A_{d+1} gmp_c)^{1/(3-d)} & \text{if} \quad d < 3, \\ p_c \exp\left(-\frac{1}{A_4 gmp_c}\right) & \text{if} \quad d = 3. \end{cases} \quad (89)$$

This result can be rewritten as

$$p_G \sim \begin{cases} p_c (A_{d+1} \tilde{g}^{d/2})^{1/(3-d)} & \text{if} \quad d < 3, \\ p_c \exp\left(-\frac{1}{A_4 \sqrt{2}\tilde{g}^{3/2}}\right) & \text{if} \quad d = 3, \end{cases} \quad (90)$$

where

$$\tilde{g} = gm\bar{n}^{1-2/d} \sim \left(\frac{p_c}{\bar{n}^{1/d}}\right)^2 \tag{91}$$

is the dimensionless coupling constant obtained by comparing the mean interaction energy per particle $g\bar{n}$ to the typical kinetic energy $1/m\bar{r}^2$, where $\bar{r} \sim \bar{n}^{-1/d}$ is the mean distance between particles [36]. A superfluid is weakly correlated if $\tilde{g} \ll 1$, that is, $p_G \ll p_c \ll \bar{n}^{1/d}$ (the characteristic momentum scale $\bar{n}^{1/d}$ does, however, not play any role in the weak-coupling limit) [37]. In this case, the Bogoliubov theory applies to a large part of the spectrum where the dispersion is linear (i.e., $|\mathbf{p}| \lesssim p_c$) and breaks down only at very low momenta $|\mathbf{p}| \lesssim p_G \ll p_c$. In the next sections, we see that the weakly correlated superfluid bears many similarities to the ordered phase of the classical O(N) model away from the critical regime. When the dimensionless coupling $\tilde{g}$ becomes of order unity, the three characteristic momentum scales $p_G \sim p_c \sim \bar{n}^{1/d}$ become of the same order. The momentum range $[p_G, p_c]$ where the linear spectrum can be described by the Bogoliubov theory is then suppressed. We expect the strong-coupling regime $\tilde{g} \gg 1$ to be governed by a single characteristic momentum scale, namely, $\bar{n}^{1/d}$.

3. *Vanishing of the anomalous self-energy*

The exact values of $\Sigma_n(p=0)$ and $\Sigma_{an}(p=0)$ can be obtained using the U(1) symmetry of the action, that is, the invariance under the field transformation $\psi(x) \to e^{i\theta}\psi(x)$ and $\psi^*(x) \to e^{-i\theta}\psi^*(x)$ [14,38]. The derivation is similar to that in Sec. II A 3. Let us consider the effective action

$$\Gamma[\phi] = -\ln Z[J_1,J_2] + \int dx[J_1\phi_1 + J_2\phi_2], \tag{92}$$

where J_i is an external source that couples linearly to the boson field ψ_i, and $\phi_i(x) = \langle\psi_i(x)\rangle_J$ the superfluid order parameter. The U(1) symmetry of the action implies that $\Gamma[\phi]$ is invariant under a uniform rotation of the vector field $(\phi_1(x),\phi_2(x))^T$. For an infinitesimal rotation angle θ, this yields

$$\int dx \sum_{i,j} \frac{\delta\Gamma[\phi]}{\delta\phi_i(x)}\epsilon_{ij}\phi_j(x) = 0, \tag{93}$$

where ϵ_{ij} is the totally antisymmetric tensor. Taking the functional derivative $\delta/\delta\phi_l(y)$ and setting $\phi_i(x) = \delta_{i,1}\sqrt{2n_0}$ leads to

$$\Gamma^{(2)}_{2l}(p=0) = 0. \tag{94}$$

For $l=2$, Eq. (94) yields the Hugenholtz-Pines theorem [12]

$$\Gamma^{(2)}_{22}(p=0) = \Sigma_n(p=0) - \Sigma_{an}(p=0) - \mu = 0. \tag{95}$$

If we now take the second-order functional derivative $\delta^{(2)}/\delta\phi_l(y)\delta\phi_m(z)$ of (93) and set $\phi_i(x) = \delta_{i,1}\sqrt{2n_0}$, we obtain the Ward identity

$$\sum_i \Gamma^{(2)}_{im}(y,z)\epsilon_{il} + \sum_i \Gamma^{(2)}_{il}(z,y)\epsilon_{im} - \sqrt{2n_0}\int dx\,\Gamma^{(3)}_{2lm}(x,y,z) = 0. \tag{96}$$

Integrating over y and z and setting $l=2$ and $m=1$, we deduce (in Fourier space)

$$\Gamma^{(3)}_{122}(0,0,0) = \frac{1}{\sqrt{\beta V}}\frac{\Gamma^{(2)}_{11}(0,0)}{\sqrt{2n_0}}, \tag{97}$$

where we have used (95).

The self-energy Σ_{11} can be written as

$$\Sigma_{11}(p) = \tilde{\Sigma}_{11}(p) - g\sqrt{\frac{n_0}{2\beta V}}\sum_q G_{22}(q)G_{22}(p+q) \times \Gamma^{(3)}_{122}(-p,-q,p+q), \tag{98}$$

where $\tilde{\Sigma}_{11}(p)$ denotes the regular part of the self-energy (i.e., the part that does not contain pairs of lines corresponding to $G_{22}G_{22}$). If we assume that the transverse propagator $G_{22}(q) \sim 1/(\omega^2 + c^2\mathbf{q}^2)$ at low energies (this result is shown in the following sections), the integral $\int_q G_{22}(q)^2$ is infrared divergent for $d \leqslant 3$. To obtain a finite self-energy $\Sigma_{11}(p=0)$, one must require that $\Gamma^{(3)}_{122}(0,0,0) = 0$. The Ward identity (97) then implies $\Gamma^{(2)}_{11}(p=0) = 0$, and in turn,

$$\begin{aligned} \Sigma_n(p=0) &= \mu + \tfrac{1}{2}\left[\Gamma^{(2)}_{11}(p=0) + \Gamma^{(2)}_{22}(p=0)\right] = \mu, \\ \Sigma_{an}(p=0) &= \tfrac{1}{2}\left[\Gamma^{(2)}_{11}(p=0) - \Gamma^{(2)}_{22}(p=0)\right] = 0. \end{aligned} \tag{99}$$

The vanishing of the anomalous self-energy $\Sigma_{an}(p=0)$ was first proven by Nepomnyashchii and Nepomnyashchii [14]. To reconcile this result with the existence of a sound mode with linear dispersion, the self-energies $\Sigma_n(p)$ and $\Sigma_{an}(p)$ must necessarily contain nonanalytic terms in the limit $p \to 0$ (Sec. III B 4).

B. Hydrodynamic approach

It was realized by Popov that the phase-density representation of the boson field $\psi = \sqrt{n}e^{i\theta}$ leads to a theory free of infrared divergences [16–18]. Popov's theory bears some similarities to the analysis of the $(\boldsymbol{\varphi}^2)^2$ theory based on the amplitude-direction representation (Sec. II B). In this section, we show how the phase-density representation can be used to obtain the infrared behavior of the propagators $G_n(p)$ and $G_{an}(p)$ without encountering infrared divergences [19]. Our approach is similar to that of Popov (with some technical differences in Sec. III B 2).

1. *Perturbative approach*

In terms of the density and phase fields, the action reads

$$S[n,\theta] = \int dx\left[in\dot{\theta} + \frac{n}{2m}(\nabla\theta)^2 + \frac{(\nabla n)^2}{8mn} - \mu n + \frac{g}{2}n^2\right]. \tag{100}$$

At the saddle-point level, $n(x) = \bar{n} = \mu/g$. Expanding the action to second order in $\delta n = n - \bar{n}$, $\dot{\theta}$ and $\nabla\theta$, we obtain

$$S[\delta n,\theta] = \int dx\left[i\delta n\dot{\theta} + \frac{\bar{n}}{2m}(\nabla\theta)^2 + \frac{(\nabla n)^2}{8m\bar{n}} + \frac{g}{2}(\delta n)^2\right]. \tag{101}$$

The higher-order terms can be taken into account within perturbation theory and only lead to finite corrections of the coefficients of the hydrodynamic action (101) [18].

We deduce the correlation functions of the hydrodynamic variables,

$$G_{nn}(p) = \langle \delta n(p)\delta n(-p)\rangle = \frac{\bar n}{m}\frac{\mathbf{p}^2}{\omega^2+E_{\mathbf{p}}^2},$$
$$G_{n\theta}(p) = \langle \delta n(p)\theta(-p)\rangle = -\frac{\omega}{\omega^2+E_{\mathbf{p}}^2}, \tag{102}$$
$$G_{\theta\theta}(p) = \langle \theta(p)\theta(-p)\rangle = \frac{\frac{\mathbf{p}^2}{4m\bar n}+g}{\omega^2+E_{\mathbf{p}}^2},$$

where $E_{\mathbf{p}}$ is the Bogoliubov excitation energy defined in Sec. III A 1. In the hydrodynamic regime $|\mathbf{p}| \ll p_c = \sqrt{2gm\bar n}$,

$$G_{nn}(p) = \frac{\bar n}{m}\frac{\mathbf{p}^2}{\omega^2+c^2\mathbf{p}^2},$$
$$G_{n\theta}(p) = -\frac{\omega}{\omega^2+c^2\mathbf{p}^2}, \tag{103}$$
$$G_{\theta\theta}(p) = \frac{mc^2}{\bar n}\frac{1}{\omega^2+c^2\mathbf{p}^2},$$

where $c=\sqrt{g\bar n/m}$ is the Bogoliubov sound mode velocity ($p_c=\sqrt{2}mc$).

2. *Exact hydrodynamic description*

In this section, we show that Eqs. (103) are exact in the low-energy limit $|\mathbf{p}|,|\omega|/c \ll p_c$ provided that c is the exact sound mode velocity and $\bar n$ the actual mean density (which may differ from μ/g). Let us consider the effective action $\Gamma[n,\theta]$ defined as the Legendre transform of the free energy $-\ln Z[J_n,J_\theta]$ (J_n and J_θ are external sources linearly coupled to n and θ) [39]. At zero temperature, $\Gamma[n,\theta]$ inherits Galilean invariance from the action (100). In a Galilean transformation (in imaginary time), $\mathbf{r}'=\mathbf{r}+i\mathbf{v}\tau$ and $\tau'=\tau$, the fields transform as

$$n'(x') = n(x),$$
$$\theta'(x') = \theta(x) - \frac{i}{2}m\mathbf{v}^2\tau - m\mathbf{v}\cdot\mathbf{r}, \tag{104}$$

where $x'=(\mathbf{r}',\tau')$. $n(x)$, $\nabla n(x)$, and $i\partial_\tau\theta+\frac{1}{2m}(\nabla\theta)^2$ are Galilean invariant [but $\partial_\tau n(x)$ is not]. $\nabla^2\theta$ is also invariant but is odd under time-reversal symmetry. Thus, to second order in derivatives, the most general effective action compatible with Galilean invariance and time-reversal symmetry reads

$$\Gamma[n,\theta] = \int dx\left\{\frac{Y(n)}{8m}(\nabla n)^2 + U(n) + \sum_{p=1}^{2} c_p(n)\left[i\partial_\tau\theta+\frac{1}{2m}(\nabla\theta)^2\right]^p\right\}, \tag{105}$$

up to an additive (field-independent) term. $Y(n)$, $U(n)$, and $c_p(n)$ are arbitrary functions of n.

To determine $c_p(n)$, we now consider the system in the presence of a fictitious vector potential $(A_0,\mathbf{A})$:

$$S[n,\theta;A_\mu] = \int dx\left[in(\partial_\tau\theta - A_0) + \frac{n}{2m}(\nabla\theta-\mathbf{A})^2 + \frac{(\nabla n)^2}{8mn} - \mu n + \frac{g}{2}n^2\right]. \tag{106}$$

The action is invariant under the local U(1) transformation $\theta\to\theta+\alpha$ and $A_\mu\to A_\mu+\partial_\mu\alpha$, where $\alpha(x)$ is an arbitrary phase. By requiring that $\Gamma[n,\theta;A_\mu]=\Gamma[n,\theta+\alpha;A_\mu+\partial_\mu\alpha]$ shares the same invariance, we deduce

$$\Gamma[n,\theta;A_\mu] = \int dx\left\{\frac{Y(n)}{8m}(\nabla n)^2 + U(n) + \sum_{p=1}^{2} c_p(n)\left[i\partial_\tau\theta - iA_0 + \frac{1}{2m}(\nabla\theta-\mathbf{A})^2\right]^p\right\}. \tag{107}$$

Noting that

$$n(x) = \frac{\delta\ln Z[J_n,J_\theta;A_\mu]}{i\delta A_0(x)} = -\frac{\delta\Gamma[n,\theta;A_\mu]}{i\delta A_0(x)}, \tag{108}$$

we must have $c_1(n)=n$ and $c_p(n)=0$ for $p\geqslant 2$. We conclude that

$$\Gamma[n,\theta] = \int dx\left\{\frac{Y(n)}{8m}(\nabla n)^2 + U(n) + n\left[i\partial_\tau\theta+\frac{(\nabla\theta)^2}{2m}\right]\right\} \tag{109}$$

to second order in derivatives.

From (109), we obtain the two-point vertex in constant fields $n(x)=\bar n$ and $\theta(x)=\text{const}$ (with $\bar n$ the actual boson density),

$$\Gamma^{(2)}(p) = \begin{pmatrix}\Gamma^{(2)}_{nn}(p) & \Gamma^{(2)}_{n\theta}(p)\\ \Gamma^{(2)}_{\theta n}(p) & \Gamma^{(2)}_{\theta\theta}(p)\end{pmatrix} = \begin{pmatrix}\frac{Y(\bar n)}{4m}\mathbf{p}^2+U''(\bar n) & \omega\\ -\omega & \frac{\bar n}{m}\mathbf{p}^2\end{pmatrix}. \tag{110}$$

By inverting $\Gamma^{(2)}(p)$, we recover the propagators (103) in the low-momentum limit $|\mathbf{p}|\ll p_c=[4mU''(\bar n)/Y(\bar n)]^{1/2}$, but with a sound mode velocity c given by

$$c = \sqrt{\frac{\bar n U''(\bar n)}{m}}. \tag{111}$$

Noting that the compressibility $\kappa=\bar n^{-2}d\bar n/d\mu$ can also be expressed as [40]

$$\kappa = \frac{1}{\bar n^2 U''(\bar n)}, \tag{112}$$

we conclude that the Bogoliubov sound mode velocity c is equal to the macroscopic sound velocity $(m\bar n\kappa)^{-1/2}$. Moreover, since the superfluid density n_s is defined by $\Gamma^{(2)}_{22}(\mathbf{p},0)=\frac{n_s}{m}\mathbf{p}^2$ for $\mathbf{p}\to 0$ [8], we find that at zero temperature $n_s=\bar n$ is given by the fluid density [13].

3. *Normal and anomalous propagators*

To compute the propagator of the ψ field, we write

$$\psi(x) = \sqrt{n_0+\delta n(x)}\,e^{i\theta(x)}, \tag{113}$$

where $n_0=|\langle\psi(x)\rangle|^2=|\langle\sqrt{n(x)}e^{i\theta(x)}\rangle|^2$ is the condensate density. For a weakly interacting superfluid, $n_0\simeq\bar n$, and we expect the fluctuations δn to be small. Let us assume that the superfluid order parameter $\langle\psi(x)\rangle = \sqrt{n_0}$ is real. Transverse and longitudinal fluctuations are then expressed as

$$\delta\psi_2 = \sqrt{2n_0}\theta + \cdots, \quad \delta\psi_1 = \frac{\delta n}{\sqrt{2n_0}} - \sqrt{\frac{n_0}{2}}\theta^2 + \cdots, \tag{114}$$

where the ellipses stand for subleading contributions to the low-energy behavior of the correlation functions. For the transverse propagator, we obtain

$$G_{22}(p) \simeq 2n_0 G_{\theta\theta}(p) = \frac{2n_0 mc^2}{\bar n}\frac{1}{\omega^2 + c^2\mathbf{p}^2} \tag{115}$$

to leading order in the hydrodynamic regime, while

$$G_{12}(p) \simeq G_{n\theta}(p) = -\frac{\omega}{\omega^2 + c^2\mathbf{p}^2}. \tag{116}$$

The longitudinal propagator is given by

$$G_{11}(x) = \frac{1}{2n_0}G_{nn}(x) + \frac{n_0}{2}\langle\theta(x)^2\theta(0)^2\rangle_c = \frac{1}{2n_0}G_{nn}(x) + n_0 G_{\theta\theta}(x)^2, \tag{117}$$

where the second line is obtained using Wick's theorem [which is justified since the Goldstone (phase) mode is effectively noninteracting in the hydrodynamic limit]. In Fourier space,

$$G_{11}(p) = \frac{\bar n}{2mn_0}\frac{\mathbf{p}^2}{\omega^2 + c^2\mathbf{p}^2} + n_0 G_{\theta\theta}\star G_{\theta\theta}(p), \tag{118}$$

where

$$G_{\theta\theta}\star G_{\theta\theta}(p) = \int_q G_{\theta\theta}(q)G_{\theta\theta}(p+q), \tag{119}$$

with the dominant contribution to the integral coming from momenta $|\mathbf{q}| \lesssim p_c$ and frequencies $|\omega'|/c \lesssim p_c$. Using (13), we find

$$G_{\theta\theta}\star G_{\theta\theta}(p) = \begin{cases} A_{d+1}c\left(\frac{m}{\bar n}\right)^2\left(\mathbf{p}^2 + \frac{\omega^2}{c^2}\right)^{(d-3)/2} & \text{if} \quad d<3, \\ \frac{A_4}{2}c\left(\frac{m}{\bar n}\right)^2 \ln\left(\frac{p_c^2}{\mathbf{p}^2 + \frac{\omega^2}{c^2}}\right) & \text{if} \quad d=3. \end{cases} \tag{120}$$

By comparing the two terms on the right-hand side of (118) with $|\mathbf{p}| = p_G$ and $|\omega| = cp_G$, we recover the Ginzburg scale, (89). For $|\mathbf{p}|,|\omega|/c \gg p_G$, the last term on the right-hand side of (118) can be neglected and we reproduce the result of the Bogoliubov theory (noting that $\bar n \simeq n_0$), while for $|\mathbf{p}|,|\omega|/c \ll p_G$, $G_{11}(p) \sim 1/(\omega^2 + c^2\mathbf{p}^2)^{(3-d)/2}$ is dominated by phase fluctuations. The longitudinal susceptibility $G_{11}(\mathbf{p},i\omega=0) \sim 1/|\mathbf{p}|^{3-d}$ for $\mathbf{p}\to 0$, in contrast to the Bogoliubov approximation, $G_{11}(\mathbf{p},i\omega=0) = 1/2mc^2$.

From these results, we deduce the hydrodynamic behavior of the normal propagator,

$$G_{\rm n}(p) = -\frac{1}{2}[G_{11}(p) - 2iG_{12}(p) + G_{22}(p)] = -\frac{n_0 mc^2}{\bar n}\frac{1}{\omega^2 + c^2\mathbf{p}^2} - \frac{i\omega}{\omega^2 + c^2\mathbf{p}^2} - \frac{1}{2}G_{11}(p), \tag{121}$$

as well as that of the anomalous propagator,

$$G_{\rm an}(p) = -\frac{1}{2}[G_{11}(p) - G_{22}(p)] = \frac{n_0 mc^2}{\bar n}\frac{1}{\omega^2 + c^2\mathbf{p}^2} - \frac{1}{2}G_{11}(p), \tag{122}$$

where $G_{11}(p)$ is given by (118). The leading-order terms in (121) and (122) agree with the results of Gavoret and Nozières [13] and are exact (see the next section). The contribution of the diverging longitudinal correlation function was first identified by Nepomnyashchii and Nepomnyashchii [15] and, later, in Refs. [19] and [41–44].

4. Normal and anomalous self-energies

To compute the self-energies $\Sigma_{\rm n}(p)$ and $\Sigma_{\rm an}(p)$, we use the relations

$$\Sigma_{\rm n}(p) = G_0^{-1}(p) - \frac{G_{\rm n}(-p)}{G_{\rm n}(p)G_{\rm n}(-p) - G_{\rm an}(p)^2}, \quad \Sigma_{\rm an}(p) = \frac{G_{\rm an}(p)}{G_{\rm n}(p)G_{\rm n}(-p) - G_{\rm an}(p)^2}, \tag{123}$$

with

$$G_{\rm n}(p)G_{\rm n}(-p) - G_{\rm an}(p)^2 = G_{11}(p)G_{22}(p) + G_{12}(p)^2 = G_{22}(p)\left[n_0 G_{\theta\theta}\star G_{\theta\theta}(p) + \frac{\bar n}{2n_0 mc^2}\right]. \tag{124}$$

Setting

$$G_{\rm n}(p) \simeq -\tfrac{1}{2}G_{22}(p), \quad G_{\rm an}(p) \simeq \tfrac{1}{2}G_{22}(p), \tag{125}$$

in the numerator of Eqs. (123), we obtain

$$\Sigma_{\rm an}(p) = \Sigma_{\rm n}(p) - G_0^{-1}(p) = \begin{cases} \frac{\bar n^2}{2A_{d+1}c^{4-d}n_0 m^2}(\omega^2 + c^2\mathbf{p}^2)^{(3-d)/2} & \text{if} \quad d<3, \\ \frac{\bar n^2}{A_4 cn_0 m^2}\left[\ln\left(\frac{c^2p_c^2}{\omega^2 + c^2\mathbf{p}^2}\right)\right]^{-1} & \text{if} \quad d=3, \end{cases} \tag{126}$$

in the infrared limit $|\mathbf{p}|,|\omega|/c \ll p_G$, where $G_0^{-1}(p) = i\omega - \epsilon_{\mathbf{p}} + \mu$. Equations (126) agree with the exact results (99) and show that $\Sigma_{\rm n}(p)$ and $\Sigma_{\rm an}(p)$ are dominated by nonanalytic terms for $p\to 0$. This nonanalyticity reflects the singular behavior of the longitudinal correlation function

$$G_{11}(p) \simeq \frac{1}{2\Sigma_{\rm an}(p)} \tag{127}$$

in the low-energy limit.

It should be noted that the singularity of the self-energies is crucial to reconcile the existence of a sound mode with a linear dispersion and the vanishing of the anomalous self-energy $\Sigma_{\rm an}(p=0)$ [14]. In the low-energy limit,

$$\Sigma_{\rm an}(p) = \Delta\Sigma(p) + \tilde\Sigma_{\rm an}(p), \quad \Sigma_{\rm n}(p) - G_0^{-1}(p) = \Delta\Sigma(p) + \tilde\Sigma_{\rm n}(p), \tag{128}$$

where $\Delta\Sigma(p)$ denotes the singular part (126), while $\tilde\Sigma_{\rm n}(p)$ and $\tilde\Sigma_{\rm an}(p)$ are regular contributions of order $\mathbf{p}^2,\omega^2$. Using

$\Delta\Sigma(p) \gg \tilde\Sigma_{\rm n}(p) - G_0^{-1}(p), \tilde\Sigma_{\rm an}(p)$ for $p \to 0$, by inverting (79) we obtain

$$\begin{aligned} G_{\rm n}(p) &\simeq -\frac{1}{2[\tilde\Sigma_{\rm n}(p) - \tilde\Sigma_{\rm an}(p)]}, \\ G_{\rm an}(p) &\simeq \frac{1}{2[\tilde\Sigma_{\rm n}(p) - \tilde\Sigma_{\rm an}(p)]}. \end{aligned} \tag{129}$$

Since both $\tilde\Sigma_{\rm n}(p)$ and $\tilde\Sigma_{\rm an}(p)$ can be expanded to order $\mathbf{p}^2,\omega^2$, we conclude that Eqs. (129) predict the existence of a sound mode with linear dispersion. Of course, Eqs. (129) are nothing but our previous Eqs. (115) and (125).

In deriving the low-energy expression (126) of the self-energies, we have assumed that the hydrodynamic description holds up to the momentum scale p_c and ignored the contribution of the nonhydrodynamic modes. In Popov's original approach [19], one introduces a momentum cutoff p_0 satisfying $p_G \ll p_0 \ll p_c$. Since $p_0 \gg p_G$, modes with momenta $|\mathbf{p}| \geqslant p_0$ can be taken into account within standard perturbation theory (see Sec. III A). In contrast, low-momentum modes $|\mathbf{p}| \leqslant p_0 \ll p_c$ are naturally treated in the hydrodynamic approach discussed in this section. The final results are independent of p_0. The only difference from our results (126) is that p_c in the expression of $\Sigma_{\rm an}(p)$ for $d = 3$ is replaced by a smaller momentum scale [45].

C. The nonperturbative renormalization group

The NPRG approach to zero-temperature interacting bosons has been discussed in detail in Refs. [8,43,44], and [46–51]. Our aim in this section is to briefly summarize the main results [52] while emphasizing the common points with the classical O(N) model studied in Sec. II D.

To implement the NPRG, we add to the action an infrared regulator term,

$$\Delta S_k[\psi^*,\psi] = \sum_p \psi^*(p) R_k(p)\psi(p), \tag{130}$$

which suppresses fluctuations with momentum/frequency below a characteristic scale k but leaves high-momentum/frequency modes unaffected. The average effective action is defined as

$$\begin{aligned} \Gamma_k[\phi^*,\phi] = &-\ln Z_k[J^*,J] + \sum_p [J^*(p)\phi(p) + \text{C.c.}] \\ &- \Delta S_k[\phi^*,\phi], \end{aligned} \tag{131}$$

where $\phi(x) = \langle\psi(x)\rangle_J$ is the superfluid order parameter. J denotes a complex external source that couples linearly to the boson field. Γ_k satisfies the RG equation, (64). As in Sec. II D, we choose the cutoff function R_k such that all fluctuations are suppressed for $k = \Lambda$ (so that $\Gamma_\Lambda[\phi^*,\phi] = S[\phi^*,\phi]$) and $R_{k=0}(p) = 0$. In practice, we take [8]

$$R_k(p) = \frac{Z_{A,k}}{2m}\left(\mathbf{p}^2 + \frac{\omega^2}{c_0^2}\right) r\left(\frac{\mathbf{p}^2}{k^2} + \frac{\omega^2}{k^2 c_0^2}\right), \tag{132}$$

where $r(Y) = (e^Y - 1)^{-1}$. The k-dependent variable $Z_{A,k}$ is defined below. A natural choice for the velocity c_0 would be the actual (k-dependent) velocity of the Goldstone mode. In the weak coupling limit, however, the Goldstone mode velocity renormalizes only weakly and is well approximated by the k-independent value $c_0 = \sqrt{g\bar n/m}$.

1. Derivative expansion and infrared behavior

The infrared regulator ensures that the vertices are regular functions of p for $|\mathbf{p}| \ll k$ and $|\omega|/c \ll k$ even when they become singular functions of $(\mathbf{p},i\omega)$ at $k = 0$ ($c \equiv c_k \simeq c_{k=0}$ is the velocity of the Goldstone mode). In the low-energy limit $|\mathbf{p}|,|\omega|/c \ll k$, we can therefore use a derivative expansion of the average effective action. We consider the ansatz

$$\begin{aligned} \Gamma_k[\phi^*,\phi] = \int dx \bigg[&\phi^*\left(Z_{C,k}\partial_\tau - V_{A,k}\partial_\tau^2 - \frac{Z_{A,k}}{2m}\nabla^2\right)\phi \\ &+ \frac{\lambda_k}{2}(n - n_{0,k})^2 \bigg] \end{aligned} \tag{133}$$

($n = |\phi|^2$), which is similar to the one used in the classical O(N) model. $n_{0,k}$ denotes the condensate density in the equilibrium state. Note that we have introduced a second-order time derivative term. Although not present in the initial average effective action Γ_Λ, we shall see that this term plays a crucial role when $d \leqslant 3$ [46,48]. As pointed out in Sec. II D, the derivative expansion gives access only to the low-energy limit $|\mathbf{p}|,|\omega|/c \ll k$ of the correlation functions. It is, however, possible to extract the p dependence of the correlation functions by stopping the flow at $k \sim (\mathbf{p}^2 + \omega^2/c^2)^{1/2}$ [8].

In a broken symmetry state with order parameter $\phi_1 = \sqrt{2n_0}$, $\phi_2 = 0$, the two-point vertex is given by

$$\begin{aligned} \Gamma^{(2)}_{k,11}(p) &= V_{A,k}\omega^2 + Z_{A,k}\epsilon_{\mathbf{p}} + 2\lambda_k n_{0,k}, \\ \Gamma^{(2)}_{k,22}(p) &= V_{A,k}\omega^2 + Z_{A,k}\epsilon_{\mathbf{p}}, \\ \Gamma^{(2)}_{k,12}(p) &= Z_{C,k}\omega. \end{aligned} \tag{134}$$

Using (82), we then find

$$\begin{aligned} \Sigma_{k,\rm n}(p) &= G_0^{-1}(p) + \tfrac{1}{2}\big[\Gamma^{(2)}_{k,11}(p) + \Gamma^{(2)}_{k,22}(p)\big] - i\Gamma^{(2)}_{k,12}(p) \\ &= \mu + V_{A,k}\omega^2 + (1 - Z_{C,k})i\omega \\ &\quad - (1 - Z_{A,k})\epsilon_{\mathbf{p}} + \lambda_k n_{0,k} \end{aligned} \tag{135}$$

and

$$\Sigma_{k,\rm an}(p) = \tfrac{1}{2}\big[\Gamma^{(2)}_{k,11}(p) - \Gamma^{(2)}_{k,22}(p)\big] = \lambda_k n_{0,k}. \tag{136}$$

At the initial stage of the flow, $Z_{A,\Lambda} = Z_{C,\Lambda} = 1$, $V_{A,\Lambda} = 0$, $\lambda_\Lambda = g$, and $n_{0,\Lambda} = \mu/g$, which reproduces the results of the Bogoliubov approximation.

Since the anomalous self-energy $\Sigma_{k=0,\rm an}(p) \sim (\omega^2 + c^2\mathbf{p}^2)^{(3-d)/2}$ is singular for $|\mathbf{p}|,|\omega|/c \ll p_G$ and $d \leqslant 3$, we expect $\Sigma_{k,\rm an}(p = 0) \sim k^{3-d}$ for $k \ll p_G$ [given the equivalence between k and $(\mathbf{p}^2 + \omega^2/c^2)^{1/2}$], that is,

$$\lambda_k \sim k^{3-d}. \tag{137}$$

The hypothesis (137) is sufficient, when combined with Galilean and gauge invariances, to obtain the exact infrared behavior of the propagator. Furthermore, we shall see that it is internally consistent. In the domain of validity of the derivative expansion, $|\mathbf{p}|^2,|\omega|^2/c^2 \ll k^2 \ll k^{3-d}$ for $k \to 0$, one obtains from (134)

$$\begin{aligned} G_{k,11}(p) &= \frac{1}{2\lambda_k n_{0,k}}, \\ G_{k,22}(p) &= \frac{1}{V_{A,k}} \frac{1}{\omega^2 + c_k^2 \mathbf{p}^2}, \\ G_{k,12}(p) &= -\frac{Z_{C,k}}{2\lambda_k n_{0,k} V_{A,k}} \frac{\omega}{\omega^2 + c_k^2 \mathbf{p}^2}, \end{aligned} \quad (138)$$

where

$$c_k = \left(\frac{Z_{A,k}/2m}{V_{A,k} + Z_{C,k}^2/2\lambda_k n_{0,k}} \right)^{1/2} \quad (139)$$

is the velocity of the Goldstone mode. From (137) and (138), we recover the divergence of the longitudinal susceptibility if we identify k with $(\mathbf{p}^2 + \omega^2/c^2)^{1/2}$.

The parameters $Z_{A,k}$, $Z_{C,k}$, and $V_{A,k}$ can be related to thermodynamic quantities using Ward identities [8,13,44,53],

$$\begin{aligned} n_{s,k} &= Z_{A,k} n_{0,k} = \bar{n}_k, \\ V_{A,k} &= -\frac{1}{2n_{0,k}} \frac{\partial^2 U_k}{\partial \mu^2}\bigg|_{n_{0,k}}, \\ Z_{C,k} &= -\frac{\partial^2 U_k}{\partial n \partial \mu}\bigg|_{n_{0,k}} = \lambda_k \frac{dn_{0,k}}{d\mu}, \end{aligned} \quad (140)$$

where $\bar{n}_k$ is the mean boson density and $n_{s,k}$ the superfluid density. Here we consider the effective potential U_k as a function of the two independent variables n and μ. The first of equations (140) states that in a Galilean invariant superfluid at zero temperature, the superfluid density is given by the full density of the fluid [13]. Equations (140) also imply that the Goldstone mode velocity c_k coincides with the macroscopic sound velocity [8,13,44], that is,

$$\frac{d\bar{n}_k}{d\mu} = \frac{\bar{n}_k}{mc_k^2}. \quad (141)$$

Since thermodynamic quantities, including the condensate "compressibility" $dn_{0,k}/d\mu$ should remain finite in the $k \to 0$ limit, we deduce from (140) that $Z_{C,k} \sim \lambda_k \sim k^{3-d}$ vanishes in the infrared limit, and

$$\lim_{k\to 0} c_k = \lim_{k\to 0} \left(\frac{Z_{A,k}}{2mV_{A,k}} \right)^{1/2}. \quad (142)$$

Both $Z_{A,k} = \bar{n}_k/n_{0,k}$ and the macroscopic sound velocity c_k being finite at $k = 0$, $V_{A,k}$ (which vanishes in the Bogoliubov approximation) takes a nonzero value when $k \to 0$.

The suppression of $Z_{C,k}$, together with the finite value of $V_{A,k=0}$, shows that the effective action (133) exhibits a "relativistic" invariance in the infrared limit and therefore becomes equivalent to that of the classical O(2) model in dimensions $d + 1$ [54]. In the ordered phase, the coupling constant of this model vanishes as $\lambda_k \sim k^{4-(d+1)}$ (see Sec. II D), which is nothing but our starting assumption (137). For $k \to 0$, the existence of a linear spectrum is due to the relativistic form of the average effective action (rather than a nonzero value of $\lambda_k n_{0,k}$ as in the Bogoliubov approximation). To neglect the term $Z_{C,k}\partial_\tau$ in the average effective action (133) (and therefore obtain a relativistic symmetry), it is necessary that $\lambda_k \gg k^2$ [8], a condition that is related to the singularity of the self-energies in the limit $p \to 0$. Thus we recover the fact that singular self-energies are crucial to obtain a linear spectrum despite the vanishing of the anomalous self-energy.

To obtain the limit $k = 0$ of the propagators (at fixed p), one should, in principle, stop the flow when $k \sim (\mathbf{p}^2 + \omega^2/c^2)^{1/2}$. Since thermodynamic quantities are not expected to flow in the infrared limit, they can be approximated by their $k = 0$ values. As for the longitudinal correlation function, its value is obtained from the replacement $\lambda_k \to C(\omega^2 + c^2\mathbf{p}^2)^{(3-d)/2}$ (with C a constant). From (138) and (140), we then deduce the exact infrared behavior of the normal and anomalous propagators (at $k = 0$),

$$\begin{aligned} G_{\rm n}(p) &= -\frac{n_0 mc^2}{\bar{n}} \frac{1}{\omega^2 + c^2\mathbf{p}^2} \\ &\quad - \frac{mc^2}{\bar{n}} \frac{dn_0}{d\mu} \frac{i\omega}{\omega^2 + c^2\mathbf{p}^2} - \frac{1}{2} G_{11}(p), \\ G_{\rm an}(p) &= \frac{n_0 mc^2}{\bar{n}} \frac{1}{\omega^2 + c^2\mathbf{p}^2} - \frac{1}{2} G_{11}(p), \end{aligned} \quad (143)$$

where

$$G_{11}(p) = \frac{1}{2n_0 C(\omega^2 + c^2\mathbf{p}^2)^{(3-d)/2}}. \quad (144)$$

The hydrodynamic approach in Sec. III B correctly predicts the leading terms in (143) but approximates $dn_0/d\mu$ by $\bar{n}/mc^2$. In contrast, it gives an explicit expression of the coefficient C in the longitudinal correlation function (144).

2. *Renormalization-group flows*

The conclusions of the preceding section can be obtained more rigorously from the RG equation satisfied by the average effective action. The dimensionless variables

$$\begin{aligned} \tilde{n}_{0,k} &= k^{-d} Z_{C,k} n_{0,k}, \\ \tilde{\lambda}_k &= k^d \epsilon_k^{-1} Z_{A,k}^{-1} Z_{C,k}^{-1} \lambda_k, \\ \tilde{V}_{A,k} &= \epsilon_k Z_{A,k} Z_{C,k}^{-2} V_{A,k} \end{aligned} \quad (145)$$

satisfy the RG equations

$$\begin{aligned} \partial_t \tilde{n}_{0,k} &= -(d + \eta_{C,k})\tilde{n}_{0,k} + \frac{3}{2}\tilde{I}_{k,ll} + \frac{1}{2}\tilde{I}_{k,tt}, \\ \partial_t \tilde{\lambda}_k &= (d - 2 + \eta_{A,k} + \eta_{C,k})\tilde{\lambda}_k \\ &\quad - \tilde{\lambda}_k^2 [9\tilde{J}_{k;ll,ll}(0) - 6\tilde{J}_{k;lt,lt}(0) + \tilde{J}_{k;tt,tt}(0)], \\ \eta_{A,k} &= 2\tilde{\lambda}_k^2 \tilde{n}_{0,k} \frac{\partial}{\partial y} [\tilde{J}_{k;ll,tt}(p) + \tilde{J}_{k;tt,ll}(p) + 2\tilde{J}_{k;lt,lt}(p)]_{p=0}, \\ \eta_{C,k} &= -2\tilde{\lambda}_k^2 \tilde{n}_{0,k} \frac{\partial}{\partial \tilde{\omega}} [\tilde{J}_{k;tt,lt}(p) - \tilde{J}_{k;lt,tt}(p) \\ &\quad - 3\tilde{J}_{k;ll,lt}(p) + 3\tilde{J}_{k;lt,ll}(p)]_{p=0}, \\ \partial_t \tilde{V}_{A,k} &= (2 - \eta_{A,k} + 2\eta_{C,k})\tilde{V}_{A,k} - 2\tilde{\lambda}_k^2 \tilde{n}_{0,k} \frac{\partial}{\partial \tilde{\omega}^2} \\ &\quad \times [\tilde{J}_{k;ll,tt}(p) + \tilde{J}_{k;tt,ll}(p) + 2\tilde{J}_{k;lt,lt}(p)]_{p=0}, \end{aligned} \quad (146)$$

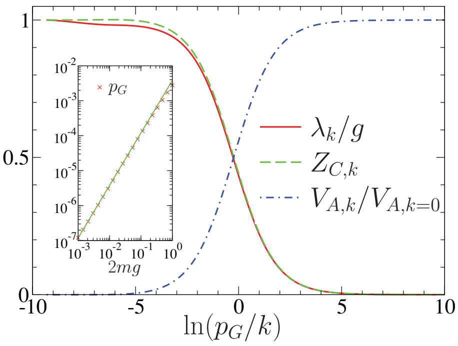

FIG. 6. (Color online) λ_k, $Z_{C,k}$, and $V_{A,k}$ vs $\ln(p_G/k)$, where $p_G = \sqrt{(gm)^3\bar{n}}/4\pi$ for $\bar{n} = 0.01$, $2mg = 0.1$, and $d = 2$ [$\ln(p_G/p_c) \simeq -5.87$]. Inset: p_G vs $2mg$ obtained from the criterion $V_{A,p_G} = V_{A,k=0}/2$. The solid (green) line is a fit to $p_G \sim (2mg)^{3/2}$.

where $\eta_{A,k} = -\partial_t \ln Z_{A,k}, \eta_{C,k} = -\partial_t \ln Z_{C,k}, y = \mathbf{p}^2/k^2$, and $\tilde{\omega} = \omega Z_{C,k}/Z_{A,k}\epsilon_k$. The definition of the threshold functions $\tilde{I}$ and $\tilde{J}$ can be found in Ref. [8].

The flow of λ_k, $Z_{C,k}$, and $V_{A,k}$ is shown in Fig. 6 for a two-dimensional system in the weak-coupling limit. We clearly see that the Bogoliubov approximation breaks down at a characteristic momentum scale $p_G \sim \sqrt{(gm)^3\bar{n}}$. In the Goldstone regime $k \ll p_G$, we find that both λ_k and $Z_{C,k}$ vanish linearly with k, in agreement with the conclusions in Sec. III C 1. Furthermore, $V_{A,k}$ takes a finite value in the limit $k \to 0$, in agreement with the limiting value (142) of the Goldstone mode velocity. Figure 7 shows the behavior of the condensate density $n_{0,k}$, the superfluid density $n_{s,k} = Z_{A,k}n_{0,k}$, and the velocity c_k. Since $Z_{A,k=0} \simeq 1.004$, the mean boson density $\bar{n}_k = n_{s,k}$ is nearly equal to the condensate density $n_{0,k}$. Apart from a slight variation at the beginning of the flow, $n_{0,k}, n_{s,k} = Z_{A,k}n_{0,k}$, and c_k do not change with k. In particular, they are not sensitive to the Ginzburg scale p_G. This result is quite remarkable for the Goldstone mode velocity c_k, whose expression, (139), involves the parameters λ_k, $Z_{C,k}$, and $V_{A,k}$, which all strongly vary when $k \sim p_G$. These findings are a nice illustration of the fact that the divergence of the longitudinal susceptibility does not affect local gauge-invariant quantities [8,44].

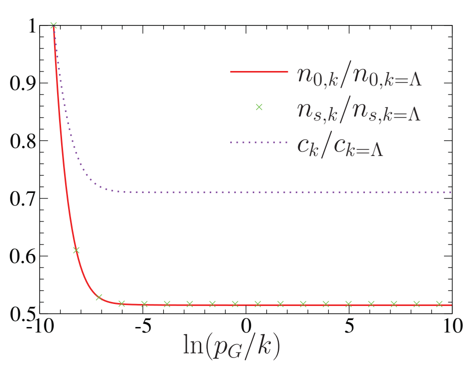

FIG. 7. (Color online) Condensate density $n_{0,k}$, superfluid density $n_{s,k}$, and Goldstone mode velocity c_k vs $\ln(p_G/k)$. Parameters are the same as in Fig. 6.

3. Analytical results in the infrared limit

In the Goldstone regime $k \ll p_G$, the physics is dominated by the Goldstone (phase) mode and longitudinal fluctuations can be ignored. If we take the regulator (132) with $r(Y) = \frac{1-Y}{Y}\Theta(Y)$, the threshold functions $\tilde{I}$ and $\tilde{J}$ can be computed exactly, and one obtains [8]

$$\begin{aligned} \partial_t \tilde{n}_{0,k} &= -(d + \eta_{C,k})\tilde{n}_{0,k}, \\ \partial_t \tilde{\lambda}_k &= (d - 2 + \eta_{C,k})\tilde{\lambda}_k + 8\frac{v_{d+1}}{d+1}\frac{\tilde{\lambda}_k^2}{\tilde{V}_{A,k}^{1/2}}, \\ \eta_{C,k} &= -8\frac{v_{d+1}}{d+1}\frac{\tilde{\lambda}_k}{\tilde{V}_{A,k}^{1/2}}, \\ \partial_t \tilde{V}_{A,k} &= (2 + 2\eta_{C,k})\tilde{V}_{A,k}, \end{aligned} \tag{147}$$

while $\eta_{A,k} \simeq 0$. The first and last of these equations can be rewritten as $n_{0,k} = n_{0,k=0}$ and $V_{A,k} = V_{A,k=0}$, respectively. From (147), we deduce

$$\partial_t \tilde{\lambda}_k = (1 - \epsilon)\tilde{\lambda}_k, \quad \partial_t \eta_{C,k} = -\epsilon \eta_{C,k} - \eta_{C,k}^2, \tag{148}$$

where $\epsilon = 3 - d$. For $d < 3$, this yields $\tilde{\lambda}_k \sim k(1 - \epsilon)$ and

$$\lim_{k\to 0} \eta_{C,k} = -\epsilon, \tag{149}$$

that is, $\lambda_k, Z_{C,k} \sim k^\epsilon$, in agreement with the numerical results in Sec. III C 2 and the analysis in Sec. III C 1. The anisotropy between time and space in the Goldstone regime $k \ll p_G$ (where the average effective action takes a relativistic form) can be eliminated by an appropriate rescaling of frequencies of fields. This leads to an isotropic relativistic model with dimensionless condensate density and coupling constant defined by [8]

$$\tilde{n}'_{0,k} = \sqrt{\tilde{V}_{A,k}}\tilde{n}_{0,k}, \quad \tilde{\lambda}'_k = \frac{\tilde{\lambda}_k}{\sqrt{\tilde{V}_{A,k}}}. \tag{150}$$

(see Fig. 8). $\tilde{\lambda}'_k$ satisfies the RG equation

$$\partial_t \tilde{\lambda}'_k = -\epsilon \tilde{\lambda}'_k + 8\frac{v_{d+1}}{d+1}\tilde{\lambda}'^2_k, \tag{151}$$

which is nothing but the RG equation of the coupling constant of the classical O(2) model in dimensions $d + 1$ [Eq. (71)].

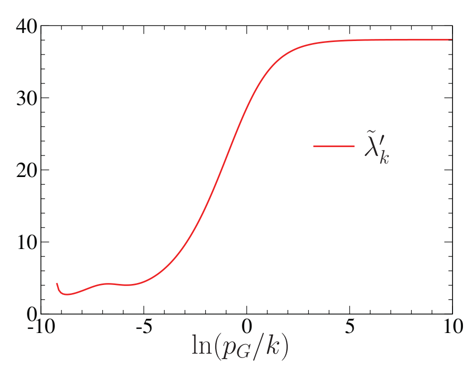

FIG. 8. (Color online) $\tilde{\lambda}'_k$ vs $\ln(p_G/k)$ [Eq. (150)]. Parameters are the same as in Fig. 6.

The corresponding fixed-point value can be deduced from (74) [55]. In the infrared limit, we find

$$\lambda_k = k^{-d}(Z_{A,k}\epsilon_k)^{3/2}V_{A,k}^{1/2}\tilde{\lambda}'_k \sim k^{\epsilon}\tilde{\lambda}'_k \qquad (152)$$

if we approximate $Z_{A,k} \simeq Z_{A,k=0}$ and $V_{A,k} \simeq V_{A,k=0}$. The vanishing of $\lambda_k \sim k^{\epsilon}$ and the divergence of the longitudinal susceptibility is therefore the consequence of the existence of a fixed point $\tilde{\lambda}'^{*}_k$ for the coupling constant of the effective $(d+1)$–dimensional O(2) model that describes the Goldstone regime $k \ll p_G$. To describe the entire hydrodynamic regime $k \ll p_c$, we should in principle relax the assumption $V_{A,k} \simeq V_{A,k=0}$, since $V_{A,k}$ strongly varies for $k \sim p_G$, which makes the analytical solution of the RG equations much more difficult. In Ref. [51], it was shown that Eq. (151) is nevertheless in good agreement with the numerical solution of the flow equations in the entire hydrodynamic regime. We can then use (76) to obtain the Ginzburg momentum scale

$$p_G \simeq \left[\frac{8v_{d+1}g}{(d+1)\epsilon}\right]^{1/\epsilon} \qquad (153)$$

in the weak-coupling limit, which agrees with the results in Secs. III A and III B.

IV. CONCLUSION

In conclusion, we have studied the classical linear O(N) model and zero-temperature interacting bosons using a variety of techniques: perturbation theory, hydrodynamic approach, large-N limit, and NPRG. We have shown that in the weak-coupling limit these two systems can be described along similar lines. They are characterized by two momentum scales, the hydrodynamic scale (or healing scale for bosons) p_c and the Ginzburg scale p_G. For momenta $|\mathbf{p}| \ll p_c$, we can use a hydrodynamic description in terms of the amplitude and direction of the vector field $\boldsymbol{\varphi}$ in the O(N) model or the density and phase in interacting boson systems. The hydrodynamic description allows us to derive the order parameter correlation function without encountering infrared divergences. In the Goldstone regime $|\mathbf{p}| \ll p_G$, amplitude (density) fluctuations no longer play a role and both the transverse and the longitudinal correlation functions are fully determined by direction (phase) fluctuations. In this momentum range, the coupling between transverse and longitudinal fluctuations leads to a divergence of the longitudinal susceptibility and singular self-energies. A direct computation of the order parameter correlation function (without relying on the hydrodynamic description) is possible, but one then has to solve the problem of infrared divergences that appear in perturbation theory when $|\mathbf{p}| \lesssim p_G$ and signal the breakdown of the Gaussian approximation. The NPRG provides a natural framework for such a calculation. In the case of bosons, it shows that in the Goldstone regime $|\mathbf{p}|,|\omega|/c \ll p_G$, the system is described by an effective action with relativistic invariance similar to that of the $(d+1)$–dimensional classical O(2) model.

These strong similarities between the classical linear O(N) model and zero-temperature interacting bosons disappear in the strong-coupling limit. For the O(N) model, this limit corresponds to the critical regime near the phase transition, which has no direct analog in zero-temperature interacting boson systems. The only approach that one can hope to extend to strongly correlated bosons is the NPRG. Recent progress in that direction, based on the Bose-Hubbard model, is reported in Ref. [56].

ACKNOWLEDGMENT

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APPENDIX : THRESHOLD FUNCTIONS

The threshold functions appearing in the NPRG equations for the O(N) model (Sec. II D) are defined by

$$\begin{aligned} I_\alpha &= -\int_{\mathbf{q}} \dot{R}(\mathbf{q})G_\alpha^2(\mathbf{q}),\\ J_{\alpha\beta}(\mathbf{p}) &= -\int_{\mathbf{q}} \dot{R}(\mathbf{q})G_\alpha^2(\mathbf{q})G_\beta(\mathbf{p}+\mathbf{q}),\\ J'_{\alpha\beta}(\mathbf{p}) &= \partial_{\mathbf{p}^2}J_{\alpha\beta}(\mathbf{p}). \end{aligned} \qquad (A1)$$

where $\alpha,\beta \in \{l,t\}$. To alleviate the notations, we drop the k index. In dimensionless form,

$$\begin{aligned} \tilde{I}_\alpha &= 2v_d\int_0^\infty dy y^{d/2}(\eta r + 2yr')\tilde{G}_\alpha^2,\\ \tilde{J}_{\alpha\beta}(0) &= 2v_d\int_0^\infty dy y^{d/2}(\eta r + 2yr')\tilde{G}_\alpha^2\tilde{G}_\beta, \end{aligned} \qquad (A2)$$

$$\begin{aligned} \tilde{J}'_{\alpha\beta}(0) = 4\frac{v_d}{d}\int_0^\infty dy y^{d/2}\{&[\eta r + (\eta+4)yr' + 2y^2r'']\tilde{G}_\alpha^2\\ &-2(1+r+yr')(\eta r+2yr')\tilde{G}_\alpha^3\}(1+r+yr')\tilde{G}_\beta^2, \end{aligned}$$

where

$$\begin{aligned} \tilde{G}_l &= \frac{1}{y(1+r)+2\tilde{\lambda}\tilde{\rho}_0},\\ \tilde{G}_t &= \frac{1}{y(1+r)}, \end{aligned} \qquad (A3)$$

and we have written the cutoff function as $R_k(\mathbf{p}) = Z_k\mathbf{p}^2r(y)$, with $y = \mathbf{p}^2/k^2$ and $r(y)$ a k independent function. For the Θ cutoff function introduced in Sec. II D 2, $r = \frac{1-y}{y}\Theta(1-y)$, and the threshold functions can be computed analytically,

$$\begin{aligned} \tilde{I}_\alpha &= -8\frac{v_d}{d}\left(1-\frac{\eta}{d+2}\right)\tilde{A}_\alpha^2,\\ \tilde{J}_{\alpha\beta}(0) &= -8\frac{v_d}{d}\left(1-\frac{\eta}{d+2}\right)\tilde{A}_\alpha^2\tilde{A}_\beta,\\ \tilde{J}'_{\alpha\beta}(0) &= 4\frac{v_d}{d}\tilde{A}_l^2, \end{aligned} \qquad (A4)$$

where

$$\tilde{A}_l = \frac{1}{1+2\tilde{\lambda}\tilde{\rho}_0}, \quad \tilde{A}_t = 1. \qquad (A5)$$

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